

# Chapter 4

Sho Shirasaka, December 1, 2022

## Errata

- p. 50:  
error:  $D_0 f^s \eta$   
correction:  $D_0 f^s(\eta, 0)$

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### Exercise 4.2.

By taking derivatives of both sides of  $f^{-1} \circ f(x) = x$ , we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \quad (0.1)$$

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### Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \text{tr}(A)x + \det(A). \quad (0.2)$$

Its discriminant is  $\text{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$ . Moreover,  $f(0) = 1 > 0$  and  $f(1) = 2 - \text{tr}(A) < 0$ . Hence the two eigenvalues  $\lambda_1, \lambda_2$  of  $A$  are real and positive and satisfy  $\lambda_1 < 1 < \lambda_2$ .

The matrix  $A$  can be diagonalized as  $A = P \text{diag}(\lambda_1, \lambda_2) P^{-1}$  since the eigenvalues are distinct. Let us define  $y = P^{-1}x$ . This conjugates  $x \mapsto Ax$  and  $y \mapsto \text{diag}(\lambda_1, \lambda_2)y$ . An orbit in the  $y$ -coordinate can be expressed as  $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$ ,  $(y_1(0), y_2(0)) \in \mathbb{R}^2$ . The orbit belongs to  $y_1 y_2 = y_1(0) y_2(0) = \text{const}$ , since  $\lambda_1 \lambda_2 = \det(A) = 1$ . This is a hyperbola if  $y_1(0) y_2(0) \neq 0$  and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix  $x = Py$  is a hyperbola (resp. a line). Thus each orbit of the linear map  $x \mapsto Ax$  belongs to a hyperbola (or a line in a degenerate situation).

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### Exercise 4.10. (WIP)

Let  $p := \dim E^s, q := \dim E^u$  and  $\xi := (\eta, g(\eta))$ .  $g(0) = 0$  holds by definition, hence

$$g(\eta) = \sum_{k=1}^{\infty} \frac{1}{k!} [\mathbf{I}_q \otimes (\eta^\top)^{\otimes j}] [D_0^{\otimes k} g] \quad (0.3)$$

where  $\mathbf{I}_q \in \mathbb{R}^{q \times q}$  is an identity matrix,  $\otimes$  is a Kronecker product and  $\eta^{\otimes k}$  is a  $k$ -th Kronecker power of  $\eta$  (see [1] for the notation and [2, Th 1.4.8] for the concrete derivation). Also,

$$\eta' = f^s(\eta, g(\eta)) = D_0 f^s \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \sum_{k=2}^{\infty} \frac{1}{k!} [\mathbf{I}_p \otimes (\xi^\top)^{\otimes j}] [D_0^{\otimes k} f^s]. \quad (0.4)$$

Here,

$$D_0 f^u \begin{pmatrix} \eta \\ g(\eta) \end{pmatrix} = D_0 f^u \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} \quad (0.5)$$

since  $\eta \in E^s$ . Therefore

$$f^u(\eta, g(\eta)) = D_0 f^u \begin{pmatrix} 0 \\ g(\eta) \end{pmatrix} + \sum_{k=2}^{\infty} \frac{1}{k!} [\mathbf{I}_q \otimes (\xi^\top)^{\otimes j}] [D_0^{\otimes k} f^u]. \quad (0.6)$$

We plug these equations into  $g(\eta') = f^u(\eta, g(\eta))$  and draw coefficients of the  $r$ -th order terms in  $\eta$  as follows.

## References

- [1] J. E. Chacón and T. Duong, "Higher order differential analysis with vectorized derivatives", arXiv:2011.01833 (2021). <https://arxiv.org/abs/2011.01833>
- [2] T. Kollo and D. von Rosen, "Advanced Multivariate Statistics with Matrices", Springer (2011).