

Lecture # 05 Set Comprehension

Set Theory-Basics

- A set is a collection of objects.
- The objects may be
 - natural numbers,
 - names of files,
 - locations of monitoring instruments,
 - user names,
 - or whatever objects are of interest to the specifier.

Set Theory- Basics

A set can be specified as a collection of objects surrounded by curly brackets. Thus

$$\{21, 7, 14, 3\}$$

is an example of a set of objects which are natural numbers and

{archiver, sorter, editor, finder}

is a set of utility programs. The important property of a set is that duplicates are not allowed. Thus

$$\{1, 3, 4, 1, 9, 3\}$$

is not an example of a set.

Set Theory- Basics

A set cannot only contain single-element objects but can also contain aggregates of objects. For example,

$$\{(1,2),(3,4),(4,5)\}$$

is a set of pairs of natural numbers and

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{(mon1, mon2, mon3), (mon2, mon4, mon5), (mon3, mon7, mon8)}
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is a set of triples which contain monitor names.

A set can be finite or can be infinite.

$$\{1, 3, 5\}$$

is an example of a finite set while the set of all natural numbers is an example of an infinite set. $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$

Set Theory- Basics

The objects which make up a set are known as members. The fact that an object is a member of a set is written as

$$x \in S$$
.

If x does not belong in a set, then this is written as

$$x \notin S$$
.

Thus,

$$3 \in \{3, 4, 7\},\$$

 $5 \in \{17, 230, 46, 5\},\$

update \in {update, write, read}

are examples of predicates which are true and

$$1 \notin \{2, 5, 7\},\$$

vdu1 ∉ {vdu3, vdu8, vdu9}

are examples of predicates which are also true.

- Unfortunately, listing a set this way has a number of disadvantages.
 - The first disadvantage is that, for large finite sets, explicit listing is tedious and, for infinite sets, impossible.
 - The second disadvantage is that such a listing does not make the relationship between the elements of a set clear.

$${n: \mathbf{N} \mid n^2 < 25 \cdot n}.$$

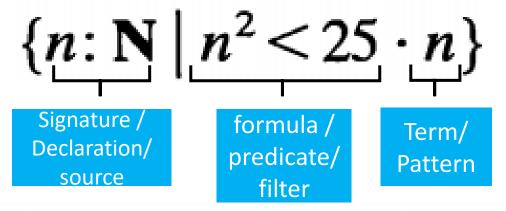
It defines a set of natural numbers whose squares are less than 25, i.e. it specifies the set

$$\{0, 1, 2, 3, 4\}.$$

• This way of defining a set in known as a comprehensive specification.

Comprehensive specification

Comprehensive specification



Thus, the comprehensive specification

$$\{n: \mathbb{N} \mid n < 20 \land n > 10 \cdot n\}$$

denotes the set of natural numbers which lie between 10 and 20, $n: \mathbb{N}$ is the signature, $n < 20 \land n > 10$ is the predicate, and n is the term.

Comprehensive specification

The predicate part of a comprehensive set specification defines the properties of the members of the set which is specified. Thus,

$${n: \mathbf{N} \mid n^3 > 10 \cdot n}$$

specifies the set of natural numbers which have the property that their cubes are greater than 10.

The term part of a comprehensive set specification defines the form of the members of the set. A term consists of an expression which, when evaluated, will deliver a value which is of the same type as the set. For example,

$$\{n: \mathbb{N} \mid n > 20 \land n < 100 \cdot n\}$$

states that a set will contain single natural numbers which satisfy the predicate $n > 20 \land n < 100$,

Comprehensive specification

$$\{x, y : \mathbf{N} \mid x + y = 100 \cdot (x, y)\}$$

specifies the set of pairs which are natural numbers whose sum is 100. i.e. $\{(0, 100), (1, 99), (2, 98), \dots, (100, 0)\}$. Thus, the term in this example defines the fact that elements of the set are pairs.

Comprehensive specification - Examples

Some more examples of comprehensive set specifications with their natural-language equivalents are now given.

Empty Set

Consider the comprehensive specification

Subset

Within any given set A there exist other sets which can be obtained by removing some of the elements of A. These are called **subsets** of the set A. For example, if A is

$$\{1, 3, 9, 14, 200\},\$$

then both $\{1, 3, 9\}$ and $\{1, 9, 14\}$ are subsets of A. The fact that a set is a subset of another set is expressed by the operators: \subset and \subseteq . The predicate

$$A \subseteq B$$

is true if A is a subset of B including being equal to B. Thus,

$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

 $\{1\} \subseteq \{1, 2, 3, 4\}$
 $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$

are all true.

Proper Subset

The predicate

$$A \subset B$$

is true when A is a **proper subset** of B, that is, A is a subset of B but not equal to B. Thus, the predicates

$$\{1, 2\} \subset \{1, 2, 3, 4\}$$

 $\{4\} \subset \{1, 2, 3, 4\}$
 $\{1, 4\} \subset \{1, 2, 3, 4\}$

are all true, while

$$\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$$

is false.

Power Set

- Given $S = \{0, 1\}$. All the possible subsets of S?
 - \bullet P(S) = { \emptyset , {0}, {1}, {0,1} }
- If A is a set then so PA, which is known as the power set of A.

$$X \in \mathbb{P}A \Leftrightarrow X \subseteq A$$

lacksquare Something is a member of $\mathbf{P}A$ iff it is a subset of A.

Thus, $\mathbb{P}\{1, 2, 3\}$ is

$$\{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\},$$

remembering, of course, that the empty set is a subset of any set.

Set Equality

If A and B are two sets of the same type, then A = B is true when each set contains the same elements. Thus,

$$\{1, 2, 4, 5\} = \{1, 2, 5, 4\}$$

 $\{9, 8, 7, 6, 5\} = \{7, 8, 6, 9, 5\}$

are true while

$$\{1, 3, 5\} = \{1, 3, 7, 11\},\$$

 $\{1, 3, 5\} = \{1, 3, 5, 9, 11\}$

are false. Again = can be defined in terms of \in and predicate calculus.

$$A = B \Leftrightarrow$$

Set Equality & Proper Subset

Given this definition of =, it is possible to define the \subset operator as

$$A \subset B \Leftrightarrow \forall a : A \cdot a \in B \land \neg (A = B)$$

which just states that A is a proper subset of B when every element of A occurs in B and A is not equal to B.

Set Union & Set Intersection

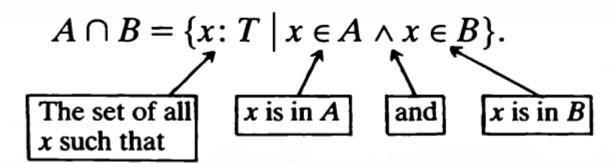
The union operator can be formally defined using \in and predicate calculus as

$$A \cup B = \{X: T \mid X \in A \lor X \in B\}$$

Where T is the type of the objects which make up the set, for example, the set N or the set of all functioning monitors.

Formally define the intersection operator \cap .

The intersection operator forms a set whose elements are in both of its arguments



Set Difference

- The notation $U \setminus V$ denotes the set consisting of all those elements of U which are not in V.
 - $\{1,2,4,8,9\} \setminus \{1,2,3\} = \{4,8,9\}$
 - $\{1,2,3\} \setminus \{1,2,3\} = \{\}$
- $N_1 == N \setminus \{0\}$

 N_I is set of non-negative numbers excluding 0

• $odds == \mathbf{N_1} \setminus evens$

{1,2,3} U {2,3,4} = {1,2,3,4}

The union operator U combines sets

The difference operator \ removes the elements of one set from another \{1,2,3,4\} \ \{2,3\} = \{1,4\}

The intersection operator \cap finds the elements common to both sets $\{1,2,3\} \cap \{2,3,4\} = \{2,3\}$

$$\mathbf{U} \setminus \mathbf{V} =$$

Cross Product

The cross-product of two sets A and B is denoted by

$$A \times B$$
.

The operator forms the set of pairs where the first element of each pair is drawn from A and the second element is drawn from B. Thus,

$$\{1, 2, 3\} \times \{4, 5\} = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

and

$$\{\text{line1}, \text{line2}\} \times \{10, 12, 3\} = \{(\text{line1}, 10), (\text{line1}, 12), (\text{line1}, 3), (\text{line2}, 10), (\text{line2}, 12), (\text{line2}, 3)\}.$$

Formally, the cross-product is defined as

$$A \times B =$$

Cardinality

The **cardinality** of a set is the number of elements in the set. For example, the cardinality of the set

is 3. In set theory the operator # is the cardinality operator. When applied to a set it gives the number of elements in the set. For example,

$$\#$$
{old, new, medium, fast, slow} = 5

and

$$^{\#}\{n: \mathbb{N} \mid n < 4\} = 4$$

are both true predicates.

REFERENCES:

• D. Ince-An Introduction to Discrete Mathematics, Formal System Specification and Z. (Oxford Applied Mathematics and Computing Science Series) – Chapter 5