Discrete Structures

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Text book

Discrete Mathematics and Its Application, 7^{th} Edition

Kenneth H. Rosen

References

Chapter 1

1. Discrete Mathematics and Its Application, 7th Edititition

by

Kenneth H. Rose

2. Discrete Mathematics with Applications

by

Thomas Koshy

Predicates and Quantifiers

- **Propositional logic**, cannot adequately express the meaning of statements in mathematics and in natural language. For example, suppose that we know that

  "Every computer connected to the university network is functioning properly."

- **No rules of propositional logic** allow us to conclude the **truth of the statement**.

- We will introduce a more powerful type of **logic** called **predicate logic.**

Predicates and Quantifiers

- **Predicate logic** can be used to express the **meaning of a wide range of statements** in **mathematics** and **computer science** in ways that permit us to reason and explore relationships between objects.

Predicates

The statements **involving the variables** such as

$x > 3$

$x + y > 0$

$x = y + 3$

are neither **true nor false** because the values of **variables are not specified**.

Predicates

Statements **involving variables**, such as

“x > 3”, “x = y + 3”, “x + y = z”, “computer x is under attack by an intruder”, and **“computer x is functioning properly”**

- These statements are **neither true nor false** when the values of the **variables are not specified**

  The statement **"x is greater than 3"** has two parts.

  Subject: (variable) x is the subject

  Predicate: is greater than 3

- **Predicate** states the property the object **x has**

  Note: Each statement consists of subject and predicate

Propositional Function

- The **statement P(x)** is also said to be the value of **the propositional function** P at **x**.

- Once a value has been assigned to the variable x, the statement **P(x)** becomes a **proposition** and has a **truth value.**

Example Let P(x) denote the statement "x > 3." What are the truth values of P(4) and P(2)?

Solution:

Let $P(x): x > 3$

$P(4): 4 > 3$

True

$P(2): 2 > 3$

False

Example Let **A(x)** denote the statement **"Computer x is under attack by an intruder."** Suppose that of the computers on campus, only **CS2** and **MATH1** are **currently under attack by intruders**. What are truth values of **A(CS1), A(CS2), and A(MATH 1)?**

Solution:

A(x) = Computer x is under attack by an intruder

A(CS1) = Computer **CS1** is under attack by an intruder

False

A(CS2) = Computer **CS2** is under attack by an intruder

True

A(MATH1) = Computer **MATH1** is under attack by an intruder

True

Example: Let **Q(x, y)** denote the statement **"x = y + 3 ."** What are the truth values of the propositions **Q(1, 2)** and **Q(3, 0)?**

Solution:

$Q(x , y): x = y + 3$

$Q(1 , 2): 1 = 2 + 3$

$Q(1 , 2): 1 = 5$

False

$Q(3, 0): 3 = 0 + 3$

$Q(3, 0): 3 = 3$

True

Predicates

- In general, a statement involving the **n variables $x_l, x_2, . . . , x_n$** can be denoted by **$P(x_l, x_2, . . . , x_n)$**

- A statement of the form $P(x_l, x_2, . . . , x_n)$ is the value of **the propositional function P** at the n-tuple $(x_l, x_2, . . . , x_n)$, and P is also called a **n-place predicate** or a **n-ary predicate**.

How to create proposition from propositional function?

- There are two ways:

- When the **variables in a propositional function** are assigned values, the **resulting statement becomes a proposition** with a certain truth value

- **Quantification:** is a process to create a **proposition** from a **propositional function**.

- In English, the words **all, some, many, none**, and **few** are used in **quantifications.**

Universe of discourse (UD) or universe or domain

The **set of all values** x can have is called the **universe of discourse (UD)**.

For example:

Set of all apples is UD

Set of all chalkboards is UD

Quantifiers

- **All** people are mortal.
- **Every** computer is 16-bit machine.
- **No** birds are black.
- **Some** people have blue eyes.
- **There exists** an even prime number.

- Each contains a word indicating such as **all, every, none, some** and **one**.
- Such words, called **quantifiers**, give us an idea about how many **objects have a certain property.**

Note: The area of logic that deals with **predicate and quantifiers** is called **the predicate calculus**

Types of quantifiers

There are two types of quantifiers: **Universal quantifier and existential quantifier**

Universal quantification, which tells us that a **predicate** is true for **every element under consideration**.

Universal quantifier

1. **Universal quantifier**: Let **p(x)** be a **propositional function** with **domain D**. For all x, p(x) is true. Symbolically $\forall x \in D, P(x)$. We read **$\forall x$ P(x)** as **"for all x P(x)"** or **"for every x P(x)".**

$\forall x P(x) = P(x_1) \land P(x_2) \ldots \land P(x_n)$

$UD = \{x_1, x_2, \ldots, x_n\}$

All elements of UD satisfy P(x).

Signal word or grammar word: **all, for all, all of, each, whole, every, given any, for arbitrary, for each, for any**

Existential Quantifier

Existential quantification, which tells us that there is **one or more element** under consideration for which the predicate is true

Existential quantifier:

- **Existential quantifier:** Let P(x) be a propositional function with domain D. For **some values** of x such P(x) is true. We read **∃xP(x)** as **"There is an x such that P(x)"**, "**There is at least one x such that P(x)"** or **"For some x P (x)".**

Symbolically **∃x∈D** such that P(x)

$\exists x P(x) = P(x_1) \vee P(x_2) \ldots \vee P(x_n)$

$UD = \{x_1, x_2, \ldots, x_n\}$

Some elements of UD satisfy P(x)

Signal word or grammar word: Some, at least, there exist, someone, few, any, exactly one

Quantifiers		
Statement	**When True ?**	**When False?**
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Assumption about the domains of discourse

Generally, an implicit assumption is made that all domains of discourse for **quantifiers are nonempty**.

1. If the **domain is empty**, then **$\forall xP(x)$ is true** for any **propositional function P(x)** because there are no elements x in the domain for which **P(x) is false**.

2. If the **domain is empty,** then **$\exists xP(x)$ is false** whenever Q(x) is a propositional function because when the domain is empty, there can be **no element in the domain for which Q(x) is true**.

Example: Let P (x) be the statement "x + 1 > x ". What is the truth value of the quantification **$\forall xP(x)$,** where the domain consists of all real numbers?

Solution:

UD = set of real numbers

Or

UD = $\mathbb{R}$

P(x) is a propositional function (pf) and P is predicate and x is a variable

P(x): x + 1 > x

P(-1): -1 + 1 > -1 (True)

P(0): 0 + 1 > 0 (True)

P(1): 1 + 1 > 1 (True)

P(x) is true for all real numbers x, the quantification

$\forall x P(x)$, is true

Example: Let **Q(x)** be the statement **"x < 2"** What is the truth value of the quantification **∀xQ(x),** where the **domain consists of all real numbers**?

Solution:

UD = set of real numbers

Or

UD = $\mathbb{R}$

Q(x) is a propositional function (pf) and Q is predicate and x is a variable

Q(x): $x < 2$

Q(-1): $-1 < 2$ (true)

Q(0): $0 < 2$ (true)

Q(1): $1 < 2$ (true)

Q(3): $3 < 2$ (false)

Therefore **$\forall x Q(x)$ is false**

Example: What is the truth value of ∀xP(x), where P(x) is the statement **"$x^2 < 10$"** and the domain consists of the positive integers not exceeding 4?

Solution: $\forall x P(x)$ or $P(1) \land P(2) \land P(3) \land P(4)$

UD = {1, 2, 3, 4}

P(x) is a propositional function (pf) and P is predicate and x is a variable

$P(x): x^2 < 1\ 0$

P(1): 1 < 10 (true)

P(2): 4 < 10 (true)

P(3): 9 < 10 (true)

P(4): 16 < 10 **(false)**

Therefore **$\forall x P(x)$ is false**

Example: Let P (x) denote the statement **"x > 3"** What is the truth value of the **quantification ∃xP(x),** where the domain consists of all real numbers?

Solution:

P(x) = "x > 3"

UD = set of real numbers

Or UD = $\mathbb{R}$

P(x) is a propositional function (pf) and P is predicate and x is a variable

P(x): x > 3

P(-1): -1 > 3 (false)

P(0): 0 > 3 (false)

P(1): 1 > 3 (false)

P(4): 4 > 3 (true)

P(x) is true for some real numbers x, the quantification

$\exists x P(x)$, is true

Example: Let **Q(x)** denote the statement **"x = x + 1".** What is the truth value of the quantification **∃xP(x),** where the domain consists of all real numbers?

UD = set of real numbers

P(x) is a propositional function (pf) and P is predicate and x is a variable

P(x): x = x + 1

P(-1): -1 = -1 + 1 (false)

P(0): 0 = 0 + 1 (false)

P(1): 1 = 1 + 1 (false)

P(x) is false for every real numbers x, the quantification

∃xP(x), is false

Example What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement **"$x^2 > 1\ 0$"** and the universe of discourse consists of the **positive integers not exceeding 4**?

Solution:

$\exists x P(x)$ or $P(1) \vee P(2) \vee P(3) \vee P(4)$

UD = {1, 2, 3, 4}

P(x) is a propositional function (pf) and P is predicate and x is a variable

$P(x): x^2 > 1\ 0$

$P(1): 1 > 10$ (false)

$P(2): 4 > 10$ (false)

$P(3): 9 > 10$ (false)

$P(4): 16 > 10$ (true)

Therefore $\exists x P(x)$ **is true**

Suggested Readings

1.4 Predicates and Quantifiers