Student names: . . . (please update)

Instructions: Update this file (or recreate a similar one, e.g. in Word) to prepare your answers to the questions. Feel free to add text, equations and figures as needed. Hand-written notes, e.g. for the development of equations, can also be included e.g. as pictures (from your cell phone or from a scanner). ***This lab is not graded. However, the lab exercises are meant as a way to familiarise with dynamical systems and to study them using Python to prepare you for the final project.*** *This file does not need to be submitted and is provided for your own benefit. The graded exercises will have a similar format.*

The file `lab#.py` is provided to run all exercises in Python. Each `exercise#.py` can be run to run an exercise individually. The list of exercises and their dependencies are shown in Figure 1. When a file is run, message logs will be printed to indicate information such as what is currently being run and and what is left to be implemented. All warning messages are only present to guide you in the implementation, and can be deleted whenever the corresponding code has been implemented correctly.

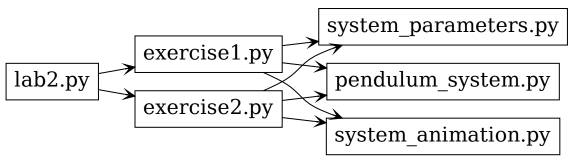


Figure 1: Exercise files dependencies. In this lab, you will be modifying `exercise1.py`, `exercise2.py` and `pendulum_system.py`.

In this exercise, you will explore the different modeling techniques that can be used to control a single joint and segment. We initially start by exploring a single joint controlled by a a single simplified pendulum model with damping(friction) (exercise1) and then extend it to pair of spring-dampers muscle models (exercise2). These only represent the passive dynamics observed in a real musculoskeletal system. You are provided with a code that can simulate a pair spring-damp muscle model .

Important note: *Both exercise use a generic class that can handle both a pair of spring damp muscles, or a single spring damp muscle sketched in Figure 2. Each muscle pair contains sping constants and resting angles, and damping coefficients. Simply set all the values of these parameters equal for the two pairs to study the behavior of a single spring-damper instead of a pair (in Exercise 1). Have a look at the specification of parameters of the pendulum system in the class PendulumParameters in `system_parameters.py`.*

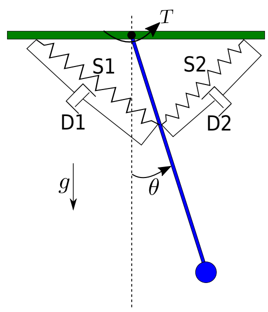


Figure 2: Pendulum model with two springs S1 and S2 and two dampers b1 and b2
T - Positive torque direction.
g - Gravity.
θ - Angle made by the pendulum

Question 1: Pendulum with friction

1.a Find the fixed points of the pendulum with friction (i.e. damping), and analyze their stability using a local linearization (briefly describe the calculation steps).

$$I\ddot{\theta} = -mgLsin(\theta) + T_{ext} \tag{1}$$

Considering Inertia $I = mL^2$, the equation of the pendulum can be written as,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} \tag{2}$$

where θ is the angle, g the gravity constant, L the length of the pendulum and d is the damping coefficient.

Fixed points:

$$\begin{aligned} \dot{x}_2 &= -\frac{g}{L}\sin x_1 - dx_2 \qquad & sin(\tilde{x}_1) = 0 \quad &\Rightarrow \quad \tilde{x}_1 = n\pi, \quad n \in Z \\ \dot{x}_1 &= x_2 = 0 & \tilde{x}_2 = 0 & \end{aligned} \tag{3}$$

Jacobian and eigenvalues:

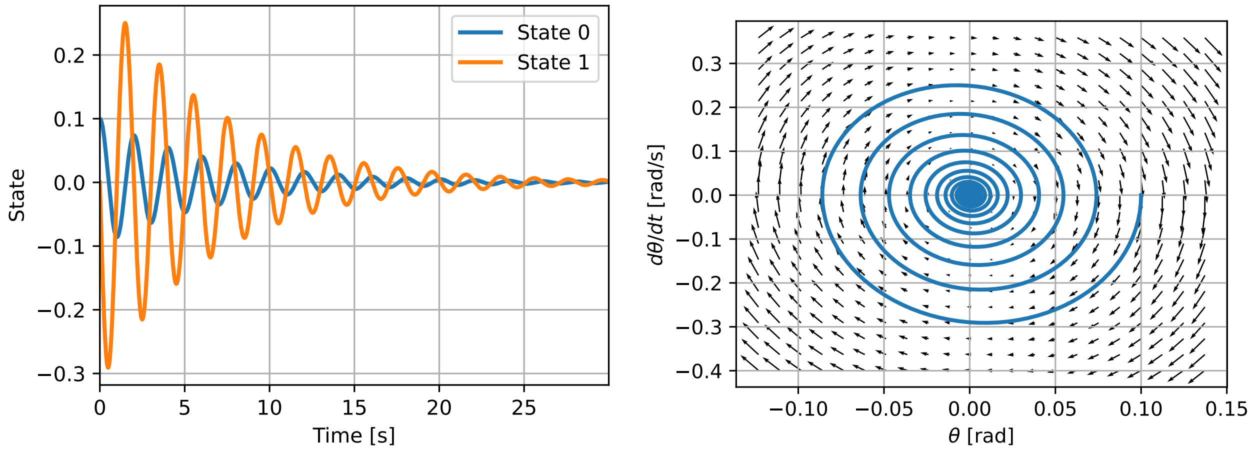
$$J = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & -d \end{pmatrix} \quad \Rightarrow \quad \lambda^2 + d\lambda + \frac{g}{L}\cos x_1 = 0 \quad \Rightarrow \quad \lambda_{\pm} = \frac{-d \pm \sqrt{d^2 - 4\frac{g}{L}\cos x_1}}{2} \tag{4}$$

In the case where $\tilde{x}_1 = 0$ (i.e. pendulum down), both eigenvalues $\lambda_{\pm} < 0$, thus the fixed point is stable. In the case $\tilde{x}_1 = \pi$, we have $\lambda_- < 0$ and $\lambda_+ > 0$, thus the fixed point is unstable (saddle point).

1.b Numerically solve the differential equations of the pendulum with different initial conditions. Show several time evolutions and phase portraits with different initial conditions that illustrate several aspects of the interesting behavior of the pendulum. Additionally, modify the damping parameter and demonstrate examples of underdampded, critically damped and overdamped behaviors. See `exercise1.py` and `system_parameters.py` and `pendulum_system.py` for help with implementation.

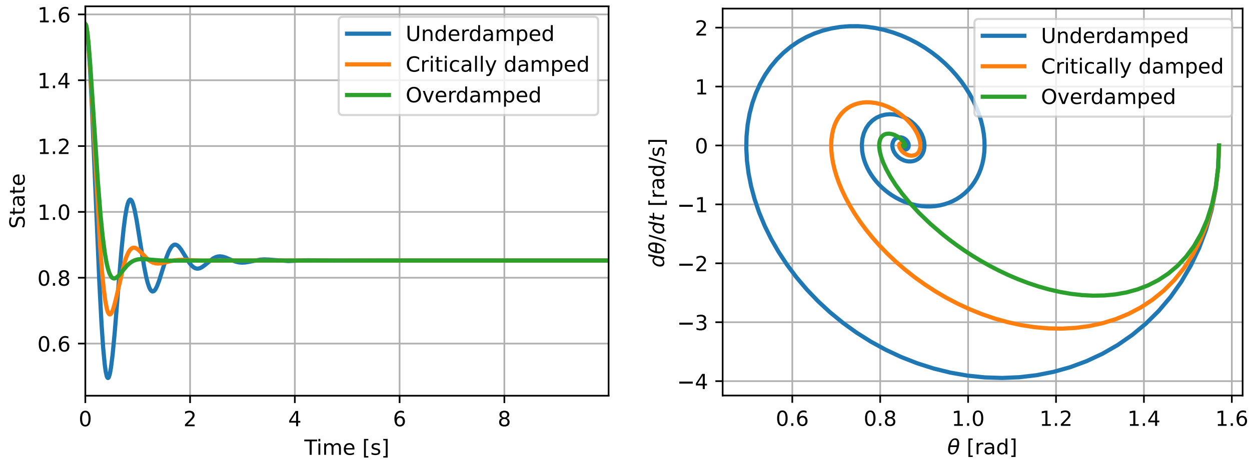
Different time evolutions should be shown. Ideally examples should include: starting and staying at unstable fixed point, simple damped oscillations around stable fixed point, complete loops due to initial high velocity that then end up at stable fixed point. Depending on the choice of the parameters, oscillations might disappear in favour of an exponential decay.

For instance:



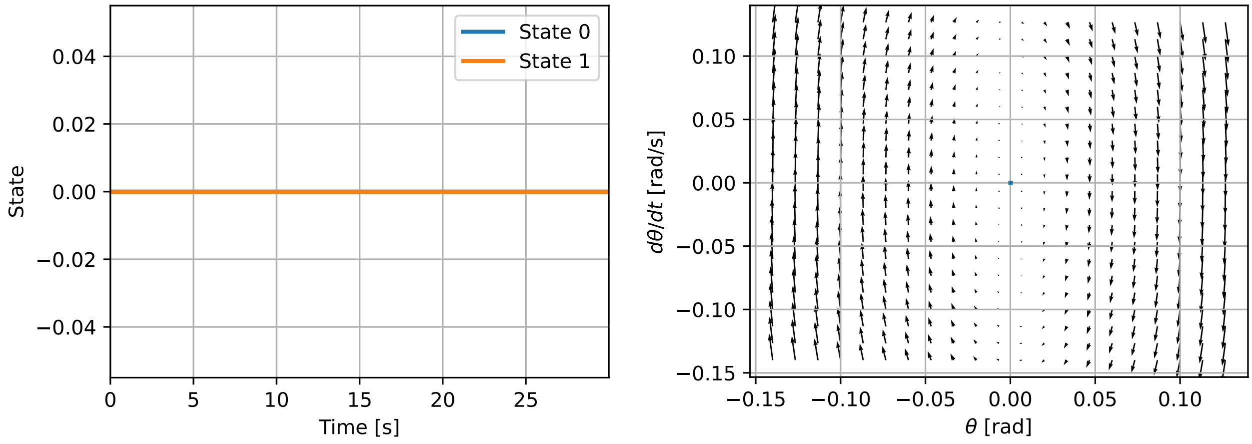
(a) Time evolution *(b) Phase portrait*

Figure 3: Basic pendulum setup ($\theta_0 = 0.1$ [rad], $\dot{\theta}_0 = 0$ [rad/s])



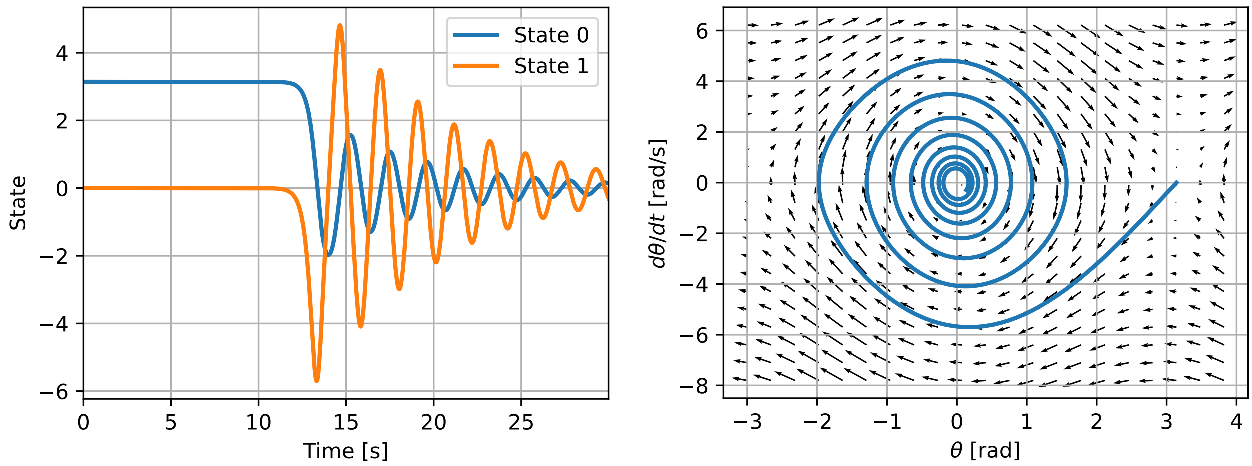
(a) Time evolution *(b) Phase portrait*

Figure 4: Different types of fixed point are obtained depending on the sign of $\Delta = d^2 - 4\frac{g}{L}$. Note that the underdamped solution shows an overshoot before settling at the equilibrium point. Also note that the critically damped solution decays faster than the overdamped (and underdamped) one.



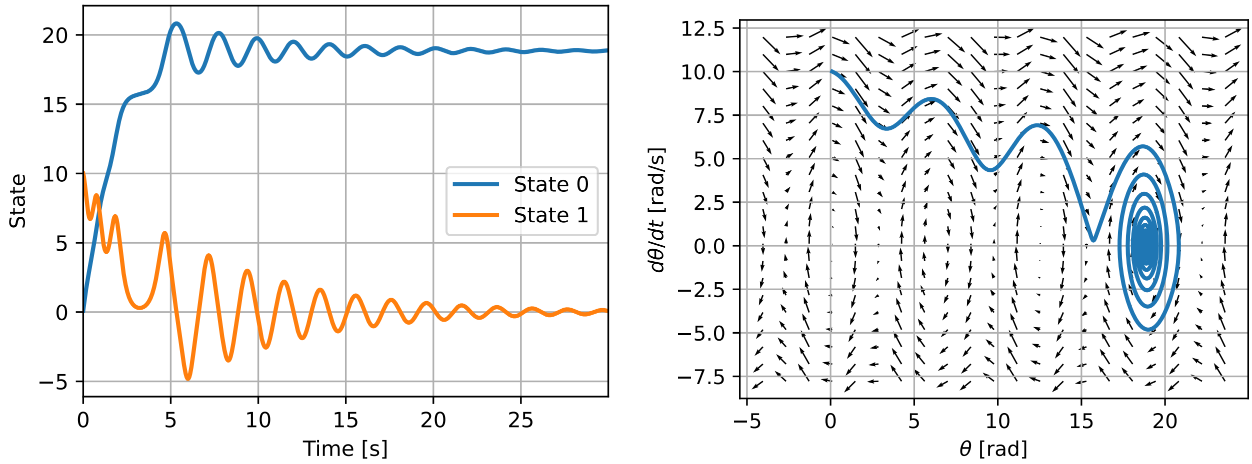
(a) Time evolution *(b) Phase portrait*

Figure 5: Stable pendulum setup ($\theta_0 = 0$ [rad], $\dot{\theta}_0 = 0$ [rad/s])



(a) Time evolution *(b) Phase portrait*

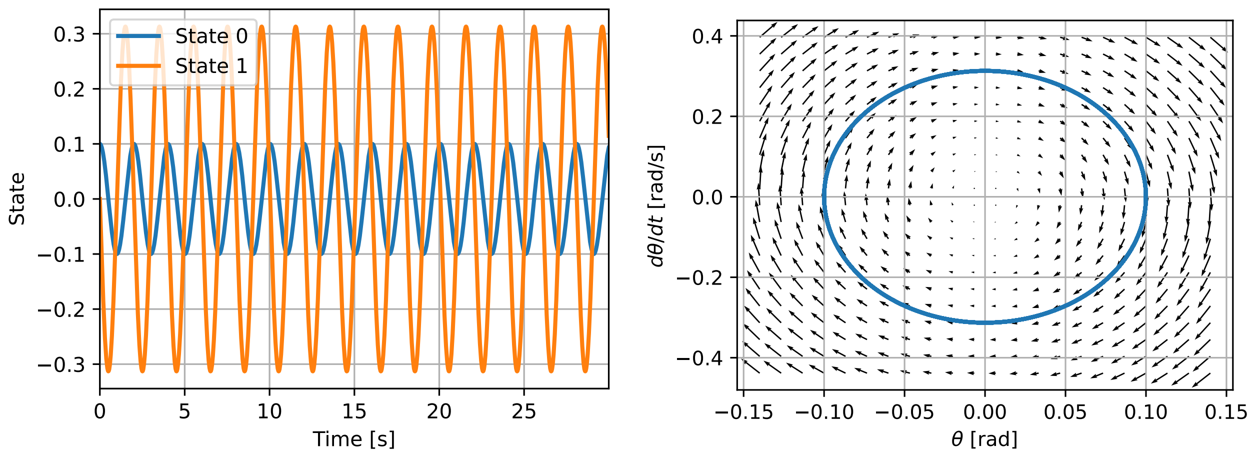
Figure 6: Unstable pendulum setup ($\theta_0 = \pi$ [rad], $\dot{\theta}_0 = 0$ [rad/s]). The pendulum stays at the unstable fixed point for a few seconds, and then is pushed away due to numerical imprecision of the integration. It ends up at the stable fixed point (0,0)



(a) Time evolution (b) Phase portrait

Figure 7: High initial velocity pendulum setup (θ_0 = 0.1 [rad], $\dot{\theta}_0$ = 10 [rad/s]). In this case, the pendulum has a high initial velocity, and therefore makes three complete rotations before converging to the stable fixed point (6 pi, 0).

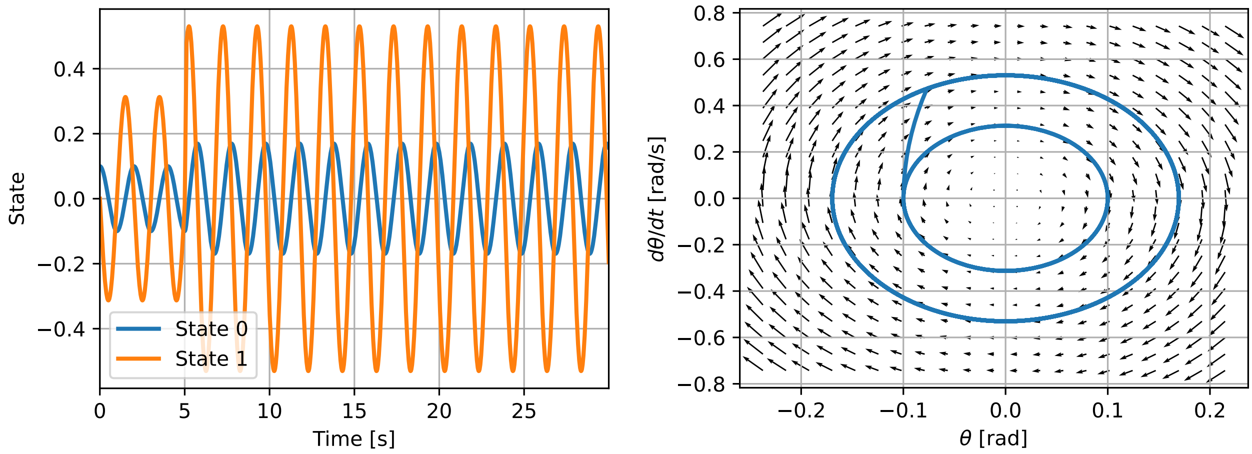
1.c Investigate and describe how the behavior of the pendulum changes if friction is zero (d=0). Show a new phase portrait.



(a) Time evolution (b) Phase portrait

Figure 8: Periodic pendulum setup (θ_0 = 0.1 [rad], $\dot{\theta}_0$ = 0 [rad/s]). The figures show that without friction the pendulum keeps oscillating without reduction of amplitude of oscillation. The fixed points, both unstable and stable, have not changed.

1.d Does the pendulum without friction (d=0) produce stable limit cycles? Discuss, and try to support your statement with some numerical simulations (show figures) and/or analytical arguments.

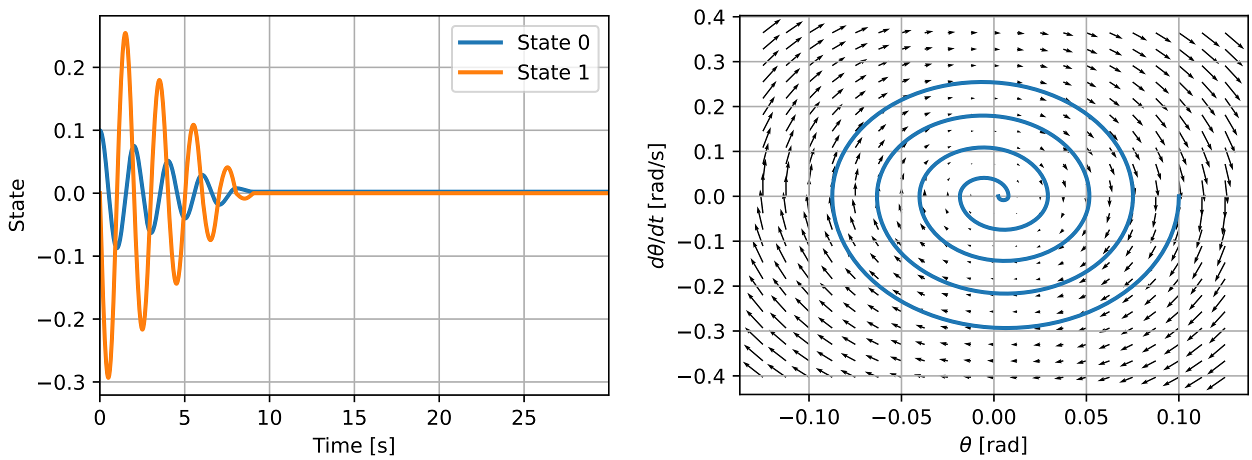


(a) Time evolution *(b) Phase portrait*

Figure 9: The pendulum without friction does not produce stable limit cycle behavior. It has multiple closed orbits that are not isolated. Perturbations lead to new orbits.

1.e Investigate how the behavior of the pendulum changes if the viscous friction term is replaced with a dry (Coulomb) friction term. Unlike viscous friction, dry friction does not depend on speed, only the direction of movement. What are the main differences between the two types of pendulum? (discuss and show some examples). And is there anything notable about the numerical integration of the pendulum with dry friction? If yes, what and why?

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - d \cdot sign(\dot{\theta}) \tag{5}$$



(a) Time evolution *(b) Phase portrait*

Figure 10: Pendulum with dry friction

This represents a stiff dynamical system, i.e. a system that is difficult to integrate numerically. Indeed, this pendulum is difficult to solve numerically because of the switch of derivatives when the angular velocity changes sign (friction jumping between d and –d). Many more integration times steps are needed than for the pendulum with viscous friction. The higher d, the more difficulties the integration method has.

Fixed points are different from the pendulum with viscous friction. Quite complex behavior: the pendulum ends in a position/angle that depends on the initial conditions (not anymore with a vertical position).

Exercise 2 : Pendulum model with passive elements

Mechanical behavior of muscle tissue can be approximated by simple passive elements such as springs and dampers. These elements, when combined properly, allow to study the behavior of muscle under compressive and tensile loads.

Consider the following equation describing the motion of simple pendulum with an external torque T_{ext},

$$I\ddot{\theta} = -mgLsin(\theta) + T_{ext} \tag{6}$$

Consider the system only for the pendulum range $\theta = [-\pi/2, \pi/2]$

Explore the pendulum model with two antagonist spring elements

In this question the goal is to add two antagonist springs to the pendulum model which you are already familiar with from lab 2 exercises. For simplicity we assume the springs directly apply a torsional force on to the pendulum. Use equation 7 to develop the spring model.

Note *: The springs can only produce force in one-direction like the muscles. That is, they can only apply a pulling force and apply a zero force when compressed. In terms of torsion this translates to, spring S1 can exert only clockwise torque and spring S2 can exert only counter-clockwise torque. You need to accommodate for this condition in the equations shown below.*

The setup for the pendulum with a pair of antagonist springs is as shown in figure 11. Use `exercise2.py`, `pendulum_system.py` and `system_parameters.py` files to complete the exercise.

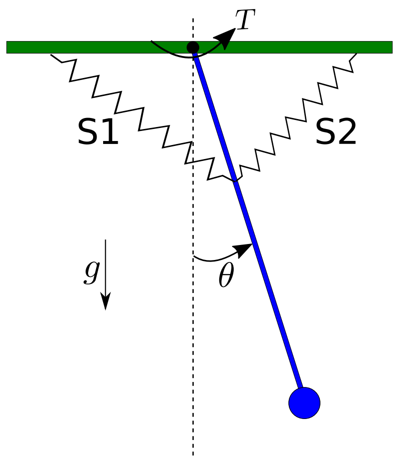


Figure 11: Pendulum model with two springs S1 and S2.
T - Positive torque direction.
g - Gravity.
θ - Angle made by the pendulum

$$T_S = k \cdot (\theta_{ref} - \theta) \tag{7}$$

Where,

- T_S : Torsional Spring force
- k : Spring Constant
- θ_{ref} : Spring reference angle
- θ : pendulum angle

Substituting the above in 2,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{T_S}{I} \tag{8}$$

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{k \cdot (\theta_{ref} - \theta)}{I} \tag{9}$$

Use the generalized form of the spring equation described in 9 to extend it to both the antagonist springs S1 and S2 with the necessary conditions to make sure springs do not produce when compressed.

Extending the above equation to both springs,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + min(\frac{k1 \cdot (\theta_{ref1} - \theta)}{I}, 0) + max(\frac{k2 \cdot (\theta_{ref2} - \theta)}{I}, 0) + \frac{T_{ext}}{I} \tag{10}$$

For all questions the initial conditions used are,

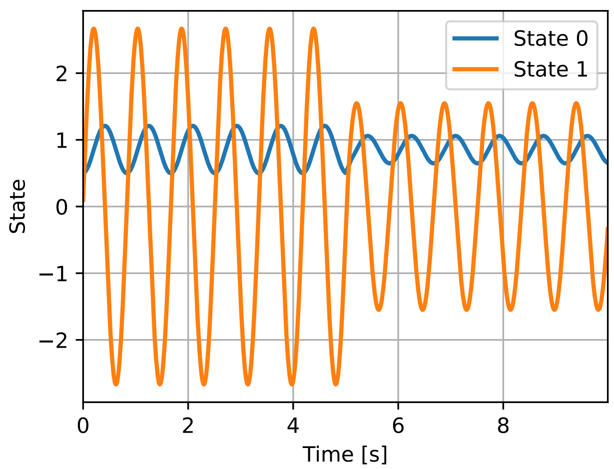
$$\theta = 0.5$$

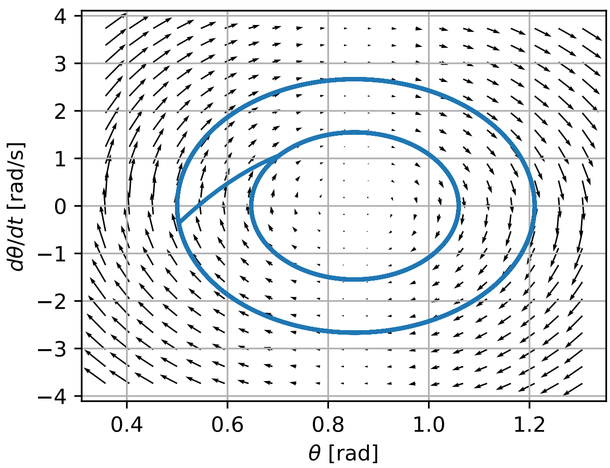
$$\dot{\theta} = 0.1$$

unless explicity specificied otherwise. Students may use different set of initial conditions

2.a Implement the dynamic equations of the pendulum with springs using equations described above in the function `pendulum_system.py::pendulum_equation`. Does the system have a stable limit cycle behavior? Describe and run an experiment to support your answer. You can use the function `exercise2.py::pendulum_perturbation` to perturb the pendulum either by changing states or applying an external torque. Use the class `system_animation.py::SystemAnimation` to visualize the pendulum. Example code can be found in `exercise2.py::exercise2`

The first requirement for a limit cycle is that the system should have a closed trajectory. The pendulum system with springs does exhibit a closed trajectory behavior. But, in order to have a stable limit cycle the system should converge to a single trajectory as time tends to either positive/negative infinity. One solution to check for stable limit cycle behavior is to use the state and phase plot like shown in figure 12 under perturbations to show that there is no stable limit cycle behavior. At t=5s a pertubation is applied to the velocity of the system ($\dot{\theta} = 2.0$). This pushes the trajectory to a new trajectory and it never returns to the original trajectory. This shows that the system does not have a stable limit cycle. Alternatively students may also use poincare map to show that there is no stable limit cycle. *Students should clearly detail the pertubation they used.*





(a) State of pendulum with spring under perturbations *(b) Phase of pendulum with spring under perturbations*

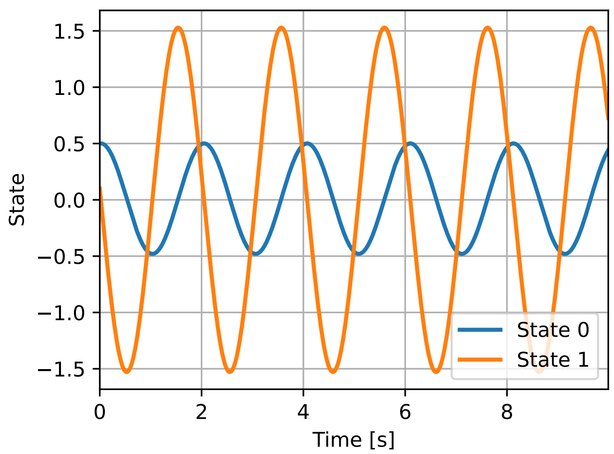
Figure 12: Pertubation approach to check to system limit cycle behavior

2.b Explore the role of spring constant (k) and spring reference angle (θ_{ref}) in terms of range of motion, amplitude and frequency of pendulum. Keep the constants equal, i.e $k_1 = k_2$ and $\theta_{ref1} = \theta_{ref2}$
Refer to `exercise2.py::exercise1` for an example

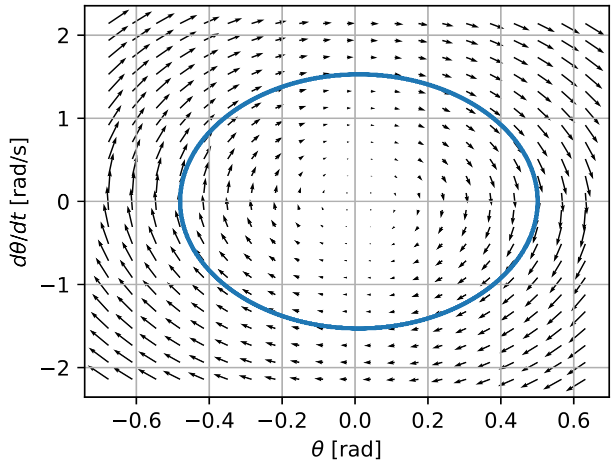
Spring constant (k): Dictates the magnitude and rate at which the pendulum oscillates. The larger the constant the faster the system oscillates. This can also be seen as the responsiveness of the system. Figure 13 shows the reponse of the pendulum with a small spring constant of $k_1 = k_2 = 0.1$. Figure 14 shows the reponse of the pendulum with a large spring constant of $k_1 = k_2 = 100$.

- For both low/high spring constant, the amplitude of the θ remains the same while the amplitude of $\dot{\theta}$ increases with increase in spring constant magnitude.
- The frequency of both θ and $\dot{\theta}$ increases with higher spring constant and vice-versa.

Reference angle: The resting angle for the spring. Since the spring like muscles act only in one direction, the resting angle dictates the angular position of the pendulum at which springs start to act. But having a symmetric spring reference angle for both springs leads to no change in amplitude, range of motion or frequency for a given set of initial conditions. Figures 15 and 16 show the state and phase plot of the system with spring references close to reference ($\theta_{ref1} = -10°$ & $\theta_{ref1} = 10°$) and far from reference ($\theta_{ref1} = -75°$ & $\theta_{ref2} = 75°$) respectively. With reference being $\theta = 0.0°$

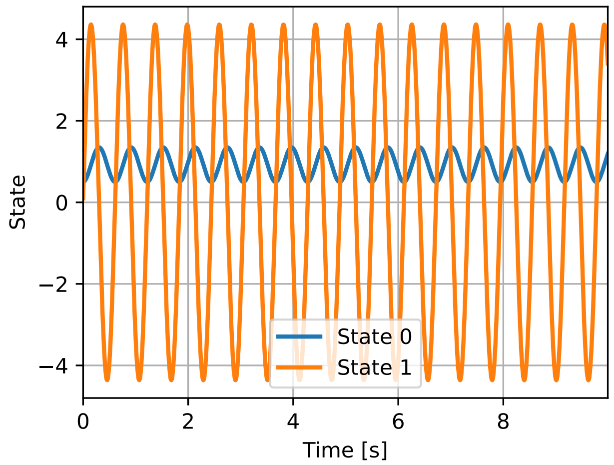


(a) State of pendulum with high spring constant

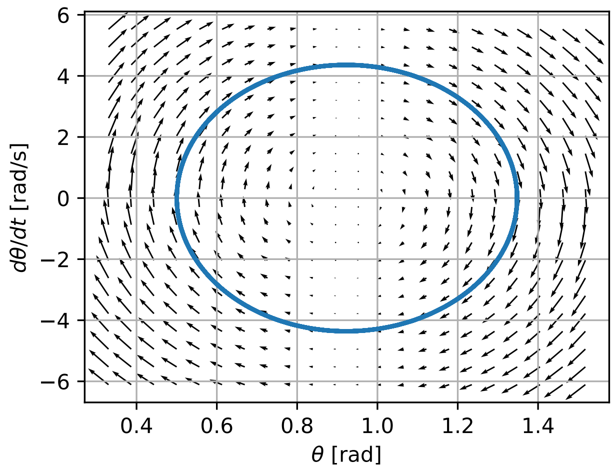


(b) Phase of pendulum with high spring constant

Figure 13: State of pendulum with spring to study the effect of spring constant

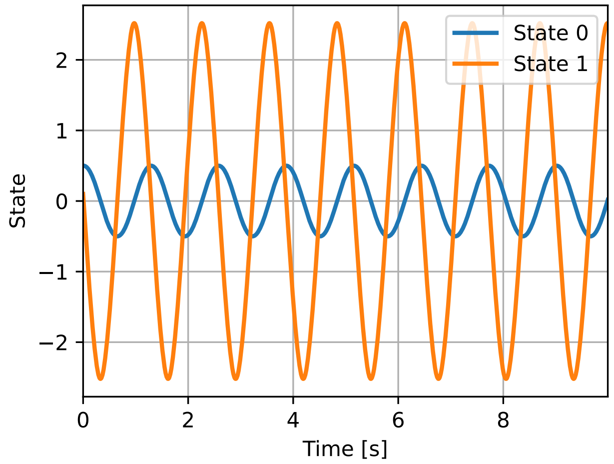


(a) State of pendulum with low spring constant

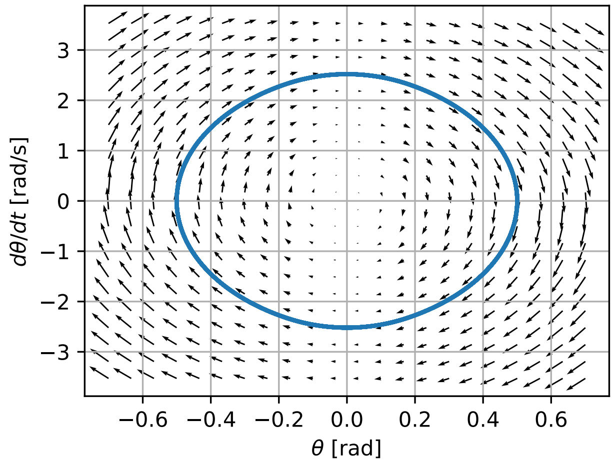


(b) Phase of pendulum with low spring constant

Figure 14: State of pendulum with spring to study the effect of spring constant

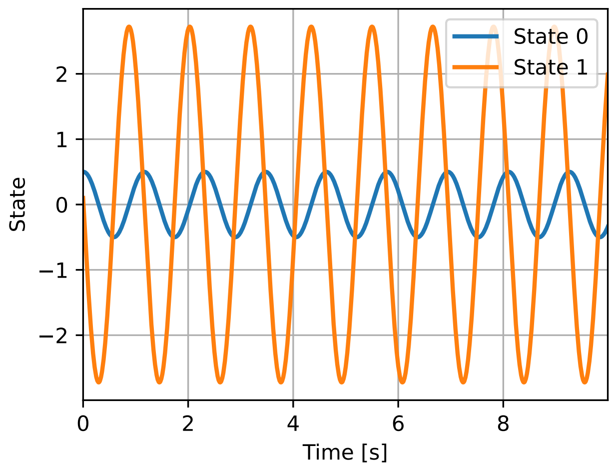


(a) State of pendulum with reference close to pendulum rest position

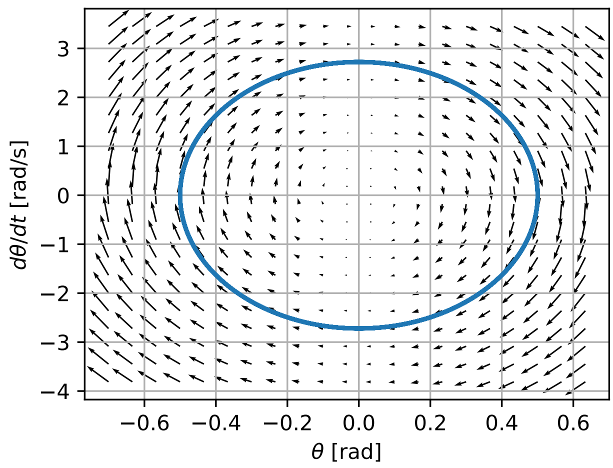


(b) Phase of pendulum with reference close to pendulum rest position

Figure 15: State of pendulum with spring to study the effect of spring reference



(a) State of pendulum with reference far to pendulum rest position



(b) Phase of pendulum with reference far to pendulum rest position

Figure 16: State of pendulum with spring to study the effect of spring reference

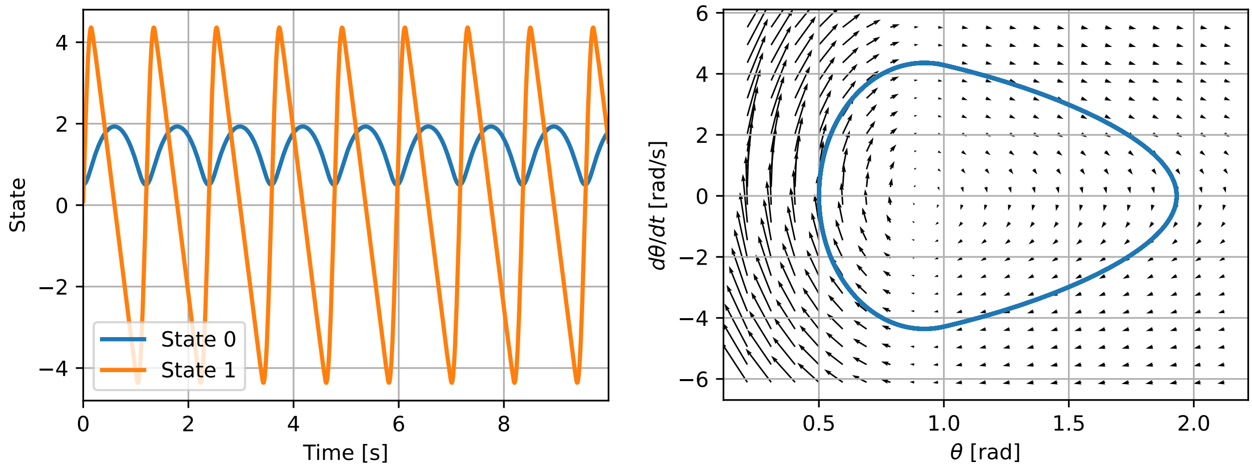
2.c Explain the behavior of the model when you have asymmetric spring constants (k) and spring reference angles (θ_{ref}), i.e. $k_1 \neq k_2$ and $\theta_{ref1} \neq \theta_{ref2}$ Support your responses with relevant plots

As we saw the previous question, changing the spring constant and reference angle yielded different behaviors. Here we introduce assymetry in the system and change parameters individually.

Variable Spring Constant (k): In figure 17 the spring constants are set to $k_1 = 1.0$ and $k_2 = 100.$ and both spring references set to $\theta_{ref1} = \theta_{ref2} = 0.0°$. With these values, it is clear from the phase plot 17b that variable spring constants introduces assymetry in the shape of the closed trajectory. The side with higher spring constant pulls the pendulum back to the reference faster. While the slower spring side is dominated more by the pendulum dynamics rather than of the spring forces.

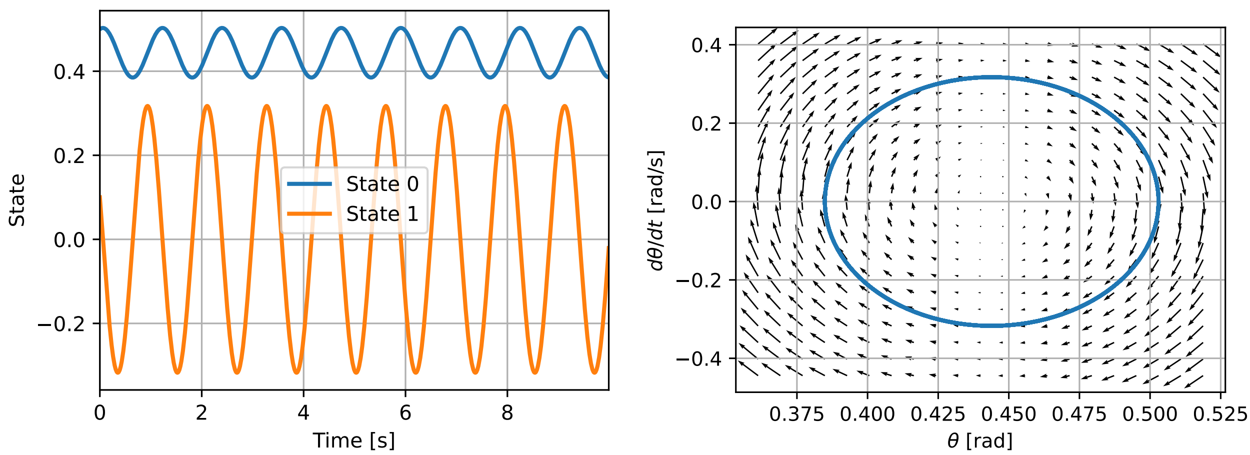
Variable Spring reference (θ_{ref}): In figure 18 the spring references are set to $\theta_{ref1} = 0.0°$ and $\theta_{ref2} = 75.°$ and both spring constants set to $k_1 = k_2 = 10.0$. With these values, the it is clear from the phase plot 18b that variable spring references changes the center of the closed trajectory. That is, the pendulum system now oscillates around a non-zero point.

Thus by having unssymetric spring values, we can produce more complex closed loop trajectories in a simple system.



(a) State of pendulum with Variable spring constant *(b) Phase of pendulum with Variable spring constant*

Figure 17: State of pendulum with spring to study the effect of Variable spring constant



(a) State of pendulum with Variable spring reference *(b) Phase of pendulum with Variable spring reference*

Figure 18: State of pendulum with spring to study the effect of variable spring reference

Explore the pendulum model with two antagonist spring and damper elements

Over time muscles lose energy while doing work. In order to account for this property, let us now add a damper in parallel to the spring model. Use equation 11 to develop the damper model.

Note *: Like the previous springs, the springs in spring-dampers can only produce a force in one-direction. However, the damper terms do not have this limitation and each damper can exert a force in both directions.*

Again use `exercise2.py`, `pendulum_system.py` and `system_parameters.py` files to complete the exercise. The setup for the pendulum model with a pair of antagonist spring and dampers in parallel is as shown in figure 19.

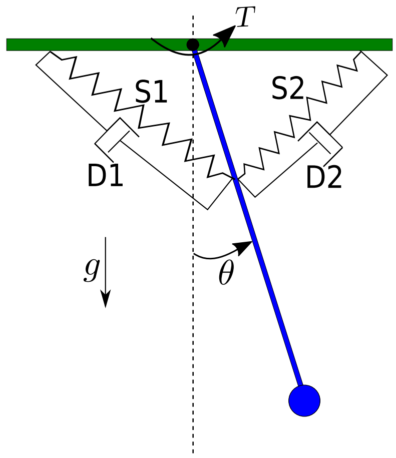


Figure 19: Pendulum model with two springs S1 and S2 and two dampers b1 and b2
T - Positive torque direction.
g - Gravity.
θ - Angle made by the pendulum

$$T_B = b \cdot \dot{\theta} \tag{11}$$

Where,

- T_B : Torsional Damper force
- b : Damping Constant
- $\dot{\theta}$: pendulum angular velocity

The combined spring damper torque is given by,

$$T_S - T_B = k \cdot (\theta_{ref} - \theta) - b \cdot \dot{\theta} \tag{12}$$

The minus for the damper comes from the fact that damper is acting against the work done by the spring.

Substituting the above in 2

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + \frac{T_S - T_B}{I} \tag{13}$$

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + (\frac{k \cdot (\theta_{ref} - \theta) - b \cdot \dot{\theta}}{I}) \tag{14}$$

Use the generalized form of the spring equation described in 14 to extend it to both the antagonist spring-damper systems (S1-D1) and (S2-D2).

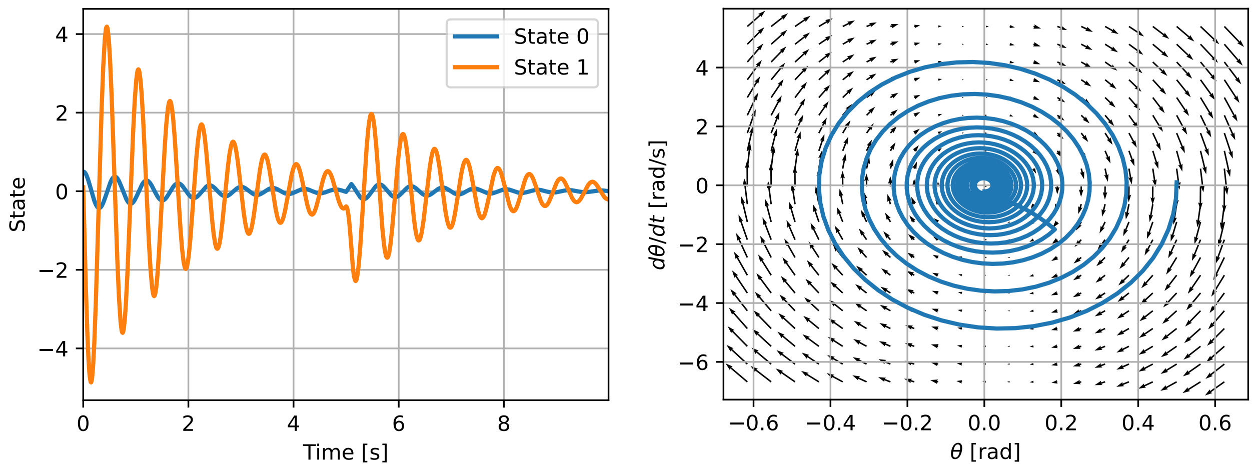
Extending the above equation for both spring and dampers,

$$\ddot{\theta} = -g\frac{sin(\theta)}{L} + \frac{T_{ext}}{I} + min\left(\frac{k1 \cdot (\theta_{ref1} - \theta)}{I}, 0\right) - \frac{b1 \cdot \dot{\theta}}{I} + max\left(\frac{k2 \cdot (\theta_{ref2} - \theta)}{I}, 0\right) - \frac{b2 \cdot \dot{\theta}}{I} \quad (15)$$

2.d Implement the dynamics equations of the pendulum to now include the damping using the equations described above. Modify `pendulum_system.py::pendulum_equation`. How does the behavior now change compared to the pendulum without dampers? Briefly explain and support your responses with relevant plots

In questions 1a-1c observed a closed loop trajectory. By adding dampers to the system introduces a fixed point behavior. The system now loses energy over time and converges to a single position. Even when the system is perturbed the pendulum is returns to the same fixed point showing that there is only one stable fixed point in the system. Figure 20 shows the behavior of the system with the following system parameters,

- $k_1 = 50.0$
- $k_2 = 50.0$
- $b_1 = 0.5$
- $b_2 = 0.5$
- $\theta_{ref1} = -45°$
- $\theta_{ref2} = 45°$



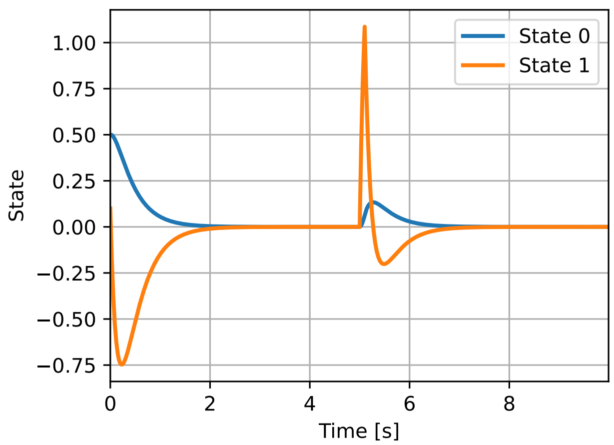
(a) State of pendulum with spring and damper _(b) Phase of pendulum with spring and damper_

Figure 20: Pendulum setup with spring and damper

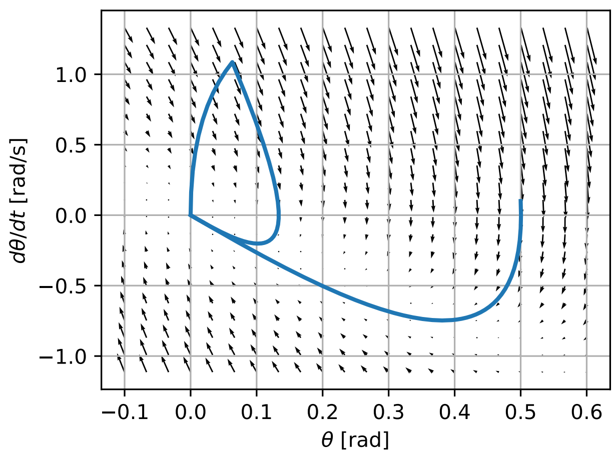
Note that, like you observed in Lab 2, the nature of the fixed point might change depending on the magnitude of the damping term. Figure 21 shows the behavior of the system with the following system parameters,

- $k_1 = 5.0$
- $k_2 = 5.0$

- $b_1 = 5.0$
- $b_2 = 5.0$
- $\theta_{ref1} = -45°$
- $\theta_{ref2} = 45°$



(a) State of pendulum with spring and large damping



(b) Phase of pendulum with spring and large damping

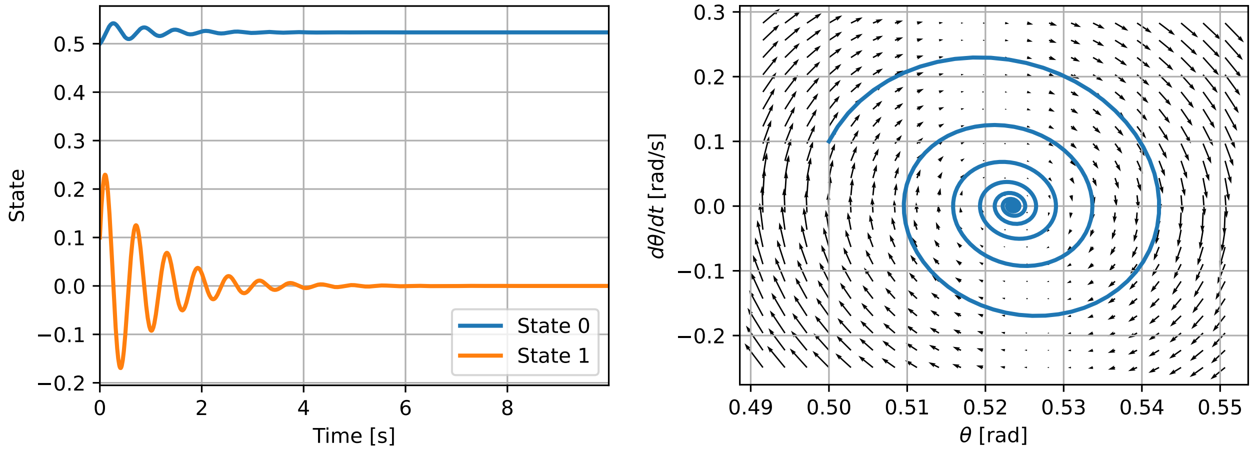
Figure 21: Pendulum setup with spring and large damping term. The fixed point now shows an overdamped behavior

2.e Can you find a combination of spring constants (k), damping constants (b) and spring reference angles (θ_{ref}) that makes the pendulum rest in a stable equilibrium at ($\theta = \pi/6$) radians? Describe how you arrive at the necessary parameters and support your response with relevant plots.

The following parameters set the pendulum at $\pi/6$

$$
\begin{aligned}
b1 &= 1. \\
b2 &= 1. \\
k1 &= 50.0 \\
k2 &= 50.0 \\
s_{theta\_ref1} &= np.deg2rad(0.0) \\
s_{theta\_ref2} &= np.deg2rad(65.6)
\end{aligned}
$$

Using the knowledge from previous questions and setting system parameters assymetrically we obtain a pendulum convergence point at $\theta = \pi/6$ as shown in figure 22.



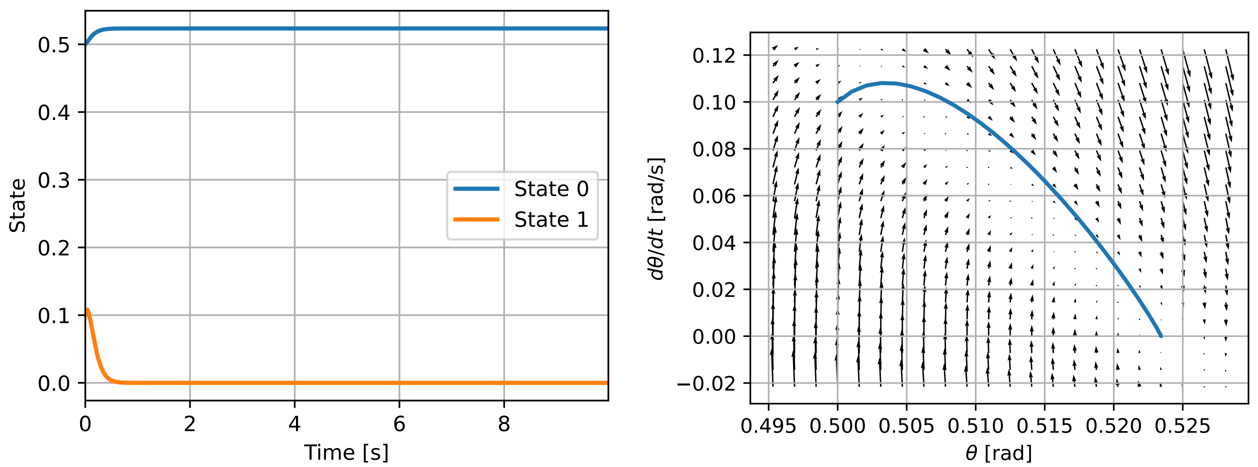
(a) State of pendulum with spring and damper for a given set point
(b) Phase of pendulum with spring and damper for a given set point

Figure 22: Pendulum setup with spring and damper for a given set point

Note that changing the damping term will not influence the position of the equilibrium point of the system. Consider the following parameters:

$$b1 = 1.$$
$$b2 = 1.$$
$$k1 = 50.0$$
$$k2 = 50.0$$
$$s_{theta\_ref1} = np.deg2rad(0.0)$$
$$s_{theta\_ref2} = np.deg2rad(65.6)$$

The system still approaches the convergence point in $\theta = \pi/6$, this time without damped oscillations, as shown in figure23.



(a) State of pendulum with spring and large damping for a given set point
(b) Phase of pendulum with spring and large damping for a given set point

Figure 23: Pendulum setup with spring and large damping term for a given set point

2.f What is the missing component between a real muscle and the muscle model with passive components that you just explored? What behavior's do you lack because of this missing component?

The missing component between a real muscle the muscle model with passive components is the active contractile element. The active contractile element can contract and produce force upon receiving an external activation. Having an active element allows for an external control to switch the behavior of the pendulum from a fixed point behavior to oscillatory and even stable limit cycle behaviors