Chapter 4

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Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0)y_2(0) = \text{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10.

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \cdots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega,\ \bigcup_{i=1}^{k} \pi_i = \omega,\ \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \tag{0.3}$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \cdots, n\}). \tag{0.4}$$

Then the Faá di Bruno's formula is given as [2, Sec. 2.4][1]

$$D^n \left(f_1 \circ f_2\right)(x)\eta_\omega = \sum_{\substack{\pi\in\mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f_1\left(f_2(x)\right)\left(D^{|\pi_1|}f_2(x)\eta_{\pi_1}, \ldots, D^{|\pi_k|}f_2(x)\eta_{\pi_k}\right), \tag{0.5}$$

where for any $\pi = (\pi_1, \cdots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \cdots, \omega_n\}$ of $\mathbb{N}$,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \cdots, \eta_{\sigma(\omega_n)}) \tag{0.6}$$

where σ is a permutation and $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n-th derivatives of the both sides of

$$g \circ f^{\mathrm{s}}(\eta, g(\eta)) = f^{\mathrm{u}}(\eta, g(\eta)) \Leftrightarrow g \circ f^{\mathrm{s}} \circ \tilde{g}(\eta) = f^{\mathrm{u}} \circ \tilde{g}(\eta) \tag{0.7}$$

[1] Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$
\begin{aligned}
&\text{(l.h.s): } D^n(g\circ f^{\mathrm{s}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k(g\circ f^{\mathrm{s}})(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\right.\\
&\qquad\qquad\left.\dots,D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right) \qquad (0.8)\\
&\text{(r.h.s): } D^n(f^{\mathrm{u}}\circ\tilde g)(0)(\eta_1,\cdots,\eta_n)\\
&=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right). \qquad (0.9)
\end{aligned}
$$

Here, we defined

$$
\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_\omega=\left(D^{|\pi_{\sigma(\omega_1)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_1)}},\cdots,D^{|\pi_{\sigma(\omega_n)}|}\tilde g(0)\eta_{\pi_{\sigma(\omega_n)}}\right), \tag{0.10}
$$

where $\sigma(\omega_1)<\cdots<\sigma(\omega_n)$. Thus we obtain

$$
\begin{aligned}
&\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}}\sum_{\substack{\tau\in\mathcal{Q}(k)\\(l:=|\tau|)}} D^l g(0)\left(D^{|\tau_1|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_1},\dots,\right.\\
&\left.D^{|\tau_l|}f^{\mathrm{s}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right)_{\tau_l}\right)=\sum_{\substack{\pi\in\mathcal{Q}(n)\\(k:=|\pi|)}} D^k f^{\mathrm{u}}(0)\left(D^{|\pi_1|}\tilde g(0)\eta_{\pi_1},\dots,D^{|\pi_k|}\tilde g(0)\eta_{\pi_k}\right).
\end{aligned} \tag{0.11}
$$

Let $D_{\mathrm{s}},D_{\mathrm{u}}$ denote differentiation w.r.t $E^{\mathrm{s}},E^{\mathrm{u}}$ respectively. Let $x_1\in E^{\mathrm{s}},x_2\in E^{\mathrm{u}}$. The linearization of the mapping f at the origin is

$$
Df(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}D_{\mathrm{s}}f^{\mathrm{s}}(0)&D_{\mathrm{u}}f^{\mathrm{s}}(0)\\D_{\mathrm{s}}f^{\mathrm{u}}(0)&D_{\mathrm{u}}f^{\mathrm{u}}(0)\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{0.12}
$$

The linearized dynamics is invariant on E^{s} and E^{u}, thus

$$
D_{\mathrm{u}}f^{\mathrm{s}}(0)=0,D_{\mathrm{s}}f^{\mathrm{u}}(0)=0. \tag{0.13}
$$

Let us denote $A_{\mathrm{s}}:=D_{\mathrm{s}}f^{\mathrm{s}}(0),A_{\mathrm{u}}:=D_{\mathrm{u}}f^{\mathrm{u}}(0)$. By definition $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A.

Let us derive a few concrete n-th order equations from (0.11).

We can see that

$$
Df^{\mathrm{s}}(0)D\tilde g(0)=(D_{\mathrm{s}}f^{\mathrm{s}}(0)\ D_{\mathrm{u}}f^{\mathrm{s}}(0))\begin{pmatrix}I\\Dg(0)\end{pmatrix}=A_{\mathrm{s}}, \tag{0.14}
$$

$$
Df^{\mathrm{u}}(0)D\tilde g(0)=A_{\mathrm{u}}Dg(0). \tag{0.15}
$$

Thus the first order equation is equivalent to the Sylvester equation

$$
Dg(0)A_{\mathrm{s}}\eta-A_{\mathrm{u}}Dg(0)\eta=0. \tag{0.16}
$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n-multilinear maps of $E_1,\cdots,E_n$ to E_0 by $\mathcal{L}(E_1,\cdots,E_n;E_0)$. For $\nu=1,\cdots,n$ let $(e_{\nu,1},\cdots,e_{\nu,\dim E_\nu})$ be an ordered basis of E_ν and let $(e^*_{\nu,1},\cdots,e^*_{\nu,\dim E_\nu})$ be its dual. Then the space $\mathcal{L}(E_1,\cdots,E_n;E_0)=E_n^*\otimes\cdots\otimes E_1^*\otimes E_0$, where $\otimes$ means a tensor product, has a basis

$$(e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0})_{(r_n,\cdots,r_0)},\quad (r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}. \tag{0.17}$$

We order the basis (0.17) lexicographically with priority to the first components of $(r_n,\cdots,r_0)$. A multilinear map $\tilde{U}\in\mathcal{L}(E_1,\cdots,E_n;E_0)$ has a coordinate representation $U\in\mathbb{R}^{\dim E_0\times\dim E_1^*\times\cdots\times\dim E_n^*}$ as

$$\tilde{U}=\sum_{(r_n\cdots,r_0)\in\prod_{k=n}^{0}\{1,\cdots,\dim E_k\}} U_{r_0r_1\cdots r_n}e^*_{n,r_n}\otimes\cdots\otimes e^*_{1,r_1}\otimes e_{0,r_0}. \tag{0.18}$$

Let $\mathrm{vec}:\mathbb{R}^{J_1\times\cdots\times J_N}\to\mathbb{R}^{\prod_{i=1}^N J_i}$ be a vectorization operator

$$(\mathrm{vec}(U))_i=U_{j_1\cdots j_N}\quad \textit{with}\quad i=1+\sum_{l=1}^{N}\left[(j_l-1)\prod_{l'=1}^{l-1}J_{l'}\right]. \tag{0.19}$$

By $U\times_n A\in\mathbb{R}^{J_1\times\cdots J_{n-1}\times I\times J_{n+1}\times\cdots\times J_N}$ we denote an n-mode product of a tensor [3] $U\in\mathbb{R}^{J_1\times\cdots\times J_N}$ with a matrix $A\in\mathbb{R}^{I\times J_n}$, defined as

$$(U\times_n A)_{j_1\cdots j_{n-1}ij_{n+1}\cdots j_N}=\sum_{j_n=1}^{J_n}U_{j_1\cdots j_N}A_{ij_n}. \tag{0.20}$$

An action of $\tilde{U}\in E_n^*\otimes\cdots\otimes E_1^*\otimes E_0$ on $(\tilde{h}_1,\cdots,\tilde{h}_n,\tilde{h}_0^*)\in\prod_{i=1}^n E^n\times E_0^*$ is rewritten in the coordinate representation as $U\times_1 h_0^*\times_2 h_1^*\times_3\cdots\times_{n+1}h_n^*$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\mathrm{vec}(U\times_1 h_0^*\times_2 h_1^*\times_3\cdots\times_{n+1}h_n^*)=(h_n^*\otimes\cdots\otimes h_1^*\otimes h_0^*)\mathrm{vec}(U), \tag{0.21}$$

where $\otimes$ denotes the Kronecker product. In the following, we identify any multilinear map and elements in $E^{\mathrm{s}},E^{\mathrm{u}}$ as their coordinate representation.

Let $p=\dim E^{\mathrm{s}},q=\dim E^{\mathrm{u}}$. From (0.16), it is obvious that for any $\eta_0^\top\in E^{\mathrm{u}*},\eta_1\in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top Dg(0)A_{\mathrm{s}}\eta_1-\eta_0^\top A_{\mathrm{u}}Dg(0)\eta_1=0\\
&\Leftrightarrow \mathrm{vec}\left(Dg(0)\times_1\eta_0^\top\times_2\eta_1^\top A_{\mathrm{s}}^\top\right)-\mathrm{vec}\left(Dg(0)\times_1\eta_0^\top A_{\mathrm{u}}\times_2\eta_1^\top\right)=0\quad(\because \mathrm{vec}(\cdot)\text{ is linear})\\
&\Leftrightarrow (\eta_1^\top A_{\mathrm{s}}^\top\otimes\eta_0^\top I_q-\eta_1^\top I_p\otimes\eta_0^\top A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because (0.21))\\
&\Leftrightarrow (\eta_1^\top\otimes\eta_0^\top)\cdot(A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because (AB)\otimes(CD)=(A\otimes C)\cdot(B\otimes D))\\
&\Leftrightarrow (A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}})\,\mathrm{vec}\,(Dg(0))=0\quad(\because \eta_0,\eta_1\text{ are arbitrary}).
\end{aligned} \tag{0.22}$$

The following theorem is well-known [4, Thm. 4.4.5].

Theorem 0.1.
If $\sigma(A)=\{\lambda_1,\cdots,\lambda_n\},\sigma(B)=\{\mu_1,\cdots,\mu_m\}$, then $\sigma(I_m\otimes A+B\otimes I_n)=\{\lambda_i+\mu_j\mid i\in\{1,\cdots,n\},j\in\{1,\cdots,m\}\}$.

This implies $A_{\mathrm{s}}^\top\otimes I_q-I_p\otimes A_{\mathrm{u}}$ is regular since $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle. Therefore $Dg(0)=0$.

The second order equation of (0.11) is

$$D^2g(0)(A_{\mathrm{s}}\eta_1,A_{\mathrm{s}}\eta_2)-Df^{\mathrm{u}}(0)D^2\tilde{g}(0)(\eta_1,\eta_2)=D^2f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1,D\tilde{g}(0)\eta_2). \tag{0.23}$$

In the second term of the l.h.s., the first p columns of $Df^{\mathrm{u}}(0)$ are zeros, so the first p mode-1 slices of $D^2\tilde{g}(0)$ do not contribute to the term, i.e., $Df^{\mathrm{u}}(0)D^2\tilde{g}(0)(\eta_1,\eta_2)=A_{\mathrm{u}}D^2g(0)(\eta_1,\eta_2)$. Also, components of $D^2f^{\mathrm{u}}(0)$ of indices larger than $p+1$ do not contribute to the r.h.s. since $Dg(0)=0$. Namely, $D^2 f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1,\eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{\mathrm{u}*}, \eta_1, \eta_2 \in E^{\mathrm{s}}$

$$\begin{aligned}
&\eta_0^\top D^2 g(0)(A_{\mathrm{s}}\eta_1, A_{\mathrm{s}}\eta_2) - \eta_0^\top A_{\mathrm{u}} D^2 g(0)(\eta_1,\eta_2) = \eta_0^\top D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1,\eta_2) \\
&\Leftrightarrow (A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}}) \operatorname{vec}\left(D^2 g(0)\right) = \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right) \quad (\because \eta_0, \eta_1, \eta_2 \text{ are arbitrary}).
\end{aligned} \tag{0.24}$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

Theorem 0.2.
If $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}, \sigma(B) = \{\mu_1, \cdots, \mu_m\}$, then $\sigma(A \otimes B) = \{\lambda_i \mu_j \mid i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}\}$.

lead to the regularity of $(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})$. Thus we obtain

$$D^2 g(0) = \operatorname{vec}^{-1}\left[(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})^{-1} \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right)\right]. \tag{0.25}$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1-y'^2) \\ y' = y + 1.6x(1-x^2) \end{cases}. \tag{0.26}$$

The jacobian of f is given as

$$Df(x,y) = \begin{pmatrix} 1 & 1.6(1-3y^2) \\ -1.6(1-3x'^2) & 1 - 1.6^2(1-3y^2)(1-3x'^2) \end{pmatrix}. \tag{0.27}$$

f is injective and $\det(Df(x,y)) \equiv 1$, so f is area-preserving. The fixed points (x_*, y_*)s satisfy simultaneous equations $y(1-y^2) = 0, x(1-x^2) = 0$, so $(x_*, y_*) \in \{0, 1, -1\}^2$. If $\operatorname{tr}(Df(x_*, y_*))^2 - 4\det(Df(x_*, y_*)) = \operatorname{tr}(Df(x_*, y_*))^2 - 4 > 0$, f is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned}
&\operatorname{tr}(Df(x_*, y_*))^2 - 4 = -1.6^2 \times 4(1-3y_*^2)(1-3x_*^2) + 1.6^4(1-3y_*^2)^2(1-3x_*^2)^2 > 0 \\
&\Leftrightarrow g(x_*, y_*) := 4 - 1.6^2(1-3y_*^2)(1-3x_*^2) \begin{cases} < 0 & \text{if } |x_*| + |y_*| \neq 1 \\ > 0 & \text{otherwise} \end{cases}. \ (\because x_*, y_* \text{ are not irrational})
\end{aligned} \tag{0.28}$$

We evaluate g on the fixed points as

$$\begin{aligned}
g(0,0) &= 4 - 1.6^2 > 0 \\
g(0,\pm 1) = g(\pm 1, 0) &= 4 + 1.6^2 \times 2 > 0 \\
g(\pm 1, \pm 1) &= 4 - 4 \times 1.6^2 < 0.
\end{aligned} \tag{0.29}$$

Therefore, $(0,0)$ is non-hyperbolic and the others are hyperbolic.

Exercise 4.20.

We take inner product with $\xi - \eta$ on both sides of $\xi = \eta + (\xi - \eta)$ as

$$\begin{aligned}
&(\xi - \eta, \xi) = (\xi - \eta, \eta) + ||\xi - \eta||^2 \\
&\Leftrightarrow 2 - 2\cos\theta = ||\xi - \eta||^2.
\end{aligned} \tag{0.30}$$

Then

$$||A\xi - A\eta||^2 = ||A\xi||^2 + ||A\eta||^2 - 2(A\xi, A\eta) \geq ||A\xi||^2 + ||A\eta||^2 - 2||A\xi|| \cdot ||A\eta|| = (\varrho^{-1} - \varrho)^2 \tag{0.31}$$

$$||A\xi - A\eta||^2 \leq ||A||_2^2 ||\xi - \eta||^2 = ||A||_2^2 (2 - 2\cos\theta) \ (\because \text{Eq. } (0.30)). \tag{0.32}$$

Thus the desired inequality is derived.

Exercise 4.21.

Let $v = \xi + \eta, \xi \in E^{\mathrm{u}}, \eta \in E^{\mathrm{s}}$. Let $\lambda_{\mathrm{s}}, C_{\mathrm{s}}$ (resp. $\lambda_{\mathrm{u}}, C_{\mathrm{u}}$) be constants which make all tangent vectors in E^{s} (resp. E^{u}) stable and define $C = \max(C_{\mathrm{s}}, C_{\mathrm{u}})$ and $\lambda = \max(\lambda_{\mathrm{s}}, \lambda_{\mathrm{u}})$. Also let $P_{\mathrm{s}}, P_{\mathrm{u}}$ be projections onto $E^{\mathrm{s}}, E^{\mathrm{u}}$.

$$\langle v, v\rangle \geq ||\xi||^2 + ||\eta||^2 \geq ||\xi + \eta||^2 = (v, v), \tag{0.33}$$

$$\begin{aligned}\langle v, v\rangle &\leq \frac{C_{\mathrm{s}}^2 ||\xi||^2}{1-\lambda_{\mathrm{s}}^{2\epsilon}} + \frac{C_{\mathrm{u}}^2 ||\eta||^2}{1-\lambda_{\mathrm{u}}^{2\epsilon}} \leq \frac{C^2}{1-\lambda^{2\epsilon}}(||\xi||^2 + ||\eta||^2) = \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}v||^2 + ||P_{\mathrm{s}}v||^2)\\ &\leq \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)(v, v).\end{aligned} \tag{0.34}$$

Thus we can take $D = \max\left(1, \frac{C^2}{1-\lambda^{2\epsilon}}(||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)\right)$.

Exercise 4.22.(WIP)

C^2 regularity of f gives rise to $||Q(x)|| \leq \mathcal{O}(1)||x||^2$, which follows from the mean value form of the remainder of the Taylor's theorem.

References

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