4. Find the closest point in the subspace spanned by v_1 and v_2 to the point x, where

$$v_1 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix}, \text{ and } x = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

Solution:

$$v_1 \cdot v_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix} = 15$$

So, $\{v_1, v_2\}$ is not orthogonal set. And because neither v_1, v_2 is 0, this means $\{v_1, v_2\}$ are linearly independent.

$$\text{Closest point} = \frac{x \cdot v_1}{v_1 \cdot v_1} \cdot v_1 + \frac{x \cdot v_2}{v_1 \cdot v_2} \cdot v_2$$

$$= \frac{9 - 20 + 2}{9 + 25 + 4} v_1 + \frac{-15 - 16 + 5}{-15 + 20 + 10} v_2$$

$$= \frac{-9}{38} v_1 - \frac{26}{15} v_2$$

$$= \frac{-9}{38} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} - \frac{26}{15} \begin{bmatrix} -5 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -27/38 \\ -45/5 \\ -18/38 \end{bmatrix} - \begin{bmatrix} -130/15 \\ \frac{-104}{15} \\ \frac{130}{15} \end{bmatrix} = \begin{bmatrix} \frac{-27}{38} + \frac{130}{15} \\ \frac{-45}{5} + \frac{104}{15} \\ \frac{-18}{38} - \frac{130}{15} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7109}{114} \\ -21/15 \\ \frac{-9.8}{57} \end{bmatrix}$$

5. Find the best approximation to z by vectors of the form $c_1 v_1 + c_2 v_2$

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix}, \quad \& \ z = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

now,

$$v_1 \cdot v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix} = 6 + 5 - 4 + 20 = 27$$

$$z = \frac{z \cdot v_1}{v_1 \cdot v_1} \cdot v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2} \cdot v_2 = c_1 v_1 + c_2 v_2$$

$$c_1 = \frac{z \cdot v_1}{v_1 \cdot v_1} = \left(\frac{-8}{30}\right) \cdot \cancel{\begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 0.267 \\ -0.533 \\ 1.067 \end{bmatrix}}$$

$$-2/15$$

$$z v_1 = [1 \ 3 \ 0 \ 2] \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix}$$

$$= 3 - 3 + 0 - 8$$

$$= -8$$

$$v_1 v_1 = [3 \ -1 \ 2 \ -4] \begin{bmatrix} 3 \\ -1 \\ 2 \\ -4 \end{bmatrix}$$

$$9 + 1 + 4 + 16$$

$$= 30$$

$$c_2 = \frac{z \cdot v_2}{v_2 \cdot v_2} = \frac{-23}{58}$$

$$z v_2 = [1 \ 3 \ 0 \ 2] \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix}$$

$$= 2 - 15 - 0 - 10$$

$$= -23$$

$$v_2 v_2 = [2 \ -5 \ -2 \ -5] \begin{bmatrix} 2 \\ -5 \\ -2 \\ -5 \end{bmatrix}$$

$$4 + 25 + 4 + 25$$

$$58$$

Hence,

$z = c_1 v_1 + c_2 v_2$ has best approximation

$$-\frac{2}{15}\begin{bmatrix}3\\-1\\2\\-4\end{bmatrix} - \frac{23}{58}\begin{bmatrix}2\\-5\\-2\\-5\end{bmatrix}$$

$$= \begin{bmatrix}-0.4\\0.133\\-0.267\\0.533\end{bmatrix} - \begin{bmatrix}0.793\\-1.982\\-0.793\\-1.982\end{bmatrix}$$

$$= \begin{bmatrix}-1.193\\2.115\\0.526\\2.515\end{bmatrix}$$

6. Verify that $\{u_1, u_2\}$ is an orthogonal set and then find the orthogonal projection of y onto span $\{u_1, u_2\}$ where:

$$u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \text{ and } y = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

Solution:

prove $\{u_1, u_2\}$ is orthogonal set

$$u_1 \cdot u_2 = u_1^T \cdot u_2 = [3, 4, 0] \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$= -12 + 12 + 0$$

$$= 0$$

Thus, $\{u_1, u_2\}$ is an orthogonal set.

The orthogonal projection of y onto span $\{u_1, u_2\}$ is

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \quad \text{—— (1)}$$

Now:

$$y \cdot u_1 = y^T \cdot u_1 = [4, 3, -2] \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

$$= 12 + 12 - 0 = 24$$

$$u_1 \cdot u_1 = u_1^T \cdot u_1 = [3 \ 4 \ 0] \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

$$= 9 + 16 + 0 = 25$$

$$y \cdot u_2 = y^T \cdot u_2 = [4 \ 3 \ -2] \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$= -7$$

$u_2 u_2 = 25$

from (1)

$$\hat{y} = \frac{24}{25}\begin{bmatrix}3\\4\\0\end{bmatrix} + \frac{-7}{25}\begin{bmatrix}-4\\3\\0\end{bmatrix}$$

$$= \begin{bmatrix}2.88\\3.84\\0\end{bmatrix} + \begin{bmatrix}1.12\\-0.84\\0\end{bmatrix}$$

$$= \begin{bmatrix}2.88+1.12\\3.84-0.84\\0+0\end{bmatrix}$$

$$= \begin{bmatrix}4\\3\\0\end{bmatrix}$$

7. let $y = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$, $u_1 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$, and $W = \text{Span}\{u_1\}$.

(a) let U be 2×1 matrix whose only column is u_1. compute $U^T U$ and $U U^T$.
Solution.

~~UTU~~

Consider U (2×1) matrix $= \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

UU^T

$$\cancel{UU}^T = \underset{2\times 1}{\begin{bmatrix} 4/5 \\ \frac{3}{5} \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 4/5 & 3/5 \end{bmatrix}}$$

$$\begin{bmatrix} \frac{4}{5} \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{\frac{20}{25}} + \frac{9}{25} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{29}{25} \end{bmatrix}$$

$$\frac{29}{25}$$

$$\frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

(b) $$U^T U = \underset{1\times 2}{\begin{bmatrix} \frac{4}{5} & \frac{3}{5} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}}$$

$$= \cancel{\frac{4}{5} \times \frac{4}{5} + \frac{3}{5} \times \frac{3}{5}}$$

$$\frac{16}{5} +$$

$$= \frac{16}{25} + \frac{9}{25}$$

$$= \frac{25}{25} = 1$$

(b) compute $proj_w y$ & $(UU^T)y$.

~~$Proj_w y = \frac{\langle y \cdot u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$~~

=

(a) $U^TU = \underset{(1\times2)}{\begin{bmatrix} 4/5 & 3/5 \end{bmatrix}} \underset{(2\times1)}{\begin{bmatrix} 4/5 \\ \frac{3}{5} \end{bmatrix}}$

$= 4/5 \times 4/5 + 3/5 \times 3/5$

$= 16/25 + 9/25$

$= \frac{25}{25}$

$= 1.$

$UU^T = \underset{2\times1}{\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}} \cdot \underset{1\times2}{\begin{bmatrix} 4/5 & 3/5 \end{bmatrix}}$

$\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$

(b) $U^TU = I_{2,1}$ the columns of U form an orthonormal basis for $W = span\{u_1\}$.

The projection of y onto $span\{u_1\}$ is given as follows:

$$\text{proj}_w y = \frac{y \cdot u_1}{u_1 \cdot u_2} \, u_2 \quad — \quad 1$$

$$y \cdot u_2 = y^T \cdot u_2$$

$$= \underset{1\times 2}{\begin{bmatrix} 9 & 4 \end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}}$$

$$= 9\times\frac{4}{5} + 4\times\frac{3}{5}$$

$$= 36/5 + 12/5$$

$$= \frac{36+12}{5} = \frac{48}{5}$$

$$u_1 \cdot u_1 = \cancel{\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}^2 = 12}$$

$$= \left(4/5\right)^2 + \left(\frac{3}{5}\right)^2$$

$$= \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

$$\text{proj}_w y = \frac{48/5}{1} \cdot \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$= \frac{48}{5} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{192}{25} \\ \frac{144}{25} \end{bmatrix}$$

$$(UU^T)y = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$2\times2 \qquad 2\times1$

$$\begin{bmatrix} \frac{16}{25} * 9 + \frac{12}{25} \times 4 \\ \frac{12}{25} \times 9 + \frac{9}{25} \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 144/25 + \frac{48}{25} \\ \frac{108}{25} + \frac{36}{25} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{144+48}{25} \\ \frac{108+36}{25} \end{bmatrix} = \begin{bmatrix} 7.68 \\ 5.76 \end{bmatrix}$$