

## Chapter 4

Sho Shirasaka, December 9, 2022

### Exercise 4.2.

By taking derivatives of both sides of  $f^{-1} \circ f(x) = x$ , we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \quad (0.1)$$


---

### Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \operatorname{tr}(A)x + \det(A). \quad (0.2)$$

Its discriminant is  $\operatorname{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$ . Moreover,  $f(0) = 1 > 0$  and  $f(1) = 2 - \operatorname{tr}(A) < 0$ . Hence the two eigenvalues  $\lambda_1, \lambda_2$  of  $A$  are real and positive and satisfy  $\lambda_1 < 1 < \lambda_2$ .

The matrix  $A$  can be diagonalized as  $A = P \operatorname{diag}(\lambda_1, \lambda_2) P^{-1}$  since the eigenvalues are distinct. Let us define  $y = P^{-1}x$ . This conjugates  $x \mapsto Ax$  and  $y \mapsto \operatorname{diag}(\lambda_1, \lambda_2)y$ . An orbit in the  $y$ -coordinate can be expressed as  $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$ ,  $(y_1(0), y_2(0)) \in \mathbb{R}^2$ . The orbit belongs to  $y_1 y_2 = y_1(0) y_2(0) = \text{const}$ , since  $\lambda_1 \lambda_2 = \det(A) = 1$ . This is a hyperbola if  $y_1(0) y_2(0) \neq 0$  and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix  $x = Py$  is a hyperbola (resp. a line). Thus each orbit of the linear map  $x \mapsto Ax$  belongs to a hyperbola (or a line in a degenerate situation).

---

### Exercise 4.10. (WIP)

This answer greatly relies on [1].

Let  $f_1, f_2$  be  $C^n$  and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \dots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega, \bigcup_{i=1}^k \pi_i = \omega, \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \quad (0.3)$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \dots, n\}). \quad (0.4)$$

Then the Faà di Bruno's formula is given as [2, Sec. 2.4]<sup>1</sup>

$$D^n (f_1 \circ f_2)(x) h_\omega = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k := |\pi|)}} D^k f_1(f_2(x)) \left( D^{|\pi_1|} f_2(x) h_{\pi_1}, \dots, D^{|\pi_k|} f_2(x) h_{\pi_k} \right), \quad (0.5)$$

where for any  $\pi = (\pi_1, \dots, \pi_k)$  let  $|\pi| = k$  denote its length and for any finite subset  $\omega = \{\omega_1, \dots, \omega_n\}$  of  $\mathbb{N}$ ,

$$h_\omega = (h_{\sigma(\omega_1)}, \dots, h_{\sigma(\omega_n)}) \quad (0.6)$$

where  $\sigma$  is a permutation and  $\sigma(\omega_1) < \dots < \sigma(\omega_n)$ .

Let  $\tilde{g}(x) := (x, g(x))$ . We take the  $n$ -th derivatives of the both sides of

$$g \circ f^s(\eta, g(\eta)) = f^u(\eta, g(\eta)) \Leftrightarrow g \circ f^s \circ \tilde{g}(\eta) = f^u \circ \tilde{g}(\eta) \quad (0.7)$$

---

<sup>1</sup>Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$\begin{aligned}
& \text{(l.h.s): } D^n(g \circ f^s \circ \tilde{g})(0)(h_1, \dots, h_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k(g \circ f^s)(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left( D^{|\tau_1|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right)_{\tau_1}, \right. \\
&\quad \left. \dots, D^{|\tau_l|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right)_{\tau_l} \right)
\end{aligned} \tag{0.8}$$

$$\begin{aligned}
& \text{(r.h.s): } D^n(f^u \circ \tilde{g})(0)(h_1, \dots, h_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right).
\end{aligned} \tag{0.9}$$

Here, we defined

$$\left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right)_\omega = \left( D^{|\pi_{\sigma(\omega_1)}|} \tilde{g}(0) h_{\pi_{\sigma(\omega_1)}}, \dots, D^{|\pi_{\sigma(\omega_n)}|} \tilde{g}(0) h_{\pi_{\sigma(\omega_n)}} \right), \tag{0.10}$$

where  $\sigma(\omega_1) < \dots < \sigma(\omega_n)$ . Thus we obtain

$$\begin{aligned}
& \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left( D^{|\tau_1|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right)_{\tau_1}, \dots, \right. \\
& \left. D^{|\tau_l|} f^s(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right)_{\tau_l} \right) = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left( D^{|\pi_1|} \tilde{g}(0) h_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) h_{\pi_k} \right).
\end{aligned} \tag{0.11}$$

We can draw a  $n$ -th order equation from (0.11) as follows:

## References

- [1] Wolf-Jürgen Beyn and Winfried Kleß. Numerical taylor expansions of invariant manifolds in large dynamical systems. *Numerische Mathematik*, 80(1):1–38, 1998.
- [2] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. *Manifolds, tensor analysis, and applications*, volume 75. Springer Science & Business Media, 2012.