

## Chapter 4

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### Exercise 4.2.

By taking derivatives of both sides of  $f^{-1} \circ f(x) = x$ , we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \quad (0.1)$$

### Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \text{tr}(A)x + \det(A). \quad (0.2)$$

Its discriminant is  $\text{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$ . Moreover,  $f(0) = 1 > 0$  and  $f(1) = 2 - \text{tr}(A) < 0$ . Hence the two eigenvalues  $\lambda_1, \lambda_2$  of  $A$  are real and positive and satisfy  $\lambda_1 < 1 < \lambda_2$ .

The matrix  $A$  can be diagonalized as  $A = P \text{diag}(\lambda_1, \lambda_2) P^{-1}$  since the eigenvalues are distinct. Let us define  $y = P^{-1}x$ . This conjugates  $x \mapsto Ax$  and  $y \mapsto \text{diag}(\lambda_1, \lambda_2)y$ . An orbit in the  $y$ -coordinate can be expressed as  $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$ ,  $(y_1(0), y_2(0)) \in \mathbb{R}^2$ . The orbit belongs to  $y_1 y_2 = y_1(0) y_2(0) = \text{const}$ , since  $\lambda_1 \lambda_2 = \det(A) = 1$ . This is a hyperbola if  $y_1(0) y_2(0) \neq 0$  and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix  $x = Py$  is a hyperbola (resp. a line). Thus each orbit of the linear map  $x \mapsto Ax$  belongs to a hyperbola (or a line in a degenerate situation).

### Exercise 4.10.

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let  $f_1, f_2$  be  $C^n$  and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \dots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega, \bigcup_{i=1}^k \pi_i = \omega, \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \quad (0.3)$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \dots, n\}). \quad (0.4)$$

Then the Faà di Bruno's formula is given as [2, Sec. 2.4]<sup>1</sup>

$$D^n (f_1 \circ f_2)(x) \eta_\omega = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k := |\pi|)}} D^k f_1(f_2(x)) \left( D^{|\pi_1|} f_2(x) \eta_{\pi_1}, \dots, D^{|\pi_k|} f_2(x) \eta_{\pi_k} \right), \quad (0.5)$$

where for any  $\pi = (\pi_1, \dots, \pi_k)$  let  $|\pi| = k$  denote its length and for any finite subset  $\omega = \{\omega_1, \dots, \omega_n\}$  of  $\mathbb{N}$ ,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \dots, \eta_{\sigma(\omega_n)}) \quad (0.6)$$

where  $\sigma$  is a permutation and  $\sigma(\omega_1) < \dots < \sigma(\omega_n)$ .

Let  $\tilde{g}(x) := (x, g(x))$ . We take the  $n$ -th derivatives of the both sides of

$$g \circ f^s(\eta, g(\eta)) = f^u(\eta, g(\eta)) \Leftrightarrow g \circ f^s \circ \tilde{g}(\eta) = f^u \circ \tilde{g}(\eta) \quad (0.7)$$

<sup>1</sup>Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$\begin{aligned}
& \text{(l.h.s): } D^n(g \circ f^s \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k(g \circ f^s)(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left( D^{|\tau_1|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \right. \\
&\quad \left. \dots, D^{|\tau_l|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right)
\end{aligned} \tag{0.8}$$

$$\begin{aligned}
& \text{(r.h.s): } D^n(f^u \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.9}$$

Here, we defined

$$\left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\omega} = \left( D^{|\pi_{\sigma(\omega_1)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_1)}}, \dots, D^{|\pi_{\sigma(\omega_n)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_n)}} \right), \tag{0.10}$$

where  $\sigma(\omega_1) < \dots < \sigma(\omega_n)$ . Thus we obtain

$$\begin{aligned}
& \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left( D^{|\tau_1|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \dots, \right. \\
& \left. D^{|\tau_l|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right) = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left( D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.11}$$

Let  $D_s, D_u$  denote differentiation w.r.t  $E^s, E^u$  respectively. Let  $x_1 \in E^s, x_2 \in E^u$ . The linearization of the mapping  $f$  at the origin is

$$Df(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_s f^s(0) & D_u f^s(0) \\ D_s f^u(0) & D_u f^u(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{0.12}$$

The linearized dynamics is invariant on  $E^s$  and  $E^u$ , thus

$$D_u f^s(0) = 0, D_s f^u(0) = 0. \tag{0.13}$$

Let us denote  $A_s := D_s f^s(0), A_u := D_u f^u(0)$ . By definition  $\sigma(A_s)$  (resp.  $\sigma(A_u)$ ) is included inside (resp. outside) the unit circle, where  $\sigma(A)$  is the spectrum of  $A$ .

Let us derive a few concrete  $n$ -th order equations from (0.11).

We can see that

$$Df^s(0) D\tilde{g}(0) = (D_s f^s(0) \ D_u f^s(0)) \begin{pmatrix} I \\ Dg(0) \end{pmatrix} = A_s, \tag{0.14}$$

$$Df^u(0) D\tilde{g}(0) = A_u Dg(0). \tag{0.15}$$

Thus the first order equation is equivalent to the Sylvester equation

$$Dg(0) A_s \eta - A_u Dg(0) \eta = 0. \tag{0.16}$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of  $n$ -multilinear maps of  $E_1, \dots, E_n$  to  $E_0$  by  $\mathcal{L}(E_1, \dots, E_n; E_0)$ . For  $\nu = 1, \dots, n$  let  $(e_{\nu,1}, \dots, e_{\nu, \dim E_\nu})$  be an

ordered basis of  $E_\nu$  and let  $(e_{\nu,1}^*, \dots, e_{\nu, \dim E_\nu}^*)$  be its dual. Then the space  $\mathcal{L}(E_1, \dots, E_n; E_0) = E_n^* \otimes \dots \otimes E_1^* \otimes E_0$ , where  $\otimes$  means a tensor product, has a basis

$$(e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0})_{(r_n, \dots, r_0)}, \quad (r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}. \quad (0.17)$$

We order the basis (0.17) lexicographically with priority to the first components of  $(r_n, \dots, r_0)$ . A multilinear map  $\tilde{U} \in \mathcal{L}(E_1, \dots, E_n; E_0)$  has a coordinate representation  $U \in \mathbb{R}^{\dim E_0 \times \dim E_1^* \times \dots \times \dim E_n^*}$  as

$$\tilde{U} = \sum_{(r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}} U_{r_0 r_1 \dots r_n} e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0}. \quad (0.18)$$

Let  $\text{vec} : \mathbb{R}^{J_1 \times \dots \times J_N} \rightarrow \mathbb{R}^{\prod_{i=1}^N J_i}$  be a vectorization operator

$$(\text{vec}(U))_i = U_{j_1 \dots j_N} \quad \text{with} \quad i = 1 + \sum_{l=1}^N \left[ (j_l - 1) \prod_{l'=1}^{l-1} J_{l'} \right]. \quad (0.19)$$

By  $U \times_n A \in \mathbb{R}^{J_1 \times \dots \times J_{n-1} \times I \times J_{n+1} \times \dots \times J_N}$  we denote an  $n$ -mode product of a tensor [3]  $U \in \mathbb{R}^{J_1 \times \dots \times J_N}$  with a matrix  $A \in \mathbb{R}^{I \times J_n}$ , defined as

$$(U \times_n A)_{j_1 \dots j_{n-1} i j_{n+1} \dots j_N} = \sum_{j_n=1}^{J_n} U_{j_1 \dots j_N} A_{i j_n}. \quad (0.20)$$

An action of  $\tilde{U} \in E_n^* \otimes \dots \otimes E_1^* \otimes E_0$  on  $(\tilde{h}_1, \dots, \tilde{h}_n, \tilde{h}_0^*) \in \prod_{i=1}^n E^n \times E_0^*$  is rewritten in the coordinate representation as  $U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*$ . The  $\text{vec}$  operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\text{vec}(U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*) = (h_n^* \otimes \dots \otimes h_1^* \otimes h_0^*) \text{vec}(U), \quad (0.21)$$

where  $\otimes$  denotes the Kronecker product. In the following, we identify any multilinear map and elements in  $E^s, E^u$  as their coordinate representation.

Let  $p = \dim E^s, q = \dim E^u$ . From (0.16), it is obvious that for any  $\eta_0^\top \in E^{u*}, \eta_1 \in E^s$

$$\begin{aligned} \eta_0^\top Dg(0) A_s \eta_1 - \eta_0^\top A_u Dg(0) \eta_1 &= 0 \\ \Leftrightarrow \text{vec} \left( Dg(0) \times_1 \eta_0^\top \times_2 \eta_1^\top A_s^\top \right) - \text{vec} \left( Dg(0) \times_1 \eta_0^\top A_u \times_2 \eta_1^\top \right) &= 0 \quad (\because \text{vec}(\cdot) \text{ is linear}) \\ \Leftrightarrow (\eta_1^\top A_s^\top \otimes \eta_0^\top I_q - \eta_1^\top I_p \otimes \eta_0^\top A_u) \text{vec}(Dg(0)) &= 0 \quad (\because (0.21)) \\ \Leftrightarrow (\eta_1^\top \otimes \eta_0^\top) \cdot (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec}(Dg(0)) &= 0 \quad (\because (AB) \otimes (CD) = (A \otimes C) \cdot (B \otimes D)) \\ \Leftrightarrow (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec}(Dg(0)) &= 0 \quad (\because \eta_0, \eta_1 \text{ are arbitrary}). \end{aligned} \quad (0.22)$$

The following theorem is well-known [4, Thm. 4.4.5].

**Theorem 0.1.**

If  $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}, \sigma(B) = \{\mu_1, \dots, \mu_m\}$ , then  $\sigma(I_m \otimes A + B \otimes I_n) = \{\lambda_i + \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ .

This implies  $A_s^\top \otimes I_q - I_p \otimes A_u$  is regular since  $\sigma(A_s)$  (resp.  $\sigma(A_u)$ ) is included inside (resp. outside) the unit circle. Therefore  $Dg(0) = 0$ .

The second order equation of (0.11) is

$$D^2 g(0)(A_s \eta_1, A_s \eta_2) - Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2). \quad (0.23)$$

In the second term of the l.h.s., the first  $p$  columns of  $Df^u(0)$  are zeros, so the first  $p$  mode-1 slices of  $D^2 \tilde{g}(0)$  do not contribute to the term, i.e.,  $Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = A_u D^2 g(0)(\eta_1, \eta_2)$ . Also, components of  $D^2 f^u(0)$  of

indices larger than  $p + 1$  do not contribute to the r.h.s. since  $Dg(0) = 0$ . Namely,  $D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_s^2 f^u(0)(\eta_1, \eta_2)$ . Then we obtain a multilinear Sylvester equation. For any  $\eta_0^\top \in E^{u*}$ ,  $\eta_1, \eta_2 \in E^s$

$$\begin{aligned} \eta_0^\top D^2 g(0)(A_s \eta_1, A_s \eta_2) - \eta_0^\top A_u D^2 g(0)(\eta_1, \eta_2) &= \eta_0^\top D_s^2 f^u(0)(\eta_1, \eta_2) \\ \Leftrightarrow (A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u) \text{vec}(D^2 g(0)) &= \text{vec}(D_s^2 f^u(0)) \quad (\because \eta_0, \eta_1, \eta_2 \text{ are arbitrary}). \end{aligned} \quad (0.24)$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

**Theorem 0.2.**

If  $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ ,  $\sigma(B) = \{\mu_1, \dots, \mu_m\}$ , then  $\sigma(A \otimes B) = \{\lambda_i \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$ .

lead to the regularity of  $(A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)$ . Thus we obtain

$$D^2 g(0) = \text{vec}^{-1} \left[ (A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)^{-1} \text{vec}(D_s^2 f^u(0)) \right]. \quad (0.25)$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

**Exercise 4.12.**

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1 - y'^2) \\ y' = y + 1.6x(1 - x^2) \end{cases}. \quad (0.26)$$

The jacobian of  $f$  is given as

$$Df(x, y) = \begin{pmatrix} 1 & 1.6(1 - 3y^2) \\ -1.6(1 - 3x'^2) & 1 - 1.6^2(1 - 3y^2)(1 - 3x'^2) \end{pmatrix}. \quad (0.27)$$

$f$  is injective and  $\det(Df(x, y)) \equiv 1$ , so  $f$  is area-preserving. The fixed points  $(x_*, y_*)$ s satisfy simultaneous equations  $y(1 - y^2) = 0, x(1 - x^2) = 0$ , so  $(x_*, y_*) \in \{0, 1, -1\}^2$ . If  $\text{tr}(Df(x_*, y_*))^2 - 4 \det(Df(x_*, y_*)) = \text{tr}(Df(x_*, y_*))^2 - 4 > 0$ ,  $f$  is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned} \text{tr}(Df(x_*, y_*))^2 - 4 &= -1.6^2 \times 4(1 - 3y_*^2)(1 - 3x_*^2) + 1.6^4(1 - 3y_*^2)^2(1 - 3x_*^2)^2 > 0 \\ \Leftrightarrow g(x_*, y_*) &:= 4 - 1.6^2(1 - 3y_*^2)(1 - 3x_*^2) \begin{cases} < 0 & \text{if } |x_*| + |y_*| \neq 1 \\ > 0 & \text{otherwise} \end{cases} \quad (\because x_*, y_* \text{ are not irrational}) \end{aligned} \quad (0.28)$$

We evaluate  $g$  on the fixed points as

$$\begin{aligned} g(0, 0) &= 4 - 1.6^2 > 0 \\ g(0, \pm 1) &= g(\pm 1, 0) = 4 + 1.6^2 \times 2 > 0 \\ g(\pm 1, \pm 1) &= 4 - 4 \times 1.6^2 < 0. \end{aligned} \quad (0.29)$$

Therefore,  $(0, 0)$  is non-hyperbolic and the others are hyperbolic.

**Exercise 4.20.**

We take inner product with  $\xi - \eta$  on both sides of  $\xi = \eta + (\xi - \eta)$  as

$$\begin{aligned} (\xi - \eta, \xi) &= (\xi - \eta, \eta) + \|\xi - \eta\|^2 \\ \Leftrightarrow 2 - 2 \cos \theta &= \|\xi - \eta\|^2. \end{aligned} \quad (0.30)$$

Then

$$\|A\xi - A\eta\|^2 = \|A\xi\|^2 + \|A\eta\|^2 - 2(A\xi, A\eta) \geq \|A\xi\|^2 + \|A\eta\|^2 - 2\|A\xi\| \cdot \|A\eta\| = (\varrho^{-1} - \varrho)^2 \quad (0.31)$$

$$\|A\xi - A\eta\|^2 \leq \|A\|_2^2 \|\xi - \eta\|^2 = \|A\|_2^2 (2 - 2 \cos \theta) \quad (\because \text{Eq. (0.30)}). \quad (0.32)$$

Thus the desired inequality is derived.

### Exercise 4.21.

Let  $v = \xi + \eta$ ,  $\xi \in E^u$ ,  $\eta \in E^s$ . Let  $\lambda_s, C_s$  (resp.  $\lambda_u, C_u$ ) be constants which make all tangent vectors in  $E^s$  (resp.  $E^u$ ) stable and define  $C = \max(C_s, C_u)$  and  $\lambda = \max(\lambda_s, \lambda_u)$ . Also let  $P_s, P_u$  be projections onto  $E^s, E^u$ .

$$\langle v, v \rangle \geq \|\xi\|^2 + \|\eta\|^2 \geq \|\xi + \eta\|^2 = \langle v, v \rangle, \quad (0.33)$$

$$\begin{aligned} \langle v, v \rangle &\leq \frac{C_s^2 \|\xi\|^2}{1 - \lambda_s^{2\epsilon}} + \frac{C_u^2 \|\eta\|^2}{1 - \lambda_u^{2\epsilon}} \leq \frac{C^2}{1 - \lambda^{2\epsilon}} (\|\xi\|^2 + \|\eta\|^2) = \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u v\|^2 + \|P_s v\|^2) \\ &\leq \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u\|_2^2 + \|P_s\|_2^2) \langle v, v \rangle. \end{aligned} \quad (0.34)$$

Thus we can take  $D = \max\left(1, \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u\|_2^2 + \|P_s\|_2^2)\right)$ .

### Exercise 4.22.

Let  $M = \max_{\|x\| \leq \epsilon} \|D^2 Q(x)\|$ ,  $N = \max_{\|x\| \leq \epsilon} \|D\varphi(x/\epsilon)\|$ . These maxima exist because  $D^2 Q, D\varphi$  are continuous and the closed ball is compact. The Taylor theorem for  $Q \in C^2$  [Sec. 4.5][5] says that

$$Q(x) = Q(0) + DQ(0)x + R_1(x) = R_1(x) \quad (\because Q(0) = 0, DQ(0) = 0), \quad (0.35)$$

$$DQ(x) = DQ(0) + R_2(x) = R_2(x), \quad (0.36)$$

where  $R_1, R_2$  are remainder terms. For  $x$  in the closed ball of radius  $\epsilon$ , the mean value inequalities for the remainder terms [Thm. 4.C][5] lead to

$$\begin{aligned} \|Q(x)\| = \|R_1(x)\| &\leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)(x, x)\| \leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)\| \cdot \|x\| \cdot \|x\| \leq \frac{M}{2} \|x\| \cdot \|x\| \\ &\leq \frac{M\epsilon^2}{2} \end{aligned} \quad (0.37)$$

$$\|DQ(x)\| = \|R_2(x)\| \leq \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)x\| \leq M\|x\| \leq M\epsilon. \quad (0.38)$$

The differential of  $Q_\epsilon$  is

$$DQ_\epsilon(x) = \frac{1}{\epsilon} D\varphi(x/\epsilon)Q(x) + \varphi(x/\epsilon)DQ(x). \quad (0.39)$$

Then

$$\|DQ_\epsilon(x)\| \leq \frac{1}{\epsilon} \|D\varphi(x/\epsilon)\| \cdot \|Q(x)\| + \|\varphi(x/\epsilon)\| \cdot \|DQ(x)\| \leq \frac{M(N+2)}{2} \epsilon. \quad (0.40)$$

### Exercise 4.23.

( $\mathcal{H}_\alpha$  is a normed vector space) If  $\beta, \gamma \in \mathbb{R}$  and  $f, g \in \mathcal{H}_\alpha$ , Then

$$\begin{aligned} \|\beta f + \gamma g\|_{\mathcal{H}_\alpha} &= \sup_{x \neq y} \frac{\|\beta(f(x) - f(y)) + \gamma(g(x) - g(y))\|}{\|x - y\|^\alpha} \leq \sup_{x \neq y} \frac{\beta\|f(x) - f(y)\|}{\|x - y\|^\alpha} + \sup_{x \neq y} \frac{\gamma\|g(x) - g(y)\|}{\|x - y\|^\alpha} \\ &< \infty. \end{aligned} \quad (0.41)$$

$$\beta f(0) + \gamma g(0) = 0. \quad (0.42)$$

So,  $\beta f + \gamma g \in \mathcal{H}_\alpha$ . Also, Eq. (0.41) for  $\beta = \gamma = 1$  includes the triangle inequality. The other conditions to be a norm is trivial. Hence  $\mathcal{H}_\alpha$  is a normed vector space.

(Completeness) Consider a Cauchy sequence  $\{f_n\}_{n=1}^\infty$  in  $\mathcal{H}_\alpha$ . This is also Cauchy in a space of continuous functions endowed with sup-norm, say,  $BC(E)$ .  $BC(E)$  is complete, so the sequence has a limit in  $BC(E)$ . We denote the limit as  $f$ .

Every Cauchy sequence in a metric space is bounded, so  $\|f_n\|_{\mathcal{H}_\alpha} \leq M$  for all  $n \in \mathbb{N}$  and some  $M \in \mathbb{R}$ . Hence

$$\|f\|_{\mathcal{H}_\alpha} = \lim_{n \rightarrow \infty} \|f_n\|_{\mathcal{H}_\alpha} (\because \text{the norm is continuous}) \leq M < \infty. \quad (0.43)$$

Therefore  $f \in \mathcal{H}_\alpha$ .

Next, we show  $f_n \rightarrow f$  in  $\mathcal{H}_\alpha$ . For any  $\epsilon > 0$ , there exists  $N > 0$  such that  $\|f_p - f_q\|_{\mathcal{H}_\alpha} < \epsilon$  whenever  $p, q \geq N$  since  $\{f_n\}$  is Cauchy. Then for any  $n \geq N$

$$\|f - f_n\|_{\mathcal{H}_\alpha} = \lim_{m \rightarrow \infty} \|f_m - f_n\|_{\mathcal{H}_\alpha} (\because \text{the norm is continuous}) \leq \epsilon. \quad (0.44)$$

Thus  $f_n \rightarrow f$  in  $\mathcal{H}_\alpha$ .

Therefore  $\mathcal{H}_\alpha$  is complete.

#### Exercise 4.24.

$R^u(0, 0) = 0, R^s(0, 0) = 0$  since  $R \in \mathcal{H}_\alpha$ . Hence

$$\begin{aligned} \Psi^u(R)(0, 0) &= (L^u)^{-1} R^u(0, 0) - (L^u)^{-1} Q_\epsilon^u(0, 0) = 0 (\because Q(0) = 0) \\ \Psi^s(R)(0, 0) &= L^s R^s(0, 0) + Q_\epsilon^s(0, 0) = 0. \end{aligned} \quad (0.45)$$

#### Exercise 4.25.

The  $C^2$  regularity is used to prove  $\|DQ_\epsilon\| \leq C\epsilon$ . This inequality is used to bound  $\|\Psi(r) - \Psi(r')\|_{\mathcal{H}_\alpha}$ . To derive the inequality regarding  $\|\Psi(r) - \Psi(r')\|_{\mathcal{H}_\alpha}$ ,  $Q_\epsilon$  need not satisfy  $\|DQ_\epsilon\| \leq C\epsilon$ . Instead, it is enough for  $Q_\epsilon$  to be Lipschitz continuous of its Lipschitz constant  $C\epsilon$ .

Let  $\mathcal{B}_\epsilon$  be a closed ball of radius  $\epsilon$  centered at the origin. Every continuous functions on a compact domain is globally Lipschitz, so  $DQ$  is globally Lipschitz on  $\mathcal{B}_\epsilon$ . Let  $K$  be the Lipschitz constant of  $DQ$ , then for all  $x \in \mathcal{B}_\epsilon$

$$\|DQ(x)\| = \|DQ(x) - DQ(0)\| \leq K\|x\| \leq K\epsilon. \quad (0.46)$$

Let  $L$  be the Lipschitz constant of  $Q$  on  $\mathcal{B}_\epsilon$ , then [6, Lem. 3.1]

$$L = \sup_{x \in \mathcal{B}_\epsilon} \|DQ(x)\| \leq K\epsilon. \quad (0.47)$$

For  $x, y \in \mathcal{B}_\epsilon$ , the mean value inequalities for the remainder terms lead to

$$\|Q(x)\| \leq \sup_{0 \leq \tau \leq 1} \|DQ(\tau x)\| \leq K\epsilon^2. \quad (0.48)$$

Let  $M$  be the global Lipschitz constant of  $\varphi$  on  $\mathcal{B}_1$ .  $Q_\epsilon$  is zero outside the closed ball, so we can limit us to consider the ball to evaluate the Lipschitz constant. For  $x, y \in \mathcal{B}_\epsilon$ ,

$$\begin{aligned} \|Q_\epsilon(x) - Q_\epsilon(y)\| &= \|\varphi(x/\epsilon)Q(x) - \varphi(y/\epsilon)Q(y)\| \\ &\leq \|\varphi(x/\epsilon) - \varphi(y/\epsilon)\| \cdot \|Q(x)\| + \|\varphi(y/\epsilon)\| \cdot \|Q(x) - Q(y)\| \\ &\leq \frac{M}{\epsilon} \|x - y\| K\epsilon^2 + 1 \cdot K\epsilon \|x - y\| = K(M + 1)\epsilon \|x - y\| \end{aligned} \quad (0.49)$$

Thus  $Q_\epsilon$  is Lipschitz continuous of its Lipschitz constant  $C\epsilon$ , where  $C$  can be taken as  $K(M + 1)$ .

## Exercise 4.35.

## References

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