Chapter 4

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Errata

- p. 50:

  error: $\mathrm{D}_0 f^s \eta$

  correction: $\mathrm{D}_0 f^s(\eta, 0)$

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$\mathrm{D}_{f(x)}(f^{-1}) \cdot \mathrm{D}_x f = \mathbf{I}. \tag{0.1}$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \mathrm{tr}(A)x + \det(A). \tag{0.2}$$

Its descriminant is $\mathrm{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \mathrm{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P\mathrm{diag}(\lambda_1, \lambda_2)P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \mathrm{diag}(\lambda_1, \lambda_2)y$. An orbit in the y-coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n\in\mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0)y_2(0) = \mathrm{const}$, since $\lambda_1\lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0)y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10. (WIP)

Let $p := \mathrm{dim}E^{\mathrm{s}}, q := \mathrm{dim}E^{\mathrm{u}}$ and $\xi := (\eta, g(\eta))$. $g(0) = 0$ holds by definition, hence

$$g(\eta) = \sum_{k=1}^{\infty} \frac{1}{k!}[\mathbf{I}_q \otimes (\eta^\top)^{\otimes k}][D_0^{\otimes k} g] \tag{0.3}$$

where $\mathbf{I}_q \in \mathbb{R}^{q\times q}$ is an identity matrix, $\otimes$ is a Kronecker product and $\eta^{\otimes k}$ is a k-th Kronecker power of η (see [1] for the notation and [2, Th 1.4.8] for the concrete derivation). Also,

$$\eta' = f^s(\eta, g(\eta)) = \mathrm{D}_0 f^s \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \sum_{k=2}^{\infty} \frac{1}{k!}[\mathbf{I}_p \otimes (\xi^\top)^{\otimes k}][D_0^{\otimes k} f^s]. \tag{0.4}$$

Here,

$$\mathrm{D}_0 f^u \begin{pmatrix} \eta \\ g(\eta) \end{pmatrix} = \mathrm{D}_0 f^u \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} \tag{0.5}$$

since $\eta \in E^{\mathrm{s}}$. Therefore

$$f^u(\eta, g(\eta)) = \mathrm{D}_0 f^u \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} + \sum_{k=2}^{\infty} \frac{1}{k!}[\mathbf{I}_q \otimes (\xi^\top)^{\otimes k}][D_0^{\otimes k} f^u]. \tag{0.6}$$

We plug these equations into $g(\eta') = f^u(\eta, g(\eta))$ and draw coefficients of the r-th order terms in η as follows.

References

[1] J. E. Chacón and T. Duong, ”Higher order differential analysis with vectorized derivatives”, arXiv:2011.01833 (2021). https://arxiv.org/abs/2011.01833

[2] T. Kollo and D. von Rosen, ”Advanced Multivariate Statistics with Matrices”, Springer (2011).