Discrete Mathematics

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Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 1

1. Discrete Mathematics and Its Application, 6th Edititition
by Kenneth H. Rose

2. Discrete Mathematics with Applications
by Thomas Koshy

3. Discrete Mathematical Structures, CS 173
by Cinda Heeren, Siebel Center

These slides contain material from the above resources.

Implication

Truth table for the conditional statement $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditionals (Bi-implication)

The Truth Table for the Biconditional $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Quantifiers with Restricted Domains

- **Example** What do the statements **$\forall x < 0\ (x^2 > 0)$** mean, where the domain in each case consists of the real numbers?

- The statement $\forall x < 0\ (x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states **"The square of a negative real number is positive."**

- This statement is the same as **$\forall x(x < 0 \rightarrow x^2 > 0)$.**

Quantifiers with Restricted Domains

- **Example:** What do the statement $\forall y \neq 0\ (y^3 \neq 0)$ mean, where the domain consists of the real numbers?

- The statement $\forall y \neq 0\ (y^3 \neq 0)$, states that for every real number y with $y \neq 0$ and, we have $y^3 \neq 0$. That is, it states **"The cube of every nonzero real number is nonzero."**

- This statement is equivalent to $\boldsymbol{\forall y(y \neq 0 \rightarrow y^3 \neq 0)}$.

Quantifiers with Restricted Domains

- **Example:** What do the statement $\exists z > 0\ (z^2 = 2)$ mean, where the domain consists of the real numbers?

- The statement $\exists z > 0\ (z^2 = 2)$, that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states **"There is a positive square root of 2"**

- This statement is equivalent to $\boldsymbol{\exists z(z > 0 \land z^2 = 2)}$.

Negating Quantified Expressions [1]

The rules for negations for quantifiers are called **De Morgan's** laws for quantifiers.

De Morgan's Laws for Quantifiers.			
Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x , P(x) is false	There is an x for which P (x) is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which P(x) is false	P (x) is true for every x .

Negating Quantified Expressions [2]

Example: What is the negation of the statement, **"Every student in your class has taken a course in calculus."**

Solution: "Every student in your class has taken a course in calculus."

UD = The students in your class

Let P(x) = x has taken a course in calculus

$\Rightarrow \forall x P(x)$

The negation of this statement is **"It is not the case that every student in your class has taken a course in calculus"**

This is equivalent to "**There is** a student in your class who has **not** taken a course in calculus."

$\Rightarrow \exists x \neg P(x)$

Negating Quantified Expressions [3]

What are the negations of the statements

"There is an honest politician“

and

"All Americans eat cheeseburgers"?

Negating Quantified Expressions [4]

Solution: "There is an honest politician"

UD = All politicians.

Let **H(x)** denote “x is honest”

$\Rightarrow \exists x H(x)$

The negation is

$\neg \exists x H(x) \equiv \forall x \neg H(x)$

$\Rightarrow$ **"Every politician is dishonest**

or

$\Rightarrow$ **"Every politician is not honest"**

Negating Quantified Expressions [5]

Solution: "All Americans eat cheeseburgers"

UD = All Americans

Let C(x) denote “x eats cheeseburgers”

$\Rightarrow \forall x C(x)$

The negation is

$\neg \forall x C(x) \equiv \exists x \neg C(x)$

$\Rightarrow$ **“There exists an American who does not eat cheeseburgers”**

Negating Quantified Expressions [6]

Example: What are the negations of the statements **$\forall x(x^2 > x)$** and **$\exists x(x^2 = 2)$?**

Negating Quantified Expressions [7]

Solution:

$\forall x(x^2 > x)$

$\neg\forall x P(x) \equiv \exists x \neg P(x)$

Negation of $\forall x(x^2 > x)$:

$\neg\forall x(x^2 > x) \equiv \exists x \neg(x^2 > x)$

$\equiv \exists x(x^2 \leq x)$

Note: The truth values of this statement depends on the domain.

Negating Quantified Expressions [8]

Solution:

$\exists x(x^2 = 2)$

$\neg\exists x P(x) \equiv \forall x \neg P(x)$

Negation of $\exists x(x^2 = 2)$:

$\neg\exists x(x^2 > x) \equiv \forall x \neg(x^2 = 2)$

$\equiv \forall x(x^2 \neq 2)$

Note: The truth value of this statement depends on the domain.

First approach

Example Express the statement **"Every student in this class has studied calculus"** using predicates and quantifiers.

Step1: Rewrite the statement

"For every student in this class, that student has studied calculus"

Step2: Introduce x in the statement

"For every student x in this class, x has studied calculus"

Domain = The students in the class

Let $C(x)$ = x has studied calculus

$\Rightarrow \forall x C(x)$

Second approach

Example Express the statement **"Every student in this class has studied calculus"** using predicates and quantifiers.

U.D = All people

Step1: Rewrite the statement

“For every person, if person is a student in this class then person has studied calculus”

Step2: Introduce x in the statement

“For every person x , if person x is a student in this class then x has studied calculus”

Let **S(x) = person x is in this class**

C(x) = x has studied calculus

$\Rightarrow$ **$\forall x(S(x) \rightarrow C\,(x))$**

Example Consider these statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument.

1. **"All lions are fierce.“**

2. **"Some lions do not drink coffee.“**

3. **"Some fierce creatures do not drink coffee. “**

Let **P(x), Q(x),** and **R(x)** be the statements **"x is a lion," "x is fierce," and "x drinks coffee,"** respectively. Assuming that the **domain consists of all creatures**, express the statements in the argument using quantifiers and P(x), Q(x), and R(x).

Given P(x) = x is a lion Q(x) = x is fierce, and **R(x) = x drinks coffee**

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink coffee."

U.D = All animals

P(x) = x is a lion

R(x) = x drinks coffee

Q(x) = x is fierce

$\forall x\,(P(x) \rightarrow Q(x))$

$\exists x(P(x) \wedge \neg R(x))$

$\exists x(Q(x) \wedge \neg R(x))$

Example

“All hummingbirds are richly colored”

“No large birds live on honey”

“Birds that do not live on honey are dull in color”

“Hummingbirds are small”

Let P(x), Q(x), R(x), and S(x) be the statements **"x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored,"** respectively.

Assuming that the **domain consists of all birds**, express the statements in the argument using quantifiers and P(x), Q(x), R (x), and S(x).

Given **P(x): x is a hummingbird**

Q(x): x is large, R(x): x lives on honey, S(x): x is richly colored

"All hummingbirds are richly colored."

U.D = All birds

P(x): x is a hummingbird

S(x): x is richly colored

$\forall x\,(P(x) \rightarrow S(x))$

"No large birds live on honey."

R(x): x lives on honey

Q(x): x is large

$\neg\exists x(Q(x) \wedge R(x))$

or

$\boldsymbol{\forall x\, \neg\, (Q(x) \wedge R(x))}$

or

$\boldsymbol{\forall x\, (\neg Q(x) \vee \neg R(x))}$

"Birds that do not live on honey are dull in color."

R(x): x lives on honey

S(x): x is richly colored

$\forall x(\neg R(x) \rightarrow \neg S(x))$

"Hummingbirds are small."

P(x): x is a hummingbird

Q(x): x is large

$\forall x(P(x) \rightarrow \neg Q(x))$

$$\neg\exists x(Q(x) \wedge R(x)) \equiv \forall x\,(\neg Q(x) \vee \neg R(x))$$

Example Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

1. **“All hummingbirds are richly colored”**
2. **“No large birds live on honey”**
3. **“Birds that do not live on honey are dull in color”**
4. **“Hummingbirds are small”**

Let **P(x), Q(x), R(x), and S(x) be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored,"** respectively. **Assuming that the domain consists of all birds**, express the statements in the argument using quantifiers and P (x), Q (x), R (x), and S(x)

Solution: We can express the statements in the argument as

1. $\forall x\,(P(x) \rightarrow S(x))$

2. $\neg\exists x(\,Q(x) \land R(x))$ or **$\forall x\,(\neg Q(x) \lor \neg R(x))$**

3. $\forall\, x(\neg R(x) \rightarrow \neg S(x))$

4. $\forall\, x(P(x) \rightarrow \neg Q(x))$

“No large birds live on honey.”

U.D = All birds

R(x): x lives on honey

Q(x): x is large

$\neg\exists x(Q(x) \wedge R(x))$

$\exists xP(x)$ means “P(x) is true for some x.”

What about $\neg\exists xP(x)$?

Not [“P(x) is true for **some x**.”]

$\equiv$ “P(x) is **not** true **for all x**.”

$\equiv \forall x \neg P(x)$

$\neg\exists xP(x) \equiv \forall x \neg P(x)$

$\boldsymbol{\forall x \neg (Q(x) \wedge R(x))}$

Or $\boldsymbol{\forall x (\neg Q(x) \vee \neg R(x))}$

Suggested Readings

1.4 Predicates and Quantifiers