General implementation of EARSM turbulence model

M. Schmelzer

February 21, 2020

1 Turbulence model k-ω SST

The incompressible and constant-density RANS equations read

$$\partial_i U_i = 0,$$
$$U_j \partial_j U_i = \partial_j \left[-\frac{1}{\rho} P + \nu \partial_j U_i - \tau_{ij} \right], \tag{1}$$

where U_i is the mean velocity, ρ is the constant density, P is the mean pressure and ν is the kinematic viscosity. The Reynolds-stress τ_{ij} is the subject of modelling. This symmetric, second-order tensor field can be decomposed into an anisotropic $a_{ij} = 2kb_{ij}$ and isotropic part $\frac{2}{3}k\delta_{ij}$

$$\tau_{ij} = 2k \left(b_{ij} + \frac{1}{3}\delta_{ij} \right), \tag{2}$$

in which the baseline model, $b^o_{ij} = -\frac{\nu_t}{k} S_{ij}$, forms a linear relation between anisotropy and the mean-strain rate tensor S_{ij} via the scalar eddy viscosity ν_t. Commonly, ν_t is computed using a transport model such as k-ω SST [1], in which k is the turbulent kinetic energy and ω the specific dissipation rate. The constitutive relation is augmented with an additive term b^Δ_{ij} leading to

$$b_{ij} = -\frac{\nu_t}{k} S_{ij} + b^\Delta_{ij}. \tag{3}$$

This additive implementation, in which the Bousinesq term is unaltered, has the benefit of enhanced numerical stability [2]. As a closure model we use the k-ω SST model

$$\partial_t k + U_j \partial_j k = P_k - \beta^* \omega k + \partial_j \left[(\nu + \sigma_k \nu_t) \partial_j k \right], \tag{4}$$
$$\partial_t \omega + U_j \partial_j \omega = \frac{\gamma}{\nu_t} P_k - \beta \omega^2 + \partial_j \left[(\nu + \sigma_\omega \nu_t) \partial_j \omega \right] + CD_{k\omega}, \tag{5}$$

in which the production of turbulent kinetic energy is augmented by b^Δ_{ij} and bounded following Menter's limiter[3]

$$P_k = \min \left(-2k(b^o_{ij} + b^\Delta_{ij}) \partial_j U_i,\ 10\beta^* \omega k \right). \tag{6}$$

The corresponding eddy viscosity is $\nu_t = \frac{a_1 k}{\max(a_1 \omega, S F_2)}$. The other standard terms of k-ω SST read

$$CD_{k\omega} = \max \left(2\sigma_{\omega 2} \frac{1}{\omega} (\partial_i k)(\partial_i \omega), 10^{-10} \right),$$
$$F_1 = \tanh \left[\left(\min \left[\max \left(\frac{\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{y^2 \omega} \right), \frac{4\sigma_{\omega 2} k}{CD_{k\omega} y^2} \right] \right)^4 \right],$$
$$F_2 = \tanh \left[\left(\max \left(\frac{2\sqrt{k}}{\beta^* \omega y}, \frac{500\nu}{y^2 \omega} \right) \right)^2 \right],$$
$$\Phi = F_1 \Phi_1 + (1 - F_1) \Phi_2, \tag{7}$$

in which the latter blends the coefficients $\Phi \to (\Phi_1, \Phi_2)$

$$\alpha = (5/9, 0.44), \beta = (3/40, 0.0828), \sigma_k = (0.85, 1.0), \sigma_\omega = (0.5, 0.856). \tag{8}$$

The remaining terms are $\beta^* = 0.09, a_1 = 0.31$ and $S = \sqrt{2 S_{ij} S_{ij}}$.

1.1 Nonlinear eddy-viscosity model for b_{ij}^{Δ}

In [4], a nonlinear generalisation of the linear eddy viscosity concept was proposed. This concept has been used in several works on data-driven turbulence modelling [5, 6]. The fundamental assumption is made that the anisotropy of the Reynolds-stress b_{ij} not only depends on the strain rate tensor $S_{ij} = \tau\frac{1}{2}(\partial_j U_i + \partial_i U_j)$ but also on the rotation rate tensor $\Omega_{ij} = \tau\frac{1}{2}(\partial_j U_i - \partial_i U_j)$ with the timescale $\tau = 1/\omega$. The Cayley-Hamilton theorem then dictates that the most general form of the anisotropic part of the Reynolds-stress can be expressed as

$$b_{ij}(S_{ij}, \Omega_{ij}) = \sum_{n=1}^{N} T_{ij}^{(n)} \alpha_n(I_1, ..., I_5), \tag{9}$$

with ten nonlinear base tensors $T_{ij}^{(n)}$ and five corresponding invariants I_m. In the following, we only consider two-dimensional flow cases, for which the first three base tensors form a linear independent basis and only the first two invariants are nonzero [7]. Our set of base tensors and invariants reads

$$\begin{aligned} T_{ij}^{(1)} &= S_{ij},\ T_{ij}^{(2)} = S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}, \\ T_{ij}^{(3)} &= S_{ik}S_{kj} - \frac{1}{3}\delta_{ij}S_{mn}S_{nm} \end{aligned} \tag{10}$$

$$I_1 = S_{mn}S_{nm},\ I_2 = \Omega_{mn}\Omega_{nm}. \tag{11}$$

Using this set for (9) we have an ansatz, which only requires functional expressions for the coefficients α_n, to model b_{ij}^{Δ}. For the present work we focus on a library, in which the primitive input features are squared and the resulting candidates are multiplied by each other leading to a maximum degree up to $D = 6$. The number of possible candidates is $\binom{M+D}{D} = \binom{8}{6} = 28$.

$$\begin{array}{lccccccc}
D=0: & 1 & & & & & & \\
D=1: & I_1 & I_2 & & & & & \\
D=2: & I_1^2 & I_1 I_2 & I_2^2 & & & & \\
D=3: & I_1^3 & I_1^2 I_2 & I_1 I_2^2 & I_2^3 & & & \\
D=4: & I_1^4 & I_1^3 I_2 & I_1^2 I_2^2 & I_1 I_2^3 & I_2^4 & & \\
D=5: & I_1^5 & I_1^4 I_2 & I_1^3 I_2^2 & I_1^2 I_2^3 & I_1 I_2^4 & I_2^5 & \\
D=6: & I_1^6 & I_1^5 I_2 & I_1^4 I_2^2 & I_1^3 I_2^3 & I_1^2 I_2^4 & I_1 I_2^5 & I_2^6
\end{array}$$

Given this set of $T_{ij}(n)$ and I_m the full expression of (9) reads

$$\begin{aligned} b_{ij}(S_{ij}, \Omega_{ij}) = {} & \theta_0 T_{ij}^{(1)} + \theta_1 I_1 T_{ij}^{(1)} + \theta_2 I_2 T_{ij}^{(1)} + \cdots + \theta_{26} I_1 I_2^5 T_{ij}^{(1)} + \theta_{27} I_2^6 T_{ij}^{(1)} \\ & + \theta_{28} T_{ij}^{(2)} + \theta_{29} I_1 T_{ij}^{(2)} + \theta_{30} I_2 T_{ij}^{(2)} + \cdots + \theta_{54} I_1 I_2^5 T_{ij}^{(2)} + \theta_{55} I_2^6 T_{ij}^{(2)} \\ & + \theta_{56} T_{ij}^{(3)} + \theta_{57} I_1 T_{ij}^{(3)} + \theta_{58} I_2 T_{ij}^{(3)} + \cdots + \theta_{82} I_1 I_2^5 T_{ij}^{(3)} + \theta_{83} I_2^6 T_{ij}^{(3)}, \end{aligned} \tag{12}$$

in which the coefficients of the vector

$$\mathbf{\Theta} = \Big[\theta_0, \theta_1, \theta_2, \ldots, \theta_{83}\Big]^T \tag{13}$$

need to be determined. The implementation of the Generalized Coefficients for EARSM (`GCEARSM`) uses a vector `Theta`, which contains all coefficients. Full overview of the terms and the corresponding indices in `Theta` are given in the following table and need to be provided in `constant/RASProperties`, e.g. using the provided `python` script.

index	term	index	term	index	term
0	$T_{ij}^{(1)}$	28	$T_{ij}^{(2)}$	56	$T_{ij}^{(3)}$
1	$I_1 T_{ij}^{(1)}$	29	$I_1 T_{ij}^{(2)}$	57	$I_1 T_{ij}^{(3)}$
2	$I_2 T_{ij}^{(1)}$	30	$I_2 T_{ij}^{(2)}$	58	$I_2 T_{ij}^{(3)}$
3	$I_1^2 T_{ij}^{(1)}$	31	$I_1^2 T_{ij}^{(2)}$	59	$I_1^2 T_{ij}^{(3)}$
4	$I_1 I_2 T_{ij}^{(1)}$	32	$I_1 I_2 T_{ij}^{(2)}$	60	$I_1 I_2 T_{ij}^{(3)}$
5	$I_2^2 T_{ij}^{(1)}$	33	$I_2^2 T_{ij}^{(2)}$	61	$I_2^2 T_{ij}^{(3)}$
6	$I_1^3 T_{ij}^{(1)}$	34	$I_1^3 T_{ij}^{(2)}$	62	$I_1^3 T_{ij}^{(3)}$
7	$I_1^2 I_2 T_{ij}^{(1)}$	35	$I_1^2 I_2 T_{ij}^{(2)}$	63	$I_1^2 I_2 T_{ij}^{(3)}$
8	$I_1 I_2^2 T_{ij}^{(1)}$	36	$I_1 I_2^2 T_{ij}^{(2)}$	64	$I_1 I_2^2 T_{ij}^{(3)}$
9	$I_2^3 T_{ij}^{(1)}$	37	$I_2^3 T_{ij}^{(2)}$	65	$I_2^3 T_{ij}^{(3)}$
10	$I_1^4 T_{ij}^{(1)}$	38	$I_1^4 T_{ij}^{(2)}$	66	$I_1^4 T_{ij}^{(3)}$
11	$I_1^3 I_2 T_{ij}^{(1)}$	39	$I_1^3 I_2 T_{ij}^{(2)}$	67	$I_1^3 I_2 T_{ij}^{(3)}$
12	$I_1^2 I_2 T_{ij}^{(1)}$	40	$I_1^2 I_2 T_{ij}^{(2)}$	68	$I_1^2 I_2 T_{ij}^{(3)}$
13	$I_1 I_2^3 T_{ij}^{(1)}$	41	$I_1 I_2^3 T_{ij}^{(2)}$	69	$I_1 I_2^3 T_{ij}^{(3)}$
14	$I_2^4 T_{ij}^{(1)}$	42	$I_2^4 T_{ij}^{(2)}$	70	$I_2^4 T_{ij}^{(3)}$
15	$I_1^5 T_{ij}^{(1)}$	43	$I_1^5 T_{ij}^{(2)}$	71	$I_1^5 T_{ij}^{(3)}$
16	$I_1^4 I_2 T_{ij}^{(1)}$	44	$I_1^4 I_2 T_{ij}^{(2)}$	72	$I_1^4 I_2 T_{ij}^{(3)}$
17	$I_1^3 I_2^2 T_{ij}^{(1)}$	45	$I_1^3 I_2^2 T_{ij}^{(2)}$	73	$I_1^3 I_2^2 T_{ij}^{(3)}$
18	$I_1^2 I_2^3 T_{ij}^{(1)}$	46	$I_1^2 I_2^3 T_{ij}^{(2)}$	74	$I_1^2 I_2^3 T_{ij}^{(3)}$
19	$I_1 I_2^4 T_{ij}^{(1)}$	47	$I_1 I_2^4 T_{ij}^{(2)}$	75	$I_1 I_2^4 T_{ij}^{(3)}$
20	$I_2^5 T_{ij}^{(1)}$	48	$I_2^5 T_{ij}^{(2)}$	76	$I_2^5 T_{ij}^{(3)}$
21	$I_1^6 T_{ij}^{(1)}$	49	$I_1^6 T_{ij}^{(2)}$	77	$I_1^6 T_{ij}^{(3)}$
22	$I_1^5 I_2 T_{ij}^{(1)}$	50	$I_1^5 I_2 T_{ij}^{(2)}$	78	$I_1^5 I_2 T_{ij}^{(3)}$
23	$I_1^4 I_2^2 T_{ij}^{(1)}$	51	$I_1^4 I_2^2 T_{ij}^{(2)}$	79	$I_1^4 I_2^2 T_{ij}^{(3)}$
24	$I_1^3 I_2^3 T_{ij}^{(1)}$	52	$I_1^3 I_2^3 T_{ij}^{(2)}$	80	$I_1^3 I_2^3 T_{ij}^{(3)}$
25	$I_1^2 I_2^4 T_{ij}^{(1)}$	53	$I_1^2 I_2^4 T_{ij}^{(2)}$	81	$I_1^2 I_2^4 T_{ij}^{(3)}$
26	$I_1 I_2^5 T_{ij}^{(1)}$	54	$I_1 I_2^5 T_{ij}^{(2)}$	82	$I_1 I_2^5 T_{ij}^{(3)}$
27	$I_2^6 T_{ij}^{(1)}$	55	$I_2^6 T_{ij}^{(2)}$	83	$I_2^6 T_{ij}^{(3)}$

References

[1] Leschziner, M., *Statistical Turbulence Modelling for Fluid Dynamics - Demystified: An Introductory Text for Graduate Engineering Students*, Imperial College Press, 2015.

[2] Weatheritt, J. and Sandberg, R. D., "Hybrid Reynolds-Averaged/Large-Eddy Simulation Methodology from Symbolic Regression: Formulation and Application," *AIAA Journal*, 2017, pp. 1–13.

[3] Menter, F. R., Kuntz, M., and Langtry, R., "Ten Years of Industrial Experience with the SST Turbulence Model Turbulence heat and mass transfer," *Turbulence, Heat and Mass Transfer 4*, 2003.

[4] Pope, S. B., "A more general effective-viscosity hypothesis," *Journal of Fluid Mechanics*, Vol. 72, No. 2, 1975, pp. 331–340.

[5] Xiao, H. and Cinnella, P., "Quantification of Model Uncertainty in RANS Simulations: A Review," *Progress in Aerospace Sciences*, 2019.

[6] Duraisamy, K., Iaccarino, G., and Xiao, H., "Turbulence Modeling in the Age of Data," *Annual Review of Fluid Mechanics*, Vol. 51, No. 1, 2019.

[7] Gatski, T. and Jongen, T., "Nonlinear eddy viscosity and algebraic stress models for solving complex turbulent flows," *Progress in Aerospace SciencesAerospace Sciences*, Vol. 36, 2000, pp. 655–682.