$P(\sigma^2 \mid \theta) = \text{InvGamma}(\frac{?}{2}, \frac{?}{2}) = \ldots$

Lec 18 Jeffrey's $P_j(\sigma^2 \mid \theta) \propto \sqrt{I(\sigma^2; \theta)}$

$$\ell'(\sigma^2; x, \theta) = \frac{-n}{2\sigma^2} \quad \frac{-(\Sigma x_i - \theta)^2}{(2\sigma^2)^2}$$

$$\ell''(\sigma^2; x, \theta) = \frac{1}{2(\sigma^2)^2} - \frac{\Sigma(x_i - \theta)^2}{(\sigma^2)^3}$$

$$I(\sigma^2; \theta) = \Sigma_x[-\ell''] = E_x\left[-\frac{n}{2(\sigma^2)^2} + \frac{\Sigma(x_i-\theta)^2}{(\sigma^2)^3}\right]$$

Definition of variance x:

$$= \left(-\frac{n}{2(\sigma^2)^2} + \frac{\Sigma E[(x_i-\theta)^2]}{(\sigma^2)^3}\right) = \rightarrow$$

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$$\left(-\frac{n}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\right)$$

$$= \frac{n}{2(\sigma^2)^2}, \quad \frac{+n}{(\sigma^2)^2} = \frac{n}{(\sigma^2)^2}\left(-\frac{1}{2}+1\right) = \frac{n}{2(\sigma)^2}$$

$$P_j(\sigma^2|\theta) \propto \sqrt{\frac{n}{2(\sigma^2)^2}} \propto \sqrt{\frac{1}{(\sigma^2)^2}} = (\sigma^2)^{-1} \propto$$

InvGamma(0,0) = Haldane

* Jeffrey and Haladine are equivalent penalized objective priors and they're the for this model.

Shrinkage $P(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right) = E[\sigma^2|\theta] = \frac{n_0\sigma_0^2}{n_0-6}$

(Valid for $n_0 > 2$)

$$\hat{\theta}^2_{MMSE} = \frac{n\hat{\theta}_{MLE} - n_0\sigma_0^2}{n+n_0-2} = \frac{n\hat{\sigma}^2_{MLE}}{n+n_0-2} + \frac{n_0\sigma_0^2}{n+n_0-2}, \frac{n_0-2}{n_0-2}$$

$$\frac{n}{n+n_0-2}\hat{\sigma}^2_{MLE} + \frac{n_0-2}{n+n_0-2} E[\sigma^2|\theta]$$

Posterior Predictive Distribution

$$P(X_* | X, \theta) = \int_0^\infty \underbrace{P(X_* | \theta, \sigma^2)}_{N(\theta, \sigma^2)} \; \underbrace{P(\sigma^2 | X, \theta)}_{InvGamma\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right)} d\sigma^2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X_* - \theta)^2} \; \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}} d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-1/2} e^{\frac{-(X_* - \theta)/2}{\sigma^2}} (\sigma^2)^{-\alpha-1} e^{\frac{-\beta}{\sigma^2}} d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\underbrace{(\alpha + 1/2)}_{\alpha'} - 1} e^{\frac{\overbrace{-(X_* - \theta)^2/2 - \beta}^{\beta'}}{\sigma^2}} d\sigma^2$$

n-Substi

$$= \frac{\Gamma(\alpha')}{\beta'^{\alpha'}} = \Gamma(\alpha') \beta'^{-\alpha'}$$

$$P(X_* | X, \theta) \propto \Gamma\left(\frac{n+n_0+1}{2}\right) \left(\frac{(x^* - \theta)^2 + \overbrace{n\hat{\theta}^2_{MLE} + n_0 \sigma_*^2}^{a}}{2}\right)^{-\frac{n+n_0+1}{2}}$$

$$\propto \left(\frac{(X_* - \theta)^2 + a}{2}\right)^{-\frac{2+1}{2}} \cdot \left(\frac{2}{a}\right)^{-\frac{2+1}{2}} \cdot \left(\frac{2}{a}\right)^{\frac{2+1}{2}}$$

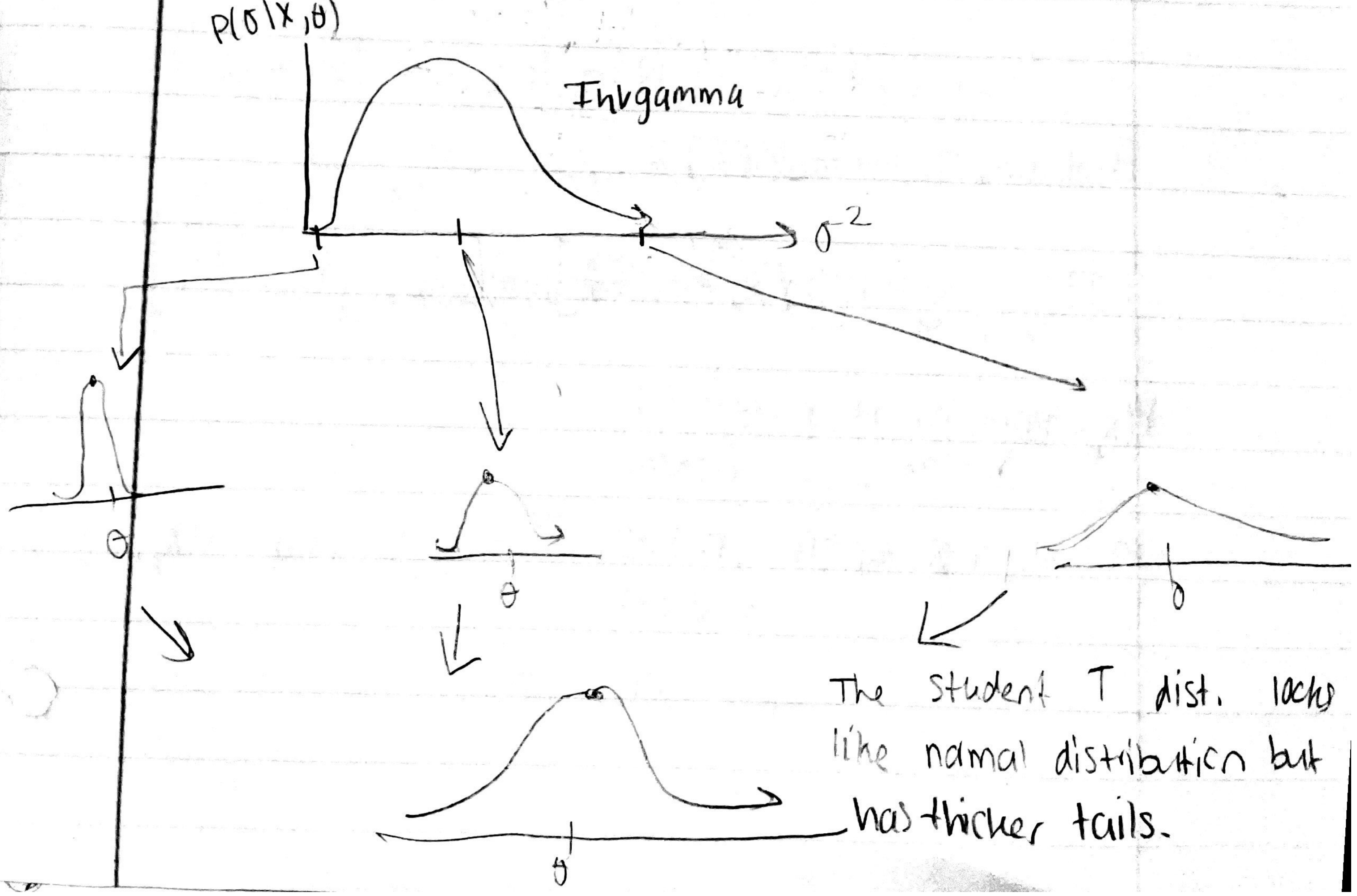
$\rightarrow$

$$\left(\frac{(X_* - \theta)^2}{a} + 1\right)^{\frac{\upsilon+1}{2}} \left(\frac{-2}{a}\right)^{\frac{\upsilon+1}{2}}$$

$$= \left(1 + \frac{1}{\upsilon}\,\frac{(X_* - \theta)^2}{\frac{a}{\upsilon}}\right)^{-\frac{\upsilon+1}{2}} \left(\frac{2}{a}\right)^{\frac{\upsilon+1}{2}}$$

$$\propto 1 + \frac{1}{\upsilon}\left(\frac{(X_* - \theta)^2}{\frac{a}{\upsilon}}\right)^{-\frac{\upsilon+1}{2}} \propto T_{n+n_0}\left(\theta,\ \frac{n\hat{\theta}_{MLE} + n_0\sigma_0^2}{n+n_0}\right)$$

This distribution is "non-standard Student's T distribution" or "Shifted and Scaled Student's T distribution".



The Student T dist. looks like normal distribution but has thicker tails.

$n+n_0 > 20$

$$\pi_{n+n_0}\left(\theta, \frac{n\hat{\theta}_{MLE} + n_0\sigma_0^2}{n+n_0}\right) \approx N\left(\theta, \frac{n\hat{\theta}_{MLE} + n_0\sigma_0^2}{n+n_0}\right)$$

if n is large

$\hat{\theta}_{MLE} \to \sigma^2$

$$\approx N(\theta, \sigma^2)$$

0.5, $n = 12, \hat{\sigma}^2_{MLE} = 0.387$, Jeffrey's prior $= n_0 = \sigma_0^2 = 0$

$$P(X_p > 8 \mid X_*, \theta) = 1 - P(X_p \leq 8\,X \mid \theta)$$

$$= 1 - \text{pt.scaled}\left(\underset{X_*}{8}, \underset{v}{12}, \underset{\theta}{5}, \underset{\sqrt{\text{scale}}}{\sqrt{\frac{12 \cdot 0.387}{12}}}\right)$$

Predictive Intervals (PI)

$$PI_{X_*, 1-\alpha_0} = \left[Q\left[X_* \mid X, \frac{\alpha_0}{2}\right], Q\left[X_*, X, 1 - \frac{\alpha_0}{2}\right]\right]$$

$$P(X_* \in PI_{X_*, 1-\alpha_0} \mid X) = 1 - \alpha_0$$

$\sigma^2 = 1.1$, $\bar{X} = 1.89$, $n = 13$, Jeffrey's prior $= P(X_* \mid X, \sigma^2)$

$$= N(\bar{X}, \sigma^2)$$

$PI_{X_{\alpha}, 95\%} = [qnorm(.025, 1.89, \sqrt{1.1}), qnorm(0.975, 1.89, \sqrt{1.1})]$

Mid 2 $\Uparrow$

Final $\Downarrow$

F: $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ where both θ, σ^2 are unknown.

thus we want inference for both. or inference for one and the other is a "nuisance parameter".

Let's assume Laplace prior.

$$P(\theta, \sigma^2 | x) \propto P(x | \theta, \sigma^2) P(\theta, \sigma^2) \propto P(x | \theta, \sigma^2)$$

$$= (2\pi)^{-n/2} (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \Sigma(x_i - \theta)^2}$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \Sigma(x_i - \theta)^2} \overset{?}{\propto} \text{InvGamma}$$

It is not invgamma since the posterior now is a 2-d distri and the invgamma is only one-dimensional

What we have above is a known distribution but to get it into canonical form, we need to do some algebra.

$$\Sigma(x_i-\theta)^2 = \Sigma((x_i-\bar{x})+(\bar{x}-\theta))^2 = \Sigma(x_i-\bar{x})^2 + 2\Sigma(x_i-\bar{x})(\bar{x}-\theta)+\Sigma(\bar{x}-\theta)^2$$

$$S^2 = \frac{1}{n-1}\Sigma(x_i-\bar{x})^2$$ the sample variance formula from MATH 241

$$\bar{x} = \frac{1}{n}\Sigma x_i \Rightarrow \Sigma x_i = n\bar{x}$$

$$= (n-1)S^2 + 2\,\Sigma(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x}-\theta)^2$$

$$= (n-1)S^2 + n(\bar{x}-\theta)^2 + 2(n\bar{x}^2 - n\bar{x}^2 - n\bar{x}\theta + n\bar{x}\theta)$$

$$P(\theta,\sigma^2|x) \propto (\sigma^2)^{-n/2+1-1} e^{-\frac{1}{2\sigma^2}((n-1)S^2 + n(\bar{x}-\theta)^2)}$$

$$= (\sigma^2)^{-(\frac{n}{2}-1)-1} e^{\frac{-(n-1)S^2/2}{\sigma^2}} e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2}$$

$$= \underbrace{e^{-\frac{1}{\frac{2\sigma^2}{n}}(\bar{x}-\theta)}}_{\propto N(\bar{x},\frac{\sigma^2}{n})} \underbrace{(\sigma^2)^{-(\frac{n}{2}+1)+1} e^{\frac{-(n-1)S^2/2}{\sigma^2}}}_{\propto \text{Invgamma}(\frac{n-2}{2},\frac{(n-1)S^2}{2})}$$

$$\propto \text{NormalInvgamma}\left(\mu=\bar{x}, \lambda=n, \alpha=\frac{n-2}{2}, \beta=\frac{(n-1)S^2}{2}\right)$$

This is the "normal-inverse-gamma" distribution with four parameter.

lec. 1a