Chapter 4

Sho Shirasaka, December 13, 2022

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1y_2 = y_1(0)y_2(0) = \mathrm{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10.

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{\pi = (\pi_1, \cdots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega,\ \bigcup_{i=1}^{k} \pi_i = \omega,\ \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j\right\}, \tag{0.3}$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \cdots, n\}). \tag{0.4}$$

Then the Faá di Bruno's formula is given as [2, Sec. 2.4][1]

$$D^n(f_1 \circ f_2)(x)\eta_\omega = \sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f_1(f_2(x)) \left(D^{|\pi_1|} f_2(x)\eta_{\pi_1}, \ldots, D^{|\pi_k|} f_2(x)\eta_{\pi_k}\right), \tag{0.5}$$

where for any $\pi = (\pi_1, \cdots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \cdots, \omega_n\}$ of $\mathbb{N}$,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \cdots, \eta_{\sigma(\omega_n)}) \tag{0.6}$$

where σ is a permutation and $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n-th derivatives of the both sides of

$$g \circ f^{\mathrm{s}}(\eta, g(\eta)) = f^{\mathrm{u}}(\eta, g(\eta)) \Leftrightarrow g \circ f^{\mathrm{s}} \circ \tilde{g}(\eta) = f^{\mathrm{u}} \circ \tilde{g}(\eta) \tag{0.7}$$

[1] Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$
\begin{aligned}
&\text{(l.h.s): } D^n(g\circ f^{\mathrm{s}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k(g\circ f^{\mathrm{s}})(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\right.\\
&\qquad\qquad\left.\dots,D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right) \qquad (0.8)\\
&\text{(r.h.s): } D^n(f^{\mathrm{u}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right). \qquad (0.9)
\end{aligned}
$$

Here, we defined

$$
\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_\omega=\left(D^{|\pi_{\sigma(\omega_1)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_1)}},\cdots,D^{|\pi_{\sigma(\omega_n)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_n)}}\right), \qquad (0.10)
$$

where $\sigma(\omega_1)<\cdots<\sigma(\omega_n)$. Thus we obtain

$$
\begin{aligned}
&\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\dots,\right.\\
&\left.D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right)=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right).
\end{aligned} \qquad (0.11)
$$

Let $D_{\mathrm{s}}, D_{\mathrm{u}}$ denote differentiation w.r.t $E^{\mathrm{s}}, E^{\mathrm{u}}$ respectively. Let $x_1\in E^{\mathrm{s}}, x_2\in E^{\mathrm{u}}$. The linearization of the mapping f at the origin is

$$
Df(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}D_{\mathrm{s}}f^{\mathrm{s}}(0) & D_{\mathrm{u}}f^{\mathrm{s}}(0)\\ D_{\mathrm{s}}f^{\mathrm{u}}(0) & D_{\mathrm{u}}f^{\mathrm{u}}(0)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \qquad (0.12)
$$

The linearized dynamics is invariant on E^{s} and E^{u}, thus

$$
D_{\mathrm{u}}f^{\mathrm{s}}(0)=0, D_{\mathrm{s}}f^{\mathrm{u}}(0)=0. \qquad (0.13)
$$

Let us denote $A_{\mathrm{s}}:=D_{\mathrm{s}}f^{\mathrm{s}}(0), A_{\mathrm{u}}:=D_{\mathrm{u}}f^{\mathrm{u}}(0)$. By definition $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A.

Let us derive a few concrete n-th order equations from (0.11).

We can see that

$$
Df^{\mathrm{s}}(0)D\tilde g(0)=(D_{\mathrm{s}}f^{\mathrm{s}}(0)\ D_{\mathrm{u}}f^{\mathrm{s}}(0))\begin{pmatrix}I\\Dg(0)\end{pmatrix}=A_{\mathrm{s}}, \qquad (0.14)
$$

$$
Df^{\mathrm{u}}(0)D\tilde g(0)=A_{\mathrm{u}}Dg(0). \qquad (0.15)
$$

Thus the first order equation is equivalent to the Sylvester equation

$$
Dg(0)A_{\mathrm{s}}\eta-A_{\mathrm{u}}Dg(0)\eta=0. \qquad (0.16)
$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n-multilinear maps of $E_1,\cdots,E_n$ to E_0 by $\mathcal{L}(E_1,\cdots,E_n;E_0)$. For $\nu=1,\cdots,n$ let $(e_{\nu,1},\cdots,e_{\nu,\dim E_\nu})$ be an ordered basis of E_ν and let $(e^*_{\nu,1},\cdots,e^*_{\nu,\dim E_\nu})$ be its dual. Then the space $\mathcal{L}(E_1,\cdots,E_n;E_0)=E^*_n\otimes\cdots\otimes E^*_1\otimes E_0$, where $\otimes$ means a tensor product, has a basis

$$(e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0})_{(r_n,\cdots,r_0)},\quad (r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}. \tag{0.17}$$

We order the basis (0.17) lexicographically with priority to the first components of $(r_n,\cdots,r_0)$. A multilinear map $\tilde U\in\mathcal{L}(E_1,\cdots,E_n;E_0)$ has a coordinate representation $U\in\mathbb{R}^{\dim E_0\times\dim E^*_1\times\cdots\times\dim E^*_n}$ as

$$\tilde U=\sum_{(r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}} U_{r_0r_1\cdots r_n}e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0}. \tag{0.18}$$

Let $\mathrm{vec}:\mathbb{R}^{J_1\times\cdots\times J_N}\to\mathbb{R}^{\prod_{i=1}^N J_i}$ be a vectorization operator

$$(\mathrm{vec}(U))_i=U_{j_1\cdots j_N}\quad \textit{with}\quad i=1+\sum_{l=1}^{N}\left[(j_l-1)\prod_{l'=1}^{l-1}J_{l'}\right]. \tag{0.19}$$

By $U\times_n A\in\mathbb{R}^{J_1\times\cdots J_{n-1}\times I\times J_{n+1}\times\cdots\times J_N}$ we denote an n-mode product of a tensor [3] $U\in\mathbb{R}^{J_1\times\cdots\times J_N}$ with a matrix $A\in\mathbb{R}^{I\times J_n}$, defined as

$$(U\times_n A)_{j_1\cdots j_{n-1}ij_{n+1}\cdots j_N}=\sum_{j_n=1}^{J_n}U_{j_1\cdots j_N}A_{ij_n}. \tag{0.20}$$

An action of $\tilde U\in E^*_n\otimes\cdots\otimes E^*_1\otimes E_0$ on $(\tilde h_1,\cdots,\tilde h_n,\tilde h^*_0)\in\prod_{i=1}^n E^n\times E^*_0$ is rewritten in the coordinate representation as $U\times_1 h^*_0\times_2 h^*_1\times_3\cdots\times_{n+1}h^*_n$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\mathrm{vec}(U\times_1 h^*_0\times_2 h^*_1\times_3\cdots\times_{n+1}h^*_n)=(h^*_n\otimes\cdots\otimes h^*_1\otimes h^*_0)\mathrm{vec}(U), \tag{0.21}$$

where $\otimes$ denotes the Kronecker product. In the following, we identify any multilinear map and elements in $E^{\mathrm{s}},E^{\mathrm{u}}$ as their coordinate representation.

Let $p=\dim E^{\mathrm{s}},q=\dim E^{\mathrm{u}}$. From (0.16), it is obvious that for any $\eta_0^\top\in E^{\mathrm{u}*},\eta_1\in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top Dg(0)A_{\mathrm{s}}\eta_1-\eta_0^\top A_{\mathrm{u}}Dg(0)\eta_1=0\\
&\Leftrightarrow \mathrm{vec}\left(Dg(0)\times_1\eta_0^\top\times_2\eta_1^\top A_{\mathrm{s}}^\top\right)-\mathrm{vec}\left(Dg(0)\times_1\eta_0^\top A_{\mathrm{u}}\times_2\eta_1^\top\right)=0\quad(\because \mathrm{vec}(\cdot)\text{ is linear})\\
&\Leftrightarrow(\eta_1^\top A_{\mathrm{s}}^\top\otimes\eta_0^\top I_q-\eta_1^\top I_p\otimes\eta_0^\top A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because(0.21))\\
&\Leftrightarrow(\eta_1^\top\otimes\eta_0^\top)\cdot(A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because(AB)\otimes(CD)=(A\otimes C)\cdot(B\otimes D))\\
&\Leftrightarrow(A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because\eta_0,\eta_1\text{ are arbitrary}).
\end{aligned}\tag{0.22}$$

The following theorem is well-known [4, Thm. 4.4.5].

Theorem 0.1.
If $\sigma(A)=\{\lambda_1,\cdots,\lambda_n\},\sigma(B)=\{\mu_1,\cdots,\mu_m\}$, then $\sigma(I_m\otimes A+B\otimes I_n)=\{\lambda_i+\mu_j\mid i\in\{1,\cdots,n\},j\in\{1,\cdots,m\}\}$.

This implies $A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}}$ is regular since $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle. Therefore $Dg(0)=0$.

The second order equation of (0.11) is

$$D^2g(0)(A_{\mathrm{s}}\eta_1,A_{\mathrm{s}}\eta_2)-Df^{\mathrm{u}}(0)D^2\tilde g(0)(\eta_1,\eta_2)=D^2f^{\mathrm{u}}(0)(D\tilde g(0)\eta_1,D\tilde g(0)\eta_2). \tag{0.23}$$

In the second term of the l.h.s., the first p columns of $Df^{\mathrm{u}}(0)$ are zeros, so the first p mode-1 slices of $D^2\tilde g(0)$ do not contribute to the term, i.e., $Df^{\mathrm{u}}(0)D^2\tilde g(0)(\eta_1,\eta_2)=A_{\mathrm{u}}D^2g(0)(\eta_1,\eta_2)$. Also, components of $D^2f^{\mathrm{u}}(0)$ of indices larger than $p+1$ do not contribute to the r.h.s. since $Dg(0)=0$. Namely, $D^2f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D^2_{\mathrm{s}}f^{\mathrm{u}}(0)(\eta_1,\eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{\mathrm{u}*}, \eta_1,\eta_2 \in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top D^2g(0)(A_{\mathrm{s}}\eta_1, A_{\mathrm{s}}\eta_2) - \eta_0^\top A_{\mathrm{u}} D^2g(0)(\eta_1,\eta_2) = \eta_0^\top D^2_{\mathrm{s}}f^{\mathrm{u}}(0)(\eta_1,\eta_2)\\
&\Leftrightarrow (A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2}\otimes A_{\mathrm{u}})\,\mathrm{vec}\left(D^2g(0)\right) = \mathrm{vec}\left(D^2_{\mathrm{s}}f^{\mathrm{u}}(0)\right) \quad (\because \eta_0,\eta_1,\eta_2 \text{ are arbitrary}).
\end{aligned} \tag{0.24}$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

Theorem 0.2.
If $\sigma(A)=\{\lambda_1,\cdots,\lambda_n\}, \sigma(B)=\{\mu_1,\cdots,\mu_m\}$, then $\sigma(A\otimes B)=\{\lambda_i\mu_j \mid i\in\{1,\cdots,n\}, j\in\{1,\cdots,m\}\}$.

lead to the regularity of $(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2}\otimes A_{\mathrm{u}})$. Thus we obtain

$$D^2g(0) = \mathrm{vec}^{-1}\left[(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2}\otimes A_{\mathrm{u}})^{-1}\,\mathrm{vec}\left(D^2_{\mathrm{s}}f^{\mathrm{u}}(0)\right)\right]. \tag{0.25}$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1-y'^2) \\ y' = y + 1.6x(1-x^2) \end{cases}. \tag{0.26}$$

The jacobian of f is given as

$$Df(x,y) = \begin{pmatrix} 1 & 1.6(1-3y^2) \\ -1.6(1-3x'^2) & 1-1.6^2(1-3y^2)(1-3x'^2) \end{pmatrix}. \tag{0.27}$$

f is injective and $\det\left(Df(x,y)\right)\equiv 1$, so f is area-preserving. The fixed points (x_*,y_*)s satisfy simultaneous equations $y(1-y^2)=0, x(1-x^2)=0$, so $(x_*,y_*)\in\{0,1,-1\}^2$. If $\mathrm{tr}\left(Df(x_*,y_*)\right)^2 - 4\det\left(Df(x_*,y_*)\right) = \mathrm{tr}\left(Df(x_*,y_*)\right)^2 - 4 > 0$, f is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned}
&\mathrm{tr}\left(Df(x_*,y_*)\right)^2 - 4 = -1.6^2\times 4(1-3y_*^2)(1-3x_*^2) + 1.6^4(1-3y_*^2)^2(1-3x_*^2)^2 > 0\\
&\Leftrightarrow g(x_*,y_*) := 4 - 1.6^2(1-3y_*^2)(1-3x_*^2)\begin{cases} <0 & \text{if } |x_*|+|y_*|\neq 1 \\ >0 & \text{otherwise} \end{cases}. \quad (\because x_*,y_* \text{ are not irrational})
\end{aligned} \tag{0.28}$$

We evaluate g on the fixed points as

$$\begin{aligned}
g(0,0) &= 4 - 1.6^2 > 0\\
g(0,\pm1) = g(\pm1,0) &= 4 + 1.6^2\times 2 > 0\\
g(\pm1,\pm1) &= 4 - 4\times 1.6^2 < 0.
\end{aligned} \tag{0.29}$$

Therefore, $(0,0)$ is non-hyperbolic and the others are hyperbolic.

References

[1] Wolf-Jürgen Beyn and Winfried Kleß. Numerical taylor expansions of invariant manifolds in large dynamical systems. *Numerische Mathematik*, 80(1):1–38, 1998.

[2] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.

[3] Tamara Gibson Kolda. Multilinear operators for higher-order decompositions. Technical report, Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA ···, 2006.

[4] Horn Roger and R Johnson Charles. Topics in matrix analysis, 1994.