Chapter 4

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Errata in the text

- p. 67, the last equation:

  error: $\prod_{E^{\mathrm{s}}_{x_{n-j}}} v_{n+j}$

  correction: $\prod_{E^{\mathrm{s}}_{x_{n-j}}} v_{n-j}$

- p. 73, Example 4.75:

  error: A_{ij} if the transition from j to i is possible

  correction1: A_{ij} if the transition from i to j is possible

  corresction2: Take a transposition of A in the following description in the text

- p. 75, caption of Fig. 4.13:

  error: mapped by the cat map f

  correction: mapped by the square root of the cat map f

- p. 76, para. c:

  error: $|j| < n - j_0$

  correction: $j \geq n - j_0$

Comments on the text

On the Theorem 4.68

This is similar to the Ω-stability theorem [1, Ch. 10], however, the statements of the theorems are surely different. I suspect the theorem does not hold in the current form.

In the statement of the theorem, C^0-perturbation of f is permitted. However, C^0-pertubations of f, say, g, can have new hyperbolic fixed points [2, Ch. 2, Fig. 2. 17]. Hence we cannot expect one to one correspondence between periodic points on nonwandering sets.

Moreover, even if we employ more stronger assumption that C^1-perturbation of f is allowed, it is not enough because the Ω-stability theorem for hyperbolic nonwandering sets requires the additional no-cycle condition [1, Ch. 10].

Lastly, I could not find the theorem and its proof in the Bowen's book [3], which is cited in the text.

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1y_2 = y_1(0)y_2(0) = \mathrm{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10.

This answer greatly relies on [4] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \cdots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega,\ \bigcup_{i=1}^{k} \pi_i = \omega,\ \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \tag{0.3}$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \cdots, n\}). \tag{0.4}$$

Then the Faá di Bruno's formula is given as [5, Sec. 2.4][1]

$$D^n (f_1 \circ f_2)(x)\eta_\omega = \sum_{\substack{\pi\in\mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f_1(f_2(x)) \left(D^{|\pi_1|} f_2(x)\eta_{\pi_1}, \ldots, D^{|\pi_k|} f_2(x)\eta_{\pi_k} \right), \tag{0.5}$$

where for any $\pi = (\pi_1, \cdots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \cdots, \omega_n\}$ of $\mathbb{N}$,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \cdots, \eta_{\sigma(\omega_n)}) \tag{0.6}$$

where σ is a permutation and $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n-th derivatives of the both sides of

$$g \circ f^{\mathrm{s}}(\eta, g(\eta)) = f^{\mathrm{u}}(\eta, g(\eta)) \Leftrightarrow g \circ f^{\mathrm{s}} \circ \tilde{g}(\eta) = f^{\mathrm{u}} \circ \tilde{g}(\eta) \tag{0.7}$$

[1] Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$\begin{aligned}
&\text{(l.h.s): } D^n(g\circ f^{\mathrm{s}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k(g\circ f^{\mathrm{s}})(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\right.\\
&\qquad\qquad\left.\dots,D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right) &(0.8)\\
&\text{(r.h.s): } D^n(f^{\mathrm{u}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right). &(0.9)
\end{aligned}$$

Here, we defined

$$\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_\omega=\left(D^{|\pi_{\sigma(\omega_1)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_1)}},\cdots,D^{|\pi_{\sigma(\omega_n)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_n)}}\right), \tag{0.10}$$

where $\sigma(\omega_1)<\cdots<\sigma(\omega_n)$. Thus we obtain

$$\begin{aligned}
&\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\dots,\right.\\
&\left.D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right)=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right).
\end{aligned}\tag{0.11}$$

Let $D_{\mathrm{s}}, D_{\mathrm{u}}$ denote differentiation w.r.t $E^{\mathrm{s}}, E^{\mathrm{u}}$ respectively. Let $x_1\in E^{\mathrm{s}}, x_2\in E^{\mathrm{u}}$. The linearization of the mapping f at the origin is

$$Df(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}D_{\mathrm{s}}f^{\mathrm{s}}(0) & D_{\mathrm{u}}f^{\mathrm{s}}(0)\\ D_{\mathrm{s}}f^{\mathrm{u}}(0) & D_{\mathrm{u}}f^{\mathrm{u}}(0)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{0.12}$$

The linearized dynamics is invariant on E^{s} and E^{u}, thus

$$D_{\mathrm{u}}f^{\mathrm{s}}(0)=0, D_{\mathrm{s}}f^{\mathrm{u}}(0)=0. \tag{0.13}$$

Let us denote $A_{\mathrm{s}}:=D_{\mathrm{s}}f^{\mathrm{s}}(0), A_{\mathrm{u}}:=D_{\mathrm{u}}f^{\mathrm{u}}(0)$. By definition $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A.

Let us derive a few concrete n-th order equations from (0.11).

We can see that

$$Df^{\mathrm{s}}(0)D\tilde g(0)=(D_{\mathrm{s}}f^{\mathrm{s}}(0)\ D_{\mathrm{u}}f^{\mathrm{s}}(0))\begin{pmatrix}I\\Dg(0)\end{pmatrix}=A_{\mathrm{s}}, \tag{0.14}$$

$$Df^{\mathrm{u}}(0)D\tilde g(0)=A_{\mathrm{u}}Dg(0). \tag{0.15}$$

Thus the first order equation is equivalent to the Sylvester equation

$$Dg(0)A_{\mathrm{s}}\eta-A_{\mathrm{u}}Dg(0)\eta=0. \tag{0.16}$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n-multilinear maps of $E_1,\cdots,E_n$ to E_0 by $\mathcal{L}(E_1,\cdots,E_n;E_0)$. For $\nu=1,\cdots,n$ let $(e_{\nu,1},\cdots,e_{\nu,\dim E_\nu})$ be an ordered basis of E_ν and let $(e^*_{\nu,1},\cdots,e^*_{\nu,\dim E_\nu})$ be its dual. Then the space $\mathcal{L}(E_1,\cdots,E_n;E_0)=E^*_n\otimes\cdots\otimes E^*_1\otimes E_0$, where $\otimes$ means a tensor product, has a basis

$$(e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0})_{(r_n,\cdots,r_0)},\quad (r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}. \tag{0.17}$$

We order the basis (0.17) lexicographically with priority to the first components of $(r_n,\cdots,r_0)$. A multilinear map $\tilde{U}\in\mathcal{L}(E_1,\cdots,E_n;E_0)$ has a coordinate representation $U\in\mathbb{R}^{\dim E_0\times\dim E_1^*\times\cdots\times\dim E_n^*}$ as

$$\tilde{U}=\sum_{(r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}} U_{r_0r_1\cdots r_n}e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0}. \tag{0.18}$$

Let $\mathrm{vec}:\mathbb{R}^{J_1\times\cdots\times J_N}\to\mathbb{R}^{\prod_{i=1}^N J_i}$ be a vectorization operator

$$(\mathrm{vec}(U))_i=U_{j_1\cdots j_N}\quad \textit{with}\quad i=1+\sum_{l=1}^{N}\left[(j_l-1)\prod_{l'=1}^{l-1}J_{l'}\right]. \tag{0.19}$$

By $U\times_n A\in\mathbb{R}^{J_1\times\cdots J_{n-1}\times I\times J_{n+1}\times\cdots\times J_N}$ we denote an n-mode product of a tensor [6] $U\in\mathbb{R}^{J_1\times\cdots\times J_N}$ with a matrix $A\in\mathbb{R}^{I\times J_n}$, defined as

$$(U\times_n A)_{j_1\cdots j_{n-1}ij_{n+1}\cdots j_N}=\sum_{j_n=1}^{J_n}U_{j_1\cdots j_N}A_{ij_n}. \tag{0.20}$$

An action of $\tilde{U}\in E^*_n\otimes\cdots\otimes E^*_1\otimes E_0$ on $(\tilde{h}_1,\cdots,\tilde{h}_n,\tilde{h}^*_0)\in\prod_{i=1}^n E^n\times E_0^*$ is rewritten in the coordinate representation as $U\times_1 h_0^*\times_2 h_1^*\times_3\cdots\times_{n+1}h_n^*$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [6, Proposition 3.7 (b)])

$$\mathrm{vec}(U\times_1 h_0^*\times_2 h_1^*\times_3\cdots\times_{n+1}h_n^*)=(h_n^*\otimes\cdots\otimes h_1^*\otimes h_0^*)\mathrm{vec}(U), \tag{0.21}$$

where $\otimes$ denotes the Kronecker product. In the following, we identify any multilinear map and elements in $E^{\mathrm{s}},E^{\mathrm{u}}$ as their coordinate representation.

Let $p=\dim E^{\mathrm{s}},q=\dim E^{\mathrm{u}}$. From (0.16), it is obvious that for any $\eta_0^\top\in E^{\mathrm{u}*},\eta_1\in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top Dg(0)A_{\mathrm{s}}\eta_1-\eta_0^\top A_{\mathrm{u}}Dg(0)\eta_1=0\\
&\Leftrightarrow \mathrm{vec}\left(Dg(0)\times_1\eta_0^\top\times_2\eta_1^\top A_{\mathrm{s}}^\top\right)-\mathrm{vec}\left(Dg(0)\times_1\eta_0^\top A_{\mathrm{u}}\times_2\eta_1^\top\right)=0\quad(\because \mathrm{vec}(\cdot)\text{ is linear})\\
&\Leftrightarrow (\eta_1^\top A_{\mathrm{s}}^\top\otimes\eta_0^\top I_q-\eta_1^\top I_p\otimes\eta_0^\top A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because (0.21))\\
&\Leftrightarrow (\eta_1^\top\otimes\eta_0^\top)\cdot(A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because (AB)\otimes(CD)=(A\otimes C)\cdot(B\otimes D))\\
&\Leftrightarrow (A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because \eta_0,\eta_1\text{ are arbitrary}).
\end{aligned} \tag{0.22}$$

The following theorem is well-known [7, Thm. 4.4.5].

Theorem 0.1.
If $\sigma(A)=\{\lambda_1,\cdots,\lambda_n\},\sigma(B)=\{\mu_1,\cdots,\mu_m\}$, then $\sigma(I_m\otimes A+B\otimes I_n)=\{\lambda_i+\mu_j\mid i\in\{1,\cdots,n\},j\in\{1,\cdots,m\}\}$.

This implies $A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}}$ is regular since $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle. Therefore $Dg(0)=0$.

The second order equation of (0.11) is

$$D^2g(0)(A_{\mathrm{s}}\eta_1,A_{\mathrm{s}}\eta_2)-Df^{\mathrm{u}}(0)D^2\tilde{g}(0)(\eta_1,\eta_2)=D^2f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1,D\tilde{g}(0)\eta_2). \tag{0.23}$$

In the second term of the l.h.s., the first p columns of $Df^{\mathrm{u}}(0)$ are zeros, so the first p mode-1 slices of $D^2\tilde{g}(0)$ do not contribute to the term, i.e., $Df^{\mathrm{u}}(0)D^2\tilde{g}(0)(\eta_1,\eta_2)=A_{\mathrm{u}}D^2g(0)(\eta_1,\eta_2)$. Also, components of $D^2f^{\mathrm{u}}(0)$ of indices larger than $p+1$ do not contribute to the r.h.s. since $Dg(0)=0$. Namely, $D^2 f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1,\eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{\mathrm{u}*}, \eta_1,\eta_2 \in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top D^2 g(0)(A_{\mathrm{s}}\eta_1, A_{\mathrm{s}}\eta_2) - \eta_0^\top A_{\mathrm{u}} D^2 g(0)(\eta_1,\eta_2) = \eta_0^\top D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1,\eta_2) \\
&\Leftrightarrow (A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}}) \operatorname{vec}\left(D^2 g(0)\right) = \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right) \quad (\because \eta_0,\eta_1,\eta_2 \text{ are arbitrary}). \qquad (0.24)
\end{aligned}$$

The theorem 0.1 and the following one [7, Thm. 4.2.12]

Theorem 0.2.
If $\sigma(A) = \{\lambda_1,\cdots,\lambda_n\}, \sigma(B) = \{\mu_1,\cdots,\mu_m\}$, then $\sigma(A\otimes B) = \{\lambda_i\mu_j \mid i\in\{1,\cdots,n\}, j\in\{1,\cdots,m\}\}$.

lead to the regularity of $(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})$. Thus we obtain

$$D^2 g(0) = \operatorname{vec}^{-1}\left[(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})^{-1} \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right)\right]. \qquad (0.25)$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1-y'^2) \\ y' = y + 1.6x(1-x^2) \end{cases}. \qquad (0.26)$$

The jacobian of f is given as

$$Df(x,y) = \begin{pmatrix} 1 & 1.6(1-3y^2) \\ -1.6(1-3x'^2) & 1-1.6^2(1-3y^2)(1-3x'^2) \end{pmatrix}. \qquad (0.27)$$

f is injective and $\det(Df(x,y)) \equiv 1$, so f is area-preserving. The fixed points (x_*,y_*)s satisfy simultaneous equations $y(1-y^2)=0, x(1-x^2)=0$, so $(x_*,y_*)\in\{0,1,-1\}^2$. If $\operatorname{tr}(Df(x_*,y_*))^2 - 4\det(Df(x_*,y_*)) = \operatorname{tr}(Df(x_*,y_*))^2 - 4 > 0$, f is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned}
&\operatorname{tr}(Df(x_*,y_*))^2 - 4 = -1.6^2\times 4(1-3y_*^2)(1-3x_*^2) + 1.6^4(1-3y_*^2)^2(1-3x_*^2)^2 > 0 \\
&\Leftrightarrow g(x_*,y_*) := 4 - 1.6^2(1-3y_*^2)(1-3x_*^2) \begin{cases} <0 & \text{if } |x_*|+|y_*| \neq 1 \\ >0 & \text{otherwise} \end{cases}. \ (\because x_*,y_* \text{ are not irrational}) \qquad (0.28)
\end{aligned}$$

We evaluate g on the fixed points as

$$\begin{aligned}
g(0,0) &= 4 - 1.6^2 > 0 \\
g(0,\pm1) = g(\pm1,0) &= 4 + 1.6^2\times 2 > 0 \\
g(\pm1,\pm1) &= 4 - 4\times 1.6^2 < 0. \qquad (0.29)
\end{aligned}$$

Therefore, $(0,0)$ is non-hyperbolic and the others are hyperbolic.

Exercise 4.20.

We take inner product with $\xi-\eta$ on both sides of $\xi = \eta + (\xi-\eta)$ as

$$\begin{aligned}
&(\xi-\eta,\xi) = (\xi-\eta,\eta) + ||\xi-\eta||^2 \\
&\Leftrightarrow 2 - 2\cos\theta = ||\xi-\eta||^2. \qquad (0.30)
\end{aligned}$$

Then

$$||A\xi - A\eta||^2 = ||A\xi||^2 + ||A\eta||^2 - 2(A\xi,A\eta) \geq ||A\xi||^2 + ||A\eta||^2 - 2||A\xi||\cdot||A\eta|| = (\varrho^{-1}-\varrho)^2 \qquad (0.31)$$

$$||A\xi - A\eta||^2 \leq ||A||_2^2 ||\xi-\eta||^2 = ||A||_2^2(2-2\cos\theta) \ (\because \text{Eq. } (0.30)). \qquad (0.32)$$

Thus the desired inequality is derived.

Exercise 4.21.

Let $v = \xi + \eta, \xi \in E^{\mathrm{u}}, \eta \in E^{\mathrm{s}}$. Let $\lambda_{\mathrm{s}}, C_{\mathrm{s}}$ (resp. $\lambda_{\mathrm{u}}, C_{\mathrm{u}}$) be constants which make all tangent vectors in E^{s} (resp. E^{u}) stable and define $C = \max(C_{\mathrm{s}}, C_{\mathrm{u}})$ and $\lambda = \max(\lambda_{\mathrm{s}}, \lambda_{\mathrm{u}})$. Also let $P_{\mathrm{s}}, P_{\mathrm{u}}$ be projections onto $E^{\mathrm{s}}, E^{\mathrm{u}}$.

$$\langle v, v\rangle \geq ||\xi||^2 + ||\eta||^2 \geq ||\xi + \eta||^2 = (v, v), \tag{0.33}$$

$$\begin{aligned}\langle v, v\rangle &\leq \frac{C_{\mathrm{s}}^2||\xi||^2}{1-\lambda_{\mathrm{s}}^{2\epsilon}} + \frac{C_{\mathrm{u}}^2||\eta||^2}{1-\lambda_{\mathrm{u}}^{2\epsilon}} \leq \frac{C^2}{1-\lambda^{2\epsilon}}(||\xi||^2 + ||\eta||^2) = \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}v||^2 + ||P_{\mathrm{s}}v||^2) \\ &\leq \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)(v, v).\end{aligned} \tag{0.34}$$

Thus we can take $D = \max\left(1, \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)\right)$.

Exercise 4.22.

Let $M = \max_{||x||\leq\epsilon}||D^2Q(x)||, N = \max_{||x||\leq\epsilon}||D\varphi(x/\epsilon)||$. These maxima exist because $D^2Q, D\varphi$ are continuous and the closed ball is compact. The Taylor theorem for $Q \in C^2$ [Sec. 4.5][8] says that

$$Q(x) = Q(0) + DQ(0)x + R_1(x) = R_1(x) \ (\because Q(0) = 0, DQ(0) = 0), \tag{0.35}$$

$$DQ(x) = DQ(0) + R_2(x) = R_2(x), \tag{0.36}$$

where R_1, R_2 are remainder terms. For x in the closed ball of radius ϵ, the mean value inequalities for the remainder terms [Thm. 4.C][8] lead to

$$\begin{aligned}||Q(x)|| &= ||R_1(x)|| \leq \frac{1}{2!}\sup_{0\leq\tau\leq1}||D^2Q(\tau x)(x, x)|| \leq \frac{1}{2!}\sup_{0\leq\tau\leq1}||D^2Q(\tau x)||\cdot||x||\cdot||x|| \leq \frac{M}{2}||x||\cdot||x|| \\ &\leq \frac{M\epsilon^2}{2}\end{aligned} \tag{0.37}$$

$$||DQ(x)|| = ||R_2(x)|| \leq \sup_{0\leq\tau\leq1}||D^2Q(\tau x)x|| \leq M||x|| \leq M\epsilon. \tag{0.38}$$

The differential of Q_ϵ is

$$DQ_\epsilon(x) = \frac{1}{\epsilon}D\varphi(x/\epsilon)Q(x) + \varphi(x/\epsilon)DQ(x). \tag{0.39}$$

Then

$$||DQ_\epsilon(x)|| \leq \frac{1}{\epsilon}||D\varphi(x/\epsilon)||\cdot||Q(x)|| + ||\varphi(x/\epsilon)||\cdot||DQ(x)|| \leq \frac{M(N+2)}{2}\epsilon. \tag{0.40}$$

Exercise 4.23.

($\mathscr{H}_\alpha$ is a normed vector space) If $\beta, \gamma \in \mathbb{R}$ and $f, g \in \mathscr{H}_\alpha$, Then

$$\begin{aligned}||\beta f + \gamma g||_{\mathscr{H}_\alpha} &= \sup_{x\neq y}\frac{||\beta(f(x)-f(y)) + \gamma(g(x)-g(y))||}{||x-y||^\alpha} \leq \sup_{x\neq y}\frac{\beta||f(x)-f(y)||}{||x-y||^\alpha} + \sup_{x\neq y}\frac{\gamma||g(x)-g(y)||}{||x-y||^\alpha} \\ &< \infty.\end{aligned} \tag{0.41}$$

$$\beta f(0) + \gamma g(0) = 0. \tag{0.42}$$

So, $\beta f + \gamma g \in \mathscr{H}_\alpha$. Also, Eq. (0.41) for $\beta = \gamma = 1$ includes the triangle inequality. The other conditions to be a norm is trivial. Hence $\mathscr{H}_\alpha$ is a normed vector space.

(Completeness) Consider a Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $\mathscr{H}_\alpha$. This is also Cauchy in a space of continuous functions endowed with sup-norm, say, $BC(E)$. $BC(E)$ is complete, so the sequence has a limit in $BC(E)$. We denote the limit as f.

Every Cauchy sequence in a metric space is bounded, so $||f_n||_{\mathscr{H}_\alpha} \leq M$ for all $n \in \mathbb{N}$ and some $M \in \mathbb{R}$. Hence

$$||f||_{\mathscr{H}_\alpha} = \lim_{n\to\infty} ||f_n||_{\mathscr{H}_\alpha} \ (\because \text{the norm is continuous}) \leq M < \infty. \tag{0.43}$$

Therefore $f \in \mathscr{H}_\alpha$.

Next, we show $f_n \to f$ in $\mathscr{H}_\alpha$. For any $\epsilon > 0$, there exists $N > 0$ such that $||f_p - f_q||_{\mathscr{H}_\alpha} < \epsilon$ whenever $p, q \geq N$ since $\{f_n\}$ is Cauchy. Then for any $n \geq N$

$$||f - f_n||_{\mathscr{H}_\alpha} = \lim_{m\to\infty} ||f_m - f_n||_{\mathscr{H}_\alpha} \ (\because \text{ the norm is continuous}) \leq \epsilon. \tag{0.44}$$

Thus $f_n \to f$ in $\mathscr{H}_\alpha$.

Therefore $\mathscr{H}_\alpha$ is complete.

Exercise 4.24.

$R^{\mathrm{u}}(0,0) = 0, R^{\mathrm{s}}(0,0) = 0$ since $R \in \mathscr{H}_\alpha$. Hence

$$\begin{aligned}
&\Psi^{\mathrm{u}}(R)(0,0) = (L^{\mathrm{u}})^{-1} R^{\mathrm{u}}(0,0) - (L^{\mathrm{u}})^{-1} Q^{\mathrm{u}}_\epsilon(0,0) = 0 \ (\because Q(0) = 0) \\
&\Psi^{\mathrm{s}}(R)(0,0) = L^{\mathrm{s}} R^{\mathrm{s}}(0,0) + Q^{\mathrm{s}}_\epsilon(0,0) = 0.
\end{aligned} \tag{0.45}$$

Exercise 4.25.

The C^2 regularity is used to prove $||DQ_\epsilon|| \leq C\epsilon$. This inequality is used to bound $||\Psi(r) - \Psi(r')||_{\mathscr{H}_\alpha}$. To derive the inequality regarding $||\Psi(r) - \Psi(r')||_{\mathscr{H}_\alpha}$, Q_ϵ need not satisfy $||DQ_\epsilon|| \leq C\epsilon$. Instead, it is enough for Q_ϵ to be Lipschitz continuous of its Lipschitz constant $C\epsilon$.

Let $\mathcal{B}_\epsilon$ be a closed ball of radius ϵ centered at the origin. Every continuous functions on a compact domain is globally Lipschitz, so DQ is globally Lipschitz on $\mathcal{B}_\epsilon$. Let K be the Lipschitz constant of DQ, then for all $x \in \mathcal{B}_\epsilon$

$$||DQ(x)|| = ||DQ(x) - DQ(0)|| \leq K||x|| \leq K\epsilon. \tag{0.46}$$

Let L be the Lipschitz constant of Q on $\mathcal{B}_\epsilon$, then [9, Lem. 3.1]

$$L = \sup_{x\in\mathcal{B}_\epsilon} ||DQ(x)|| \leq K\epsilon. \tag{0.47}$$

For $x, y \in \mathcal{B}_\epsilon$, the mean value inequalities for the remainder terms lead to

$$||Q(x)|| \leq \sup_{0\leq\tau\leq1} ||DQ(\tau x)x|| \leq K\epsilon^2. \tag{0.48}$$

Let M be the global Lipschitz constant of φ on $\mathcal{B}_1$. Q_ϵ is zero outside the closed ball, so we can limit us to consider the ball to evaluate the Lipschitz constant. For $x, y \in \mathcal{B}_\epsilon$,

$$\begin{aligned}
&||Q_\epsilon(x) - Q_\epsilon(y)|| = ||\varphi(x/\epsilon)Q(x) - \varphi(y/\epsilon)Q(y)|| \\
&\leq ||\varphi(x/\epsilon) - \varphi(y/\epsilon)|| \cdot ||Q(x)|| + ||\varphi(y/\epsilon)|| \cdot ||Q(x) - Q(y)|| \\
&\leq \frac{M}{\epsilon}||x - y||K\epsilon^2 + 1 \cdot K\epsilon||x - y|| = K(M + 1)\epsilon||x - y||
\end{aligned} \tag{0.49}$$

Thus Q_ϵ is Lipschitz continuous of its Lipschitz constant $C\epsilon$, where C can be taken as $K(M + 1)$.

Exercise 4.35.

Let us assume Λ is compact (at the beginning of this chapter, the compactness of M is assumed but seems frequently violated (for example, $M = \mathbb{R}^n$)). In fact, in many literatures a uniformly hyperbolic set is defined to be compact.

Let $||\cdot||_{\mathrm{T}_xM}$ be an arbitrary norm and $||\cdot||_{*,\mathrm{T}_xM}$ be is associated Lyapunov metric. Let $n_0 \in \mathbb{N}$ such that $C\lambda^{n_0} < 1$ in the Definition 4. 16 in the text[2]. Then, as in the solution of Exercise 4.20, we obtain

$$1 - \cos\theta_x \geq \frac{(C^{-1}\lambda^{-n_0} - C\lambda^{n_0})^2}{||D_x f^{n_0}||_2^2}, \tag{0.50}$$

where θ_x is the angle between E^{s}_x and E^{u}_x. $||D_x f^{n_0}||_2^2$ is bounded above on Λ since Λ is compact. Hence the r.h.s of (0.50) is uniformly bounded away from zero on Λ. Let $\mu \in (0,1)$ be a (x-independent) lower bound of the r.h.s.

Let $v = \xi + \eta, \xi \in E^{\mathrm{u}}_x, \eta \in E^{\mathrm{s}}_x$. As in the solution of Exercise 4.21,

$$||v||_{\mathrm{T}_xM} \leq ||v||_{*,\mathrm{T}_xM}. \tag{0.51}$$

Also,

$$\begin{aligned} ||v||_{\mathrm{T}_xM} &= ||\xi||^2_{\mathrm{T}_xM} + ||\eta||^2_{\mathrm{T}_xM} - 2\cos\theta_x ||\xi||_{\mathrm{T}_xM}||\eta||_{\mathrm{T}_xM} \\ &\geq ||\xi||^2_{\mathrm{T}_xM} + ||\eta||^2_{\mathrm{T}_xM} - 2(\mu-1)||\xi||_{\mathrm{T}_xM}||\eta||_{\mathrm{T}_xM} = (1-\mu)||\xi-\eta||^2_{\mathrm{T}_xM} + \mu(||\xi||^2_{\mathrm{T}_xM} + ||\eta||^2_{\mathrm{T}_xM}) \\ &\geq \mu(||\xi||^2_{\mathrm{T}_xM} + ||\eta||^2_{\mathrm{T}_xM}). \end{aligned} \tag{0.52}$$

Then

$$||v||_{*,\mathrm{T}_xM} \leq \frac{C^2}{1-\lambda^{2\epsilon}}(||\xi||^2_{\mathrm{T}_xM} + ||\eta||^2_{\mathrm{T}_xM}) \leq \frac{C^2}{\mu(1-\lambda^{2\epsilon})}||v||_{\mathrm{T}_xM}. \tag{0.53}$$

$C, \mu, \lambda, \epsilon$ are independent of $x \in \Lambda$. Hence, from (0.52, 0.53), the Lyapunov metric is uniformly equivalent to the original one.

Exercise 4.61.

Let us define $\mathcal{B}, ||\cdot||_{\mathcal{B}}, A$ as in the proof of the Theorem 4.63 in the text. Let $U_{\epsilon'}$ be a ϵ'-neighborhood of Λ. Then, for any $\mathbf{y} \in U_{\epsilon'}^{\mathbb{Z}}$, there exists $\mathbf{x} \in \Lambda^{\mathbb{Z}}$ such that

$$||\mathbf{y} - \mathbf{x}||_{\mathcal{B}} \leq \epsilon'. \tag{0.54}$$

Suppose $\mathbf{y}$ is an ϵ'-pseudo orbit lying in $U_{\epsilon'}$, i.e.,

$$||A\mathbf{y} - \mathbf{y}||_{\mathcal{B}} \leq \epsilon'. \tag{0.55}$$

$||Df||$ attains is maximum in Λ since Df is continuous and Λ is compact. We denote the maximum as M. Then

$$\begin{aligned} &\sup_{i\in\mathbb{Z}} ||f(x_i) - f(y_i)|| = \sup_{i\in\mathbb{Z}} ||f(x_i) - f(x_i - x_i + y_i)|| \\ &\leq \sup_{i\in\mathbb{Z}} \sup_{\tau\in[0,1]} ||Df(x_i)|| \cdot ||\tau(y_i - x_i)|| \ (\because \text{mean value inequality for the Taylor's theorem}) \leq M\epsilon'. \end{aligned} \tag{0.56}$$

So

$$||A\mathbf{x} - \mathbf{x}||_{\mathcal{B}} = ||A\mathbf{x} - A\mathbf{y} + A\mathbf{y} - \mathbf{y} + \mathbf{y} - \mathbf{x}||_{\mathcal{B}} \leq (M+2)\epsilon', \tag{0.57}$$

namely, $\mathbf{x}$ is a $(M+2)\epsilon'$-pseudo orbit. Let

$$\epsilon = \min\left(\frac{\delta}{2}, \epsilon_{\frac{\delta}{2}}\right), \quad \epsilon' = \frac{\epsilon}{M+2}, \tag{0.58}$$

where $\epsilon_{\frac{\delta}{2}}$ corresponds to the choice of ϵ for $\frac{\delta}{2}$ in the shadowing lemma. Then for any ϵ'-pseudo orbit $\mathbf{y}$ lying in $U_{\epsilon'}$, there exists an ϵ-pseudo orbit $\mathbf{x}$ lying in Λ, which satisfies (0.54) and can be $\delta/2$-shadowed by z_0. Obviously $\mathbf{y}$ is δ-shadowed by z_0.

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[2] Note that $C > 0$ is independent of $x \in \Lambda$ for a uniformly hyperbolic set. This seems implicitly assumed in the text.

Exercise 4.64.

From the chain rule,

$$\mathrm{D}_{x_{n-1}}\mathrm{D}_{x_{n-1-j}}f^j=\mathrm{D}_{x_{n-1-j}}f^{j+1},\quad \mathrm{D}_{x_{n-1}}\mathrm{D}_{x_{n-1+j}}f^{-j}=\mathrm{D}_{x_{n-1+j}}f^{-j+1}. \tag{0.59}$$

Hence

$$\begin{aligned}
&[(I-D_{\mathbf{x}}A)\circ(L_{\mathbf{x}})v]_n=\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}\\
&-D_{x_{n-1}}f\left[\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-1-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-1-j}}}v_{n-1-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n-1+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n-1+j}}}v_{n-1+j}\right]\\
&=\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}\\
&-\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-1-j}}f^{j+1}\Pi_{E^{\mathrm{s}}_{x_{n-1-j}}}v_{n-1-j}+\sum_{j=1}^{\infty}\mathrm{D}_{x_{n-1+j}}f^{-j+1}\Pi_{E^{\mathrm{u}}_{x_{n-1+j}}}v_{n-1+j}\quad(\because\ \text{Eq. 0.59})\\
&=\Pi_{E^{\mathrm{s}}_{x_n}}v_n+\sum_{j=1}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}\\
&-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}+\Pi_{E^{\mathrm{u}}_{x_n}}v_n+\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}=v_n.
\end{aligned} \tag{0.60}$$

Thus $(I-D_{\mathbf{x}}A)\circ(L_{\mathbf{x}})=\mathrm{id}$.

$E^{\mathrm{s}}_x, E^{\mathrm{u}}_x$ are covariant w.r.t an action of the differential, i.e., $\mathrm{D}_{x_n}f^i(E^{\mathrm{s}}_{x_n})=E^{\mathrm{s}}_{x_{n+i}}, \mathrm{D}_{x_n}f^i(E^{\mathrm{u}}_{x_n})=E^{\mathrm{u}}_{x_{n+i}}$. Therefore

$$\Pi_{E^{\mathrm{s}}_{x_{n-j}}}\mathrm{D}_{x_{n-j-1}}f=\mathrm{D}_{x_{n-j-1}}f\ \Pi_{E^{\mathrm{s}}_{x_{n-j-1}}},\quad \Pi_{E^{\mathrm{u}}_{x_{n+j}}}\mathrm{D}_{x_{n+j-1}}f=\mathrm{D}_{x_{n+j-1}}f\ \Pi_{E^{\mathrm{s}}_{x_{n+j-1}}}. \tag{0.61}$$

Also,

$$\mathrm{D}_{x_{n-j}}f^j\mathrm{D}_{x_{n-1-j}}f=\mathrm{D}_{x_{n-1-j}}f^{j+1},\quad \mathrm{D}_{x_{n+j}}f^{-j}\mathrm{D}_{x_{n-1+j}}f=\mathrm{D}_{x_{n-1+j}}f^{-j+1}. \tag{0.62}$$

Then

$$\begin{aligned}
&[(L_{\mathbf{x}})\circ(I-D_{\mathbf{x}}A)v]_n=\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}\\
&-\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}\mathrm{D}_{x_{n-1-j}}v_{n-1-j}+\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}\mathrm{D}_{x_{n-1+j}}v_{n-1+j}\\
&=\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-j}}f^j\Pi_{E^{\mathrm{s}}_{x_{n-j}}}v_{n-j}-\sum_{j=1}^{\infty}\mathrm{D}_{x_{n+j}}f^{-j}\Pi_{E^{\mathrm{u}}_{x_{n+j}}}v_{n+j}\\
&-\sum_{j=0}^{\infty}\mathrm{D}_{x_{n-1-j}}f^{j+1}\Pi_{E^{\mathrm{s}}_{x_{n-1-j}}}v_{n-1-j}+\sum_{j=1}^{\infty}\mathrm{D}_{x_{n-1+j}}f^{-j+1}\Pi_{E^{\mathrm{u}}_{x_{n-1+j}}}v_{n-1+j}\quad(\because\ \text{Eqs. 0.61, 0.62})\\
&=v_n.
\end{aligned} \tag{0.63}$$

Thus $(L_{\mathbf{x}})\circ(I-D_{\mathbf{x}}A)=\mathrm{id}$.

Exercise 4.70.

Let x_*, y_* are fixed points of f, g respectively and $\Phi(x_*) = y_*$. Then

$$\mathrm{D}(\Phi \circ f)(x_*) = \mathrm{D}(g \circ \Phi)(x_*) \Leftrightarrow \mathrm{D}\Phi(x_*)\mathrm{D}f(x_*) = \mathrm{D}g(y_*)\mathrm{D}\Phi(x_*) \ (\because f(x_*) = x_*, \Phi(x_*) = y_*). \tag{0.64}$$

Therefore $\mathrm{D}f(x_*)$ and $\mathrm{D}g(y_*)$ are similar.

Let x_*, y_* are fixed points of f^n, g^n respectively and $\Phi(x_*) = y_*$. Then

$$\mathrm{D}(\Phi \circ f^n)(x_*) = \mathrm{D}(g^n \circ \Phi)(x_*) \Leftrightarrow \mathrm{D}\Phi(x_*)\mathrm{D}f^n(x_*) = \mathrm{D}g^n(y_*)\mathrm{D}\Phi(x_*) \ (\because f^n(x_*) = x_*, \Phi(x_*) = y_*). \tag{0.65}$$

Therefore $\mathrm{D}f^n(x_*)$ and $\mathrm{D}g^n(y_*)$ are similar.

Exercise 4.76.

We introduce a coordinate z (resp. w) in direction of the long (resp. short) side of the red rectangle, where the unit length is defined as that of the long (resp. short) side. In the following, we consider in the $z - w$ coordinate.

Let R be the red rectangle and

$$D_n = R \cap \left(\bigcap_{i=1}^{n} f^{-i}(R) \right), \quad D_0 = R. \tag{0.66}$$

The set of points whose forward codes are the same is described as $\lim_{n \to \infty} D_n$. We can easily see that

$$D_{n+1} = R \cap f^{-1}(D_n). \tag{0.67}$$

Let $\lambda = (3 - \sqrt{5})/2$, $T_{\mathrm{L}} = \lambda z \times [0, 1)$, $T_{\mathrm{R}} = (\lambda z - (\lambda - 1)) \times [0, 1)$. The recursive relation (0.67) is translated as

$$D_{n+1} = T_{\mathrm{L}}(D_n) \cup T_{\mathrm{R}}(D_n). \tag{0.68}$$

This decreasing series of set converges to the desired Cantor set.

Exercise 4.79.

See [3, pp. 53–55].

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