$$P = \begin{bmatrix} 4 & -9 & 5 \\ -3 & -1 & 6 \\ 9 & -2 & -6 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

(a) Find basis $\{a_1, a_2, a_3\}$ for $\mathbb{R}^3$ such that P is the change-of-coordinates matrix from $\{a_1, a_2, a_3\}$ to the basis $\{b_1, b_2, b_3\}$.

Solution

We know that to find the change of basis matrix P from (a_1, a_2, a_3) to $\{b_1, b_2, b_3\}$, we have to write each element in the basis $B = \{a_1, a_2, a_3\}$ in linear combination of elements of basis $C = \{b_1, b_2, b_3\}$ and $P = [[a_1]_C, [a_2]_C, [a_3]_C]$ and C is the basis of $\{b_1, b_2, b_3\}$.

Therefore the basis is obtained as follows:

$$a_j = [a_1 \ a_2 \ a_3][b_j]_C$$

$$\Rightarrow [a_1 \ a_2 \ a_3] = [b_1 \ b_2 \ b_3][[b_1]_C \ [b_2]_C \ [b_3]_C]$$

$$\Rightarrow [a_1 \ a_2 \ a_3] = [b_1 \ b_2 \ b_3]P$$

$$\Rightarrow [a_1, a_2, a_3] = \begin{bmatrix} 0 & 4 & 3 \\ -1 & 5 & 3 \\ 3 & -4 & -6 \end{bmatrix} \begin{bmatrix} 4 & -9 & 5 \\ -3 & -1 & 6 \\ 9 & -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -10 & 6 \\ 8 & -2 & 7 \\ -30 & -11 & 27 \end{bmatrix}$$

$$\Rightarrow a_1 = \begin{bmatrix} 15 \\ 8 \\ -30 \end{bmatrix} \quad a_2 = \begin{bmatrix} -10 \\ -2 \\ -11 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 6 \\ 7 \\ 27 \end{bmatrix}$$

Here, upon further processing, it can be shown that P is the change-of-coordinates matrix from $\{a_1 \ a_2 \ a_3\}$ to basis $\{b_1, b_2, b_3\}$.

2. Find the vector x determined by the given coordinate vector $[x]_B$ and basis B.

$$B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}; [x]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution

By the definition of (B-coordinate of x):
The vector x determined by the coordinate vector $[x]_B$ and basis $B = \{B_1, B_2\}$ is

$$x = c_1 B_1 + c_2 B_2$$

Here, $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\{B_1, B_2\} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$.

$$x = c_1 B_1 + c_2 B_2$$

$$= 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 \\ -2+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

3. Find the coordinate vector $[x_B]$ of x,

$$B = \left\{ \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -5 \end{bmatrix} \right\}, \quad x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

<u>Solution</u>

To find the coordinate vector $[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, of x relative to basis $B = \{b_1, b_2, b_3\}$.
Solve the equation

$$c_1 \cdot \vec{b}_1 + c_2 \cdot \vec{b}_2 + c_3 \cdot \vec{b}_3 = x \text{ for } c_1, c_2, c_3 \text{ f}$$

write them in column-wise.

$$b_1 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -3 \\ 1 \\ -5 \end{bmatrix} \text{ and } x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3c_1 + 6c_2 - 3c_3 \\ 3c_1 \quad 0 \quad c_3 \\ -6c_1 \quad 8c_2 \quad -5c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -3 \\ 3 & 0 & 1 \\ -6 & 8 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$\downarrow$ Augmented matrix

$$\begin{bmatrix} 3 & 6 & -3 & 3 \\ 3 & 0 & 1 & 2 \\ -6 & 8 & -5 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 3/7 \\ 0 & 1 & 0 & 9/14 \\ 0 & 0 & 1 & 5/7 \end{bmatrix}$$

Thus: $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 9/14 \\ 5/7 \end{bmatrix} \quad [x_B] = \begin{bmatrix} 3/7 \\ 9/14 \\ 5/7 \end{bmatrix}$

4 solution.

These three column vectors defines a 3×3 matrix.

$$P = \begin{bmatrix} 3 & 6 & -3 \\ 3 & 0 & 1 \\ -6 & 8 & -5 \end{bmatrix}$$

which is a matrix of linear map.

$$Id : (\mathbb{R}^3, B) \to (\mathbb{R}^3, E)$$

This means in particular that whenever we right multiply it by a column vector (x_1, x_2, x_3) where x_j are the coordinates of a vector $x = x_1B_1 + x_2B_2 + x_3B_3$ with respect to the Basis B, we obtain the coordinates of x in the canonical basis E.

What we want is the matrix of:

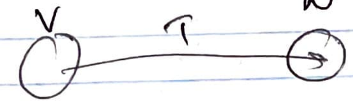
$$Id : (\mathbb{R}^3, E) \to (\mathbb{R}^3, B).$$

That is P^{-1} (the inverse of matrix above). This will transform, by right multiplication, the coordinate of vector with respect to E into its coordinate with respect to B.

$$P^{-1} = \begin{bmatrix} 4/21 & -1/7 & -1/7 \\ -3/14 & 1/14 & 2/7 \\ -4/7 & 10/7 & 3/7 \end{bmatrix}$$

5. Give a specific example to show that a plane in R^n NOT going through the origin is not isomorphic to $\mathbb{R}^2$.

Def: Two vectors V & W are isomorphic iff $\exists T : V \to W$ linear, 1-1, ONTO



If the plane in R^n that doesn't going through the origin will have different dimension. So, they have different dimensions and they're not isomorphic.

→ For instance, let V and W be two finite dimensional vector space, then when W is not going through the origin, and V does, then they are dimensionally different.

$V \neq M_{2\times2}(\mathbb{R})$: ~~and so~~ not dim = 4.

and V can be written as:

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R}) : a+d=0\right\} = \left\{\begin{pmatrix} -d & b \\ c & d \end{pmatrix} : b, c, d \in \mathbb{R}\right\}$$

It follows $\dim(V) = 3$, so V and W are not isomorphic since they are not have not same dimension.

6 Determine the dimension of the following set of vectors.

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-4\\5\end{bmatrix}, \begin{bmatrix}-2\\0\\4\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\0\\5\end{bmatrix}, \begin{bmatrix}5\\4\\8\\-15\end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} 0 & -1 & -2 & -3 & 5 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & -4 & 4 & 0 & 8 \\ 0 & 5 & 0 & 5 & -15 \end{bmatrix}$

The dimension of set of vector above is same as the dimension of the column space of A.
Convert A to ~~row~~ reduced echelon form.
we get:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 4 \\ 0 & \boxed{-1} & -2 & -3 & 5 \\ 0 & 0 & \boxed{12} & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It has three pivot columns. Therefore, the basis for subspace is the column in A with respective pivot columns of reduced echelon form of A.
Since, the number of elements in the subspace is called dimension.

Therefore the dimension of the above set of ~~vectors~~ is 3

7. Determine the dimension of Nul A, Col A, and Col A^T for the matrix formed by the vector given in the previous problem.

Solution:

let

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 & 5 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & -4 & 4 & 0 & 8 \\ 0 & 5 & 0 & 5 & -15 \end{bmatrix}$$

Solution:

The number of pivot columns in any $m \times n$ matrix S, gives the dimension of Col of S. and the number of free variables in the equation $AX=0$ gives the dimension of Nul S

Change A to reduced row echelon form

$$A \xrightarrow{ref} \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & -1 & -2 & -3 & 5 \\ 0 & 0 & 12 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three pivot columns

Thus, the dimension of Col A = 3

The system can be written as
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & -1 & -2 & -3 & 5 \\ 0 & 0 & 12 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 4x_5 = 0$ (i)

$-x_2 - 2x_3 - 3x_4 + 5x_5 = 0$ (ii)

$12x_3 + 12x_4 - 12x_5 = 0$ (iii)

$x_3 + x_4 - x_5 = 0$ (iii)

Solve these equations

$x_1 = -4x_5$ (a) $\rightarrow x_5 = -\frac{1}{4}x_1$

$x_2 = 2x_3 + 3x_4 - x_5$

$x_2 = 2x_3 + 3x_4 + \frac{1}{4}x_1$ (b)

$x_3 + x_4 - x_5 = 0$

$x_5 = x_3 + x_4$

from (b)

$x_2 = 2x_3 + 3x_4 - x_3 - x_4$

$x_2 = x_3 + 2x_4$

from (a)

$x_1 = -4(x_3 + x_4)$

$x_1 = -4x_3 - 4x_4$

So,

$x_1 = -4x_3 - 4x_4$

$x_2 = x_3 + 2x_4$

$x_5 = x_3 + x_4$

x_1, x_2 and x_5 are the linear combination of x_3 & x_4.

So, x_3 and x_4 are free variable.

As there are two free variables in Null A, So, the number of vectors in basis for Null A is 2.

So, dimension $= \boxed{2}$

$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -4 & 5 \\ -2 & 0 & 4 & 0 \\ -3 & 0 & 0 & 5 \\ 5 & 4 & 8 & -15 \end{bmatrix}$$

$$A^T \xrightarrow{rref} \begin{bmatrix} \boxed{-1} & 0 & -4 & 5 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{12} & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row echelon form

There are three pivot colums.
Thus, the dimension of col $A^T = \boxed{3}$

10

Let $S = \left\{ \begin{bmatrix} 2b+3c \\ a+b-2c \\ 4a+b \\ 3a-b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

For any vector x in S,

$$x = \begin{bmatrix} 2b+3c \\ a+b-2c \\ 4a+b \\ 3a-b-c \end{bmatrix} \text{ for some } a, b, c \in \mathbb{R}$$

$$x = \begin{bmatrix} 0 \\ a \\ 4a \\ 3a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ b \\ -b \end{bmatrix} + \begin{bmatrix} 3c \\ -2c \\ 0 \\ -c \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} a + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} b + \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} c$$

Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}$

Thus, every vector in S is a linear combination of vectors v_1, v_2, and v_3.
Therefore, set of vectors $\{v_1, v_2, v_3\}$ spans the set S.
Clearly, v_1 is non-zero vector. The vector v_2 is not the scalar multiple of v_1.