Discrete Structures

Dr. Syed Faisal Bukhari
Associate Professor
Department of Data Science, Faculty of Computing and Information Technology, University of the Punjab

Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 1

1. Discrete Mathematics and Its Application, 7th Edition
by Kenneth H. Rose

2. Discrete Mathematics with Applications
by Thomas Koshy

3. Discrete Mathematical Structures, CS 173
by Cinda Heeren, Siebel Center

These slides contain material from the above resources.

Nested Quantifiers

- Two quantifiers are nested if one is within the scope of the other

  $\forall x \exists y(x + y = 0)$.

- Everything within the scope of a quantifier can be thought of as a **propositional function**.

  $\forall x \exists y(x + y = 0)$ is the same thing as $\forall x Q(x)$

$\Longrightarrow \forall x Q(x)$

  where

  $Q(x): \exists y P(x, y)$

  $P(x, y): x + y = 0$

- Nested quantifiers commonly occur in mathematics and computer science

Quantifications of Two Variables.

Statement	When true?	When false?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	P(x, y) is true for every pair x, y.	There is a pair x , y for which P(x, y) is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which P(x, y) is true.	There is an x such that P(x, y) is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which P(x, y) is true for every y	For every x there is a y for which P(x, y) is false
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which P(x, y) is true	P(x, y) is false for every pair x, y.

- **Example** Assume that the domain for the variables x and y consists of all real numbers. The statement **$\forall x \forall y(x + y = y + x)$** says that **$x + y = y + x$** for all real numbers x and y.

- **This is the commutative law for addition of real numbers**

- **Example** Assume that the domain for the variables x and y consists of all real numbers. **$\forall x \exists y(x + y = 0)$** says that for every real number x there is a real number y such that **$x + y = 0$**.

- **This states that every real number has an additive inverse**

Example Assume that the domain for the variables x, y, and consists of all real numbers.

$\forall \mathbf{x} \forall \mathbf{y} \forall \boldsymbol{z}(\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z})$ is the associative law for addition of real numbers.

Example Translate into English the statement

$\forall x\ \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0))$ where the domain for both variables consists of all real numbers.

Solution: $\forall x \, \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

U.D = Both variables consists of all real numbers.

This statement says that for every real number x and for every real number y, **if $x > 0$ and $y < 0$, then $xy < O$**.

That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative.

This can be stated more succinctly as **"The product of a positive real number and a negative real number is always a negative real number."**

Thinking of Quantification as Loops [1]

- In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops.

1. **$\forall x \forall y P(x,y)$** is true, we loop through the values for x , and for each x we loop through the values for y. If we find that P (x , y) is true for all values for x and y, we have determined **$\forall x \forall y P(x,y)$** is **true**. If we ever hit a value x for which we hit a value y for which P (x , y) is **false**, we have shown that **$\forall x \forall y P(x,y)$** is **false**.

2. **$\forall x \exists y P(x,y)$** is true, we loop through the values for x. For each x we loop through the values for y until we find a y for which P (x , y) is **true**. If for every x we hit such a y, then $\forall x \exists y P(x,y)$ is true; if for some x we never hit such a y, then **$\forall x \exists y P(x,y)$** is **false**.

Thinking of Quantification as Loops [2]

3. **$\exists x \forall y P(x,y)$** is true, we loop through the values for x until we find an x for which P (x , y) is always **true** when we loop through all values for y. Once we find such an x , we know that **$\exists x \forall y P(x,y)$** is **true.** If we never hit such an x , then we know **$\exists x \forall y P(x,y)$** is **false**.

4. **$\exists x \exists y P(x,y)$** is true, we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which P (x , y) is **true**. The statement **$\exists x \exists y P(x,y)$** is **false** only if we never hit an x for which we hit a y such that P (x , y) is **true**.

The Order of Quantifiers [1]

1. **$\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$** are **logically equivalent**.

2. **$\exists x \exists y P(x,y)$ and $\exists y \exists x P(x,y)$** are **logically equivalent**.

3. **$\forall x \exists y P(x,y)$** and **$\exists y \forall x P(x,y)$** are **not logically equivalent**.

The Order of Quantifiers [2]

- **$\exists y \forall x P(x,y)$** is true if and only if **there is a y** that makes P(x , y) true for **every x** So, for this statement to be true, there must be a **particular value of y** for which P(x , y) is true regardless of the choice of x. **y is a constant independent of x**

- If **$\exists y \forall x P(x,y)$** is true then **$\forall x \exists y P(x,y)$** must also be true.

- If **$\forall x \exists y P(x,y)$** is true then it is not necessary for **$\exists y \forall x P(x,y)$** to be true

The Order of Quantifiers [3]

If $\exists y \forall x P(x,y)$ is true then $\forall x \exists y P(x,y)$ must also be true.

If $\forall x \exists y P(x,y)$ is true then it is not necessary for $\exists y \forall x P(x,y)$ to be true

Predicates - the meaning of multiple quantifiers

Suppose P(x,y) = "x's favorite class is y."

1. $\forall x \forall y P(x,y)$

P(x,y) true for all x, y pairs.

2. $\exists x \exists y P(x,y)$

P(x,y) true for at least one x, y pair.

3. $\forall x \exists y P(x,y)$

For every value of x we can find a (possibly different) y so that P(x,y) is true.

4. $\exists x \forall y P(x,y)$

There is at least one x for which P(x,y) is always true.

The Order of Quantifiers [2]

- If $\exists y \forall x P(x,y)$ is true then $\forall x \exists y P(x,y)$ must also be true.
- If $\forall x \exists y P(x,y)$ is true then it is not necessary for $\exists y \forall x P(x,y)$ to be true

The Order of Quantifiers [3]

- **Example** Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

- The quantification **$\forall x \forall y$ P(x , y)** denotes the proposition “For all real numbers x , for all real numbers y, x + y = y + x .’’

- Since P (x , y) is true for all real numbers x and y, **$\forall x \forall y$P (x , y) is true**.

- The quantification **$\forall y \forall x$ P(x , y)** denotes the proposition ``For all real numbers y , for all real numbers x, x + y = y + x .’’

- Since **P(x , y)** is true for all real numbers x and y, **$\forall y \forall x$ P (x , y) is true.**

Example Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution:

UD = Set of real numbers

Q(x , y) = "x + y = 0."

Truth value of ∃y∀xQ(x, y)?

- The quantification **∃y∀xQ(x, y)** denotes the proposition **"There is a real number y such that for every real number x, Q(x , y)."**

- **No matter what value of y is chosen**, there is only one value of x for which x + y = O. Because there is no real number y such that x + y = 0 for all real numbers x , the statement **∃y∀xQ(x , y) is false**

Solution

UD = Set of real numbers

$Q(x, y) = "x + y = 0."$

Truth value of $\forall x \exists y Q(x, y)$?

- The quantification $\forall x \exists y Q(x, y)$, denotes the proposition **"For every real number x there is a real number y such that Q (x , y)."**

- Given a real number x , there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement **$\forall x \exists y Q(x, y)$, is true**.

Translating into Nested Quantifiers

Example Translate the statement "The sum of two positive integers is always positive" into a **logical expression**.

Dr. Faisal Bukhari, Department of Data Science, PU

First approach:

Step 1: Rewrite so that the implied quantifiers and a domain are shown **"For every two integers, if these integers are both positive, then the sum of these integers is positive"**

Step 2: Introduce the variables x and y

"For all positive integers x and y, x + y is positive"

U.D = All integers (both variables)

$\Rightarrow$ $\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$

Second approach:

Step 1: Rewrite so that the implied quantifiers and a domain are shown **"For every two positive integers, the sum of these integers is positive."**

Step 2: Introduce the variables x and y

"For all positive integers x and y, x + y is positive

U.D = All positive integers (both variables)

$\Rightarrow \forall x \forall y(x + y > 0)$

Example Translate the statement **"Every real number except zero has a multiplicative inverse."** (A multiplicative inverse of a real number x is a real number y such that x y = 1 .)

Step 1: Rewrite

"For every real number x except zero, x has a multiplicative inverse"

Step 2: Introduce the variables x and y

"For every real number x , if $x \neq 0$, then there exists a real number y such that x y = 1"

U.D = Set of real numbers

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$$

Translating from Nested Quantifiers into English

Example Translate the statement

$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$ into English, where

C (x) = "x has a computer"

F (x , y) = "x and y are friends"

and the domain for both x and y consists of all students in your school.

Solution:

$\forall x(C(x) \vee \exists y(C(y) \wedge F(x , y)))$

U.D = x and y consists of all students in your school.

C (x) = "x has a computer"

F (x , y) = "x and y are friends"

The statement says that for every student x in your school, **x has a computer or there** is a student y such that **y has a computer and x and y are friends**.

In other words, every student in your school has a computer or has a friend who has a computer.

Translating English Sentences into Logical Expressions

Example Express the statement **"If a person is female and is a parent, then this person is someone's mother"** as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connective

“If a person is female and is a parent, then this person is someone's mother”

Step1: Rewrite

“For every person, if person is female and person is a parent, then there exists a person such that person is the mother of person”

Step2: Introduce x and y

“For every person x , if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y”

We introduce the propositional functions

U.D = All people

F(x) = x is female

P(x) = x is a parent

M (x , y) = x is the mother of y

$\forall x((F(x) \land P(x)) \rightarrow \exists y M(x,y))$

Negating Nested Quantifiers

Example: Express the negation o f the statement **$\forall x \exists y(xy = 1)$** so that no negation precedes a quantifier.

$$\neg\forall x \exists y (xy = 1) \;\; = \exists x \forall y \,\neg(xy = 1)$$
$$= \exists x \forall y (xy \neq 1)$$

Suggested Readings

1.4 Predicates and Quantifiers