lec. 20

$$P(X_*|X) \propto \int_0^\infty \int_{\mathbb{R}} K(X_*|\theta,\sigma^2)\, K(\theta,\sigma^2|X)\, d\theta\, d\sigma^2$$

$$= \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_*-\theta)^2} (\sigma^2)^{-n/2-1} e^{\frac{-(n-1)S^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2} d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-1/2} (\sigma^2)^{-n/2-1} e^{\frac{-(n-1)S^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left(\underbrace{(X_*-\theta)^2 + n(\theta-\bar{X})^2}_{X_*^2 - 2X_*\theta + n\theta^2 - 2n\theta\bar{x} + n\bar{x}^2}\right)} d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{\frac{-(n+1)S^2/2 + X_*^2/2 + n\bar{X}^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{\overbrace{\frac{X^* + n\bar{X}}{\sigma^2}\theta - \frac{(n+1)}{2\sigma^2}\theta^2}} \to e^{p\theta - b\theta^2}$$

$\longrightarrow$

$$= \int_0^\infty \sigma^{2\frac{-n+1}{2}-1} \sqrt{\frac{\pi}{\frac{n+1}{2\sigma^2}}} \; e^{\left(\frac{x^* + n\bar{x}}{\sigma^2}\right)^2 / 4^2 \, \frac{n+1}{2\sigma^2}}$$

$$\downarrow$$

$$\frac{(x^* + n\bar{x})^2}{2\sigma^2 (n+1)}$$

$$\int_0^\infty \sqrt{\frac{2\pi}{n+1}} \, (\sigma^2)^{\frac{-n+1}{2}-1} \, e^{-\frac{(n-1)s^2/2 + x_*^2/2 + n\bar{x}^2/2 - (x^* + n\bar{x})^2/(2(n+1))}{\sigma^2}} (\sigma^2)^{1/2} \, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{\overset{\alpha}{\widetilde{-n/2}}-1} e^{-\frac{\beta}{\sigma^2}} \, d\sigma^2 = \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\beta^{-\alpha} \propto \beta^{-\alpha}$$

$$= \left(\frac{(n-1)s^2}{2} + \frac{x_*^2}{2} + \frac{n\bar{x}^2}{2} - \frac{\overbrace{(x_* + n\bar{x})^2}^{x_*^2 + 2nx_*\bar{x} + n^2x^2}}{2n+2}\right)^{-\frac{n}{2}} = (ax_*^2 + bx_* + c)^{-n/2}$$

$$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$$

$$b = -\frac{2n\bar{x}}{2n+2} = -\frac{n\bar{x}}{n+1}, \quad c = \frac{(n-1)s^2}{2} + \frac{n\bar{x}^2}{2} - \frac{n^2\bar{x}^2}{2n+2} = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$$

$$= \left(\frac{1}{a}\right)^{n/2} \left(\frac{1}{a}\right)^{-n/2} (ax_*^2 + bx_* + c)^{-n/2} = \left(\frac{1}{a}\right)^{-n/2} \left(x_*^2 - \frac{b}{a}x_* + \frac{c}{a}\right)^{-n/2}$$

$$\propto \left(x_*^2 - \frac{b}{a}x_* + \frac{c}{a}\right)^{-n/2} = \left(\left(x^* + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)^{-n/2} \left(\frac{1}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-n/2}$$

$$\longrightarrow$$

$$\propto \left(1 + \frac{(x^* + \frac{b}{2a})^2}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{\frac{-(n-1)+1}{2}} = \left(1 + \underset{\nu}{\frac{1}{n-1}} \frac{(x^* - \overset{\mu}{\frac{-b}{2a}})^2}{\underbrace{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)(n-1)}_{S}}\right)^{\frac{-(n-1)+1}{2}}$$

$$\propto T_\nu(\mu, s^2) = T_{n-1}\left(\bar{X}, \frac{n+1}{n} s^2\right) \approx N(\bar{X}, s^2)$$

$$= \frac{-b}{2a} = \frac{\frac{n\bar{X}}{n+1}}{\frac{n}{n+1}} = \bar{X}$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{\frac{1}{2}\left((n-1)s^2 + n\bar{X}^2 - \frac{n^2\bar{X}^2}{n+1}\right)}{\frac{1}{2}\frac{n}{n+1}} =$$

$$= \frac{(n-1)(n+1)}{n} s^2 + (n+1)\bar{X}^2 - n\bar{X}^2$$

$$\downarrow$$

$$n\bar{X}^2 + \bar{X}^2$$

$$= \frac{-b^2}{4a^2} = -\left(\frac{b}{2a}\right)^2 = -\left(\frac{-b}{2a}\right)^2 = -\bar{X}^2 \Rightarrow \frac{c}{a} - \frac{b^2}{4a^2}$$

$$= \frac{(n-1)(n+1)}{n} s^2$$

$\rightarrow$

$$S_0^{\,2} = \frac{\frac{(n-1)(n+1)s^2}{n}}{n-1} = \frac{n+1}{n} s^2$$

under Jeffrey prior...

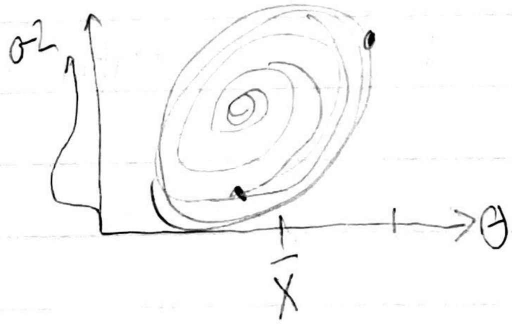
$P(\theta,\sigma^2|x) = P(\theta|x,\sigma^2)\,P(\sigma^2|x)$ By def of conditional probability

NormInvGamma(; ; ;)

$N(\bar{x}, \frac{\sigma^2}{n})$

$\text{InvGamma}(\frac{n-1}{2}, \frac{(n-1)s^2}{2})$

You can think of a normal Inverse gamma as first sampling from an InverseGamma$\left(\frac{(n-1)}{2}, \frac{(n-1)s^2}{2}\right)$ to get a Sigsq value and then you use that value sigsq to draw a theta from $N(\bar{x}, \frac{\sigma^2}{n})$ and return the two-dimension point $[\theta]$



$P(\theta,\sigma^2|X)$

This can also be done the other way...

$$P(\theta, \sigma^2|x) = P(\sigma^2|x, \theta), P(\theta|x)$$

$\downarrow$ $\text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$ $\qquad \rightarrow T_{n-1}\left(\bar{x}, \frac{s^2}{2}\right)$

If we decompose the first way, we draw theta from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ and thus σ^2 must be known.

What if we break this by instead of using the Jeffrey's prior, use

$$P(\theta) = N(\mu_0, \tau^2) \text{ and } P(\sigma^2) = \text{Invgamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

These were the two prior we began with when we started investing the normal likelihood model. However, it's important to note we are not allowing $\tau^2 = \frac{\sigma^2}{n_0}$

what happens?

$$P(\theta, \sigma^2) = P(\theta)P(\sigma^2) \text{ not } P(\theta|\sigma^2)P(\sigma^2)$$

= The two priors are disconnected

Derive the posterior under this two-dim prior

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P(\theta,\sigma^2) = P(X|\theta,\sigma^2)\,P(\theta)\,P(\sigma^2)$$

$$\propto k(X|\theta,\sigma^2)\,k(\theta)\,k(\sigma^2)$$

$$= (\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\theta)^2)}\, e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2}\, (\theta^2)^{-\frac{n_0}{2}-1}\, e^{\frac{n_0\sigma_0^2}{\sigma^2}}$$

$$= (\sigma^2)^{-n/2-\frac{n_0}{2}-1}\, e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n_0\sigma_0^2 + n\bar{x}^2)}\, e^{\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau}\right)\theta - \left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)\theta^2}$$