Bahria University
Discovering Knowledge

Computer Architecture and Logic Design (CALD)

Lecture 09

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Digital Systems and Binary Numbers

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Outline of Chapter 1

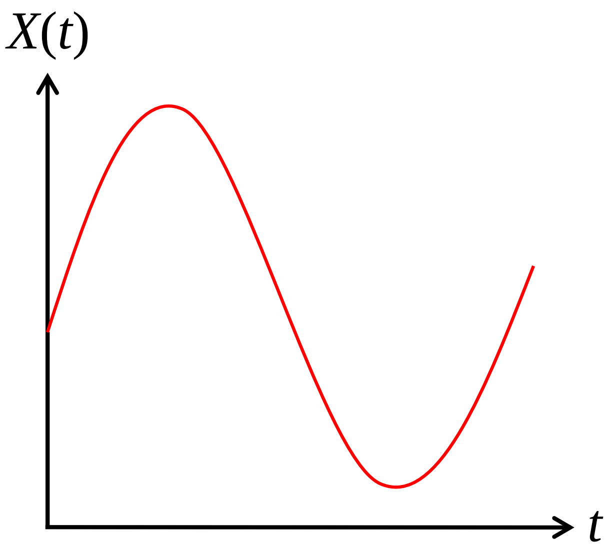
- 1.1 Digital Systems
- 1.2 Binary Numbers
- 1.3 Number-base Conversions
- 1.4 Octal and Hexadecimal Numbers
- 1.5 Complements
- 1.6 Binary Storage and Registers
- 1.7 Binary Logic

Digital Systems and Binary Numbers

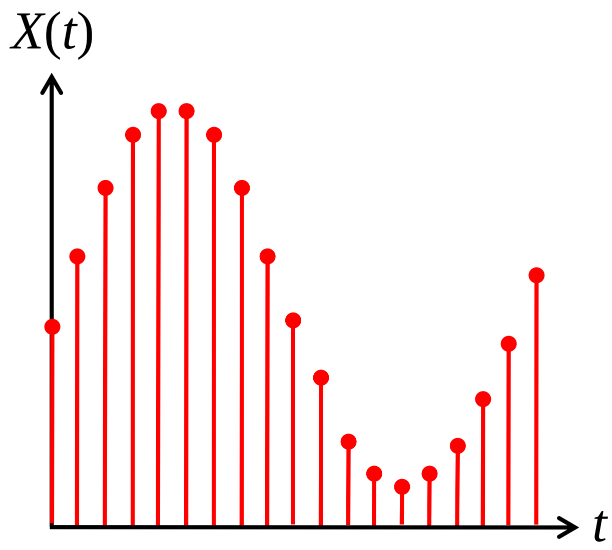
- Digital age and information age
- Digital computers
  - General purposes
  - Many scientific, industrial and commercial applications
- Digital systems
  - Telephone switching exchanges
  - Digital camera
  - Electronic calculators, PDA's
  - Digital TV
- Discrete information-processing systems
  - Manipulate discrete elements of information
  - For example, {1, 2, 3, …} and {A, B, C, …}…

Analog and Digital Signal

- Analog system
  - The physical quantities or signals may vary continuously over a specified range.
- Digital system
  - The physical quantities or signals can assume only discrete values.
  - Greater accuracy



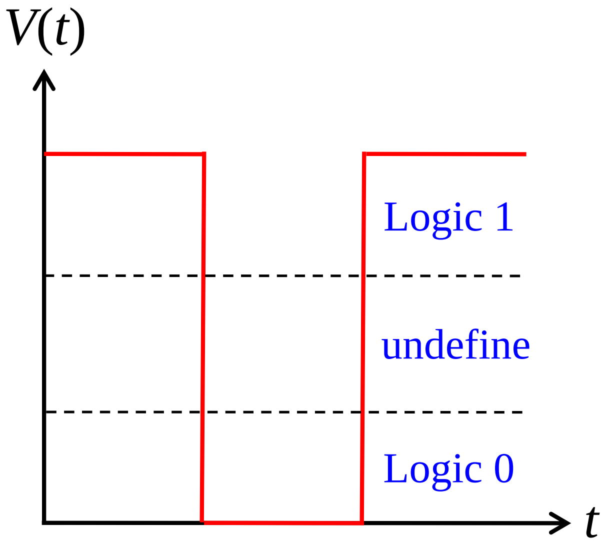
Analog signal



Digital signal

Binary Digital Signal

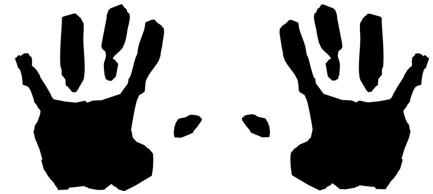
- An information variable represented by physical quantity.
- For digital systems, the variable takes on discrete values.
  - Two level, or binary values are the most prevalent values.
- Binary values are represented abstractly by:
  - Digits 0 and 1
  - Words (symbols) False (F) and True (T)
  - Words (symbols) Low (L) and High (H)
  - And words On and Off
- Binary values are represented by values or ranges of values of physical quantities.



Binary digital signal

Decimal Number System

- Base (also called radix) = 10
  - 10 digits { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Digit Position
  - Integer & fraction
- Digit Weight
  - Weight = $(Base)^{Position}$
- Magnitude
  - Sum of "*Digit* x *Weight*"
- Formal Notation



2	1	0		-1	-2
5	1	2	.	7	4
100	10	1		0.1	0.01
			.		
500	10	2		0.7	0.04

$$d_2{*}B^2+d_1{*}B^1+d_0{*}B^0+d_{-1}{*}B^{-1}+d_{-2}{*}B^{-2}$$

$$(512.74)_{10}$$

Octal Number System

- Base = 8
  - 8 digits { 0, 1, 2, 3, 4, 5, 6, 7 }
- Weights
  - Weight = (*Base*) $^{\textit{Position}}$
- Magnitude
  - Sum of "*Digit* x *Weight*"
- Formal Notation

64	8	1		1/8	1/64
5	1	2	.	7	4
2	1	0		-1	-2

$$5 *8^{2}+1 *8^{1}+2 *8^{0}+7 *8^{-1}+4 *8^{-2}$$

$$=(330.9375)_{10}$$

$$(512.74)_{8}$$

Binary Number System

- Base = 2
  - 2 digits { 0, 1 }, called *b*inary dig*its* or "*bits*"
- Weights
  - Weight = (*Base*) ^{Position}
- Magnitude
  - Sum of "*Bit* x *Weight*"
- Formal Notation
- Groups of bits   4 bits = *Nibble*

  8 bits = *Byte*

4	2	1		1/2	1/4
1	0	1	•	0	1
2	1	0		-1	-2

$1 * 2^{2} + 0 * 2^{1} + 1 * 2^{0} + 0 * 2^{-1} + 1 * 2^{-2}$

$= (5.25)_{10}$

$(101.01)_{2}$

1 0 1 1

1 1 0 0 0 1 0 1

Hexadecimal Number System

- Base = 16
  - 16 digits { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F }
- Weights
  - Weight = $(Base)^{Position}$
- Magnitude
  - Sum of "*Digit* x *Weight*"
- Formal Notation

256	16	1		1/16	1/256
1	E	5	•	7	A
2	1	0		-1	-2

$$1 * 16^2 + 14 * 16^1 + 5 * 16^0 + 7 * 16^{-1} + 10 * 16^{-2}$$

$$=(485.4765625)_{10}$$

$$(1E5.7A)_{16}$$

The Power of 2

n	2^n
0	$2^0=1$
1	$2^1=2$
2	$2^2=4$
3	$2^3=8$
4	$2^4=16$
5	$2^5=32$
6	$2^6=64$
7	$2^7=128$

n	2^n	
8	$2^8=256$	
9	$2^9=512$	
10	$2^{10}=1024$	Kilo
11	$2^{11}=2048$	
12	$2^{12}=4096$	
20	$2^{20}=1M$	Mega
30	$2^{30}=1G$	Giga
40	$2^{40}=1T$	Tera

Addition

- Decimal Addition

	1	1		← Carry
		5	5	
+		5	5	
	1	1	0	

Binary Addition

- Column Addition

	1	1	1	1	1	1		
		1	1	1	1	0	1	= 61
+			1	0	1	1	1	= 23
	1	0	1	0	1	0	0	= 84

Binary Subtraction

- Borrow a “Base” when needed

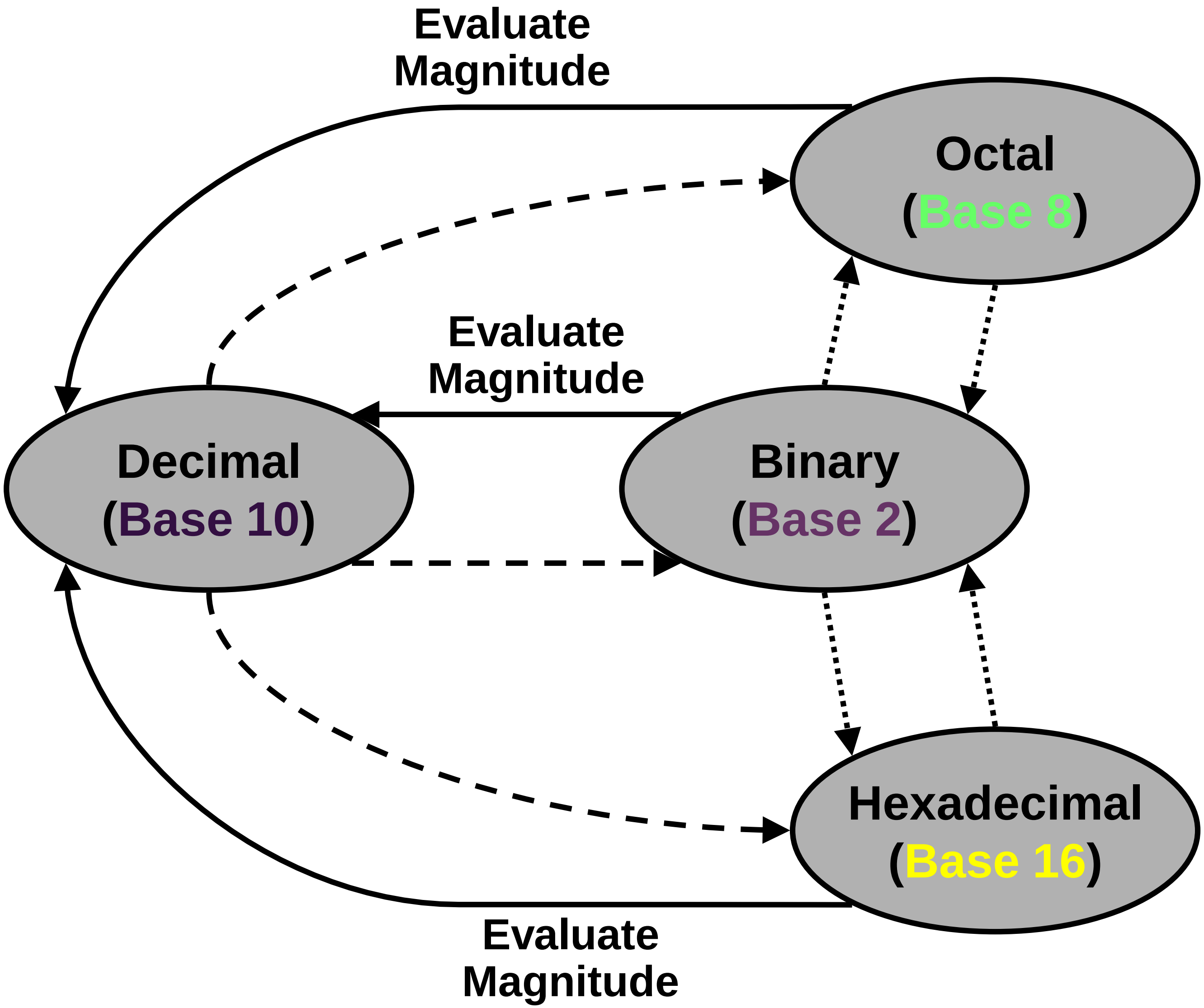
$$
\begin{array}{cccccccccl}
 & & 1 & & & 2 & & & & = (10)_2 \\
 & 0 & \cancel{2} & 2 & 0 & 0 & 2 & & & \\
 & \cancel{1} & 0 & 0 & \cancel{1} & \cancel{1} & 0 & 1 & & = 77 \\
- & & & 1 & 0 & 1 & 1 & 1 & & = 23 \\
\hline
 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & & = 54
\end{array}
$$

Binary Multiplication

- Bit by bit

				1	0	1	1	1
x					1	0	1	0
				0	0	0	0	0
			1	0	1	1	1	
		0	0	0	0	0		
	1	0	1	1	1			
	1	1	1	0	0	1	1	0

Number Base Conversions



Decimal (*Integer*) to Binary Conversion

- Divide the number by the 'Base' (=2)
- Take the remainder (either 0 or 1) as a coefficient
- Take the quotient and repeat the division

Example: $(13)_{10}$

	Quotient	Remainder	Coefficient
13/ 2 =	6	1	$a_0 = 1$
6 / 2 =	3	0	$a_1 = 0$
3 / 2 =	1	1	$a_2 = 1$
1 / 2 =	0	1	$a_3 = 1$

Answer: $(13)_{10} = (a_3\ a_2\ a_1\ a_0)_2 = (1101)_2$

MSB LSB

Decimal (*Fraction*) to Binary Conversion

- Multiply the number by the 'Base' (=2)
- Take the integer (either 0 or 1) as a coefficient
- Take the resultant fraction and repeat the division

Example: $(0.625)_{10}$

	Integer		Fraction	Coefficient
$0.625 * 2 =$	1	.	25	$a_{-1} = 1$
$0.25 * 2 =$	0	.	5	$a_{-2} = 0$
$0.5 * 2 =$	1	.	0	$a_{-3} = 1$

Answer: $(0.625)_{10} = (0.a_{-1}\, a_{-2}\, a_{-3})_2 = (0.101)_2$

MSB **LSB**

Decimal to Octal Conversion

Example: $(175)_{10}$

	Quotient	Remainder	Coefficient
175 / 8 =	21	7	$a_0 = 7$
21 / 8 =	2	5	$a_1 = 5$
2 / 8 =	0	2	$a_2 = 2$

Answer: $(175)_{10} = (a_2\, a_1\, a_0)_8 = (257)_8$

Example: $(0.3125)_{10}$

	Integer		Fraction	Coefficient
0.3125 * 8 =	2	.	5	$a_{-1} = 2$
0.5 * 8 =	4	.	0	$a_{-2} = 4$

Answer: $(0.3125)_{10} = (0.a_{-1}\, a_{-2}\, a_{-3})_8 = (0.24)_8$

Binary – Octal Conversion

- $8 = 2^3$
- Each group of 3 bits represents an octal digit

Octal	Binary
0	0 0 0
1	0 0 1
2	0 1 0
3	0 1 1
4	1 0 0
5	1 0 1
6	1 1 0
7	1 1 1

Example:

Assume Zeros

$$(\;1\;0\;1\;1\;0\;.\;0\;1\;)_2$$

$$(\;2\quad 6\;.\;2\;)_8$$

Works **<u>both</u>** **ways (*Binary* to *Octal* & *Octal* to *Binary*)**

Binary – Hexadecimal Conversion

- $16 = 2^4$
- Each group of 4 bits represents a hexadecimal digit

Hex	Binary
0	0 0 0 0
1	0 0 0 1
2	0 0 1 0
3	0 0 1 1
4	0 1 0 0
5	0 1 0 1
6	0 1 1 0
7	0 1 1 1
8	1 0 0 0
9	1 0 0 1
A	1 0 1 0
B	1 0 1 1
C	1 1 0 0
D	1 1 0 1
E	1 1 1 0
F	1 1 1 1

Example:

Assume Zeros

$$(\ \underbrace{1}\ \underbrace{0\,1\,1\,0}\ .\ \underbrace{0\,1}\ )_2$$

$$(\ 1\ \ \ 6\ \ .\ \ 4\ )_{16}$$

Works both ways (*Binary* to *Hex* & *Hex* to *Binary*)

Octal – Hexadecimal Conversion

- Convert to Binary as an intermediate step

Example:

$$(2\quad 6\quad .\quad 2)_8$$

Assume Zeros

$$(0\,1\,0\,1\,1\,0\,.\,0\,1\,0)_2$$

Assume Zeros

$$(1\quad 6\quad .\quad 4)_{16}$$

Works both ways (*Octal* to *Hex* & *Hex* to *Octal*)

Decimal, Binary, Octal and Hexadecimal

Decimal	Binary	Octal	Hex
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

1.5 Complements

- There are two types of complements for each base-r system: the radix complement and diminished radix complement.

- **Diminished Radix Complement - (r-1)'s Complement**
  - Given a number N in base r having n digits, the $(r-1)$'s complement *of* N is defined as:

$$(r^n - 1) - N$$

- **Example for 6-digit <u>decimal</u> numbers**:
  - 9's complement is $(r^n - 1) - N = (10^6 - 1) - N = 999999 - N$
  - 9's complement of 546700 is 999999–546700 = 453299

- **Example for 7-digit <u>binary</u> numbers:**
  - 1's complement is $(r^n - 1) - N = (2^7 - 1) - N = 1111111 - N$
  - 1's complement of 1011000 is 1111111–1011000 = 0100111

- **Observation:**
  - Subtraction from $(r^n - 1)$ will never require a borrow
  - Diminished radix complement can be computed digit-by-digit
  - For binary: 1 – 0 = 1 and 1 – 1 = 0

Complements

- 1's Complement (*Diminished Radix* Complement)
  - All '0's become '1's
  - All '1's become '0's

  Example $(10110000)_2$

  ⇨ $(01001111)_2$

  If you add a number and its 1's complement …

$$
\begin{array}{r}
1\,0\,1\,1\,0\,0\,0\,0 \\
+\;\;0\,1\,0\,0\,1\,1\,1\,1 \\
\hline
1\,1\,1\,1\,1\,1\,1\,1
\end{array}
$$

Complements

- Radix Complement

The r's complement of an n-digit number N in base r is defined as $r^n - N$ for $N \neq 0$ and as 0 for $N = 0$. Comparing with the $(r - 1)$'s complement, we note that the r's complement is obtained by adding 1 to the $(r - 1)$'s complement, since $r^n - N = [(r^n - 1) - N] + 1$.

- Example: Base-10

The 10's complement of 012398 is 987602
The 10's complement of 246700 is 753300

- Example: Base-2

The 2's complement of 1101100 is 0010100
The 2's complement of 0110111 is 1001001

Complements

- 2's Complement (*Radix* Complement)
  - Take 1's complement then add 1
  - Toggle all bits to the left of the first '1' from the right

OR

Example:

Number:	1 0 1 1 0 0 0 0	1 0 1 1 0 0 0 0
1's Comp.:	0 1 0 0 1 1 1 1	
	+ 1	
	0 1 0 1 0 0 0 0	0 1 0 1 0 0 0 0

Complements

- **Subtraction with Complements**
  - The subtraction of two n-digit unsigned numbers $M - N$ in base r can be done as follows:

1. Add the minuend M to the r's complement of the subtrahend N. Mathematically, $M + (r^n - N) = M - N + r^n$.
2. If $M \geqq N$, the sum will produce and end carry r^n, which can be discarded; what is left is the result $M - N$.
3. If $M < N$, the sum does not produce an end carry and is equal to $r^n - (N - M)$, which is the r's complement of $(N - M)$. To obtain the answer in a familiar form, take the r's complement of the sum and place a negative sign in front.

Complements

- Example 1.5
  - Using 10's complement, subtract 72532 – 3250.

$$
\begin{aligned}
M &= \quad 72532 \\
\text{10's complement of } N &= \underline{+96750} \\
\text{Sum} &= \quad 169282 \\
\text{Discard end carry } 10^5 &= \underline{-100000} \\
\text{Answer} &= \quad 69282
\end{aligned}
$$

- Example 1.6
  - Using 10's complement, subtract 3250 – 72532.

$$
\begin{aligned}
\text{M} &= \quad 03250 \\
\text{10's complement of N} &= \underline{+27468} \\
\text{Sum} &= \quad 30718
\end{aligned}
$$

There is no end carry.

Therefore, the answer is – (10's complement of 30718) = – 69282.

Complements

- Example 1.7
  - Given the two binary numbers X = 1010100 and Y = 1000011, perform the subtraction (a) $X - Y$; and (b) $Y - X$, by using 2's complement.

$$
\begin{array}{lrr}
\text{(a)} & X = & 1010100 \\
 & \text{2's complement of } Y = & \underline{+0111101} \\
 & \text{Sum} = & 10010001 \\
 & \text{Discard end carry } 2^7 = & \underline{-10000000} \\
 & \text{Answer. } X - Y = & 0010001
\end{array}
$$

$$
\begin{array}{lrr}
\text{(b)} & Y = & 1000011 \\
 & \text{2's complement of } X = & +\underline{0101100} \\
 & \text{Sum} = & 1101111
\end{array}
$$

There is no end carry. Therefore, the answer is Y – X = – (2's complement of 1101111) = – 0010001.

Complements

- Subtraction of unsigned numbers can also be done by means of the (r – 1)'s complement. Remember that the (r – 1) 's complement is one less then the r's complement.

- Example 1.8
  - Repeat Example 1.7, but this time using 1's complement.

$$
\begin{array}{lrr}
\text{(a)} \quad X - Y = 1010100 - 1000011 & & \\
 & X = & 1010100 \\
 & \text{1's complement of } Y = & \pm\,0111100 \\
 & \text{Sum} = & 10010000 \\
 & \text{End-around carry} = & \underline{+ \qquad 1} \\
 & \text{Answer. } X - Y = & 0010001
\end{array}
$$

$$
\begin{array}{lrr}
\text{(b)}\, Y - X = 1000011 - 1010100 & & \\
 & Y = & 1000011 \\
 & \text{1's complement of } X = & \underline{+\,0101011} \\
 & \text{Sum} = & 1101110
\end{array}
$$

There is no end carry, Therefore, the answer is Y – X = – (1's complement of 1101110) = – 0010001.

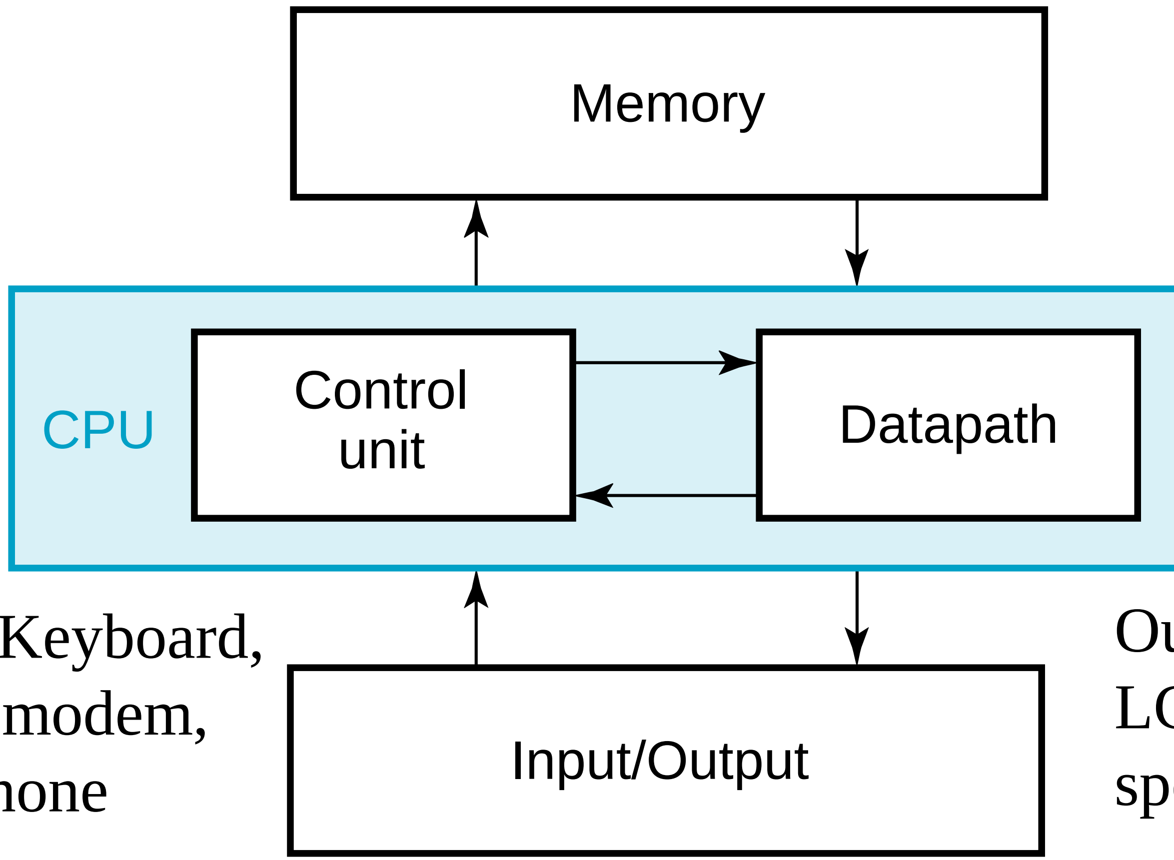
1.8 Binary Storage and Registers

- Registers
  - A binary cell is a device that possesses two stable states and is capable of storing one of the two states.
  - A register is a group of binary cells. A register with *n* cells can store any discrete quantity of information that contains *n* bits.

n cells → 2^n possible states

- A binary cell
  - Two stable state
  - Store one bit of information
  - Examples: flip-flop circuits, capacitor
- A register
  - A group of binary cells
  - AX in x86 CPU
- Register Transfer
  - A transfer of the information stored in one register to another.
  - One of the major operations in digital system.

A Digital Computer Example



Inputs: Keyboard, mouse, modem, microphone

Outputs: CRT, LCD, modem, speakers

Synchronous or Asynchronous?

Transfer of information

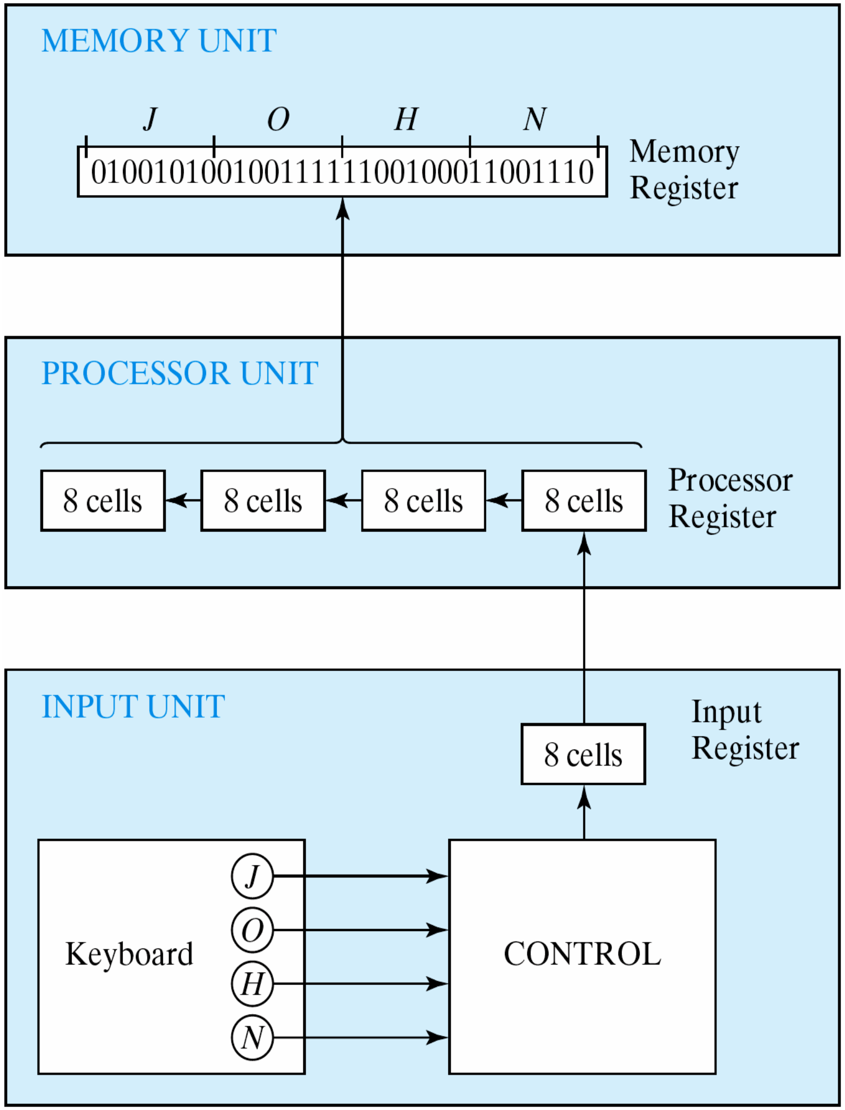


Figure 1.1 Transfer of information among register

Transfer of information

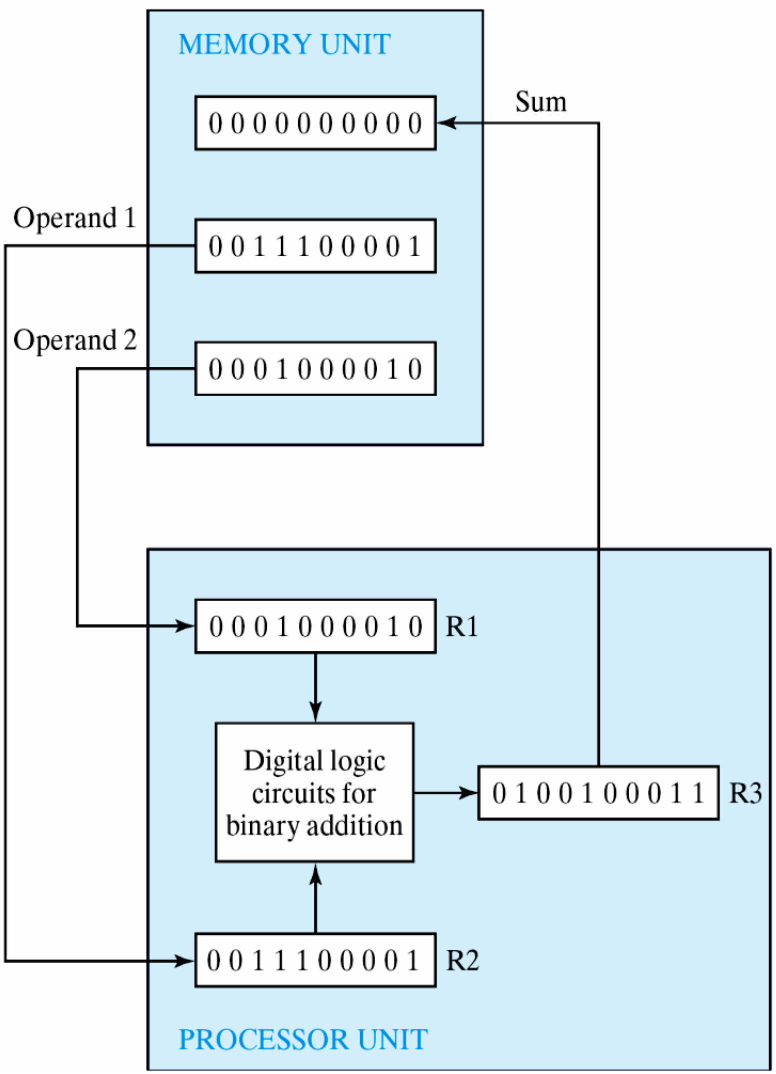


Figure 1.2 Example of binary information processing

- The other major component of a digital system
  - Circuit elements to manipulate individual bits of information
  - Load-store machine

    LD R1;

    LD R2;

    ADD R3, R2, R1;

    SD R3;

1.9 Binary Logic

- Definition of Binary Logic
  - Binary logic consists of binary variables and a set of logical operations.
  - The variables are designated by letters of the alphabet, such as A, B, C, x, y, z, etc, with each variable having two and only two distinct possible values: 1 and 0,
  - Three basic logical operations: AND, OR, and NOT.

1.9 Binary Logic

1. AND: This operation is represented by a dot or by the absence of an operator. For example, $x \cdot y = z$ or $xy = z$ is read "x AND y is equal to z," The logical operation AND is interpreted to mean that $z = 1$ if only $x = 1$ and $y = 1$; otherwise $z = 0$. (Remember that x, y, and z are binary variables and can be equal either to 1 or 0, and nothing else.)
2. OR: This operation is represented by a plus sign. For example, $x + y = z$ is read "x OR y is equal to z," meaning that $z = 1$ if $x = 1$ or $y = 1$ or if both $x = 1$ and $y = 1$. If both $x = 0$ and $y = 0$, then $z = 0$.
3. NOT: This operation is represented by a prime (sometimes by an overbar). For example, $x' = z$ (or $\bar{x} = z$) is read "not x is equal to z," meaning that z is what z is not. In other words, if $x = 1$, then $z = 0$, but if $x = 0$, then $z = 1$, The NOT operation is also referred to as the complement operation, since it changes a 1 to 0 and a 0 to 1.

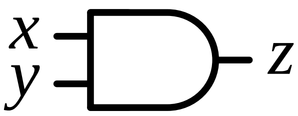
Binary Logic

- Truth Tables, Boolean Expressions, and Logic Gates

AND

x	y	z
0	0	0
0	1	0
1	0	0
1	1	1

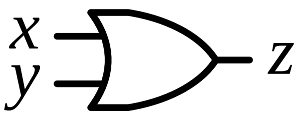
$$z = x \bullet y = x\,y$$



OR

x	y	z
0	0	0
0	1	1
1	0	1
1	1	1

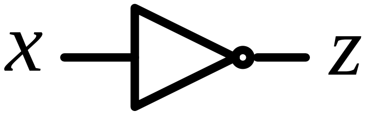
$$z = x + y$$



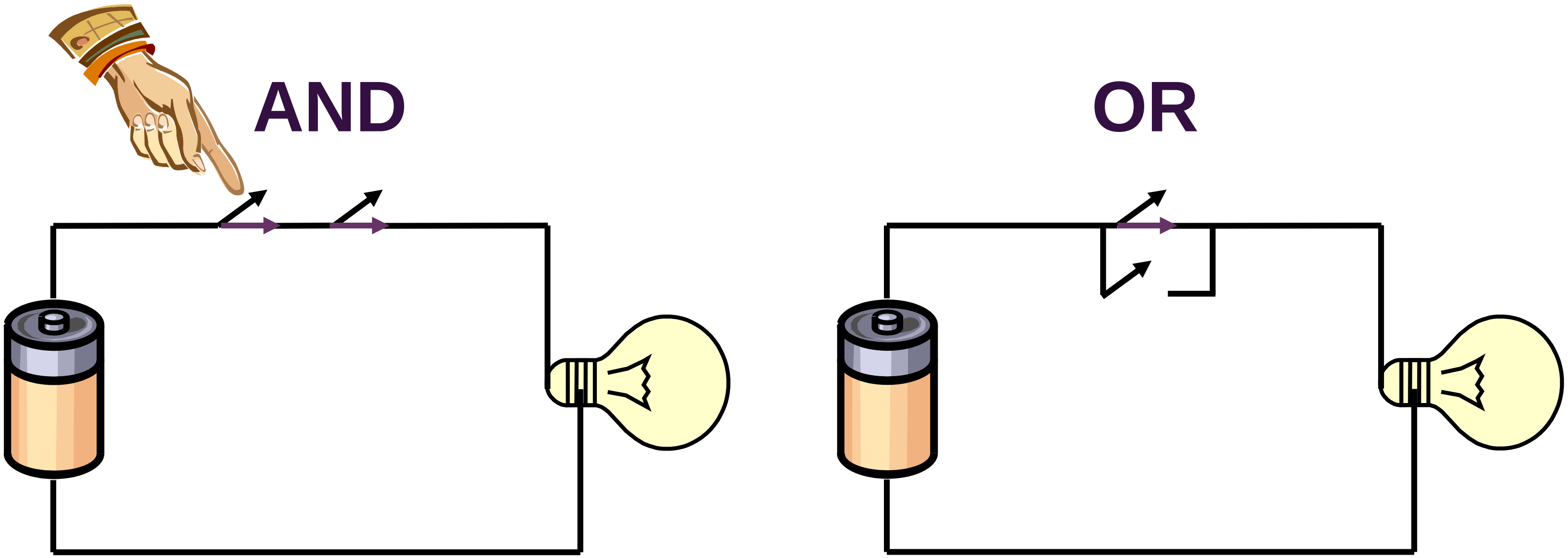
NOT

x	z
0	1
1	0

$$z = \overline{x} = x'$$

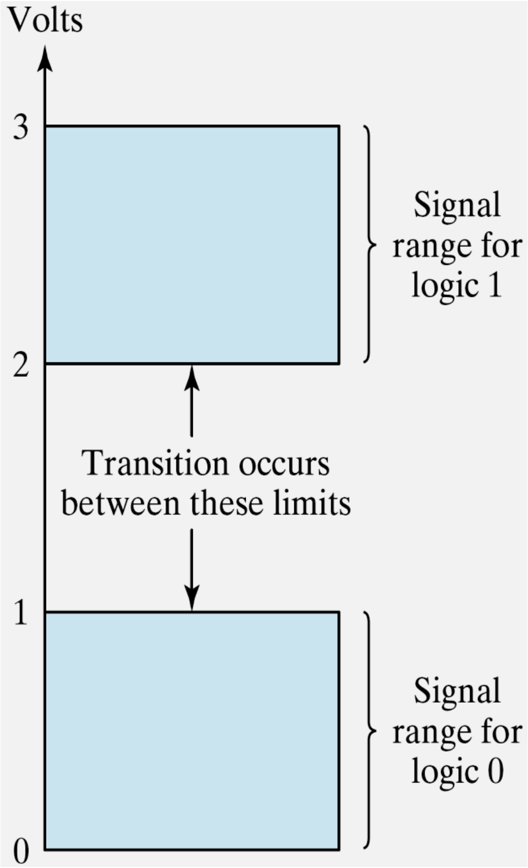


Switching Circuits



Binary Logic

- Logic gates
  - Example of binary signals



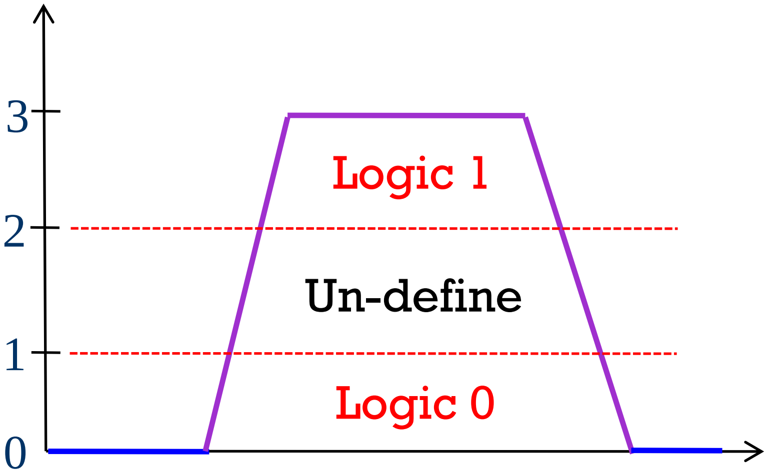
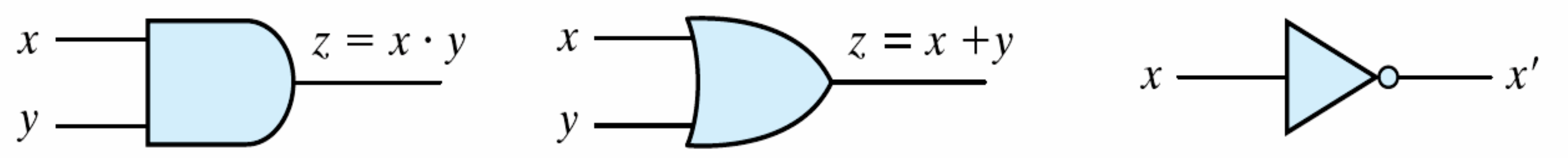


Figure 1.3 Example of binary signals

Binary Logic

- Logic gates
  - Graphic Symbols and Input-Output Signals for Logic gates:



(a) Two-input AND gate (b) Two-input OR gate (c) NOT gate or inverter

Fig. 1.4 Symbols for digital logic circuits

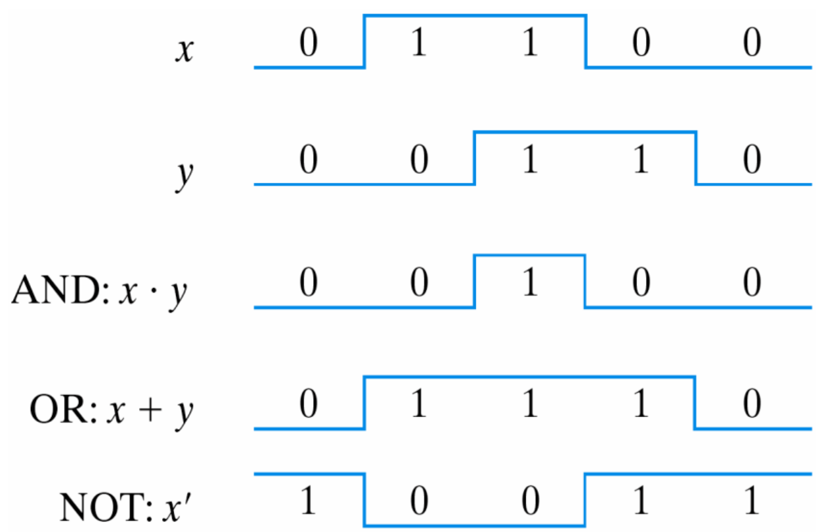
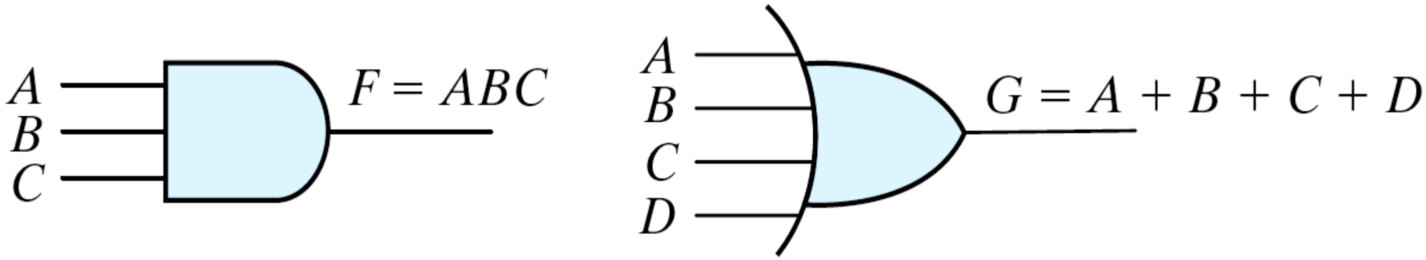


Fig. 1.5 Input-Output signals for gates

Binary Logic

- Logic gates
  - Graphic Symbols and Input-Output Signals for Logic gates:



(a) Three-input AND gate (b) Four-input OR gate

Fig. 1.6 Gates with multiple inputs