lec 13

Curly F: Poission (θ). θ is our unknown parameter of interest. It is not the same θ in the binomial. Let's try to find the conjugate prior for this parametric model (likelihood).

pattern matching

$$P(\theta|x) \propto P(x|\theta)P(\theta) = \frac{e^{-\theta}\theta^{x}}{x!} P(\theta) \propto e^{-\theta}\theta^{x} k(\theta) =$$

* We want to find $P(\theta)$ of the same distribution as $P(\theta|x)$.

$$e^{-\theta}\theta^{x} \underbrace{e^{-\theta b}\theta^{\alpha}}_{} = \underbrace{e^{-\theta(b+1)}\theta^{x+b}}_{}$$

same kernel

same rv

Let's figure out $P(\theta)$ from $K(\theta)$

$$\int_0^\infty K(\theta)\,d\theta = \int_0^\infty e^{-\theta b}\,\theta^{a}\,d\theta = \int_0^\infty e^{a+1-1}\,e^{-b\theta}\,d\theta = \ldots$$

u-sub

$$\downarrow$$

$$\frac{\Gamma(a+1)}{\beta^{a+1}}$$

$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\,\theta^{a+1-1}\,e^{-b\theta} = \text{Gamma}(a+1, b)$$

Back to probability Class ....

$$\underbrace{\quad\ldots\quad}\uparrow f(y)$$

$$Y \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\,y^{\alpha-1}\,e^{-\beta y} \propto y^{\alpha-1}\,e^{-\beta y} = k(y)$$

$\text{Supp}[Y] = (0, \infty)$, Param space: $\alpha > 0$, $\beta > 0$

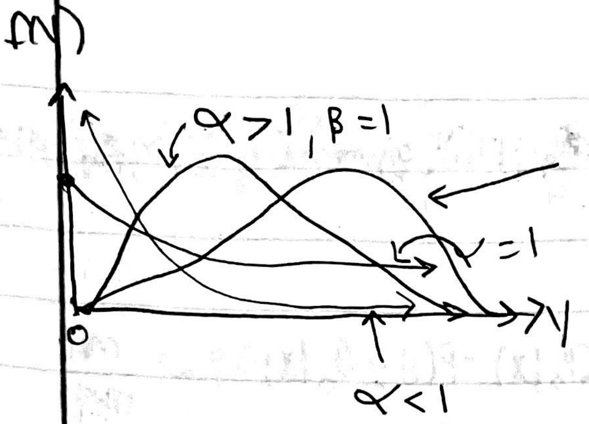
$$E[Y] = \int_0^\infty y\,f(y)\,dy = \ldots \text{ u-substitution } = \frac{\alpha}{\beta}$$

$$\text{mode}[Y] = \ldots \text{ calc } = \frac{\alpha-1}{\beta} \text{ if } \alpha > 1.$$

median →

$\text{Med}[Y] = q$ such that $\int_0^q f(y)\,dy = \frac{1}{2}$ ... no closed form expression is possible so, we use a computer to do numerical integration

$$= \text{qgamma}(0.5, \alpha, \beta)$$



Let's go back to inference for θ in the Poisson model. Now, we consider

$$F: \text{iid Poisson}(\theta)$$

$$P(\theta|X) \propto P(X|\theta)\,P(\theta) = P(\theta)\prod_{i=1}^{n}\frac{e^{-\theta}\theta^{x_i}}{x_i!} = P(\theta)\,\frac{e^{-n\theta}\,\theta^{\Sigma x_i}}{\Pi\, x_i!} \propto e^{-n\theta}\,\theta^{\Sigma x_i}\,K(\theta)$$

↓

pattern matching

↓

$$= e^{-n\theta}\,\theta^{\Sigma x_i}\,\underbrace{\theta^{\alpha-1}e^{-\beta\theta}}_{\text{gamma kernel}} = \underbrace{\theta^{\alpha+\Sigma x_i-1}\,e^{-(\beta+n)\theta}}_{\text{gamma kernel}}$$

$\propto \text{Gamma}(\alpha + \Sigma x_i, \beta+n)$

# of events in the data

x_0, # of pseudoevents

# of observation in data

n_0, # of pseudo-observation

$$= \hat{\theta}_{MMSE} = \frac{\alpha+\Sigma x_i}{\beta+n}, \quad \hat{\theta}_{MAP} = \frac{\alpha+\Sigma x_i - 1}{\beta+n} \text{ only if } \alpha+\Sigma x_i \geq 1$$

$$\hat{\theta}_{MMASE} = \text{qgamma}(0.5, \alpha+\Sigma x_i, \beta+n).$$

$$CR_{\theta,1-\alpha_0} = [qgamma(\tfrac{\alpha_0}{2}, \Sigma x_i ; \beta+n), qgamma(1-\tfrac{\alpha_0}{2}, \alpha+\Sigma x_i, \beta+n)$$

$H_a: \theta > \theta_0$ v.s. $H_0: \theta \leq \theta_0$, $P_{val} = P(H_0|X) = P(\theta \leq \theta_0 | X)$

$$= \int_0^{\theta_0} P(\theta|X)\, d\theta$$

$$= pgamma(\theta_0, \alpha + \Sigma x_i, \beta + n)$$

Let's derive the MLE:

$$L(\theta;x) = \frac{e^{-n\theta}\, \theta^{\Sigma x_i}}{\Pi x_i!}$$

$$= l(\theta;x) = -n\theta + \Sigma x_i \ln(\theta) - \ln(\Pi x_i!)$$

$$= l'(\theta;x) = -n + \frac{\Sigma x_i}{\theta} \overset{set}{=} 0$$

$$= \frac{\Sigma x_i}{\theta} = n$$

$$= \hat{\theta}_{MLE} = \frac{\Sigma x_i}{n} = \bar{X}$$

Let's prove that $\hat{\theta}_{MMSE}$ is a shrinkage estimator

and let's find the value of P. $\rightarrow$

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$$\hat{\theta}_{MMSE} = \frac{\alpha+\sum x_i}{\beta+n} = \frac{\alpha}{\beta+n}\cdot\frac{\beta}{\beta} + \frac{\sum x_i}{\beta+n}\cdot\frac{n}{n} = \overbrace{\frac{\beta}{\beta+n}}^{e}\,\overbrace{\frac{\alpha}{\beta}}^{E[\theta]} + \overbrace{\frac{n}{\beta+n}}^{1-e}\,\overbrace{\frac{\sum x_i}{n}}^{\hat{\theta}_{MLE}}$$

$$\frac{\beta}{\beta+n} = \frac{n_0}{n+n_0}$$

$$\lim_{n\to\infty} e = 0$$