

Chapter 4

Sho Shirasaka, December 13, 2022

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \quad (0.1)$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \operatorname{tr}(A)x + \det(A). \quad (0.2)$$

Its discriminant is $\operatorname{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \operatorname{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P \operatorname{diag}(\lambda_1, \lambda_2) P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \operatorname{diag}(\lambda_1, \lambda_2)y$. An orbit in the y -coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0) y_2(0) = \operatorname{const}$, since $\lambda_1 \lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0) y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10. (WIP)

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \dots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega, \bigcup_{i=1}^k \pi_i = \omega, \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \quad (0.3)$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \dots, n\}). \quad (0.4)$$

Then the Faà di Bruno's formula is given as [2, Sec. 2.4]¹

$$D^n (f_1 \circ f_2)(x) \eta_\omega = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k := |\pi|)}} D^k f_1(f_2(x)) \left(D^{|\pi_1|} f_2(x) \eta_{\pi_1}, \dots, D^{|\pi_k|} f_2(x) \eta_{\pi_k} \right), \quad (0.5)$$

where for any $\pi = (\pi_1, \dots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \dots, \omega_n\}$ of \mathbb{N} ,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \dots, \eta_{\sigma(\omega_n)}) \quad (0.6)$$

where σ is a permutation and $\sigma(\omega_1) < \dots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n -th derivatives of the both sides of

$$g \circ f^s(\eta, g(\eta)) = f^u(\eta, g(\eta)) \Leftrightarrow g \circ f^s \circ \tilde{g}(\eta) = f^u \circ \tilde{g}(\eta) \quad (0.7)$$

¹Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$\begin{aligned}
& \text{(l.h.s): } D^n(g \circ f^s \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k(g \circ f^s)(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \right. \\
&\quad \left. \dots, D^{|\tau_l|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right)
\end{aligned} \tag{0.8}$$

$$\begin{aligned}
& \text{(r.h.s): } D^n(f^u \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.9}$$

Here, we defined

$$\left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\omega} = \left(D^{|\pi_{\sigma(\omega_1)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_1)}}, \dots, D^{|\pi_{\sigma(\omega_n)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_n)}} \right), \tag{0.10}$$

where $\sigma(\omega_1) < \dots < \sigma(\omega_n)$. Thus we obtain

$$\begin{aligned}
& \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \dots, \right. \\
& \left. D^{|\tau_l|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right) = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.11}$$

Let D_s, D_u denote differentiation w.r.t E^s, E^u respectively. Let $x_1 \in E^s, x_2 \in E^u$. The linearization of the mapping f at the origin is

$$Df(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_s f^s(0) & D_u f^s(0) \\ D_s f^u(0) & D_u f^u(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{0.12}$$

The linearized dynamics is invariant on E^s and E^u , thus

$$D_u f^s(0) = 0, D_s f^u(0) = 0. \tag{0.13}$$

Let us denote $A_s := D_s f^s(0), A_u := D_u f^u(0)$. By definition $\sigma(A_s)$ (resp. $\sigma(A_u)$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A .

Let us derive a few concrete n -th order equations from (0.11).

We can see that

$$Df^s(0) D\tilde{g}(0) = (D_s f^s(0) \ D_u f^s(0)) \begin{pmatrix} I \\ Dg(0) \end{pmatrix} = A_s, \tag{0.14}$$

$$Df^u(0) D\tilde{g}(0) = A_u Dg(0). \tag{0.15}$$

Thus the first order equation is equivalent to the Sylvester equation

$$Dg(0) A_s \eta - A_u Dg(0) \eta = 0. \tag{0.16}$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n -multilinear maps of E_1, \dots, E_n to E_0 by $\mathcal{L}(E_1, \dots, E_n; E_0)$. For $\nu = 1, \dots, n$ let $(e_{\nu,1}, \dots, e_{\nu, \dim E_\nu})$ be an

ordered basis of E_ν and let $(e_{\nu,1}^*, \dots, e_{\nu, \dim E_\nu}^*)$ be its dual. Then the space $\mathcal{L}(E_1, \dots, E_n; E_0) = E_n^* \otimes \dots \otimes E_1^* \otimes E_0$, where \otimes means a tensor product, has a basis

$$(e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0})_{(r_n, \dots, r_0)}, \quad (r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}. \quad (0.17)$$

We order the basis (0.17) lexicographically with priority to the first components of (r_n, \dots, r_0) . A multilinear map $\tilde{U} \in \mathcal{L}(E_1, \dots, E_n; E_0)$ has a coordinate representation $U \in \mathbb{R}^{\dim E_0 \times \dim E_1^* \times \dots \times \dim E_n^*}$ as

$$\tilde{U} = \sum_{(r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}} U_{r_0 r_1 \dots r_n} e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0}. \quad (0.18)$$

Let $\text{vec} : \mathbb{R}^{J_1 \times \dots \times J_N} \rightarrow \mathbb{R}^{\prod_{i=1}^N J_i}$ be a vectorization operator

$$(\text{vec}(U))_i = U_{j_1 \dots j_N} \quad \text{with} \quad i = 1 + \sum_{l=1}^N \left[(j_l - 1) \prod_{l'=1}^{l-1} J_{l'} \right]. \quad (0.19)$$

By $U \times_n A \in \mathbb{R}^{J_1 \times \dots \times J_{n-1} \times I \times J_{n+1} \times \dots \times J_N}$ we denote an n -mode product of a tensor [3] $U \in \mathbb{R}^{J_1 \times \dots \times J_N}$ with a matrix $A \in \mathbb{R}^{I \times J_n}$, defined as

$$(U \times_n A)_{j_1 \dots j_{n-1} i j_{n+1} \dots j_N} = \sum_{j_n=1}^{J_n} U_{j_1 \dots j_N} A_{i j_n}. \quad (0.20)$$

An action of $\tilde{U} \in E_n^* \otimes \dots \otimes E_1^* \otimes E_0$ on $(\tilde{h}_1, \dots, \tilde{h}_n, \tilde{h}_0^*) \in \prod_{i=1}^n E^n \times E_0^*$ is rewritten in the coordinate representation as $U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\text{vec}(U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*) = (h_n^* \otimes \dots \otimes h_1^* \otimes h_0^*) \text{vec}(U), \quad (0.21)$$

where \otimes denotes the Kronecker product. In the following, we identify any multilinear map and elements in E^s, E^u as their coordinate representation.

Let $p = \dim E^s, q = \dim E^u$. From (0.16), it is obvious that for any $\eta_0^\top \in E^{u*}, \eta_1 \in E^s$

$$\begin{aligned} \eta_0^\top Dg(0) A_s \eta_1 - \eta_0^\top A_u Dg(0) \eta_1 &= 0 \\ \Leftrightarrow \text{vec} \left(Dg(0) \times_1 \eta_0^\top \times_2 \eta_1^\top A_s^\top \right) - \text{vec} \left(Dg(0) \times_1 \eta_0^\top A_u \times_2 \eta_1^\top \right) &= 0 \quad (\because \text{vec}(\cdot) \text{ is linear}) \\ \Leftrightarrow (\eta_1^\top A_s^\top \otimes \eta_0^\top I_q - \eta_1^\top I_p \otimes \eta_0^\top A_u) \text{vec} (Dg(0)) &= 0 \quad (\because (0.21)) \\ \Leftrightarrow (\eta_1^\top \otimes \eta_0^\top) \cdot (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec} (Dg(0)) &= 0 \quad (\because (AB) \otimes (CD) = (A \otimes C) \cdot (B \otimes D)) \\ \Leftrightarrow (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec} (Dg(0)) &= 0 \quad (\because \eta_0, \eta_1 \text{ are arbitrary}). \end{aligned} \quad (0.22)$$

The following theorem is well-known [4, Thm. 4.4.5].

Theorem 0.1.

If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}, \sigma(B) = \{\mu_1, \dots, \mu_m\}$, then $\sigma(I_m \otimes A + B \otimes I_n) = \{\lambda_i + \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

This implies $A_s^\top \otimes I_q - I_p \otimes A_u$ is regular since $\sigma(A_s)$ (resp. $\sigma(A_u)$) is included inside (resp. outside) the unit circle. Therefore $Dg(0) = 0$.

The second order equation of (0.11) is

$$D^2 g(0)(A_s \eta_1, A_s \eta_2) - Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2). \quad (0.23)$$

In the second term of the l.h.s., the first p columns of $Df^u(0)$ are zeros, so the first p mode-1 slices of $D^2 \tilde{g}(0)$ do not contribute to the term, i.e., $Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = A_u D^2 g(0)(\eta_1, \eta_2)$. Also, components of $D^2 f^u(0)$ of

indices larger than $p + 1$ do not contribute to the r.h.s. since $Dg(0) = 0$. Namely, $D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_s^2 f^u(0)(\eta_1, \eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{u*}$, $\eta_1, \eta_2 \in E^s$

$$\begin{aligned} \eta_0^\top D^2 g(0)(A_s \eta_1, A_s \eta_2) - \eta_0^\top A_u D^2 g(0)(\eta_1, \eta_2) &= \eta_0^\top D_s^2 f^u(0)(\eta_1, \eta_2) \\ \Leftrightarrow (A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u) \text{vec}(D^2 g(0)) &= \text{vec}(D_s^2 f^u(0)) \quad (\because \eta_0, \eta_1, \eta_2 \text{ are arbitrary}). \end{aligned} \quad (0.24)$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

Theorem 0.2.

If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, $\sigma(B) = \{\mu_1, \dots, \mu_m\}$, then $\sigma(A \otimes B) = \{\lambda_i \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

lead to the regularity of $(A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)$. Thus we obtain

$$D^2 g(0) = \text{vec}^{-1} \left[(A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)^{-1} \text{vec}(D_s^2 f^u(0)) \right]. \quad (0.25)$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

References

- [1] Wolf-Jürgen Beyn and Winfried Kleß. Numerical taylor expansions of invariant manifolds in large dynamical systems. *Numerische Mathematik*, 80(1):1–38, 1998.
- [2] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.
- [3] Tamara Gibson Kolda. Multilinear operators for higher-order decompositions. Technical report, Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA ..., 2006.
- [4] Horn Roger and R Johnson Charles. Topics in matrix analysis, 1994.