Dynamic Programming

Dynamic Programming

- An algorithm design technique (like divide and conquer)
- Divide and conquer
  - Partition the problem into independent subproblems
  - Solve the subproblems recursively
  - Combine the solutions to solve the original problem

Dynamic Programming

- Applicable when subproblems are **not** independent
  - Subproblems share subsubproblems

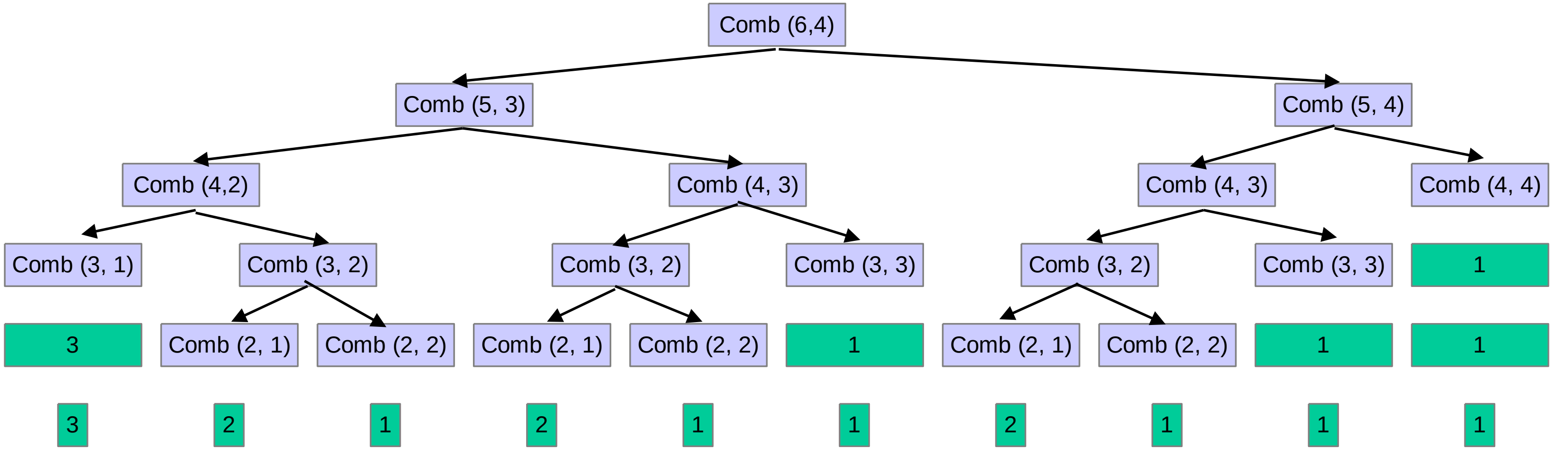
E.g.: Combinations:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

  - A divide and conquer approach would repeatedly solve the common subproblems
  - Dynamic programming solves every subproblem just once and stores the answer in a table

Example: Combinations



$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Dynamic Programming

- Used for **optimization problems**
  - A set of choices must be made to get an optimal solution
  - Find a solution with the optimal value (minimum or maximum)
  - There may be many solutions that lead to an optimal value
  - Our goal: **find an optimal solution**

Dynamic Programming Algorithm

1. **Characterize** the structure of an optimal solution
2. **Recursively** define the value of an optimal solution
3. **Compute** the value of an optimal solution in a bottom-up fashion
4. **Construct** an optimal solution from computed information (not always necessary)

Example 1: Fibonacci numbers

- Recall definition of Fibonacci numbers:

$$F(n) = F(n\text{-}1) + F(n\text{-}2)$$
$$F(0) = 0$$
$$F(1) = 1$$

- Computing the n^{th} Fibonacci number recursively (top-down):

$$F(n)$$

$$F(n\text{-}1) \quad + \quad F(n\text{-}2)$$

$$F(n\text{-}2) \quad + \quad F(n\text{-}3) \qquad\qquad F(n\text{-}3) \quad + \quad F(n\text{-}4)$$

...

Example 1: Fibonacci numbers (cont.)

Computing the n^{th} Fibonacci number using bottom-up iteration and recording results:

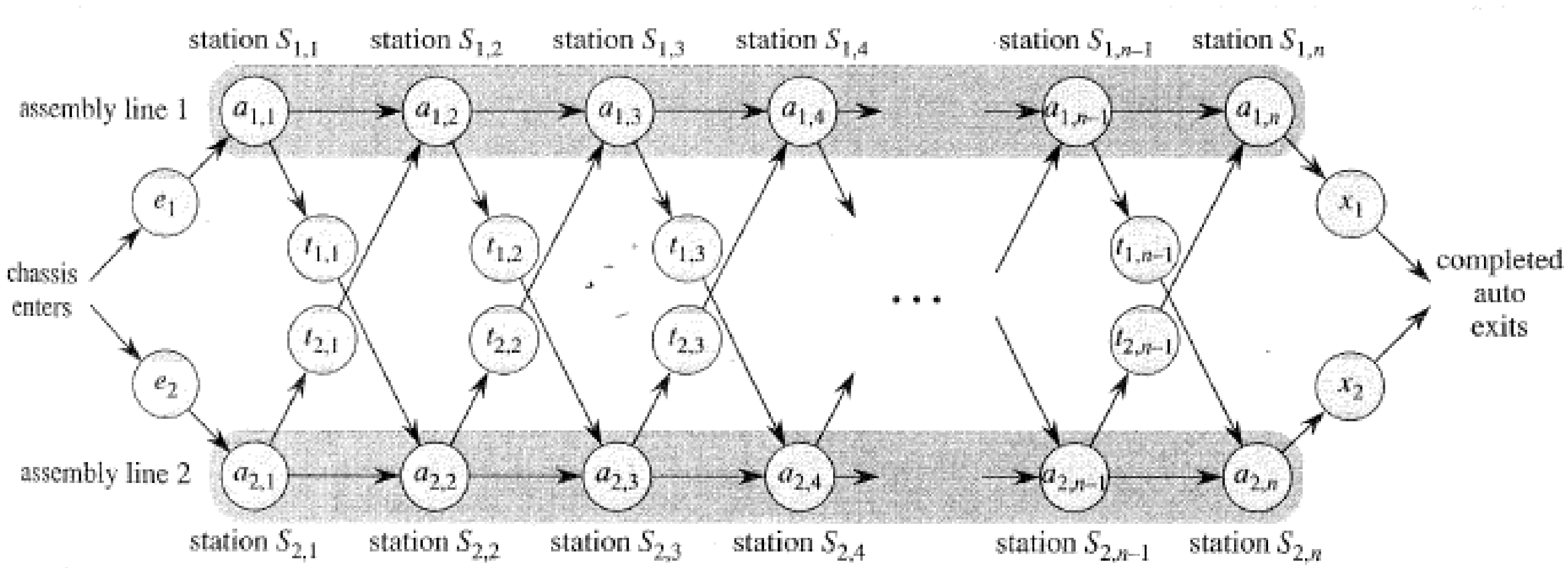
$F(0) = 0$
$F(1) = 1$
$F(2) = 1+0 = 1$
…
$F(n\text{-}2) =$
$F(n\text{-}1) =$
$F(n) = F(n\text{-}1) + F(n\text{-}2)$

0	**1**	**1**	**. . .**	***F*(*n*-2)**	***F*(*n*-1)**	***F*(*n*)**

Efficiency:
- time
- space

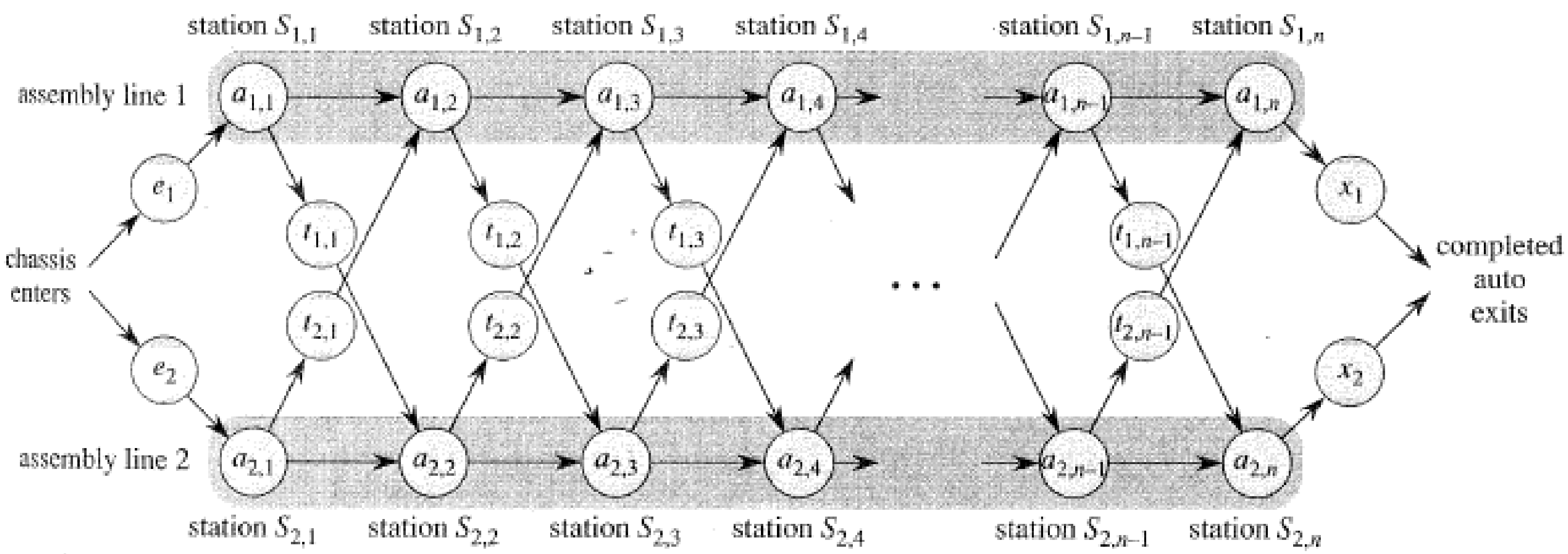
Assembly Line Scheduling

- Automobile factory with two assembly lines
  - Each line has n stations: $S_{1,1}, \ldots, S_{1,n}$ and $S_{2,1}, \ldots, S_{2,n}$
  - Corresponding stations $S_{1,j}$ and $S_{2,j}$ perform the same function but can take different amounts of time $a_{1,j}$ and $a_{2,j}$
  - Entry times are: e_1 and e_2; exit times are: x_1 and x_2



Assembly Line Scheduling

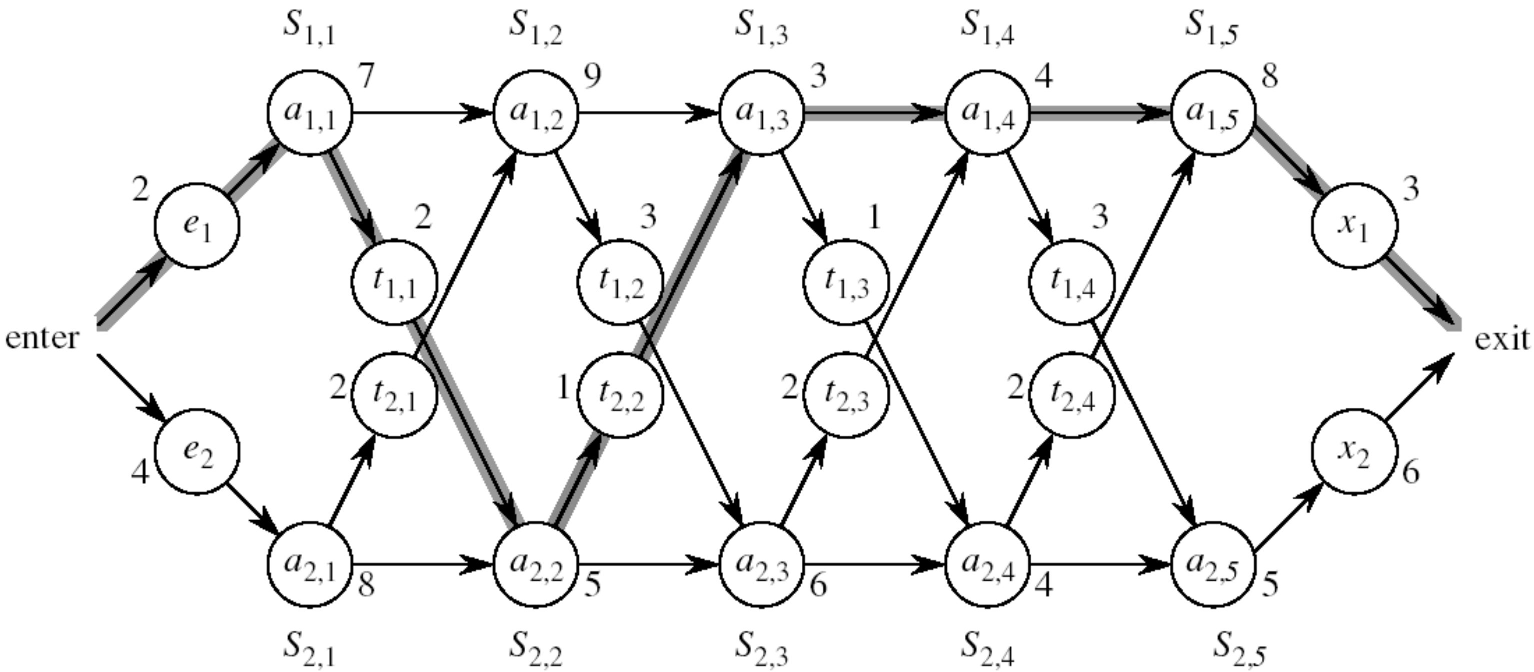
- After going through a station, can either:
  – stay on same line at no cost, or
  – transfer to other line: cost after $S_{i,j}$ is $t_{i,j}$, $j = 1, . . . , n - 1$



Assembly Line Scheduling

- Problem:

  what stations should be chosen from line 1 and which from line 2 in order to minimize the total time through the factory for one car?



One Solution

- Brute force
  - Enumerate all possibilities of selecting stations
  - Compute how long it takes in each case and choose the best one
- Solution:

1	2	3	4				n
1	0	0	1	- - -	- - -	- - -	1

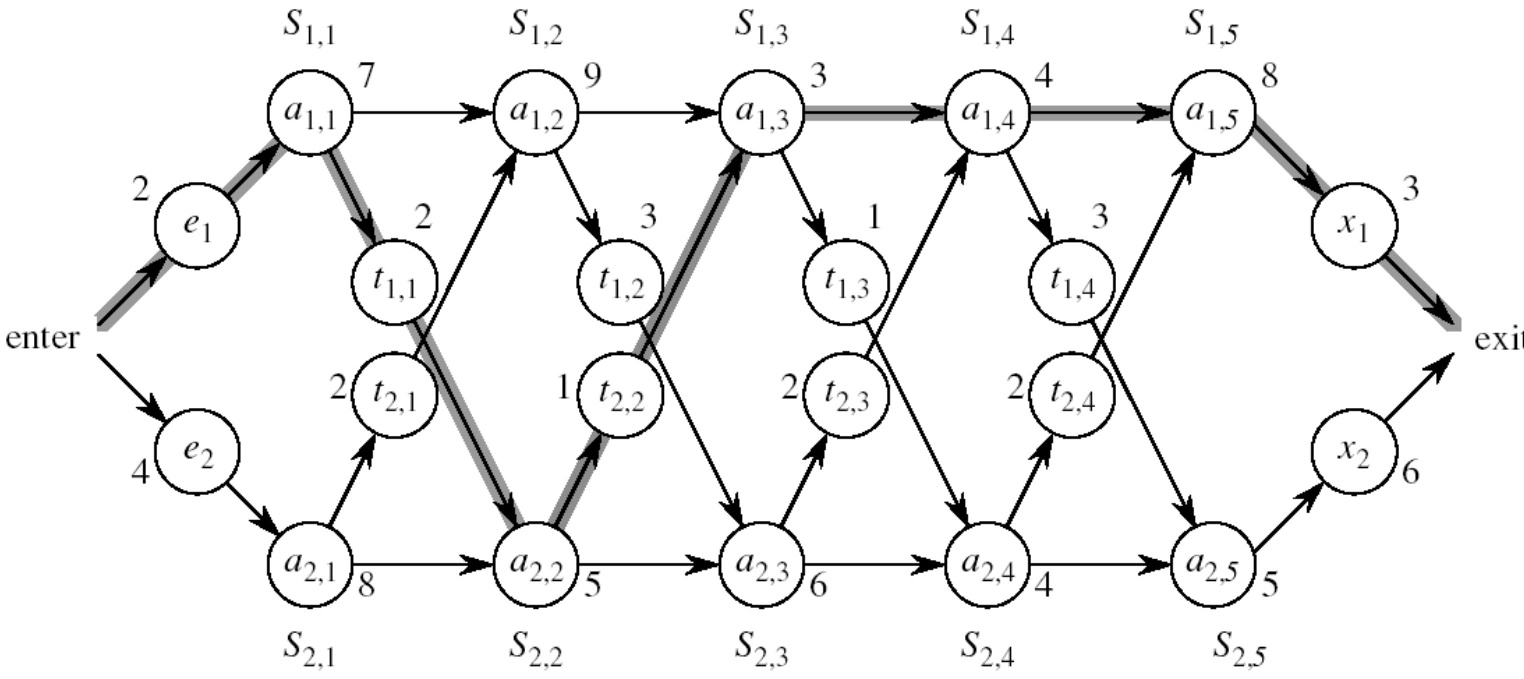
0 if choosing line 2
at step j (= 3)

1 if choosing line 1
at step j (= n)

  - There are 2^n possible ways to choose stations
  - Infeasible when n is large!!

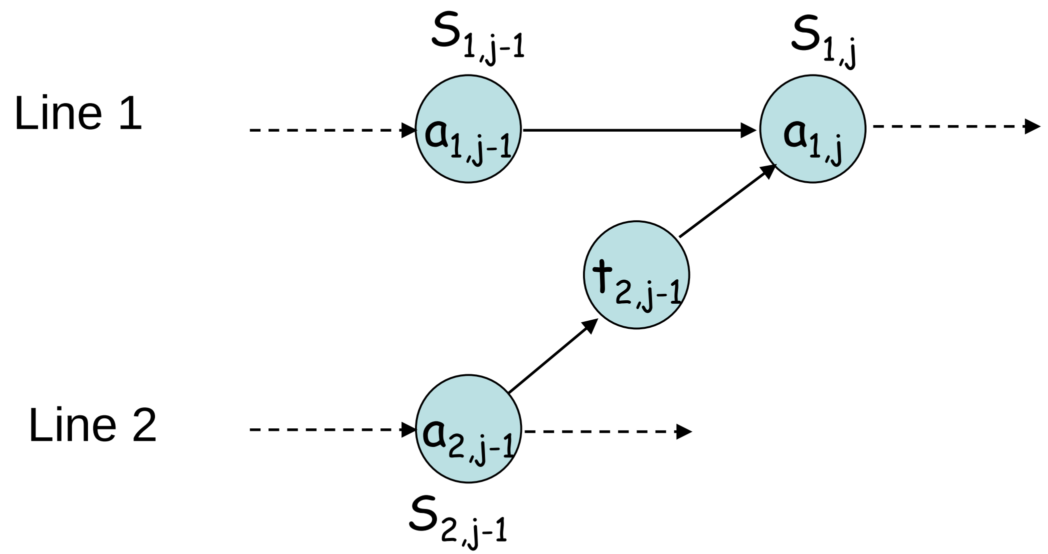
1. Structure of the Optimal Solution

- How do we compute the minimum time of going through a station?



1. Structure of the Optimal Solution

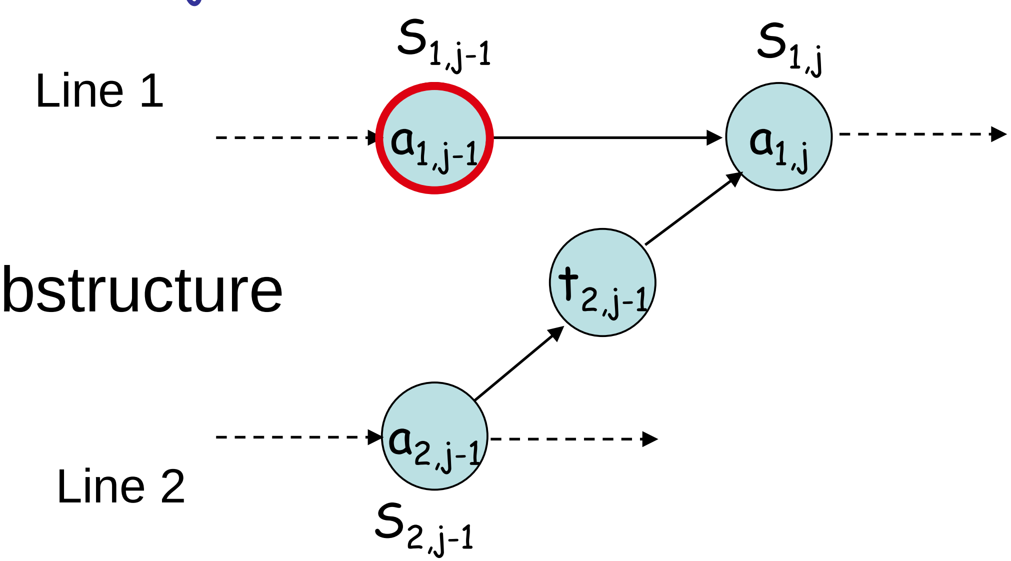
- Let's consider all possible ways to get from the starting point through station $S_{1,j}$
  - We have two choices of how to get to $S_{1,j}$:
    - Through $S_{1,j-1}$, then directly to $S_{1,j}$
    - Through $S_{2,j-1}$, then transfer over to $S_{1,j}$



1. Structure of the Optimal Solution

- Suppose that the fastest way through $S_{1,\,j}$ is through $S_{1,\,j-1}$
  - We must have taken a fastest way from entry through $S_{1,\,j-1}$
  - If there were a faster way through $S_{1,\,j-1}$, we would use it instead
- Similarly for $S_{2,\,j-1}$

Optimal Substructure

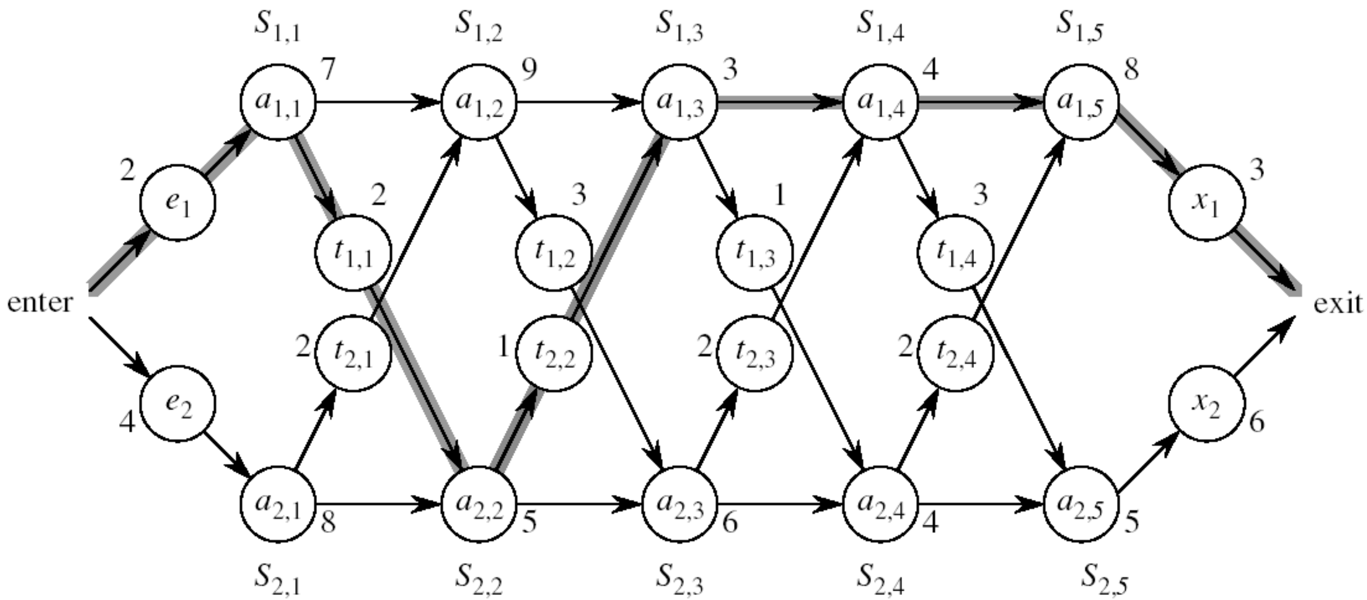


Optimal Substructure

- **Generalization**: an optimal solution to the problem "*find the fastest way through* $S_{1,j}$" contains within it an optimal solution to subproblems: "*find the fastest way through* $S_{1,j-1}$ *or* $S_{2,j-1}$".
- This is referred to as the **optimal substructure** property
- We use this property to construct an optimal solution to a problem from optimal solutions to subproblems

2. A Recursive Solution

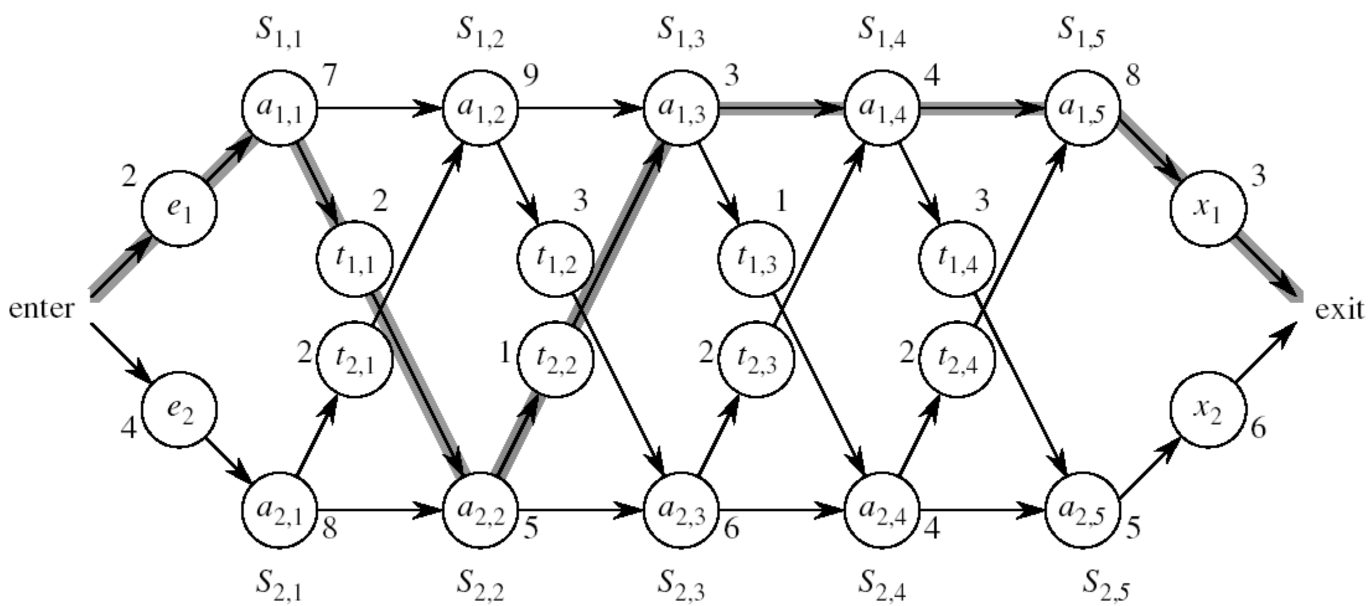
- Define the value of an optimal solution in terms of the optimal solution to subproblems



2. A Recursive Solution (cont.)

- Definitions:
  - f^* : the fastest time to get through the entire factory
  - $f_i[j]$: the fastest time to get from the starting point through station $S_{i,j}$

$$f^* = \min(f_1[n] + x_1, f_2[n] + x_2)$$

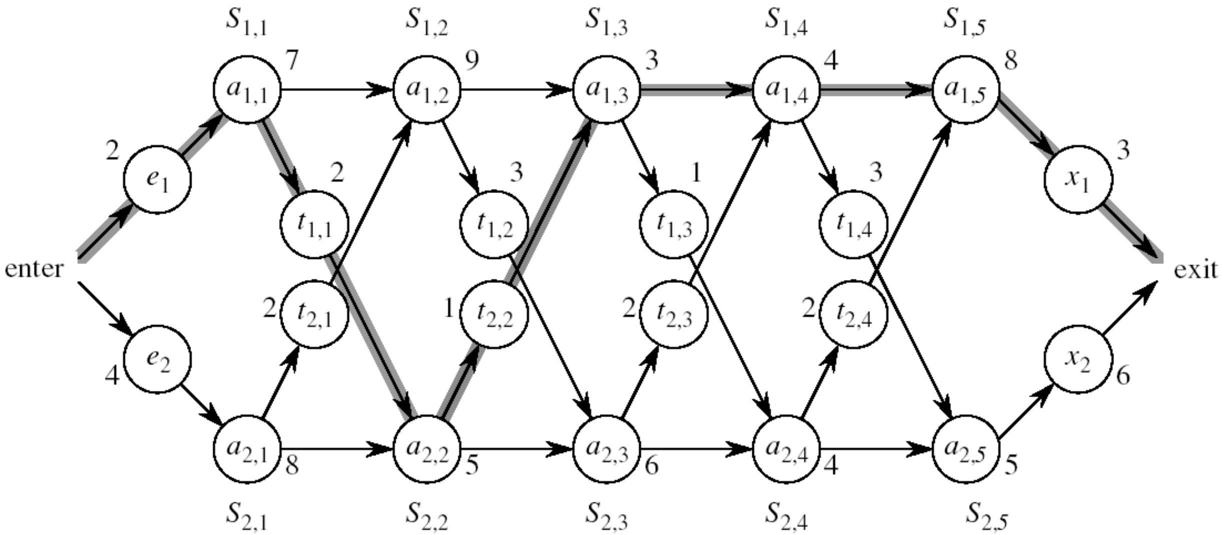


2. A Recursive Solution (cont.)

- Base case: $j = 1, i=1,2$ (getting through station 1)

  $f_1[1] = e_1 + a_{1,1}$

  $f_2[1] = e_2 + a_{2,1}$



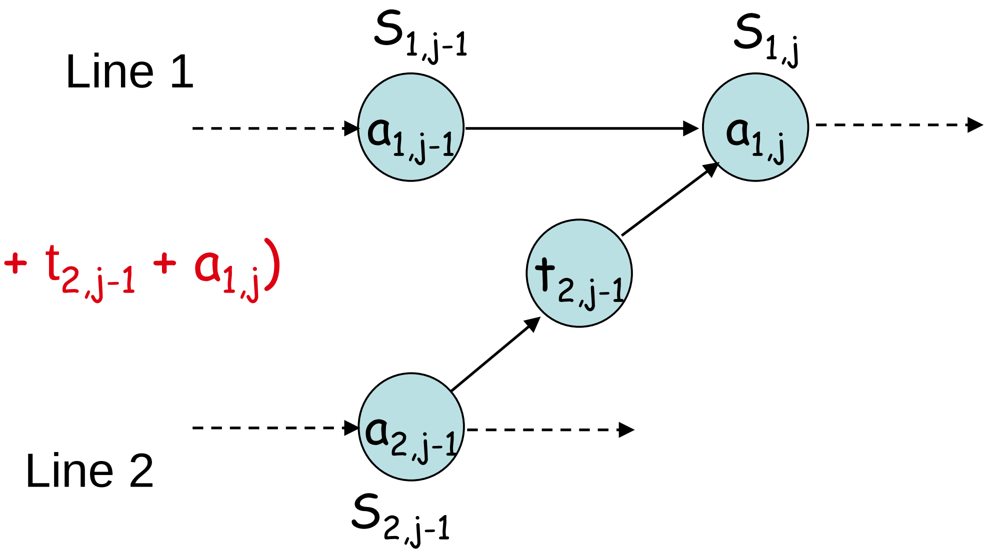
2. A Recursive Solution (cont.)

- <u>General Case</u>: $j = 2, 3, \ldots, n$, and $i = 1, 2$
- Fastest way through $S_{1,j}$ is either:
  - the way through $S_{1,j-1}$ then directly through $S_{1,j}$, or

    $f_1[j - 1] + a_{1,j}$
  - the way through $S_{2,j-1}$, transfer from line 2 to line 1, then through $S_{1,j}$

    $f_2[j - 1] + t_{2,j-1} + a_{1,j}$

$$f_1[j] = \min(f_1[j - 1] + a_{1,j}, f_2[j - 1] + t_{2,j-1} + a_{1,j})$$



2. A Recursive Solution (cont.)

$$f_1[j] = \begin{cases} e_1 + a_{1,1} & \text{if } j = 1 \\ \min(f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}) & \text{if } j \geq 2 \end{cases}$$

$$f_2[j] = \begin{cases} e_2 + a_{2,1} & \text{if } j = 1 \\ \min(f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}) & \text{if } j \geq 2 \end{cases}$$

3. Computing the Optimal Solution

$$f^* = \min\left(f_1[n] + x_1,\ f_2[n] + x_2\right)$$

$$f_1[j] = \min\left(f_1[j-1] + a_{1,j},\ f_2[j-1] + t_{2,j-1} + a_{1,j}\right)$$

$$f_2[j] = \min\left(f_2[j-1] + a_{2,j},\ f_1[j-1] + t_{1,j-1} + a_{2,j}\right)$$

	1	2	3	4	5
$f_1[j]$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$f_1(5)$
$f_2[j]$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$f_2(5)$

4 times 2 times

- Solving top-down would result in exponential running time

3. Computing the Optimal Solution

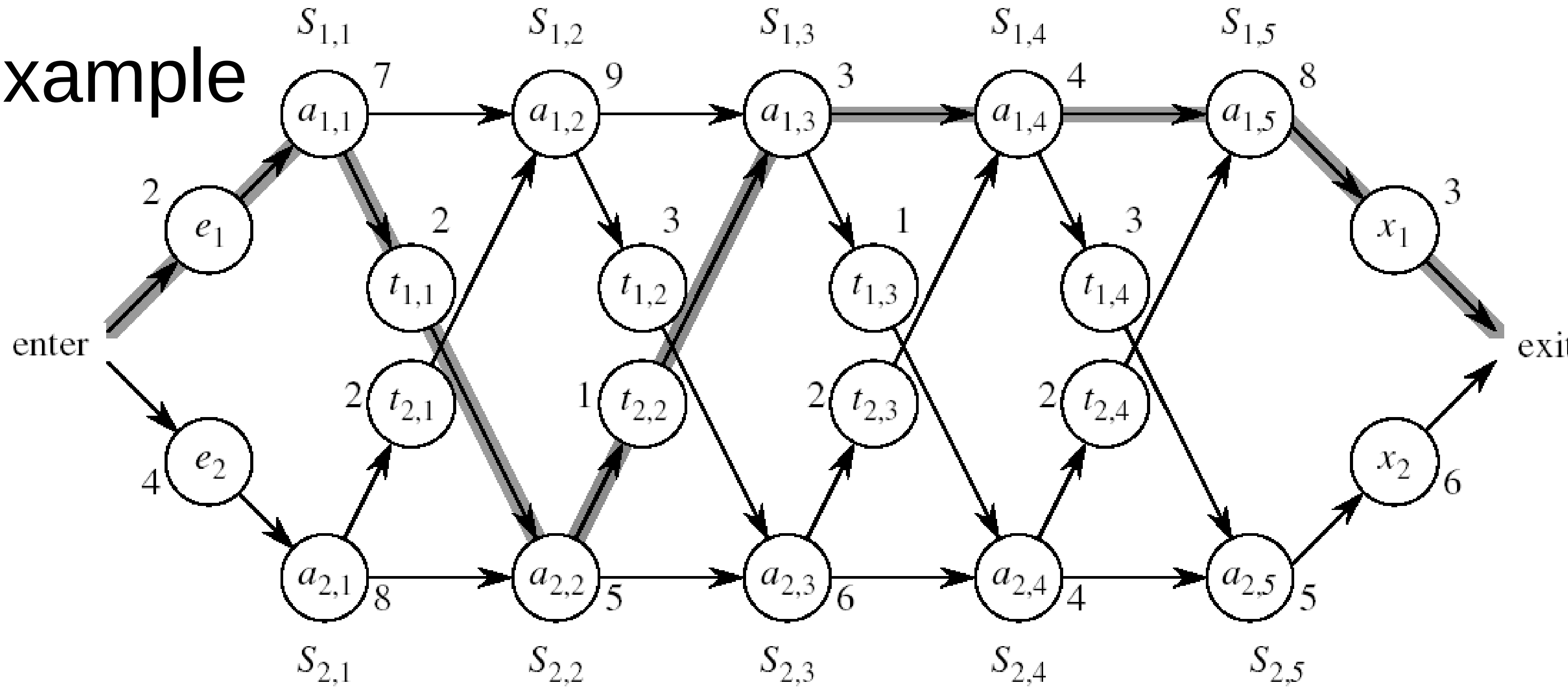
- For $j \geq 2$, each value $f_i[j]$ depends only on the values of $f_1[j - 1]$ and $f_2[j - 1]$
- Idea: compute the values of $f_i[j]$ as follows:

in increasing order of j

	1	2	3	4	5
$f_1[j]$					
$f_2[j]$					

- Bottom-up approach
  - First find optimal solutions to subproblems
  - Find an optimal solution to the problem from the subproblems

Example



$$f_1[j] = \begin{cases} e_1 + a_{1,1}, & \text{if } j = 1 \\ \min(f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}) & \text{if } j \geq 2 \end{cases}$$

	1	2	3	4	5
$f_1[j]$	9	18[1]	20[2]	24[1]	32[1]
$f_2[j]$	12	16[1]	22[2]	25[1]	30[2]

f* = 35[1]

FASTEST-WAY(a, t, e, x, n)

$f_1[1] \leftarrow e_1 + a_{1,1}$

$f_2[1] \leftarrow e_2 + a_{2,1}$

Compute initial values of f_1 and f_2

for $j \leftarrow 2$ **to** n

O(N)

    do if $f_1[j-1] + a_{1,j} \le f_2[j-1] + t_{2,j-1} + a_{1,j}$

        then $f_1[j] \leftarrow f_1[j-1] + a_{1,j}$

            $l_1[j] \leftarrow 1$

        else $f_1[j] \leftarrow f_2[j-1] + t_{2,j-1} + a_{1,j}$

            $l_1[j] \leftarrow 2$

Compute the values of $f_1[j]$ and $l_1[j]$

    if $f_2[j-1] + a_{2,j} \le f_1[j-1] + t_{1,j-1} + a_{2,j}$

        then $f_2[j] \leftarrow f_2[j-1] + a_{2,j}$

            $l_2[j] \leftarrow 2$

        else $f_2[j] \leftarrow f_1[j-1] + t_{1,j-1} + a_{2,j}$

            $l_2[j] \leftarrow 1$

Compute the values of $f_2[j]$ and $l_2[j]$

FASTEST-WAY(a, t, e, x, n) (cont.)

if $f_1[n] + x_1 \leq f_2[n] + x_2$

  then $f^* = f_1[n] + x_1$

    $l^* = 1$

  else $f^* = f_2[n] + x_2$

    $l^* = 2$

Compute the values of the fastest time through the entire factory

4. Construct an Optimal Solution

Alg.: PRINT-STATIONS(l, n)

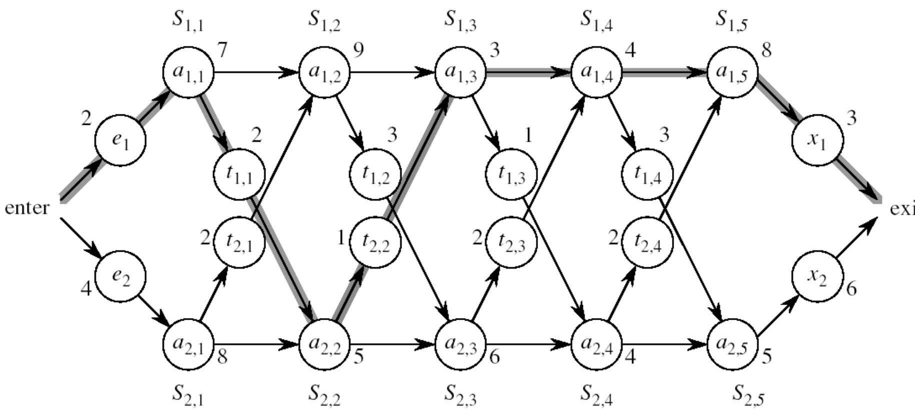
i ← l*

print "line " i ", station " n

for j ← n **downto** 2

do i ← $l_i[j]$

print "line " i ", station " j - 1



	1	2	3	4	5
$f_1[j]/l_1[j]$	9	18[1]	20[2]	24[1]	32[1]
$f_2[j]/l_2[j]$	12	16[1]	22[2]	25[1]	30[2]

l* = 1