# Chapter 4

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### Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1 - y'^2) \\ y' = y + 1.6x(1 - x^2) \end{cases} . \tag{0.1}$$

The jacobian of $f$ is given as

$$Df(x, y) = \begin{pmatrix} 1 & 1.6(1 - 3y^2) \\ -1.6(1 - 3x'^2) & 1 - 1.6^2(1 - 3y^2)(1 - 3x'^2) \end{pmatrix}. \tag{0.2}$$

$f$ is injective and $\det(Df(x, y)) \equiv 1$, so $f$ is area-preserving. The fixed points $(x_*, y_*)$s satisfy simultaneous equations $y(1 - y^2) = 0, x(1 - x^2) = 0$, so $(x_*, y_*) \in \{0, 1, -1\}^2$. If $\mathrm{tr}\,(Df(x_*, y_*))^2 - 4\det(Df(x_*, y_*)) = \mathrm{tr}\,(Df(x_*, y_*))^2 - 4 > 0$, $f$ is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned} &\mathrm{tr}\,(Df(x_*, y_*))^2 - 4 = -1.6^2 \times 4(1 - 3y_*^2)(1 - 3x_*^2) + 1.6^4(1 - 3y_*^2)^2(1 - 3x_*^2)^2 > 0 \\ &\Leftrightarrow \mathrm{tr}\,(Df(x_*, y_*)) + 2 < 0 \ (\because x_*, y_* \text{ are not irrational}) \\ &\Leftrightarrow g(x_*, y_*) := 4 - 1.6^2(1 - 3y_*^2)(1 - 3x_*^2) < 0. \end{aligned} \tag{0.3}$$

We evaluate $g$ on the fixed points as

$$\begin{aligned} g(0, 0) &= 4 - 1.6^2 > 0 \\ g(0, \pm 1) = g(\pm 1, 0) &= 4 + 1.6^2 \times 2 > 0 \\ g(\pm 1, \pm 1) &= 4 - 4 \times 1.6^2 < 0. \end{aligned} \tag{0.4}$$

Therefore, $(0, 0), (0, \pm 1), (\pm 1, 0)$ are non-hyperbolic and $(\pm 1, \pm 1)$ are hyperbolic.

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