Existence and Uniqueness of Solutions using Contraction Mapping

Shamanth Sreekanth

Theorem

Existence and Uniqueness via Contraction Mapping
Consider a system of n linear equations with n variables, represented by:

$$Ax = b$$

where $x, b \in \mathbb{R}^n$ and A is an $n \times n$ real matrix. Let the vector space $\mathbb{R}^n$ consist of vector norm $|| \cdot ||$. If the matrix norm of $(I - A)$ is strictly less than 1,

$$||I - A|| < 1$$

then a solution for the system $Ax = b$ exists and is unique.

Assumptions and Definitions

- X is a non-empty set.
- $T : X \to X$ is a mapping. There exists a fixed point $x \in X$ such that

  $$T(x) = x.$$

- Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction if

  $$d(Tx, Ty) \leq L\, d(x, y), \quad \forall x, y \in X,$$

  where $L \in [0, 1)$ and d is the distance function.
- If (X, d) is a complete metric space, i.e., all Cauchy sequences converge to points strictly within the space (X, d), then $T : X \to X$ has a unique fixed point $x^* \in X$.

  $$\lim_{n\to\infty} T^n x = x^*,$$

  $$d(T^n x, x^*) \leq \frac{L^n}{1 - L} d(Tx, x).$$

  This is called the Banach fixed point theorem.

Proof

Consider a system of linear equations:

$$Ax = b$$

To use a fixed point theorem, we need to successively iterate the transformation T on x. This needs rearranging:

$$\begin{aligned}
0 &= b - Ax \\
x &= x + b - Ax && \text{(Adding } x \text{ on both sides)} \\
x &= Ix + b - Ax && \text{(Identity matrix property)} \\
x &= (I - A)x + b && (1)
\end{aligned}$$

Let us define a mapping $T : X \to X, T(x) = x$ such that

$$Tx = (I - A)x + b \tag{2}$$

(1) can be written in equation form as

$$\begin{aligned}
x_1 &= (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\
x_2 &= -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n + b_2 \\
&\vdots \\
x_n &= -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1 - a_{nn})x_n + b_n
\end{aligned} \tag{3}$$

Through (2), we can solve the system of equations by finding the fixed points of T. Let us assume two arbitrary vectors in (X, d), x and x', undergoing mapping T.

$$\begin{aligned}
Tx &= (I - A)x + b \\
Tx' &= (I - A)x' + b \\
Tx - Tx' &= (I - A)(x - x')
\end{aligned} \tag{4}$$

From the definition of a contraction, $Ax = b$ has a unique solution if

$$||I - A||_\infty \leq L < 1 \quad (L \neq 0)$$

We define the distance function d as

$$d(x, x') = \sup_{1 \leq i \leq n} ||x_i - x'_i||_\infty,$$

Then, we have from (4)

$$\begin{aligned}
d(Tx, Tx') &= \sup_{1 \leq i \leq n} ||Tx_i - Tx'_i|| \\
&= \sup_{1 \leq i \leq n} \left|\left| \sum_{j=1}^{n} (I - A)_{ij}(x_j - x'_j) \right|\right| \\
&\leq \sup_{1 \leq i \leq n} (\sum_{j=1}^{n} ||(I - A)_{ij}||.||x_j - x'_j||) \\
&\leq \left(\sup_{1 \leq i \leq n} \sum_{j=1}^{n} ||(I - A)_{ij}|| \right) d(x, x') \\
&\leq L \cdot d(x, x')
\end{aligned}$$

Therefore, $d(Tx, Tx')$, where $\leq Ld(x, x'), \quad 0 \leq L < 1$. This proves that T is a contraction mapping, and that the linear system has a unique solution at the fixed point.