Discrete Structures

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References

Chapter 1

1. Discrete Mathematics and Its Application, 7th Edition
by Kenneth H. Rose

2. Discrete Mathematics with Applications
by Thomas Koshy

3. Discrete Mathematical Structures, CS 173
by Cinda Heeren, Siebel Center

These slides contain material from the above resources.

Quantifications of Two Variables.

Statement	When true?	When false?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x , y for which $P(x, y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x, y)$ is true for every y	For every x there is a y for which $P(x, y)$ is false
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x, y)$ is true	$P(x, y)$ is false for every pair x, y.

Predicates - multiple quantifiers

To bind many variables, use many quantifiers!

Example: $P(x,y)$ = "x > y"

UD = $\mathbb{N}$

- $\forall x\, P(x,y)$

  c)

- $\forall x \forall y\, P(x,y)$

  b)

- $\forall x \exists y\, P(x,y)$

  b)

- $\forall x\, P(x,3)$

  b)

a) True proposition

b) False proposition

c) Not a proposition

d) No clue

Rules of Inference [1]

- **Proofs** in mathematics are **valid arguments** that establish the truth of mathematical statements

- **Argument**: a **sequence of statements** that end with a **conclusion.**

- **Valid**: **the conclusion**, or **final statement** of the argument, must follow from the **truth of the preceding statements**, or **premises**, of the argument.

- That is, an argument is **valid** if and only if it is **impossible** for all the **premises to be true** and the **conclusion to be false**.

Rules of Inference [2]

- To **deduce** new statements from statements we already have, we use **rules of inference** which are **templates for constructing valid arguments**.

- Some common forms of incorrect reasoning, called **fallacies**, which lead to **invalid arguments.**

Rules of Inference for Propositional Logic

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Valid Arguments in Propositional Logic

Example: Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"**If** you have a current password, **then** you can log onto the network"

"You have a current password"

Therefore, "You can log onto the network"

"**If** you have a current password, **then** you can log onto the network"

"You have a current password"

Therefore, "You can log onto the network"

Let **p**: "you have a current password"

and **q**: "you can log onto the network"

Then, the argument has the form

$p \rightarrow q$

p

$\therefore q$

Note: What is the validity of this particular argument?

$((p \rightarrow q) \land p) \rightarrow q$ is a tautology

The argument is valid using Modus ponens (MP).

Argument is valid because **whenever all its premises** (all statements in the argument other than the final one, the conclusion) **are true**, the conclusion must also be true.

"If **you have access to the network**, then **you can change your grade**"

"You can change your grade"

∴"You have access to the network"

Is the above argument is valid using rules of inference?

Let **p: "you have access to the network"**

And **q: "you can change your grade"**

Then, the argument has the form

$$\begin{array}{l} q \\ \underline{p \rightarrow q} \\ \therefore ? \end{array}$$

$$\begin{array}{l} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

The argument we obtained is a **valid argument**, but because **one of the premises**, namely the first **premise, is false**, we cannot conclude that the conclusion is true. (Most likely, this conclusion is false.)

- An **argument** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called **premises** and the final proposition is called the **conclusion**.

- An argument is valid if the truth of all its premises implies that the conclusion is true.

From the definition of a valid argument form we see that the argument form with **premises $p_1, p_2, \ldots, p_n$** and **conclusion q** is valid, when

$(p_1 \land p_2 \land \ldots \land p_n) \rightarrow q$ is a tautology.

Example Suppose that the conditional statement **"If it snows today, then we will go skiing"** and its hypothesis, **"It is snowing today,"** are true. Then, by **modus ponens**, it follows that the conclusion of the conditional statement, "**We will go skiing**," is true.

Solution:

Let p: “it snows today”

and q: “we will go skiing”

Then, the argument has the form

$$
\begin{array}{l}
p \rightarrow q \\
\underline{P \qquad} \\
\therefore q
\end{array}
$$

Then, by **modus ponens**, it follows that the conclusion of the conditional statement, “We will go skiing” is true.

Determine whether the argument given here is valid and determine whether its conclusion must be true because of the validity of the argument.

If $\sqrt{2} > \frac{3}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$.

Let p: $\sqrt{2} > \frac{3}{2}$ **(False)** $\because$ $\sqrt{2}$ **= 1.4142**

and q: $(\sqrt{2})^2 > (\frac{3}{2})^2$

Then, the argument has the form

P

$p \to q$

$\therefore q$

This argument is valid because it is constructed by using modus ponens, a valid argument form. However, one of its premises, $\sqrt{2} > \frac{3}{2}$ **false.**

Consequently, we **cannot conclude** that the conclusion is true. Furthermore, note that the conclusion of this **argument is false**, because $(\sqrt{2})^2 < (\frac{3}{2})^2$

Example State which **rule of inference** is the basis of the following argument: "**It is below freezing now**". Therefore, it is either **below freezing** or **raining now**"

Solution

Let p: “It is below freezing now”

and q: “It is raining now”

Then this argument is of the form

$$\begin{array}{l} \quad p \\ \hline \therefore p \vee q \end{array}$$

This is an argument that uses the **addition rule**.

Example State which **rule of inference is used in the argument**: If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Suppose
p: it rains today
q: we will have a barbecue today
r: we will have a barbecue tomorrow

then this argument is of the form

$p \rightarrow \neg q$

$\underline{\neg q \rightarrow r}$

$\therefore p \rightarrow r$

Hence, this argument is a hypothetical syllogism (HS).

Using Rules of Inference to Build Arguments

Example: Show that the hypotheses "It is not sunny this afternoon **and** it is colder than yesterday," "We will go swimming **only if** it is sunny this afternoon," "**If** we do not go swimming, **then** we will take a canoe trip," and "**If** we take a canoe trip, **then** we will be home by sunset" lead to the conclusion "We will be home by sunset."

Suppose
p: It is sunny this afternoon
q: It is colder than yesterday
r: We will go swimming
s: We will take a canoe trip
t: We will be home by sunset

We need to give a valid argument with hypotheses $\neg p \land q$, $r \to p$, $\neg r \to s$, **and** $s \to t$ and conclusion t

"p only if q" $\equiv p \to q$

Modus Tollens

$\neg q$

$p \to q$

$\therefore \neg p$

Modus Ponens

p

$p \to q$

$\therefore q$

- We need to give a valid argument with hypotheses $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t

Step	Reason	
1. $\neg p \wedge q$	**Hypotheses**	
2. $\neg p$	$\dfrac{p \wedge q}{\therefore p}$	**Simplification on 1**
3. $r \rightarrow p$ 4. $\neg r$	**Hypotheses** $\neg q$ $\dfrac{p \rightarrow q}{\therefore \neg p}$	**Modus tollens on 2 and 3**
5. $\neg r \rightarrow s \equiv \neg s \rightarrow r$	**Hypotheses**	
6. $\neg(\neg s) = s$		**Modus tollens on 4 and 5**
7. $s \rightarrow t$	**Hypotheses**	
8. t	p $\dfrac{p \rightarrow q}{\therefore q}$	**Modus ponens on 6 and 7**

- Alternative method

- We need to give a valid argument with hypotheses $\neg p \land q$, $r \to p$, $\neg r \to s$, and $s \to t$ and conclusion t

Step	Reason	
1. $\neg p \land q$	**Hypotheses**	
2. $\neg p$	$p \land q$ $\therefore p$	**Simplification on 1**
3. $r \to p$ 4. $\neg r$	**Hypotheses** $\neg q$ $p \to q$ $\therefore \neg p$	**Modus tollens on 2 and 3**
5. $\neg r \to s$	**Hypotheses**	
6. s		**Modus ponens on 4 and 5**
7. $s \to t$	**Hypotheses**	
8. t	p $p \to q$ $\therefore q$	**Modus ponens on 6 and 7**

Resolution

- Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a **rule of inference** known as **resolution**. This rule of inference is based on the tautology

  $((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$

- The final disjunction in the resolution rule, $q \lor r$, is called the **resolvent**

Fallacies

- Several common fallacies arise in incorrect arguments. These fallacies resemble **rules of inference** but are based on **contingencies** rather than **tautologies**. These are discussed here to show the distinction between correct and incorrect reasoning.

In class quiz [1]

I am Shahid Afridi
If I am Shahid Afridi, then I am a batting all rounder.

$\therefore$ I am a batting all rounder!

Let p: I am Shahid Afridi
and q: I am a batting all rounder

$$\frac{\begin{array}{l} p \\ p \rightarrow q \end{array}}{\therefore q}$$

Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Inference Rule:

Modus Ponens

In class quiz [2]

I am not a great batsman.
If I am Baber Azam, then I am a great batsman.
$\therefore$ I am not Baber Azam.

Let p: I am Muhammad Asif
and q: I am a great batsman

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

Tautology:

$(\neg q \land (p \to q)) \to \neg p$

Inference Rule:

Modus Tollens

In class quiz [3]

I am a CSP officer.

$\therefore$ I am a CSP officer or I am tall.

Let p: I am a CSP officer
and q: I am tall

$$\frac{p}{\therefore\ p \vee q}$$

Tautology:

$p \rightarrow (p \vee q)$

Inference Rule:

Addition

In class quiz [4]

I am not a great skater and you are sleepy.

$\therefore$ you are sleepy.

Let $\neg p$: I am not a great skater
and q: you are sleepy

Inference Rule:

Simplification

$$\frac{\neg p \wedge q}{\therefore q}$$

Tautology:

$(p \wedge q) \rightarrow p$

Suggested Readings

1.5 Rules of Inference