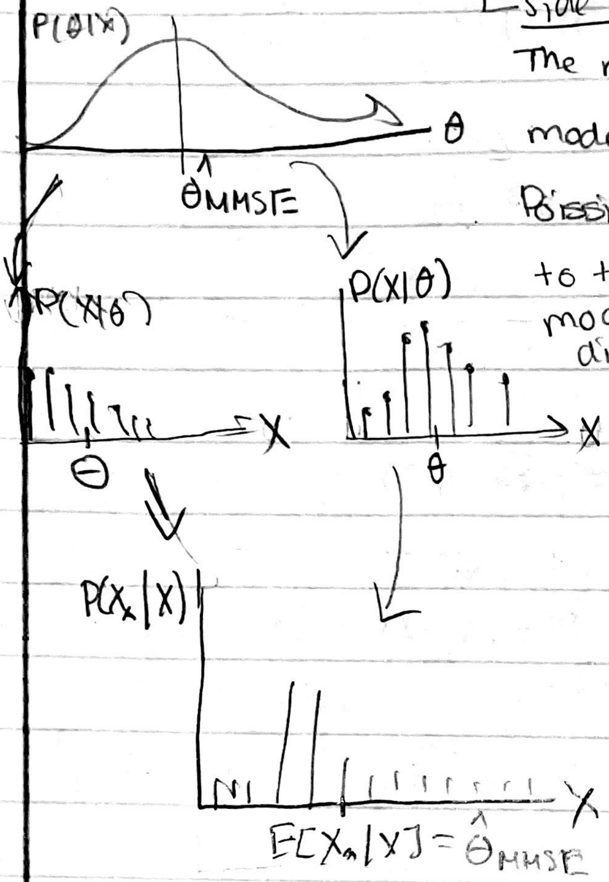
From 368... the negative binomial is the sum of iid geometric random variable. Since the expectation of the geometric rv is $(1-p)/p$, the expectation of the negative binomial by linearity is

$$P(X_*|X) = \text{ExtNegBin}(r,p) \Rightarrow E[X_*|X] = r\,\frac{1-p}{p}$$

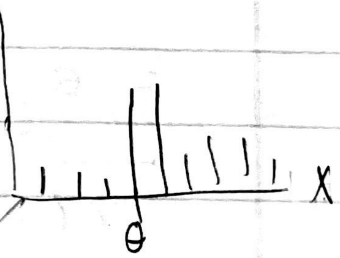
Lec 15



Side Note: The negative binomial model is a "dispersed" Poisson analogous to the betabinomial model being a dispersed binomial

$$= \left(\sum x_i + \alpha\right)\frac{\frac{1}{n+\beta+1}}{\frac{n+\beta}{n+\beta+1}}$$

$$= \frac{\sum x_i + \alpha}{n+\beta} = \hat\theta_{MMSE}$$



$$Var[X_*|X] = r\,\frac{1-p}{r} = \hat\theta_{MMSE} \cdot \frac{1}{p} \quad \frac{1}{p} = \frac{\beta+n+1}{\beta+n}$$

$$= \hat\theta_{MMSE}$$

Curly F: one (N θ, σ^2 $= P(x|\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma}(x-\theta)^2}$

let σ^2 be fixed / known in advance

$$P(X|\theta,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(X-\theta)^2} = e^{\frac{-1}{2\sigma^2}(X^2 - 2X\theta + \theta^2)}$$

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$$e^{-\frac{x}{2\sigma}} e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\theta^2}{-2\sigma^2}} \propto e^{\frac{-x}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}}$$

$$= e^{ax-bx^2}, \quad a = \frac{\theta}{\sigma^2}, \; b = \frac{1}{2\sigma^2}$$

$$P(\theta \mid x, \sigma^2) \propto e^{-\frac{1}{2\sigma}(x-\theta)^2} = e^{\frac{-x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}}$$

$$\propto e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\sigma^2}{2\sigma^2}} = e^{a\theta - b\theta^2}, \quad a = \frac{x}{\sigma^2}, \; b = \frac{1}{2\sigma^2}$$

$$E[\theta] = x = \frac{a}{2b}, \; Var[\theta] = \sigma^2 = \frac{1}{2b}$$

$$E[X] = \theta = \frac{a}{2b}$$

$$= \frac{\frac{\theta}{\sigma^2}}{2\left(\frac{1}{2\sigma^2}\right)} = \theta$$

$$Var[X] = \sigma^2 = \frac{1}{2b}$$

$$= \frac{1}{2\left(\frac{1}{2\sigma}\right)} = \sigma^2$$

F: $\overset{iid}{\sim} N(\theta, \sigma^2)$ $x_1, \ldots, x_n$

$$P(x \mid \theta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$$

$$\sum(x_i - \theta)^2 = \sum x_i^2 - 2x_i\theta + \theta^2$$

$$= \sum x_i^2 - 2n\bar{x}\theta + n\theta^2$$

$$(2\pi\sigma^2)^{-n/2} e^{\frac{-\sum x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}}$$

$$= \mathcal{L}(\theta; x, \sigma^2)$$

$$\propto e^{\frac{-\Sigma x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}}$$

$$P(\theta | x, \sigma^2) \propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} = e^{a\theta - b\sigma^2}, \quad a = \frac{n\bar{x}}{\sigma^2}, \quad b = \frac{n}{2\sigma^2}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N\left(\frac{\frac{n\bar{x}}{\sigma^2}}{\frac{n}{\sigma^2}}, \frac{1}{\frac{n}{\sigma^2}}\right) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

All we've done thus far is probability theory and we seemingly just made random computations for fun. Now we'll do Bayesian...

$F: x_1 \dots x_n \overset{iid}{\sim} N(\theta, \sigma^2)$ with σ^2 known. Let's find posterior $P(\theta | X, \sigma^2) \propto P(\theta | \sigma^2)$

$$= (2\pi\sigma^2)^{-n/2} e^{\frac{-\Sigma x_i}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta | \theta^2)$$

$$\propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta | \sigma^2) = P(\theta | \sigma^2) = N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right)$$

Traditionally ...

$$\alpha = \frac{u_0}{z^2}, \quad \beta = \frac{1}{2z^2} \text{ "au"}$$

$$\propto e^{a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2}$$

$$P(\theta | \sigma^2) = N(u_0, z^2)$$

$$= e^{(a+\alpha)\theta - (b+\beta)\sigma^2}$$

$$\propto N\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \alpha}{\frac{n}{\sigma^2} + 2\beta}, \frac{1}{\frac{n}{\sigma^2} + 2\beta}\right)$$

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{u_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}, \frac{1}{\frac{n}{\sigma} + \frac{1}{z^2}}\right)$$

The normal-normal conjugate model (where sigsq is assumed fixed)

$$P(\theta \mid \sigma^2) = N(u_0, z^2) = P(\theta \mid x, \sigma^2) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{u_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{z^2}}\right)$$

Point estimation

$$\hat{\theta}_{MMSE} = E[\theta \mid x, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}$$

$$\hat{\theta}_{MMAE} = Med[\theta \mid x, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}$$

$$\hat{\theta}_{MAP} = Med[\theta \mid x, \sigma^2] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}$$

Credible Regions

$$CR_{\theta, 1-\alpha_0} = \left[qnorm\left(\frac{\alpha_0}{2}, \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{z^2}}{\frac{n}{\sigma^2} + \frac{1}{z^2}}\right), qnorm\left(1 - \frac{\alpha_0}{2}, mean, var\right)\right]$$

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$H_a : \theta < \theta \Rightarrow H_0 : \theta \geq \theta_0$

$$P_{val} = P(H_0 | X, \sigma^2) = \int_{\theta_*}^{\infty} P(\theta | X, \sigma^2) d\theta = 1 - \text{pnorm}(0_0, \text{mean}, \text{var})$$

Let's calculate the MLE (this was on HW1)

$$L(\theta; x, \sigma^2) = (2\pi\sigma^2)^{\frac{-n}{2}} e^{\frac{-\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\theta^2}}$$

$$\ell(\theta; x, \sigma^2) = \ln(\;) - \frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma}$$

$$\ell'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \overset{\text{set}}{=} 0 \Rightarrow n\bar{x} = n\theta \Rightarrow \hat{\theta}_{MLE} = \bar{X}$$