Chapter 4

Sho Shirasaka, December 2, 2022

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0)y_2(0) = \mathrm{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10. (WIP)