Chapter 4

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Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0)y_2(0) = \mathrm{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10. (WIP)

This answer greatly relies on [1] but somewhat concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{\pi = (\pi_1, \cdots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega,\ \bigcup_{i=1}^{k} \pi_i = \omega,\ \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j\right\}, \tag{0.3}$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \cdots, n\}). \tag{0.4}$$

Then the Faá di Bruno's formula is given as [2, Sec. 2.4][1]

$$D^n (f_1 \circ f_2)(x)\eta_\omega = \sum_{\substack{\pi\in\mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f_1(f_2(x)) \left(D^{|\pi_1|} f_2(x)\eta_{\pi_1}, \ldots, D^{|\pi_k|} f_2(x)\eta_{\pi_k}\right), \tag{0.5}$$

where for any $\pi = (\pi_1, \cdots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \cdots, \omega_n\}$ of $\mathbb{N}$,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \cdots, \eta_{\sigma(\omega_n)}) \tag{0.6}$$

where σ is a permutation and $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n-th derivatives of the both sides of

$$g \circ f^{\mathrm{s}}(\eta, g(\eta)) = f^{\mathrm{u}}(\eta, g(\eta)) \Leftrightarrow g \circ f^{\mathrm{s}} \circ \tilde{g}(\eta) = f^{\mathrm{u}} \circ \tilde{g}(\eta) \tag{0.7}$$

[1] Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$
\begin{aligned}
&\text{(l.h.s): } D^n(g\circ f^{\mathrm{s}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k(g\circ f^{\mathrm{s}})(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\right.\\
&\qquad\qquad\left.\dots,D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right) \qquad (0.8)\\
&\text{(r.h.s): } D^n(f^{\mathrm{u}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right). \qquad (0.9)
\end{aligned}
$$

Here, we defined

$$
\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_\omega=\left(D^{|\pi_{\sigma(\omega_1)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_1)}},\cdots,D^{|\pi_{\sigma(\omega_n)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_n)}}\right), \qquad (0.10)
$$

where $\sigma(\omega_1)<\cdots<\sigma(\omega_n)$. Thus we obtain

$$
\begin{aligned}
&\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\dots,\right.\\
&\left.D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right)=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right).
\end{aligned} \qquad (0.11)
$$

Let $D_{\mathrm{s}},D_{\mathrm{u}}$ denote differentiation w.r.t $E^{\mathrm{s}},E^{\mathrm{u}}$ respectively. Let $x_1\in E^{\mathrm{s}},x_2\in E^{\mathrm{u}}$. The linearization of the mapping f at the origin is

$$
Df(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}D_{\mathrm{s}}f^{\mathrm{s}}(0) & D_{\mathrm{u}}f^{\mathrm{s}}(0)\\ D_{\mathrm{s}}f^{\mathrm{u}}(0) & D_{\mathrm{u}}f^{\mathrm{u}}(0)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \qquad (0.12)
$$

The linearized dynamics is invariant on E^{s} and E^{u}, thus

$$
D_{\mathrm{u}}f^{\mathrm{s}}(0)=0,D_{\mathrm{s}}f^{\mathrm{u}}(0)=0. \qquad (0.13)
$$

Let us denote $A_{\mathrm{s}}:=D_{\mathrm{s}}f^{\mathrm{s}}(0),A_{\mathrm{u}}:=D_{\mathrm{u}}f^{\mathrm{u}}(0)$. By definition $\sigma_{A_{\mathrm{s}}}$ (resp. $\sigma_{A_{\mathrm{u}}}$) is included inside (resp. outside) the unit circle, where σ_A is the spectrum of A.

Let us derive a few concrete n-th order equations from (0.11).

We can see that

$$
Df^{\mathrm{s}}(0)D\tilde g(0)=(D_{\mathrm{s}}f^{\mathrm{s}}(0)\ D_{\mathrm{u}}f^{\mathrm{s}}(0))\begin{pmatrix}I\\Dg(0)\end{pmatrix}=A_{\mathrm{s}}, \qquad (0.14)
$$

$$
Df^{\mathrm{u}}(0)D\tilde g(0)=A_{\mathrm{u}}Dg(0). \qquad (0.15)
$$

Thus the first order equation is equivalent to the Sylvester equation

$$
Dg(0)A_{\mathrm{s}}\eta-A_{\mathrm{u}}Dg(0)\eta=0. \qquad (0.16)
$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n-multilinear maps of $E_1,\cdots,E_n$ to E_0 by $\mathcal{L}(E_1,\cdots,E_n;E_0)$. For $\nu=1,\cdots,n$ let $(e_{\nu,1},\cdots,e_{\nu,\dim E_\nu})$ be an ordered basis of E_ν and let $(e^*_{\nu,1}, \cdots, e^*_{\nu,\dim E_\nu})$ be its dual. Then the space $\mathcal{L}(E_1, \cdots, E_n; E_0) = E^*_n \otimes \cdots \otimes E^*_1 \otimes E_0$, where $\otimes$ means a tensor product, has a basis

$$(e^*_{n,r_n} \otimes \cdots \otimes e^*_{1,r_1} \otimes e_{0,r_0})_{(r_n \cdots, r_0)}, \quad (r_n \cdots, r_0) \in \prod_{k=n}^{0} \{1, \cdots, \dim E_k\}. \tag{0.17}$$

We order the basis (0.17) lexicographically with priority to the first components of $(r_n \cdots, r_0)$. A multilinear map $\tilde{U} \in \mathcal{L}(E_1, \cdots, E_n; E_0)$ has a coordinate representation $U \in \mathbb{R}^{\dim E^*_n \times \cdots \times \dim E^*_1 \times \dim E_0}$ as

$$\tilde{U} = \sum_{(r_n \cdots, r_0) \in \prod_{k=n}^{0} \{1, \cdots, \dim E_k\}} U_{r_n \cdots r_1 r_0} e^*_{n,r_n} \otimes \cdots \otimes e^*_{1,r_1} \otimes e_{0,r_0}. \tag{0.18}$$

Let $\mathrm{vec} : \mathbb{R}^{I_1 \times \cdots \times I_N} \to \mathbb{R}^{\prod_{i=1}^{N} I_i}$ be a vectorization operator

$$(\mathrm{vec}(U))_i = U_{i_1 \cdots i_N} \quad with \quad i = 1 + \sum_{l=1}^{N} \left[(i_l - 1) \prod_{l'=1}^{l-1} I_{l'} \right]. \tag{0.19}$$

By $U \times_n A \in \mathbb{R}^{I_1 \times \cdots I_{n-1} \times I \times I_{n+1} \times I_N}$ we denote an n-mode product of a tensor [3] $U \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ with a matrix $A \in \mathbb{R}^{I \times I_n}$, defined as

$$(U \times_n A)_{i_1 \cdots i_{n-1} i i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} U_{i_1 \cdots i_N} A_{i i_n}. \tag{0.20}$$

An action of $\tilde{U} \in E^*_n \otimes \cdots \otimes E^*_1 \otimes E_0$ on $(\tilde{h}_1, \cdots, \tilde{h}_n, \tilde{h}^*_0) \in \prod_{i=1}^{n} E^n \times E^*_0$ is rewritten in the coordinate representation as $U \times_1 h^*_0 \times_2 h^*_1 \times_3 \cdots \times_{n+1} h^*_n$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\mathrm{vec}(U \times_1 h^*_0 \times_2 h^*_1 \times_3 \cdots \times_{n+1} h^*_n) = (h^*_n \otimes \cdots \otimes h^*_1 \otimes h^*_0)\mathrm{vec}(U), \tag{0.21}$$

where $\otimes$ denotes the Kronecker product. In the following, we identify any multilinear map and elements in $E^{\mathrm{s}}, E^{\mathrm{u}}$ as their coordinate representation.

References

[1] Wolf-Jürgen Beyn and Winfried Kleß. Numerical taylor expansions of invariant manifolds in large dynamical systems. *Numerische Mathematik*, 80(1):1–38, 1998.

[2] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.

[3] Tamara Gibson Kolda. Multilinear operators for higher-order decompositions. Technical report, Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA ⋯, 2006.