Chapter 2

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Errata

- p. 11, Sec. 2.3, para. 2:
  error: $h : \Omega \to \mathbb{R}^t$
  correction: $h : \Omega \to \mathbb{R}$
- p. 15:
  error: $\varrho = \sqrt{u^2 + v^2}$
  correction: $\varrho = \sqrt{u^2 + v^2} - 1$
- p. 15:
  error: $\varrho + 0.4\cos(\varphi)$
  correction: $0.25\varrho + 0.4\cos(\varphi)$

Exercise 2.5.

($\Rightarrow$) Let φ_t, ψ_t be flows of $\mathbf{X}, \mathbf{Y}$ respectively. The definition of the flow leads to

$$\dot{\varphi}_t(\boldsymbol{x}) = \mathbf{X}(\varphi_t(\boldsymbol{x})), \quad \dot{\psi}_t(\boldsymbol{x}) = \mathbf{Y}(\psi_t(\boldsymbol{x})). \tag{0.1}$$

Suppose a diffeomorphism $\mathbf{\Phi}$ conjugates the flows, i.e., $\mathbf{\Phi} \circ \varphi_t(\boldsymbol{x}) = \psi_t \circ \mathbf{\Phi}(\boldsymbol{x})$. By differentiating both sides of this equation w.r.t. t,

$$\begin{aligned} &[\mathrm{D}\mathbf{\Phi}(\varphi_t(\boldsymbol{x}))]\,\dot{\varphi}_t(\boldsymbol{x}) = \dot{\psi}_t \circ \mathbf{\Phi}(\boldsymbol{x}) \\ \Leftrightarrow\, &[\mathrm{D}\mathbf{\Phi}(\varphi_t(\boldsymbol{x}))]\,\mathbf{X}(\varphi_t(\boldsymbol{x})) = \mathbf{Y}(\psi_t \circ \mathbf{\Phi}(\boldsymbol{x})) \end{aligned} \tag{0.2}$$

Let $t = 0$, then we obtain

$$\mathrm{D}_x\mathbf{\Phi} \cdot \mathbf{X}(\boldsymbol{x}) = \mathbf{Y}(\mathbf{\Phi}(\boldsymbol{x})), \tag{0.3}$$

since φ_0, ψ_0 are identity maps.

($\Leftarrow$) Consider a diffeomorphism $\mathbf{\Phi}$ which satisfies Eq. (0.3). Let $\boldsymbol{y}(t) = \mathbf{\Phi} \circ \varphi_t(\boldsymbol{x})$. Then

$$\begin{aligned} \dot{\boldsymbol{y}}(t) = D\mathbf{\Phi}(\varphi_t(\boldsymbol{x}))\mathbf{X}(\varphi_t(\boldsymbol{x})) &= \mathbf{Y} \circ \mathbf{\Phi} \circ \varphi_t(\boldsymbol{x}) \quad (\because\ \text{eq. } (0.3)) \\ &= \mathbf{Y}(\boldsymbol{y}(t)). \end{aligned} \tag{0.4}$$

Thus $\boldsymbol{y}(t)$ is a solution of the ODE of the vector field $\mathbf{Y}$ and of the initial condition $\mathbf{\Phi}(\boldsymbol{x})$. That is, $\boldsymbol{y}(t) = \psi_t \circ \mathbf{\Phi}(\boldsymbol{x})$.

Exercise 2.14.

In the (ϱ, φ, w) coordinate system, the torus is written as

$$(\varrho, \varphi, w)^\top = (0.8\sin\vartheta, \varphi, 0.8\cos\vartheta)^\top. \tag{0.5}$$

Let $\boldsymbol{S}$ be the solenoid map and $\boldsymbol{p}_1 = (\varrho_1, \varphi_1, w_1)^\top, \boldsymbol{p}_2 = (\varrho_2, \varphi_2, w_2)^\top$. Suppose $\boldsymbol{S}(\boldsymbol{p}_1) = \boldsymbol{S}(\boldsymbol{p}_2)$. Then $\varphi_1 = \varphi_2$ follows from the second component of this equation. The other components provide $\varrho_1 = \varrho_2, w_1 = w_2$. Therefore the solenoid map is injective. The solenoid map operates as

$$\boldsymbol{S}(\boldsymbol{p}) = (0.2\sin\vartheta + 0.4\cos\varphi, 2\varphi, 0.2\cos\vartheta + 0.4\sin\varphi)^\top. \tag{0.6}$$

Then we obtain

$$S(\boldsymbol{p})_1^2 + S(\boldsymbol{p})_3^2 = 0.2 + 0.16\sin\vartheta\cos\varphi + 0.16\cos\vartheta\sin\varphi \le 0.52 < 0.64 = (\boldsymbol{p})_1^2 + (\boldsymbol{p})_3^2. \tag{0.7}$$

Thus the injection is strict.

Exercise 2.16.

Let $J_f := \{I_{f,j}\}_j = \{[a_{f,j}, a_{f,j+1})\}_j$ be a set of intervals associated with a piecewise expanding map f.

Lemma 0.1.
Let f, g be piecewise expanding maps. The phase space of $g \circ f$ can be decomposed into subintervals of the elements of J_f such that $g \circ f$ is monotone and C^2 in each of the subintervals.

Proof.
Let K be a minimal set which satisfies $f(I_{f,j}) \subset \cup_{k \in K} I_{g,k}$. Since f is monotone on $I_{f,j}$,

$$f(x) = a_{g,\min K + k'} \tag{0.8}$$

have unique solution in $I_{f,j}$. Let $b_{j,k'}$ $(1 \leq k' \leq |K| - 1)$ be such solutions. Consider a finite sequence $a_{f,j} = b_{j,0} < b_{j,1} < \cdots < b_{j,|K|} = a_{f,j+1}$ and let $L_{j,p} := [b_{j,p}, b_{j,p+1})$. f, g are monotone C^2 maps on each $L_{j,p}$, so its composition $f \circ g$ is. □

Suppose f^n be a piecewise expanding map. The Lemma. 0.1 tells us that the phase space of f^{n+1} can be decomposed into intevals in each of which f^{n+1} is monotone and C^2. Also,

$$|(f^{(n+1)m})'| \geq c^{n+1} > 1 \tag{0.9}$$

holds. Therefore, f^{n+1} is a piecewise expanding map. The proposition follows by induction.

Exercise 2.19.

$$\begin{aligned}
&f_2 \circ \Phi = 1 - 2\sin^2 \frac{\pi x}{2} = \cos \pi x \\
&\Phi \circ g = \sin \frac{\pi}{2}(1 - 2|x|) = \sin\left(\frac{\pi}{2} - \pi|x|\right) = \cos \pi|x| = \cos \pi x \\
&\therefore f_2 \circ \Phi = \Phi \circ g
\end{aligned} \tag{0.10}$$

Exercise 2.21.

$$\mathcal{S}^{-1}(\mathbf{x})_j = x_{j-1} \tag{0.11}$$

Exercise 2.22.

Let $\bar{\ }$ denotes a periodic part of a dyadic representation.
If a number x has a finite dyadic representation, it give rise to two representations as follows:

$$\begin{aligned}
x &= 0.a_1 a_2 a_3 \cdots a_m 1\bar{0} \\
&= 0.a_1 a_2 a_3 \cdots a_m 0\bar{1}.
\end{aligned} \tag{0.12}$$

Thus its dyadic representation is not unique.
Suppose that a number x, which admits no finite dyadic representation, have two different dyadic representation:

$$\begin{aligned}
x_a &= 0.a_1 a_2 a_3 \cdots \\
x_b &= 0.b_1 b_2 b_3 \cdots
\end{aligned} \tag{0.13}$$

Let n be minumum i that satisfies $a_i \neq b_i$. We can assume $a_n = 1, b_n = 0$ w.l.o.g. This means $x_a \geq x_b$ and the equality holds iff $a_{n'} = 0, b_{n'} = 1$ $(n' \geq n + 1)$. This contradicts our assumption that x admits no finite dyadic representation. Thus such number has a unique dyadic representation.
Therefore, the set of numbers of nonunique dyadic representation coincides with that of finite dyadic representation. The number of finite dyadic representation is rational, since it is a sum of rationals. Thus the set of numbers of nonunique dyadic representation is countable, since it is a subset of $\mathbb{Q}$.

Exercise 2.23.

Let x_* be a 32 bit (approximated) representation of $x \in [0,1)$. The inteval map act as a unilateral shift in the representation. Hence, the bits are moved to the left in order, during which the leftmost ones are discarded. When the map is applied 32 times, the least significant bit of x_* is discarded at last, then bits of zeros remain. This is apparently the fixed point.

The same applies to the case of 64 bit representation.

Exercise 2.24.

Let $I_j := [a_j, a_{j+1})$.

(well-definedness) Since $f^n(x)$ belongs to Ω for any $n \in \mathbb{N}$, it is in one of the intervals $\{I_j\}_j$. Thus σ_n is uniquely determined. That is, the code is well-defined.

(invertibility) Suppose two different numbers $x_1, x_2 \in \Omega, x_1 < x_2$ share the same code, i.e., $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\ (= \boldsymbol{\sigma})$.

Let us show that for any $n \in \mathbb{Z}^+$, f^n is monotone and C^2 on $[x_1, x_2]$. The base case ($n = 0$) holds by definition. Suppose that f^k is monotone C^2 map on $[x_1, x_2]$ and that f^{k+1} fails to have such regularity on the interval. f^{k+1} lacks such regularity only when $f^k(x_1)$ and $f^k(x_2)$ belong to different intervals $I_l, I_{l'}, l \neq l'$. This violates the assumption that $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2$. Thus f^{k+1} is monotone C^2 map on $[x_1, x_2]$. Hence the proposition follows by induction.

Since f is piecewise expanding map, there exists $m > 0$ and $c > 1$ such that $|(f^m)'| > c$. Let $p \in \mathbb{N}$ satisfy $(x_2 - x_1)c^p > 1$. Using the mean value theorem and the monotonicity of f^{mp}, we obtain $|f^{mp}(x_2) - f^{mp}(x_1)| > 1$. This contradicts the fact that f^{mp} is a piecewise expanding map.

The argument above shows that the coding is injective. The coding is defined so that it is surjective. Hence the coding is invertible.

(conjugacy) Let Φ be the coding map, i.e., $\Phi(x) = \boldsymbol{\sigma}$ and $\mathcal{A} := \{0, 1, 2, \cdots, k-1\}$. Let us introduce

$$\Omega_s := \left\{ \mathbf{x} \in \mathcal{A}^{\mathbb{Z}^+} \mid M_{x_j, x_{j+1}} = 1, \quad \forall j \in \mathbb{Z}^+ \right\}, \tag{0.14}$$

where the transition matrix M is defined as

$$M_{ij} = \begin{cases} 1 & \text{if} \quad I_j \subset f(I_i) \\ 0 & \quad \text{otherwise} \end{cases}. \tag{0.15}$$

Let us show $\Phi([0,1)) = \Omega_s$. If $\boldsymbol{\sigma} \in \Phi([0,1))$, $I_{\sigma_{j+1}} \subset f(I_{\sigma_j})$ for all $j \in \mathbb{Z}^+$ due to the Markov property. Hence $\Phi([0,1)) \subset \Omega_s$. Suppose there exists $\boldsymbol{\sigma} \in \Omega_s$ such that $\boldsymbol{\sigma} \notin \Phi([0,1))$. This implies that there exists a minimum q such that $I_{\sigma_q} \cap f^q(I_{\sigma_0}) = \emptyset$. $f^r(I_j)$ $(\forall r \in \mathbb{Z}^+)$ can be written as $\cup_{j' \in \mathcal{B}} I_{j'}$ for some $\mathcal{B} \subset \mathcal{A}$ due to the Markov property. Hence $I_{\sigma_{q-1}} \cap f^{q-1}(I_{\sigma_0}) = I_{\sigma_{q-1}}$ since it is not empty. Then we obtain

$$I_{\sigma_q} \subset f(I_{\sigma_{q-1}}) = f(I_{\sigma_{q-1}} \cap f^{q-1}(I_{\sigma_0})) \subseteq f(I_{\sigma_{q-1}}) \cap f^q(I_{\sigma_0}). \tag{0.16}$$

Thus I_{σ_q} is a subset of $f^q(I_{\sigma_0})$ and then $I_{\sigma_q} \cap f^q(I_{\sigma_0}) = I_{\sigma_q} \neq \emptyset$. This contradicts the assumption and thus we obtain $\Phi([0,1)) \supset \Omega_s$.

Thus the coding provides a conjugacy between f and the unilateral shift of finite type defined using M.

Exercise 2.26.

The inverse of the baker's map is given as

$$f^{-1}(x,y) = \begin{cases} (x/2, 2y) & \text{if} \quad y < 1/2 \\ ((1+x)/2, 2y-1) & \text{if} \quad y \geq 1/2 \end{cases}. \tag{0.17}$$

Let $\Phi : [0,1)^2 \to \{0,1\}^{\mathbb{Z}}$ be the coding map. Suppose two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in [0,1)^2$ share the same code, i.e., $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\ (= \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}_i := \Phi(\boldsymbol{x}_i)$. Let $\boldsymbol{\sigma}^{\geq}$ (resp. $\boldsymbol{\sigma}^{\leq}$) be a subcode of $\boldsymbol{\sigma}$ of positive (resp. negative) indices, i.e., $\boldsymbol{\sigma}^{\geq} := (\sigma_i)_{i \geq 0}$ (resp. $\boldsymbol{\sigma}^{\leq} := (\sigma_i)_{i \leq 0}$). We call it a positive semi-infinite (resp. negative) subcode.

Two points which share their y-component have the same positive semi-infinite subcode since the y-component does not affect the positive time-evolution of the x-component. Let $\boldsymbol{x}_{2*} := (x_{2,1}, x_{1,2})$, then $\boldsymbol{\sigma}_1^{\geq} = \boldsymbol{\sigma}_2^{\geq} = \boldsymbol{\sigma}_{2*}^{\geq}$ follows. $\boldsymbol{\sigma}_1^{\geq} = \boldsymbol{\sigma}_{2*}^{\geq}$ requires $x_{1,1} = x_{2,1}$ because $g(x) := f(x, x_{1,2})$ is a piecewise expanding map. Similarly, we can show $x_{1,2} = x_{2,2}$ by considering the negative semi-infinite subcode. Thus Φ is injective.

Let Ω_s denote the phase space of the full bilateral shift over the two symbols. Let us show $\Phi([0,1)^2) = \Omega_s$. Obviously $\Phi([0,1)^2) \subset \Omega_s$ since the shift is full. Let $I_0 = [0, 1/2) \times [0,1), I_1 = [1/2, 1) \times [0, 1)$. Suppose there exists $\boldsymbol{\sigma} \in \Omega_s$ such that $\boldsymbol{\sigma} \notin \Phi([0,1)^2)$. This implies that there exists $q \in \mathbb{Z}$ such that $I_{\sigma_q} \cap f^q(I_{\sigma_0}) = \emptyset$. Apparently $\boldsymbol{\sigma} \notin \Phi([0,1)^2)$ holds if and only if $\mathcal{S}^r(\boldsymbol{\sigma}) \notin \Phi([0,1)^2)$ for all $r \in \mathbb{Z}$. We take an r which satisfies $r < q$ and denote $q' := q - r$. Then $I_{\mathcal{S}^r(\boldsymbol{\sigma})_{q'}} \cap f^{q'}(I_{\mathcal{S}^r(\boldsymbol{\sigma})_0}) = \emptyset$. This cannot hold because the positive time-evolution of the x-component by the baker's map is given by a piecewise expanding Markov map (see the solution of Exercise 2.24) of its transition matrix $M_{i,j} = 1, \forall i, j \in \{0, 1\}$. Thus $\Phi([0,1)^2) = \Omega_s$, namely, Φ is bijective. Hence the coding Φ provides a desired conjugacy.

Exercise 2.28.

Let us denote $f(x, y) = 2x + y \pmod 1, g(x, y) = x + y \pmod 1$. We obtain $f_x = 2, f_y = 1, g_x = 1, g_y = 1$ on $(0,1)^2$. The derivatives coincide on the gluing boundaries $\{0\} \times [0, 1), \{1\} \times [0, 1)$ and $[0, 1) \times \{0\}, [0, 1) \times \{1\}$ and they are continuous. Thus the cat map is C^1. The second or higher partial derivatives are all zeros, so they coincide on the gluing boundaries and are continuous. Thus the cat map is C^∞.

Let us denote $p(x, y) = x - y \pmod 1, q(x, y) = -x + 2y \pmod 1$. We obtain $p_x = 1, p_y = -1, q_x = -1, q_y = 2$ on $(0,1)^2$. The derivatives coincide on the gluing boundaries $\{0\} \times [0, 1), \{1\} \times [0, 1)$ and $[0, 1) \times \{0\}, [0, 1) \times \{1\}$ and they are continuous. Thus the inverse map is C^1. The second or higher partial derivatives are all zeros, so they coincide on the gluing boundaries and are continuous. Thus the inverse map is C^∞.

Exercise 2.30.