Answers

1 Exploring data

1.1 Measures of central tendency

1 (i) Mean $= \frac{5+3+8+1+12+3+2+3+7}{9}$

$= \frac{44}{9} = 4.8$ or 4.89 (3 s.f.)

1 2 3 3 3 5 7 8 12

Median is 3

Mode is 3

(ii) Median is good as it is not influenced by outlier of 12. Arguments for all three!

2 Mean $= \frac{90}{12} = 7.5$

5 6 6 7 7 7 8 8 8 8 9 11

Median is 7.5

Mode is 8

3 (i) Discrete

(ii) Continuous

(iii) Discrete

(iv) Category

(v) Discrete

(vi) Continuous

4 (i) 3 3 3 3 3

0 1 3 3 8

1 2 3 3 6

There are other possibilities.

(ii) 1 2 3 4 5 6

0 1 3 4 6 7

There are other possibilities.

5 $6.50 = \frac{\sum x_b}{18} \Rightarrow \sum x_b = 117$

$8 = \frac{\sum x_g}{12} \Rightarrow \sum x_g = 96$

$\bar{x} = \frac{117+96}{30} = \frac{213}{30} = \$7.10$

6 $86.5 = \frac{\sum x}{18} \Rightarrow \sum x = 1557$

$86 = \frac{\sum x}{19} \Rightarrow \sum x = 1634$

New players weight $= 1634 - 1557 = 77\,\text{kg}$

7

Time, x	Frequency, f	xf
2	28	56
2.5	22	55
3	n	$3n$
Total	$50 + n$	$111 + 3n$

$\bar{x} = \frac{\sum xf}{\sum f} \Rightarrow 2.4 = \frac{111+3n}{50+n}$

$2.4(50+n) = 111+3n$

$120+2.4n = 111+3n$

$9 = 0.6n$

$n = 15$

1.2 Frequency distributions

1 (i)

Number of people, x	Frequency, f	xf
1	42	42
2	15	30
3	6	18
4	7	28
5	1	5
6	0	0
7	4	28
	$\sum f = 75$	$\sum xf = 151$

(ii) Mean $= \frac{\sum xf}{\sum f} = \frac{151}{75} = 2.01$ (3 s.f.)

Median $= 1$

Mode $= 1$

2 (i) $3+6+3+2+4+1+1 = 20$

(ii)

No. of pieces of fruit (x)	Frequency (f)	xf
0	3	0
1	6	6
2	3	6
3	2	6
4	4	16
5	1	5
6	0	0
7	0	0
8	1	8
	$\sum f = 20$	$\sum xf = 47$

Mean $= \frac{\sum xf}{\sum f} = \frac{47}{20} = 2.35$ (3 s.f.)

Median = 2

Mode = 1

$= \sqrt{\frac{640}{8} - 6.5^2}$

$= \sqrt{37.75}$

$= 6.14$ (3 s.f.)

1.3 Grouped data

1 (i)

Test score	Frequency, f	Midpoint, x	xf
0–4	7	2	14
5–9	18	7	126
10–14	14	12	168
15–20	11	17.5	192.5
	$\sum f = 50$		$\sum xf = 500.5$

(ii) Mean $= \frac{\sum xf}{\sum f} = \frac{500.5}{50} = 10.01$

(iii) 5–9 class

2

Length	Midpoint	Frequency
$0 \leqslant x < 2$	1	9
$2 \leqslant x < 4$	3	21
$4 \leqslant x < 8$	6	f
$8 \leqslant x \leqslant 10$	9	6

$$\frac{1 \times 9 + 3 \times 21 + 6 \times f + 9 \times 6}{9 + 21 + f + 6} = 4.125$$

$$\frac{126 + 6f}{36 + f} = 4.125$$

$$126 + 6f = 4.125(36 + f)$$

$$126 + 6f = 148.5 + 4.125f$$

$$1.875f = 22.5$$

$$f = 12$$

So the total number of earthworms is 48.

1.4 Measures of spread (variation)

1 Range = 21 – 0 = 21

Arranging in order,

0 2 2 5 6 7 9 21

LQ Median UQ

IQR = UQ – LQ = 8 – 2 = 6

$$s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

$$= \sqrt{\frac{0^2 + 2^2 + 2^2 + 5^2 + 6^2 + 7^2 + 9^2 + 21^2}{8} - 6.5^2}$$

2 $\bar{h} = \frac{\sum h}{n} = \frac{9150}{60} = 152.5\,\text{cm}$

$$s_h = \sqrt{\frac{\sum (h - \bar{h})^2}{n}} = \sqrt{\frac{9077.4}{60}} = 12.3\text{ cm}$$

3 (i)

Number of faults, x	Frequency, f	xf	x^2f
0	22	0	0
1	34	34	34
2	25	50	100
3	12	36	108
4	7	28	112
	$\sum f = 100$	$\sum xf = 148$	$\sum x^2 f = 354$

(ii) Mean $= \frac{148}{100} = 1.48$

(iii) $s = \sqrt{\frac{354}{100} - 1.48^2} = 1.16$ (3 s.f.)

(iv) Range = 4 – 0 = 4

IQR = 2 – 1 = 1

4 (i)

Time (min)	Frequency, f	x	xf	x^2f
$0 \leqslant x < 6$	6	3	18	54
$6 \leqslant x < 12$	8	9	72	648
$12 \leqslant x < 20$	24	16	384	6144
$20 \leqslant x < 30$	7	25	175	4375
$30 \leqslant x \leqslant 40$	3	35	105	3675
	$\sum f = 48$		$\sum xf = 754$	$\sum x^2 f = 14\,896$

(ii) Mean $= \bar{x} = \frac{754}{48} = 15.7\,\text{min}$ (3 s.f.)

(iii) $sd = \sqrt{\frac{14896}{48} - 15.7^2} = 7.97\,\text{min}$ (3 s.f.)

5 (i)

Score	f	x	xf	x^2f
0–4	7	2	14	28
5–9	18	7	126	882
10–14	14	12	168	2016
15–20	11	17.5	192.5	3368.75
	$\sum f = 50$		$\sum xf = 500.5$	$\sum x^2 f = 6294.75$

(ii) Mean $\approx \frac{500.5}{50} = 10.01$

$$sd = \sqrt{\frac{6294.75}{50} - 10.01^2} = 5.07 \text{ (3 s.f.)}$$

6 Mean $= \bar{x} = \frac{1170}{36} = 32.5$ min

$$sd = \sqrt{\frac{38370}{36} - 32.5^2} = 3.10 \text{ min (3 s.f.)}$$

7 $\bar{x} = \frac{\sum x}{4} = 2 \Rightarrow \sum x = 8$

$$\bar{x}_{new} = \frac{\sum x_{new}}{5} = 3 \Rightarrow \sum x_{new} = 15$$

So the number added is 7.

$$sd = \sqrt{\frac{\sum x^2}{4} - 2^2} = 4$$

$$\frac{\sum x^2}{4} - 2^2 = 16$$

$$\sum x^2 = 80$$

$$\sum x^2_{new} = 80 + 7^2 = 129$$

$$sd_{new} = \sqrt{\frac{129}{5} - 3^2} = 4.10 \text{ (3 s.f.)}$$

8 (i) Since the standard deviation is 0, all the activities Lara did must cost the same, which is the mean of \$12.

(ii) $\bar{x}_s = \frac{6 \times 12 + 4x}{10} = 11$

$72 + 4x = 110 \Rightarrow 4x = 38 \Rightarrow x = \$9.50$

$$sd = \sqrt{\frac{\sum x^2 f}{\sum f} - \bar{x}^2}$$

$$sd = \sqrt{\frac{12^2 \times 4 + 12^2 \times 2 + 9.5^2 \times 4}{10} - 11^2}$$

$$= \sqrt{\frac{1225}{10} - 11^2} = 1.22 \text{ (3 s.f.)}$$

1.5 Working with an assumed mean

1 Let $Y = T - 100$

$\sum Y = 24$ and $\sum Y^2 = 330$

$$\bar{Y} = \frac{24}{12} = 2 \Rightarrow \bar{T} = 100 + 2 = 102°\text{C}$$

$$sd_Y = \sqrt{\frac{330}{12} - 2^2} = 4.85°\text{C (3 s.f.)} \Rightarrow sd_T = 4.85°\text{C}$$

2 (i) Let $y = x - 50$ so $\sum y = -240$

$$\bar{y} = \frac{\sum y}{n} = \frac{-240}{100} = -2.4$$

$$\Rightarrow \bar{x} = -2.4 + 50 = 47.6\text{kg}$$

(ii) Since $sd_y = sd_x$

$$sd_y = \sqrt{\frac{\sum y^2}{n} - \bar{y}^2}$$

$$\Rightarrow 3.5 = \sqrt{\frac{\sum y^2}{100} - (-2.4)^2}$$

$$\Rightarrow \sum y^2 = \left(3.5^2 + (-2.4)^2\right) \times 100 = 1801$$

$$\Rightarrow \sum (x - 50)^2 = 1801$$

3 (i) Let $y = x - 50$

$$\bar{y} = \frac{496}{80} = 6.2$$

So $\bar{x} = 6.2 + 50 = 56.2$ km/h

(ii) Let $z = x - 40$

$$\bar{z} = \bar{x} - 40 = 16.2$$

$$\bar{z} = \frac{\sum z}{80} = 16.2 \Rightarrow \sum z = 1296$$

So $\sum (x - 40) = 1296$

(iii) $sd_x = sd_y = sd_z = 6.7$

$$sd_z = 6.7 = \sqrt{\frac{\sum z^2}{n} - \bar{z}^2}$$

$$6.7 = \sqrt{\frac{\sum z^2}{80} - 16.2^2}$$

$$6.7^2 = \frac{\sum z^2}{80} - 16.2^2$$

$$\sum z^2 = 80\left(6.7^2 + 16.2^2\right) = 24\,586.4$$

$$\sum (x - 40)^2 = 24\,586.4$$

4 (i) $\bar{y} = \frac{\sum y}{n} = \frac{-189}{35} = -5.4 \text{ min}$

$\bar{x} = -5.4 + 120 = 114.6 \text{ min}$

(ii) $sd_x = sd_x = 12.8$

$12.8 = \sqrt{\frac{\sum y^2}{35} - (-5.4)^2}$

$\sum y^2 = 35 \times \left(12.8^2 + 5.4^2\right) = 6755$

$\sum(x - 120)^2 = 6755$

Further practice

1

Number of siblings, x	Frequency, f	xf
0	7	$0 \times 7 = 0$
1	18	$1 \times 18 = 18$
2	12	$2 \times 12 = 24$
3	9	$3 \times 9 = 27$
4	4	$4 \times 4 = 16$
	$\sum f = 50$	$\sum xf = 85$

Mean $= \bar{x} = \frac{\sum xf}{\sum f} = \frac{85}{50} = 1.7$

The median is the $\left(\frac{50+1}{2}\right) = \left(\frac{51}{2}\right) = 25.5$th value.

Hence the median is $\frac{1+2}{2} = 1.5$

The mode is 1 (it occurs 18 times).

2 $\bar{x}_1 = \frac{310}{25} = 12.4 \text{ min}$

$\bar{x}_2 = \frac{496}{n} = 12.4 \text{ min} \Rightarrow n = 40$

3 (i) Range $= 18 - 2 = 16$

IQR $= 13 - 2.5 = 10.5$

$\bar{x} = \frac{2+3+5+8+18}{5} = 7.2$

$sd = \sqrt{\frac{2^2+3^2+5^2+8^2+18^2}{5} - 7.2^2}$

$= 5.78$ (3 s.f.)

(ii) Range $= 140 - 21 = 119$

IQR $= 131 - 36 = 95$

$\bar{x} = \frac{21+25+36+\ldots+140}{10} = 92.8$

$sd = \sqrt{\frac{21^2+25^2+36^2+\ldots+140^2}{5} - 92.8^2}$

$= 45.4$ (3 s.f.)

4 (i) $\bar{x} = \frac{\sum xf}{\sum f} = \frac{410}{60} = 6.83$ (3 s.f.)

(ii) The modal class is $4 \leqslant x < 6$.

(iii) $sd = \sqrt{\frac{\sum x^2 f}{\sum f} - \bar{x}^2} = \sqrt{\frac{3682}{60} - 6.83^2} = 3.83$ (3 s.f.)

5 (i) Let $y = x - 70$

$\bar{y} = \frac{\sum y}{n} \Rightarrow 4 = \frac{108}{n} \Rightarrow n = \frac{108}{4} = 27$

(ii) $sd_y = \sqrt{\frac{\sum y^2}{n} - \bar{y}^2} = \sqrt{\frac{3025}{27} - 4^2} = 9.80$ (3 s.f.)

Let $z = x - 74$

Since $sd_x = sd_y = sd_z$

$sd_z = \sqrt{\frac{\sum z^2}{n} - \bar{z}^2} \Rightarrow 9.80 = \sqrt{\frac{\sum z^2}{27} - 0^2}$

$\Rightarrow \sum z^2 = 2593$

$\sum(x - 74)^2 = 2593$

6 $\bar{x} = \frac{\sum x}{n} = \frac{4308}{20} = \$215.40$

$sd = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\frac{948551}{20} - 215.40^2} = \$32.10$

7 Let the boys marks be b and the girls marks be g.

$\bar{b} = \frac{\sum b}{nb} \Rightarrow 28 = \frac{\sum b}{12} \Rightarrow \sum b = 28 \times 12 = 336$

$\bar{g} = \frac{\sum g}{ng} \Rightarrow 30 = \frac{\sum g}{18} \Rightarrow \sum g = 30 \times 18 = 540$

$\overline{b+g} = \frac{\sum b + \sum g}{nb+g} = \frac{336+540}{30} = 29.2$

$sd_b = \sqrt{\frac{\sum b^2}{nb} - \bar{b}^2}$

$\Rightarrow 3.1 = \sqrt{\frac{\sum b^2}{12} - 28^2}$

$\Rightarrow \sum b^2 = \left(3.1^2 + 28^2\right) \times 12 = 9523.32$

$sd_g = \sqrt{\frac{\sum g^2}{ng} - \bar{g}^2}$

$\Rightarrow 2.5 = \sqrt{\frac{\sum g^2}{18} - 30^2}$

$\Rightarrow \sum g^2 = \left(2.5^2 + 30^2\right) \times 18 = 16\,312.5$

$sd_{b+g} = \sqrt{\frac{\sum b^2 + \sum g^2}{n} - \overline{b+g}^2}$

$= \sqrt{\frac{9523.32 + 16\,312.5}{30} - 29.2^2}$

$= 2.92$ (3 s.f.)

8 (i) Let $y = x - k$

$\bar{y} = \frac{\sum y}{n} \Rightarrow \bar{y} = \frac{252}{45} = 5.6$

$sd_x = sd_y = \sqrt{\frac{\sum y^2}{n} - \bar{y}^2}$

$= \sqrt{\frac{1895}{45} - 5.6^2} = 3.28$ (3 s.f.)

(ii) $78 - k = 5.6$ so $k = 72.4$

Past exam questions

1 (i) Let $y = x - 60$

$$\bar{y} = \frac{\sum y}{n} = \frac{245}{70} = 3.5$$

$$\bar{x} = 3.5 + 60 = 63.5 \text{ km/h}$$

(ii) $\sum(x-60) = \sum x - \sum 60 = \sum x - 70 \times 6$

$\sum x - 4200 = 245 \Rightarrow \sum x = 4445$

$\sum(x-50) = \sum x - \sum 50$

$= 4445 - 70 \times 50$

$= 945$

(iii) Let $z = x - 50$

$sd_x = sd_y = sd_z = 10.6$

$$\bar{z} = \frac{945}{70} = 13.5$$

$$sd_z = \sqrt{\frac{\sum z^2}{n} - \bar{z}^2}$$

$$10.6 = \sqrt{\frac{\sum z^2}{70} - 13.5^2}$$

$\sum z^2 = 20\,622.7$

$\sum(x-50)^2 = 20\,622.7 = 20\,600$ (3 s.f.)

2 (i) $\bar{x} = \dfrac{\sum x}{n} \Rightarrow 172.6 = \dfrac{\sum x}{28} \Rightarrow \sum x = 4832.8$

Sum of heights after person has left is

$4832.8 - 161.8 = 4671$ cm

$$\bar{x} = \frac{4671}{27} = 173 \text{ cm}$$

(ii) $sd_x = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2} \Rightarrow 4.58 = \sqrt{\dfrac{\sum x^2}{28} - 172.6^2}$

$\Rightarrow \sum x^2 = 834\,728.6$

Subtracting the person who left

$\sum x^2 = 834728.6 - 161.8^2 = 808\,549.4$

$$sd_x(new) = \sqrt{\frac{808\,549.4}{27} - 173^2} = 4.16 \text{ cm (3 s.f.)}$$

3 $\sum x - 100n = 216$

$2416 - 100n = 216$

$2200 = 100n$

$n = 22$

OR

$$\frac{2416}{n} = \frac{216}{n} + 100$$

$2416 = 216 + 100n$

$2200 = 100n$

$n = 22$

4 (i) $\bar{x} = \dfrac{\sum x}{n} \Rightarrow 2205 = \dfrac{\sum x}{9} \Rightarrow \sum x = 19\,845$

$\sum(x-c) = 1845$

$\sum x - 9c = 1845$

$19845 - 9c = 1845$

$c = 2000$

(ii) Variance of x is the same as the variance of $y = x - 2000$

$$\text{Var}(y) = \frac{\sum y^2}{n} - \bar{y}^2$$

$$= \frac{477\,450}{9} - \left(\frac{1845}{9}\right)^2$$

$= 11\,025$

(iii) $\bar{x}(new) = \dfrac{\sum x}{10}$

$$2120.50 = \frac{\sum x}{10}$$

$\sum x = 21205$

Rental price of new house

$= 21205 - 19845$

$= \$1360$

5 (i) Total cost $= 29 \times 8 = \$232$

Cost of vests and coat

$= 4 \times \$5.50 + \$90 = \$112$

Cost of sweater $= \dfrac{232 - 112}{3} = \$40$

(ii) If the sd = 0, then the price of a hat must be the same as the price of a shirt.

So each of the 5 items must cost \$26.

4 items (the shirts) must cost

$4 \times \$26 = \$104$

Stretch and challenge

1 $\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} = \dfrac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$

LHS :

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

$$= \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - 2\bar{x} \times n\bar{x} + n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2$$

$= RHS$

2 **(i)** W and Y

(ii) Z

3 $\bar{x} = \frac{\sum x}{n} \Rightarrow 124 = \frac{\sum x}{n}$ and $125 = \frac{\sum x + 159}{n+1}$

$$124n = \sum x \text{ so } 125 = \frac{124n + 159}{n+1}$$

$$125(n+1) = 124n + 159$$

$$125n + 125 = 124n + 159$$

$$n = 34$$

$$\sum x = 4216$$

$$126 = \frac{\sum x}{36} \Rightarrow \sum x = 4536$$

Score needed $= 4536 - (4216 + 159) = 161$

4 $\bar{x} = \frac{\sum x}{n} \Rightarrow 15 = \frac{\sum x}{4} \Rightarrow \sum x = 60$

The lowest possible score will occur when the other three students all score the maximum mark of 20, giving a total of 60 marks. The lowest possible mark is therefore 0.

5 Many answers!

6 $\bar{x} = \frac{\sum x}{n} \Rightarrow 8 = \frac{\sum x}{11} \Rightarrow \sum x = 88$

$$sd = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} \Rightarrow 2 = \sqrt{\frac{\sum x^2}{11} - 8^2} \Rightarrow \sum x^2 = 748$$

If we remove one 8 $\sum x = 88 - 8 = 80$

$$\bar{x} = \frac{80}{10} = 8$$

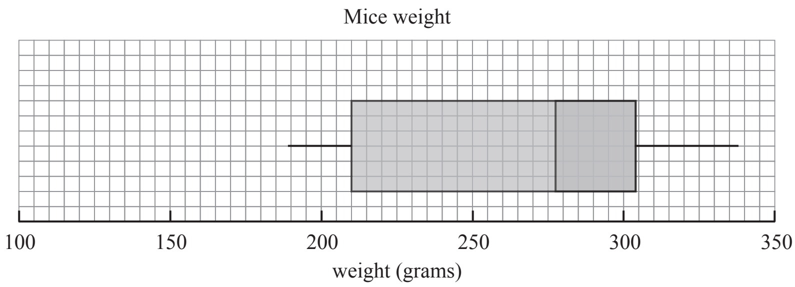
$$\sum x^2 = 748 - 8^2 = 684$$

$$sd = \sqrt{\frac{684}{10} - 8^2} = 2.10 \text{ (3 s.f.)}$$

2 Representing and interpreting data

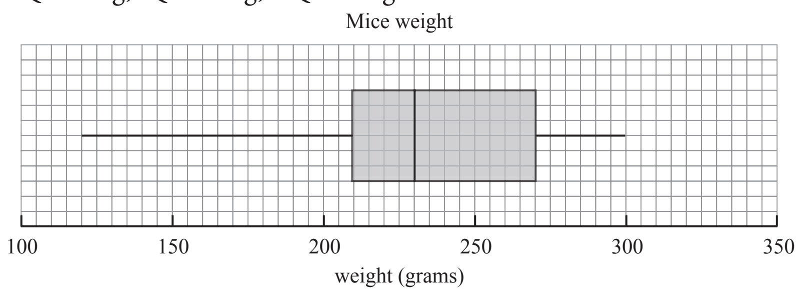
2.1 Statistical graphs – box-and-whisker plots, stem-and-leaf diagrams, etc.

1 **(i)**



(ii) Minimum 120 g, range of 180 g. Maximum is 300 g.

LQ = 210 g, IQR = 60 g, UQ = 270 g



(iii) The median weight of the second sample of mice is lower than the first.

The range of weights for the second sample is larger than the first.

The interquartile range of the second sample is smaller than the first.

In general, the weights of the first sample were higher than the weights of the second sample.

The weights in the first sample are skewed towards the higher values.

The weights in the second sample are skewed towards the lower values.

2 (i)

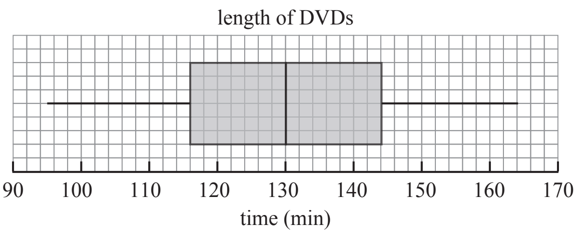
16	0	3	4	
15				
14	4			
13	0	4	5	8
12	2	7		
11	2	6	9	
10	9			
9	5			

(ii) Median = 130 min

UQ = 144 min

LQ = 116 min

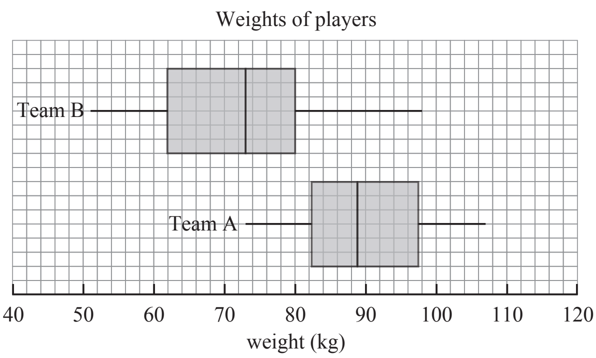
(iii)



3 (i)

	Team A	Team B
LQ	82.5	62
Median	89	73
UQ	97.5	80

(ii)



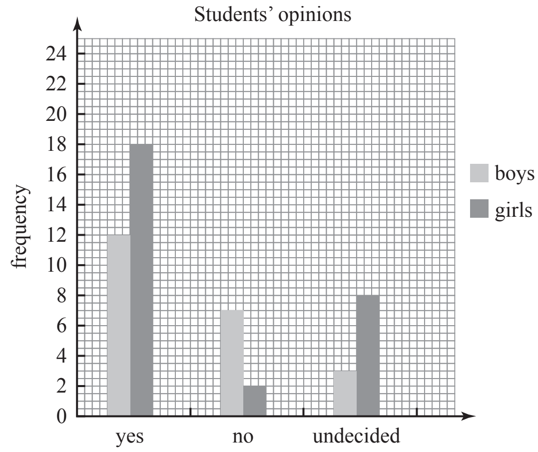
(iii) The median weight for Team A is higher than Team B.

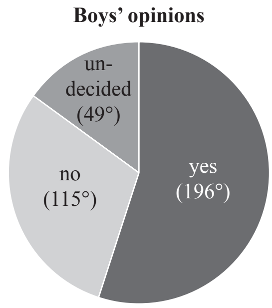
In general, the weight of the people in Team A is greater.

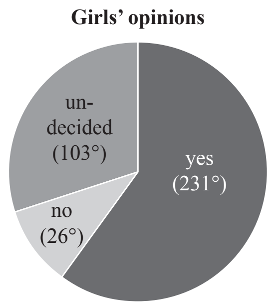
The range of weights in Team B is greater than Team A.

The IQR is similar for both teams.

4 A side-by-side bar chart would work well here. Two pie charts, one for boys and one for girls, would also work. There are many ways to represent this data!







5 (i) The scale on the vertical axis does not start at zero.

(ii) The axes are not labeled, the 3D perspective makes the 'Yes' category look larger.

(iii) Line graph is not appropriate, and the scale is not uniform.

(iv) Using 3-D objects is not appropriate, making the width and height twice as large makes the 200 g barrel appear much more than twice the 'size'.

6 (i) 23

(ii) $w = x = 0$

(iii) $z = 8$ or 9 so UQ is 38 or 39

(iv) Stem-and-leaf diagrams show the shape of the distribution of the data more clearly.

All of the data values are seen in a stem-and-leaf diagram.

Simple to see the mode.

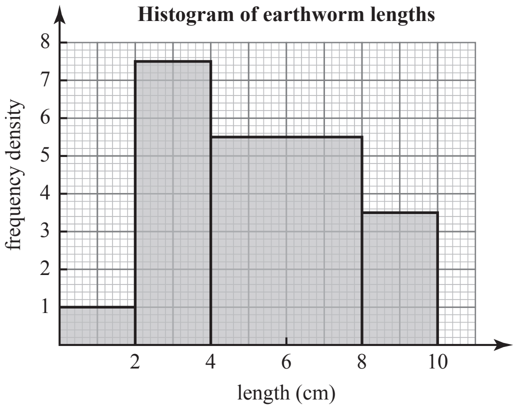
(v) Box-and-whisker plots show the summary values of the median and quartiles without any calculation needed.

2.2 Histograms and cumulative frequency graphs

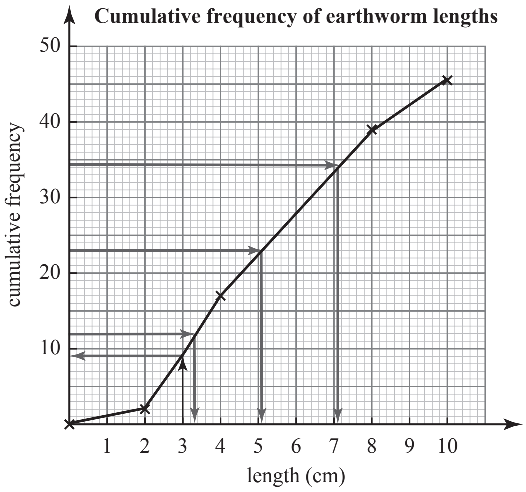
1 (i)

Length (x cm)	Frequency, f	Frequency density	Cumulative frequency
$0 \leqslant x < 2$	2	1	2
$2 \leqslant x < 4$	15	7.5	17
$4 \leqslant x < 8$	22	5.5	39
$8 \leqslant x \leqslant 10$	7	3.5	46
	$\sum f = 46$		

(ii)



(iii)



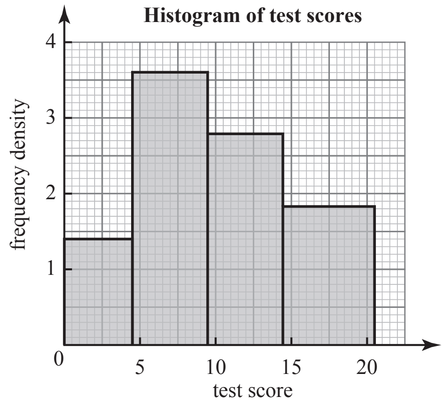
(iv) LQ = 3.3 Median = 5.1 UQ = 7.1

(v) $46 - 9 = 37$

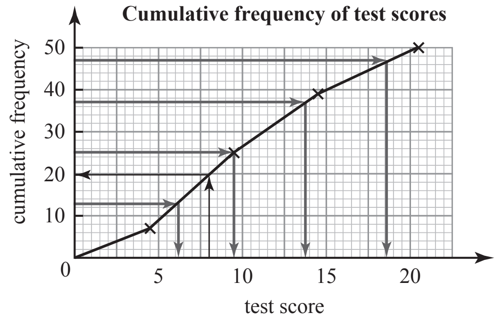
2 (i)

Score	f	Boundaries	Width	FD	CF
0–4	7	$-0.5 < x < 4.5$	5	1.4	7
5–9	18	$4.5 < x < 9.5$	5	3.6	25
10–14	14	$9.5 < x < 14.5$	5	2.8	39
15–20	11	$14.5 < x < 20.5$	6	1.83	50

(ii)



(iii)



Median ≈ 9.5 LQ ≈ 6.2 UQ ≈ 13.8

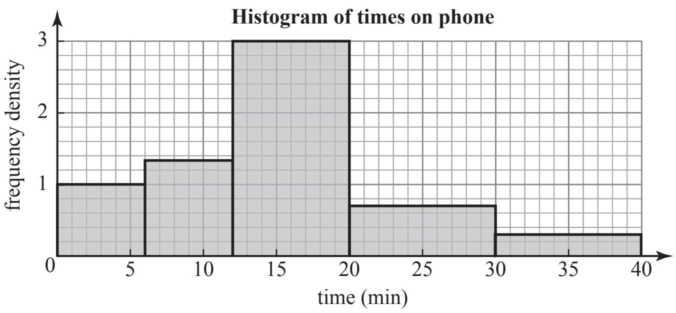
(iv) 19

(v) 20 students

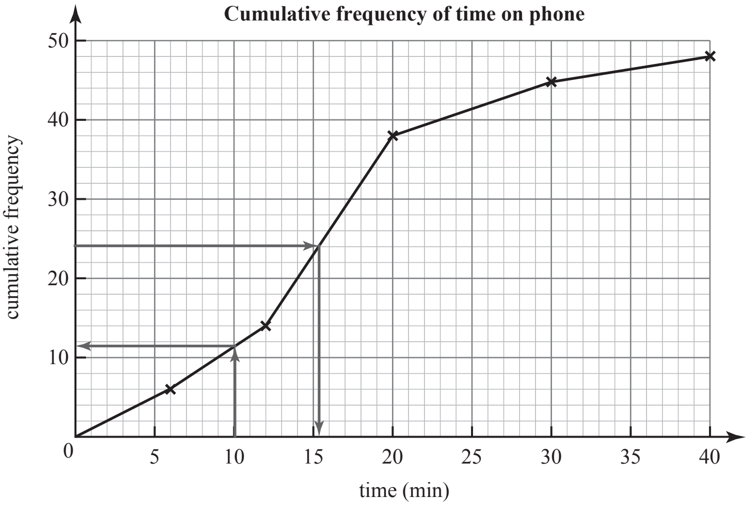
3 (i)

Time (min)	Frequency, f	Frequency density	Cumulative frequency
$0 \leqslant x < 6$	6	1	6
$6 \leqslant x < 12$	8	1.33	14
$12 \leqslant x < 20$	24	3	38
$20 \leqslant x < 30$	7	0.7	45
$30 \leqslant x \leqslant 40$	3	0.3	48
	$\sum f = 48$		

(ii)



(iii)



(iv) Median ≈ 15.2

(v) 48 − 11 = 37 spend more than 10 minutes.

4 **(i)**

Time (t minutes)	$5 < t \leqslant 20$	$20 < t \leqslant 30$	$30 < t \leqslant 35$	$35 < t \leqslant 45$	$45 < t \leqslant 60$	$60 < t \leqslant 80$
Frequency	9	32	63	50	30	16

(ii) Mean $= \dfrac{\sum xf}{\sum xf} = \dfrac{9 \times 12.5 + 32 \times 25 + 63 \times 32.5 + 50 \times 40 + 30 \times 52.5 + 16 \times 70}{9 + 32 + 63 + 50 + 30 + 16}$

$= \dfrac{7655}{200} = 38.275$ minutes $= 38.3$ minutes (3 s.f.)

5

Runs	Frequency	Frequency density	Cumulative frequency
0 – 4	8	1.6	8
5 – 25	21	1	29
26 – 49	48	2	77
50 – 99	20	0.4	97
100 – 120	3	0.14	100
	100		

6 **(i)** IQR = 69 − 45 = 24

(accept 23.5–24.5)

(ii) 60% of 1200 = 720

$x = 63$ or 64

(iii) CF at 68 marks is 860

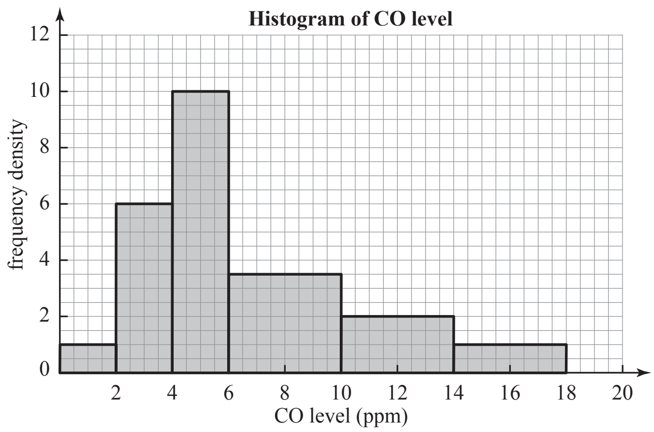
Number of students above 68 = 1200 − 860

= 340

Further practice

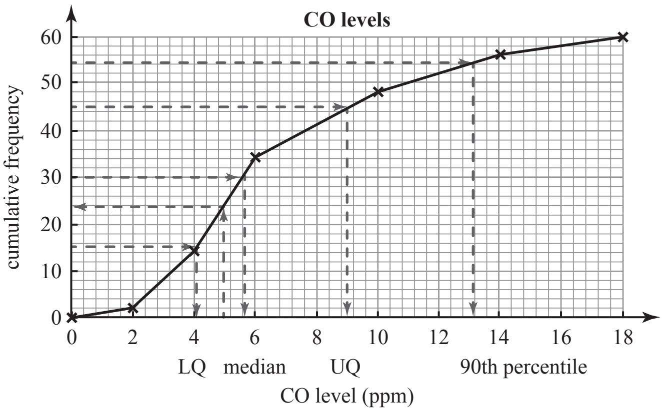
1 **(i)**

CO level (ppm)	Frequency, f	Frequency density
$0 \leqslant x < 2$	2	$\frac{2}{2} = 1$
$2 \leqslant x < 4$	12	$\frac{12}{2} = 6$
$4 \leqslant x < 6$	20	$\frac{20}{2} = 10$
$6 \leqslant x < 10$	14	$\frac{14}{4} = 3.5$
$10 \leqslant x < 14$	8	$\frac{8}{4} = 2$
$14 \leqslant x \leqslant 18$	4	$\frac{4}{4} = 1$



(ii) (a)

CO level (ppm)	Frequency, f	Cumulative frequency
$0 \leqslant x < 2$	2	2
$2 \leqslant x < 4$	12	14
$4 \leqslant x < 6$	20	34
$6 \leqslant x < 10$	14	48
$10 \leqslant x < 14$	8	56
$14 \leqslant x \leqslant 18$	4	60

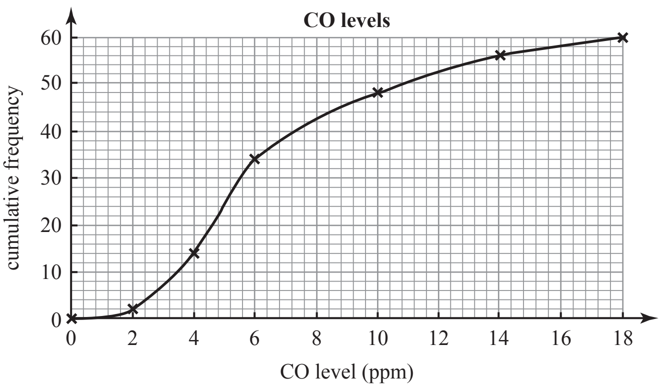


(b) From the graph, the median ≈ 5.6, LQ ≈ 4.1, UQ ≈ 9.

(c) 90th percentile ≈ 13.2

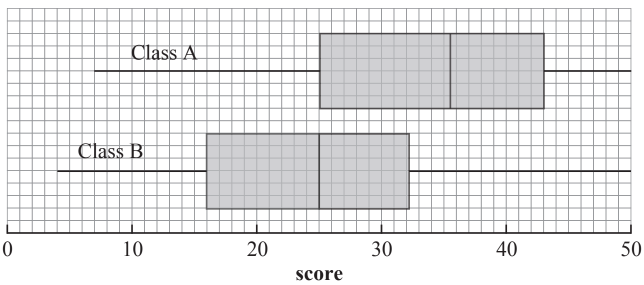
(d) Drawing a line from 5 ppm to the curve and across to the vertical axis gives a value of about 23. So the number of days when the reading is *above* 5 ppm is $60 - 23 = 37$

The graph can be drawn with curved lines also…



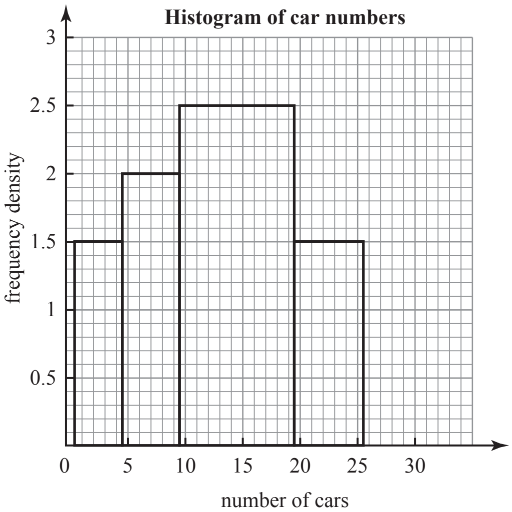
2 (i) Median = 35.5 LQ = 25 UQ = 43

(ii)

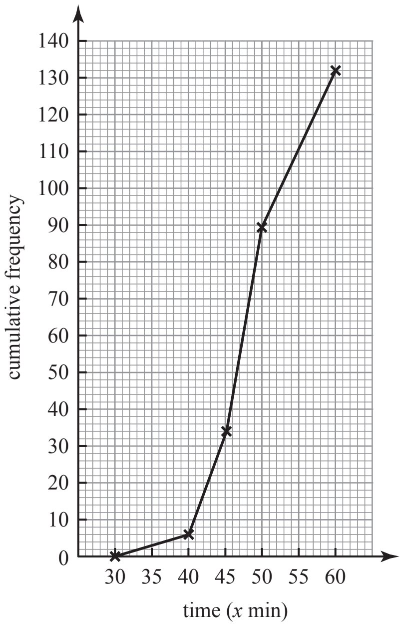


3

No. of cars, c	Frequency, f	Class boundaries	Class width	Frequency density
1–4	6	$0.5 \leqslant c < 4.5$	4	$\frac{6}{4} = 1.5$
5–9	10	$4.5 \leqslant c < 9.5$	5	$\frac{10}{5} = 2$
10–19	25	$9.5 \leqslant c < 19.5$	10	$\frac{25}{10} = 2.5$
20–25	9	$19.5 \leqslant c < 25.5$	6	$\frac{9}{6} = 1.5$



4 (i)



(ii) $k = 49.2$ min

(iii)

Time (x min)	f	Midpoint (m)	fm	fm^2
$30 \leqslant x < 40$	5	35	175	6125
$40 \leqslant x < 45$	29	42.5	1232.5	52381.25
$45 \leqslant x < 50$	55	47.5	2612.5	124093.75
$50 \leqslant x < 60$	43	55	2365	130075
	132		6385	312675

$$\text{Mean} = \frac{6385}{132} = 48.4\ \text{min}\ (3\ \text{s.f.})$$

$$\text{SD} = \sqrt{\frac{312675}{132} - 48.4^2} = 5.38\ \text{min}\ (3\ \text{s.f.})$$

5

Time (t hours)	FD	Frequency	Midpoint (m)	fm	fm^2
$0 \leqslant x < 1$	8	8	0.5	4	2
$1 \leqslant x < 2$	20	20	1.5	30	45
$2 \leqslant x < 4$	25	50	3	150	450
$4 \leqslant x < 6$	12	24	5	120	600
$6 \leqslant x < 10$	4.5	18	8	144	1152
		120		448	2249

(i) $\text{Mean} = \frac{448}{120} = 3.73$ hours

(ii) $\text{SD} = \sqrt{\frac{2249}{120} - 3.73...^2} = 2.19$ hours

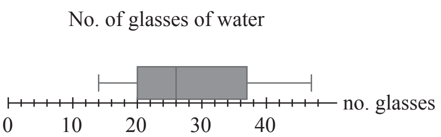
Past exam questions

1 (i)

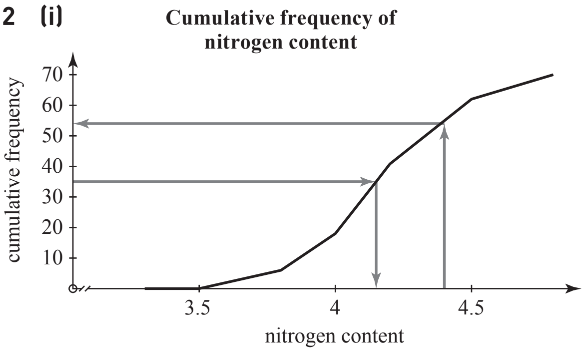
4	1 2 5 6 7
3	0 2 6 8
2	1 2 2 3 4 5 6 6 8 8
1	4 5 7 8 9 9

Key: 2|1 represents 21 glasses of water

(ii)



2 (i)



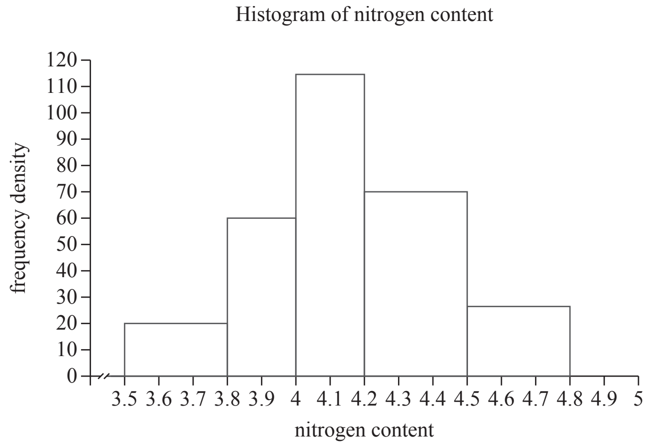
(ii) CF at 4.4 is approximately 55,
hence there are $70 - 55 = 15$ samples above 4.4

$$\text{Percentage greater than } 4.4 = \frac{15}{70} \times 100\% = 21.4\%$$

(iii) Median = 4.15

(iv)

Nitrogen content	Frequency	Frequency density
$x \leqslant 3.5$	0	0
$3.5 < x \leqslant 3.8$	6	20
$3.8 < x \leqslant 4.0$	12	60
$4.0 < x \leqslant 4.2$	23	115
$4.2 < x \leqslant 4.5$	21	70
$4.5 < x \leqslant 4.8$	8	26.7



3 (i) Weights of rugby players

Team A		Team B
2	12	
6 5	11	1 3 5
9 7 4 0	10	1 8
9 8 7 6 1	9	4 5 6
4 4 2	8	2 3 4 6
	7	5 7 9

Key: 5 |11| 1 represents 115 kg Team A and 111 kg Team B

(ii) IQR = 109 − 91 = 18 kg

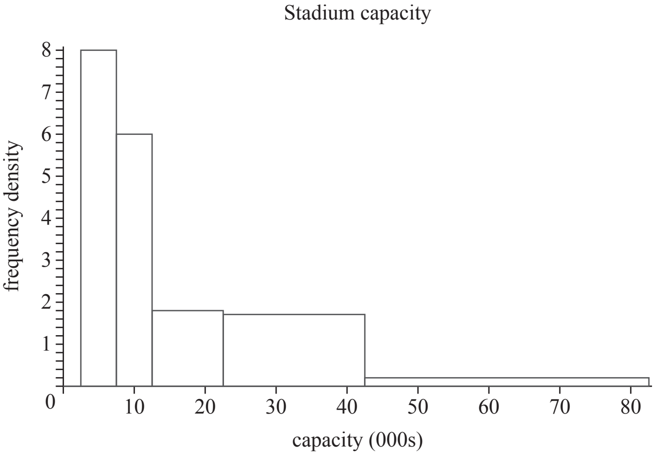
(iii) Total weight of 15 players = 1399 kg

Total weight of 16 players = 93.9 × 16 = 1502.4 kg

Weight of new player = 1502.4 − 1399 = 103.4 kg

4 (i)

Capacity (000s)	Frequency	Frequency density
2.5–	40	8
7.5–	30	6
12.5–	18	1.8
22.5–	34	1.7
42.5–82.5	8	0.2



(ii)

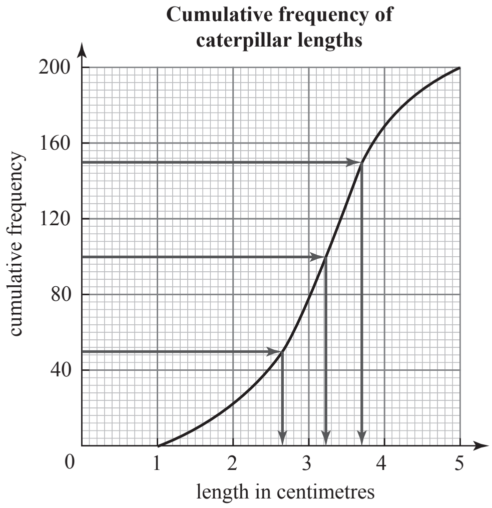
Capacity (000s)	f	Midpoint (m)	fm
2.5–	40	5	200
7.5–	30	10	300
12.5–	18	17.5	315
22.5–	34	32.5	1105
42.5–82.5	8	62.5	500
	130		2420

Mean $= \frac{2420}{130} = 18.6$ thousand $= 18600$

(iii) Median is between the 65th and 66th values so in the 8000–12 000 class.

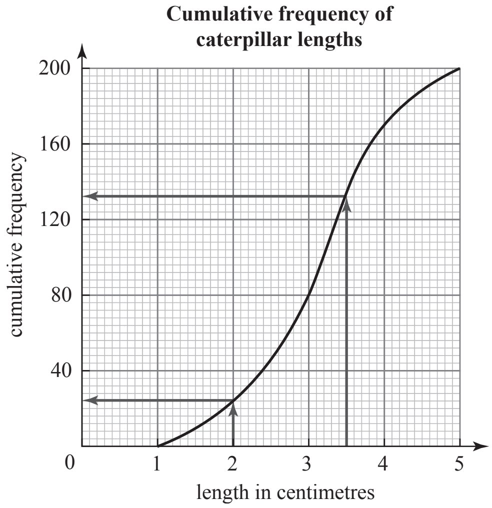
LQ is the 33rd value so in the 3000–7000 class.

5 (i)



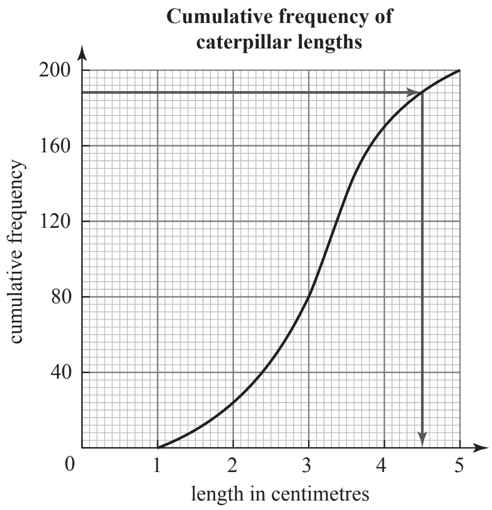
Median = 3.2 cm, IQR = 3.7 − 2.6 = 1.1 cm

(ii)



Number of caterpillars between 3.5 cm and 2 cm is approximately 134 − 24 = 110

(iii)



Longest 6% is 94% of 200 = 188th length
Approximately 4.5 cm

Stretch and challenge

1 (i) C **(ii)** A

(iii) B **(iv)** E

(v) D

3 Probability

3.1 Basic probability

1 (i) $\frac{13}{52} = \frac{1}{4}$

(ii) $\frac{4}{52} = \frac{1}{13}$

(iii) $\frac{1}{52}$

(iv) $\frac{16}{52} = \frac{4}{13}$

(v) $\frac{8}{52} = \frac{2}{13}$

2 (i)

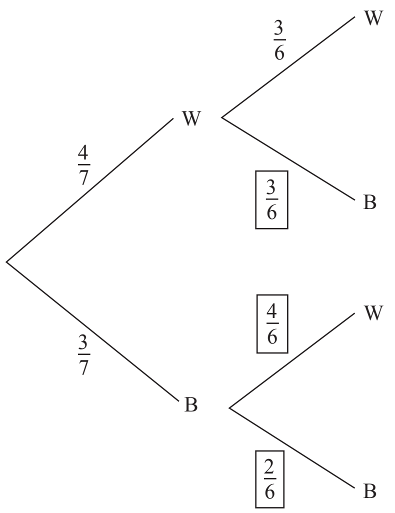
	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

(ii) $\frac{1}{12}$

(iii) $\frac{7}{12}$

(iv) $\frac{3}{12} = \frac{1}{4}$

3 (i)



(ii) (a) $\frac{4}{7}\times\frac{3}{6} = \frac{2}{7}$

(b) $\frac{4}{7}\times\frac{3}{6}+\frac{3}{7}\times\frac{4}{6} = \frac{4}{7}$

4 (i) 112, 121, 211, 321

(ii) Total no. outcomes $= 6 \times 6 \times 6 = 216$

so P(total of 4) $= \frac{3}{216} = \frac{1}{72}$

(iii) No. ways of getting 6:

114, 141, 411, 123, 132, 213, 231, 312, 321

so P(total of 6) $= \frac{10}{216} = \frac{5}{72}$

5 (i)

Difference	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

(ii) P(difference less than 2) $= \frac{10}{16} = \frac{5}{8}$

(iii) P(difference at least 2) $= \frac{6}{16} = \frac{3}{8}$

No. games won $= \frac{3}{8}\times 40 = 15$

6 P(Bryan wins)

$$= \left(\frac{5}{6}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)$$

$$= \frac{5}{6^2}+\frac{5^3}{6^4}+\frac{5^5}{6^6}+\ldots$$

This is a geometric series with $a = \frac{5}{36}$ and

$$r = \frac{5^3}{6^4} \div \frac{5}{6^2} = \frac{5^2}{6^2} = \frac{25}{36}$$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{5}{36}}{1-\frac{25}{36}} = \frac{5}{11}$$

7 Probability that Cody wins on her first turn

$$= \frac{1}{2}\times\frac{2}{3}\times\frac{1}{6} = \frac{1}{18}$$

Probability that she wins on her second turn

$$= \frac{1}{2}\times\frac{2}{3}\times\frac{5}{6}\times\frac{1}{2}\times\frac{2}{3}\times\frac{1}{6} = \frac{1}{18}\times\frac{5}{18} = \frac{5}{324}$$

Continuing the pattern we get

$$= \frac{1}{18}+\left(\frac{1}{18}\times\frac{5}{18}\right)+\left(\frac{1}{18}\times\left(\frac{5}{18}\right)^2\right)+\ldots$$

$$= \frac{1}{18}\left(1+\frac{5}{18}+\left(\frac{5}{18}\right)^2+\ldots\right)$$

$$= \frac{1}{18}\left(\frac{1}{1-\frac{5}{18}}\right)$$

$$= \frac{1}{18}\times\frac{18}{13}$$

$$= \frac{1}{13}$$

Alternately, let the probability of Cody winning be p.

The probability she wins on her first turn is $\frac{1}{18}$.

The probability that no one wins on the first round is $\frac{1}{2}\times\frac{2}{3}\times\frac{5}{6} = \frac{5}{18}$

The game then starts again so $p = \frac{1}{18}+\frac{5}{18}p$

$$p - \frac{5}{18}p = \frac{1}{18}$$

$$\frac{13}{18}p = \frac{1}{18}$$

$$p = \frac{1}{13}$$

8 P(at least one 6) $= 1 -$ P(no 6s)

$$= 1-\left(\frac{5}{6}\right)^n$$

$$1-\left(\frac{5}{6}\right)^n \geqslant 0.99$$

$$0.01 \geqslant \left(\frac{5}{6}\right)^n$$

Using guess and check, $n = 26$

9 (i) The first packet bought obviously contains a new card. The probability that the second packet bought contains a different card is $\frac{2}{3}$ so it takes, on average, $\frac{3}{2}$ packets to get the next card. Similarly, after Olivia has the first two cards, the probability that any randomly chosen packet has the last card is $\frac{1}{3}$ so on average she would need to buy 3 packets.

The total number of packets is $1+\frac{3}{2}+3=5\frac{1}{2}$

(ii) Using the same logic the number of packets needed on average to collect all 6 cards is

$1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}=14.7$

(iii) Looking at the expression above

$1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}$ can be written as

$6\left(\frac{1}{6}+\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)$

So in general the number of packets needed for a full set of n cards is $n\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n}\right)$

3.2 Independent and dependent events

1 (i)

		Die 1					
	M	1	2	3	4	5	6
Die 2	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

(ii) $\frac{14}{36}=\frac{7}{18}$

(iii) $\frac{19}{36}$

(iv) $\frac{11}{36}$

(v) $\frac{26}{36}=\frac{13}{18}$

$\mathrm{P}(S \text{ or } T)=\mathrm{P}(S)+\mathrm{P}(T)-\mathrm{P}(S \text{ and } T)$

$=\frac{1}{2}+\frac{19}{36}-\frac{11}{36}$

$=\frac{13}{18}$

(vi) If independent then

$\mathrm{P}(S \text{ and } T)=\mathrm{P}(S)\times\mathrm{P}(T)$

but $\frac{11}{36}\neq\frac{1}{2}\times\frac{19}{36}$

so S and T are not independent.

Since $\mathrm{P}(S \text{ and } T)\neq 0$

S and T are not mutually exclusive.

2 (i) $0.4\times 0.6=0.24$

(ii) $0.6\times(1-0.4)=0.36$

(iii) $\mathrm{P}(D \text{ or } C)=\mathrm{P}(D)+\mathrm{P}(C)-\mathrm{P}(D \text{ and } C)$

$=0.4+0.6-0.24$

$=0.76$

(iv) No they are not mutually exclusive as $\mathrm{P}(D \text{ and } C)\neq 0$.

If two events are independent then $\mathrm{P}(D \text{ and } C)=\mathrm{P}(D)\times\mathrm{P}(C)$ so $\mathrm{P}(D \text{ and } C)\neq 0$

So if the events are independent they cannot be mutually exclusive.

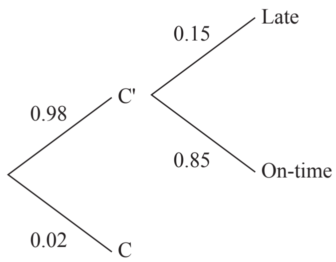
3 (i) $\mathrm{P}(A \text{ and } B)=0.32\times 0.5=0.16$

(ii) $\mathrm{P}(A \text{ or } B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \text{ and } B)$

$=0.32+0.5-0.16$

$=0.66$

4 (i)



(ii) $0.98\times 0.85=0.833$

(iii) $0.02+0.98\times 0.15=0.167$

Or $1-0.833=0.167$

5 $\mathrm{P}(A)=\frac{1}{6}$ and $\mathrm{P}(B)=\frac{1}{6}$

If independent then

$\mathrm{P}(A \text{ and } B)=\mathrm{P}(A)\times\mathrm{P}(B)$

$\frac{1}{36}=\frac{1}{6}\times\frac{1}{6}$

So A and B are independent.

Since $\mathrm{P}(A \text{ and } B)\neq 0$, the events are not mutually exclusive.

3.3 Conditional probability

1 (i) $\mathrm{P}(A \text{ and } B)=0.4\times 0.8=0.32$

(ii) $\mathrm{P}(A|B)=\frac{\mathrm{P}(A \text{ and } B)}{\mathrm{P}(B)}=\frac{0.32}{0.8}=0.4$

or, since A and B are independent

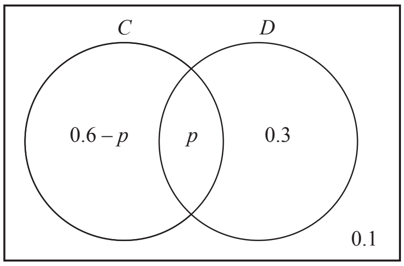
$\mathrm{P}(A|B)=\mathrm{P}(A)=0.4$

(iii) $\mathrm{P}(B|A)=\frac{\mathrm{P}(B \text{ and } A)}{\mathrm{P}(A)}=\frac{0.32}{0.4}=0.8$

or, since A and B are independent

$\mathrm{P}(B|A)=\mathrm{P}(B)=0.8$

2 (i)



$P(D) = 0.3 + p$

(ii) (a) $P(D|C) = \dfrac{P(D \cap C)}{P(C)} = \dfrac{p}{0.6} = 0.4 \Rightarrow p = 0.24$

(b) $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{p}{p+0.3} = \dfrac{1}{4}$

$$4p = p + 0.3$$
$$3p = 0.3$$
$$p = 0.1$$

(c) C and D mutually exclusive $\Rightarrow P(C \cap D) = 0$

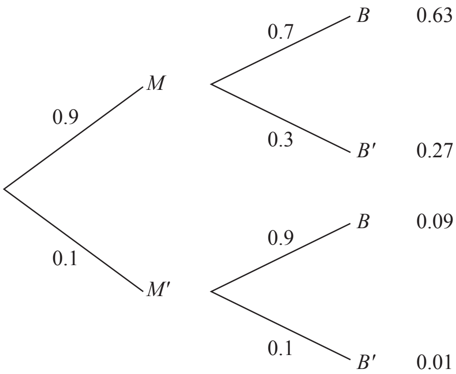
$p = 0$

(d) C and D independent

$\Rightarrow P(C \cap D) = P(C) \times P(D)$

$$p = 0.6 \times (p + 0.3)$$
$$p = 0.6p + 0.18$$
$$0.4p = 0.18$$
$$p = 0.45$$

3 (i)



(ii) $0.63 + 0.09 = 0.72$

(iii) $P(M|B) = \dfrac{P(M \text{ and } B)}{P(B)}$

$= \dfrac{0.63}{0.72}$

$= 0.875$ or $\dfrac{7}{8}$

(iv) If M and B are independent then

$P(M \text{ and } B) = P(M) \times P(B)$

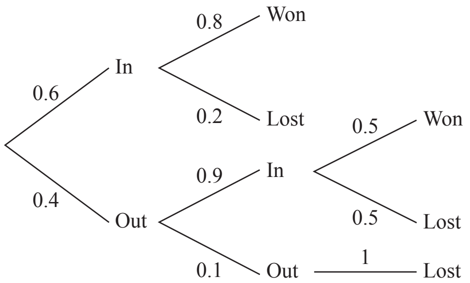
but $0.63 \neq 0.9 \times 0.72$

So M and B are not independent.

Or

Since $P(M|B) \neq P(M)$ the events are not independent.

4 (i)



(ii) $P(\text{Win}) = 0.6 \times 0.8 + 0.4 \times 0.9 \times 0.5$

$= 0.66$

(iii) $P(F|W) = \dfrac{P(F \text{ and } W)}{P(W)}$

$= \dfrac{0.6 \times 0.8}{0.66}$

$= 0.727$ (3 s.f.)

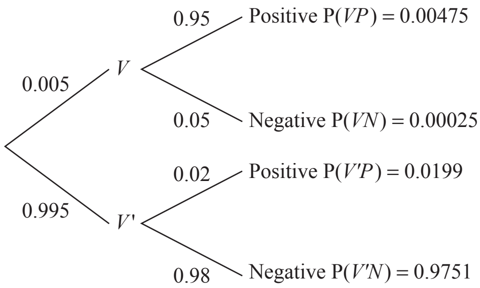
(iv) If F and W are independent then

$P(F \text{ and } W) = P(F) \times P(W)$

but $0.48 \neq 0.6 \times 0.66$

so F and W are *not* independent.

5 (i)



(ii) P(incorrect)

$= 0.005 \times 0.05 + 0.995 \times 0.02$

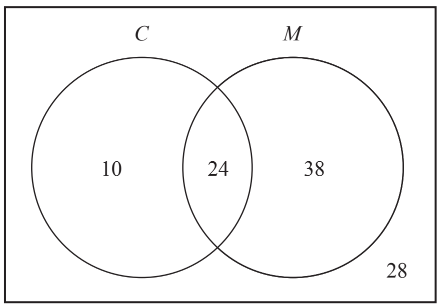
$= 0.02015$

(iii) $P(V|P) = \dfrac{P(V \text{ and } P)}{P(P)}$

$= \dfrac{0.00475}{0.00475 + 0.0199}$

$= 0.193$ (3 s.f.)

6 (i)



(ii) (a) $P(C') = 0.66$

(b) $P(M|C) = \dfrac{24}{34} = \dfrac{12}{17} = 0.706$ (3 s.f.)

(c) $P(C|M') = \dfrac{10}{38} = \dfrac{5}{19} = 0.263$ (3 s.f.)

7

	Maths	English	Total
Boys	105	92	197
Girls	84	112	196
Total	189	204	393

(i) $P(B) = \frac{197}{393} = 0.501$ (3 s.f.)

(ii) $P(M') = \frac{204}{393} = 0.519$ (3 s.f.)

(iii) $P(B|M) = \frac{105}{189} = 0.556$ (3 s.f.)

(iv) $P(M|B') = \frac{84}{196} = 0.429$ (3 s.f.)

(v) If M and B are independent then

$P(M \text{ and } B) = P(M) \times P(B)$

but $\frac{105}{393} \neq \frac{189}{393} \times \frac{197}{393}$

So M and B are *not* independent.

Or

Since $P(B|M) \neq P(B)$ the events are *not* independent.

8 **(i)** $P(CT) = 0.05 + 0.15 + 0.28 + 0.1 = 0.58$

$$P(S|CT) = \frac{P(S \text{ and } CT)}{P(CT)} = \frac{0.05}{0.58} = \frac{5}{58} \approx 0.0862$$

(ii) If the events were independent then

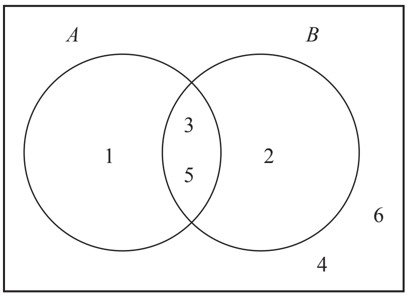
$P(V \cap CT) = P(V) \times P(CT)$

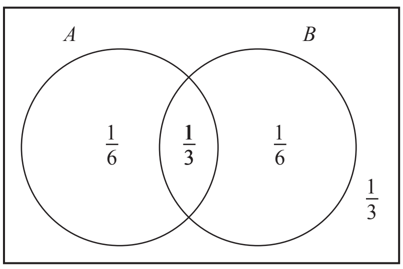
but $0.28 \neq 0.35 \times 0.58$

So the events are *not* independent.

Further practice

1 $A = \{1, 3, 5\}$ and $B = \{2, 3, 5\}$





(i) $P(A \cap B) = \frac{1}{3}$

(ii) $P(A \cup B) = \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = \frac{2}{3}$ (from diagram)

or $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$ (using the formula)

(iii) Since $P(A \cap B) \neq P(A) \times P(B)$ i.e. $\frac{1}{3} \neq \frac{1}{2} \times \frac{1}{2}$,

A and B are NOT independent.

Since $P(A \cap B) \neq 0$,

A and B are NOT mutually exclusive.

2 **(i)** Since F and G are independent then

$P(F \cap G) = P(F) \times P(G)$

$0.1 = 0.25 \times P(G)$

$P(G) = \frac{0.1}{0.25} = 0.4$

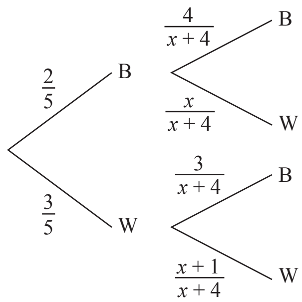
(ii) $P(F \cup G) = P(F) + P(G) - P(F \cap G)$

$= 0.25 + 0.4 - 0.1$

$= 0.55$

3 **(i)** After choosing a blue ball from box A and placing it in box B, there are now $x + 4$ balls in total in box B and x of them are white.

(ii)



(iii) $\frac{4}{x+4} = \frac{1}{5}$

$20 = x + 4 \Rightarrow x = 16$

(iv)
$$P(AW|BW) = \frac{P(AW \text{ and } BW)}{P(BW)}$$
$$= \frac{\frac{3}{5} \times \frac{x+1}{x+4}}{\frac{3}{5} \times \frac{x+1}{x+4} + \frac{2}{5} \times \frac{x}{x+4}}$$
$$= \frac{\frac{3(x+1)}{5(x+4)}}{\frac{3x+3}{5(x+4)} + \frac{2x}{5(x+4)}}$$
$$= \frac{3(x+1)}{5x+3}$$
$$= \frac{3x+3}{5x+3}$$

4

		Die 1			
		1	1	3	4
Die 2	1	2	2	4	5
	1	2	2	4	5
	3	4	4	6	7
	4	5	5	7	8

(i) $\frac{8}{16} = \frac{1}{2}$

(ii) $\frac{6}{16} = \frac{3}{8}$

(iii) To find if two events are independent, see if $P(A \cap B) = P(A) \times P(B)$. In this case $P(A \cap B) = \frac{1}{16}$, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{16}$

Clearly $\frac{1}{16} \neq \frac{1}{4} \times \frac{1}{16}$ so the events are *not* independent.

In fact, if A does not happen, B cannot happen so they are certainly not independent.

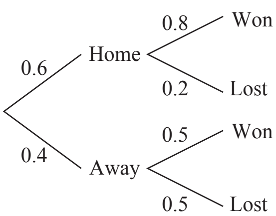
In this case $P(A \cap B) \neq 0$, so the events are *not* mutually exclusive.

5 $P(P|A) = \frac{P(P \text{ and } A)}{P(A)}$

$= \frac{0.1}{0.4}$

$= \frac{1}{4}$ or 0.25

6 **(i)**



(ii) $P(\text{Won}) = (0.6 \times 0.8) + (0.4 \times 0.5) = 0.68$

(iii) $P(H|W) = \frac{P(H \cap W)}{P(W)} = \frac{0.48}{0.68} = \frac{12}{17} \approx 0.706$ (3 s.f.)

7 **(i)** $P(B|A) = P(B) = 0.2$

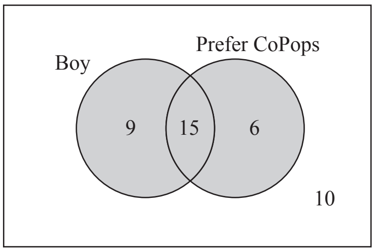
(ii) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 0.75 + 0.2 - 0.75 \times 0.2$

$= 0.8$

(iii) Since $P(A \text{ and } B) \neq 0$, A and B are *not* mutually exclusive.

8 **(i)** Using a Venn diagram:



$P(\text{Boy or CoPops}) = \frac{9+15+6}{40} = \frac{30}{40} = \frac{3}{4}$

OR

$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and } C)$

$= \frac{24}{40} + \frac{21}{40} - \frac{15}{40}$

$= \frac{30}{40} = \frac{3}{4}$

(ii) G: person chosen is a girl.

C: person chosen prefers CoPops.

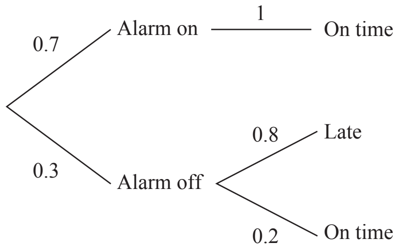
If independent,

$P(G \text{ and } C) = P(G) \times P(C)$

but $\frac{6}{40} \neq \frac{16}{40} \times \frac{21}{40}$

So the events are *not* independent.

9



(i) $P(\text{on time}) = 0.7 + 0.3 \times 0.2 = 0.76$

(ii) $P(\text{Alarm on}|\text{on time})$

$= \frac{P(\text{Alarm on } \textit{and} \text{ on time})}{P(\text{on time})}$

$= \frac{0.7}{0.76} = \frac{35}{38} = 0.921$ (3 s.f.)

10 **(i)** $P(\text{at least one six}) = 1 - P(\text{no sixes})$

$= 1 - \left(\frac{5}{6}\right)^{10}$

$= 1 - 0.16150...$

$= 0.838$ (3 s.f.)

(ii) $1 - \left(\frac{5}{6}\right)^n > 0.99$

$\left(\frac{5}{6}\right)^n < 0.01$

Using trial and error, $n > 26$

(iii) P(Russell wins)

= P(A blue, R red)

+ P(A blue, R blue, A blue, R red)

+ P(A blue, R blue, A blue, R blue, A blue, R red)

$= \left(\frac{5}{9}\times\frac{4}{8}\right)+\left(\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{4}{6}\right)+\left(\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times 1\right)$

$= \frac{23}{63} \approx 0.365$ (3 s.f.)

Past exam questions

1 **(i)** $24 = 24 \times 1, 12 \times 2, 8 \times 3, 6 \times 4$

Q: (12,2) or (2,12), (8,3) or (3,8), (6,4) or (4,6)

$P(Q) = \frac{6}{144} = \frac{1}{24}$

(ii) $P(\text{one die} > 8) = \frac{4}{12} = \frac{1}{3}$

$P(\text{both dice} > 8) = \frac{1}{3}\times\frac{1}{3} = \frac{1}{9}$

OR

List all 16 possibilities:

(9, 9) (9, 10) (9, 11) (9, 12) (10, 9) etc…

(iii) Since $P(Q \cap R) = 0$, Q and R are exclusive.

(iv) If independent, then

$P(Q \cap R) = P(Q) \times P(R)$

but $0 \neq \frac{1}{24}\times\frac{1}{9}$

So Q and R are *not* independent.

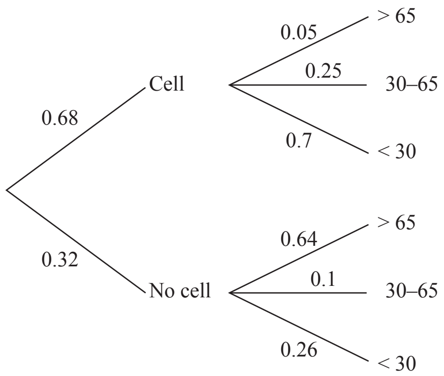
OR

If the events were independent then

$P(Q|R) = P(Q) = \frac{1}{24}$.

But $P(Q|R) = 0$ so the events are *not* independent.

2 **(i)**



(ii) $P(C|30-65) = \frac{P(C \cap 30-65)}{P(30-65)}$

$= \frac{0.68 \times 0.25}{0.68 \times 0.25 + 0.32 \times 0.1}$

$= 0.842$ (3 s.f.)

3 **(i)** $0.7 \times 0.7 = 0.49$

(ii) P(Ben champion)

$= P(BB) + P(BTB) + P(TBB)$
$= 0.49 + 0.7 \times 0.3 \times 0.7 + 0.3 \times 0.7 \times 0.7$

$= 0.784$

(iii) P(Tom won 2nd | Tom champion)

$= \frac{P(\text{Tom won 2nd and champion})}{P(\text{Tom champion})}$

$= \frac{P(WW) + P(LWW)}{1 - 0.784}$

$= \frac{0.3 \times 0.3 + 0.7 \times 0.3 \times 0.3}{0.216}$

$= \frac{17}{24} = 0.708$ (3 s.f.)

4 **(i)**

	Kitchen left in a mess	Kitchen not left in a mess	Total
Meal served on time	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
Meal not served on time	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{4}{5}$
Total	$\frac{3}{5}$	$\frac{2}{5}$	1

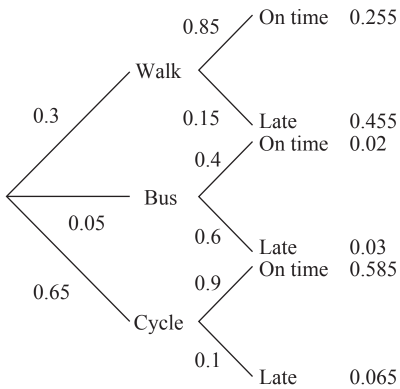
(ii) P(Meal not on time | Kitchen left in a mess)

$= \frac{P(\text{Meal not on time and kitchen left in a mess})}{P(\text{Kitchen left in a mess})}$

$= \frac{\frac{1}{2}}{\frac{3}{5}}$

$= \frac{5}{6} = 0.833$ (3 s.f.)

5



$P(C|L) = \frac{P(C \cap L)}{P(L)}$

$= \frac{0.065}{0.045 + 0.03 + 0.065}$

$= 0.464$ (3 s.f.)

Stretch and challenge

1 Assume that the probabilities of being born in each month are all equal, i.e. $\frac{1}{2}$. P(at least 2 born in same month)

$= 1 - \text{P(none born in the same month)}$

$= 1 - \left(\frac{12}{12} \times \frac{11}{12} \times \frac{10}{12} \times \frac{9}{12} \times \frac{8}{12}\right)$

$= 1 - 0.382\ldots$

$= 0.618\ldots$

2 $\text{P}(2K \mid \geqslant 1K) = \dfrac{\text{P}(2K \cap \geqslant 1K)}{\text{P}(\geqslant 1K)}$

$= \dfrac{\text{P}(2K)}{1 - \text{P}}$

$= \dfrac{\frac{4}{52} \times \frac{3}{51}}{1 - \frac{48}{52} \times \frac{47}{51}}$

≈ 0.0303 (3 s.f.)

3 **(i)** P(Cam wins)

$= \text{P(TTH)} + \text{P(TTTTTH)} + \text{P(TTTTTTTTH)} + \ldots$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \ldots$

GP, $a = \left(\frac{1}{2}\right)^3$, $r = \left(\frac{1}{2}\right)^3$

$S_\infty = \dfrac{\left(\frac{1}{2}\right)^3}{\left(1 - \frac{1}{2}\right)^3} = \left(\frac{1}{7}\right)$

(ii) P(Win at least one game)

$= 1 - \text{P(Win no games)}$

$= 1 - \left(\frac{5}{7}\right)^{10}$

$= 0.965$ (3 s.f.)

4 $0.7 \times \frac{1}{3} + 0.3 = \frac{8}{15}$

5 **(i)** P(Sahil will win)

$= \text{P(Win in 3 sets)} + \text{P(Win in 4 sets)} + \text{P(Win in 5 sets)}$

$= \left(\frac{3}{5}\right)^3 + 3 \times \left(\frac{3}{5}\right)^3 \times \left(\frac{2}{5}\right)^1 + 6 \times \left(\frac{3}{5}\right)^3 \times \left(\frac{2}{5}\right)^2$

$= 0.68256$

$= 0.683$ (3 s.f.)

(ii) P(win from 2 – 0)

$= \text{P(W)} + \text{P(LW)} + \text{P(LLW)}$

$= \left(\frac{3}{5}\right) + \left(\frac{2}{5}\right) \times \left(\frac{3}{5}\right) + \left(\frac{2}{5}\right)^2 \times \left(\frac{3}{5}\right)$

$= 0.936$

So Sahil should get \$936 000.

6 P(undetected)

$= \text{P(King finds no false coins)}$

$= \left(\frac{99}{100}\right)^{100}$

$= 0.366$

7 Consider the case where $\text{P}(A) = 0$. Then $\text{P}(A) \times \text{P}(B) \times \text{P}(C) = 0$ and also $\text{P}(A \cap B \cap C) = 0$. Now consider that B and C are not independent so that $\text{P}(B \cap C) \neq \text{P}(B) \times \text{P}(C)$. So the three events are not independent but the equality holds.

8 Yes it is true.

$\text{P}(A' \cap B') = 1 - \text{P}(A \cup B)$

$= 1 - (\text{P}(A) + \text{P}(B) - \text{P}(A \cap B))$

Since A and B are independent this becomes

$= 1 - (\text{P}(A) + \text{P}(B) - \text{P}(A)\text{P}(B))$

$= 1 - \text{P}(A) - \text{P}(B) + \text{P}(A)\text{P}(B)$

$= (1 - \text{P}(A))(1 - \text{P}(B))$

$= \text{P}(A') \times \text{P}(B')$

9 **(i)** P(Ravi wins first game)

$= \frac{1}{3} \times 0.2 + \frac{1}{3} \times 0.5 + \frac{1}{3} \times 0.8$

$= 0.5$

(ii) $\text{P}(W_2 \mid W_1) = \dfrac{\text{P}(W_2 \cap W_1)}{\text{P}(W_1)}$

$= \dfrac{\frac{1}{3}\left[\begin{array}{l}\text{P}(\text{Win both} \mid \text{good}) + \text{P}(\text{Win both} \mid \text{average}) \\ + \text{P}(\text{Win both} \mid \text{beginner})\end{array}\right]}{0.5}$

$= \dfrac{\frac{1}{3}\left(0.2^2 + 0.5^2 + 0.8^2\right)}{0.5}$

$= \frac{31}{50} = 0.62$

10 P(Green 1st draw)

$= \dfrac{g}{g + r}$

P(Green 2nd draw)

$= \text{P(RG)} + \text{P(GG)}$

$= \dfrac{r}{g + r} \times \dfrac{g}{g + r - 1} + \dfrac{g}{g + r} \times \dfrac{g - 1}{g + r - 1}$

$= \dfrac{rg}{(g + r)(g + r - 1)} + \dfrac{g(g - 1)}{(g + r)(g + r - 1)}$

$= \dfrac{g^2 + rg - g}{(g + r)(g + r - 1)}$

$= \dfrac{g(g + r - 1)}{(g + r)(g + r - 1)}$

$= \dfrac{g}{g + r}$

4 Discrete random variables

4.1 Discrete random variables

1 (i) $p = 0.62$

(ii) $p + 2p + 3p + 4p = 1$

$$10p = 1$$

$$p = \frac{1}{10} = 0.1$$

(iii) $p^2 + 2p + 2p^2 = 1$

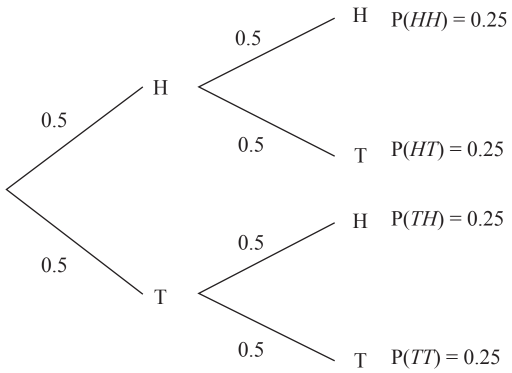
$$3p^2 + 2p - 1 = 0$$

$$(3p - 1)(p + 1) = 0$$

$$p = \frac{1}{3} \text{ or } -1$$

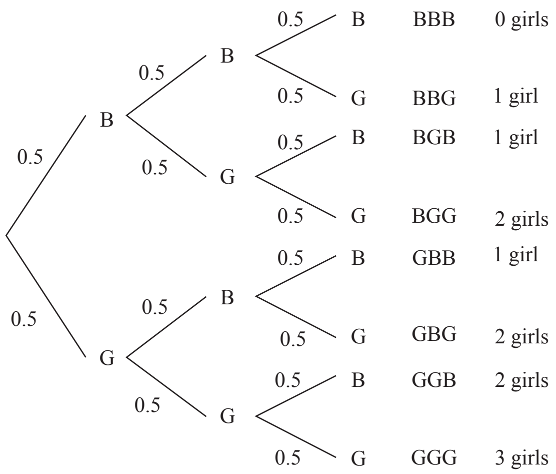
Since probabilities cannot be negative $p = \frac{1}{3}$

2 (i) We could use a probability tree:



h	0	1	2
$\mathrm{P}(H = h)$	0.25	0.5	0.25

(ii)



g	0	1	2	3
$\mathrm{P}(G = g)$	0.125	0.375	0.375	0.125

(iii) Sample space:

		Die 1					
	D	1	2	3	4	5	6
Die 2	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

d	0	1	2	3	4	5
$\mathrm{P}(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

3 (i) We want to find the value of r so that $\frac{12}{r} = 3$.

So $r = 4$.

The probability $X = 3$ is then $\frac{4}{10} = \frac{2}{5}$

(ii)

r	1	2	3	4
x	$\frac{12}{1} = 12$	$\frac{12}{2} = 6$	$\frac{12}{3} = 4$	$\frac{12}{4} = 3$
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{2}{10} = \frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

So the final table with the values of X in numerical order would look like this:

x	3	4	6	12
$\mathrm{P}(X = x)$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$

4

Number of 4s	1	2	3	4
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	1	1	1	2

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

5 (i) When $X = 24$, $\frac{120}{r} = 24 \Rightarrow r = 5$

So $P(X = 24) = \frac{5}{15} = \frac{1}{3}$

(ii)

r	1	2	3	4	5
x	120	60	40	30	24
$P(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{1}{3}$

(iii) The modal value of X is 24

(iv) $P(35 < X < 90) = P(X = 40 \text{ or } 60)$

$= \frac{2}{15} + \frac{1}{5} = \frac{1}{3}$

6 (i) If the correct order is ABCD, we can look at each of the 4! = 24 different arrangements to see how many letters go in the correct envelope.

ABCD	4	BACD	2	CABD	1	DABC	0
ABDC	2	BADC	0	CADB	0	DACB	1
ACBD	2	BCAD	1	CBAD	1	DBAC	1
ACDB	1	BCDA	0	CBDA	1	DBCA	2
ADBC	1	BDAC	0	CDAB	0	DCAB	0
ADCB	2	BDCA	1	CDBA	0	DCBA	0

$P(X = 0) = \frac{9}{24} = \frac{3}{8}$

(ii)

x	0	1	2	3	4
$P(X = x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{24}$	0	$\frac{1}{24}$

(iii) The modal values are 0 and 1.

4.2 Expectation and variance

1 (i) $E(X) = 0 \times \frac{1}{10} + 5 \times \frac{2}{5} + 10 \times \frac{1}{2} = 7$

$\text{Var}(X) = \left(0^2 \times \frac{1}{10} + 5^2 \times \frac{2}{5} + 10^2 \times \frac{1}{2}\right) - 7^2$

$= 11$

$\sigma = \sqrt{11} = 3.32$ (3 s.f.)

(ii) $E(X) = -1 \times 0.15 + 0 \times 0.35 + 2 \times 0.5 = 0.85$

$\text{Var}(X) = \left((-1)^2 \times 0.15 + 0^2 \times 0.35 + 2^2 \times 0.5\right) - 0.85^2$

$= 1.4275$

$\sigma = \sqrt{1.4275} = 1.19$ (3s.f.)

2 (i) $E(X) = -1 \times 0.65 + 4 \times 0.35 = 0.75$

$\text{Var}(X) = \left((-1)^2 \times 0.65 + 4^2 \times 0.35\right) - 0.75^2$

$= 5.6875$

(ii) $P(X > \mu) = P(X > 0.75) = P(X = 4) = 0.35$

3 $E(X) = 5 \times p + 10 \times (1 - p) = 8$

$5p + 10 - 10p = 8$

$10 - 5p = 8$

$-5p = -2$

$p = \frac{2}{5}$ or 0.4

4 $a + 0.2 + b + 0.4 = 1$

$a + b = 0.4$

$E(X) = -3 \times a + 1 \times 0.2 + 2 \times b + 4 \times 0.4 = 2.1$

$-3a + 0.2 + 2b + 1.6 = 2.1$

$-3a + 2b = 0.3$

Solve simultaneously

$a + b = 0.4$ ... ①

$-3a + 2b = 0.3$ ... ②

$3a + 3b = 1.2$ ... ① × 3

$5b = 1.5 \Rightarrow b = 0.3$

$a = 0.1$

5 $E(X) = 2 \times p + 3 \times (1 - p)$

$= 2p + 3 - 3p$

$= 3 - p$

$\text{Var}(X) = 2^2 \times p + 3^2 \times (1 - p) - (3 - p)^2$

$= 4p + 9 - 9p - (9 - 6p + p^2)$

$= p - p^2$

$p - p^2 = 0.24$

$p^2 - p + 0.24 = 0$

Using the quadratic formula,

$p = 0.4$ or 0.6

6

x	$p - 400000$	p
$P(X = x)$	0.0012	0.9988

$E(X) = 500$

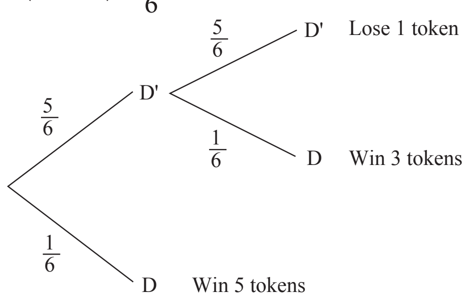
$(p - 400000) \times 0.0012 + p \times 0.9988 = 500$

$0.0012p - 480 + 0.9988p = 500$

$p - 480 = 500$

$p = \$980$

7 (i) $P(\text{double}) = \frac{1}{6}$



x	-1	3	5
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{36}$	$\frac{1}{6}$

(ii) $E(X) = -1 \times \frac{25}{36} + 3 \times \frac{5}{36} + 5 \times \frac{1}{6}$

$= \frac{5}{9}$ of a token

If he plays the game 50 times, his expected profit is $50 \times \frac{5}{9} = 27\frac{7}{9}$ tokens.

8 **(i)** $10 \times 1\frac{1}{2} = 15$ so profit is 5 points

(ii)

		White					
	Die	1	2	3	4	5	6
Die	**Score**	0.5	1	1.5	2	2.5	3
Red 1	2	1	2	3	4	5	6
2	4	2	4	6	8	10	12
3	6	3	6	9	12	15	18
4	8	4	8	12	16	20	24
5	10	5	10	15	20	25	30
6	12	6	12	18	24	30	36

Points gain	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**
2	–10	–10	–10	–10	–10	–10
4	–10	–10	–10	–10	0	2
6	–10	–10	–10	2	5	8
8	–10	–10	2	6	10	14
10	–10	0	5	10	15	20
12	–10	2	8	14	20	26

Since there are 4 entries with a gain of 2 points, $P(Y = 2) = \frac{4}{36} = \frac{1}{9}$.

(iii)

y	–10	0	2	5	6	8	10	14	15	20	26
$P(Y = y)$	$\frac{17}{36}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$

(iv) $E(Y) = -10 \times \frac{17}{36} + 0 \times \frac{1}{18} + 2 \times \frac{1}{9} + 5 \times \frac{1}{18} + 6 \times \frac{1}{36} + 8 \times \frac{1}{18} + 10 \times \frac{1}{18} + 14 \times \frac{1}{18} + 15 \times \frac{1}{36} + 20 \times \frac{1}{18} + 26 \times \frac{1}{36}$

$= -\frac{1}{36} \approx -0.0278$ (3 s.f.) points

Further practice

1 **(i)** $E(X) = \sum xp = -2 \times 0.4 + 1 \times 0.1 + 2 \times 0.5 = 0.3$

(ii) $\text{Var}(X) = \sum x^2 p - \{E(X)\}^2$

$= (-2)^2 \times 0.4 + 1^2 \times 0.1 + 2^2 \times 0.5 - 0.3^2$

$= 3.7 - 0.3^2$

$= 3.61$

Standard deviation $= \sqrt{\text{Var}(X)} = \sqrt{3.61} = 1.9$

(iii) Since $(-2)^2 = 2^2 = 4$ you can combine the probabilities for the first and last values of X.

x^2	1	4
$P(X^2 = x^2)$	0.1	0.9

2 **(i)** $P(S = 5) = P(2 \text{ then } 3) + P(3 \text{ then } 2)$

$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}$

$= \frac{1}{3}$

Another way to find the possible sums is to write out all the possibilities (since there are only ${}^3C_2 = 3$ possible sums)

Since each sum is equally likely, $P(S = 5) = \frac{1}{3}$

(ii)

s	3	4	5
$P(S = s)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(iii) $E(S) = 3 \times \frac{1}{3} + 4 \times \frac{1}{3} + 5 \times \frac{1}{3} = 4$

$\text{Var}(S) = 3^2 \times \frac{1}{3} + 4^2 \times \frac{1}{3} + 5^2 \times \frac{1}{3} - 4^2$

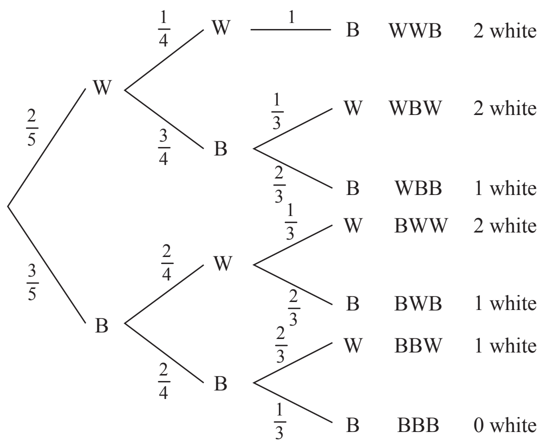
$= \frac{50}{3} - 16 = \frac{2}{3}$

(iv)

l	2	3
$P(L = l)$	$\frac{1}{3}$	$\frac{2}{3}$

$\mu = E(L) = 2 \times \frac{1}{3} + 3 \times \frac{2}{3} = \frac{8}{3} = 2\frac{2}{3}$

3 **(i)**



$P(W = 0) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

(ii) $P(W=1) = P(WBB) + P(BWB) + P(BBW)$

$= \left(\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\right) + \left(\frac{3}{5}\times\frac{1}{2}\times\frac{2}{3}\right) + \left(\frac{3}{5}\times\frac{1}{2}\times\frac{2}{3}\right)$

$= \frac{1}{5}+\frac{1}{5}+\frac{1}{5} = \frac{3}{5}$

Once we know two of the probabilities, the other one can be found easily since the sum of the probabilities is 1.

$P(W=2) = 1 - \left(\frac{1}{10}+\frac{3}{5}\right) = \frac{3}{10}$

w	0	1	2
$P(W=w)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

(iii) $E(W) = 0\times\frac{1}{10} + 1\times\frac{3}{5} + 2\times\frac{3}{10} = 1\frac{1}{5} = 1.2$

(iv) $P(W < E(W)) = P(W < 1.2)$

$= P(W=0) + P(W=1)$

$= \frac{1}{10} + \frac{3}{5}$

$= \frac{7}{10}$

4 (i) $\frac{8}{20}\times\frac{5}{16} = \frac{1}{8}$

(ii) $P(G_A|G_B) = \frac{P(G_A \cap G_B)}{P(G_B)}$

$= \frac{\frac{8}{20}\times\frac{11}{16}}{\frac{8}{20}\times\frac{11}{16}+\frac{12}{20}\times\frac{10}{16}}$

$= \frac{0.275}{0.65}$

$= 0.423$ (3 s.f.)

(iii) $P(X=0) = P(GG) = \frac{8}{20}\times\frac{11}{16} = 0.275$

$P(X=1) = P(GB) + P(BG)$

$= \frac{8}{20}\times\frac{5}{16} + \frac{12}{20}\times\frac{10}{16}$

$= 0.5$

$P(X=2) = P(BB) = \frac{12}{20}\times\frac{6}{16} = 0.225$

x	0	1	2
$P(X=x)$	0.275	0.5	0.225

5 X: number on die

x	1	2	3	4	5	6
$P(X=x)$	0.18	0.18	0.18	0.18	0.18	0.1

P: points scored

p	−3	5	10
$P(P=p)$	0.72	0.18	0.1

$E(P) = -3\times0.72 + (5\times0.18) + (10\times0.1) = -0.26$

Expected score after 100 rolls is $100\times(-0.26) = -26$

Past exam questions

1 (i)

x	0	1	2
$P(X=x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

(ii) $E(X) = 0\times\frac{1}{7} + 1\times\frac{4}{7} + 2\times\frac{2}{7} = \frac{8}{7}$

$\mathrm{Var}(X) = 0^2\times\frac{1}{7} + 1^2\times\frac{4}{7} + 2^2\times\frac{2}{7} - \left(\frac{8}{7}\right)^2$

$= 0.408$ (3 s.f.)

(iii) $P(G|A') = \frac{P(G\cap A')}{P(A')}$

$= \frac{\frac{2}{5}\times\frac{1}{4}}{\frac{2}{5}\times\frac{1}{4}+\frac{3}{5}\times\frac{9}{10}}$

$= \frac{5}{32} = 0.156$ (3 s.f.)

2 (i) P(one white rabbit)

$= P(WW') + P(W'W)$

$= \frac{6}{9}\times\frac{3}{8} + \frac{3}{9}\times\frac{6}{8}$

$= \frac{1}{2}$

(ii) P(no white rabbits) $= \frac{3}{9}\times\frac{2}{8} = \frac{6}{72} = \frac{1}{12}$

P(two white rabbits) $= \frac{6}{9}\times\frac{5}{8} = \frac{30}{72} = \frac{5}{12}$

X: number of white rabbits selected

x	0	1	2
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{5}{12}$

(iii) $E(X) = 0\times\frac{1}{12} + 1\times\frac{1}{2} + 2\times\frac{5}{12} = \frac{4}{3} = 1.33$

3 (i)

		Spinner A			
		1	2	3	3
Spinner B	−3	−2	−1	0	0
	−2	−1	0	1	1
	−1	0	1	2	2
	1	2	3	4	4

(ii)

x	–2	–1	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{8}$

(iii) $E(X) = -2\times\frac{1}{16} + -1\times\frac{1}{8} + 0\times\frac{1}{4} +$

$1\times\frac{3}{16} + 2\times\frac{3}{16} + 3\times\frac{1}{16} + 4\times\frac{1}{8}$

$= 0.75$

$$\mathrm{Var}(X) = \begin{bmatrix} (-2)^2\times\frac{1}{16} + (-1)^2\times\frac{1}{8} + 0^2\times\frac{1}{4} + \\ 1^2\times\frac{3}{16} + 2^2\times\frac{3}{16} + 3^2\times\frac{1}{16} + 4^2\times\frac{1}{8} \end{bmatrix} - 1^2$$

$= \frac{62}{16} - 1^2$

$= 2.875$

(iv) $P(X \text{ is even} \mid X \text{ is positive})$

$= \frac{P(X \text{ is even and } X \text{ is } +)}{P(X \text{ is } +)}$

$= \frac{\frac{5}{16}}{\frac{9}{16}}$

$= \frac{5}{9}$

4 **(i)**

		Die 1					
	S	1	2	3	4	5	6
Die 2	1	0	1	1	1	1	1
	2	1	0	2	2	2	2
	3	1	2	0	3	3	3
	4	1	2	3	0	4	4
	5	1	2	3	4	0	5
	6	1	2	3	4	5	0

s	0	1	2	3	4	5
$P(S=s)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

(ii) $E(S) = 0\times\frac{6}{36} + 1\times\frac{10}{36} + 2\times\frac{8}{36} +$

$3\times\frac{6}{36} + 4\times\frac{4}{36} + 5\times\frac{2}{36}$

$= \frac{35}{18} \approx 1.94$ (3 s.f.)

5 **(i)** $P(X = -2) = k(-2)^2 = 4k$

$P(X = 2) = k(2)^2 = 4k$

(ii)

x	–2	–1	2	4
$P(X=x)$	$4k$	k	$4k$	$16k$

$4k + k + 4k + 16k = 1$

$25k = 1$

$k = 0.04$

(iii) $E(X) = -2\times 4k + -1\times k + 2\times 4k + 4\times 16k$

$= 63k$

$= 63\times 0.04$

$= 2.52$

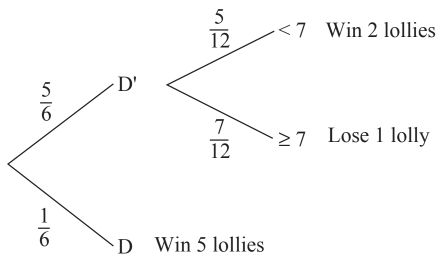
OR

$E(X) = -2\times 0.16 + -1\times 0.04 + 2\times 0.16 + 4\times 0.64$

$= 2.52$

Stretch and challenge

1 **(i)** $P(\text{Double}) = \frac{1}{6}$, $P(\text{sum} < 7) = \frac{15}{36} = \frac{5}{12}$



x	–1	2	5
$P(X=x)$	$\frac{35}{72}$	$\frac{25}{72}$	$\frac{1}{6}$

$E(X) = -1\times\frac{35}{72} + 2\times\frac{25}{72} + 5\times\frac{1}{6} = \frac{75}{72} = 1\frac{1}{24}$

(ii) Many answers possible, for example,

Forfeit = 3 lollies

Prize for rolling double = $2\frac{1}{2}$ lollies

Prize for < 7 on 2nd roll = 3 lollies

If the forfeit is a lollies, the prize for < 7 on 2nd roll is b lollies and the prize for double on 1st roll is c lollies, we need any values of a, b and c such that

$-a\times\frac{35}{72} + b\times\frac{25}{72} + c\times\frac{12}{72} = 0$

$-35a + 25b + 12c = 0$

2 $E(N) = 1$

Consider when $n = 1$, it is obvious that $E(N) = 1$.

Consider when $n = 2$, there is a 50% chance that the first person will choose their own name so

$E(N) = 0\times\frac{1}{2} + 2\times\frac{1}{2} = 1$

Consider when $n = 3$. The probability distribution will be

n	0	1	3
$\mathrm{P}(N=n)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

So $\mathrm{E}(N) = 0\times\frac{1}{3}+1\times\frac{1}{2}+3\times\frac{1}{6}=1$

Similarly for $n = 4$.

n	0	1	2	4
$\mathrm{P}(N=n)$	$\frac{9}{24}$	$\frac{8}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

$\mathrm{E}(N) = 0\times\frac{9}{24}+1\times\frac{8}{24}+2\times\frac{6}{24}+4\times\frac{1}{24}=1$

This is essentially the same as in question 6 in exercise 4.1 where these probabilities are tabulated.

3 Obviously $\mathrm{P}(H=1)=p$

$\mathrm{P}(H=2)=p(1-p), \mathrm{P}(H=3)=p(1-p)^2$

$\mathrm{P}(H=h)=p(1-p)^{h-1}$

4 $\mathrm{E}(X)=p+3(1-p)=3-2p$

$$\begin{aligned}\mathrm{Var}(X)&=1^2\times p+3^2(1-p)-(3-2p)^2\\&=p+9-9p-(9-12p+4p^2)\\&=4p-4p^2\end{aligned}$$

$$\begin{aligned}\mathrm{E}(X)&=2\mathrm{Var}(X)\\3-2p&=2(4p-4p^2)\\3-2p&=8p-8p^2\\8p^2-10p+3&=0\\(4p-3)(2p-1)&=0\\p&=\tfrac{1}{2}\text{ or }\tfrac{3}{4}\end{aligned}$$

5 Permutations and combinations

5.1 Permutations and factorials

1 **(i)** $6! = 720$

(ii) $5! = 120$

(iii) $2 \times 5! = 240$

(iv) $2! \times 5! = 240$

(v) $6 \times 5 \times 4$ or ${}^6P_3 = 120$

2 **(i)** $\frac{5!}{2!} = 60$

(ii) $\frac{2}{5}\times 60 = 24$ or $\frac{4!}{2!}\times 2 = 24$

(iii) $\frac{3!}{2!}\times 3! = 18$

3 **(i)** $\underline{5}\times\underline{5}\times\underline{5}\times\underline{5}\times\underline{5}=3125$

(ii) $\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1} = 5! = {}^5P_5 = 120$

(iii) $\underline{5}\times\underline{4}\times\underline{3} = {}^5P_3 = 60$

(iv) $\underline{4}\times\underline{3}\times\underline{2}\times\underline{2}=48$

(v) $\underline{3}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{2} = 36$

4 **(i)** There are 8 letters in the word, with three Es and two Ss. So the number of arrangements is $\frac{8!}{3!\times 2!} = 3360$

(ii) There are 6 digits, with two 1s and two 9s, so the number of arrangements is $\frac{6!}{2!\times 2!} = 180$

5 **(i)** $\frac{6!}{2!\times 2!} = 180$

(ii) $\frac{5!}{2!} = 60$

(iii) $180 - 60 = 120$

6 $2\times 2\times 2 = 8$

7 **(i)** If you treat John and Rachel as a single 'object' there are now 5 objects to arrange which can be done in 5! ways. In each of these arrangements, John and Rachel could swap around, JR A B C D, so the final answer is:

$5! \times 2 = 240$

(ii) To find the number of ways where they are *not* together, subtract the number of ways they *are* together from the total number of ways the group of 6 could be arranged:

$$6! - (5! \times 2) = 480$$

8 **(i)** $\frac{13!}{8!\times 3!} = 25\,740$

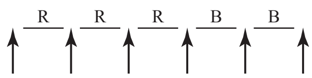
(ii) $\frac{6!}{3!} = 120$

(iii) $\frac{11!}{8!} = 990$

9 **(i)** $\frac{8!}{3!3!2!} = 560$

(ii) Grouping the red blocks as one object, you now have 6 objects – the red blocks, three green blocks and two blue blocks $\frac{6!}{3!2!} = 60$

(iii) First you can arrange the 3 red and 2 blue blocks in $\frac{5!}{3!\times 2!} = 10$ ways. In each of those arrangements we can place the green blocks in the 'gaps' in $\frac{6\times5\times4}{3!} = 20$ ways



So the total number of ways is $10\times 20 = 200$

(iv) You need to consider the different cases of the top and bottom blocks being green, red or blue.

If the top and bottom blocks are green (or red), you have 6 blocks to arrange in the middle, with two identical blocks and three identical blocks in the middle.

Number of ways with green top and bottom

$= \frac{6!}{3!2!} = 60$

If the top and bottom blocks are red, we will also have 60 arrangements.

If the top and bottom blocks are blue, we will have 6 blocks to arrange in the middle – 3 identical red and three identical green blocks.

Number of ways with blue blocks top and bottom $= \frac{6!}{3!3!} = 20$

Total number of arrangements $= 60 + 60 + 20 = 140$

Another method involves finding the probability that the arrangement will have two blocks at the top and bottom the same colour, then multiplying this answer by the total number of arrangements.

P(same top and bottom) = P(both green or red or blue)

$= \frac{3}{8} \times \frac{2}{7} + \frac{3}{8} \times \frac{2}{7} + \frac{2}{8} \times \frac{1}{7} = \frac{7}{28}$

Number of ways $= \frac{7}{28} \times 560 = 140$

(v) Grouping red blocks as one object, there are now 6 objects and $\frac{6!}{3!2!} = 60$ arrangements. Of those 60 arrangements, there are $\frac{5!}{3!} = 20$ arrangements where the blue blocks are together, so the number of arrangements where the three red blocks are together and the two blue blocks are not next to each other is $60 - 20 = 40$.

10 **(i)** $6^4 = 1296$

(ii) $6 \times 5 \times 4 \times 3 = {}^6P_4 = 360$

(iii) Total number – number with *no* red

$= 1296 - 5^4 = 671$

(iv) There are 6 colours to choose from for the ends, then 6 choices for both middle spots.

$6 \times 6 \times 6 = 216$

5.2 Combinations

1 **(i)** ${}^9C_3 = 84$

(ii) ${}^4C_3 = 4$ (or ${}^4C_3 \times {}^5C_0 = 4$)

(iii) ${}^5C_2 \times {}^4C_1 = 40$

(iv) 1 woman or 2 women or 3 women

$= {}^4C_1 \times {}^5C_2 + {}^4C_2 \times {}^5C_1 + {}^4C_3 \times {}^5C_0$

$= 40 + 30 + 4$

$= 74$

OR 84 – no. ways with no women

$= 84 - {}^5C_3$

$= 84 - 10$

$= 74$

2 **(i)** ${}^{15}C_{11} = 1365$

(ii) ${}^7C_6 \times {}^6C_4 \times {}^2C_1 = 210$

(iii) ${}^{14}C_{10} = 1001$

(iv) ${}^2C_1 \times {}^{13}C_{10} = 572$

3 ${}^9C_5 = {}^9C_4 = 126$

The group of five could go in either car:

$126 \times 2 = 252$

4 4 different colours: $5 \times 4 \times 3 \times 2 = 5! = 120$

3 different colours:

First choose the colours 5 ways.

If the colours chosen were B, W, R:

2B 1W 1R or 1B 2W 1R or 1B 1W 2R

each in $\frac{4!}{2!} = 12$ arrangements, total 36 ways.

Total for 3 different colours $= 36 \times {}^5C_3 = 360$

2 different colours:

First choose the colours in ${}^5C_2$ ways

If the colours chosen were blue and green we could have

BBBG in $\frac{4!}{3!} = 4$, GGGB in $\frac{4!}{3!} = 4$ arrangements

BBGG in $\frac{4!}{2!2!} = 6$, total of 14 arrangements

Total for 2 different colours $= {}^5C_2 \times 14 = 140$

Bronson only has 3 pegs of any colour, so he cannot choose all 4 the same.

Total number $= 120 + 360 + 140 = 620$

5 **(i)** ${}^7C_4 = 35$

(ii) 2 or 3 tropical fruits

$= {}^3C_2 \times {}^4C_2 + {}^3C_3 \times {}^4C_1 = 22$

6 **(i)** $3! \times 2! \times 5! \times 4! = 34\,560$

(ii) $7! \times 8 \times 7 \times 6 \times 5 = 8\,467\,200$

(iii) ${}^2C_1 \times {}^5C_1 \times {}^4C_1 = 40$

(iv) 2H 2F 1N or 2H 1F 2N or 1H 2F 2N or 1H 3F 1N or 1H 1F 3N

$$\left({}^2C_2 \times {}^5C_2 \times {}^4C_1\right) + \left({}^2C_2 \times {}^5C_1 \times {}^4C_2\right) + \left({}^2C_1 \times {}^5C_2 \times {}^4C_2\right) + \left({}^2C_1 \times {}^5C_3 \times {}^4C_1\right) + \left({}^2C_1 \times {}^5C_1 \times {}^4C_3\right)$$

$= 40 + 30 + 120 + 80 + 40$

$= 310$

5.3 Probability with arrangements

1 **(i)** $\dfrac{{}^3C_2}{{}^7C_2} = \dfrac{3}{21} = \dfrac{1}{7}$

(ii) $1 - \text{P(no funk CDs)}$

$= 1 - \dfrac{{}^3C_0 \times {}^4C_2}{{}^7C_2} = 1 - \dfrac{6}{21} = \dfrac{5}{7}$

(iii) $\dfrac{{}^1C_1 \times {}^6C_1}{{}^7C_2} = \dfrac{6}{21} = \dfrac{2}{7}$

2 $\dfrac{3! \times 2! \times 2! \times 2!}{6!} = \dfrac{48}{720} = \dfrac{1}{15}$

3 **(i)** $\dfrac{\frac{4!}{2!} \times 2}{\frac{5!}{2!}} = \dfrac{24}{60} = \dfrac{2}{5}$ OR $\dfrac{\frac{2}{5} \times \frac{5!}{2!}}{\frac{5!}{2!}} = \dfrac{2}{5}$

OR

There are two numbers out of the 5 that are odd so P(end in odd) $= \dfrac{2}{5}$

(ii) Number of ways starting with 2:

$2\_\_\_\_ = 4! = 24$

Number of ways starting with 4:

$4\_\_\_\_ = \dfrac{4!}{2!} = 12$

Total number of ways = 36

P(start with an even number) $= \dfrac{36}{60} = \dfrac{3}{5}$

OR

There are 5 numbers, 3 of them are even so

P (start with an even number) $= \dfrac{3}{5}$

(iii) To be greater than 30 000, the number must start with a 3 or 4 (2 choices)

$\text{P}(> 30\,000) = \dfrac{2 \times \frac{4!}{2!}}{60} = \dfrac{24}{60} = \dfrac{2}{5}$

4 **(i)** $\text{P(no boys)} = \dfrac{{}^4C_0 \times {}^5C_4}{{}^9C_4} = \dfrac{5}{126}$

(ii) P(at least 1 girl) = 1 – P(no girls)

$= 1 - \dfrac{{}^5C_0 \times {}^4C_4}{{}^9C_4}$

$= \dfrac{125}{126}$

(iii) $\dfrac{{}^2C_2 \times {}^7C_2}{{}^9C_4} = \dfrac{21}{126} = \dfrac{1}{6}$

(iv) $\dfrac{{}^2C_1 \times {}^7C_3}{{}^9C_4} = \dfrac{70}{126} = \dfrac{5}{9}$

(v) The girls can be arranged in 5! ways and the 4 boys go in the 'gaps' at the two ends and between the girls.

$\dfrac{5! \times 6 \times 5 \times 4 \times 3}{9!} = \dfrac{43\,200}{362\,880} = \dfrac{5}{42} = 0.119$

5 **(i)** $\dfrac{5!}{2!} = 60$

(ii) $\dfrac{4!}{60} = \dfrac{2}{5} = 0.4$

(iii) P(E first) + P(E second)

$= \dfrac{2}{5} \times \dfrac{3}{4} \times 1 \times 1 \times 1 + \dfrac{3}{5} \times \dfrac{2}{4} \times 1 \times 1 \times 1$

$= \dfrac{3}{5}$

OR

$= \dfrac{\frac{2 \times 3 \times 3 \times 2 \times 1}{2!} + \frac{3 \times 2 \times 3 \times 2 \times 1}{2!}}{60}$

$= \dfrac{3}{5}$

6 **(i)** $\dfrac{{}^6C_6}{{}^{40}C_6} = \dfrac{1}{3838380} \approx 0.000\,000\,261$ (3 s.f.)

(ii) $\dfrac{{}^6C_3 \times {}^{34}C_3}{{}^{40}C_6} = \dfrac{119\,680}{3\,838\,380} \approx 0.0312$ (3 s.f.)

(iii) P(at least one even number)

= 1 – P(no even numbers)

$= 1 - \dfrac{{}^{20}C_6}{{}^{40}C_6}$

$= 1 - \dfrac{38760}{3838380} \approx 0.990$ (3 s.f.)

(iv) $\dfrac{{}^3C_3}{{}^{37}C_3} = \dfrac{1}{7770} \approx 0.000\,129$ (3 s.f.)

7 **(i)** $\dfrac{6}{6^5} = \dfrac{1}{1296} \approx 0.000\,772$ (3 s.f.)

(ii) $\dfrac{{}^5C_4 \times 6 \times 5}{6^5} = \dfrac{150}{7776} = \dfrac{25}{1296} \approx 0.0193$ (3 s.f.)

(iii) $\dfrac{2\times 5!}{6^5} = \dfrac{5}{162} \approx 0.0309$ (3 s.f.)

(iv) There are 3 possible ways to get two 6s in at most two rolls.

6 = getting a six

6′ = *not* getting a six

(66) or (66′ then 6) or (6′6′ then 66)

$= \left(\frac{1}{6}\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\right)$

$+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$

$= 0.0934$ (3 s.f.)

8 **(i)** P(Full house)

$= \dfrac{{}^{13}C_1\times{}^{4}C_3\times{}^{12}C_1\times{}^{4}C_2}{{}^{52}C_5} = \dfrac{3744}{2\,598\,960}$

$\approx 0.001\,44$ (3 s.f.)

(ii) P(Two pairs)

$= \dfrac{{}^{13}C_2\times\left({}^{4}C_2\right)^2\times{}^{11}C_1\times{}^{4}C_1}{{}^{52}C_5} = \dfrac{123\,552}{2\,598\,960}$

≈ 0.0475 (3 s.f.)

(iii) P(Four of a kind)

$= \dfrac{{}^{13}C_1\times{}^{4}C_4\times{}^{12}C_1\times{}^{4}C_1}{{}^{52}C_5} = \dfrac{624}{2\,598\,960}$

$\approx 0.000\,240$ (3 s.f.)

(iv) P(Royal flush)

$= \dfrac{{}^{4}C_1}{{}^{52}C_5} = \dfrac{4}{2\,598\,960} \approx 0.000\,001\,54$ (3 s.f.)

Further practice

1 **(i)** You must consider that the two 5s are identical.

The answer is $\dfrac{5!}{2!} = 60$

(ii) If the number ends with the 3, the number of arrangements is $2\times\dfrac{3!}{2!}\times 1 = 6$

If the number ends in one of the 5s, the number of arrangements is $2\times\dfrac{3!\times 2}{2!} = 12$

So the final answer is 6 + 12 = 18

2 **(i)** P(one of each)

$= \dfrac{{}^{5}C_1\times{}^{3}C_1\times{}^{4}C_1}{{}^{12}C_3} = \dfrac{3}{11} \approx 0.273$ (3 s.f.)

(ii) P(one apple) $= \dfrac{{}^{5}C_1\times{}^{7}C_2}{{}^{12}C_3} = \dfrac{105}{220} = \dfrac{21}{44}$

(iii) $P(X=0) = \dfrac{{}^{5}C_0\times{}^{7}C_3}{{}^{12}C_3} = \dfrac{7}{44}$

$P(X=2) = \dfrac{{}^{5}C_2\times{}^{7}C_1}{{}^{12}C_3} = \dfrac{7}{22}$

$P(X=3) = \dfrac{{}^{5}C_3\times{}^{7}C_0}{{}^{12}C_3} = \dfrac{1}{22}$

OR

$P(X=3) = 1-\left[\dfrac{7}{44}+\dfrac{21}{44}+\dfrac{7}{22}\right] = \dfrac{1}{22}$

x	0	1	2	3
P($X = x$)	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{7}{22}$	$\frac{1}{22}$

3 **(i)** 9! = 362 880

(ii) There are now 6 separate groups:

$\underline{B_1\ B_2\ B_3\ B_4}$ $\underline{G_1}$ $\underline{G_2}$ $\underline{G_3}$ $\underline{G_4}$ $\underline{G_5}$

These 6 groups can be arranged in 6! ways. In each of these ways, the boys can be arranged in 4! ways. The total number of arrangements is: 6! × 4! = 17 280

(iii)

G_1		B_1		G_2		B_2		G_3		B_3		G_4		B_4		G_5
5	×	4	×	4	×	3	×	3	×	2	×	2	×	1	×	1

The 5 girls can be arranged in 5! ways, the 4 boys can be arranged in 4! ways. The number of arrangements is:

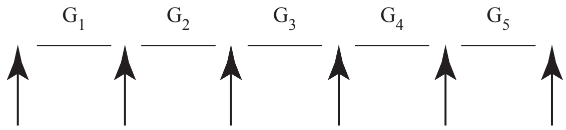
5! × 4! = 2880

(iv) There is one choice for the first position, the rest of the students can be arranged in 8! = 40 320 ways.

Front $\underline{1}$ × $\underline{8}$ × $\underline{7}$ × $\underline{6}$ × $\underline{5}$ × $\underline{4}$ × $\underline{3}$ × $\underline{2}$ × $\underline{1}$

(v) It would be wrong to say the answer is the difference between (i) and (ii) … this leaves other cases where 2 boys are standing next to each other and the others are separated.

The way to approach this problem is to slot the boys in the gaps between the girls:



The girls can be arranged in 5! ways. There are 6 spots for the first boy to go, 5 spots for the second, 4 for the third and 3 for the fourth boy. The total number of arrangements is:

5! × 6 × 5 × 4 × 3 = 43 200

4 (i) The first person has 8 seats to choose from, then the second has 7 to choose from and so on.
$^{8}P_7 = 40\,320$

(ii) There are 2 choices for the driver seat, leaving 6 other people to fill the 7 remaining seats.
$2 \times {}^{7}P_6 = 2 \times 5040 = 10\,080$

(iii) Find the number of ways where they are next to each other and subtract from the total.
There are two choices for the driver seat and four choices of two seats together. In each of these four arrangements, the two children can be interchanged.
Number of ways together =
$2! \times (4 \times 2) \times {}^{5}P_4 = 1920$
Number of ways apart = 40320 − 1920 = 38 400

(iv) There are 5 window seats, so the number of ways the girls can be seated at a window is $5 \times 4 = 20$.
The driver seat is taken, which leaves 5 seats left to choose for the other 4 family members in $^{5}P_4 = 120$ ways.
Answer is $20 \times 120 = 2400$

5 (i) $^{7}C_4 = 35$

(ii) There are now three places left on the team and 6 people to choose from so $^{6}C_3 = 20$

(iii) Choose the boys and girls separately and multiply to get the total number of ways
$^{4}C_2 \times {}^{3}C_2 = 18$

(iv) More boys than girls means 3 boys *or* 4 boys in the team. The word *or* here means we add the number of combinations for each case:
No. teams with 3 boys = $^{4}C_3 \times {}^{3}C_1 = 12$
No. teams with 4 boys = $^{4}C_4 \times {}^{3}C_0 = 1$
Total number of teams = 12 + 1 = 13

6 (i)

S	SUV	U	
3	1	1	$^{8}C_3 \times {}^{4}C_1 \times {}^{6}C_1 = 1344$
2	2	1	$^{8}C_2 \times {}^{4}C_2 \times {}^{6}C_1 = 1008$
2	1	2	$^{8}C_2 \times {}^{4}C_1 \times {}^{6}C_2 = 1680$
1	2	2	$^{8}C_1 \times {}^{4}C_2 \times {}^{6}C_2 = 720$

Total = 1344 + 1008 + 1680 + 720 = 4752

(ii) 4 objects to arrange, then the sedans and the trucks can be swapped.
$4! \times 2! \times 2! = 96$

(iii) Sedans at end 2!
3! to arrange vehicles in the centre
6 possible ways to place the block of vehicles between the spaces
$6 \times 3! \times 2! = 72$

7 (i)

C	D	K	
3	2	2	$^{6}C_3 \times {}^{4}C_2 \times {}^{3}C_2 = 360$
2	3	2	$^{6}C_2 \times {}^{4}C_3 \times {}^{3}C_2 = 180$
2	2	3	$^{6}C_2 \times {}^{4}C_2 \times {}^{3}C_3 = 90$

Total = 360 + 180 + 90 = 630

(ii) $3! \times 3! \times 2! \times 2! = 144$

(iii) $4! \times 5 \times 4 \times 3 = 1440$

Past exam questions

1 (i) $9! = 362\,880$

(ii) Number of ways with pink and green together
$= 8! \times 2 = 80\,640$
Number of ways apart
$= 362\,880 - 80\,640$
$= 28\,2240$

(iii) $^{9}P_3 = 504$ OR $^{9}C_3 \times 3! = 504$

(iv) $^{1}C_1 \times {}^{8}C_2 \times 3! = 168$

(v) Number of ways with pink next to green
= PG_ or GP_ or _ PG or _ GP
$= 7 + 7 + 7 + 7$ (or $7 \times 2! \times 2$)
$= 28$
Number of ways apart = 504 − 28 = 476

2 (i) $\dfrac{10!}{5! \times 4!} = 1260$

(ii) (a) $^{8}P_4$ or $^{8}C_4 \times 4! = 1680$

(b) $^{2}C_2 \times {}^{6}C_2 \times 4! = 360$

(iii)

A	B	C	
7	1	1	$^{9}C_7 \times {}^{2}C_1 \times 3 = 216$
5	3	1	$^{9}C_5 \times {}^{4}C_3 \times {}^{1}C_1 \times 3! = 3024$
3	3	3	$^{9}C_3 \times {}^{6}C_3 \times {}^{3}C_3 = 1680$

Total = 216 + 3024 + 1680 = 4920

3 (i) $^{15}P_5 = 360\,360$

(ii) $5 \times 10 \times 4 \times 9 \times 3 = 5400$

(iii) Number of ways with 3 or 4 or 5 males
$= {}^{5}C_3 \times {}^{10}C_2 + {}^{5}C_4 \times {}^{10}C_1 + {}^{5}C_5 \times {}^{10}C_0$
$= 450 + 50 + 1$
$= 501$

(iv) Need a further 3 performers.
Number of ways with 2 or 3 males from the remaining 13 people – 9 female and 4 male
$= {}^{4}C_2 \times {}^{9}C_1 + {}^{4}C_3 \times {}^{9}C_0$
$= 54 + 4$
$= 58$

4 **(i)** $\dfrac{9!}{4!\times 2!} = 7560$

(ii) $2\times\dfrac{7!}{4!\times 2!} = 210$

(iii) Group all the Es as one object, there are now 6 objects.

$\dfrac{6!}{2!} = 360$

(iv) $({}^4C_0 \times {}^2C_1 \times {}^3C_2) \div 2 = 3$

(v) Number with no Rs: ${}^3C_3 = 1$

Number with one R: 3 (see (**iv**))

Number with two Rs: ${}^2C_2 \times {}^3C_1 = 3$

Total = 7

5 **(i)** Number of ways with Book A at ends

$= \dfrac{9!}{5!\times 2!\times 2!} = 756$

Number of ways with Book B at ends

$= \dfrac{9!}{4!\times 5!} = 126$

Total ways = 756 + 126 = 882

(ii) With all the Book A together there are 6 objects with the As and Cs

↑ AAAA ↑ C ↑ C ↑ C ↑ C ↑ C ↑

7 gaps for the two B's

Number of arrangements is

$\dfrac{6!}{5!}\times {}^7C_2 = 126$

Stretch and challenge

1

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$$

$$\begin{aligned}
LHS &= \frac{n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!}\\
&= \frac{n!(r+1)+n!(n-r)}{(n-r)!(r+1)!}\\
&= \frac{n!r+n!+n!n-n!r}{(n-r)!(r+1)!}\\
&= \frac{n!(n+1)}{(n-r)!(r+1)!}\\
&= \frac{(n+1)!}{((n+1)-(r+1))!(r+1)!}\\
&= \binom{n+1}{r+1}\\
&= RHS
\end{aligned}$$

2 For any one of the players, there are 11 opponents he could play. Consider a third player, there are 9 people he could play. Continuing like this we get

$11\times 9\times 7\times 5\times 3\times 1 = 10\,395$

OR

$\dfrac{12!}{6!\times 2^6} = 10395$

There are 12! ways to arrange all 12 players in an imaginary line, as 6 pairs; but there's a lot of double counting, as the 6! arrangements of the pairs are equivalent, and so are the swaps of each pair, hence 2^6

3 Let A be the event that at least two people share the same birthday.

A' is the event that no one shares the same birthday.

$$\begin{aligned}
P(A) &= 1-P(A')\\
&= 1-\left(\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\ldots\times\frac{365-k+1}{365}\right)\\
&= 1-\frac{\frac{365!}{(365-k)!}}{365^k}\\
&= 1-\frac{365!}{365^k(365-k)!}
\end{aligned}$$

6 Discrete probability distributions

6.1 The binomial distribution

1 **(i)** Not appropriate, the number of trials is not fixed.

(ii) Appropriate, $X \sim B(3, 0.5)$

(iii) Not appropriate, the number of trials is not fixed.

(iv) Not appropriate, the probability changes at each trial.

(v) Appropriate, e.g. $X \sim B(20, 0.9)$

(vi) Appropriate, $X \sim B\left(5, \frac{1}{6}\right)$

(vii) Could be appropriate if P(1st) is constant in each race e.g. $X \sim B(10, 0.2)$

(viii) Not appropriate, breakage of eggs would not be independent.

2 **(i)** 0.6

(ii) $0.4 \times 0.4 = 0.16$

(iii) $C \sim B(5, 0.4)$

(iv) $P(C=3) = \binom{5}{3}\times 0.4^3 \times 0.6^2 = 0.2304$

(v)
$$\begin{aligned}
P(C \leqslant 2) &= P(C=0)+P(C=1)+P(C=2)\\
&= \binom{5}{0}\times 0.4^0\times 0.6^5+\binom{5}{1}\times 0.4^1\times 0.6^4\\
&\quad+\binom{5}{2}\times 0.4^2\times 0.6^3\\
&= 0.077\,76+0.2592+0.3456\\
&= 0.682\,56 = 0.683\text{ (3 s.f.)}
\end{aligned}$$

(vi) $P(C \geqslant 1) = 1 - P(C = 0)$

$$= 1 - \binom{5}{0} \times 0.4^0 \times 0.6^5$$

$$= 0.922 \text{ (3 s.f.)}$$

(vii) $\binom{n}{3} \times 0.4^3 \times 0.6^{n-3} = \binom{n}{4} \times 0.4^4 \times 0.6^{n-4}$

$$\frac{n!}{3!(n-3)!} \times 0.4^3 \times 0.6^{n-3} = \frac{n!}{4!(n-4)!} \times 0.4^4 \times 0.6^{n-4}$$

$$\frac{4!}{3!} \times \frac{0.6^{n-3}}{0.6^{n-4}} = \frac{0.4^4}{0.4^3} \times \frac{(n-3)!}{(n-4)!}$$

$$4 \times 0.6 = 0.4 \times (n-3)$$

$$2.4 = 0.4n - 1.2$$

$$3.6 = 0.4n$$

$$n = 9$$

3 **(i)** $P(\text{Holden car}) = \frac{2}{3}$

(ii) Two outcomes – Holden or Ford.

Probability constant – $\frac{2}{3}$ of cars are Holden.

Each car is independent.

Fixed number of trials (10).

(iii) $P(H = 4) = \binom{10}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^6$

$$= 0.0569 \text{ (3 s.f.)}$$

(iv) $P(H \geqslant 8) = P(H = 8 \text{ or } 9 \text{ or } 10)$

$$= \binom{10}{8} \times \left(\frac{2}{3}\right)^8 \times \left(\frac{1}{3}\right)^2 + \binom{10}{9} \times \left(\frac{2}{3}\right)^9 \times \left(\frac{1}{3}\right)^1 + \binom{10}{10} \times \left(\frac{2}{3}\right)^{10} \times \left(\frac{1}{3}\right)^0$$

$$= 0.1951 + 0.0867 + 0.0173$$

$$= 0.299 \text{ (3 s.f.)}$$

4 **(i)** There are two outcomes – blue or white.

The probability is constant – the first ball is replaced.

Each draw is independent.

There are a fixed number of trials – 2 draws.

(ii) $n = 2,\ p = \frac{3}{5} = 0.6$

So $X \sim B(2, 0.6)$

(iii) $P(X = 1) = {}^2C_1 \times 0.6^1 \times (1 - 0.6)^{2-1} = 0.48$

(iv) $P(X \geqslant 1) = P(X = 1) + P(X = 2)$

$$= 1 - P(X = 0)$$

$$= 1 - {}^2C_0 \times 0.6^0 \times (1 - 0.6)^{2-0}$$

$$= 1 - 0.16$$

$$= 0.84$$

5 $\binom{10}{4} \times p^4 \times (1-p)^6 = 2 \times \binom{10}{6} \times p^6 \times (1-p)^4$

$$210 \times \frac{(1-p)^6}{(1-p)^4} = 420 \times \frac{p^6}{p^4}$$

$$(1-p)^2 = 2p^2$$

$$1 - 2p + p^2 = 2p^2$$

$$0 = p^2 + 2p - 1$$

Using the quadratic formula, $p = 0.414$ (3 s.f.)

6 X: no. of thick crust pizzas in next n orders

$X \sim B(n, 0.32)$

$$P(X \geqslant 1) \geqslant 0.99$$

$$1 - P(X = 0) \geqslant 0.99$$

$$P(X = 0) \leqslant 0.01$$

$$\binom{n}{0} 0.32^0 0.68^n \leqslant 0.01$$

$$0.68^n \leqslant 0.01$$

$$n \geqslant 12$$

7 X : no. of puzzles solved each week

$X \sim B(7, 0.85)$

(i) $P(X > 5) = P(X = 6 \text{ or } 7)$

$$= \binom{7}{6}(0.85)^6(0.15)^1 + \binom{7}{7}(0.85)^7(0.15)^0$$

$$= 0.717 \text{ (3 s.f.)}$$

(ii) $(0.717...)^3 = 0.368$ (3 s.f.)

8 X : no. times the train is late on any day

$X \sim B(4, 0.12)$

$P(X > 1) = P(X = 2 \text{ or } 3 \text{ or } 4)$

$$= 1 - P(X = 0 \text{ or } 1)$$

$$= 1 - \left[\binom{4}{0}(0.12)^0(0.88)^4 + \binom{4}{1}(0.12)^1(0.88)^3\right]$$

$$= 1 - 0.9268...$$

$$= 0.0732 \text{ (3 s.f.)}$$

Y : no. days the train is late more than once in the next week

$Y \sim B(7, 0.0732)$

$$P(Y = 2) = \binom{7}{2}(0.0732)^2(1 - 0.0732)^5$$

$$= 0.0769$$

9 **(i)** X: depth of randomly chosen drum

x	8	10
$P(X = x)$	0.75	0.25

$E(X) = 8 \times 0.75 + 10 \times 0.25 = 8.5$

$\mathrm{Var}(X) = 8^2 \times 0.75 + 10^2 \times 0.25 - 8.5^2 = 0.75$

(ii) Either 10” then 10” or 8” then 8”
P(both the same) $= 0.25 \times 0.25 + 0.75 \times 0.75 = 0.625$

(iii) For a depth of 34” we need exactly three 8” drums and one 10” drum
Y : number of 8” drums chosen out of 4
$Y \sim B(4, 0.75)$
$P(Y = 3) = {}^4C_3 \times 0.75^3 \times 0.25^1 = 0.422$ (3 s.f.)

6.2 The expectation and variance of B(n, p)

1 $C \sim B(8, 0.2)$

(i) $E(C) = 8 \times 0.2 = 1.6$

(ii) $Var(C) = 8 \times 0.2 \times (1 - 0.2) = 1.28$

2 $H \sim B(40, 0.9)$

(i) $E(H) = 40 \times 0.9 = 36$

(ii) $\sigma(H) = \sqrt{40 \times 0.9 \times (1 - 0.9)} = 1.90$ (3 s.f.)

3 **(i)** $n = 3$ (3 balls drawn) $p = \frac{4}{10} = 0.4$

(ii) $E(X) = 3 \times 0.4 = 1.2$

(iii) $Var(X) = 3 \times 0.4 \times (1 - 0.4) = 0.72$

(iv) $\sigma(X) = \sqrt{0.72} = 0.849$ (3 s.f.)

4
$$np = 4$$
$$np(1 - p) = 3.92$$
$$\Rightarrow 4(1 - p) = 3.92$$
$$(1 - p) = \frac{3.92}{4}$$
$$p = 0.02 = 2\%$$
$$n = 200$$

5 X: no. broken eggs in a carton
$X \sim B(12, p)$
$E(X) = np = 0.48 \Rightarrow 12p = 0.48 \Rightarrow p = 0.04$
$P(1 \leqslant X \leqslant 3)$
$$= \binom{12}{1}0.04^1 0.96^{11} + \binom{12}{2}0.04^2 0.96^{10} + \binom{12}{3}0.04^3 0.96^9$$
$= 0.386$ (3 s.f.)

6.3 The geometric distribution

1 **(i)** $X \sim B(5, 0.5)$

(ii) $X \sim Geo(0.5)$

(iii) $X \sim Geo(0.5)$

(iv) $X \sim B(10, 0.5)$

(v) $X \sim B(8, 0.05)$

(vi) $X \sim Geo(0.05)$

(vii) $X \sim Geo\left(\frac{1}{6}\right)$

2 **(i)** X: number of attempts for first success
$X \sim Geo(0.8)$
$P(X = 3) = (1 - 0.8)^2\, 0.8 = 0.032$

(ii) P (at most four attempts)
= 1– P (fails on all of first four attempts)
$= 1 - 0.2^4$
$= 0.998$ (3 s.f.)

3 X: number of children until they have a girl
$X \sim Geo(0.5)$

(i) $P(X = 4) = (1 - 0.5)^3\, 0.5 = 0.0625$

(ii) $E(X) = \frac{1}{0.5} = 2$

4 X: number of tests needed to pass $X \sim Geo(0.7)$

(i) $P(X = 2) = (1 - 0.7)^1\, 0.7 = 0.21$

(ii) $E(X) = \frac{1}{0.7} = 1.43$ (3 s.f.)

5 **(i)** $E(Y) = \frac{1}{p} \Rightarrow 4 = \frac{1}{p} \Rightarrow p = 0.25$

(ii) $P(Y = 6) = (1 - 0.25)^5\, 0.25 = 0.0593$ (3 s.f.)

(iii) $P(Y = 8 \mid Y \geqslant 4) = P(Y > 3) = P(Y = 5)$
$= (1 - 0.25)^4\, 0.25 = 0.0791$ (3 s.f.)

6 $X \sim Geo(0.05)$

(i) $E(X) = \frac{1}{0.05} = 20$

(ii) $P(X = 10) = (1 - 0.05)^9\, 0.05 = 0.0315$ (3 s.f.)

7 X: number of rolls to get a double $X \sim Geo\left(\frac{1}{6}\right)$

(i) $P(X = 3) = \left(1 - \frac{1}{6}\right)^2 \times \frac{1}{6} = 0.116$ (3 s.f.)

(ii) P(At most three attempts) $= P(X \leqslant 3)$
= 1 – P(fails on all of first three attempts)
$= 1 - \left(\frac{5}{6}\right)^3 = 0.421$ (3 s.f.)

Further practice

1 **(i)** $E(X) = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} = \frac{7}{8}$

$Var(X)$
$$= 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{8} + 3^2 \times \frac{1}{8} - \left(\frac{7}{8}\right)^2$$
$= \frac{71}{64} \approx 1.11$ (3 s.f.)

(ii) X: number of of 1s from 10 throws, $X \sim B\left(10, \frac{1}{4}\right)$
$P(X \leq 3)$
$= P(X = 0, 1, 2, 3)$
$$= \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9 + \binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8 + \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7$$
$= 0.776$ (3 s.f.)

(iii) X: number of 2s from 10 throws, $X \sim B\left(10, \frac{1}{8}\right)$

$$P(X=3) = \binom{10}{3}\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^7$$
$$= 0.0920 \text{ (3 s.f.)}$$

2 $np = 18$ and $np(1-p) = 12$

$18(1-p) = 12$

$$1 - p = \frac{12}{18}$$
$$1 - p = \frac{2}{3}$$
$$p = \frac{1}{3}$$
$$n \times \frac{1}{3} = 18 \Rightarrow n = 54$$

3 X: number of questions answered until the first correct answer

$X \sim \text{Geo}(0.25)$

$P(X=6) = (1-0.25)^5\,0.25 = 0.0593$ (3 s.f.)

4 (i) Possible multiples of 4:

8, 12, 16, 20, 24 out of 20 possible years.

Since each is equally likely,

$P(\text{multiple of } 4) = \frac{5}{20} = \frac{1}{4} = 0.25$

$X \sim B(8, 0.25)$

(ii) $P(2 \leqslant X \leqslant 4)$

$$= \binom{8}{2}0.25^2 0.75^6 + \binom{8}{3}0.25^3 0.75^5 + \binom{8}{4}0.25^4 0.75^4$$

$= 0.606$ (3 s.f.)

(iii) $X \sim B(n, 0.25)$

$$P(X=0) < 0.02 \Rightarrow \binom{n}{0}0.25^0 0.75^n < 0.02$$
$$\Rightarrow 0.75^n < 0.02$$
$$\Rightarrow n > 13.6$$

The least possible value of n is 14.

5 (i) $P(X=4) = (1-0.4)^3\,0.4 = 0.0864$

(ii) $P(X>4) = P(X \geqslant 5)$
$$= 1 - P(X \leqslant 4)$$
$$= 1 - (1 - 0.6^4)$$
$$= 0.6^4$$
$$= 0.1296$$

6 (i) $P(\text{Win on any spin}) = \frac{2}{10} = \frac{1}{5} = 0.2$

X: number of times she spins a number greater than 8 in 6 spins

$X \sim B(6, 0.2)$

$$P(X>1) = 1 - P(X = 0,1)$$
$$= 1 - \left[\binom{6}{0}0.2^0 0.8^6 + \binom{6}{1}0.2^1 0.8^5\right]$$
$$= 1 - 0.65536$$
$$= 0.345 \text{ (3 s.f.)}$$

(ii) $P(1 \text{ spin} \leqslant k) = \frac{k}{10}$

$X \sim B\left(n, \frac{k}{10}\right)$

$$E(X) = 30 \Rightarrow \frac{nk}{10} = 30$$
$$\text{Var}(X) = 12 \Rightarrow \frac{nk}{10}\left(1 - \frac{k}{10}\right) = 12$$
$$\Rightarrow 30\left(1 - \frac{k}{10}\right) = 12$$
$$\Rightarrow \left(1 - \frac{k}{10}\right) = \frac{12}{30}$$
$$\Rightarrow \frac{k}{10} = \frac{3}{5}$$
$$\Rightarrow k = 6$$

Substituting into $\frac{nk}{10} = 30$,

$$\frac{6n}{10} = 30 \Rightarrow n = 50$$

7 (i) X: number of of flawed glasses in sample of 10

$X \sim B(10, 0.08)$

$$P(X=1) = \binom{10}{1}0.08^1 0.92^9 = 0.378 \text{ (3 s.f.)}$$

(ii) $X \sim \text{Geo}(0.08)$

$P(X=5) = (1-0.08)^{-4} \times 0.08 = 0.0573$ (3 s.f.)

(iii) Y: number of boxes with at least one flawed glass

$$P(X=0) = \binom{10}{0}0.08^0 0.92^{10} = 0.434 \text{ (3 s.f.)}$$

$P(X \geqslant 1) = 1 - P(X=0) = 0.566$ (3 s.f.)

$Y \sim B(50, 0.566)$

$$P(Y=20) = \binom{50}{20}0.566^{20}(1-0.566)^{30}$$
$$= 0.00724 \text{ (3 s.f.)}$$

(iv) In 50 boxes there are 500 glasses.

Z: number of flawed glasses in 50 boxes.

$Z \sim B(500, 0.08)$

$$P(Z=20) = \binom{500}{20}0.08^{20}0.92^{480} = 0.000128 \text{ (3 s.f.)}$$

Past exam questions

1 X: number of puzzles completed
$X \sim \mathrm{B}(n, 0.75)$

(i) $\mathrm{P}(X = n) < 0.06$
$0.75^n < 0.06$
By guess and check, $n = 10$

(ii) $X \sim \mathrm{B}(14, 0.75)$
$\mathrm{E}(X) = 14 \times 0.75 = 10.5$
$$\mathrm{P}(X = 10) = \binom{14}{10} \times (0.75)^{10} \times (0.25)^4 = 0.220 \text{ (3 s.f.)}$$
$$\mathrm{P}(X = 11) = \binom{14}{11} \times (0.75)^{11} \times (0.25)^3 = 0.240 \text{ (3 s.f.)}$$
$X = 11$ has the highest probability.

(iii) $\mathrm{P}(X > 11) = \mathrm{P}(X = 12, 13, 14)$
$$= \binom{14}{12} 0.75^{12} 0.25^2 + \binom{14}{13} 0.75^{13} 0.25^1 + \binom{14}{14} 0.75^{14} 0.25^0$$
$= 0.281$ (3 s.f.)

Y: number of months when Sue completes more than 11 puzzles
$Y \sim \mathrm{B}(5, 0.281)$
$$\mathrm{P}(Y = 3) = \binom{5}{3} 0.281^3 (1 - 0.281)^2$$
$= 0.115$ (3 s.f.)

2 **(i)** X: number of fireworks that fail to work
$X \sim \mathrm{B}(20, 0.05)$
$\mathrm{P}(X > 1)$
$= 1 - \mathrm{P}(X = 0 \text{ or } 1)$
$$= 1 - \left[0.95^{20} + \binom{20}{1} 0.05^1 0.95^{19}\right]$$
$= 0.264$ (3 s.f.)

(ii) P: profit for company
Profit if no refund $= 450 \times 10 - 20 \times 24 = \$4020$
Expenditure $= 20 \times 24 = \$480$
Profit if refund $= -\$480$

p	4020	-480
$\mathrm{P}(P = p)$	0.736	0.264

$\mathrm{E}(P) = 4020 \times 0.736 + -480 \times 0.264$
$= \$2832$

3 **(i)** X: number of integers generated $\leqslant 4$
$X \sim \mathrm{B}\left(5, \frac{4}{9}\right)$
$\mathrm{P}(X \geqslant 2)$
$= 1 - \mathrm{P}(X \leqslant 1)$
$$= 1 - \left[\binom{5}{0}\left(\frac{4}{9}\right)^0\left(\frac{5}{9}\right)^5 + \binom{5}{1}\left(\frac{4}{9}\right)^1\left(\frac{5}{9}\right)^4\right]$$
$= 1 - 0.265$
$= 0.735$ (3 s.f.)

(ii) X: number of integers generated $\leqslant k$
$X \sim \mathrm{B}\left(n, \frac{k}{9}\right)$
$$\mathrm{E}(X) = \frac{nk}{9} = 96 \Rightarrow nk = 864$$
$$\mathrm{Var}(X) = n\left(\frac{k}{9}\right)\left(1 - \frac{k}{9}\right) = 32$$
$$\frac{nk}{9}\left(1 - \frac{k}{9}\right) = 32$$
Substituting into 2nd equation,
$$96\left(1 - \frac{k}{9}\right) = 32$$
$$1 - \frac{k}{9} = \frac{1}{3}$$
$$\frac{k}{9} = \frac{2}{3}$$
$$k = 6$$
$$nk = 864 \Rightarrow n \times 6 = 864$$
$$n = 144$$

4 **(i)** $n = 12$ is the greatest number
$\mathrm{P}(X = 12) = 0.7^{12} = 0.0138$ (3 s.f.)

(ii) $\mathrm{P}(X < 10) = \mathrm{P}(X \leqslant 9)$
$= 1 - \mathrm{P}(X = 10, 11, 12)$
$$= 1 - \left[\binom{12}{10}(0.7)^{10}(0.3)^2 + \binom{12}{11}(0.7)^{11}(0.3)^1 + 0.7^{12}\right]$$
$= 1 - 0.2528...$
$= \frac{15}{64} = 0.747$ (3 s.f.)

Stretch and challenge

1 $X \sim \mathrm{B}\left(3, \frac{1}{2}\right)$
$$\mathrm{P}(X = 0) = \left(\frac{1}{2}\right)^3 = 0.125$$
$$\mathrm{P}(X = 1) = \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 0.375$$
$$\mathrm{P}(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 0.375$$

$$\begin{aligned}P(\text{sum} = 2) &= P(0, 2 + P(1, 1) + P(2, 0)\\ &= 0.125\times0.375+0.375\times0.375\\ &\quad +0.375\times0.125\\ &= 0.234 \text{ (3 s.f.)}\end{aligned}$$

2 X: number of patients that need further surgery from a sample of n patients.

$X \sim B(n, 0.02)$

$P(X = 0) \approx 0.7$

$\binom{n}{0}0.02^0 0.98^n = 0.7$

$0.98^n = 0.7$

$n = 17.7$

Sample size of 18

3 **(i)** P(at least one accident in 7 days)

$= 1 - $ P(no accidents in 7 days)

$= 1 - 0.88^7$

$= 0.591$ (3 s.f.)

(ii) X: number of weeks where there is at least one accident on the freeway

$X \sim B(10, 0.591\ldots)$

$P(X \geqslant 3)$

$= 1 - P(X = 0,1,2)$

$$= 1 - \left[\binom{10}{0}0.591^0 0.409^{10} + \binom{10}{1}0.591^1 0.409^9 + \binom{10}{2}0.591^2 0.409^8\right]$$

$= 1 - 0.0143$

$= 0.986$ (3 s.f.)

4 $E(X) = p$

$\mathrm{Var}(X) = p - p^2 = 0.16$

$p^2 - p + 0.16 = 0$

$p = 0.2$ or 0.8

5 X: number of people who fail to arrive

$X \sim B(224, 0.04)$

P(not enough seats)

$= P(X \leqslant 3)$

$$= \binom{224}{0}0.04^0 0.96^{224} + \binom{224}{1}0.04^1 0.96^{223} + \binom{224}{2}0.04^2 0.96^{222} + \binom{224}{3}0.04^3 0.96^{221}$$

$= 0.0200$ (3 s.f.)

The assumption of independence may not be satisfied as it is likely that there will be pairs or groups of people who are late for the same reason.

7 The normal distribution

7.1 Using normal distribution tables

1 **(i)** $P(Z < 1.92)$

$= 0.9726$

(ii) $P(0 \leqslant Z \leqslant 0.765) = 0.7779 - 0.5$

$= 0.2779$

(iii) $P(Z > 1.058)$

$= 1 - 0.8550$ (3 s.f.)

$= 0.1450$ (3 s.f.)

(iv) $P(Z \leqslant -0.257)$

$= 1 - 0.6014$

$= 0.3986$

(v) $P(-1.26 < Z < 2.417)$

$= 0.3962 + 0.4921$

$= 0.8883$

(vi) $P(Z \geqslant -1.172)$

$= 0.8794$

(vii) $P(0.068 < Z \leqslant 1.925)$

$= 0.9729 - 0.5271$

$= 0.4458$

(viii) $P(-1.818 < Z < -0.844)$

$= 0.9655 - 0.8006$

$= 0.1649$

(ix) $P(-0.522 < Z \leqslant 1.263)$

$= 0.3968 + 0.1992$

$= 0.5960$

7.2 Solving normal distribution problems

1 **(i)** $P(X \geqslant 38)$

$= P\left(Z \geqslant \frac{38-30}{\sqrt{25}}\right)$

$= P(Z \geqslant 1.6)$

$= 1 - 0.9452$

$= 0.0548$

(ii) $P(X < 23)$

$= P\left(Z < \frac{23-30}{\sqrt{25}}\right)$

$= P(Z < -1.4)$

$= 1 - 0.9192$

$= 0.0808$

(iii) $P(X \geqslant 21.8)$

$= P\left(Z \geqslant \frac{21.8 - 30}{\sqrt{25}}\right)$

$= P(Z \geqslant -1.64)$

$= 0.9495$

(iv) $P(24.7 \leqslant X < 40.25)$

$= P\left(\frac{24.7 - 30}{\sqrt{25}} \leqslant Z \leqslant \frac{40.25 - 30}{\sqrt{25}}\right)$

$= P(-1.06 \leqslant Z \leqslant 2.05)$

$= 0.4798 + 0.3554$

$= 0.8352$

2 (i) X: waiting time to order

$z = \frac{4 - 3.5}{0.8} = 0.625$

$P(X > 4) = P(Z > 0.625)$

$= 1 - 0.7340$

$= 0.2660$ (3 s.f.)

(ii) $P(X < 2) = P\left(Z < \frac{2 - 3.5}{0.8}\right)$

$= P(Z < -1.875)$

$= 1 - 0.9697$

$= 0.0303$

(iii) $P(2.8 < X < 5.2)$

$= P\left(\frac{2.8 - 3.5}{0.8} < Z < \frac{5.2 - 3.5}{0.8}\right)$

$= P(-0.875 < Z < 2.125)$

$= (0.9832 - 0.5) + (0.8092 - 0.5)$

$= 0.7924$

3 (i) $P(X \geqslant 395)$

$= P\left(Z \geqslant \frac{395 - 400}{3.3}\right)$

$= P(Z \geqslant -1.515)$

$= 0.9351$

(ii) $P(X > 402)$

$= P\left(Z > \frac{402 - 400}{3.3}\right)$

$= P(Z > 0.606)$

$= 1 - 0.7276$

$= 0.2724$

(iii) $P(404 < X < 408)$

$= P\left(\frac{404 - 400}{3.3} \leqslant Z \leqslant \frac{408 - 400}{3.3}\right)$

$= P(1.212 \leqslant Z \leqslant 2.424)$

$= 0.9923 - 0.8873$

$= 0.1050$ (3 s.f.)

4 (i) (a) $P(H < 167)$

$= P\left(Z < \frac{167 - 172}{\sqrt{47}}\right)$

$= P(Z < -0.729)$

$= 1 - 0.7669$

$= 0.2331$

(b) $P(H > 183)$

$= P\left(Z > \frac{183 - 172}{\sqrt{47}}\right)$

$= P(Z > 1.605)$

$= 1 - 0.9457$

$= 0.0543$

(c) $P(155 \leqslant H \leqslant 163)$

$= P\left(\frac{155 - 172}{\sqrt{47}} \leqslant Z \leqslant \frac{163 - 172}{\sqrt{47}}\right)$

$= P(-2.480 \leqslant Z \leqslant -1.313)$

$= 0.9934 - 0.9054$

$= 0.0880$ (3 s.f.)

(ii) $P(H < 180)$

$= P\left(Z < \frac{180 - 172}{\sqrt{47}}\right)$

$= P(Z < 1.167)$

$= 0.8784$

No. students less than 180 cm

$\approx 86 \times 0.8784 = 75.5$

So 75 or 76 students.

5 (i) $P(T < 15)$

$= P\left(Z < \frac{15 - 20}{\sqrt{10}}\right)$

$= P(Z < -1.581)$

$= 1 - 0.9430$

$= 0.0570$ (3 s.f.)

(ii) X: number of times she prepares the meal in under 15 minutes in the next week

$X \sim B(7, 0.057)$

$P(X < 2) = P(X = 0 \text{ or } 1)$

$= \binom{7}{0} 0.057^0 (1 - 0.057)^7 + \binom{7}{1} 0.057^1 (1 - 0.057)^6$

$= 0.9440$ (3 s.f.)

7.3 Inverse normal

1 **(i)** $a = 0.810$ (3 s.f.)

(ii) $b = -0.539$

(iii) $c = 2.054$ or 2.055

(iv) $d = -1.439$

(v) $e = 2.576$

2 **(i)** $P(Z < k') = 0.635 \Rightarrow k' = 0.345$

$$0.345 = \frac{k-12}{2.5}$$

$$k = 12.8625 = 12.9 \text{ (3 s.f.)}$$

(ii) $P(Z \leqslant k') = 0.218 \Rightarrow k' = -0.779$

$$-0.779 = \frac{k-12}{2.5}$$

$$k = 10.0525 = 10.1 \text{ (3 s.f.)}$$

3 $P(Z < k') = 0.03 \Rightarrow k' = -1.881$

$$-1.881 = \frac{k-22}{0.6}$$

$$k = 20.8714 = 20.9 \text{ s (3 s.f.)}$$

4 $P(Z \geqslant k') = 0.04 \Rightarrow k' = 1.751$

$$1.751 = \frac{k-45}{4.8}$$

$$k = 53.4048 = 53.4 \text{ m (3 s.f.)}$$

5 **(i)** $P(Z \geqslant k') = 0.005 \Rightarrow k' = 2.576$

$$2.576 = \frac{k-56}{13}$$

$$k = 89.488 = 89.5\% \text{ (3 s.f.)}$$

(ii) $P(Z < k') = 0.25 \Rightarrow k' = -0.674$

$$-0.674 = \frac{k-56}{13}$$

$$k = 47.238 = 47.2\% \text{ (3 s.f.)}$$

7.4 Finding the mean and variance

1 $P(Z < k') = 0.3 \Rightarrow k' = -0.524$

$$-0.524 = \frac{12-\mu}{5}$$

$\mu = 14.62 = 14.6$ (3 s.f.)

2 $P(Z > k') = 0.9 \Rightarrow k' = -1.282$

$$-1.282 = \frac{6-8}{\sigma}$$

$\sigma = 1.5600 = 1.56$ (3 s.f.)

3 $P(Z < k') = 0.03 \Rightarrow k' = -1.881$

$$-1.881 = \frac{9.5-\mu}{\sqrt{15}}$$

$$\mu = 16.785 = 16.8 \text{ (3 s.f.)}$$

4 $P(Z < z) = 0.843 \Rightarrow z = 1.007$

$$1.007 = \frac{30-\mu}{\sqrt{20}}$$

$$\mu = 25.496 = 25.5 \text{ (3 s.f.)}$$

5 T: time for trip

$P(T > 30) = \frac{1}{5}$

$P(Z > k') = 0.2 \Rightarrow k' = 0.842$

$$0.842 = \frac{30-25}{\sigma}$$

$\sigma = 5.9382 = 5.94$ (3 s.f.)

6 $P(Z > k') = 0.15 \Rightarrow k' = 1.036$

$P(Z < k'') = 0.01 \Rightarrow k'' = -2.326$

$$1.036 = \frac{1.5-\mu}{\sigma} \Rightarrow 1.036\sigma = 1.5-\mu$$

$$-2.326 = \frac{1-\mu}{\sigma} \Rightarrow -2.326\sigma = 1-\mu$$

Subtract the two equations

$3.362\sigma = 0.5$

$\sigma = 0.149$ m (3 s.f.)

$\mu = 1.35$ m (3 s.f.)

7 M: mark in exam

(i) $P(M < 46)$

$$= P\left(Z < \frac{46-56}{13}\right)$$

$= P(Z < -0.769)$

$= 1 - 0.7791$

$= 0.2209 \approx 22.1\%$ (3 s.f.)

(ii) $100\% - 22.1\% = 77.91\%$

$$\frac{77.91\%}{2} = 38.955\%$$

So you want the top 38.955%

$P(Z > k') = 0.38955 \Rightarrow k' = 0.28$

$$0.280 = \frac{m-56}{13}$$

$m = 59.64\% \approx 60\%$

8 X: height of a randomly chosen Year 12 student.

$$X \sim N\mu, \left(\frac{1}{15}\mu\right)^2$$

(i) First find the z value so that $P(Z > z) = 0.04$ or $P(Z < z) = 0.96$.

$$z = 1.751$$

$$1.751 = \frac{182-\mu}{\frac{1}{15}\mu}$$

$$1.751 \times \frac{1}{15}\mu = 182 - \mu$$

$$0.11673\mu = 182 - \mu$$

$$\mu + 0.11673\mu = 182$$
$$1.11673\mu = 182$$
$$\mu = 163\text{ cm (3 s.f.)}$$

(ii) First find the probability that any one student is over 180 cm tall.

$$\sigma = \frac{1}{15}\mu = \frac{1}{15}\times 163 = 10.9\text{ cm}$$

$$\begin{aligned}\mathrm{P}(X > 180) &= \mathrm{P}\left(Z > \frac{180-163}{10.9}\right)\\ &= \mathrm{P}(Z > 1.567)\\ &= 1 - 0.9414\\ &= 0.0586\end{aligned}$$

The problem now *switches* to a binomial distribution problem, where

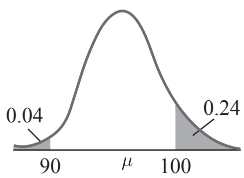
Y: number of students over 180 cm tall in a group of 5. $Y \sim \mathrm{B}(5, 0.0586)$

$$\begin{aligned}&\mathrm{P}(Y > 1)\\ &= 1 - \mathrm{P}(Y = 0 \text{ or } 1)\\ &= 1 - \left[\binom{5}{0}0.0586^0 0.9414^5 + \binom{5}{1}0.0586^1 0.9414^4\right]\\ &= 1 - (0.9695)\\ &= 0.0305 \text{ (3 s.f.)}\end{aligned}$$

9 $S \sim \mathrm{N}(\mu, \sigma^2)$

$\mathrm{P}(S > 100) = 0.24$

$\mathrm{P}(S < 90) = 0.04$



$\mathrm{P}(Z > z_1) = 0.24 \Rightarrow z_1 = 0.707$

$\mathrm{P}(Z < z_2) = 0.04 \Rightarrow z_2 = -1.751$

$$0.707 = \frac{100-\mu}{\sigma} \Rightarrow 0.707\sigma = 100 - \mu$$

$$-1.751 = \frac{90-\mu}{\sigma} \Rightarrow -1.751\sigma = 90 - \mu$$

Subtract the two equations

$$2.458\sigma = 10$$
$$\sigma = 4.07\text{ km/h}$$

Substitute

$$0.707 \times 4.07 = 100 - \mu$$
$$\mu = 97.1\text{ km/h (3 s.f.)}$$

10 $4\sigma^2 = 3\mu \Rightarrow \mu = \dfrac{4\sigma^2}{3}$

$\mathrm{P}(Z < k) = 0.0309 \Rightarrow k = -1.867$

$$\mathrm{P}\left(H < \tfrac{1}{2}\mu\right) = \mathrm{P}\left(Z < \frac{\frac{1}{2}\mu - \mu}{\sigma}\right)$$

$$\frac{\frac{1}{2}\mu - \mu}{\sigma} = \frac{-\frac{1}{2}\mu}{\sigma} = \frac{-\mu}{2\sigma} = \frac{-\frac{4\sigma^2}{3}}{2\sigma} = \frac{-4\sigma}{6}$$

$$\frac{-4\sigma}{6} = -1.867 \Rightarrow \sigma = 2.80$$

$$\mu = \frac{4\sigma^2}{3} = \frac{4\times 2.80^2}{3} = 10.5\text{ m}$$

7.5 Using the normal distribution as an approximation for the binomial distribution

1 **(i)** A: number of shoes of brand A sold

$A \sim \mathrm{B}(6, 0.3)$

B: number of shoes of brand B sold

$B \sim \mathrm{B}(6, 0.7)$

(a) $\mathrm{P}(B = 3) = \binom{6}{3}0.7^3 0.3^3 = 0.185$ (3 s.f.)

(b) $\mathrm{P}(A > 3) = \mathrm{P}(A = 4,5,6)$

$$= \binom{6}{4}0.3^4 0.7^2 + \binom{6}{5}0.3^5 0.7^1 + \binom{6}{6}0.3^6 0.7^0$$

$= 0.0705$ (3 s.f.)

(ii) X: number of shoes of brand B sold in a week

$X \sim \mathrm{B}(48, 0.7)$

$\mu = 48 \times 0.7 = 33.6$

$\sigma^2 = 48 \times 0.7 \times 0.3 = 10.08$

$X \approx \mathrm{N}(33.6, 10.08)$

$$\begin{aligned}&\mathrm{P}(X \geqslant 30)\\ &\approx \mathrm{P}\left(Z \geqslant \frac{29.5-33.6}{\sqrt{10.08}}\right)\\ &= \mathrm{P}(Z \geqslant -1.291)\\ &= 0.9017\end{aligned}$$

2 F: number of flat batteries from 10 sampled

$F \sim \mathrm{B}(10, 0.125)$

(i) $\mathrm{P}(F < 2) = \mathrm{P}(F = 0,1)$

$$= \binom{10}{0}0.125^0 0.875^{10} + \binom{10}{1}0.125^1 0.875^9$$

$= 0.639$ (3 s.f.)

(ii) X : number of flat batteries in a box of 200

$X \sim B(200, 0.125)$

$\mu = 200 \times 0.125 = 25$

$\sigma^2 = 200 \times 0.125 \times 0.875 = 21.875$

$X \approx N(25, 21.875)$

$P(X < 20)$

$\approx P\left(Z \leqslant \frac{19.5 - 25}{\sqrt{21.875}}\right)$

$= P(Z \leqslant -1.176)$

$= 1 - 0.8802$

$= 0.1198$

3 X: number of students who use the internet for more than 24 h per week from the 950 students

$p = \frac{16}{50} = 0.32$

$X \sim B(950, 0.32)$

$\mu = 950 \times 0.32 = 304$

$\sigma^2 = 950 \times 0.32 \times (1 - 0.32) = 206.72$

$X \approx N(304, 206.72)$

$P(275 \leqslant X \leqslant 318)$

$\approx P\left(\frac{274.5 - 304}{\sqrt{206.72}} \leqslant Z \leqslant \frac{318.5 - 304}{\sqrt{206.72}}\right)$

$= P(-2.052 \leqslant Z \leqslant 1.009)$

$= 0.4799 + 0.3434$

$= 0.8233$

4 **(i)** $P(Z < k) = 0.35 \Rightarrow k = -0.385$

$-0.385 = \frac{10 - \mu}{\frac{1}{2}\mu}$

$-0.385 \times \frac{1}{2}\mu = 10 - \mu$

$-0.1925\mu = 10 - \mu$

$0.8075\mu = 10$

$\mu = 12.4$

(ii) $P(X < 2\mu) = P(X < 2 \times 12.4...) = P(X < 24.8...)$

$P(X < 2\mu)$

$= P\left(Z < \frac{2\mu - \mu}{\frac{1}{2}\mu}\right)$

$= P(Z < 2)$

$= 0.977$

(iii) Y: number of observations less than 10

$Y \sim B(120, 0.35)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$\mu = 120 \times 0.35 = 42$

$\sigma = \sqrt{120 \times 0.35 \times (1 - 0.35)} = \sqrt{27.3}$

$Y \approx N(42, 27.3)$

$P(Y < 40) \approx P\left(Z < \frac{39.5 - 42}{\sqrt{27.3}}\right)$

$= P(Z < -0.478)$

$= 1 - 0.6837$

$= 0.316$ (3 s.f.)

5 X: number of drivers who pass at the first attempt

$X \sim B(1200, 0.82)$

$np > 5$ and $n(1 - p) > 5$

So normal approximation is appropriate

$\mu = 1200 \times 0.82 = 984$

$\sigma^2 = 1200 \times 0.82 \times (1 - 0.82) = 177.12$

$X \approx N(984, 177.12)$

$P(X < 950$ becomes $P(X < 949.5)$ with a continuity correction

$P(X < 949.5) \approx P\left(Z < \frac{949.5 - 984}{\sqrt{177.12}}\right)$

$= P(Z < -2.592)$

$= 1 - 0.9952$

$= 0.0048$

Assuming that the probability that any person passes on the first attempt is 0.82, there is only a very small chance (less than 0.5%) that less than 950 people will pass out of 1200.

The figure shows that the testing procedures are too hard or rigorous at this centre.

Even though the percentage who passed is 79% – not much difference from 82%, the probability of 0.0048 shows that there is only an extremely small chance that this would happen.

Further practice

1 **(i)** $C \sim N\left(\mu, \frac{2}{3}\mu\right)$

$P(C > 0) = P\left(Z > \frac{0 - \mu}{\frac{2}{3}\mu}\right)$

$= P\left(Z > -\frac{3}{2}\right)$

$= 0.9332$

(ii) $H \sim N\left(175, \sigma^2\right)$

$$\mathrm{P}(H > 195) = \frac{182}{6000} = 0.0303$$
$$\mathrm{P}(Z > k') = 0.0303 \Rightarrow k' = 1.875$$
$$1.875 = \frac{195 - 175}{\sigma}$$
$$1.875\sigma = 20$$
$$\sigma = 10.7 \text{ cm (3 s.f.)}$$

2 $D \sim \mathrm{N}\mu, \left(\frac{1}{4}\mu\right)^2$

$$\mathrm{P}(Z \leqslant k') = 0.045 \Rightarrow k' = -1.695$$
$$-1.695 = \frac{10 - \mu}{\frac{1}{4}\mu}$$
$$-0.42375\mu = 10 - \mu$$
$$0.57625\mu = 10$$
$$\mu = 17.4 \text{ (3 s.f.)}$$

3 W: weight of randomly chosen apple.

$W \sim \mathrm{N}\left(160, 6^2\right)$

(i) $\mathrm{P}(W < 150) \approx \mathrm{P}\left(Z < \frac{150 - 160}{6}\right)$

$$= \mathrm{P}(Z < -1.667)$$
$$= 1 - 0.9522$$
$$= 0.0478$$

(ii) Proportion of medium and large apples

$= 1 - 0.0478 = 0.9522$

Medium : Large

3 : 1

0.714 : 0.238

$$\mathrm{P}(W > w) = 0.238$$
$$\mathrm{P}(Z > k') = 0.238 \Rightarrow k' = 0.713$$
$$0.713 = \frac{w - 160}{6}$$
$$w = 164.3\text{ g}$$

4 X: number of babies born before their due date

$X \sim \mathrm{B}(160, 0.25)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$\mu = 160 \times 0.25 = 40$

$\sigma = \sqrt{160 \times 0.25 \times (1 - 0.25)} = \sqrt{30}$

$X \approx \mathrm{N}(40, 30)$

$$\mathrm{P}(X > 50) \approx \mathrm{P}\left(Z > \frac{50.5 - 40}{\sqrt{30}}\right)$$
$$= \mathrm{P}(Z > 1.917)$$
$$= 1 - 0.9723$$
$$= 0.0277 \text{ (3 s.f.)}$$

5 T: time that players arrive at training relative to start time.

(i) $T \sim \mathrm{N}(-5, 7^2)$

$$\mathrm{P}(T < 0) \approx \mathrm{P}\left(Z < \frac{0 - (-5)}{7}\right)$$
$$= \mathrm{P}(Z < 0.714)$$
$$= 0.7623$$

(ii) F: number of forwards on time

$F \sim \mathrm{B}(8, 0.762)$

$\mathrm{P}(F \geqslant 7)$

$= \mathrm{P}(F = 7 \text{ or } 8)$

$= \binom{8}{7} 0.762^7\, 0.238^1 + \binom{8}{8} 0.762^8\, 0.238^0$

$= 0.399$ (3 s.f.)

(iii) B: number of backs on time

$B \sim \mathrm{B}(25, 0.762)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$\mu = 25 \times 0.762 = 19.05$

$\sigma = \sqrt{25 \times 0.762 \times (1 - 0.762)} = \sqrt{4.52...}$

$B \approx \mathrm{N}(19.05, 4.52...)$

$$\mathrm{P}(B \geqslant 18) = \mathrm{P}\left(Z \geqslant \frac{17.5 - 19.05}{\sqrt{4.52...}}\right)$$
$$= \mathrm{P}(Z \geqslant -0.729)$$
$$= 1 - 0.7669$$
$$= 0.233 \text{ (3 s.f.)}$$

Past exam questions

1 $X \sim \mathrm{N}(\mu, \sigma^2)$

(i) $\mathrm{P}(X < 2\mu)$

$$= \mathrm{P}\left(Z < \frac{2\mu - \mu}{\frac{3\mu}{5}}\right)$$
$$= \mathrm{P}\left(Z < \frac{5}{3}\right)$$
$$= 0.952 \text{ (3 s.f.)}$$

(ii) $P\left(X < \frac{1}{3}\mu\right)$

$= P\left(Z < \dfrac{\frac{1}{3}\mu - \mu}{\sigma}\right)$

$= P\left(Z < \dfrac{-\frac{2}{3}\mu}{\sigma}\right)$

$\dfrac{-\frac{2}{3}\mu}{\sigma} = 1.047$

$-2\mu = 3.141\sigma$

$\mu = -1.57\sigma$

2 X: time spent visiting dentist

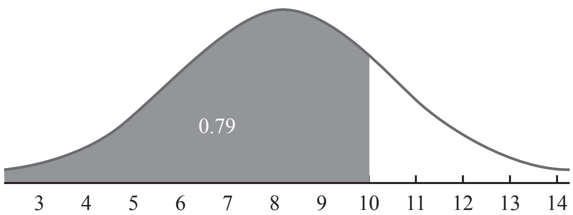
$X \sim N(8.2, \sigma^2)$

(i) $P(X < 10) = 0.79$

$P\left(Z < \dfrac{10 - 8.2}{\sigma}\right) = 0.79$

$\dfrac{10 - 8.2}{\sigma} = 0.807$

$\sigma = 2.23$ (3 s.f.)



(ii) $P(X < 7.2) + P(X > 9.2)$

$= 2 \times P\left(Z > \dfrac{9.2 - 8.2}{2.23}\right)$

$= 2 \times P(Z > 0.448)$

$= 2 \times (1 - 0.6729)$

$= 0.654$

(iii) $P(X > 10) = 0.21$

Y: number of people whose visits last longer than 10 min.

$Y \sim B(6, 0.21)$

$P(Y > 2)$

$= 1 - P(Y = 0,1,2)$

$$= 1 - \left[\binom{6}{0}0.21^0 0.79^6 + \binom{6}{1}0.21^1 0.79^5 + \binom{6}{2}0.21^2 0.79^4\right]$$

$= 1 - 0.888$

$= 0.112$

(iv) T: number of people with visits lasting less than 8.2 min from sample of 35 people.

$T \sim B(35, 0.5)$

$\mu = 35 \times 0.5 = 17.5$

$\sigma^2 = 35 \times 0.5 \times (1 - 0.5) = 8.75$

Since $np > 5$ and $n(1 - p) > 5$ normal approximation is appropriate.

$X \approx N(17.5, 8.75)$

$P(T < 16)$

$= P\left(Z < \dfrac{15.5 - 17.5}{\sqrt{8.75}}\right)$

$= P(Z < -0.676)$

$= 1 - 0.7505$

$= 0.250$

3 X: no. of cameras that are substandard

$X \sim B(300, 0.072)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$\mu = 300 \times 0.072 = 21.6$

$\sigma = \sqrt{300 \times 0.072 \times (1 - 0.072)}$

$= \sqrt{20.0448} = 4.48$ (3 s.f.)

$X \approx N(21.6, 20.0448)$

$P(X < 18) \approx P\left(Z < \dfrac{17.5 - 21.6}{\sqrt{20.0448}}\right)$

$= P(Z < -0.916)$

$= 1 - 0.8201$

$= 0.180$

4 X: heights of school desks

$X \sim N(69, \sigma^2)$

(i) $P(X > 70) = 0.155$

$P(Z > k') = 0.155 \Rightarrow k' = 1.015$

$1.015 = \dfrac{70 - 69}{\sigma}$

$\sigma = 0.985$

(ii) Desk is comfortable for Jodu if $X > 67$

$P(X > 67) = P\left(Z > \dfrac{67 - 69}{0.9852}\right)$

$= P(Z > -2.03)$

$= 0.9788$

Number of comfortable desks

$= 300 \times 0.9788 = 293.64$

So about 293 desks

5 X: time taken to fit a tow bar

(i) $X \sim \text{N}(m, 0.35)$

$\text{P}(Z > z) = 0.95 \Rightarrow z = -1.645$

$$-1.645 = \frac{0.9 - m}{0.35}$$

$$-1.645 \times 0.35 = 0.9 - m$$

$$m = 1.48 \text{ hours (3 s.f.)}$$

(ii) $\text{P}(X < 2) = \text{P}\left(Z < \frac{2 - 1.47575}{0.35}\right)$

$= \text{P}(Z < 1.498)$

$= 0.9330$

P(no cars take longer than 2 hours)

$= 0.933^4$

$= 0.758$

(iii) Times at another garage $X \sim \text{N}\left(\mu, \left(\frac{\mu}{3}\right)^2\right)$

$$\text{P}(X > 0.6\mu) = \text{P}\left(Z > \frac{0.6\mu - \mu}{\frac{\mu}{3}}\right)$$

$= \text{P}(Z > -1.2)$

$= 0.885$

Stretch and challenge

1 X: diameter of a randomly chosen bolt

$X \sim \text{N}(18, 0.2^2)$

P(rejected)

$= \text{P}(X < 17.68) + \text{P}(X > 18.32)$

$= \text{P}\left(Z < \frac{17.68 - 18}{0.2}\right) + \text{P}\left(Z > \frac{18.32 - 18}{0.2}\right)$

$= \text{P}(Z < -1.6) + \text{P}(Z > 1.6)$

$= 2(1 - 0.9452)$

$= 0.1096 = 0.110$ (3 s.f.)

$\text{P}(X > 18.5 = \text{P}\left(Z > \frac{18.5 - 18}{0.2}\right)$

$= \text{P}(Z > 2.5)$

$= 1 - 0.9938$

$= 0.0062$

$$\text{P}(X > 18.5 \mid \text{rejected}) = \frac{\text{P}(X > 18.5)}{\text{P}(\text{rejected})}$$

$$= \frac{0.0062}{0.1096}$$

$= 0.0566$ (3 s.f.)

2 X: IQ score, $X \sim \text{N}(100, 15^2)$

$\text{P}(100 < X < 105)$

$= \text{P}\left(0 < Z < \frac{105 - 100}{15}\right)$

$= \text{P}(0 < Z < 0.333)$

$= 0.1304$

$\text{P}(105 < X < k) = 0.12$

$\Rightarrow \text{P}(100 < X < k) = 0.1304 + 0.12 = 0.2504$

$\text{P}\left(0 < Z < \frac{k - 100}{15}\right) = 0.2504$

$\Rightarrow \text{P}\left(Z < \frac{k - 100}{15}\right) = 0.7504$

$\frac{k - 100}{15} = 0.676 \Rightarrow k = 110.14$

$k = 110$ (nearest whole number)

3 X: number of correct guesses
Assuming the person is guessing,
$X \sim \text{B}(60000, 0.2)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$\mu = 60000 \times 0.2 = 12000$

$\sigma = \sqrt{60000 \times 0.2 \times (1 - 0.2)} = \sqrt{9600}$

$X \approx \text{N}(12000, 9600)$

$$\text{P}(X > 12284) \approx \text{P}\left(Z > \frac{12283.5 - 12000}{\sqrt{9600}}\right)$$

$= \text{P}(Z > 2.893)$

$= 1 - 0.9981$

$= 0.0019$ (3 s.f.)

Since the probability of picking the correct card that many times is very low, you can conclude the experiment shows evidence of ESP.