CALCULUS

Limit :- A number L is said to be limit of the func^n f(x) as $x \to a$, if $\forall\ \epsilon > 0$ (however small) then there exist a, $\delta > 0$ such that

$$|f(x) - l| < \epsilon \quad \forall \quad |x - a| < \delta .$$

$$\left(\lim_{x \to a} f(x) = l \right)$$

$$l - \epsilon < f(a) < l + \epsilon \quad \forall \quad a - \delta < x < a + \delta$$

Left limit

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$$

$$\forall \quad a - \delta < x < a$$

Right Limit

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$$

$$\forall \quad a < x < a + \delta$$

// Existence of limit :-

The limit of f(x) exist if both LHS limit & R.H.S limit exist and are equal.

eg:- $\underset{x\to a}{lt}\left(\frac{1}{x-a}\right)$

$$\underset{x\to a^-}{lt} f(x) = \underset{h\to 0}{lt} f(a-h) = \frac{1}{\cancel{a}-h-\cancel{a}} = -\frac{1}{h} = -\frac{1}{0} = -\infty$$

$$\underset{x\to a^+}{lt} f(x) = \underset{h\to 0}{lt} f(a+h) = \frac{1}{\cancel{a}+h-\cancel{a}} = +\frac{1}{h} = +\frac{1}{0} = +\infty$$

LHS Limit $\neq$ RHS limit

Limit doesn't exist at $x \to a$

eg:- $\underset{x\to 0}{lt}\ 2^{-1/x^2}$

$$\underset{x\to 0^-}{lt} f(x) = \underset{h\to 0}{lt} f(0-h)$$

$$= \underset{h\to 0}{lt}\ 2^{-1/(-h)^2}$$

$$= \underset{h\to 0}{lt}\ 2^{-1/h^2}$$

$$= 2^{-\infty}$$

$$= 0$$

$$\underset{x\to 0^+}{lt} f(x) = \underset{h\to 0}{lt} f(0+h)$$

$$= \underset{h\to 0}{lt}\ 2^{-1/h^2}$$

$$= 2^{-\infty}$$

$$= 0$$

LHS lt = RHS lt

Limit exist

Some shortcut Formulae:-

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{e^{mx} - 1}{x} = m$$

$$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$$

$$\lim_{x \to 0} \frac{\cos ax - \cos bx}{x^2} = \frac{b^2 - a^2}{2}$$

$0/0$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\frac{\infty}{0}$, 1^∞, etc

are indeterminant form

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x\to 0} \frac{a^x - b^x}{x} = \log(a/b)$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

$$\lim_{x\to 0} (1+ax)^{1/x} = e^a$$

$$\lim_{x\to 0} (1+ax)^{b/x} = e^{ab}$$

$$\lim_{x\to 0} (1+ax)^{1/bx} = e^{a/b}$$

$$\lim_{x\to \infty} (1+1/x)^x = e$$

$$\lim_{x\to \infty} \left(1+\frac{n}{x}\right)^x = e^n$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = 1$$

If $0/0$, ∞/∞ then apply L-Hospital rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} \cdots\cdots$$

$$\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(f(x)-1)}$$

$\lim_{x\to\infty} \frac{f(x)}{g(x)}$, if both $f(x)$ and $g(x)$ are algebric function then,

If degree of $f(x) > g(x)$

$= \infty$

eg:: $\left(\lim_{x\to\infty} \frac{x^2+1}{x+1} = \infty\right)$

If degree of $f(x) < g(x)$

$= 0$

eg:- $\left(\lim_{x\to\infty} \frac{x^2+1}{x^3+2} = 0\right)$

If degree of $f(x) = g(x)$

$= \frac{\text{cofficient of } N^r}{\text{cofficient of } D^r}$

eg:- $\lim_{x\to\infty} \frac{2x^2+3x+4}{3x^2+2} = 2/3$

$|x| = x$ if $x > 0$

$|x| = -x$ if $x < 0$

$[x]$ = greatest integer $[2.5] = 2$

$[3.5] = 3$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \quad \text{(L-Hospital)}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x\cdot \sin x} = \frac{1}{2} \quad \text{(2 times L-Hospital)}$$

Method 1:-

$$\lim_{x\to 0} \frac{1-\cos x}{x\cdot\left(\frac{\sin x}{x}\right)\cdot x} = \lim_{x\to 0} \frac{1-\cos x}{x^2\cdot(1)} = \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

Method 2:-

$\sin x = 2\sin x/2 \cos x/2$

$1-\cos x = 2\sin^2 x/2$

$1+\cos x = 2\cos^2 x/2$

$$\lim_{x\to 0} \frac{\cancel{2}\sin^{\cancel{2}} x/2}{x\cdot \cancel{2}\,\cancel{\sin x/2}\cdot \cos x/2} = \frac{\tan x/2}{2(x/2)}$$

$$= \left(\frac{1}{2}\right)$$

Method 3:-

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x\sin x}\times\frac{1+\cos x}{1+\cos x}\right) = \lim_{x\to 0} \frac{1-\cos^2 x}{x\sin x(1+\cos x)}$$

$$\lim_{x\to 0} \frac{\sin^{\cancel{2}} x}{x\,\cancel{\sin x}(1+\cos x)} = \lim_{x\to 0} \frac{\sin x/x}{1+\cos x} = \left(\frac{1}{2}\right)$$

$$\lim_{x\to 0}\left(\frac{\tan x-\sin x}{\sin^3 x}\right)=\lim_{x\to 0}\frac{\frac{\cancel{\sin x}}{\cos x}-\cancel{\sin x}}{\cancel{\sin x}\,(\sin^2 x)}$$

$$\lim_{x\to 0}\frac{1-\cos x}{\cos x\,(1-\cos^2 x)}=\frac{\frac{\cancel{(1-\cos x)}}{\cos x}}{\cancel{(1-\cos x)}\,(1+\cos x)}$$

$$\lim_{x\to 0}=\frac{\cos x}{1+\cos x}=\frac{1}{1+1}=\left(\frac{1}{2}\right)$$

$$\lim_{x\to 1}\frac{\cos\frac{\pi}{2}x}{1-\sqrt{x}}$$ apply L- Hospital

$$\lim_{x\to 1}\frac{\cancel{-}\frac{\pi}{2}\sin\frac{\pi}{2}x}{0\cancel{-}\frac{1}{2\sqrt{x}}}=\frac{\frac{\pi}{2}\sin\frac{\pi}{2}}{\frac{1}{2\sqrt{1}}}=1$$

$$=\lim_{x\to 0}\left(\frac{\sinh x-\sin x}{x\sin^2 x}\right)$$

$$\left[\begin{array}{l} d(\sinh x)=\cosh x\\ d(\cosh x)=\sinh x\end{array}\right]$$

$$=\lim_{x\to 0}\frac{\sinh x-\sin x}{x\left(\frac{\sin^2 x}{x^2}\right)\times x^2}$$

$$=\lim_{x\to 0}\frac{\sinh x-\sin x}{x^3}$$ apply L-Hospital

$$=\lim_{x\to 0}\frac{\cosh x+\cos x}{6}=\frac{1+1}{6}=\frac{2}{6}=\frac{1}{3}$$

$$\underset{x\to 0}{lt} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} = \frac{1-1}{0} \quad \text{Apply L-Hospital}$$

$$\underset{x\to 0}{lt} \frac{-\left(\frac{\cancel{2x}}{2\sqrt{1-x^2}} + \frac{\cancel{2x}}{2\sqrt{1+x^2}}\right)}{\cancel{2x}} = -\frac{1}{2} - \frac{1}{2} = -1$$

$$\underset{x\to 0}{lt} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} = \frac{0}{0}$$

$$\underset{x\to 0}{lt} \frac{\frac{1}{3}(1+x)^{\frac{1}{3}-1} + \frac{1}{3}(1-x)^{\frac{1}{3}-1}}{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\underset{x\to \infty}{lt} \left[e^{1/5x} - 1\right]\left[5x + \frac{x}{5}\sin(1/x)\right] = 0 \times \infty$$

$$\left(\underset{x\to \infty}{lt} \frac{e^{1/x} - 1}{1/x} = 1\right)$$

$$\underset{x\to \infty}{lt} \frac{e^{1/5x} - 1}{1/5x} \times \frac{1}{5x}\left[5x + x/5 \ \sin(1/x)\right]$$

$$\underset{x\to \infty}{lt} \left(1 + \frac{1}{25}\sin 1/x\right) = 1$$

$$\lim_{x\to a} f(x)\cdot g(x) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$$

$$\lim_{x\to 0} 2^{8\cos x}\left(\frac{\sin^8(\pi/6+x) - \sin^8 \pi/6}{8x}\right)$$

$$\lim_{x\to 0} 2^{8\cos(0)}\left(\frac{\cancel{8}\sin^7(\pi/6+x)\cos(\pi/6+x) - 0}{\cancel{8}}\right)$$

$$= \cancel{2^8} \times \frac{1}{\cancel{2^7}} \times \frac{\sqrt{3}}{\cancel{2}} = \sqrt{3}$$

$$\lim_{x\to 0} (1+2x)^{1/3x} = e^{2/3}$$

$$\lim_{x\to\infty} \left(\frac{x-1}{x-2}\right)^x = e$$

apply, $e^{\lim_{x\to\infty} g(x)\left[f(x)-1\right]}$

$$e^{\lim_{x\to\infty} x\left[\frac{\cancel{x}-1-\cancel{x}+2}{x-2}\right]}$$

$$e^{\lim_{x\to\infty} \frac{1x}{1x-2}}$$

$$e^{1/1} = e$$

$$\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{n^2}\right) = \frac{\cancel{n}(n+1)/2}{n^{\cancel{2}}}$$

$$\lim_{n\to\infty}\left(\frac{n+1}{2n}\right) = \left(\frac{1}{2}\right)$$

$$\lim_{n\to\infty}\left(\frac{1+4+9+\cdots+n^2}{n^3}\right) = \frac{\cancel{n}(n+1)(2n+1)/6}{\cancel{n^3}\, n^2}$$

$$= \frac{2n^2+n+2n+1}{6n^2} = \left(\frac{1}{3}\right)$$

$$\lim_{n\to\infty}\frac{n\left[1^3+2^3+\cdots+n^3\right]^2}{\left[1^2+2^2+\cdots+n^2\right]^3}$$

$$\lim_{n\to\infty} = \frac{n\left(\left(\frac{n(n+1)}{2}\right)^2\right)^2}{\left(\frac{n(n+1)(2n+1)}{6}\right)^3} = \frac{n\left(\frac{n^2(n+1)^2}{4}\right)^2}{\frac{n^3(n+1)^3(2n+1)^3}{6^3}}$$

$$\lim_{n\to\infty}\frac{\frac{n^5(n+1)^4}{\cancel{16}}\times 6^3}{16\, n^3(n+1)^3(2n+1)^3} = \left(\frac{27}{16}\right)$$

$$\underset{n\to\infty}{\text{lt}} \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\cdots+\frac{1}{n(n+1)}$$

$$\underset{n\to\infty}{\text{lt}} \sum_{n}^{\infty} \frac{1}{n(n+1)} = \underset{n\to\infty}{\text{Lt}} \sum_{n}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\underset{n\to\infty}{\text{lt}} \left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{3}}+\cdots\cdots -\frac{1}{n+1}\right)$$

$$\underset{n\to\infty}{\text{lt}} \left(1-\frac{1}{n+1}\right) = (1)$$

$$\underset{x\to 0}{\text{lt}} \frac{1-\cos 2x}{1-\cos 3x} = \frac{2^2}{3^2} = \left(\frac{4}{9}\right)$$

$$\underset{x\to 0}{\text{lt}} \frac{\cos 2x-\cos 3x}{x^2} = \frac{3^2-2^2}{2} = \left(\frac{5}{2}\right)$$

Continuity of func^n:-

A func^n $f(x)$ is said to be continuous at $x=a$, if $\left(\underset{x\to a}{\text{lt}} f(x) = f(a)\right)$ otherwise it is said to a discontinuous func^n.

Types of discontinuous funcⁿ :-

(i) **Discontinuity of I Type** :- A funcⁿ $f(x)$ is said to
(Jump discontinuity) be jump discontinuity, if

$$\left(\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \right)$$

(ii) **Discontinuity of II type** :- If one of the LHS limit or R.H.S limit doesn't exist.

(iii) **Discontinuity III type** :-
(Removable discontinuity)

$$\left(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a) \right)$$

Standard Continuous funcⁿ :-

- 'x^n' is a continuous funcⁿ $\forall x$ when n is +ve i.e $n>0$ $(x, x^2, x^3, \ldots)$
- x^n is continuous funcⁿ $\forall x$ except $x=0$ for $n<0$

- $|x|$ is a continuous func^n $\forall\ x$

- $\log x$ is continuous func^n $\forall\ x > 0$

- Every exponential func^n a^x is continuous

- $\sin x$ & $\cos x$ are continuous $\forall\ x$

- $\tan x$ & $\sec x$ are continuous $\forall\ x$ except $\cos x = 0$
  i.e $x = (2n+1)\pi/2$

- $\cot x$ & $\operatorname{cosec} x$ are continuous $\forall x$ except $\sin x = 0$
  i.e $x = n\pi$

Q. Check continuity :

$$f(x) = \frac{x\left(e^{1/x}-1\right)}{e^{1/x}+1} \quad ; \quad f(0) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} \frac{-h\left(e^{1/-h}-1\right)}{e^{-1/h}+1} = \frac{-h(0-1)}{0+1} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{h\left(e^{1/h}-1\right)}{e^{1/h}+1} = \frac{h\,\cancel{e^{1/h}}\left(1-\frac{1}{e^{1/h}}\right)}{\cancel{e^{1/h}}\left(1+\frac{1}{e^{1/h}}\right)}$$

$$= \frac{h(1-0)}{(1+0)} = 0$$

$$\left(\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \right)$$

Continuous function

Q. A $f(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{2} - x & 0 < x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{3}{2} - x & \frac{1}{2} < x < 1 \\ 1 & x \ge 1 \end{cases}$

which of the following is True :

(a) continuous at $x = 0$ — False

(b) discontinous at $x = \frac{1}{2}$ — True

(c) continuous at $x = 1$ — False

(d) All are true — False

$\lim_{x \to 0} f(x)$

$x < 0$	$x = 0$	$x > 0$
0	0	$\frac{1}{2}$

discontinuous

$\lim_{x \to \frac{1}{2}} f(x)$

$x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
0	$\frac{1}{2}$	1

discontinuous

Q.

$$f(x) = \begin{cases} -x^2 , & x \le 0 \\ 5x-4 , & 0 < x \le 1 \\ 4x^2-3x , & 1 \le x \le 2 \\ 3x+4 , & x > 2 \end{cases}$$

check continuity at $x = \overset{\times}{0}, \overset{\checkmark}{1}, \overset{\checkmark}{2}$

$\lim_{x \to 0} f(x)$

$x<0$	$x=0$	$x>0$
0	0	-4

discontinuous

$\lim_{x \to 1} f(x)$

$x<1$	$x=1$	$x>1$
1	1	1

$\lim_{x \to 2} f(x)$

$x<2$	$x=2$	$x>2$
10	10	10

continuous

⇒ Differentiability of a func^n:-

A func^n $f(x)$ is said to be differentiable at $x=a$ if both $L.H.D = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ and

$RHD = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists & equal.

Formulae:-

$d(x^n) = n.x^{n-1}$

$d(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$d(1/\sqrt{x}) = \frac{-1}{2x\sqrt{x}}$

$d(1/x) = -1/x^2$

$d(1/x^2) = -2/x^3$

$d(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$d(\tan^{-1}x) = \frac{1}{1+x^2}$

$d(\sinh^{-1}x) = \frac{-1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$d(a^x) = a^x \log a$

$d(e^{ax}) = a.e^{ax}$

$d(\log x) = \frac{1}{x}$

$d(\sin x) = \cos x$

$d(\cos x) = -\sin x$

$d(\tan x) = \sec^2 x$

$d(\sec x) = \sec x.\tan x$

$d(\sinh x) = \cosh x$

$d(\cosh x) = \sinh x$

$d(x^x) = x^x(1+\log x)$

$d(x^{1/x}) = x^{1/x}\frac{1}{x^2}(1-\log x)$

Q. $y = x^{x^{x^{x^{\cdots}}}}$, then $\frac{dy}{dx} =$ ______

$\Rightarrow y = x^y$

$\Rightarrow \log y = y \log x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}$

$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = y/x$

$$\left(\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}\right)$$

Q. if $x^y = e^{x-y}$, then $\frac{dy}{dx} =$ ______.

$y \log x = x-y$

$y(\log x+1) = x \Rightarrow y = \frac{x}{\log x+1}$

$\frac{dy}{dx}(\log x+1) - \frac{y}{x} = x$

$$\frac{dy}{dx} = \frac{(\log x + 1) - 1}{(\log x + 1)^2}$$

$$\left(\frac{dy}{dx} = \frac{\log x}{(\log x + 1)^2}\right)$$

Q. If $x^m \cdot y^n = a^{m+n}$; $\frac{dy}{dx}$ ———

$$m \log x + n \log y = m+n \log a$$

$$m/x + n/y \frac{dy}{dx} = 0$$

$$\frac{m}{x} = \frac{n}{y}\frac{dy}{dx}$$

$$-m/nx \cdot y = \frac{dy}{dx}$$

$$\left(\frac{dy}{dx} = -\frac{m}{n}(y/x)\right)$$

Q. If $x^m \cdot y^n = (x+y)^{m+n}$; $\frac{dy}{dx}$ ———

$$m \log x + n \log y = m+n \left(\log (x+y) \right)$$

$$m/x + n/y \, y' = \frac{m+n}{x+y} \left(1 + y' \right)$$

$$m/x = \left(\frac{m+n}{x+y} - \frac{n}{y} \right) y' + \frac{m+n}{x+y}$$

$$m/x - \frac{m+n}{x+y} = \left(\frac{m+n}{x+y} - n/y \right) y'$$

$$\left(y' = y/x \right)$$

Q. if $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x} \cdots \infty}}$; $\frac{dy}{dx}$ ______

$\Rightarrow \quad y = \sqrt{\sin x + y}$

$\Rightarrow \quad y^2 = \sin x + y$

$\Rightarrow \quad (2y)\, y' = \cos x + y'$

$\Rightarrow \quad \left(y' = \frac{\cos x}{2y-1} \right)$

STANDARD FORM:-

$$y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \cdots \infty}}}$$

$$\left(\frac{dy}{dx} = \frac{f'(x)}{2y-1} \right)$$

Q. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ $\Rightarrow (1-x^2) \cdot y' - xy =$ ______

(a) 0 (b) 1 (c) y (d) $-y$

$$\sqrt{1-x^2}\cdot y = \sin^{-1}x$$

$$(1-x^2)\,y^2 = (\sin^{-1}x)^2$$

$$-\cancel{2}xy^2 + \cancel{2}yy'(1-x^2) = \cancel{2}\,\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\cancel{y}y'(1-x^2) - xy^{\cancel{2}} = \cancel{y}$$

$$\left(y'(1-x^2) - xy = 1\right)$$

Q. If $y = a\cos(\log x) + b\sin(\log x)$

then $x^2y'' + xy'$ ______

$$y' = -a\sin(\log x)\left(\frac{1}{x}\right) + b\cos(\log x)\left(\frac{1}{x}\right)$$

$$xy' = -a\sin(\log x) + b\cos(\log x)$$

$$y' + xy'' = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$xy' + x^2y'' = -\left(a\cos(\log x) + b\sin(\log x)\right)$$

$$\left(xy' + x^2y'' = -y\right)$$

Q. If $y = \tan^{-1}\left[\frac{(3-x)\sqrt{x}}{1-3x}\right] \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} =$ ______

Let $\sqrt{x} = \tan\theta$

so, $y = \tan^{-1}\left(\frac{(3-\tan^2\theta)\tan\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$= 3\theta$

$= 3\sqrt{x}$

$y' = \left[\frac{3}{2\sqrt{x}}\right]_1$

$= \left(\frac{3}{2}\right)$

Q: If $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, $\frac{dy}{dx} =$ ______

$$y = \tan^{-1}\left(\frac{\sin(\pi/2 - x)}{1 + \cos(\pi/2 - x)}\right)$$

$$y = \tan^{-1}\left(\frac{\cancel{2}\sin(\pi/4 - x/2)\cancel{\cos(\pi/4 - x/2)}}{\cancel{2}\cos^{\cancel{2}}(\pi/4 - x/2)}\right)$$

$$y = \tan^{-1}\left(\tan(\pi/4 - x/2)\right)$$

$$y = \pi/4 - x/2$$

$$\left(y' = 0 - 1/2 = -1/2\right)$$

Q: If $x = a\cos^3\theta$; $y = a\sin^3\theta$ find $\frac{dy}{dx}$ ______

$$\frac{dy/d\theta}{dx/d\theta} = \frac{3\cancel{a}\sin^{\cancel{2}}\theta \cdot \cancel{\cos\theta}}{3\cancel{a}\cos^{\cancel{2}}\theta(-\cancel{\sin\theta})} = \left(-\tan\theta\right)$$

Q: $x = 3\cos\theta - \cos^3\theta$ $\qquad \frac{dy}{dx}$ —

$y = 3\sin\theta - \sin^3\theta$

$$\frac{dy}{dx} = \frac{\cancel{3}\cos\theta - \cancel{3}\sin^2\theta\cos\theta}{-\cancel{3}\sin\theta + \cancel{3}\cos^2\theta\sin\theta} = -\cot\theta\left(\frac{1-\sin^2\theta}{1-\cos^2\theta}\right)$$

$$\left(\frac{dy}{dx} = -\cot^3\theta\right)$$

Q. $x = a(\theta + \sin\theta)$ $\qquad \frac{dy}{dx}$ —

$y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{\cancel{a}(\sin\theta)}{\cancel{a}(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\left(\frac{dy}{dx} = \tan\theta/2\right)$$

Q: $x = a[\theta\sin\theta + \cos\theta]$ $\qquad \frac{dy}{dx}$ —

$y = a[\sin\theta - \theta\cdot\cos\theta]$

$$\frac{dy}{dx} = \frac{\cancel{\cos\theta} - (\cancel{\cos\theta} - \theta\sin\theta)}{\cancel{\sin\theta} + \theta\cos\theta - \cancel{\sin\theta}}$$

$$\left(\frac{dy}{dx} = \tan\theta\right)$$

Q. $x = a(\theta - \sin\theta)$ $\qquad \frac{d^2y}{dx^2} \longrightarrow$

$y = a(1 - \cos\theta)$

$$\frac{dy}{dx} = \frac{-\sin\theta}{1 - \cos\theta} = \cancel{-\tan\theta/2}$$

$$\frac{dy}{d\theta} = -a\sin\theta \quad ; \quad \frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{dx} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d(\cot\theta/2)}{d\theta} \times \frac{d\theta}{dx}$$

$$= -\frac{1}{2}\operatorname{cosec}^2\theta/2 \times \frac{1}{a(1 - \cos\theta)}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{1}{4a}\operatorname{cosec}^4\theta/2\right)$$

- $|x|$ is continous $\forall x$ exept $x = 0$

eg:-

$f(x) = |2-3x|$ $\forall x$ exept $x = 2/3$

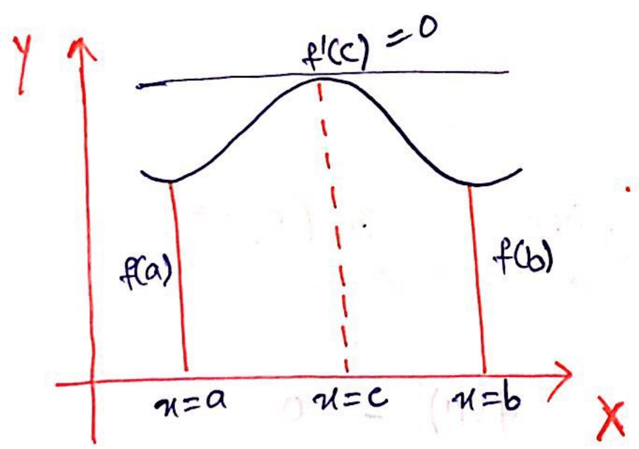
$f(x) = |x-1| + |x-2|$ $\forall x$ exept $x = 1$ and 2

⇒ Mean Value Theorem :-

(i) Rolle's mean value theorem :-

Let $f(x)$ be a func^n defined such that -

- $f(x)$ is continuous in $[a,b]$
- $f(x)$ is differentiable in (a,b)
- $f(a) = f(b)$ then their atleast one value $c \in (a,b)$ such that $f'(c) = 0$



Q. Discuss the applicability of rolle's mean value theorem :

$$f(x) = x^3(1-x)^2 \text{ in } [0,1]$$

$f(0) = 0$

$f(1) = 0$

f(x) is differentiable b/w 0,1

so it is continuous also

$$f(x) = x^3(x^2-2x+1)$$

$$= x^5 - 2x^4 + x^3$$

$$f'(x) = 5x^4 - 8x^3 + 3x^2$$

$$= x^2(5x^2 - 8x + 3)$$

$x = 0, 3/5, 1$

$(c = 3/5)$

Q. $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$

$f(0) = \dfrac{0}{1}$ $\quad$ $f(\pi) = 0$

$$f'(x) = \frac{\cos x\, e^x - \sin x\, e^x}{(e^x)^2}$$

$$= \cos x - \sin x / e^x$$

$$\frac{\cos x - \sin x}{e^x} = 0$$

$\cos x = \sin x$ $\quad x = \pi/4$

so, $(c = \pi/4)$

Q. $f(x) = x^3 - 4x$ in $[-2, 2]$

$f(-2) = -8 + 8 = 0$ $\quad f(2) = 8 - 8 = 0$

$f'(x) = 3x^2 - 4$ $\quad \left(x = \pm 2/\sqrt{3}\right)$

$3x^2 - 4 = 0$

$x^2 = \frac{\pm 2}{\sqrt{3}}$

Q. $f(x) = \frac{x^2 + ab}{x(a+b)}$, $[a, b]$ $\quad a > 0 \quad b > 0$

$$\frac{2x^2(\cancel{a+b}) - (x^2+ab)(\cancel{a+b})}{x^2(a+b)^{\cancel{2}}} = 0$$

$2x^2 = x^2 + ab$ $\quad (c = \sqrt{ab})$

$x^2 = ab$

$x = \pm\sqrt{ab}$

Means

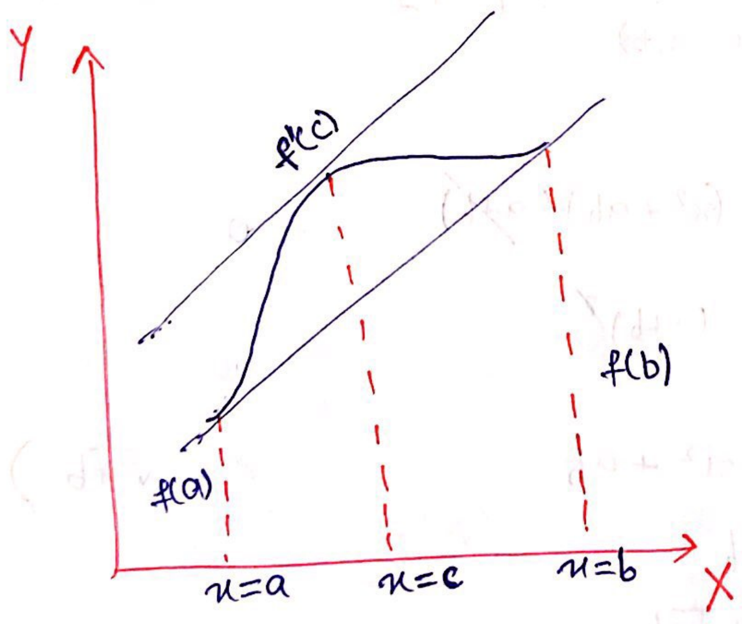
Arthematic	Geometric	Harmonic
$\left(\frac{a+b}{2}\right)$	$\left(\sqrt{ab}\right)$	$\left(\frac{2ab}{a+b}\right)$

⇒ Lagrange's Mean value Theorem :-

Let $f(x)$ be a func^n defined such that :

- $f(x)$ is continuous in $[a,b]$
- $f(x)$ is differentiable in (a,b)
- $f(a) \neq f(b)$ then there must be atleast one value $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Q. $f(x) = \log x$ in $[1, e]$, find c.

$f(1) = 0$

$f(e) = 1$

$f'(c) = \dfrac{1-0}{e-1} = \dfrac{1}{e-1}$

$\dfrac{1}{x} = \dfrac{1}{e-1} \Rightarrow (x = e-1)$

Q. $f(x) = x^3 - 6x^2 + 11x - 6$ in $(0, 4)$

$f(0) = -6$

$f(4) = 64 - 96 + 44 - 6 = 6$

$f'(c) = \dfrac{6+6}{4} = \underline{3}$

$f'(x) = 3x^2 - 12x + 11 = 3$

$3x^2 - 12x + 8 = 0$

$f'(c) = 3c^2 - 12c + 8 = 0$

$3c^2 - 12c + 8$

$c = \dfrac{12 \pm \sqrt{144 - 4(3)(8)}}{6}$

$c = \dfrac{12 \pm \sqrt{144 - 96}}{6} = \left(2 \pm 2/\sqrt{3}\right)$

Cauchy's Mean value Theorem :-

Let $f(x)$ and $g(x)$ be two functions defined such that :

- $f(x)$ and $g(x)$ are continuous in $[a,b]$
- f and g are differentiable in (a,b)
- $g'(x) \neq 0 \ \forall \ x \in (a,b)$ then there must be atleast one value $c \in (a,b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Q. $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \pi/2]$

$f(0) = 0 \qquad g(0) = 1$

$f(\pi/2) = 1 \qquad g(\pi/2) = 0$

$$f(c) = \frac{1-0}{0-1} = -1$$

$$\frac{f'(x)}{g'(x)} = -\frac{\cos x}{\sin x} = -1$$

$$\cos x = \sin x \qquad (x = \pi/4)$$

Q. $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

$$f'(c) = \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{\cancel{e^b - e^a}}{\frac{\cancel{e^a - e^b}}{e^{a+b}}}$$

$$f'(c) = -e^{a+b}$$

$$\cancel{-}\frac{e^x}{e^{-x}} = \cancel{-}e^{a+b}$$

$$e^{2x} = e^{a+b} \qquad \boxed{c = \frac{a+b}{2}}$$

Q. $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$

$$f'(c) = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = -\sqrt{ab}$$

$$\frac{f'(x)}{g'(x)} = \frac{\cancel{-}\frac{1}{\cancel{2}\sqrt{x}}}{\cancel{-}\frac{1}{\cancel{2}}\sqrt{x}\sqrt{x}} \qquad -\frac{1}{2}x^{-1/2+1} \qquad -\frac{1}{2}\sqrt{x}$$

$$= \cancel{-}\frac{x\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = \cancel{-}\sqrt{ab}$$

$$(x = \sqrt{ab})$$

Q. $f(x) = \frac{1}{x^2}$, $g(x) = -1/x$ in $[a, b]$

$$f'(c) = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{a} - \frac{1}{b}} = \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{b-a}{ab}}$$

$$f'(c) = -\frac{(a+b)}{ab} = -\frac{(b+a)}{ab}$$

$$\frac{f'(x)}{g'(x)} = \frac{-2/x^3}{1/x^2} = \cancel{-}2/x = \cancel{-}\frac{(b+a)}{ab}$$

$$\left(x = \frac{2ab}{a+b}\right)$$

⇒ Taylor's Mean value Theorem :-

Let $f(x)$ be a func^n defined such that :

- $f(x)$ is continuous in $[a, a+h]$
- $f^{n-1}(x)$ is differentiable in $(a, a+h)$

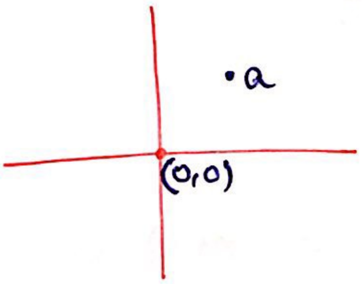
then atleast one value $\theta \in (0, 1)$ such that

$$\left(f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \text{-----} \frac{h^n}{n!} f^n(a + \theta h) \right)$$

Let $a+h = x$ so $h = x-a$

Now, enhanced Taylor's equation (series)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a)$$
$$+ \cdots\cdots \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^{n}(a)$$



Maclurin consider origin instead of a

so put $a = 0$

Maclurin series:-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(0) + \cdots\cdots + \frac{x^n}{n!} f^n(0).$$

Q: Expand e^x at $x = 0$.

$f'(x) = e^x = e^{(0)} = 1$

$f^2(x) = e^x = e^{(0)} = 1$

$\vdots$

$f^n(x) = e^0 = 1$

$$\left(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}\right)$$

Q. Expand $f(x) = \sin x$ at $x = 0$

$f^1(x) = \cos x = \cos(0) = 1$

$f^2(x) = -\sin x = -\sin 0 = 0$

$f^3(x) = -\cos x = -1$

$f^4(x) = +\sin x = 0$

$f^5(x) = \cos x = 1$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0)$$

$$+ \frac{x^5}{5!}(1) + \frac{x^6}{6!}(0) + \frac{x^7}{7!}(-1) + \cdots$$

$$\left(\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots\right)$$

similarly,

$$\left(\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)$$

Q: Expand $\log(1+x)$ at $x = 0$.

- $\log(1+x) \rightarrow \log 1 \rightarrow 0$

$f'(0) = \frac{1}{1+x} = 1$

$f^2(0) = \frac{-1}{(1+x)^2} = -1$

$f^3(0) = \frac{2}{(1+x)^3} = 2$

$\vdots$

$$\log(1+x) = 0 + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(0) + \cdots$$

$$= 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!}(2) + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \infty$$

Q. Find the expansion of $f(x) = \frac{x}{1+x}$ about $x=0$

$f(x) = 1 - \frac{1}{1+x}$ $\quad f(0) = 0$

$f'(x) = \frac{1}{(1+x)^2}$ $\quad f'(0) = 1$

$f^2(x) = \frac{-2}{(1+x)^3}$ $\quad f^2(0) = -2$

$f^3(x) = \frac{+6}{(1+x)^4}$ $\quad f^3(0) = 6$

$\vdots$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f^2(0) + \frac{x^3}{3!} f^3(0) + \cdots\cdots \infty$$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(6) + \cdots$$

$$f(x) = x - x^2 + x^3 - x^4 + \cdots \infty$$

Q. Find the cofficient of x^2 in the expansions of $\cos^2 x$ at $x = 0$

$f(x) = \cos^2 x$ at $x = 0$

$f'(x) = -2\cos x \sin x = -\sin 2x$

$f^2(x) = -2\cos 2x$

$$f^2(0) = -2$$

$\cos^2 x = \cos x \times \cos x$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= -\frac{x^2}{2!} - \frac{x^2}{2!} = (-1)x^2$$

(cofficient $= -1$)

Q. Find the cofficeint of $(x-2)^4$ in the expansion of $f(x) = e^x$ about $x = 2$.

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!} f^2(2) + \frac{(x-2)^3}{3!} f^3(2)$$

$$+ \frac{(x-2)^4}{4!} f^4(2)$$

$$\frac{e^2}{4!}$$

Q. Find the expansion of $f(x) = x^2$ about $x = 1$

$f(1) = 1$

$f'(1) = 2$

$f^2(1) = 2$

"Taylor's expansion can't be applied on algebric expression (function)"

so, $f(x) = x^2$

* $f^{n-1}(x^2)$ doesn't exist, so we can't expand.

Q. Expand $f(x) = \log\left(\frac{1+x}{1-x}\right)$ about $x=0$

$f(0) = 0$

$f'(0) = \frac{1}{1+x} - \frac{1}{1-x} = 0$

$f''(0) = \frac{-1}{(1+x)^2} - \frac{1}{(1-x)^2} = -2$

$f'''(0) = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3} = 4$

$\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \infty$$

$$\log(1-x) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \cdots \infty$$

Q. $f(x) = \sin x$ about $x = \pi/6$

$f(x) = \sin x = 1/2$

$f'(x) = \cos x = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x = -1/2$

$f'''(x) = -\cos x = -\sqrt{3}/2$

$$\sin x = \sin \pi/6 + (x - \pi/6)\left(\sqrt{3}/2\right) + \frac{(x-\pi/6)^2}{2!}\left(-1/2\right) + \frac{(x-\pi/6)^3}{3!}\left(-\sqrt{3}/2\right) + \cdots \infty$$

Q. $f(x) = \dfrac{\sin x}{(x-\pi)}$ about $x = \pi$

Let $x - \pi = t$ so when $x \to \pi$ then $t \to 0$

$$f(x) = \frac{\sin(t+\pi)}{t} = -\frac{\sin t}{t}$$

$$-\frac{\sin t}{t} = -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right]$$

$$= \left(-1 + \frac{t^2}{3!} - \frac{t^4}{5!} - \cdots\right)$$

Partial & Total Derivatives :-

Ordinary Derivative	Partial Derivative
$y = f(x)$	$z = f(u,v)$
$x = f(y)$	$u = f(x,y,z)$

$z = x^2 - xy + y^2$; $\frac{\partial z}{\partial x} = 2x - y$; $\frac{\partial^2 z}{\partial y \partial x} = -1$

$\frac{\partial z}{\partial y} = -x + 2y$ $\quad \frac{\partial^2 z}{\partial x \partial y} = -1$

$$\left(\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \right)$$

Homogeneous func^n :- A function $f(x,y)$ is said to be homogeneous of degree 'n' in x and y if

$$\left(f(kx, ky) = k^n \cdot f(x,y) \right)$$

eg:- $f(x,y) = x^2 - xy + y^2$

$f(kx, ky) = k^2x^2 - k^2xy + k^2y^2$

$= k^2(x^2 - xy + y^2)$

$= k^2 f(x,y)$

- Product of two homogenous $func^n$ is a homogeneous $func^n$.

$d=3$ $\quad d=2$

$$f(x,y) = (x^3-y^3)(x^2-y^2)$$

$d=5$

$$= x^5 - x^3y^2 - x^2y^3 + y^5$$

degree = 5 (sum of two)

- If the $func^n$ is in rational form, both numerator & denominator are homogeneous, then rational $func^n$ is also homogeneous.

$$f(x,y) = \frac{x^3+y^3}{x^2-y^2} \qquad f(x,y) = \frac{x^{1/5}+y^{1/5}}{x^{1/4}-y^{1/4}}$$

$d=1$ (numerator $d=3$, denominator $d=2$) $\qquad d=-1/20$ (numerator $d=1/5$, denominator $d=1/4$)

* Degree can be negative

Euler's theorem :-

If 'z' is a homogeneous $func^n$ of degree n in x, y then

$$z=f(x,y) \qquad \left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \right)$$

$$u=f(x,y,z) \qquad \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \right)$$

- This formula is applicable directly if $f(x,y)$ is algebric func^n.

eg:-

- $z = (x^2+y^2)^{1/3}$ $\qquad n = 2 \times 1/3$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2}{3}z$$

- $z = \dfrac{x^3y^3}{x^2+y^2}$ $\qquad n = 6-2 = 4$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4z$$

- $u = (x^2+y^2+z^2)^{-1/2}$ $\qquad n = 2 \times -\frac{1}{2} = -1$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$$

Q. $f = a_0x^n + a_1x^{n-1}\cdot y + a_2x^{n-2}y^2 + \ldots$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

If z is not algebric function, let $\phi(z)$ is algebric and homogeneous of degree n in x & y then,

$$\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\frac{\phi(z)}{\phi'(z)} \right)$$

eg:-

$$z = \log(x^2+y^2)$$

$$\phi(z) = e^z = x^2+y^2 \qquad n=2$$

$$\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2\cancel{e^z}}{\cancel{e^z}} = 2 \right)$$

eg:- $z = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

$$\phi(z) = \sin z = \frac{x^2+y^2}{x-y} \qquad d = 2-1 = 1$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\sin z}{\cos z} = \tan z$$

eg:- $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\phi(z) = \tan z = \frac{x^3+y^3}{x-y} \Rightarrow n=2$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\tan z}{\sec^2 z} = \sin 2z$$

Q. if $z = x^2 \sin^{-1}(y/x) + y^2 \tan^{-1}(x/y)$

$$z = x^2 \underbrace{\sin^{-1}(y/x)}_{\text{angle}^{\circ},\ d=0} + y^2 \underbrace{\tan^{-1}(x/y)}_{}$$

so, $n = 2$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Q. if $z = (x^3+y^3)\, \underbrace{e^{-x/y}}_{d=0}$

so, $n = 3$

Application of Euler's theorem:-

- If z is a homogeneous funcⁿ of degree n in x, y then
$$\left(x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)\, z\right)$$

- It is applicable for algebric expression

NOTE:-

If z is not algebric ; and let $\phi(z)$ is algebric & homogeneous of degree n in x and y then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = F(z)\left[F'(z) - 1\right]$$

$$\text{where } F(z) = n\frac{\phi(z)}{\phi'(z)}$$

Explicit function :-

Any function which we can express in the form of either $y = f(x)$ or $x = g(y)$ etc is said to be an explicit function.

$$y = ax^2 + bx + c \qquad dy/dx = 2ax + b$$

Implicit function :-

Any func^n which is not explicit is said to be Implicit func^n.

$$ax^2 + 2bxy + by^2 = 0$$

$$\boxed{dy/dx = -\frac{\partial f/\partial x}{\partial f/\partial y}}$$

$$z = f(x,y)$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} \qquad s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$ax^2 + 2hxy + by^2 = 1$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}}$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{(q^2r - 2pq + pt^2)}{q^3}}$$

Composite function:-

Any funcⁿ inside another funcⁿ as input is known as composposite function.

$$gof(x) = g[f(x)]$$

If $z = f(x,y)$ and $x(t)$ and $y(t)$

then $z = f(t)$ composite funcⁿ in t.

Total derivative of composite funcⁿ:-

If $z = f(x,y)$ and $x = g(t)$, $y = h(t)$ then total derivative of z w.r.t 't' is denoted by $\frac{dz}{dt}$:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

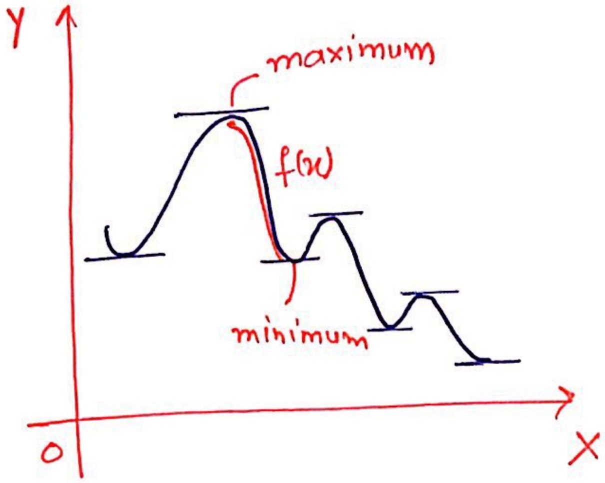
eg:- $z = e^x \sin y$, $x = \log t$, $y = t^2$ — $\frac{dz}{dt}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^x \sin y \cdot 1/t + e^x \cos y \, (2t)$$

$$= \frac{e^x}{t} (\sin y + 2t^2 \cos y)$$

Maxima & Minima



- If a continuous func^n f(x) increases to a certain value & then decreases, that value is called <u>maximum value</u>

- If a continuous func^n f(x) decreases to a certain value & then increases, that value is called <u>minimum value</u>

- Maxima & minima occur alternatively. A func^n can have several maximum and minimum value.

- The minimum value may be greater than the maximum value.

- The least minimum value is called the <u>global or universal minimum.</u>

  The highest maximum value is called <u>global maximum or universal maximan.</u>

for a maxima or minima:-

$f'(x) = 0$ we get $x = \alpha, \beta, \gamma, \ldots$ etc

then, put

$(x = \alpha)$

$f''(\alpha) > 0$, then minima at $x = \alpha$

Min value $f(\alpha)$

$(x = \beta)$

$f''(\beta) < 0$, then maxima at $x = \beta$

Maximum value $f(\beta)$

$(x = \gamma)$

$f''(\gamma) = 0$, then f is stationary at γ

$f'''(\gamma) \neq 0$, then γ is said inflection point

eg:- $f(x) = x^x$

$f'(x) = x^x(1 + \log x)$

$f''(x) = x^x(1/x) + x^x(1 + \log x)^2$

$= x^{x-1} + x^x(1 + \log x)^2$

$f'(x) = 0$

$\log x = -1$ $(x = 1/e)$

$$f''(1/e) = (1/e)^{1/e-1} + (1/e)^{1/e}(1-1)^2$$

$$= \underbrace{(1/e)^{1/e-1}}_{\text{exponential always +ve}}$$

$f''(1/e) > 0$

so, (f will be minimum at $x = 1/e$)

$$\left(f_{min} = -e^{-1/e} \right)$$

f^n	Min value	Max value	At point
x^x	$e^{-1/e}$	—	$x = 1/e$
$x^{1/x}$	—	$e^{-1/e}$	$x = e$

eg:- $f(x) = \dfrac{\log x}{x}$