Chapter 4

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Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \text{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\text{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \text{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\text{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \text{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0)y_2(0) = \text{const}$, since $\lambda_1 \lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10.

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \cdots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega, \ \bigcup_{i=1}^{k} \pi_i = \omega, \ \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \tag{0.3}$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \cdots, n\}). \tag{0.4}$$

Then the Faá di Bruno's formula is given as [2, Sec. 2.4][1]

$$D^n (f_1 \circ f_2)(x)\eta_\omega = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f_1 (f_2(x)) \left(D^{|\pi_1|} f_2(x)\eta_{\pi_1}, \ldots, D^{|\pi_k|} f_2(x)\eta_{\pi_k} \right), \tag{0.5}$$

where for any $\pi = (\pi_1, \cdots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \cdots, \omega_n\}$ of $\mathbb{N}$,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \cdots, \eta_{\sigma(\omega_n)}) \tag{0.6}$$

where σ is a permutation and $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n-th derivatives of the both sides of

$$g \circ f^{\text{s}}(\eta, g(\eta)) = f^{\text{u}}(\eta, g(\eta)) \Leftrightarrow g \circ f^{\text{s}} \circ \tilde{g}(\eta) = f^{\text{u}} \circ \tilde{g}(\eta) \tag{0.7}$$

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[1] Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

(l.h.s): $D^n(g \circ f^{\mathrm{s}} \circ \tilde{g})(0)(\eta_1, \cdots, \eta_n)$

$$= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k(g \circ f^{\mathrm{s}})(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)$$

$$= \sum_{\substack{\pi \in \mathcal{Q}(n) \ \tau \in \mathcal{Q}(k) \\ (k:=|\pi|) \ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^{\mathrm{s}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)_{\tau_1}, \right.$$

$$\left. \ldots, D^{|\tau_l|} f^{\mathrm{s}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)_{\tau_l} \right) \tag{0.8}$$

(r.h.s): $D^n(f^{\mathrm{u}} \circ \tilde{g})(0)(\eta_1, \cdots, \eta_n)$

$$= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^{\mathrm{u}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right). \tag{0.9}$$

Here, we defined

$$\left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)_\omega = \left(D^{|\pi_{\sigma(\omega_1)}|}\tilde{g}(0)\eta_{\pi_{\sigma(\omega_1)}}, \cdots, D^{|\pi_{\sigma(\omega_n)}|}\tilde{g}(0)\eta_{\pi_{\sigma(\omega_n)}} \right), \tag{0.10}$$

where $\sigma(\omega_1) < \cdots < \sigma(\omega_n)$. Thus we obtain

$$\sum_{\substack{\pi \in \mathcal{Q}(n) \ \tau \in \mathcal{Q}(k) \\ (k:=|\pi|) \ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^{\mathrm{s}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)_{\tau_1}, \ldots, \right.$$

$$\left. D^{|\tau_l|} f^{\mathrm{s}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right)_{\tau_l} \right) = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^{\mathrm{u}}(0) \left(D^{|\pi_1|}\tilde{g}(0)\eta_{\pi_1}, \ldots, D^{|\pi_k|}\tilde{g}(0)\eta_{\pi_k} \right). \tag{0.11}$$

Let $D_{\mathrm{s}}, D_{\mathrm{u}}$ denote differentiation w.r.t $E^{\mathrm{s}}, E^{\mathrm{u}}$ respectively. Let $x_1 \in E^{\mathrm{s}}, x_2 \in E^{\mathrm{u}}$. The linearization of the mapping f at the origin is

$$Df(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_{\mathrm{s}} f^{\mathrm{s}}(0) & D_{\mathrm{u}} f^{\mathrm{s}}(0) \\ D_{\mathrm{s}} f^{\mathrm{u}}(0) & D_{\mathrm{u}} f^{\mathrm{u}}(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{0.12}$$

The linearized dynamics is invariant on E^{s} and E^{u}, thus

$$D_{\mathrm{u}} f^{\mathrm{s}}(0) = 0, \, D_{\mathrm{s}} f^{\mathrm{u}}(0) = 0. \tag{0.13}$$

Let us denote $A_{\mathrm{s}} := D_{\mathrm{s}} f^{\mathrm{s}}(0), A_{\mathrm{u}} := D_{\mathrm{u}} f^{\mathrm{u}}(0)$. By definition $\sigma(A_{\mathrm{s}})$ (resp. $\sigma(A_{\mathrm{u}})$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A.

Let us derive a few concrete n-th order equations from (0.11).

We can see that

$$Df^{\mathrm{s}}(0)D\tilde{g}(0) = (D_{\mathrm{s}} f^{\mathrm{s}}(0) \, D_{\mathrm{u}} f^{\mathrm{s}}(0)) \begin{pmatrix} I \\ Dg(0) \end{pmatrix} = A_{\mathrm{s}}, \tag{0.14}$$

$$Df^{\mathrm{u}}(0)D\tilde{g}(0) = A_{\mathrm{u}} Dg(0). \tag{0.15}$$

Thus the first order equation is equivalent to the Sylvester equation

$$Dg(0)A_{\mathrm{s}}\eta - A_{\mathrm{u}}Dg(0)\eta = 0. \tag{0.16}$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n-multilinear maps of $E_1, \cdots, E_n$ to E_0 by $\mathcal{L}(E_1, \cdots, E_n; E_0)$. For $\nu = 1, \cdots, n$ let $(e_{\nu,1}, \cdots, e_{\nu,\dim E_\nu})$ be an

ordered basis of E_ν and let $(e^*_{\nu,1}, \cdots, e^*_{\nu,\dim E_\nu})$ be its dual. Then the space $\mathcal{L}(E_1, \cdots, E_n; E_0) = E^*_n \otimes \cdots \otimes E^*_1 \otimes E_0$, where $\otimes$ means a tensor product, has a basis

$$(e^*_{n,r_n} \otimes \cdots \otimes e^*_{1,r_1} \otimes e_{0,r_0})_{(r_n, \cdots, r_0)}, \quad (r_n \cdots, r_0) \in \prod_{k=n}^{0} \{1, \cdots, \dim E_k\}. \tag{0.17}$$

We order the basis (0.17) lexicographically with priority to the first components of $(r_n, \cdots, r_0)$. A multilinear map $\tilde{U} \in \mathcal{L}(E_1, \cdots, E_n; E_0)$ has a coordinate representation $U \in \mathbb{R}^{\dim E_0 \times \dim E^*_1 \times \cdots \times \dim E^*_n}$ as

$$\tilde{U} = \sum_{(r_n \cdots, r_0) \in \prod_{k=n}^{0} \{1, \cdots, \dim E_k\}} U_{r_0 r_1 \cdots r_n} e^*_{n,r_n} \otimes \cdots \otimes e^*_{1,r_1} \otimes e_{0,r_0}. \tag{0.18}$$

Let $\mathrm{vec} : \mathbb{R}^{J_1 \times \cdots \times J_N} \to \mathbb{R}^{\prod_{i=1}^{N} J_i}$ be a vectorization operator

$$(\mathrm{vec}(U))_i = U_{j_1 \cdots j_N} \quad \text{with} \quad i = 1 + \sum_{l=1}^{N} \left[(j_l - 1) \prod_{l'=1}^{l-1} J_{l'} \right]. \tag{0.19}$$

By $U \times_n A \in \mathbb{R}^{J_1 \times \cdots J_{n-1} \times I \times J_{n+1} \times \cdots \times J_N}$ we denote an n-mode product of a tensor [3] $U \in \mathbb{R}^{J_1 \times \cdots \times J_N}$ with a matrix $A \in \mathbb{R}^{I \times J_n}$, defined as

$$(U \times_n A)_{j_1 \cdots j_{n-1} i j_{n+1} \cdots j_N} = \sum_{j_n=1}^{J_n} U_{j_1 \cdots j_N} A_{i j_n}. \tag{0.20}$$

An action of $\tilde{U} \in E^*_n \otimes \cdots \otimes E^*_1 \otimes E_0$ on $(\tilde{h}_1, \cdots, \tilde{h}_n, \tilde{h}^*_0) \in \prod_{i=1}^{n} E^n \times E^*_0$ is rewritten in the coordinate representation as $U \times_1 h^*_0 \times_2 h^*_1 \times_3 \cdots \times_{n+1} h^*_n$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\mathrm{vec}(U \times_1 h^*_0 \times_2 h^*_1 \times_3 \cdots \times_{n+1} h^*_n) = (h^*_n \otimes \cdots \otimes h^*_1 \otimes h^*_0) \mathrm{vec}(U), \tag{0.21}$$

where $\otimes$ denotes the Kronecker product. In the following, we identify any multilinear map and elements in E^s, E^u as their coordinate representation.

Let $p = \dim E^s, q = \dim E^u$. From (0.16), it is obvious that for any $\eta^\top_0 \in E^{u*}, \eta_1 \in E^s$

$$\eta^\top_0 Dg(0) A_s \eta_1 - \eta^\top_0 A_u Dg(0) \eta_1 = 0$$
$$\Leftrightarrow \mathrm{vec}\left(Dg(0) \times_1 \eta^\top_0 \times_2 \eta^\top_1 A^\top_s\right) - \mathrm{vec}\left(Dg(0) \times_1 \eta^\top_0 A_u \times_2 \eta^\top_1\right) = 0 \quad (\because \mathrm{vec}(\cdot) \text{ is linear})$$
$$\Leftrightarrow (\eta^\top_1 A^\top_s \otimes \eta^\top_0 I_q - \eta^\top_1 I_p \otimes \eta^\top_0 A_u) \mathrm{vec}(Dg(0)) = 0 \quad (\because (0.21))$$
$$\Leftrightarrow (\eta^\top_1 \otimes \eta^\top_0) \cdot (A^\top_s \otimes I_q - I_p \otimes A_u) \mathrm{vec}(Dg(0)) = 0 \quad (\because (AB) \otimes (CD) = (A \otimes C) \cdot (B \otimes D))$$
$$\Leftrightarrow (A^\top_s \otimes I_q - I_p \otimes A_u) \mathrm{vec}(Dg(0)) = 0 \quad (\because \eta_0, \eta_1 \text{ are arbitrary}). \tag{0.22}$$

The following theorem is well-known [4, Thm. 4.4.5].

Theorem 0.1.
If $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}, \sigma(B) = \{\mu_1, \cdots, \mu_m\}$, then $\sigma(I_m \otimes A + B \otimes I_n) = \{\lambda_i + \mu_j \mid i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}\}$.

This implies $A^\top_s \otimes I_q - I_p \otimes A_u$ is regular since $\sigma(A_s)$ (resp. $\sigma(A_u)$) is included inside (resp. outside) the unit circle. Therefore $Dg(0) = 0$.

The second order equation of (0.11) is

$$D^2 g(0)(A_s \eta_1, A_s \eta_2) - Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2). \tag{0.23}$$

In the second term of the l.h.s., the first p columns of $Df^u(0)$ are zeros, so the first p mode-1 slices of $D^2\tilde{g}(0)$ do not contribute to the term, i.e., $Df^u(0) D^2\tilde{g}(0)(\eta_1, \eta_2) = A_u D^2 g(0)(\eta_1, \eta_2)$. Also, components of $D^2 f^u(0)$ of

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indices larger than $p+1$ do not contribute to the r.h.s. since $Dg(0) = 0$. Namely, $D^2 f^{\mathrm{u}}(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1, \eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{\mathrm{u}*}, \eta_1, \eta_2 \in E^{\mathrm{s}}$

$$\eta_0^\top D^2 g(0)(A_{\mathrm{s}}\eta_1, A_{\mathrm{s}}\eta_2) - \eta_0^\top A_{\mathrm{u}} D^2 g(0)(\eta_1, \eta_2) = \eta_0^\top D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)(\eta_1, \eta_2)$$
$$\Leftrightarrow (A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}}) \operatorname{vec}\left(D^2 g(0)\right) = \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right) \quad (\because \eta_0, \eta_1, \eta_2 \text{ are arbitrary}). \tag{0.24}$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

Theorem 0.2.
If $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}, \sigma(B) = \{\mu_1, \cdots, \mu_m\}$, then $\sigma(A \otimes B) = \{\lambda_i \mu_j \mid i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}\}$.

lead to the regularity of $(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})$. Thus we obtain

$$D^2 g(0) = \operatorname{vec}^{-1}\left[(A_{\mathrm{s}}^\top \otimes A_{\mathrm{s}}^\top \otimes I_q - I_{p^2} \otimes A_{\mathrm{u}})^{-1} \operatorname{vec}\left(D_{\mathrm{s}}^2 f^{\mathrm{u}}(0)\right)\right]. \tag{0.25}$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1 - y'^2) \\ y' = y + 1.6x(1 - x^2) \end{cases}. \tag{0.26}$$

The jacobian of f is given as

$$Df(x, y) = \begin{pmatrix} 1 & 1.6(1 - 3y^2) \\ -1.6(1 - 3x'^2) & 1 - 1.6^2(1 - 3y^2)(1 - 3x'^2) \end{pmatrix}. \tag{0.27}$$

f is injective and $\det(Df(x, y)) \equiv 1$, so f is area-preserving. The fixed points (x_*, y_*)s satisfy simultaneous equations $y(1 - y^2) = 0, x(1 - x^2) = 0$, so $(x_*, y_*) \in \{0, 1, -1\}^2$. If $\operatorname{tr}(Df(x_*, y_*))^2 - 4\det(Df(x_*, y_*)) = \operatorname{tr}(Df(x_*, y_*))^2 - 4 > 0$, f is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\operatorname{tr}(Df(x_*, y_*))^2 - 4 = -1.6^2 \times 4(1 - 3y_*^2)(1 - 3x_*^2) + 1.6^4(1 - 3y_*^2)^2(1 - 3x_*^2)^2 > 0$$
$$\Leftrightarrow g(x_*, y_*) := 4 - 1.6^2(1 - 3y_*^2)(1 - 3x_*^2) \begin{cases} < 0 & \text{if } |x_*| + |y_*| \neq 1 \\ > 0 & \text{otherwise} \end{cases}. \ (\because x_*, y_* \text{ are not irrational}) \tag{0.28}$$

We evaluate g on the fixed points as

$$g(0, 0) = 4 - 1.6^2 > 0$$
$$g(0, \pm 1) = g(\pm 1, 0) = 4 + 1.6^2 \times 2 > 0$$
$$g(\pm 1, \pm 1) = 4 - 4 \times 1.6^2 < 0. \tag{0.29}$$

Therefore, $(0, 0)$ is non-hyperbolic and the others are hyperbolic.

Exercise 4.20.

We take inner product with $\xi - \eta$ on both sides of $\xi = \eta + (\xi - \eta)$ as

$$(\xi - \eta, \xi) = (\xi - \eta, \eta) + ||\xi - \eta||^2$$
$$\Leftrightarrow 2 - 2\cos\theta = ||\xi - \eta||^2. \tag{0.30}$$

Then

$$||A\xi - A\eta||^2 = ||A\xi||^2 + ||A\eta||^2 - 2(A\xi, A\eta) \geq ||A\xi||^2 + ||A\eta||^2 - 2||A\xi|| \cdot ||A\eta|| = (\varrho^{-1} - \varrho)^2 \tag{0.31}$$
$$||A\xi - A\eta||^2 \leq ||A||_2^2 ||\xi - \eta||^2 = ||A||_2^2(2 - 2\cos\theta) \ (\because \text{Eq. } (0.30)). \tag{0.32}$$

Thus the desired inequality is derived.

Exercise 4.21.

Let $v = \xi + \eta, \xi \in E^{\mathrm{u}}, \eta \in E^{\mathrm{s}}$. Let $\lambda_{\mathrm{s}}, C_{\mathrm{s}}$ (resp. $\lambda_{\mathrm{u}}, C_{\mathrm{u}}$) be constants which make all tangent vectors in E^{s} (resp. E^{u}) stable and define $C = \max(C_{\mathrm{s}}, C_{\mathrm{u}})$ and $\lambda = \max(\lambda_{\mathrm{s}}, \lambda_{\mathrm{u}})$. Also let $P_{\mathrm{s}}, P_{\mathrm{u}}$ be projections onto $E^{\mathrm{s}}, E^{\mathrm{u}}$.

$$\langle v, v \rangle \geq ||\xi||^2 + ||\eta||^2 \geq ||\xi + \eta||^2 = (v, v), \tag{0.33}$$

$$\langle v, v \rangle \leq \frac{C_{\mathrm{s}}^2 ||\xi||^2}{1 - \lambda_{\mathrm{s}}^{2\epsilon}} + \frac{C_{\mathrm{u}}^2 ||\eta||^2}{1 - \lambda_{\mathrm{u}}^{2\epsilon}} \leq \frac{C^2}{1 - \lambda^{2\epsilon}} (||\xi||^2 + ||\eta||^2) = \frac{C^2}{1 - \lambda^{2\epsilon}} (||P_{\mathrm{u}} v||^2 + ||P_{\mathrm{s}} v||^2)$$

$$\leq \frac{C^2}{1 - \lambda^{2\epsilon}} (||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)(v, v). \tag{0.34}$$

Thus we can take $D = \max\left(1, \frac{C^2}{1 - \lambda^{2\epsilon}}(||P_{\mathrm{u}}||_2^2 + ||P_{\mathrm{s}}||_2^2)\right)$.

Exercise 4.22.

Let $M = \max_{||x|| \leq \epsilon} ||D^2 Q(x)||, N = \max_{||x|| \leq \epsilon} ||D\varphi(x/\epsilon)||$. These maxima exist because $D^2 Q, D\varphi$ are continuous and the closed ball is compact. The Taylor theorem for $Q \in C^2$ [Sec. 4.5][5] says that

$$Q(x) = Q(0) + DQ(0)x + R_1(x) = R_1(x) \ (\because Q(0) = 0, DQ(0) = 0), \tag{0.35}$$

$$DQ(x) = DQ(0) + R_2(x) = R_2(x), \tag{0.36}$$

where R_1, R_2 are remainder terms. For x in the closed ball of radius ϵ, the mean value inequalities for the remainder terms [Thm. 4.C][5] lead to

$$||Q(x)|| = ||R_1(x)|| \leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} ||D^2 Q(\tau x)(x, x)|| \leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} ||D^2 Q(\tau x)|| \cdot ||x|| \cdot ||x|| \leq \frac{M}{2} ||x|| \cdot ||x||$$

$$\leq \frac{M\epsilon^2}{2} \tag{0.37}$$

$$||DQ(x)|| = ||R_2(x)|| \leq \sup_{0 \leq \tau \leq 1} ||D^2 Q(\tau x) x|| \leq M ||x|| \leq M\epsilon. \tag{0.38}$$

The differential of Q_ϵ is

$$DQ_\epsilon(x) = \frac{1}{\epsilon} D\varphi(x/\epsilon) Q(x) + \varphi(x/\epsilon) DQ(x). \tag{0.39}$$

Then

$$||DQ_\epsilon(x)|| \leq \frac{1}{\epsilon} ||D\varphi(x/\epsilon)|| \cdot ||Q(x)|| + ||\varphi(x/\epsilon)|| \cdot ||DQ(x)|| \leq \frac{M(N+2)}{2} \epsilon. \tag{0.40}$$

Exercise 4.23.

($\mathscr{H}_\alpha$ is a normed vector space) If $\beta, \gamma \in \mathbb{R}$ and $f, g \in \mathscr{H}_\alpha$, Then

$$||\beta f + \gamma g||_{\mathscr{H}_\alpha} = \sup_{x \neq y} \frac{||\beta(f(x) - f(y)) + \gamma(g(x) - g(y))||}{||x - y||^\alpha} \leq \sup_{x \neq y} \frac{\beta ||f(x) - f(y)||}{||x - y||^\alpha} + \sup_{x \neq y} \frac{\gamma ||g(x) - g(y)||}{||x - y||^\alpha}$$

$$< \infty. \tag{0.41}$$

$$\beta f(0) + \gamma g(0) = 0. \tag{0.42}$$

So, $\beta f + \gamma g \in \mathscr{H}_\alpha$. Also, Eq. (0.41) for $\beta = \gamma = 1$ includes the triangle inequality. The other conditions to be a norm is trivial. Hence $\mathscr{H}_\alpha$ is a normed vector space.

(Completeness) Consider a Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $\mathscr{H}_\alpha$. This is also Cauchy in a space of continuous functions endowed with sup-norm, say, $BC(E)$. $BC(E)$ is complete, so the sequence has a limit in $BC(E)$. We denote the limit as f.

Every Cauchy sequence in a metric space is bounded, so $||f_n||_{\mathscr{H}_\alpha} \leq M$ for all $n \in \mathbb{N}$ and some $M \in \mathbb{R}$. Hence

$$||f||_{\mathscr{H}_\alpha} = \lim_{n \to \infty} ||f_n||_{\mathscr{H}_\alpha} \ (\because \text{the norm is continuous}) \leq M < \infty. \tag{0.43}$$

Therefore $f \in \mathscr{H}_\alpha$.

Next, we show $f_n \to f$ in $\mathscr{H}_\alpha$. For any $\epsilon > 0$, there exists $N > 0$ such that $||f_p - f_q||_{\mathscr{H}_\alpha} < \epsilon$ whenever $p, q \geq N$ since $\{f_n\}$ is Cauchy. Then for any $n \geq N$

$$||f - f_n||_{\mathscr{H}_\alpha} = \lim_{m \to \infty} ||f_m - f_n||_{\mathscr{H}_\alpha} \ (\because \text{ the norm is continuous}) \leq \epsilon. \tag{0.44}$$

Thus $f_n \to f$ in $\mathscr{H}_\alpha$.

Therefore $\mathscr{H}_\alpha$ is complete.

Exercise 4.24.

$R^{\mathrm{u}}(0,0) = 0, R^{\mathrm{s}}(0,0) = 0$ since $R \in \mathscr{H}_\alpha$. Hence

$$\Psi^{\mathrm{u}}(R)(0,0) = (L^{\mathrm{u}})^{-1} R^{\mathrm{u}}(0,0) - (L^{\mathrm{u}})^{-1} Q_\epsilon^{\mathrm{u}}(0,0) = 0 \ (\because Q(0) = 0)$$
$$\Psi^{\mathrm{s}}(R)(0,0) = L^{\mathrm{s}} R^{\mathrm{s}}(0,0) + Q_\epsilon^{\mathrm{s}}(0,0) = 0. \tag{0.45}$$

Exercise 4.25.

The C^2 regularity is used to prove $||DQ_\epsilon|| \leq C\epsilon$. This inequality is used to bound $||\Psi(r) - \Psi(r')||_{\mathscr{H}_\alpha}$. To derive the inequality regarding $||\Psi(r) - \Psi(r')||_{\mathscr{H}_\alpha}$, Q_ϵ need not satisfy $||DQ_\epsilon|| \leq C\epsilon$. Instead, it is enough for Q_ϵ to be Lipschitz continuous of its Lipschitz constant $C\epsilon$.

Q is C^1, so for x in the closed ball of radius ϵ, the mean value inequalities for the remainder terms [Thm. 4.C][5] lead to

$$||Q(x)|| \leq M\epsilon \tag{0.46}$$

for some $M > 0$. Q_ϵ is zero outside the closed ball, so we can limit us to consider the ball to evaluate the Lipschitz constant. For x, y in the closed ball,

$$||Q_\epsilon(x) - Q_\epsilon(y)|| = ||\varphi(x/\epsilon) Q(x) - \varphi(y/\epsilon) Q(y)||$$
$$\leq ||\varphi(x/\epsilon) - \varphi(y/\epsilon)|| \cdot ||Q(x)|| + ||\varphi(y/\epsilon)|| \cdot ||Q(x) - Q(y)||$$
$$\leq (||\varphi(x/\epsilon)|| + ||\varphi(y/\epsilon)||) \cdot ||Q(x)|| + ||\varphi(y/\epsilon)|| \cdot (||Q(x)|| + ||Q(y)||)$$
$$\leq 4M\epsilon. \tag{0.47}$$

Thus Q_ϵ is Lipschitz continuous of its Lipschitz constant $C\epsilon$, where C can be taken as $4M$.

Exercise 4.35.

References

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