Asg: 9

1. Find the closest point in the subspace spanned by v_1 and v_2 to the point x, where

$$v_1 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix}, \quad \& \ x = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

Solution:

Check for orthogonality:

$$v_1 \cdot v_2 = v_1^T \cdot v_2 = \begin{bmatrix} 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix} = -15 + 20 + 10 = 15$$

$\{v_1, v_2\}$ is non-orthogonal, since v_1 nor v_2 is 0, that means $\{v_1, v_2\}$ is linearly independent.

$$\text{Closest point} = \frac{x v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} \cdot v_2$$

$$x v_1 = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 9 - 20 + 2 = -9$$

$$v_1 \cdot v_1 = \begin{bmatrix} 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 9 + 25 + 4 = 38$$

$$x v_2 = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix} = -15 - 16 + 5 = -26$$

$$v_2 \cdot v_2 = \begin{bmatrix} -5 & 4 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix} = 25 + 16 + 20 = 61$$

$$= \frac{-9}{38} v_1 + \left(\frac{-26}{61}\right) v_2$$

$$= \frac{-9}{38} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} + \left(-\frac{26}{61}\right) \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -0.710526 \\ -1.184210 \\ -0.473684 \end{bmatrix} + \begin{bmatrix} 2.13114754 \\ -0.0776581 \\ -0.03882655 \end{bmatrix}$$

$$\begin{bmatrix} 1.4206 \\ -1.261 \\ -0.512 \end{bmatrix}$$

2.

According to the Gram-Schmidt process:

$$\vec{u_k} = \vec{v_k} - \sum_{j=1}^{k-1} proj_{u_j}(\vec{v_k}), \text{ where}$$

$proj_{\vec{u_j}}(\vec{v_k}) = \dfrac{u_j \cdot v_k}{|\vec{u_j}|^2}\vec{u_j}$, is a projection vector

The normalized vector is $\vec{e_k} = \dfrac{\vec{u_k}}{|\vec{u_k}|}$

$$\vec{u_1} = \vec{v_1} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ -4 \end{bmatrix}$$

$$\vec{e_1} = \frac{\vec{u_1}}{|\vec{u_1}|} = \begin{bmatrix} 2\sqrt{58}/29 \\ 5\sqrt{58}/58 \\ -\sqrt{58}/58 \\ -2\sqrt{58}/29 \end{bmatrix}$$

$$u_2 = \vec{v_2} - proj_{\vec{u_1}}(\vec{v_2}) = \begin{bmatrix} -75/29 \\ -14/29 \\ 26/29 \\ -99/29 \end{bmatrix}$$

$$\vec{e_2} = \frac{\vec{u_2}}{|\vec{u_2}|} = \begin{bmatrix} \frac{-75\sqrt{16298}}{16298} \\ \frac{-7\sqrt{16298}}{8149} \\ \frac{13\sqrt{16298}}{8149} \\ \frac{-99\sqrt{16298}}{16298} \end{bmatrix}$$

Therefor, set of orthonormal vector is:

$$\left\{ \begin{bmatrix} \frac{2\sqrt{58}}{29} \\ \frac{5\sqrt{58}}{58} \\ -\frac{\sqrt{58}}{58} \\ -\frac{2\sqrt{58}}{29} \end{bmatrix}, \begin{bmatrix} -\frac{75\sqrt{16298}}{16298} \\ -\frac{7\sqrt{16298}}{8149} \\ \frac{13\sqrt{16298}}{8149} \\ -\frac{99\sqrt{16298}}{16298} \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & -2 \\ -2 & 0 & 6 \end{bmatrix}$$

want:

$A = QR$

orthogonal matrix (orthonormal columns)

upper triangular matrix

3// Step1:

columns of A ~> orthonormal set (using Graham Smidt process).

Use Gram-schmidt process to find an orthogonal basis for the column space W of A. Segregate, the columns of matrix as $x_1, x_2, \& x_3$.

$A = QR$, multiply both sides of matrix equation with Q^T.

$Q^T A = Q^T Q R$

$Q^T A = I R$, since columns of Q are ortho..

$Q^T A = R$

Q from python script:

$$Q = \begin{bmatrix} -0.33333333 & -0.2981424 & 0.89442719 \\ 0.66666667 & -0.74535599 & 0.0 \\ 0.66666667 & 0.59628479 & 0.4472136 \end{bmatrix}$$

transpose of Q:

$$Q^T = \begin{bmatrix} -0.33333333 & 0.66666667 & 0.66666667 \\ -0.2981424 & -0.74535599 & 0.59628479 \\ 0.89442719 & \cancel{0.59628479}\ 0.0 & 0.4472136 \end{bmatrix}$$

Determine: $Q^T A$ to get R:

$$Q^T A = \begin{bmatrix} 0.33333333 & 0.6666667 & 0.66666667 \\ 0.2981424 & -0.74535599 & 0.596284 \\ 0.89442719 & 0.0 & 0.447213 \end{bmatrix}$$

$$QP_k \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & -2 \\ -2 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & -2 \\ 2 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3.00000001 & 2.00000001 & 1.33333336 \\ 0 & -2.23606797 & 3.87585112 \\ 0.0000001 & 0 & 6.26099035 \end{bmatrix}$$

this is equivalent to the R produced by the python script using scipy.linalg.qr.

<u>verified</u>

3. ~~Compute QR factorization of the given matrix A using~~

4. Find the least squares solution given A and b.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Solution:

Compute $A^T A$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

The equation $A^T A x = A^T b$

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7. Find a basis for the eigenspace corresponding to each listed eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \lambda = 1, 2, 3$$

Solution:

An Eigen vector of $m \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. The λ is Eigen value and x Eigen vector corresponding λ.

Consider the matrix: $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

The scalar 1 is Eigen value iff the equation $Ax = 3x$, has a non-trivial solution.

$(A - 1I)x = 0$

The matrix $A - 1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A - 1I = \begin{bmatrix} 4-1 & 0 & 1 \\ -2 & 1-1 & 0 \\ -2 & 0 & 1-1 \end{bmatrix}$$

$$A - 1I = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$R_2 - R_1 \rightarrow R_2$

$$A - 1I \quad \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. On row reducing:

$$A - 2I = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free variables : 1

$x_3 = k$ & $-2x_1 - x_2 = 0$

$-x_2 + x_3 = 0$

$x_2 = x_3$

$-2x_1 - k = 0$

$x_1 = -k/2$

so, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -2k \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

The scalar 3 is ~~equivalent~~ Eigen value if & only if equation $Ax = 3x$, has a non-trivial solution.

$(A - 3I)x = 0$

The matrix: $A - 3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

On row-reduction:

$$A - 3I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

one free variable:

$x_3 = k$, $x_1 + x_3 = 0$

$x_2 - x_3 = 0$

$x_2 = x_3$

$x_1 + k = 0$

$x_1 = -k$

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

The basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.