lec. 16

$$\hat{\theta}_{MMSE} = \frac{n\bar{x}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} + \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$= \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \hat{\theta}_{MLE} + \frac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$E[\theta | \sigma^2]$

$\rightarrow$

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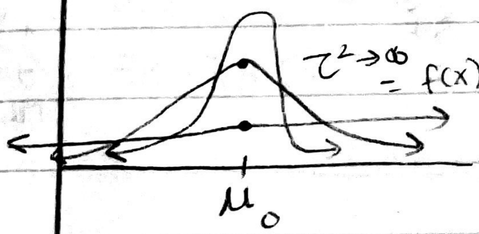
$$\frac{n\tau^2}{n\tau^2} \quad \frac{1}{1+\frac{\sigma^2}{n\tau^2}}\hat{\theta}_{MLE} + \frac{1}{\frac{n\tau^2}{\sigma^2}+1} E[\theta|\sigma^2] \quad \frac{\tau^2}{\tau^2}$$

$$= \underbrace{\frac{n\tau^2}{n\tau^2+\sigma^2}}_{1-e}\hat{\theta}_{MLE} + \underbrace{\frac{\sigma^2}{n\tau^2+\sigma^2}}_{e} E[\theta|\sigma^2]$$

what is Laplace prior? $P(\theta|\sigma^2) \propto 1$ see lec. 15

$$P(\theta|x,\sigma^2) \propto P(X|\theta,\sigma^2)P(\theta|\sigma^2) \propto P(X|\theta,\sigma^2) \propto N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} + \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right) = \frac{\sigma^2}{n} = \frac{1}{\frac{n}{\sigma^2}+\cancel{\frac{1}{\tau^2}}} = \tau^2 = \infty$$



$$\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \bar{x} = \mu_0$$

could be any value...
by convention we let
it equal zero

$\Rightarrow P(\theta|\sigma^2) = N(0, \infty)$ is Laplace's prior (improper)

But is the posterior proper? Yes, always!

Jeffrey's prior: $P_J(\theta|\sigma^2) \propto \sqrt{I(\theta|\sigma^2)} = \sqrt{\frac{n}{\sigma^2}} \propto 1 \propto N(0, \infty)$

$$\ell'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \Rightarrow \ell''(\theta; x, \sigma^2) = \frac{-n}{\sigma^2}$$

$$= I(\theta|\sigma^2) = E_x\left[-\ell''(\theta; x, \sigma)\right] = E_x\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

We want a pseudocount interpretation of the hyperparameters mu_0 and tau^2. The best way to do this is to do a small reparameterization of the prior's tau^2. Recall that we know sigsq.

$$\tau^2 = \frac{\sigma^2}{n_0} = P(\theta | x, \sigma^2) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{n_0\mu_0}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{n_0}{\sigma^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{n_0}{\sigma^2}}\right)$$

$$= N\left(\frac{n\bar{x} + n_0\mu_0}{n + n_0}, \frac{\sigma^2}{n + n_0}\right)$$

So, n_0 represents number of pseudoobservation. What does mu_0 represent?

Let $Y_1, Y_2, \ldots, Y_{n_0}$ be the "pseudodata". Let $\mu_0 = \bar{Y}$, be the sample average of the psedodata and $n_0 \cdot \mu_0$ is the sum of the pseudodata.

What's the Haldane prior of total ignorance?

$$n_0 = 0 \Rightarrow N\left(n_0, \frac{\sigma^2}{n_0}\right) = N(0, \infty) \propto 1$$

This means all three objective priors we studied are the same.

What is the posterior predictive distribution for $n_* = 1$ observation?

$$P(x_* \mid x, \sigma^2) = \int_\Theta \underbrace{P(x_* \mid \theta, \sigma^2)}_{N(\theta, \sigma^2)} \overbrace{P(\theta \mid x, \sigma^2)}^{N(\theta_p, \sigma_p^2)} d\theta$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2} \frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \theta_p)^2} d\theta$$

$$\propto \int_{\mathbb{R}} e^{\frac{-x_*^2}{2\sigma^2}} e^{\frac{x_*\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}} e^{\frac{-\theta^2}{2\sigma_p^2}} e^{\frac{\theta\theta_p}{\sigma_p^2}} e^{\frac{-\theta_p^2}{2\sigma_p}} d\theta$$

$$\propto e^{\frac{-x_*^2}{2\sigma^2}} \int_{\mathbb{R}} e^{a\theta - b\theta^2} d\theta \qquad a = \frac{x_*}{\sigma^2} + \frac{\theta_p}{\sigma_p^2}$$

$$b = \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)$$

Review this section again

$$P(\theta) = N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\left(\theta^2 - \frac{\theta a}{b} + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\theta^2 + \theta a - \frac{a^2}{4b}} = \sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}} e^{a\theta - b\theta^2}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \frac{1}{c} \int_{\mathbb{R}} \underbrace{c e^{a\theta - b\theta^2} d\theta}_{1} = e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}}$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \left(\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)\right)^{-\frac{1}{2}} e^{\frac{\left(\frac{x_*}{\sigma^2} + \frac{\theta_p}{\sigma_p^2}\right)}{2\left(\frac{1}{\sigma} + \frac{1}{\sigma_p}\right)}}$$

let $A = \left(\frac{1}{\sigma} + \frac{1}{\sigma_p}\right)$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*^2}{2A\sigma^4}} e^{\frac{x_*\theta_p}{A\sigma^2\sigma_p^2}} e^{\frac{\theta_p^2}{2A\sigma_p^4}} \propto e^{\frac{\theta_p}{A\sigma^2\sigma_p^2} x_* - \left(\frac{1}{2\sigma^2} - \frac{1}{2A\sigma^4}\right)x_*^2}$$

$$A\sigma^2\sigma_p^2 = \sigma^2\sigma_p^2\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right) = \sigma_p^2 + \sigma^2, \quad A\sigma^4 = \sigma^4\left(\frac{1}{\sigma^4} + \frac{1}{\sigma_p^2}\right) = \sigma^2 \frac{\sigma^4}{\sigma_p^2}$$

$$\propto N\left(\frac{n}{2v}, \frac{1}{2v}\right) = N\left(\frac{\frac{\theta_p}{\sigma_p^2 - \sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma + \frac{\sigma^4}{\sigma_p^2}}}, \quad \sigma_p^2 + \theta^2\right) = N(\theta_p, \sigma_p^2 + \sigma^2)$$

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$$2v = \frac{1}{\sigma} - \frac{1}{\sigma^2 + \frac{\sigma^4}{\sigma^2_p}} = \frac{1}{\sigma^2} - \frac{\sigma^2}{6^2\sigma^2_p + \sigma^4} = \frac{\sigma^2\sigma^2_p + \sigma^4 - \sigma^2_p\,\sigma^2}{\sigma^2(\sigma^2\sigma^2_p + \sigma^4)}$$

$$\sigma^2$$

$$= \frac{\sigma^2\, 1}{\sigma^2(\sigma^2_p + \sigma^2)} = \frac{1}{2v} = \sigma^2_p + \sigma^2$$

Dec. 17