

Chapter 4

Sho Shirasaka, December 16, 2022

Exercise 4.2.

By taking derivatives of both sides of $f^{-1} \circ f(x) = x$, we obtain

$$D_{f(x)}(f^{-1}) \cdot D_x f = \mathbf{I}. \quad (0.1)$$

Exercise 4.5.

Consider a characteristic polynomial

$$f(x) = x^2 - \operatorname{tr}(A)x + \det(A). \quad (0.2)$$

Its discriminant is $\operatorname{tr}(A)^2 - 4\det(A) > 2^2 - 4 = 0$. Moreover, $f(0) = 1 > 0$ and $f(1) = 2 - \operatorname{tr}(A) < 0$. Hence the two eigenvalues λ_1, λ_2 of A are real and positive and satisfy $\lambda_1 < 1 < \lambda_2$.

The matrix A can be diagonalized as $A = P \operatorname{diag}(\lambda_1, \lambda_2) P^{-1}$ since the eigenvalues are distinct. Let us define $y = P^{-1}x$. This conjugates $x \mapsto Ax$ and $y \mapsto \operatorname{diag}(\lambda_1, \lambda_2)y$. An orbit in the y -coordinate can be expressed as $(\lambda_1^n y_1(0), \lambda_2^n y_2(0))_{n \in \mathbb{Z}}$, $(y_1(0), y_2(0)) \in \mathbb{R}^2$. The orbit belongs to $y_1 y_2 = y_1(0) y_2(0) = \operatorname{const}$, since $\lambda_1 \lambda_2 = \det(A) = 1$. This is a hyperbola if $y_1(0) y_2(0) \neq 0$ and a line otherwise. An image of a linear transformation of a hyperbola (resp. a line) by a regular matrix $x = Py$ is a hyperbola (resp. a line). Thus each orbit of the linear map $x \mapsto Ax$ belongs to a hyperbola (or a line in a degenerate situation).

Exercise 4.10.

This answer greatly relies on [1] but somewhat elementalized and concretized.

Let f_1, f_2 be C^n and

$$\mathcal{Q}_k(\omega) = \left\{ \pi = (\pi_1, \dots, \pi_k) \mid \emptyset \neq \pi_i \subset \omega, \bigcup_{i=1}^k \pi_i = \omega, \pi_i \cap \pi_j = \emptyset \text{ and } \min \pi_i < \min \pi_j \text{ for all } i < j \right\}, \quad (0.3)$$

$$\mathcal{Q}(\omega) = \bigcup_{k=1}^{|\omega|} \mathcal{Q}_k(\omega), \quad \mathcal{Q}(\emptyset) = \{\emptyset\}, \quad \mathcal{Q}(n) = \mathcal{Q}(\{1, \dots, n\}). \quad (0.4)$$

Then the Faà di Bruno's formula is given as [2, Sec. 2.4]¹

$$D^n (f_1 \circ f_2)(x) \eta_\omega = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k := |\pi|)}} D^k f_1(f_2(x)) \left(D^{|\pi_1|} f_2(x) \eta_{\pi_1}, \dots, D^{|\pi_k|} f_2(x) \eta_{\pi_k} \right), \quad (0.5)$$

where for any $\pi = (\pi_1, \dots, \pi_k)$ let $|\pi| = k$ denote its length and for any finite subset $\omega = \{\omega_1, \dots, \omega_n\}$ of \mathbb{N} ,

$$\eta_\omega = (\eta_{\sigma(\omega_1)}, \dots, \eta_{\sigma(\omega_n)}) \quad (0.6)$$

where σ is a permutation and $\sigma(\omega_1) < \dots < \sigma(\omega_n)$.

Let $\tilde{g}(x) := (x, g(x))$. We take the n -th derivatives of the both sides of

$$g \circ f^s(\eta, g(\eta)) = f^u(\eta, g(\eta)) \Leftrightarrow g \circ f^s \circ \tilde{g}(\eta) = f^u \circ \tilde{g}(\eta) \quad (0.7)$$

¹Note that how the sum is taken in the last expression in the page. 97 is somewhat vague.

and apply the rule (0.5).

$$\begin{aligned}
& \text{(l.h.s): } D^n(g \circ f^s \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k(g \circ f^s)(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \right. \\
&\quad \left. \dots, D^{|\tau_l|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right)
\end{aligned} \tag{0.8}$$

$$\begin{aligned}
& \text{(r.h.s): } D^n(f^u \circ \tilde{g})(0)(\eta_1, \dots, \eta_n) \\
&= \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.9}$$

Here, we defined

$$\left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\omega} = \left(D^{|\pi_{\sigma(\omega_1)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_1)}}, \dots, D^{|\pi_{\sigma(\omega_n)}|} \tilde{g}(0) \eta_{\pi_{\sigma(\omega_n)}} \right), \tag{0.10}$$

where $\sigma(\omega_1) < \dots < \sigma(\omega_n)$. Thus we obtain

$$\begin{aligned}
& \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} \sum_{\substack{\tau \in \mathcal{Q}(k) \\ (l:=|\tau|)}} D^l g(0) \left(D^{|\tau_1|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_1}, \dots, \right. \\
& \left. D^{|\tau_l|} f^s(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right)_{\tau_l} \right) = \sum_{\substack{\pi \in \mathcal{Q}(n) \\ (k:=|\pi|)}} D^k f^u(0) \left(D^{|\pi_1|} \tilde{g}(0) \eta_{\pi_1}, \dots, D^{|\pi_k|} \tilde{g}(0) \eta_{\pi_k} \right).
\end{aligned} \tag{0.11}$$

Let D_s, D_u denote differentiation w.r.t E^s, E^u respectively. Let $x_1 \in E^s, x_2 \in E^u$. The linearization of the mapping f at the origin is

$$Df(0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} D_s f^s(0) & D_u f^s(0) \\ D_s f^u(0) & D_u f^u(0) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{0.12}$$

The linearized dynamics is invariant on E^s and E^u , thus

$$D_u f^s(0) = 0, D_s f^u(0) = 0. \tag{0.13}$$

Let us denote $A_s := D_s f^s(0), A_u := D_u f^u(0)$. By definition $\sigma(A_s)$ (resp. $\sigma(A_u)$) is included inside (resp. outside) the unit circle, where $\sigma(A)$ is the spectrum of A .

Let us derive a few concrete n -th order equations from (0.11).

We can see that

$$Df^s(0) D\tilde{g}(0) = (D_s f^s(0) \ D_u f^s(0)) \begin{pmatrix} I \\ Dg(0) \end{pmatrix} = A_s, \tag{0.14}$$

$$Df^u(0) D\tilde{g}(0) = A_u Dg(0). \tag{0.15}$$

Thus the first order equation is equivalent to the Sylvester equation

$$Dg(0) A_s \eta - A_u Dg(0) \eta = 0. \tag{0.16}$$

To analyze and solve the Sylvester equation, we resort to the trick of the Kronecker form. Let us denote a space of n -multilinear maps of E_1, \dots, E_n to E_0 by $\mathcal{L}(E_1, \dots, E_n; E_0)$. For $\nu = 1, \dots, n$ let $(e_{\nu,1}, \dots, e_{\nu, \dim E_\nu})$ be an

ordered basis of E_ν and let $(e_{\nu,1}^*, \dots, e_{\nu, \dim E_\nu}^*)$ be its dual. Then the space $\mathcal{L}(E_1, \dots, E_n; E_0) = E_n^* \otimes \dots \otimes E_1^* \otimes E_0$, where \otimes means a tensor product, has a basis

$$(e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0})_{(r_n, \dots, r_0)}, \quad (r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}. \quad (0.17)$$

We order the basis (0.17) lexicographically with priority to the first components of (r_n, \dots, r_0) . A multilinear map $\tilde{U} \in \mathcal{L}(E_1, \dots, E_n; E_0)$ has a coordinate representation $U \in \mathbb{R}^{\dim E_0 \times \dim E_1^* \times \dots \times \dim E_n^*}$ as

$$\tilde{U} = \sum_{(r_n, \dots, r_0) \in \prod_{k=n}^0 \{1, \dots, \dim E_k\}} U_{r_0 r_1 \dots r_n} e_{n,r_n}^* \otimes \dots \otimes e_{1,r_1}^* \otimes e_{0,r_0}. \quad (0.18)$$

Let $\text{vec} : \mathbb{R}^{J_1 \times \dots \times J_N} \rightarrow \mathbb{R}^{\prod_{i=1}^N J_i}$ be a vectorization operator

$$(\text{vec}(U))_i = U_{j_1 \dots j_N} \quad \text{with} \quad i = 1 + \sum_{l=1}^N \left[(j_l - 1) \prod_{l'=1}^{l-1} J_{l'} \right]. \quad (0.19)$$

By $U \times_n A \in \mathbb{R}^{J_1 \times \dots \times J_{n-1} \times I \times J_{n+1} \times \dots \times J_N}$ we denote an n -mode product of a tensor [3] $U \in \mathbb{R}^{J_1 \times \dots \times J_N}$ with a matrix $A \in \mathbb{R}^{I \times J_n}$, defined as

$$(U \times_n A)_{j_1 \dots j_{n-1} i j_{n+1} \dots j_N} = \sum_{j_n=1}^{J_n} U_{j_1 \dots j_N} A_{i j_n}. \quad (0.20)$$

An action of $\tilde{U} \in E_n^* \otimes \dots \otimes E_1^* \otimes E_0$ on $(\tilde{h}_1, \dots, \tilde{h}_n, \tilde{h}_0^*) \in \prod_{i=1}^n E^n \times E_0^*$ is rewritten in the coordinate representation as $U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*$. The vec operation on this representation provides a convenient Kronecker form (this is a special case of [3, Proposition 3.7 (b)])

$$\text{vec}(U \times_1 h_0^* \times_2 h_1^* \times_3 \dots \times_{n+1} h_n^*) = (h_n^* \otimes \dots \otimes h_1^* \otimes h_0^*) \text{vec}(U), \quad (0.21)$$

where \otimes denotes the Kronecker product. In the following, we identify any multilinear map and elements in E^s, E^u as their coordinate representation.

Let $p = \dim E^s, q = \dim E^u$. From (0.16), it is obvious that for any $\eta_0^\top \in E^{u*}, \eta_1 \in E^s$

$$\begin{aligned} & \eta_0^\top Dg(0) A_s \eta_1 - \eta_0^\top A_u Dg(0) \eta_1 = 0 \\ & \Leftrightarrow \text{vec} \left(Dg(0) \times_1 \eta_0^\top \times_2 \eta_1^\top A_s^\top \right) - \text{vec} \left(Dg(0) \times_1 \eta_0^\top A_u \times_2 \eta_1^\top \right) = 0 \quad (\because \text{vec}(\cdot) \text{ is linear}) \\ & \Leftrightarrow (\eta_1^\top A_s^\top \otimes \eta_0^\top I_q - \eta_1^\top I_p \otimes \eta_0^\top A_u) \text{vec}(Dg(0)) = 0 \quad (\because (0.21)) \\ & \Leftrightarrow (\eta_1^\top \otimes \eta_0^\top) \cdot (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec}(Dg(0)) = 0 \quad (\because (AB) \otimes (CD) = (A \otimes C) \cdot (B \otimes D)) \\ & \Leftrightarrow (A_s^\top \otimes I_q - I_p \otimes A_u) \text{vec}(Dg(0)) = 0 \quad (\because \eta_0, \eta_1 \text{ are arbitrary}). \end{aligned} \quad (0.22)$$

The following theorem is well-known [4, Thm. 4.4.5].

Theorem 0.1.

If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}, \sigma(B) = \{\mu_1, \dots, \mu_m\}$, then $\sigma(I_m \otimes A + B \otimes I_n) = \{\lambda_i + \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

This implies $A_s^\top \otimes I_q - I_p \otimes A_u$ is regular since $\sigma(A_s)$ (resp. $\sigma(A_u)$) is included inside (resp. outside) the unit circle. Therefore $Dg(0) = 0$.

The second order equation of (0.11) is

$$D^2 g(0)(A_s \eta_1, A_s \eta_2) - Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2). \quad (0.23)$$

In the second term of the l.h.s., the first p columns of $Df^u(0)$ are zeros, so the first p mode-1 slices of $D^2 \tilde{g}(0)$ do not contribute to the term, i.e., $Df^u(0) D^2 \tilde{g}(0)(\eta_1, \eta_2) = A_u D^2 g(0)(\eta_1, \eta_2)$. Also, components of $D^2 f^u(0)$ of

indices larger than $p + 1$ do not contribute to the r.h.s. since $Dg(0) = 0$. Namely, $D^2 f^u(0)(D\tilde{g}(0)\eta_1, D\tilde{g}(0)\eta_2) = D_s^2 f^u(0)(\eta_1, \eta_2)$. Then we obtain a multilinear Sylvester equation. For any $\eta_0^\top \in E^{u*}$, $\eta_1, \eta_2 \in E^s$

$$\begin{aligned} \eta_0^\top D^2 g(0)(A_s \eta_1, A_s \eta_2) - \eta_0^\top A_u D^2 g(0)(\eta_1, \eta_2) &= \eta_0^\top D_s^2 f^u(0)(\eta_1, \eta_2) \\ \Leftrightarrow (A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u) \text{vec}(D^2 g(0)) &= \text{vec}(D_s^2 f^u(0)) \quad (\because \eta_0, \eta_1, \eta_2 \text{ are arbitrary}). \end{aligned} \quad (0.24)$$

The theorem 0.1 and the following one [4, Thm. 4.2.12]

Theorem 0.2.

If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, $\sigma(B) = \{\mu_1, \dots, \mu_m\}$, then $\sigma(A \otimes B) = \{\lambda_i \mu_j \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$.

lead to the regularity of $(A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)$. Thus we obtain

$$D^2 g(0) = \text{vec}^{-1} \left[(A_s^\top \otimes A_s^\top \otimes I_q - I_{p^2} \otimes A_u)^{-1} \text{vec}(D_s^2 f^u(0)) \right]. \quad (0.25)$$

Similarly, the higher order equation of (0.11) lead to a multilinear Sylvester equation of a unique solution, which can be explicitly written using a more complicated Kronecker form.

Exercise 4.12.

Inverse map is obtained as

$$\begin{cases} x' = x - 1.6y'(1 - y'^2) \\ y' = y + 1.6x(1 - x^2) \end{cases}. \quad (0.26)$$

The jacobian of f is given as

$$Df(x, y) = \begin{pmatrix} 1 & 1.6(1 - 3y^2) \\ -1.6(1 - 3x'^2) & 1 - 1.6^2(1 - 3y^2)(1 - 3x'^2) \end{pmatrix}. \quad (0.27)$$

f is injective and $\det(Df(x, y)) \equiv 1$, so f is area-preserving. The fixed points (x_*, y_*) s satisfy simultaneous equations $y(1 - y^2) = 0, x(1 - x^2) = 0$, so $(x_*, y_*) \in \{0, 1, -1\}^2$. If $\text{tr}(Df(x_*, y_*))^2 - 4 \det(Df(x_*, y_*)) = \text{tr}(Df(x_*, y_*))^2 - 4 > 0$, f is hyperbolic at the fixed point and non-hyperbolic otherwise.

$$\begin{aligned} \text{tr}(Df(x_*, y_*))^2 - 4 &= -1.6^2 \times 4(1 - 3y_*^2)(1 - 3x_*^2) + 1.6^4(1 - 3y_*^2)^2(1 - 3x_*^2)^2 > 0 \\ \Leftrightarrow g(x_*, y_*) &:= 4 - 1.6^2(1 - 3y_*^2)(1 - 3x_*^2) \begin{cases} < 0 & \text{if } |x_*| + |y_*| \neq 1 \\ > 0 & \text{otherwise} \end{cases} \quad (\because x_*, y_* \text{ are not irrational}) \end{aligned} \quad (0.28)$$

We evaluate g on the fixed points as

$$\begin{aligned} g(0, 0) &= 4 - 1.6^2 > 0 \\ g(0, \pm 1) &= g(\pm 1, 0) = 4 + 1.6^2 \times 2 > 0 \\ g(\pm 1, \pm 1) &= 4 - 4 \times 1.6^2 < 0. \end{aligned} \quad (0.29)$$

Therefore, $(0, 0)$ is non-hyperbolic and the others are hyperbolic.

Exercise 4.20.

We take inner product with $\xi - \eta$ on both sides of $\xi = \eta + (\xi - \eta)$ as

$$\begin{aligned} (\xi - \eta, \xi) &= (\xi - \eta, \eta) + \|\xi - \eta\|^2 \\ \Leftrightarrow 2 - 2 \cos \theta &= \|\xi - \eta\|^2. \end{aligned} \quad (0.30)$$

Then

$$\|A\xi - A\eta\|^2 = \|A\xi\|^2 + \|A\eta\|^2 - 2(A\xi, A\eta) \geq \|A\xi\|^2 + \|A\eta\|^2 - 2\|A\xi\| \cdot \|A\eta\| = (\varrho^{-1} - \varrho)^2 \quad (0.31)$$

$$\|A\xi - A\eta\|^2 \leq \|A\|_2^2 \|\xi - \eta\|^2 = \|A\|_2^2 (2 - 2 \cos \theta) \quad (\because \text{Eq. (0.30)}). \quad (0.32)$$

Thus the desired inequality is derived.

Exercise 4.21.

Let $v = \xi + \eta$, $\xi \in E^u$, $\eta \in E^s$. Let λ_s, C_s (resp. λ_u, C_u) be constants which make all tangent vectors in E^s (resp. E^u) stable and define $C = \max(C_s, C_u)$ and $\lambda = \max(\lambda_s, \lambda_u)$. Also let P_s, P_u be projections onto E^s, E^u .

$$\langle v, v \rangle \geq \|\xi\|^2 + \|\eta\|^2 \geq \|\xi + \eta\|^2 = \langle v, v \rangle, \quad (0.33)$$

$$\begin{aligned} \langle v, v \rangle &\leq \frac{C_s^2 \|\xi\|^2}{1 - \lambda_s^{2\epsilon}} + \frac{C_u^2 \|\eta\|^2}{1 - \lambda_u^{2\epsilon}} \leq \frac{C^2}{1 - \lambda^{2\epsilon}} (\|\xi\|^2 + \|\eta\|^2) = \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u v\|^2 + \|P_s v\|^2) \\ &\leq \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u\|_2^2 + \|P_s\|_2^2) \langle v, v \rangle. \end{aligned} \quad (0.34)$$

Thus we can take $D = \max\left(1, \frac{C^2}{1 - \lambda^{2\epsilon}} (\|P_u\|_2^2 + \|P_s\|_2^2)\right)$.

Exercise 4.22.

Let $M = \max_{\|x\| \leq \epsilon} \|D^2 Q(x)\|$, $N = \max_{\|x\| \leq \epsilon} \|D\varphi(x/\epsilon)\|$. These maxima exist because $D^2 Q, D\varphi$ are continuous and the closed ball is compact. The Taylor theorem for $Q \in C^2$ [Sec. 4.5][5] says that

$$Q(x) = Q(0) + DQ(0)x + R_1(x) = R_1(x) \quad (\because Q(0) = 0, DQ(0) = 0), \quad (0.35)$$

$$DQ(x) = DQ(0) + R_2(x) = R_2(x), \quad (0.36)$$

where R_1, R_2 are remainder terms. For x in the closed ball of radius ϵ , the mean value inequalities for the remainder terms [Thm. 4.C][5] lead to

$$\begin{aligned} \|Q(x)\| = \|R_1(x)\| &\leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)(x, x)\| \leq \frac{1}{2!} \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)\| \cdot \|x\| \cdot \|x\| \leq \frac{M}{2} \|x\| \cdot \|x\| \\ &\leq \frac{M\epsilon^2}{2} \end{aligned} \quad (0.37)$$

$$\|DQ(x)\| = \|R_2(x)\| \leq \sup_{0 \leq \tau \leq 1} \|D^2 Q(\tau x)x\| \leq M\|x\| \leq M\epsilon. \quad (0.38)$$

The differential of Q_ϵ is

$$DQ_\epsilon(x) = \frac{1}{\epsilon} D\varphi(x/\epsilon)Q(x) + \varphi(x/\epsilon)DQ(x). \quad (0.39)$$

Then

$$\|DQ_\epsilon(x)\| \leq \frac{1}{\epsilon} \|D\varphi(x/\epsilon)\| \cdot \|Q(x)\| + \|\varphi(x/\epsilon)\| \cdot \|DQ(x)\| \leq \frac{M(N+2)}{2} \epsilon. \quad (0.40)$$

Exercise 4.23.

(\mathcal{H}_α is a normed vector space) If $\beta, \gamma \in \mathbb{R}$ and $f, g \in \mathcal{H}_\alpha$, Then

$$\begin{aligned} \|\beta f + \gamma g\|_{\mathcal{H}_\alpha} &= \sup_{x \neq y} \frac{\|\beta(f(x) - f(y)) + \gamma(g(x) - g(y))\|}{\|x - y\|^\alpha} \leq \sup_{x \neq y} \frac{\beta\|f(x) - f(y)\|}{\|x - y\|^\alpha} + \sup_{x \neq y} \frac{\gamma\|g(x) - g(y)\|}{\|x - y\|^\alpha} \\ &< \infty. \end{aligned} \quad (0.41)$$

$$\beta f(0) + \gamma g(0) = 0. \quad (0.42)$$

So, $\beta f + \gamma g \in \mathcal{H}_\alpha$. Also, Eq. (0.41) for $\beta = \gamma = 1$ includes the triangle inequality. The other conditions to be a norm is trivial. Hence \mathcal{H}_α is a normed vector space.

(Completeness) Consider a Cauchy sequence $\{f_n\}_{n=1}^\infty$ in \mathcal{H}_α . This is also Cauchy in a space of continuous functions endowed with sup-norm, say, $BC(E)$. $BC(E)$ is complete, so the sequence has a limit in $BC(E)$. We denote the limit as f .

Every Cauchy sequence in a metric space is bounded, so $\|f_n\|_{\mathcal{H}_\alpha} \leq M$ for all $n \in \mathbb{N}$ and some $M \in \mathbb{R}$. Hence

$$\|f\|_{\mathcal{H}_\alpha} = \lim_{n \rightarrow \infty} \|f_n\|_{\mathcal{H}_\alpha} (\because \text{the norm is continuous}) \leq M < \infty. \quad (0.43)$$

Therefore $f \in \mathcal{H}_\alpha$.

Next, we show $f_n \rightarrow f$ in \mathcal{H}_α . For any $\epsilon > 0$, there exists $N > 0$ such that $\|f_p - f_q\|_{\mathcal{H}_\alpha} < \epsilon$ whenever $p, q \geq N$ since $\{f_n\}$ is Cauchy. Then for any $n \geq N$

$$\|f - f_n\|_{\mathcal{H}_\alpha} = \lim_{m \rightarrow \infty} \|f_m - f_n\|_{\mathcal{H}_\alpha} (\because \text{the norm is continuous}) \leq \epsilon. \quad (0.44)$$

Thus $f_n \rightarrow f$ in \mathcal{H}_α .

Therefore \mathcal{H}_α is complete.

Exercise 4.24.

$R^u(0, 0) = 0, R^s(0, 0) = 0$ since $R \in \mathcal{H}_\alpha$. Hence

$$\begin{aligned} \Psi^u(R)(0, 0) &= (L^u)^{-1} R^u(0, 0) - (L^u)^{-1} Q_\epsilon^u(0, 0) = 0 (\because Q(0) = 0) \\ \Psi^s(R)(0, 0) &= L^s R^s(0, 0) + Q_\epsilon^s(0, 0) = 0. \end{aligned} \quad (0.45)$$

Exercise 4.25.

The C^2 regularity is used to prove $\|DQ_\epsilon\| \leq C\epsilon$. This inequality is used to bound $\|\Psi(r) - \Psi(r')\|_{\mathcal{H}_\alpha}$. To derive the inequality regarding $\|\Psi(r) - \Psi(r')\|_{\mathcal{H}_\alpha}$, Q_ϵ need not satisfy $\|DQ_\epsilon\| \leq C\epsilon$. Instead, it is enough for Q_ϵ to be Lipschitz continuous of its Lipschitz constant $C\epsilon$.

Let \mathcal{B}_ϵ be a closed ball of radius ϵ centered at the origin. Every continuous functions on a compact domain is globally Lipschitz, so DQ is globally Lipschitz on \mathcal{B}_ϵ . Let K be the Lipschitz constant of DQ , then for all $x \in \mathcal{B}_\epsilon$

$$\|DQ(x)\| = \|DQ(x) - DQ(0)\| \leq K\|x\| \leq K\epsilon. \quad (0.46)$$

Let L be the Lipschitz constant of Q on \mathcal{B}_ϵ , then [6, Lem. 3.1]

$$L = \sup_{x \in \mathcal{B}_\epsilon} \|DQ(x)\| \leq K\epsilon. \quad (0.47)$$

For $x, y \in \mathcal{B}_\epsilon$, the mean value inequalities for the remainder terms lead to

$$\|Q(x) - Q(y)\| \leq \sup_{0 \leq \tau \leq 1} \|DQ(\tau x + (1-\tau)y)\| \|x - y\| \leq K\epsilon^2. \quad (0.48)$$

Let M be the global Lipschitz constant of φ on \mathcal{B}_1 . Q_ϵ is zero outside the closed ball, so we can limit us to consider the ball to evaluate the Lipschitz constant. For $x, y \in \mathcal{B}_\epsilon$,

$$\begin{aligned} \|Q_\epsilon(x) - Q_\epsilon(y)\| &= \|\varphi(x/\epsilon)Q(x) - \varphi(y/\epsilon)Q(y)\| \\ &\leq \|\varphi(x/\epsilon) - \varphi(y/\epsilon)\| \cdot \|Q(x)\| + \|\varphi(y/\epsilon)\| \cdot \|Q(x) - Q(y)\| \\ &\leq \frac{M}{\epsilon} \|x - y\| K\epsilon^2 + 1 \cdot K\epsilon \|x - y\| = K(M + 1)\epsilon \|x - y\| \end{aligned} \quad (0.49)$$

Thus Q_ϵ is Lipschitz continuous of its Lipschitz constant $C\epsilon$, where C can be taken as $K(M + 1)$.

Exercise 4.35.

Let us assume Λ is compact (at the beginning of this chapter, the compactness of M is assumed but seems frequently violated (for example, $M = \mathbb{R}^n$)). In fact, in many literatures a uniformly hyperbolic set is defined to be compact.

Let $\|\cdot\|_{T_x M}$ be an arbitrary norm and $\|\cdot\|_{*, T_x M}$ be associated Lyapunov metric. Let $n_0 \in \mathbb{N}$ such that $C\lambda^{n_0} < 1$ in the Definition 4.16 in the text². Then, as in the solution of [Exercise 4.20](#), we obtain

$$1 - \cos \theta_x \geq \frac{(C^{-1}\lambda^{-n_0} - C\lambda^{n_0})^2}{\|D_x f^{n_0}\|_2^2}, \quad (0.50)$$

where θ_x is the angle between E_x^s and E_x^u . $\|D_x f^{n_0}\|_2^2$ is bounded above on Λ since Λ is compact. Hence the r.h.s. of (0.50) is uniformly bounded away from zero on Λ . Let $\mu \in (0, 1)$ be a (x -independent) lower bound of the r.h.s.

Let $v = \xi + \eta$, $\xi \in E_x^u$, $\eta \in E_x^s$. As in the solution of [Exercise 4.21](#),

$$\|v\|_{T_x M} \leq \|v\|_{*, T_x M}. \quad (0.51)$$

Also,

$$\begin{aligned} \|v\|_{T_x M} &= \|\xi\|_{T_x M}^2 + \|\eta\|_{T_x M}^2 - 2 \cos \theta_x \|\xi\|_{T_x M} \|\eta\|_{T_x M} \\ &\geq \|\xi\|_{T_x M}^2 + \|\eta\|_{T_x M}^2 - 2(\mu - 1) \|\xi\|_{T_x M} \|\eta\|_{T_x M} = (1 - \mu) \|\xi - \eta\|_{T_x M}^2 + \mu(\|\xi\|_{T_x M}^2 + \|\eta\|_{T_x M}^2) \\ &\geq \mu(\|\xi\|_{T_x M}^2 + \|\eta\|_{T_x M}^2). \end{aligned} \quad (0.52)$$

Then

$$\|v\|_{*, T_x M} \leq \frac{C^2}{1 - \lambda^{2\epsilon}} (\|\xi\|_{T_x M}^2 + \|\eta\|_{T_x M}^2) \leq \frac{C^2}{\mu(1 - \lambda^{2\epsilon})} \|v\|_{T_x M}. \quad (0.53)$$

$C, \mu, \lambda, \epsilon$ are independent of $x \in \Lambda$. Hence, from (0.52, 0.53), the Lyapunov metric is uniformly equivalent to the original one.

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²Note that $C > 0$ is independent of $x \in \Lambda$ for a uniformly hyperbolic set. This seems implicitly assumed in the text.