Chapter 3

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Exercise 3.26.

Every $[0,1)$ in this answer is homeomorphic to a circle, namely, $[0,1) = \mathbb{R}/\mathbb{Z}$. Note that $\mathbb{R}/\mathbb{Z}$ is compact.

Let $T_{\mathrm{L}}(y) = y/4, T_{\mathrm{R}}(y) = (2+y)/3$ and $C_0 = [0,1), C_n = T_{\mathrm{L}}(C_{n-1}) \cup T_{\mathrm{R}}(C_{n-1})$. $(C_n)_n$ is a monotone decreasing series of sets, so its limit exists. We denote the limit as $\mathcal{C} := \lim_{n\to\infty} C_n$.

$C_n \setminus C_{n-1}$ is an open set, so $\mathcal{C}$ is a complement of a union of open sets. Hence $\mathcal{C}$ is closed since a union of open sets is open. Thus $\mathcal{C}$ is compact. Let $\mathcal{A} := [0,1) \times \mathcal{C}$. This is also a compact set and is the product of a Cantor set $\mathcal{C}$ by a segment.

$f^n(\mathcal{A})_y = \mathcal{C}$ follows from the definition of C_n and $\mathcal{C}$. Thus $f^n(\mathcal{A}) \subset \mathcal{A}$, i.e., $\mathcal{A}$ is invariant.

Any neighborhood U of $\mathcal{A}$ contains an open neighborhood $\bar{U}$ of $\mathcal{A}$. For all $x \in [0,1), y \in \mathcal{C}$, by $\mathcal{B}_1(x), \mathcal{B}_2(x)$ we denote fundamental systems of neighborhoods at x, y in $[0,1), \mathcal{C}$ composed of open balls respectively. Any open neighborhood of $\mathcal{A}$ can be expressed as

$$\bar{U} = \bigcup_{(x,y)\in\mathcal{A}} \bigcup_{B_1(x)\in\mathcal{B}_1^{\bar{U}}(x), B_2(y)\in\mathcal{B}_2^{\bar{U}}(y)} B_1(x) \times B_2(y), \tag{0.1}$$

where $\mathcal{B}_1^{\bar{U}}(x) \subset \mathcal{B}_1(x), \mathcal{B}_2^{\bar{U}}(y) \subset \mathcal{B}_2(y)$. There is a subcover

$$\tilde{U} = \bigcup_{(x,y)\in\tilde{\mathcal{A}}} \bigcup_{B_1(x)\in\tilde{\mathcal{B}}_1^{\bar{U}}(x), B_2(y)\in\tilde{\mathcal{B}}_2^{\bar{U}}(y)} B_1(x) \times B_2(y), \tag{0.2}$$

of $\mathcal{A}$ of $\bar{U}$, where $|\tilde{\mathcal{A}}| \cdot |\tilde{\mathcal{B}}_1^{\bar{U}}(x)| \cdot |\tilde{\mathcal{B}}_2^{\bar{U}}(y)|$ is finite, since $\mathcal{A}$ is compact. Let r be a minimum radius of $\{\tilde{\mathcal{B}}_2^{\bar{U}}(y)\}_{(x,y)\in\tilde{\mathcal{A}}}$. This is nonzero due to the finiteness of $|\tilde{\mathcal{A}}| \cdot |\tilde{\mathcal{B}}_2^{\bar{U}}(y)|$. We can see that $f^n([0,1)^2 \subset \tilde{U} \subset \bar{U} \subset U$ for all $n > n_U$, where n_U is a number such that $(1/3)^{n_U} < r$. Hence $\mathcal{A}$ is an attracting set.

Apparently f is invertible on $\mathcal{A}$. Let $I_0 = [0,1/3) \times [0,1), I_1 = [1/3,1) \times [0,1)$ and let $\Phi_{\mathcal{R}} = (\mathcal{A}, \Omega_s, R)$ be the coding relation, where

$$\Omega_s := \left\{ \mathbf{x} \in \mathcal{D}^{\mathbb{Z}} \mid M_{x_j,x_{j+1}} = 1, \quad \forall j \in \mathbb{Z} \right\}, \quad \mathcal{D} := \{0,1\}, \tag{0.3}$$

$$M_{ij} = 1 \quad \forall (i,j) \in \mathcal{D}^2, \tag{0.4}$$

$$R = \{(x, \boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \Omega_s \text{ and } f^n(x) \in I_{\sigma_n} \text{ for all } n \in \mathbb{Z}\}. \tag{0.5}$$

$\Phi_{\mathcal{R}}$ is obviously left-total. We can also show that $\Phi_{\mathcal{R}}$ is one-to-many (resp. right-total) by the similar discussion of the paragraph 3 (resp. 4) of the solution of Exercise 2.26. Thus $\Psi := \Phi_{\mathcal{R}}^{-1}$ is surjective mapping on Ω_s, which provides a semiconjugacy

$$f \circ \Psi = \Psi \circ \mathcal{S}, \tag{0.6}$$

where $\mathcal{S}$ is a full shift on Ω_s.

We claim that, to prove an orbit of $x \in \mathcal{A}$ is dense, it is enough to show that there is a code $\boldsymbol{\sigma} \in \Omega_s$ such that it includes all strings of the form $(b_n, b_{n-1}, \cdots, b_2, b_1, a_1, a_2, \cdots, a_{n-1}, a_n)$. Let $z \in \mathcal{A}$ and $\boldsymbol{\sigma}_z \in \Psi^{-1}(z)$. For all codes $\tilde{\boldsymbol{\sigma}}$ which satisfies $(\sigma_i)_{i=-n}^n = (\sigma_{z,i})_{i=-n}^n$, z and $\Psi(\tilde{\boldsymbol{\sigma}})$ are in the same rectangle, whose side lengths are at most $(2/3)^n, (1/3)^n$. Thus such code $\boldsymbol{\sigma}$ is associated with $x \in \mathcal{A}$ whose orbit has element arbitrary close to any element in $\mathcal{A}$. Let $\|$ be a concatenation operator of strings. We define

$$(\sigma_i)_{i\leq 0} \equiv 0 \tag{0.7}$$

$$(\sigma_i)_{i\geq 0} = \mathop{\Big\|}_{n=1}^{\infty} \mathop{\Big\|}_{\mathbf{x}\in\mathcal{D}^{2n}} \mathbf{x}. \tag{0.8}$$

Then the orbit of $\Psi(\boldsymbol{\sigma})$ is dense in $\mathcal{A}$. Therefore, $\mathcal{A}$ is an attractor.