2.5 Solutions by Substitutions

HOMOGENEOUS EQUATIONS If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous.

A first-order DE in differential

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is said to be **homogeneous** if both coefficient functions M and N are homogeneous functions of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha}M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^{\alpha}N(x, y).$$

In addition, if M and N are homogeneous functions of degree α, we can also write

$$M(x, y) = x^{\alpha}M(1, u) \qquad \text{and} \qquad N(x, y) = x^{\alpha}N(1, u), \quad \text{where } u = y/x, \tag{2}$$

and

$$M(x, y) = y^{\alpha}M(v, 1) \qquad \text{and} \qquad N(x, y) = y^{\alpha}N(v, 1), \quad \text{where } v = x/y. \tag{3}$$

Solution procedure for solving homogeneous differential equation

The homogeneous DE the substitutions $y = ux$

$M(x, y)\,dx + N(x, y)\,dy = 0$ can be rewritten as

$$x^{\alpha}M(1, u)\,dx + x^{\alpha}N(1, u)\,dy = 0$$

$$M(1, u)\,dx + N(1, u)\,dy = 0,$$

By substituting the differential $dy = u\,dx + x\,du$

$$M(1, u)\,dx + N(1, u)[u\,dx + x\,du] = 0$$

$$[M(1, u) + uN(1, u)]\,dx + xN(1, u)\,du = 0$$

$$\frac{dx}{x} + \frac{N(1, u)\,du}{M(1, u) + uN(1, u)} = 0.$$

The proof that the substitutions $x = vy$ and $dx = v\,dy + y\,dv$ also lead to a separable equation follows in an analogous manner from (3).

EXAMPLE 1 Solving a Homogeneous DE

Solve $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$.

SOLUTION Inspection of $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$, then $dy = u\,dx + x\,du$, so after substituting, the given equation becomes

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$

$$x^2(1 + u)\,dx + x^3(1 - u)\,du = 0$$

$$\frac{1 - u}{1 + u}\,du + \frac{dx}{x} = 0$$

Example **Solve**

$$2x^3y\,dx + \left(x^4 + y^4\right)dy = 0$$

BERNOULLI'S EQUATION The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2.$

SOLUTION We begin by rewriting the equation in the form given in (4) by dividing by x:

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2.$$

With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

REDUCTION TO SEPARATION OF VARIABLES A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C) \tag{5}$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C$, $B \neq 0$. Example 3 illustrates the technique.

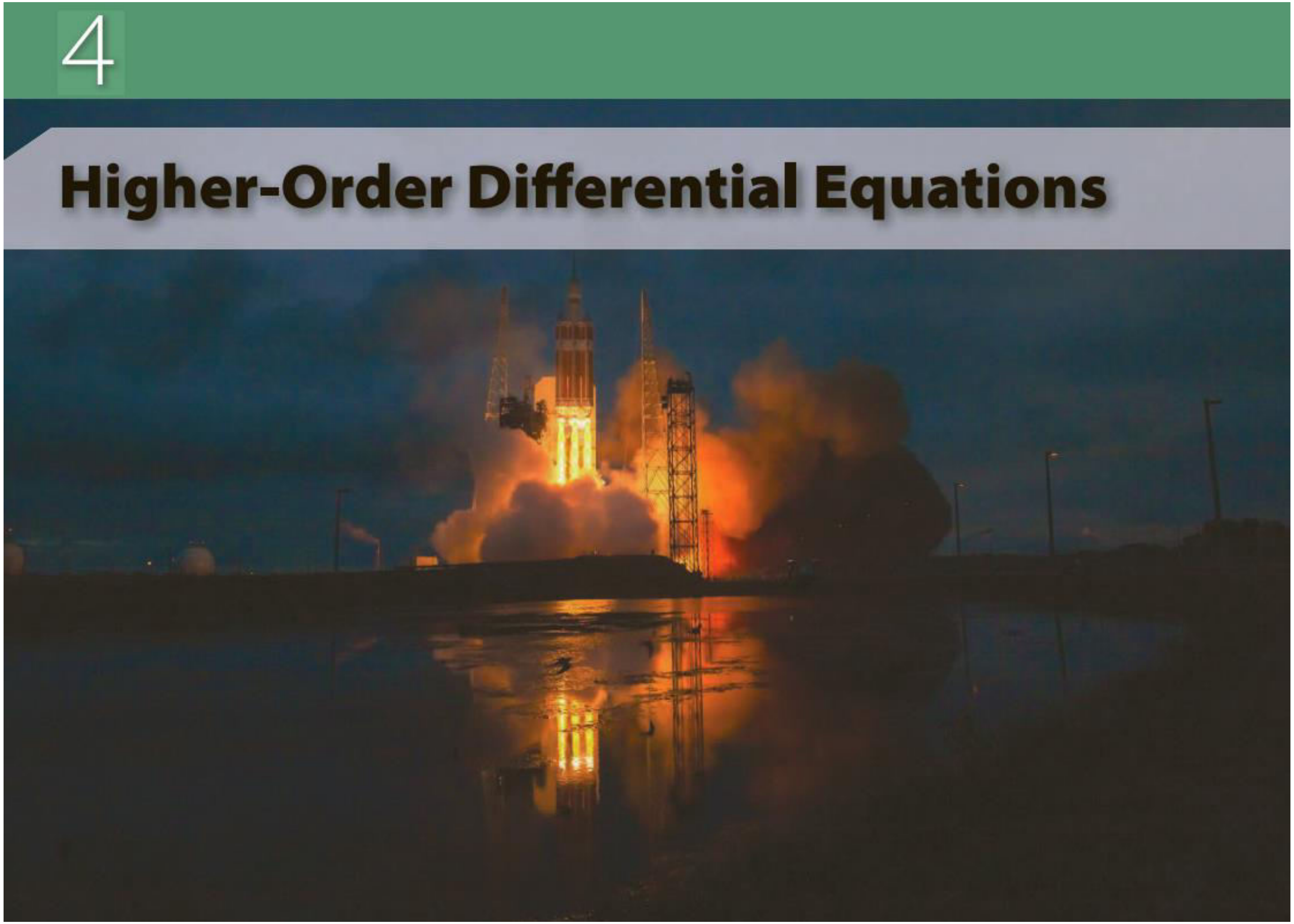
EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$

EXERCISES 2.5

Riccati's equation.

The differential equation $dy/dx = P(x) + Q(x)y + R(x)y^2$ is known as **Riccati's equation.**


4
Higher-Order Differential Equations

4.1	Preliminary Theory—Linear Equations
4.2	Reduction of Order
4.3	Homogeneous Linear Equations with Constant Coefficients
4.4	Undetermined Coefficients—Superposition Approach
4.5	Undetermined Coefficients—Annihilator Approach
4.6	Variation of Parameters
4.7	Cauchy-Euler Equations

4.1 Preliminary Theory—Linear Equations

INTRODUCTION In Chapter 2 we were able to solve a few first-order differential equations by recognizing them as separable, linear, exact, or having homogeneous coefficients. Even though the solutions obtained were in the form of a one-parameter family, this family, with one exception, did not represent the *general solution* of the differential equation. Recall that a **general solution** is a family of solutions defined on some interval I that contains *all* solutions of the DE that are defined on I. Only in the case of *linear* first-order differential equations were we able to obtain general solutions by paying close attention to certain continuity conditions imposed on the coefficients in the equation. Because our primary goal in this chapter is to find general solutions of linear higher-order DEs, we first need to examine some of the basic theory of linear equations.

4.1.1 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

INITIAL-VALUE PROBLEM In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation an **nth-order initial-value problem (IVP)** is

Solve: $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

Subject to: $y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$

BOUNDARY-VALUE PROBLEM Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. A problem such as

$$\textit{Solve:} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\textit{Subject to:} \quad y(a) = y_0, \quad y(b) = y_1$$

is called a **boundary-value problem (BVP).** The prescribed values $y(a) = y_0$ an $y(b) = y_1$ are called **boundary conditions (BC).** A solution of the foregoing probler is a function satisfying the differential equation on some interval I, containing a an b, whose graph passes through the two points (a, y_0) and (b, y_1). See Figure 4.1.1.

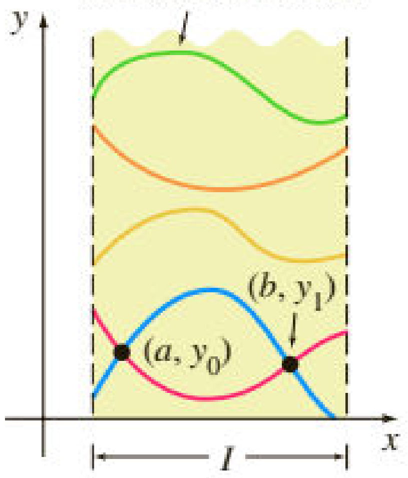


FIGURE 4.1.1 Solution curves of a BVP that pass through two points

For a second-order differential equation other pairs of boundary conditions could be

$$y'(a) = y_0, \qquad y(b) = y_1$$
$$y(a) = y_0, \qquad y'(b) = y_1$$
$$y'(a) = y_0, \qquad y'(b) = y_1,$$

where y_0 and y_1 denote arbitrary constants. These three pairs of conditions are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2.$$

THEOREM 4.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

EXAMPLE 1 Unique Solution of an IVP

The initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

possesses the trivial solution $y = 0$. Because the third-order equation is linear with constant coefficients, it follows that all the conditions of Theorem 4.1.1 are fulfilled. Hence $y = 0$ is the *only* solution on any interval containing $x = 1$. ■

EXAMPLE 2 Unique Solution of an IVP

You should verify that the function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the initial-value problem

$$y'' - 4y = 12x, \quad y(0) = 4, \quad y'(0) = 1.$$

Now the differential equation is linear, the coefficients as well as $g(x) = 12x$ are continuous, and $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$. We conclude from Theorem 4.1.1 that the given function is the unique solution on I. ■

The requirements in Theorem 4.1.1 that $a_i(x)$, $i = 0, 1, 2, \ldots, n$ be continuous and $a_n(x) \neq 0$ for every x in I are both important. Specifically, if $a_n(x) = 0$ for some x in the interval, then the solution of a linear initial-value problem may not be unique or even exist. For example, you should verify that the function $y = cx^2 + x + 3$ is a solution of the initial-value problem

$$x^2y'' - 2xy' + 2y = 6, \quad y(0) = 3, \quad y'(0) = 1$$

on the interval $(-\infty, \infty)$ for any choice of the parameter c. In other words, there is no unique solution of the problem. Although most of the conditions of Theorem 4.1.1 are satisfied, the obvious difficulties are that $a_2(x) = x^2$ is zero at $x = 0$ and that the initial conditions are also imposed at $x = 0$.

EXAMPLE 3 A BVP Can Have Many, One, or No Solutions

In Example 9 of Section 1.1 we saw that the two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is

$$x = c_1 \cos 4t + c_2 \sin 4t. \tag{2}$$

(a) Suppose we now wish to determine the solution of the equation that further satisfies the boundary conditions $x(0) = 0, \ x(\pi/2) = 0.$

boundary-value problem

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 0 \tag{3}$$

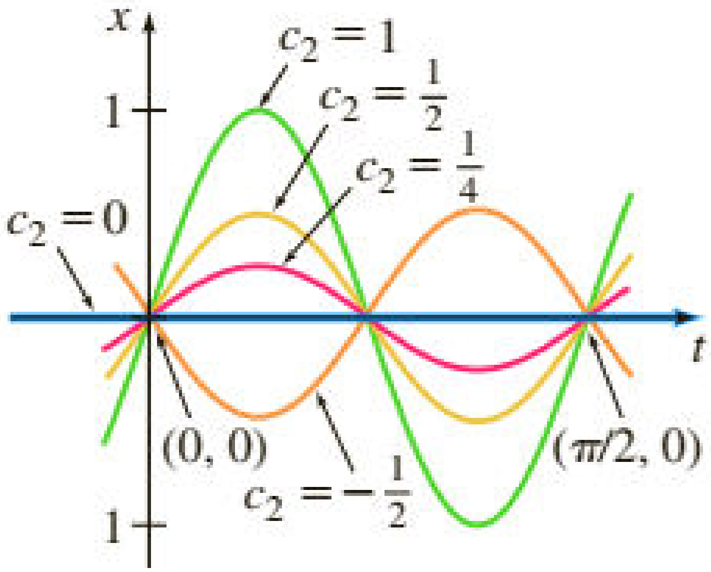


Figure 4.1.2 shows the graphs of some of the members of the one-parameter family $x = c_2 \sin 4t$ that pass through the two points $(0, 0)$ and $(\pi/2, 0)$.

(b) If the boundary-value problem in (3) is changed to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{8}\right) = 0,$$

(c) Finally, if we change the problem to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 1, \tag{5}$$

4.1.2 HOMOGENEOUS EQUATIONS

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{6}$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{7}$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.** For example,

$2y'' + 3y' - 5y = 0$ ⟹ homogeneous linear second-order differential equation,

$x^3y''' + 6y' + 10y = e^x$ ⟹ nonhomogeneous linear third-order differential equation

Please remember these two assumptions.

- the coefficient functions $a_i(x)$, $i = 0, 1, 2, \ldots, n$ and $g(x)$ are continuous;
- $a_n(x) \neq 0$ for every x in the interval.

THEOREM 4.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x),$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

EXAMPLE 4 Superposition—Homogeneous DE

The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3 y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle the linear combination

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a solution of the equation on the interval. ■

EXAMPLE 4 Superposition—Homogeneous DE

The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle the linear combination

$$y = c_1x^2 + c_2x^2 \ln x$$

is also a solution of the equation on the interval. ■

The function $y = e^{7x}$ is a solution of $y'' - 9y' + 14y = 0$. Because the differential equation is linear and homogeneous, the constant multiple $y = ce^{7x}$ is also a solution. For various values of c we see that $y = 9e^{7x}$, $y = 0$, $y = -\sqrt{5}e^{7x}$, . . . are all solutions of the equation.

COROLLARIES TO THEOREM 4.1.2

(A) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

(B) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

DEFINITION 4.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

EXAMPLE 5 Linearly Dependent Set of Functions

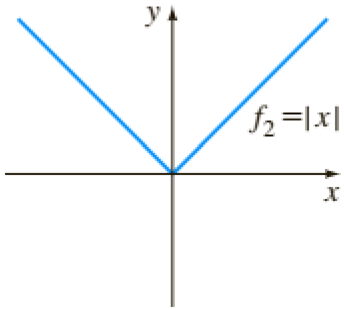
The set of functions $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$ is linearly dependent on the interval $(-\pi/2, \pi/2)$ because

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

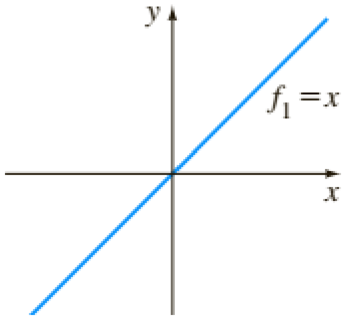
for every real number x in the interval when $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 1$. We used here $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$. ■

if a set of two functions is linearly dependent, then one function is simply a constant multiple of the other. $f_1(x) = c_2 f_2(x)$

For example, the set of functions $f_1(x) = \sin 2x$, $f_2(x) = \sin x \cos x$ is linearly dependent on $(-\infty, \infty)$ because $f_1(x)$ is a constant multiple of $f_2(x)$.



the set of functions $f_1(x) = x$, $f_2(x) = |x|$ is linearly independent on $(-\infty, \infty)$.



A set of n functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly dependent on an interval I if at least one of the functions can be expressed as a linear combination of the remaining functions. For example, three functions $f_1(x), f_2(x)$, and $f_3(x)$ are linearly dependent on I if at least one of these functions is a linear combination of the other two, say,

$$f_3(x) = c_1 f_1(x) + c_2 f_2(x)$$

for all x in I. A set of n functions is linearly independent on I if no one function is a linear combination of the other functions.

EXAMPLE 6 Linearly Dependent Set of Functions

The set of functions $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$, $f_4(x) = x^2$ is linearly dependent on the interval $(0, \infty)$ because f_2 can be written as a linear combination of f_1, f_3, and f_4. Observe that

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$$

for every x in the interval $(0, \infty)$. ■

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

DEFINITION 4.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

THEOREM 4.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

EXAMPLE 7 General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y = c_1e^{3x} + c_2e^{-3x}$ is the general solution of the equation on the interval. ■

EXAMPLE 9 General Solution of a Homogeneous DE

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third-order equation $y''' - 6y'' + 11y' - 6y = 0$. Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every real value of x, the functions y_1, y_2, and y_3 form a fundamental set of solutions on $(-\infty, \infty)$. We conclude that $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the general solution of the differential equation on the interval. ■

THEOREM 4.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x),$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

$$\begin{aligned} y &= \textit{complementary function} + \textit{any particular solution} \\ &= y_c + y_p. \end{aligned}$$

EXAMPLE 10 General Solution of a Nonhomogeneous DE

By substitution the function $y_p = -\frac{11}{12} - \frac{1}{2}x$ is readily shown to be a particular solution of the nonhomogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x. \tag{11}$$

To write the general solution of (11), we must also be able to solve the associated homogeneous equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

But in Example 9 we saw that the general solution of this latter equation on the interval $(-\infty, \infty)$ was $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Hence the general solution of (11) on the interval is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x.$$

THEOREM 4.1.7 Superposition Principle—Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (7) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x), \qquad (12)$$

where $i = 1, 2, \ldots, k$. Then

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x) \qquad (13)$$

is a particular solution of

$$\begin{aligned} a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} &+ \cdots + a_1(x)y' + a_0(x)y \\ &= g_1(x) + g_2(x) + \cdots + g_k(x). \end{aligned} \qquad (14)$$

EXAMPLE 11 Superposition—Nonhomogeneous DE

You should verify that

$$
\begin{aligned}
y_{p_1} &= -4x^2 && \text{is a particular solution of} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8,\\
y_{p_2} &= e^{2x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2e^{2x},\\
y_{p_3} &= xe^{x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2xe^{x} - e^{x}.
\end{aligned}
$$

It follows from (13) of Theorem 4.1.7 that the superposition of y_{p_1}, y_{p_2}, and y_{p_3},

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x},$$

is a solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^{x} - e^{x}}_{g_3(x)}.$$

EXERCISES 4.1

4.2 Reduction of Order

INTRODUCTION In the preceding section we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I.

The basic idea described in this section is that

equation (1) *can be reduced to a linear first-order DE by means of a substitution* involving the known solution y_1. A second solution y_2 of (1) is apparent after this first-order differential equation is solved.

REDUCTION OF ORDER Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I. Recall from Section 4.1 that if y_1 and y_2 are linearly independent, then their quotient y_2/y_1 is nonconstant on I—that is, $y_2(x)/y_1(x) = u(x)$ or $y_2(x) = u(x)y_1(x)$. The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation. This method is called **reduction of order** because we must solve a linear first-order differential equation to find u.

GENERAL CASE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

EXAMPLE 1 A Second Solution by Reduction of Order

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval $(0, \infty)$.

4.3 Homogeneous Linear Equations with Constant Coefficients

In this section we will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order DEs,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$.

AUXILIARY EQUATION We begin by considering the special case of the second-order equation

$$ay'' + by' + cy = 0, \tag{2}$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2 e^{mx}$, equation (2) becomes

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \qquad \text{or} \qquad e^{mx}(am^2 + bm + c) = 0.$$

As in the introduction we argue that because $e^{mx} \neq 0$ for all x, it is apparent that the only way $y = e^{mx}$ can satisfy the differential equation (2) is when m is chosen as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (2) corresponding to the three cases:

- m_1 and m_2 real and distinct $(b^2 - 4ac > 0)$,
- m_1 and m_2 real and equal $(b^2 - 4ac = 0)$, and
- m_1 and m_2 conjugate complex numbers $(b^2 - 4ac < 0)$.

We discuss each of these cases in turn.

CASE I: DISTINCT REAL ROOTS Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

CASE II: REPEATED REAL ROOTS When $m_1 = m_2$,

$$y = c_1e^{m_1x} + c_2xe^{m_1x}.$$

CASE III: CONJUGATE COMPLEX ROOTS If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real and $i^2 = -1$.

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

EXAMPLE 2 An Initial-Value Problem

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

HIGHER-ORDER EQUATIONS In general, to solve an nth-order differential equation (1), where the a_i, $i = 0, 1, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0. \tag{12}$$

If all the roots of (12) are real and distinct, then the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

CASE II: REPEATED REAL ROOTS

$$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

CASE III:

EXAMPLE 3 Third-Order DE

Solve $y''' + 3y'' - 4y = 0$.

EXAMPLE 4 Fourth-Order DE

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$.

EXAMPLE 5 Finding Rational Roots

Solve $3y''' + 5y'' + 10y' - 4y = 0$.

SOLUTION To solve the equation we must solve the cubic polynomial auxiliary equation $3m^3 + 5m^2 + 10m - 4 = 0$. With the identifications $a_0 = -4$ and $a_3 = 3$ then the integer factors of a_0 and a_3 are, respectively, $p: \pm 1, \pm 2, \pm 4$ and $q: \pm 1, \pm 3$. So the possible rational roots of the cubic equation are

$$\frac{p}{q}: \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}.$$

Each of these numbers can then be tested—say, by synthetic division. In this way we discover both the root $m_1 = \frac{1}{3}$ and the factorization

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \tfrac{1}{3}\right)(3m^2 + 6m + 12).$$

The quadratic formula applied to $3m^2 + 6m + 12 = 0$ then yields the remaining two roots $m_2 = -1 - \sqrt{3}i$ and $m_3 = -1 + \sqrt{3}i$. Therefore the general solution of the given differential equation is $y = c_1 e^{x/3} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$. ■

EXERCISES 4.3

4.4 Undetermined Coefficients—Superposition Approach*

INTRODUCTION To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \qquad (1)$$

we must do two things:

- find the complementary function y_c and
- find *any* particular solution y_p of the nonhomogeneous equation (1).

Then, as was discussed in Section 4.1, the general solution of (1) is $y = y_c + y_p$. The complementary function y_c is the general solution of the associated homogeneous DE of (1), that is,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In Section 4.3 we saw how to solve these kinds of equations when the coefficients were constants. Our goal in the present section is to develop a method for obtaining particular solutions.

METHOD OF UNDETERMINED COEFFICIENTS The first of two ways we shall consider for obtaining a particular solution y_p for a nonhomogeneous linear DE is called the **method of undetermined coefficients.** The underlying idea behind this method is a conjecture about the form of y_p, an educated guess really, that is motivated by the kinds of functions that make up the input function $g(x)$. The general method is limited to linear DEs such as (1) where

- the coefficients a_i, $i = 0, 1, \ldots, n$ are constants and
- $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, a sine or cosine function $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

EXAMPLE 1 General Solution Using Undetermined Coefficients

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$ (2)

EXAMPLE 2 Particular Solution Using Undetermined Coefficients

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

EXAMPLE 3 Forming y_p by Superposition

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

EXAMPLE 4 A Glitch in the Method

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

TABLE 4.4.1 **Trial Particular Solutions**

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

CASE I No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

If $g(x)$ consists of a sum of, say, m terms of the kind listed in the table, then (as in Example 3) the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

EXAMPLE 6 Finding y_p by Superposition—Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 8xe^{6x}.$$

CASE II A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

EXAMPLE 7 Particular Solution—Case II

Find a particular solution of $y'' - 2y' + y = e^x$.

Multiplication Rule for Case II *If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.*

EXAMPLE 9 Using the Multiplication Rule

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

SOLUTION The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$. And so, based on entries 3 and 7 of Table 4.4.1, the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

EXERCISES 4.4

4.5 Undetermined Coefficients—Annihilator Approach

INTRODUCTION We saw in Section 4.1 that an nth-order differential equation can be written

$$a_nD^ny + a_{n-1}D^{n-1}y + \cdots + a_1Dy + a_0y = g(x), \tag{1}$$

where $D^ky = d^ky/dx^k$, $k = 0, 1, \ldots, n$. When it suits our purpose, (1) is also written as $L(y) = g(x)$, where L denotes the linear nth-order differential, or polynomial, operator

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0. \tag{2}$$

Not only is the operator notation a helpful shorthand, but also on a very practical level the application of differential operators enables us to justify the somewhat mind-numbing rules for determining the form of particular solution y_p that were presented in the preceding section. In this section there are no special rules; the form of y_p follows almost automatically once we have found an appropriate linear differential operator that *annihilates* $g(x)$ in (1).

ANNIHILATOR OPERATOR If L is a linear differential operator with constant coefficients and f is a sufficiently differentiable function such that

$$L(f(x)) = 0,$$

then L is said to be an **annihilator** of the function. For example, a constant function $y = k$ is annihilated by D, since $Dk = 0$. The function $y = x$ is annihilated by

the differential operator D^2 since the first and second derivatives of x are 1 and 0, respectively. Similarly, $D^3x^2 = 0$, and so on.

The differential operator D^n annihilates each of the functions

$$1, \quad x, \quad x^2, \quad \ldots, \quad x^{n-1}. \tag{3}$$

The differential operator $(D - \alpha)^n$ annihilates each of the functions

$$e^{\alpha x}, \quad xe^{\alpha x}, \quad x^2 e^{\alpha x}, \quad \ldots, \quad x^{n-1} e^{\alpha x}. \tag{5}$$

EXAMPLE 1 Annihilator Operators

Find a differential operator that annihilates the given function.

(a) $1 - 5x^2 + 8x^3$ **(b)** e^{-3x} **(c)** $4e^{2x} - 10xe^{2x}$

SOLUTION **(a)** From (3) we know that $D^4x^3 = 0$, so it follows from (4) that

$$D^4(1 - 5x^2 + 8x^3) = 0.$$

(b) From (5), with $\alpha = -3$ and $n = 1$, we see that

$$(D + 3)e^{-3x} = 0.$$

(c) From (5) and (6), with $\alpha = 2$ and $n = 2$, we have

$$(D - 2)^2(4e^{2x} - 10xe^{2x}) = 0.$$

The differential operator $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates each of the functions

$$\begin{aligned} &e^{\alpha x}\cos\beta x, \quad xe^{\alpha x}\cos\beta x, \quad x^2e^{\alpha x}\cos\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\cos\beta x, \\ &e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \quad x^2e^{\alpha x}\sin\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\sin\beta x. \end{aligned} \tag{7}$$

EXAMPLE 2 Annihilator Operator

Find a differential operator that annihilates $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$.

When $\alpha = 0$ and $n = 1$, a special case of (7) is

$$(D^2 + \beta^2)\begin{cases}\cos \beta x \\ \sin \beta x\end{cases} = 0. \qquad (8)$$

For example, $D^2 + 16$ will annihilate any linear combination of $\sin 4x$ and $\cos 4x$.

NOTE The differential operator that annihilates a function is not unique. We saw in part (b) of Example 1 that $D + 3$ will annihilate e^{-3x}, but so will differential operators of higher order as long as $D + 3$ is one of the factors of the operator. For example, $(D + 3)(D + 1)$, $(D + 3)^2$, and $D^3(D + 3)$ all annihilate e^{-3x}. (Verify this.) As a matter of course, when we seek a differential annihilator for a function $y = f(x)$, we want the operator of *lowest possible order* that does the job.

UNDETERMINED COEFFICIENTS This brings us to the point of the preceding discussion. Suppose that $L(y) = g(x)$ is a linear differential equation with constant coefficients and that the input $g(x)$ consists of finite sums and products of the functions listed in (3), (5), and (7)—that is, $g(x)$ is a linear combination of functions of the form

$$k \text{ (constant)}, \quad x^m, \quad x^m e^{\alpha x}, \quad x^m e^{\alpha x} \cos \beta x, \quad \text{and} \quad x^m e^{\alpha x} \sin \beta x,$$

where m is a nonnegative integer and α and β are real numbers. We now know that such a function $g(x)$ can be annihilated by a differential operator L_1 of lowest order, consisting of a product of the operators D^n, $(D - \alpha)^n$, and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$. Applying L_1 to both sides of the equation $L(y) = g(x)$ yields $L_1L(y) = L_1(g(x)) = 0$. By solving the *homogeneous higher-order* equation $L_1L(y) = 0$, we can discover the *form* of a particular solution y_p for the original *nonhomogeneous* equation $L(y) = g(x)$. We then substitute this assumed form into $L(y) = g(x)$ to find an explicit particular solution. This procedure for determining y_p, called the **method of undetermined coefficients,**

EXAMPLE 3 General Solution Using Undetermined Coefficients

Solve $y'' + 3y' + 2y = 4x^2$. (9)

EXAMPLE 4 General Solution Using Undetermined Coefficients

Solve $y'' - 3y' = 8e^{3x} + 4 \sin x$. (14)

EXAMPLE 7 Form of a Particular Solution

Determine the form of a particular solution for

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}.$$

EXERCISES 4.5

4.6 VARIATION OF PARAMETERS

Joseph Louis Lagrange (1736–1813)

SUMMARY OF THE METHOD

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x).$$

$$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x),$$

$$u_k' = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n,$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x),$$

$$y_c = c_1y_1(x) + c_2y_2(x),$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W},$$

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

When $n = 3$,

$$y_p = u_1y_1 + u_2y_2 + u_3y_3,$$

$$u_1' = \frac{W_1}{W}, \qquad u_2' = \frac{W_2}{W}, \qquad u_3' = \frac{W_3}{W}, \tag{15}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}.$$

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

INTEGRAL-DEFINED FUNCTIONS We have seen several times in the preceding sections and chapters that when a solution method involves integration we may encounter nonelementary integrals. As the next example shows, sometimes the best we can do in constructing a particular solution (7) of a linear second-order differential equation is to use the integral-defined functions

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)f(t)}{W(t)}\,dt \qquad \text{and} \qquad u_2(x) = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt.$$

Here we assume that the integrand is continuous on the interval $[x_0, x]$.

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \frac{1}{x}$.

Final course

4.7 Cauchy-Euler Equations

CAUCHY-EULER EQUATION A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x), \tag{1}$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known as a **Cauchy-Euler equation.**

The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation $d^k y/dx^k$:

same same

$$a_n x^{n} \frac{d^{n} y}{dx^{n}} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots.$$

the homogeneous second-order equation

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

nonhomogeneous equation $ax^2y'' + bxy' + cy = g(x)$ by variation of parameters, once we have determined the complementary function y_c.

METHOD OF SOLUTION We try a solution of the form $y = x^m$, where m is to be determined. Analogous to what happened when we substituted e^{mx} into a linear equation with constant coefficients, when we substitute x^m, each term of a Cauchy-Euler equation becomes a polynomial in m times x^m, since

$$a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)(m-2)\cdots(m-k+1)x^{m-k} = a_k m(m-1)(m-2)\cdots(m-k+1)x^m.$$

For example, when we substitute $y = x^m$, the second-order equation (2) becomes

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \qquad \text{or} \qquad am^2 + (b-a)m + c = 0. \tag{3}$$

CASE I: DISTINCT REAL ROOTS Let m_1 and m_2 denote the real roots of (3) such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence the general solution of (2) is

$$y = c_1 x^{m_1} + c_2 x^{m_2}. \tag{4}$$

EXAMPLE 1 Distinct Roots

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = 0.$

CASE II: REPEATED REAL ROOTS

The general solution of (2) is then

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

For higher-order equations, if m_1 is a root of multiplicity k, then it can be shown that

$$x^{m_1}, \quad x^{m_1} \ln x, \quad x^{m_1}(\ln x)^2, \ \ldots, \quad x^{m_1}(\ln x)^{k-1}$$

EXAMPLE 2 Repeated Roots

Solve $4x^2 \dfrac{d^2 y}{dx^2} + 8x \dfrac{dy}{dx} + y = 0$.

CASE III: CONJUGATE COMPLEX ROOTS If the roots of (3) are the conjugate pair $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real, then a solution is

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)].$$

EXAMPLE 3 An Initial-Value Problem

Solve $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$.

EXAMPLE 4 Third-Order Equation

Solve $x^3 \dfrac{d^3y}{dx^3} + 5x^2 \dfrac{d^2y}{dx^2} + 7x \dfrac{dy}{dx} + 8y = 0.$

NONHOMOGENEOUS EQUATIONS The method of undetermined coefficients described in Sections 4.4 and 4.5 does not carry over, *in general,* to nonhomogeneous linear differential equations with variable coefficients. Consequently, in our next example the method of variation of parameters is employed.

EXAMPLE 5 Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

REDUCTION TO CONSTANT COEFFICIENTS The similarities between the forms of solutions of Cauchy-Euler equations and solutions of linear equations with constant coefficients are not just a coincidence. For example, when the roots of the auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1e^{m_1x} + c_2e^{m_2x} \qquad \text{and} \qquad y = c_1x^{m_1} + c_2x^{m_2}, \quad x > 0. \tag{7}$$

In view of the identity $e^{\ln x} = x$, $x > 0$, the second solution given in (7) can be expressed in the same form as the first solution:

$$y = c_1e^{m_1 \ln x} + c_2e^{m_2 \ln x} = c_1e^{m_1t} + c_2e^{m_2t},$$

where $t = \ln x$. This last result illustrates the fact that any Cauchy-Euler equation can *always* be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The idea is to solve the new differential equation in terms of the variable t, using the methods of the previous sections, and, once the general solution is obtained, resubstitute $t = \ln x$. This method, illustrated in the last example, requires the use of the Chain Rule of differentiation.

EXAMPLE 6 Changing to Constant Coefficients

Solve $x^2y'' - xy' + y = \ln x$.

A DIFFERENT FORM A second-order equation of the form

$$a(x - x_0)^2 \frac{d^2y}{dx^2} + b(x - x_0)\frac{dy}{dx} + cy = 0 \tag{8}$$

is also a Cauchy-Euler equation. Observe that (8) reduces to (2) when $x_0 = 0$.

We can solve (8) as we did (2), namely, seeking solutions of $y = (x - x_0)^m$ and using

$$\frac{dy}{dx} = m(x - x_0)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = m(m-1)(x - x_0)^{m-2}.$$

EXERCISES 4.7

6

Series Solutions of Linear Equations

POWER SERIES Recall from calculus that **power series** in $x - a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series is also said to be a **power series centered at *a*.** For example, the power series $\sum_{n=0}^{\infty}(x+1)^n$ is centered at $a = -1$. In the next section we will be concerned principally with power series in x, in other words, power series that are centered at $a = 0$. For example,

$$\sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + \cdots$$

is a power series in x.

IMPORTANT FACTS The following bulleted list summarizes some important facts about power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

- **Convergence** A power series is **convergent** at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is, $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists. If the limit does not exist at x, then the series is said to be **divergent.**

- **Interval of Convergence** Every power series has an **interval of convergence.** The interval of convergence is the set of *all* real numbers x for which the series converges. The center of the interval of convergence is the center a of the series.
- **Radius of Convergence** The radius R of the interval of convergence of a power series is called its **radius of convergence.** If $R > 0$, then a power series converges for $|x - a| < R$ and diverges for $|x - a| > R$. If the series converges only at its center a, then $R = 0$. If the series converges for all x, then we write $R = \infty$. Recall, the absolute-value inequality $|x - a| < R$ is equivalent to the simultaneous inequality $a - R < x < a + R$. A power series may or may not converge at the endpoints $a - R$ and $a + R$ of this interval.

- **Absolute Convergence** In the interior of its interval of convergence a power series **converges absolutely.** In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty}|c_n(x-a)^n|$ converges. See Figure 6.1.1.

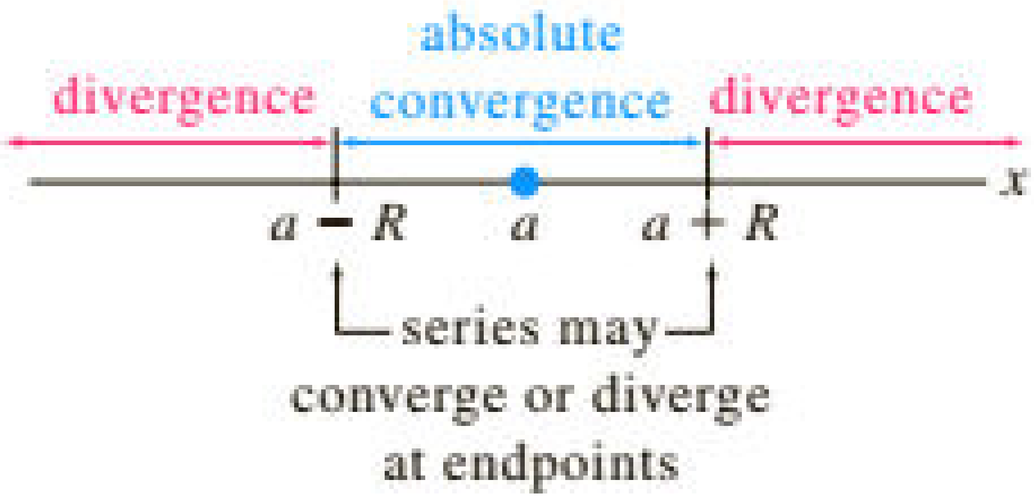


FIGURE 6.1.1 Absolute convergence within the interval of convergence and divergence outside of this interval

- **Ratio Test** Convergence of power series can often be determined by the **ratio test.** Suppose $c_n \neq 0$ for all n in $\sum_{n=0}^{\infty} c_n(x-a)^n$, and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L.$$

If $L < 1$, the series converges absolutely; if $L > 1$ the series diverges; and if $L = 1$ the test is inconclusive. The ratio test is always inconclusive at an endpoint $a \pm R$.

EXAMPLE 1 Interval of Convergence

Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$.

- **A Power Series Defines a Function** A power series defines a function, that is, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is $R > 0$ or $R = \infty$, then f is continuous, differentiable, and integrable on the intervals $(a - R, a + R)$ or $(-\infty, \infty)$, respectively. Moreover, $f'(x)$ and $\int f(x)\,dx$ can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If

$$y = \sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

is a power series in x, then the first two derivatives are $y' = \sum_{n=0}^{\infty} nx^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$. Notice that the first term in the first derivative and the first two terms in the second derivative are zero. We omit these zero terms and write

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots \\ y'' &= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots. \end{aligned} \tag{1}$$

Be sure you understand the two results given in (1); especially note where the index of summation starts in each series. These results are important and will be used in all examples in the next section.

- **Identity Property** If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $R > 0$, for all numbers x in some open interval, then $c_n = 0$ for all n.
- **Analytic at a Point** A function f is said to be **analytic at a point** $\boldsymbol{a}$ if it can be represented by a power series in $x - a$ with either a positive

or an infinite radius of convergence. In calculus it is seen that infinitely differentiable functions such as e^x, $\sin x$, $\cos x$, $e^x \ln(1 + x)$, and so on, can be represented by Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{1!}(x-a)^2 + \cdots$$

or by a Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{1!}x^2 + \cdots.$$

Maclaurin Series	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$ (2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$

EXAMPLE 2 Multiplication of Power Series

Find a power series representation of $e^x \sin x$.

SOLUTION We use the power series for e^x and $\sin x$:

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots.$$

SHIFTING THE SUMMATION INDEX For the three remaining sections of this chapter, it is crucial that you become adept at simplifying the sum of two or more power series, each series expressed in summation notation, to an expression with a single Σ. As the next example illustrates, combining two or more summations as a single summation often requires a reindexing, that is, a shift in the index of summation.

EXAMPLE 3 Addition of Power Series

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

as one power series.

EXAMPLE 4 A Power Series Solution

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the differential equation $y' + y = 0$.

EXERCISES 6.1

Answers to selected odd-numbered problems begin on page ANS-9.

6.2 Solutions About Ordinary Points

A DEFINITION If we divide the homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

by the lead coefficient $a_2(x)$ we obtain the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

We have the following definition.

DEFINITION 6.2.1 Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential of the differential equation (1) if both coefficients $P(x)$ and $Q(x)$ in the standard form (2) are analytic at x_0. A point that is *not* an ordinary point of (1) is said to be a **singular point** of the DE.

EXAMPLE 1 Ordinary Points

(a) A homogeneous linear second-order differential equation with constant coefficients, such as

$$y'' + y = 0 \quad \text{and} \quad y'' + 3y' + 2y = 0,$$

can have no singular points. In other words, every finite value* of x is an ordinary point of such equations.

(b) Every finite value of x is an ordinary point of the differential equation

$$y'' + e^x y' + (\sin x)y = 0.$$

Specifically $x = 0$ is an ordinary point of the DE, because we have already seen in (2) of Section 6.1 that both e^x and $\sin x$ are analytic at this point. ■

EXAMPLE 2 Singular Points

(a) The differential equation

$$y'' + xy' + (\ln x)y = 0$$

is already in standard form. The coefficient functions are

$$P(x) = x \quad \text{and} \quad Q(x) = \ln x.$$

Now $P(x) = x$ is analytic at every real number, and $Q(x) = \ln x$ is analytic at every *positive* real number. However, since $Q(x) = \ln x$ is discontinuous at $x = 0$ it cannot be represented by a power series in x, that is, a power series centered at 0. We conclude that $x = 0$ is a singular point of the DE.

(b) By putting $xy'' + y' + xy = 0$ in the standard form

$$y'' + \frac{1}{x}y' + y = 0,$$

we see that $P(x) = 1/x$ fails to be analytic at $x = 0$. Hence $x = 0$ is a singular point of the equation. ■

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

A number $x = x_0$ is an ordinary point of (1) if $a_2(x_0) \neq 0$, whereas $x = x_0$ is a singular point of (1) if $a_2(x_0) = 0$.

EXAMPLE 3 Ordinary and Singular Points

(a) The only singular points of the differential equation

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other values of x are ordinary points.

(b) Inspection of the Cauchy-Euler

$$\downarrow a_2(x) = x^2 = 0 \text{ at } x = 0$$

$$x^2y'' + y = 0$$

shows that it has a singular point at $x = 0$. All other values of x are ordinary points.

(c) Singular points need not be real numbers. The equation

$$(x^2 + 1)y'' + xy' - y = 0$$

has singular points at the solutions of $x^2 + 1 = 0$—namely, $x = \pm i$. All other values of x, real or complex, are ordinary points. ■

THEOREM 6.2.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form of a power series centered at x_0, that is,

$$y = \sum_{n=0}^{\infty} c_n(x - x_0)^n.$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

EXAMPLE 5 Power Series Solutions

Solve $y'' - xy = 0$.

EXAMPLE 7 Three-Term Recurrence Relation

If we seek a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ for the differential equation

$$y'' - (1 + x)y = 0,$$

EXAMPLE 8 DE with Nonpolynomial Coefficients

Solve $y'' + (\cos x)y = 0$.

6.3 Solutions About Singular Points

INTRODUCTION The two differential equations

$$y'' - xy = 0 \qquad \text{and} \qquad xy'' + y = 0$$

A DEFINITION A singular point x_0 of a linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is further classified as either regular or irregular. The classification again depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

DEFINITION 6.3.1 Regular and Irregular Singular Points

A singular point $x = x_0$ is said to be a **regular singular point** of the differential equation (1) if the functions $p(x) = (x - x_0)\,P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

EXAMPLE 1 Classification of Singular Points

It should be clear that $x = 2$ and $x = -2$ are singular points of

$$(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0.$$

EXAMPLE $x^3y'' - 2xy' + 8y = 0$ $x = 0$ is an irregular singular point

THEOREM 6.3.1 Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}, \tag{4}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

EXAMPLE 2 Two Series Solutions

Because $x = 0$ is a regular singular point of the differential equation

$$3xy'' + y' - y = 0, \tag{5}$$

INDICIAL EQUATION Equation (6) is called the **indicial equation** of the problem, and the values $r_1 = \frac{2}{3}$ and $r_2 = 0$ are called the **indicial roots,** or **exponents,** of the singularity $x = 0$.

It is possible to obtain the indicial equation in advance of substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation. If $x = 0$ is a regular singular point of (1), then by Definition 6.3.1 both functions $p(x) = xP(x)$ and $q(x) = x^2 Q(x)$, where P and Q are defined by the standard form (2), are analytic at $x = 0$; that is, the power series expansions

$$p(x) = xP(x) = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{and} \qquad q(x) = x^2 Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots \tag{12}$$

$$y'' + P(x)y' + Q(x)y = 0.$$

are valid on intervals that have a positive radius of convergence. By multiplying (2) by x^2, we get the form given in (3):

$$x^2y'' + x[xP(x)]y' + [x^2Q(x)]y = 0. \tag{13}$$

$$(x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0, \tag{3}$$

we find the general indicial equation to be

$$r(r - 1) + a_0r + b_0 = 0,$$

THREE CASES For the sake of discussion let us again suppose that $x = 0$ is a regular singular point of equation (1) and that the indicial roots r_1 and r_2 of the singularity are real. When using the method of Frobenius, we distinguish three cases corresponding to the nature of the indicial roots r_1 and r_2. In the first two cases the symbol r_1 denotes the largest of two distinct roots, that is, $r_1 > r_2$. In the last case $r_1 = r_2$.

CASE I: If r_1 and r_2 are distinct and the difference $r_1 - r_2$ is not a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0.$$

$n = 0$

This is the case illustrated in Examples 2 and 3.

CASE II: If r_1 and r_2 are distinct and the difference $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0, \tag{19}$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0, \tag{20}$$

where C is a constant that could be zero.

CASE III: If r_1 and r_2 are equal, then there always exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0, \tag{21}$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}. \tag{22}$$

EXAMPLE 5 Example 4 Revisited Using a CAS

Find the general solution of $xy'' + y = 0$.

EXAMPLE 5 Example 4 Revisited Using a CAS

Find the general solution of $xy'' + y = 0$.

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots,$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int 0\,dx}}{[y_1(x)]^2}\,dx = y_1(x)\int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots\right]^2}$$

$$= y_1(x)\int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots\right]} \quad \leftarrow \text{after squaring}$$

$$= y_1(x)\int \left[\frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots\right] dx \quad \leftarrow \text{after long division}$$

$$= y_1(x)\left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right] \quad \leftarrow \text{after integrating}$$

$$= y_1(x)\ln x + y_1(x)\left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \cdots\right],$$

or $$y_2(x) = y_1(x)\ln x + \left[-1 - \frac{1}{2}x + \frac{1}{2}x^2 + \cdots\right].$$ $\leftarrow$ after multiplying out

On the interval $(0, \infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$. ■

11

Fourier Series

DEFINITION 11.2.1 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) \tag{8}$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx. \tag{11}$$

EXAMPLE 1 Expansion in a Fourier Series

Expand

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases} \tag{12}$$

in a Fourier series.

EXERCISES 11.2

THEOREM 11.3.1 Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

11.3 Fourier Cosine and Sine Series

DEFINITION 11.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function f defined on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x, \tag{1}$$

(*continued*)

where
$$a_0 = \frac{2}{p}\int_0^p f(x)\, dx \tag{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x) \cos \frac{n\pi}{p} x\, dx. \tag{3}$$

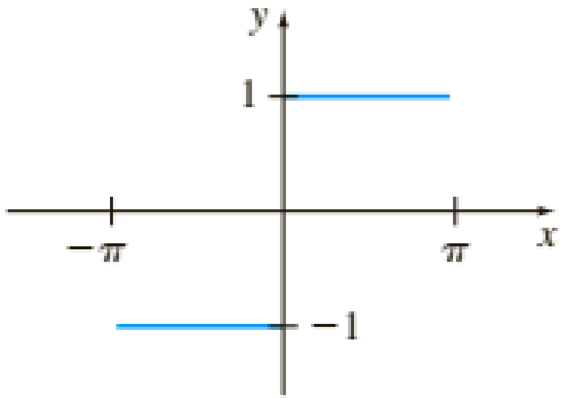
(*ii*) The Fourier series of an odd function f defined on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x, \tag{4}$$

where
$$b_n = \frac{2}{p}\int_0^p f(x) \sin \frac{n\pi}{p} x\, dx. \tag{5}$$

EXAMPLE 2 Expansion in a Sine Series

The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi, \end{cases}$ sl



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