Lecture # 05

Set Comprehension

Set Theory- Basics

- A set is a collection of objects.
- The objects may be
  - natural numbers,
  - names of files,
  - locations of monitoring instruments,
  - user names,

  or whatever objects are of interest to the specifier.

Set Theory- Basics

A set can be specified as a collection of objects surrounded by curly brackets. Thus

$$\{21, 7, 14, 3\}$$

is an example of a set of objects which are natural numbers and

$$\{\text{archiver}, \text{sorter}, \text{editor}, \text{finder}\}$$

is a set of utility programs. The important property of a set is that **duplicates are not allowed**. Thus

$$\{1, 3, 4, 1, 9, 3\}$$

is not an example of a set.

Set Theory- Basics

A set cannot only contain single-element objects but can also contain aggregates of objects. For example,

$$\{(1,2),(3,4),(4,5)\}$$

is a set of pairs of natural numbers and

$$\{(\mathrm{mon1},\mathrm{mon2},\mathrm{mon3}),(\mathrm{mon2},\mathrm{mon4},\mathrm{mon5}),\\ (\mathrm{mon3},\mathrm{mon7},\mathrm{mon8})\}$$

is a set of triples which contain monitor names.
A set can be **finite** or can be **infinite**.

$$\{1,3,5\}$$

is an example of a finite set while the set of all natural numbers is an example of an infinite set. $\mathbb{N} = \{0,1,2,3,4,5,\ldots\}$

Set Theory- Basics

The objects which make up a set are known as **members**. The fact that an object is a member of a set is written as

$$x \in S.$$

If x does not belong in a set, then this is written as

$$x \notin S.$$

Thus,

$$3 \in \{3, 4, 7\},$$
$$5 \in \{17, 230, 46, 5\},$$
$$\text{update} \in \{\text{update, write, read}\}$$

are examples of predicates which are true and

$$1 \notin \{2, 5, 7\},$$
$$\text{vdu1} \notin \{\text{vdu3, vdu8, vdu9}\}$$

are examples of predicates which are also true.

- Unfortunately, listing a set this way has a number of disadvantages.
  - The first disadvantage is that, for large finite sets, explicit listing is tedious and, for infinite sets, impossible.
  - The second disadvantage is that such a listing does not make the relationship between the elements of a set clear.

$$\{n : \mathbf{N} \mid n^2 < 25 \cdot n\}.$$

It defines a set of natural numbers whose squares are less than 25, i.e. it specifies the set

$$\{0, 1, 2, 3, 4\}.$$

- This way of defining a set in known as a comprehensive specification.

Comprehensive specification

- Comprehensive specification

$$\{\underbrace{n:\mathbf{N}}_{\text{Signature / Declaration/ source}} \mid \underbrace{n^2 < 25}_{\text{formula / predicate/ filter}} \cdot \underbrace{n}_{\text{Term/ Pattern}}\}$$

Thus, the comprehensive specification

$$\{n:\mathbf{N} \mid n < 20 \land n > 10 \cdot n\}$$

denotes the set of natural numbers which lie between 10 and 20, $n:\mathbf{N}$ is the signature, $n < 20 \land n > 10$ is the predicate, and n is the term.

Comprehensive specification

The predicate part of a comprehensive set specification defines the properties of the members of the set which is specified. Thus,

$$\{n : \mathbf{N} \mid n^3 > 10 \cdot n\}$$

specifies the set of natural numbers which have the property that their cubes are greater than 10.

The term part of a comprehensive set specification defines the form of the members of the set. A term consists of an expression which, when evaluated, will deliver a value which is of the same type as the set. For example,

$$\{n : \mathbf{N} \mid n > 20 \wedge n < 100 \cdot n\}$$

states that a set will contain single natural numbers which satisfy the predicate $n > 20 \wedge n < 100$,

Comprehensive specification

$$\{x, y : \mathbf{N} \mid x + y = 100 \cdot (x, y)\}$$

specifies the set of pairs which are natural numbers whose sum is 100. i.e. $\{(0, 100), (1, 99), (2,98), \ldots, (100, 0)\}$. Thus, the term in this example defines the fact that elements of the set are pairs.

Comprehensive specification - Examples

Some more examples of comprehensive set specifications with their natural-language equivalents are now given.

Empty Set

Consider the comprehensive specification

Subset

Within any given set A there exist other sets which can be obtained by removing some of the elements of A. These are called **subsets** of the set A. For example, if A is

$$\{1, 3, 9, 14, 200\},$$

then both $\{1, 3, 9\}$ and $\{1, 9, 14\}$ are subsets of A. The fact that a set is a subset of another set is expressed by the operators: $\subset$ and $\subseteq$. The predicate

$$A \subseteq B$$

is true if A is a subset of B including being equal to B. Thus,

$$\{1, 2\} \subseteq \{1, 2, 3, 4\}$$

$$\{1\} \subseteq \{1, 2, 3, 4\}$$

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

are all true.

Proper Subset

The predicate

$$A \subset B$$

is true when A is a **proper subset** of B, that is, A is a subset of B but not equal to B. Thus, the predicates

$$\{1, 2\} \subset \{1, 2, 3, 4\}$$
$$\{4\} \subset \{1, 2, 3, 4\}$$
$$\{1, 4\} \subset \{1, 2, 3, 4\}$$

are all true, while

$$\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$$

is false.

Power Set

- Given S = {0, 1}. All the possible subsets of S?
  - P(S) = { ∅, {0}, {1}, {0,1} }
- If ***A*** is a set then so **P** ***A***, which is known as the power set of ***A*.**

$$X \in \mathbb{P}A \Leftrightarrow X \subseteq A$$

- Something is a member of **P** ***A*** iff it is a subset of ***A***.

Thus, $\mathbb{P}\{1, 2, 3\}$ is

$$\{\{\,\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

remembering, of course, that the empty set is a subset of *any* set.

Set Equality

If A and B are two sets of the same type,

then $A = B$ is true when each set contains the same elements. Thus,

$$\{1, 2, 4, 5\} = \{1, 2, 5, 4\}$$
$$\{9, 8, 7, 6, 5\} = \{7, 8, 6, 9, 5\}$$

are true while

$$\{1, 3, 5\} = \{1, 3, 7, 11\},$$
$$\{1, 3, 5\} = \{1, 3, 5, 9, 11\}$$

are false. Again $=$ can be defined in terms of $\in$ and predicate calculus.

$$A = B \Leftrightarrow$$

Set Equality & Proper Subset

Given this definition of $=$, it is possible to define the $\subset$ operator as

$$A \subset B \Leftrightarrow \forall a : A \cdot a \in B \wedge \neg(A = B)$$

which just states that A is a proper subset of B when every element of A occurs in B and A is not equal to B.

Set Union & Set Intersection

The union operator can be formally defined using $\in$ and predicate calculus as

$$A \cup B = \{X: T \mid X \in A \vee X \in B\}$$

Where T is the type of the objects which make up the set, for example, the set **N** or the set of all functioning monitors.

Formally define the intersection operator $\cap$.

The intersection operator forms a set whose elements are in both of its arguments

$$A \cap B = \{\underbrace{x:}_{\text{The set of all } x \text{ such that}} T \mid \underbrace{x \in A}_{x \text{ is in } A} \underbrace{\wedge}_{\text{and}} \underbrace{x \in B}_{x \text{ is in } B}\}.$$

Set Difference

- The notation $\boldsymbol{U} \setminus \boldsymbol{V}$ denotes the set consisting of all those elements of $\boldsymbol{U}$ which are not in $\boldsymbol{V}$.
  - $\{1,2,4,8,9\} \setminus \{1,2,3\} = \{4,8,9\}$
  - $\{1,2,3\} \setminus \{1,2,3\} = \{\ \}$
- $\boldsymbol{N_1} == \boldsymbol{N} \setminus \{0\}$

  $\boldsymbol{N_1}$ is set of non-negative numbers excluding 0
- $odds == \mathbf{N_1} \setminus evens$

$\mathbf{U} \setminus \mathbf{V} =$

The union operator U combines sets
{1,2,3} U {2,3,4} = {1,2,3,4}

**The difference operator ** removes the elements of one set from another
{1,2,3,4} \\ {2,3} = {1,4}

The intersection operator ∩ finds the elements common to both sets
{1,2,3} ∩ **{2,3,4} = {2,3}**

Cross Product

The cross-product of two sets A and B is denoted by

$$A \times B.$$

The operator forms the set of pairs where the first element of each pair is drawn from A and the second element is drawn from B. Thus,

$$\{1, 2, 3\} \times \{4, 5\} = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

and

{line1, line2) × {10, 12, 3} = {(line1, 10), (line1, 12), (line1, 3), (line2, 10), (line2, 12), (line2,3)}.

Formally, the cross-product is defined as

$$A \times B =$$

Cardinality

The **cardinality** of a set is the number of elements in the set. For example, the cardinality of the set

$$\{\text{tax, update, oldupdate}\}$$

is 3. In set theory the operator $\#$ is the cardinality operator. When applied to a set it gives the number of elements in the set. For example,

$$\#\{\text{old, new, medium, fast, slow}\} = 5$$

and

$$\#\{n : \mathbf{N} \mid n < 4\} = 4$$

are both true predicates.

REFERENCES :

- D. Ince-An Introduction to Discrete Mathematics, Formal System Specification and Z. (Oxford Applied Mathematics and Computing Science Series) – Chapter 5