Numerical Linear Algebra Workshop

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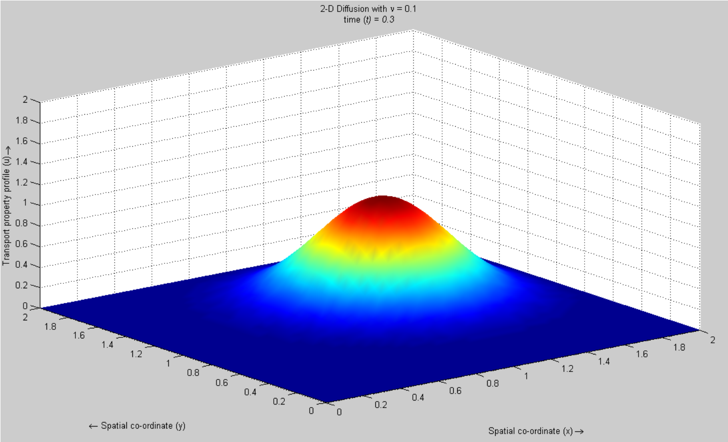
Overview

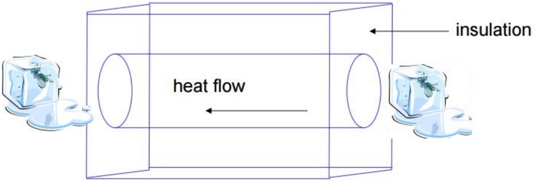
Diffusion Equations

Diffusive Transport: Dye in water, pollution, heat, perfume...

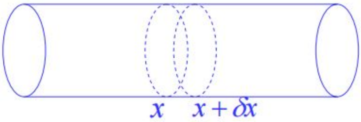


Diffusion Equations Cont'd

- Consider temperature in a long thin tube of constant cross section.
- The tube is perfectly insulated laterally. Heat only flow along the tube.
- Its ends maintain at zero temperature.



Diffusion Equations Cont'd

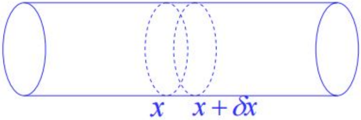


Suppose

- thermal conductivity in the wire is K.
- Cross sectional area is A.
- Material density ρ.
- Heat capacity is σ.
- Temperature at point x at time t is $u(x, t)$.

Then the heat flow into bar across face at x : $-KA\frac{\partial u}{\partial x}|_x$.
At the face $x + \delta x$: $-KA\frac{\partial u}{\partial x}|_{x+\delta x}$

Diffusion Equations Cont'd



- The net flow out is: $KA\frac{\partial^2 u}{\partial x^2}\delta x$
- $Q = \sigma m \Delta T$
- So, the conservation of heat gives: $KA\frac{\partial^2 u}{\partial x^2}\delta x = \sigma\rho A\frac{\partial u}{\partial t}\delta x$

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$

Boundary Conditions

- Homogeneous Dirichlet Boundary Condition: $u(0,t) = u(L,t) = 0$
- Homogeneous Neumann Boundary Condition: $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

Miscellaneous Equations

- Navier-Stokes Equation: $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \gamma \nabla^2 \mathbf{u} + \frac{1}{\rho}\mathbf{F}$
- Fisher's Equation: $\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2}$
- Nonlinear Schrodinger Equation: $i\partial_t \phi = -\frac{1}{2}\partial_x^2 \phi + \kappa |\phi|^2 \phi$
- Black-Scholes Equation: $\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$
- etc

Finite Difference Method

We consider the two-point boundary value problem:

$$Au: \quad = -au'' + bu' + cu = f \ \ in\Omega = (0,1), \tag{1}$$

$$u(0) = u_0, \ \ u(1) = u_1 \tag{2}$$

where the coefficients $a = a(x), b = b(x)$,and $c = c(x)$ are smooth functions satisfying $a(x) > 0$ and $c(x) \geq 0$ in $\overline{\Omega}$. And f, u_0, u_1 are given.

To find numerical solution of (2) we introduce $M+1$ grid points $0 = x_0 < x_1 < ... < x_M = 1$ by setting $x_j = jh, j = 0, ..., M$, where $h = 1/M$. We denote the approximation of $u(x_j)$ by U_j and use the following finite difference approximation for derivatives.

$$
\begin{aligned}
\partial U_j &= \frac{U_{j+1} - U_j}{h}, (forward\ \ difference) \\
\partial \overline{U_j} &= \frac{U_j - U_{j-1}}{h}, (backward\ \ difference) \\
\widehat{\partial} U_j &= \frac{U_{j+1} - U_{j-1}}{2h}, (central\ \ difference) \\
\partial \overline{\partial} U_j &= \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}
\end{aligned}
$$

Setting also $a_j = a(x_j), b_j = b(x_j), c_j = c(x_j), f_j = f(x_j)$, we now define a finite difference approximation of (2) by

$$\begin{aligned} A_h U_j &:= -a_j \partial\bar{\partial} U_j + b_j \widehat{\partial} U_j + c_j U_j = f_j, \text{for } j = 1, \ldots, M-1, \quad (3) \\ & U_0 = u_0,\ U_M = u_1. \quad (4) \end{aligned}$$

Then, after simplification, the equation at the interior point x_j may be written as

$$(2a_j + h^2 c_j)U_j - (a_j - \frac{1}{2}hb_j)U_{j+1} - (a_+ \frac{1}{2}hb_j)U_{j-1} = h^2 f_j \quad (5)$$

for all j.

Put (5)into a matrix form:

$$\boxed{AU = g}$$

We finally comes to LINEAR ALGEBRA!! OH YEAH!!

In our system $AU = g$:
$U = (U_1, \ldots, U_{M-1})^T$ and the first and last components of the vector $g = (g_1, \ldots, g_{M-1})^T$ contain contributions from the boundary values u_0, u_1 as well as f_1 and f_{M-1}, respectively. The $(M-1) \times (M-1)$ matrix A is tridiagonal and diagonally dominant for h sufficiently small.

Finite Element Method for BVP

We consider the special case $b = 0$ of the two-point boundary value problem of (2),

$$Au := -(au')' + cu = f \ in \ \Omega := (0, 1), \ with u(0) = u(1) = 0,$$

where $a = a(x), c = c(x)$ are smooth functions with $a(x) \geq q_0 > 0$, $c(x) \geq 0$ in $\overline{\Omega}$ and $f \in L_2 = L_2(\Omega)$.

Recall the variational formulation of this problem is to find $u \in H_0^1$ such that

$$a(u, \phi) = (f, \phi), \ \forall \phi \in H_0^1,$$

where

$$a(v, w) = \int_\Omega (av'w' + cvw)dx \ and \ (f, v) = \int_\Omega fvdx,$$

and that this problem has a unique solution $u \in H^2$.

For the purpose of finding an approximate solution of (15) we introduce a partition of Ω,

$$0 = x_0 < x_1 < \ldots < x_M = 1,$$

and set

$$h_j = x_j - x_{j-1}, K_j = [x_{j-1}, x_j], \ \textit{for} \ j = 1, \ldots, M, \ \textit{and} \ h = \max_j h_j.$$

The discrete solution will be sought in the finite-dimensional space of functions

$$S_h = v \in C = C(\overline{\Omega}) : v \ linear \ on \ each \ K_j, v(0) = v(1) = 0.$$

The set $\{\Phi_i\}_{i=1}^{M-1} \subset S_h$ is defined by

$$\Phi_i(x_j) = \begin{cases} 1 & if \ \ i = j \\ 0 & if \ \ i \neq j \end{cases}$$

and any $v \in S_h$ may be written as

$$v(x) = \sum_{i=1}^{M-1} v_i \Phi_i(x), \ with \ v_i = v(x_i).$$

Now we pose the finite-dimensional problem to find $u_h \in S_h$ such that

$$a(u_h, \chi) = (f, \chi), \ \forall \chi \in S_h. \tag{6}$$

In terms of the basis $\{\Phi_i\}_{i=1}^{M-1}$ we write $u_h(x) = \sum_{j-1}^{M-1} U_j \Phi_j(x)$ and insert this into (6) to find that this equation is equivalent to

$$\sum_{j=1}^{M-1} U_j a(\Phi_j, \Phi_i) = (f, \Phi_i), \ for \ i = 1, \ldots, M-1.$$

This linear system of equations could be expressed in matrix form as

$$\boxed{AU = b}$$

Finished!!!

In our system $U = (U_j), A = (a_{ij})$ is the stiffness matrix with elements $a_{ij} = a(\Phi_j, \Phi_i)$, and $b = (b_i)$ with elements $b_i = (f, \Phi_i)$. The matrix A is symmetric and positive definite, because for $V = (V_i)$ and $v(x) = \sum_{i=1}^{M-1} V_i \Phi_i(x)$ we have

$$V^T AV = \sum_{i,j=1}^{M-1} V_i a_{ij} V_j = a\Big(\sum_{j=1}^{M-1} V_j \Phi_j, \sum_{i=1}^{M-1} V_i \Phi_i\Big) = a(v, v) \geq a_0 ||v'||^2,$$

and hence $V^T AV = 0$ implies $v' = 0$, so that v is 0. Matrix A is tridiagonal since$a_{ij} = 0$ when x_i and x_j are not neighbors, i.e., when $|i - j| \geq 2$.

Miscellaneous Numerical Methods

- Finite Volume Method
- Spectral Method
- Meshfree Methods
- Multigrid
- etc.

$$\boxed{AU = b}$$

Blocks of Highlighted Text

Block 1

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Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

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Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
\frametitle{Theorem}
\begin{theorem}[Mass--energy equivalence]
$E = mc^2$
\end{theorem}
\end{frame}
```

Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Erwin Kreyszig, 2014].

References

Erwin Kreyszig, Advanced Engineering Mathematics, 8th Edn, Sections 11.4b

The End