(LINEAR ALGEBRA

Properties of determinants:-

(i) If any 2 rows or 2 columns of a matrix are identical then determinant is zero

(ii) If 2 rows or 2 columns of a matrix are interchange then sign of determinant changes.

(iii) If 3 rows or 3 colums of a matrix are inter- -changed then no change in determinant sign.

(iv) In the determinant of a matrix, any column containing sum or difference of 2 elements then it can be split into sum or difference of two seperate determinants.

$$\begin{vmatrix} a & a^2 & a^2+1 \\ b & b^2 & b^2+1 \\ c & c^2 & c^2+1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^2 \\ b & b^2 & b^2 \\ c & c^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \; ; \quad \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - cb)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \; ; \quad \Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$- \; a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \; +$$

$$a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

= Traingular matrix :-

Lower

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

all elements above the principal diagonal are 0.

Upper

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

all elements below the principal diagonal are 0.

NOTE:- If a matrix is either Lower or upper triangle then determinant is the product of the principal diagonal element.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \quad ; \quad |A| = 1 \times 3 \times 6 = \underline{18}$$

(V) Determinant of a matrix and its transpose is same.

$$(|A| = |A^T|)$$

Elementary operations:-

eg:-

$$A \begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix} = \quad |A| = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$C_3 \to C_2 + C_3$ $\Rightarrow$ $\begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$

$$|A| = (a+b+c) \cdot \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$$|A| = (a+b+c) \times 0 = 0$$

$$\underline{|A| = 0}$$

eg:-

$$|A| = \begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & b & ca \\ \frac{1}{c} & c & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} 1 & a^2 & abc \\ 1 & b^2 & abc \\ 1 & c^2 & abc \end{vmatrix}$$

$$= \frac{abc}{abc} \begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} = 0$$

eg:-

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} \quad \underline{R_2 \to R_2 - R_1} \quad \underline{R_3 \to R_3 - R_1}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} = a \times b \times 1 = \underline{ab}$$

eg:-

$$|A| = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc \begin{vmatrix} 1+1/a & 1/a & 1/a \\ 1/b & 1+1/b & 1/b \\ 1/c & 1/c & 1+1/c \end{vmatrix}$$

$$(R_1 \to R_1+R_2+R_3) \quad = abc\left(1+1/a+1/b+1/c\right) \begin{vmatrix} 1 & 1 & 1 \\ 1/b & 1+1/b & 1/b \\ 1/c & 1/c & 1+1/c \end{vmatrix}$$

$$\begin{matrix}(C_2 \to C_2 - C_1) \\ (C_3 \to C_3 - C_1)\end{matrix} \quad = abc\left(1+1/a+1/b+1/c\right) \begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix}$$

$$|A| = \boxed{abc\left(1+1/a+1/b+1/c\right)}$$

⇒ shortcut for determinant:- (3×3 matrix)

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3}$$

$$|A| = \begin{array}{ccccc} c_0 & c_1 & c_2 & c_0 & c_1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \end{array} \quad = \begin{matrix} 2\times1\times0 \\ + \\ 1\times3\times3 \\ + \\ 2\times1\times1 \end{matrix} = \underline{11}$$

$$\begin{array}{ccccc} 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \end{array} \quad = 12+0+1 = 13$$

$$(|A| = 11 - 13 = -2)$$

eg:-

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{matrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{matrix} = (1+8+8)-(4+4+4)$$

$$= 5$$

$$(|A| = 5)$$

eg:-

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

$$|A| = \begin{matrix} 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{matrix} = (2+8+15)-(5+4+12)$$

$$= 4$$

$$|A| = 4$$

Singular & Non singular matrix :-

$|A| = 0$ (singular)

$|A| \neq 0$ (Non- singular)

Inverse of Matrix :- (2X2)

Only non- singular matrix has inverse.

$$\boxed{A^{-1} = \frac{Adj\, A}{\Delta}}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

eg:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

eg:-

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Adjoint of Matrix :-

for higher order matrix (3x3 or higher)

(Adj A = (Transpose of cofactors of matrix))

(cofactors of an element = $(-1)^{i+j}$ Minor)

(Minor of an element = determinant of square sub-matrix in which row and column of particular elements are not included.)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} M_{21}$$

$$= (-1) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

eg:-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2$$

$$C_{32} = (-1)^{3+2}(-2)$$

$$C_{32} = 2$$

Shortcut for Inverse of Matrix (3X3) :-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c|cccc} 0 & 1 & 2 & 0 & 1 \\ \hline 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

go row-wise
write column wise

$$\text{Adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\left(A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} \right)$$

eg:-

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|A| = 1(3\ ) - 2(-1) - 2(2)$$
$$= 3 + 2 - 4$$
$$= 1$$

$$\begin{array}{c|ccccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

$$\text{Adj } A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

eg:-

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta$$
$$= 1$$

$$
\begin{array}{c|cccc}
\cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\
\hline
\sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta \\
0 & 0 & 1 & 0 & 0 \\
\cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\
\sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta
\end{array}
$$

$$
A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

eg:-

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (|A| = -1)
$$

$$
\begin{array}{c|cccc}
0 & 1 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{array}
$$

$$
A^{-1} = -\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

$$
A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Rank of matrix :-

If all minors of order $(r+1)$ are zeros, but there is atleast one non-zero minor of order 'r', if exists it is called the rank of matrix and is denoted by $\boxed{\rho(A) = r}$.

Properties of rank :-

- If 'A' is null matrix, then rank of A is 0

$$(\rho(A) = 0)$$

- If 'A' is a non-zero matrix then rank of A :

$$(\rho(A) \geq 1)$$

- If 'I' is a unit or identity matrix of or $n \times n$ then

$$(\rho(I_n) = n)$$

$$\rho(I_2) = 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- If 'A' is a matrix of order $m \times n$, then

$$\left(\rho(A) \le \min(m,n)\right)$$

- $$\left(\rho(AB) \le \min\left(\rho(A), \rho(B)\right)\right)$$

eg:-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3} \qquad 0 < \rho(A) \le 2$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \qquad \text{so} \quad \rho(A) < 2$$

$\Delta = 0$ $\qquad \Delta = 0$ $\qquad \Delta = 0$ $\qquad$ i.e $\boxed{\rho(A) = 1}$

eg:-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3} \qquad 0 < \rho(A) \le 3$$

since $|A| = 0$ so $\rho(A) < 3$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \Delta = -2 \quad \text{so,} \quad \boxed{\rho(A) = 2}$$

eg:-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}_{3\times3} \qquad 0 < e(A) \leq 3$$

As $|A| \neq 0$ so $e(A) = 3$

eg:-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}_{3\times3} \qquad 0 < e(A) \leq 3$$

$$A = {}_{2\times3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3} \qquad \text{so, } |A| = 0$$

$0 < e(A) \leq 2$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times2}$ $\Delta = 0$ all 2×2 matrix are same

so, $e(A) \not< 2$

so, $e(A) = 1$

eg:-

$$A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}_{3\times4} \qquad 1 \leq e(A) \leq 3$$

$R_3 \to R_3 - 3R_1$

$$|A| = \begin{vmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -10 & -4 & 0 & 2 \end{vmatrix}_{3\times 4}$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & -1 \\ 5 & 2 & -1 \\ -10 & -4 & 2 \end{vmatrix}$$

since all 3×3 matrix has $\Delta = 0$

so, $\rho(A) < 3$

$$\Delta_2 = \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} \neq 0$$ so $\boxed{\rho(A) = 2}$

eg:-

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}_{3\times 4}$$

$1 \leq \rho(A) \leq 3$

$$A = 3\begin{bmatrix} 1 & 3 & 4 & 3 \\ 1 & 3 & 4 & 1 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

all 3×3 matrices have $\Delta = 0$

so, $\rho(A) < 3$

$$\Delta_2 = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2 \neq 0 \quad \text{so,} \quad \boxed{\rho(A) = 2}$$

eg:-

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4} \qquad 1 \le \rho(A) \le 4$$

$$R_4 \rightarrow R_4 - (R_1 + R_2 + R_3)$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad |A| = 0,$$

so $\rho(A) < 4$

$$R_2 \rightarrow R_2 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Delta_3 \neq 0$$

so, $\boxed{\rho(A) = 3}$

eg:-

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}_{4\times4} \qquad 1 \le \rho(A) \le 4$$

$$R_4 \to R_4 - (R_1 + R_2 + R_3)$$

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad |A| = 0$$

so $\rho(A) < 4$

$$A = -\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \; ; \; R_3 \to R_3 - 3R_1$$

$$A = -\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Delta_3 \neq 0$$

so $\rho(A) = 3$

eg:-

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$R_4 \to R_3 - (R_1 + R_3)$$

$$R_3 \to R_3 - (R_1 + R_2)$$

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) < 4$

$\rho(A) < 3$

$$A_2 = \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} \neq 0$$

$\rho(A) = 2$

⇒ Consistency & Inconsistency of the system of equations:-

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

cofficient matrix — constant matrix — variable matrix

$$[AB] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

Augmented matrix

$AX = B$

Consistent system

System has solution

- unique solution

  $\rho(AB) = \rho(A) =$ No. of unknown (variables)

- more than one solution (infinite)

  $\rho(AB) = \rho(A) <$ No. of variables

In-consistent system

system has no solution

$\rho(AB) \neq \rho(A)$

eg:-

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

(a) 0

(b) 1 ✓

(c) 2

(d) ∞

NOTE:- If augmented matrix is rectangular matrix then don't make 0's in last column (column excluded)

If augmented matrix is square matrix then we can make 0's in last column (column included)

- column excluded – gives rank of both A & AB
- column included – gives rank of AB only

$$AB = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]_{3\times 4} \qquad \rho(AB) \leq 3$$

$\Delta \rightarrow 3$

so, $\rho(AB) = \rho(A) = 3 =$ No. of variable

Consistent & unique solution.

eg:-

$x - 2y + 3z = 2$

$2x - 3z = 3$

$x + y + z = 0$

$$AB = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right]_{3\times 4} \qquad \rho(AB) \leq 3$$

$\Delta \rightarrow 1(-3) + 2(5) + 3(2) = -3 + 16 = 13$

$\Delta \neq 0$

$\rho(AB) = \rho(A) = 3 =$ No. of variable

Consistent & unique solution

eg:-

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

$\rho(A) \leq 3$

$$AB = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}_{4\times 4} \quad \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \\ 3 & 3 & 2 & 1 \end{bmatrix}_{4\times 4}$$

$\rho(AB) \leq 4$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \\ 0 & -3 & 2 & -11 \end{bmatrix}$$

$$|AB| = 1\left(10(17) - 3(77-9) - 2(-14-3)\right)$$
$$= (170 - 3(68) + 34)$$
$$= 0$$

so $\rho(AB) < 4$

$$\Delta_3 = 1(-10+21)$$

$$\Delta_3 \neq 0$$

so, $\rho(AB) = \rho(A) = 3$ = No. of variables

(Consistent & unique solution)

Calculation of solutions

Unique solution

(i) $X = A^{-1}B$

(ii) Crammer's rule :

$x = \frac{\Delta_1}{\Delta}$; $y = \frac{\Delta_2}{\Delta}$; $z = \frac{\Delta_3}{\Delta}$

$\Delta_1 \rightarrow$ determinant (replace first column with constant matrix)

$\Delta \rightarrow$ determinant of A

(iii) Gauss - Jordan :

$$AB = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

X

Infinite solution

Apply elementary op^n so that more no. of 0 will appear in AB matrix, then apply simple substitution method.

eg:-

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{bmatrix} \rightarrow AB$$

$x = 1$

$1 + y - 3z = -1$

$y + 2 = 3z$

$z = K$ (Let)

so, $y = 3K - 2$

eg:-

$$2x - y + z = 4$$
$$3x - y + z = 6$$
$$4x - y + 2z = 7$$
$$-x + y - z = 9$$

$$AB = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{A} & \\ -1 & 1 & -1 & 9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array}\right]$$

Apply, $R_2 \to R_2 + R_1$
$R_3 \to R_3 + R_1$
$R_4 \to R_4 + R_1$

$$AB = \begin{bmatrix} -1 & 1 & -1 & 9 \\ 1 & 0 & 0 & 13 \\ 2 & 0 & 0 & 15 \\ 3 & 0 & 1 & 16 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 9 \\ 0 & 1 & 0 & 13 \\ 0 & 2 & 0 & 15 \\ 0 & 3 & 1 & 16 \end{bmatrix}$$

$\Delta \neq 0$

so $\rho(AB) = 4$

whereas $\rho(A) \leq 3$

so, $\rho(AB) \neq \rho(A)$

Inconsistent system

eg:-

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

Find value of μ & λ for:

(i) no solution

(ii) unique solution

(iii) infinite solution

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case 1:- $\lambda - 3 = 0$; $\mu - 10 \neq 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & \mu-10 \end{array}\right]$$

$|A| = 0$ $\qquad \rho(A) < 3$

$$AB = \left[\begin{array}{c|ccc} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & \mu-10 \end{array}\right]$$

$\mu-10(3-2) = \mu-10$

$|AB| \neq 0$

so, $e(AB) = \cancel{4}\,3$

$e(AB) \neq e(A)$ **No - solution**

Case 2 :-

$\lambda-3 = 0$ and $\mu-10 = 0$

so,
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Both
$e(A) = e(AB) = 2 < 3$

Infinite solutions

Case 3 :-

$\lambda-3 \neq 0$

$$AB = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] = \lambda-3(2-1) = \lambda-3$$

$|AB| = |A| \neq 0$

so, $e(AB) = e(A) = 3 =$ no. of variables

Unique solution

⇒ Eigen values & Eigen vectors :-

- **Characteristic equation :-** Let A be the square matrix of order $n \times n$ and I be the unit matrix of order $n \times n$ then $|A - \lambda I| = 0$ is called the characteristic equation where λ is a parameter.

- The roots of the characteristic equation are known as characteristic roots, latent roots or Eigen value or proper values.

- The value of $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ or $[x_1 \; x_2 \cdots x_n]^T$

which satisfy matrix eq. $[A - \lambda I] X = 0$

is called the corresponding eigen vector of matrix.

NOTE :- (i) The sum of the eigen values of any matrix is equal to sum of the principal diagonal elements.

Sum of eigen value = sum of Principal diagonal element
- (Trace of A)

(ii) The product of eigen values of any matrix is equal to its determinant.

$$\lambda_1, \lambda_2 \cdots \lambda_n = |A|$$

(iii) Eigen values of a symmetric matrix $(A^T = A)$ are purely real.

eg:- $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ $\lambda = 1, 5$ (purely real)

(iv) Eigen values of a skew symmetric matrix $(A^T = -A)$ are either purely imaginary or maybe 0's.

eg:- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda = i, -i$ (purely imaginary)

(v) If a matrix is either lower or upper traingular, then principal diagonal elements are eigen values.

eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ $\lambda = 1, -4, 7$

(vi) The eigen values of the Hermitian matrix ($A^{\theta} = A$)

$$A^{\theta} = (\bar{A})^{T} = A$$

are purely real.

eg:-

$$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$$

(If M is hermitian, then iM is skew-hermitian)

(vii) Eigen values of skew hermitian matrix are purely imaginary

(viii) If λ is eigen value of A

then
λ^2 — A^2

λ^3 — A^3

⋮

λ^n — A^n

$1/\lambda$ — A^{-1}

Q. Find the eigen values & the corresponding eigen vectors of matrix A:

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

so, $|A - \lambda I| = (\lambda-2)(\lambda-5) - 4$

$= \lambda^2 - 7\lambda + 10 - 4$

$= (\lambda^2 - 7\lambda + 6)$ characteristic equation

$= \lambda^2 - 7\lambda + 6$

$= \lambda^2 - 6\lambda - \lambda + 6$

$= \lambda(\lambda - 6) - 1(\lambda - 6)$

so, $\boxed{\lambda = 1, 6}$

for $\lambda = 1$:-

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

(Eigen vector = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$)

$4x_1 + 4x_2 = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

for $\lambda = 6$:-

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$-x_1 + 4x_2 = 0$

$x_1 - 4x_2 = 0$

$(x_1 = 4x_2)$

$$\left(\text{Eigen vector} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)$$

Q. Find characteristic roots of matrix A:

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(a) 3, 7, 18

(b) 2, 2, 14

(c) 0, 3, 15 ✓

(d) 1, 4, 9

$|A| = 0$ so, $\lambda_1 \lambda_2 \lambda_3 = 0$

Q. Find Latent roots of:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

(a) 3, 3, 6

(b) 2, 2, 8 ✓

(c) 1, 4, 7

(d) 3, 4, 5

$|A| = 32$

Q. The proper value of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ are -

(a) 1, 2, 3

(b) 0, 2, 4

(c) 1, 1, 4

(d) 2, 2, 2

$|A| = 1(4) - (-2)$

$= \underline{6}$

Q. characteristic roots of : $\begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$

(a) a, a, a

(b) $-a, 2a, 3a$

(c) $0, a, 2a$

(d) $a, a-\sqrt{2}, a+\sqrt{2}$

$\Delta = a(a^2-1) - 1(a)$

$= a^3 - a - a$

$= a^3 - 2a$

$= a(a^2-2)$

$= a(a+\sqrt{2})(a-\sqrt{2})$

Q. The eigen vectors of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

$\lambda = 1, -4, 7$

for $\lambda = 1$:-

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_2 + 3x_3 = 0$

$-5x_2 + 2x_3 = 0$

$6x_3 = 0$

$\boxed{x_3 = 0 \quad x_2 = 0}$

since x_1 is independent so take $\boxed{x_1 = 1}$

it can't be 0

$$\left(\text{vector} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) X_1$$

for $\lambda = -4$:-

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$5x_1 + 2x_2 + 3x_3 = 0$

$2x_3 = 0$

$11x_3 = 0$

so, $x_3 = 0$

$5x_1 + 2x_2 = 0$

$5x_1 = -2x_2$

$$\frac{x_1}{-2} = \frac{x_2}{5}$$

$$\frac{x_1}{2} = \frac{x_2}{-5}$$

$$\left(X_2 = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} \right)$$

for $\lambda = 7$:-

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-6x_1 + 2x_2 + 3x_3 = 0$

$-11x_2 + 2x_3 = 0$

$$\frac{x_2}{2} = \frac{x_3}{11}$$

$-6x_1 + 2(2) + 3(11) = 0$

$x_1 = 37/6$

$$\left(X_3 = \begin{bmatrix} 37 \\ 12 \\ 66 \end{bmatrix} \right)$$

Q: For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen value is -2, which of the following is an eigen vector

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (d) ✓ $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_3 = 0$

Q. For the matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ which one of the following is an eigen vector.

(a) $[1 \quad -1 \quad 1]^T$

(b) $[1 \quad 2 \quad 1]^T$ ✓

(c) $[1 \quad 1 \quad -2]^T$

(d) $[2 \quad -1 \quad 1]^T$

$\lambda = 1, 2, 3$

$\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = x_3 = 0$

not possible

$\lambda = 2$

$$\begin{bmatrix} & & \\ 0 & 0 & 2 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = x_2 = 0$

Not possible

$\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\frac{x_1}{1} = \frac{x_2}{2}$

Option (b)

Q. The vector, $x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, then one of the eigen value is :

(a) 1 (b) 2 (c) 5 ✓ (d) 7

$$[A - \lambda I]X = 0 \qquad AX = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\quad A \qquad\qquad\quad X$$

$$AX - \lambda X = 0$$

$$\underbrace{AX} = \underbrace{\lambda X}$$

so,

$$\underline{AX} = \begin{bmatrix} -2+4+3 \\ 2+2+6 \\ -1-4+0 \end{bmatrix} = \underset{\lambda}{5} \underset{X}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \qquad (\lambda = 5)$$

Q. A matrix has eigen value $\lambda_1 = -1$ and $\lambda_2 = -2$ and corresponding eigen vectors are $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. what is matrix.

(a) $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (d) ✓ $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{X_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \underbrace{-1}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{X_2}$$

it should be X_1

so, it is not possible.

$$A\underline{X_1} = \lambda_1 \underline{X_1}$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{X_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{-1}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{X_1}$$

it is possible

Q. For matrix $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, which of the following is eigen vector:

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ✓ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -8 \\ 18 \end{bmatrix}}_{X'}$$ not possible $(X \neq X')$

$$\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 25 \end{bmatrix}$$ not possible

$$\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = -5 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

possible

Q. Find eigen vector pair for given matrix:

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(a) ✓ $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ ✓

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$ ✓

Q. A real $n \times n$ matrix is defined as follows:

$$A = [a_{ij}]$$

$$a_{ij} = \begin{cases} i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

then the sum of all the eigen value of A.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}$$

So it's a traingular matrix

So, $\lambda = 1, 2, 3, 4, \cdots n$

sum of eigen values $= \frac{n(n+1)}{2}$

⇒ Cayley - Hamilton Theorem :-

- Every square matrix satisfy its own characteristic equation.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

Characteristic eq $= (\lambda-2)(\lambda-5) - 4$

$= \lambda^2 - 7\lambda + 10 - 4$

$= (\lambda^2 - 7\lambda + 6)$

According to cayley hamilton theorem :-

$$A^2 - 7A + 6I = 0$$

$$(A^2 = 7A - 6I)$$

we can find A^n

$$6I = 7A - A^2$$

$$6IA^{-1} = 7AA^{-1} - A^2 \cdot A^{-1}$$

$$6A^{-1} = 7I - A$$

$$\left(A^{-1} = \frac{1}{6}[7I - A]\right)$$

we can find A^{-1} also

so,

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

Q. Let M be a 3X3 matrix with characteristic equation $x^3 + 1 = 0$, then the Inverse of M i.e M^{-1} is $\underline{-M^2}$.

$$M^3 + I = 0$$

$$I = -M^3$$

$$IM^{-1} = -M^3 \cdot M^{-1}$$

$$\left(M^{-1} = -M^2\right)$$

Q. Let P be 3×3 matrix with C.E $= \lambda^3+\lambda^2+2\lambda+1=0$, then $P^{-1} =$ ______

(a) P^2+P+I (b) $-(P^2+P+I)$

(c) P^2+P+2I (d) $-(P^2+P+2I)$ ✓

$$-(P^3+P^2+2P) = I$$

$$P^{-1} = -(P^2+P+2I)$$

Q. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and if $a+d = ad-bc = 1$

then A^3 $\underline{\ -I\ }$.

$\lambda_1+\lambda_2 = a+d = 1$ $\qquad$ $\lambda_1 = 1-\lambda_2$

$\lambda_1 \cdot \lambda_2 = ad-bc = 1$ $\qquad$ $\lambda_2(1-\lambda_2) = 1$

$\lambda_2 - \lambda_2^2 = 1$

$$\lambda^2+1 = \lambda$$

$$A^2+I = A$$

$$A^2 = A - I$$

$$A^3 = [A-I][A]$$

$$= A^2 - A$$

$$= \cancel{A} - I - \cancel{A}$$

so, $\boxed{A^3 = -I}$

Q. If $-1, 0, 1$ are the eigen values of 3×3 matrix A, then $|A^{100} + 3I| =$ ______ .

$A \to -1, 0, 1$ so, $A^{100} \to (-1)^{100} = 1,\ 0^{100} = 0,\ 1^{100} = 1$

$I \to 1, 1, 1$ so, $3I \to 3, 3, 3$

so, eigen values of $A^{100} + 3I \to$

$1 + 3 = 4\ \lambda_1$

$0 + 3 = 3\ \lambda_2$

$1 + 3 = 4\ \lambda_3$

now $|A^{100} + 3I| = \lambda_1 \lambda_2 \lambda_3 = 4 \times 3 \times 4$

$= \underline{48}$

Q. If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$ then Trace of A^{102}

$=$ ______

(a) 0 (b) 1 (c) 2 (d) 3 ✓

$\lambda = 1, \omega, \omega^2$ where $\omega = \frac{-1 + i\sqrt{3}}{2}$

$\omega^2 = \frac{-1 - i\sqrt{3}}{2}$

Trace of A = Sum of λ

$= 1 + \omega + \omega^2$

$= 1 + \left(\frac{-1 + i\sqrt{3}}{2}\right) - \left(\frac{1 + i\sqrt{3}}{2}\right)$

$= \underline{0}$

Trace of A^{102} $= 1^{102} + \omega^{102} + (\omega^2)^{102}$

$= 1 + (\omega^3)^{34} + (\omega^3)^{68}$

$= 1 + 1 + 1$

$= 3$

Q. If $\alpha = e^{2\pi i/5}$ and $M = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & \alpha^2 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^4 \\ 0 & 0 & 0 & 0 & \alpha^4 \end{bmatrix}$

Then trace of $I + M + M^2 = \underline{\quad 5 \quad}$

$$1 = \cos 2\pi + i \sin 2\pi$$

$$\cos x + i \sin x = e^{ix}$$

$$1+\alpha^2+\alpha^3+\alpha^4 = 0$$

$$(1)^{1/5} = (\cos 2\pi + i \sin 2\pi)^{1/5}$$

$$(\alpha^5 = 1)$$

$$= e^{2\pi i/5}$$

Then trace of $I+M+M^2 = 5 + (1+\alpha+\alpha^2+\alpha^3+\alpha^4)$

$$+ \alpha(1+\alpha^2+\alpha^4+\alpha^6+\alpha^8)$$

$$1+\alpha^2+\alpha^4+\alpha^5\cdot\alpha+\alpha^3\alpha^5$$

$$= 5+0+0$$

$$= \underline{5}$$

Q. Let A, B, C, D be $n \times n$ matrices, each with non zero determinant, and if $ABCD = I$ then

$B^{-1} =$ ______.

(a) $D^{-1}C^{-1}A^{-1}$ (b) CDA

(c) ADC (d) does not exists

$$ABCD = I$$

$$ABCDD^{-1} = D^{-1}$$

$$ABCC^{-1} = D^{-1}C^{-1}$$

$$AB = D^{-1}C^{-1}$$

$$(ABCD)^{-1} = I^{-1}$$

$$D^{-1}C^{-1}B^{-1}A^{-1} = I$$

$$\cancel{D}\cdot\cancel{D^{-1}}C^{-1}B^{-1}A^{-1} = D$$

$$\cancel{C}\cancel{C^{-1}}B^{-1}A^{-1} = CD$$

$$B^{-1}A^{-1}\cdot A = CD\cdot A$$

$$\boxed{B^{-1} = CDA}$$

Q. The value of λ for which the following system of equation, have non-trivial solution.

$$\lambda x + 3y + 5z = 0$$

$$2x - 4\lambda y + \lambda z = 0$$

$$-4x + 18y + 7z = 0$$

(a) -1 or 3 (b) 1 or -3 (c) -1 or -3 (d) 1 or 3

NOTE:- For the system of eq. $AX = 0$, $X = 0$ is always a solution, and this solution is called trivial solution.

For a non-trivial solution, determinant of the coffiecient matrix is always zero i.e $|A| = 0$

$$\Delta = \begin{vmatrix} \lambda & 3 & 5 \\ 2 & -4\lambda & \lambda \\ -4 & 18 & 7\lambda \end{vmatrix} = 0$$

$$\lambda = 1 \text{ or } -3$$

Q. The rank of $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{3\times 4}$ is ______

(a) 1 (b) 2 ✓ (c) 3 (d) 4

$= R_2 - 4R_1$

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ -10 & -5 & 0 & -5 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{3\times 4} = -5 \begin{bmatrix} 4 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

all $\Delta_3 = 0$ so $\rho(A) < 3$ $\quad \Delta_2 = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \neq 0$

$$\boxed{\rho(A) = 2}$$

Q. The rank and nullity of a matrix A :

$$A = \begin{bmatrix} 2 & -3 & -1 & 1 \\ 3 & 4 & -4 & -3 \\ 0 & 17 & -5 & -9 \end{bmatrix}$$

$\rho(A) = 2$

(a) 1, 3 (b) 3, 1 (c) 2, 2 (d) 4, 0

Nullity = No. of columns − Rank

$= (4 - 2 = 2)$

Q. The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & K \end{bmatrix}$ is 2

Find K.

for $\rho(A) = 2$ $|A| = 0$

so, $K = 5$

Q. The inverse of $\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$ is ______

For diagonal matrix inverse,

substitute diagonal element with there inverse

so, $A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Q. The rank of $A = \begin{bmatrix} 3 & 0 & 0 & & \\ & -3 & & & \\ & 0 & 4 & 0 & 0 \\ & & & & -4 \end{bmatrix}_{9\times 9}$

0

0

The rank of a diagonal matrix = no. of non-zero elements in diagonal

so, $\boxed{\rho(A) = 4}$

Q. How many of the following matrices have an eigen value 1.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

✓ ✗ ✗ ✗

$= (\lambda-1)^2 + 1$

$= \lambda^2 - 2\lambda + 1 + 1$

$= \lambda^2 - 2\lambda + 2$

$(\lambda \neq 1)$

$= (\lambda+1)^2$

$= (\lambda+1)(\lambda+1)$

$\lambda = -1$

not 1

Q. $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}_{3\times 3}$ and $B = 2A^2$ then $|B| = \_\_\_\_$

(a) 16 (b) 32 (c) 64 (d) 128 ✓

$|A| = 2 \times -2 \times -1 = 4$

$|A^2| = 4^2 = 16$

$|B| = |2A^2|$

$= 2^3 |A^2| = 2^3(16) = 128$

Q: Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$ then

$a+b = \underline{7/20}$

$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$ $\underline{a = 0.1/6}$ $\underline{b = 1/3}$

Q. Let A be a 3×4 real matrix and $AX = B$ is an inconsistent system of equations then the heighest possible rank of A is ___.

(a) 1 (b) 2 ✓ (c) 3 (d) 4

$[A]_{3\times4}$ $\varrho(A) \le 3$ $[AB]_{3\times5} = \varrho(A) \le 3$

for $\varrho(AB) \neq \varrho(A)$

$(\varrho(A)_{max} = 2)$

Q. If the following determinant represent the eqn. of a line, then the line passing through the point :

(a) 0,0 (b) 3,4 (c) 4,3 (d) 4,4

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Delta = x(8) - 2(y) + 4(y-8) = 0$$

$$8x + 2y = 32$$

Q. Given an orthogonal matrix A :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}_{4\times4}$$

then $(AA^T)^{-1} =$ ________

$AA^T = I$ (orthogonal matrix)

so, $(AA^T)^{-1} = I^{-1} = I$

NOTE:- A matrix is said be orthogonal if

$(AA^T = I)$

Q. Consider the following system of eq:

$$2x_1 - x_2 + 3x_3 = 1$$

$$3x_1 + 2x_2 + 5x_3 = 2$$

$$-x_1 + 4x_2 + x_3 = 3$$

This system has a:

(a) no solution ✓(b) unique solution

(c) 2 solution (d) infinite solution.

$$AB = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 3 & 2 & 5 & 2 \\ -1 & 4 & 1 & 3 \end{bmatrix}_{3\times 4}$$

$\Delta = 2(-18) + 1(3+5) + 3(12+2)$

$-36 + 8 + 42$

$\Delta \neq 0$

so, $\underline{\rho(AB) = \rho(A) = 3}$

Q. For what value of 'a' if any such value will the following system of equation :

$$2x + 3y = 4$$
$$x + y + z = 4$$
$$x + 2y - z = a \quad \text{have a solution ?}$$

(a) any real number

(b) 0

(c) 1

(d) No such value.

Let, $e(AB) = e(A)$

$$AB = \begin{array}{c} A \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \\ 2 & 3 & 0 & 4 \end{array}\right]_{3\times 4} \end{array}$$

$|A| = 0$

$e(A) < 3$

$e(A) = 2$

so, $\begin{vmatrix} 1 & 1 & 4 \\ 2 & -1 & a \\ 3 & 0 & 4 \end{vmatrix} = 0$

for $e(AB) < 3$

$e(AB) = 2$

$1(-4) - 1(8-3a) + 4(3) = 0$

$\cancel{-4} + 3a - \cancel{8} + \cancel{12} = 0$ $\qquad (a = 0)$

Q. For the matrix $M = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$ the transpose is equal to inverse, then the value of x ____.

$$M^T = M^{-1}$$

$$\begin{bmatrix} 3/5 & x \\ 4/5 & 3/5 \end{bmatrix} = \frac{1}{9/25 - \frac{4x}{5}} \begin{bmatrix} 3/5 & -4/5 \\ -x & 3/5 \end{bmatrix}$$

$$9/25 - \frac{4x}{5} = 1$$

$$\frac{4x}{5} = \frac{9}{25} - 1$$

$$\frac{4x}{\cancel{5}} = \frac{9-25}{\cancel{25}\,5}$$

$$\boxed{x = -4/5}$$

Q. The no. of different $n \times n$ symmetric matrices each element is being 0 or 1

(a) 2^n (b) $2n$ (c) $2^{\frac{n^2-n}{2}}$ (d) $2^{\frac{n^2+n}{2}}$ ✓

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$ has a certain complex number $\lambda \neq 0$ as an eigen value.

which of the following must also be an eigen value of matrix A.

(a) $\lambda + 20$ (b) $\lambda - 20$

(c) $20 - \lambda$ (d) $-20 - \lambda$

since $|A| = 0$ so, one of the eigen value must be 0,

so, $\lambda_1, 0, \lambda_3$

$$\lambda + 0 + \lambda_3 = \text{sum of diagonal elements}$$

$$\lambda + 0 + \lambda_3 = 20$$

$$(\lambda_3 = 20 - \lambda)$$

Q. Let A be 3×3 matrix such that $|A - I| = 0$ the trace of $A = 13$ and $|A| = 32$ then the sum of the squares of eigen values of A.

CE = $|A - \lambda I| = 0$ and $|A - I| = 0$ (given)

so, $(\lambda_1 = 1)$

$$\lambda_1 + \lambda_2 + \lambda_3 = 13 \qquad \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 32$$

$$\lambda_2 + \lambda_3 = 12 \qquad \lambda_2 \lambda_3 = 32$$

$$\lambda_2 = 4, \quad \lambda_3 = 8$$

$$\boxed{\lambda = 1, 4, 8}$$

$$= 1^2 + 4^2 + 8^2$$

$$= 1 + 16 + 64 = \underline{81}$$

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