CS499: Undergraduate Project **(Due: 10th May 2021)**

No-wastage Resource Allocation without money

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Abstract

In this work, we propose mechanisms for resource allocation ensuring truthfulness and no resource wastage. We only consider mechanisms which do not involve any monetary transfers. Mechanisms are provided for homogeneous valuations and cardinal single-peaked valuations. We finally provide inefficiency bounds in terms of social welfare for these mechanisms.

1 Introduction

Resource allocation is widely studied in the literature pertaining to mechanism design. These mechanisms, more often than not, involve monetary transfers among agents in order to achieve desirable properties such as truthfulness. The widely used VCG mechanism[Gro73, Vic61, Cla71] is an incentive-compatible, individually rational and efficient algorithm for allocating resources involving payments among agents which are decided by the mechanism. This mechanism, however, is not budget balanced which means that the mechanism generates surplus payments which have to be compensated by the resource allocator. Green and Laffont [GL79] proved that all the 3 properties - incentive-compatibility, efficiency and budget balance cannot be achieved by a mechanism. To ensure budget balance in required scenarios, sink mechanism has been proposed by Faltings [FFRA04] which compromises with efficiency but keeps the mechanism truthful. This mechanism, like others, necessitates payments among agents. There are certain scenarios where resource allocation has to be carried out in absence of money.Cole et al.[CGG13a] have proposed a truthful mechanism which allocates resources without money but wastes resource to enforce truthfulness. We wish to explore mechanisms which do not waste resource (equivalent to budget balance in mechanisms using money), are truthful and least inefficient in terms of social welfare.

2 Homogeneous valuations

In this section, we discuss the large class of homogeneous valuations and present a mechanism which works for this valuation domain. We first discuss some other mechanisms from the literature which are used as sub-modules in our proposed mechanism.

2.1 Partial Allocation mechanism

Cole et al [CGG13a] focus on allocating divisible goods using proportional fairness (PF) solution as benchmark. This solution approach aims at finding allocation which maximizes the product of agent's valuations i.e an allocation which maximizes $\prod_i [v_i(x)]$ or equivalently $\sum_i \log v_i(x)$.

PF solution(x*) can be alternatively interpreted as an allocation if the aggregate proportional change to the valuations for any other allocation (x') is not positive, i.e.

$$\sum_{i \in N} \frac{v_i(x') - v_i(x*)}{v_i(x*)} \leq 0 \tag{1}$$

The PF allocation lies between the two extremes of utilitarian and egalitarian by providing a significant fairness guarantee without neglecting efficiency. The Partial Allocation mechanism is a novel way to distribute divisible items to bidders in a truthful manner as follows:

- Compute the PF allocation x* based on the reported bids.
- For each player i, remove the player and compute the PF allocation $x*_{-i}$ that would arise in absence.

- Allocate to each player i a fraction f_i of everything that she receives according to x*, where

$$f_i = \frac{\prod_{i' \neq i} v_{i'}(x*)}{\prod_{i' \neq i} v_{i'}(x_{-i}*)} \tag{2}$$

This mechanism is truthful by virtue of resource fraction that is thrown away as a form of non-monetary "payment" in exchange of the resource. The mechanism in itself does not involve money but wastes resource which is revoked from bidders as payment so as to ensure truthfulness. This is analogous to payment in monetary settings. We can ask the question of eliminating resource wastage (equivalent to budget balance) in this setting as well.

This mechanism has interesting connections with the domain of mechanism design with money, specifically the VCG mechanism as illustrated below:

The valuation of player i for the PA mechanism outcome is $v_i(x) = f_i v_i(x*)$

$$v_i(x) = \frac{\prod_{i' \neq i} v_{i'}(x*)}{\prod_{i' \neq i} v_{i'}(x_{-i}*)} v_i(x*) \tag{3}$$

Taking a logarithm on both sides, we have:

$$\log v_i(x) = \log v_i(x*) - (\sum_{i' \neq i} \log v_{i'}(x_{-i}*) - \sum_{i' \neq i} \log v_{i'}(x*)) \tag{4}$$

Now, if we consider $u_i = \log v_i$ as bidder's surrogate valuation in Equation 4, it is exactly the VCG net utility (valuation - VCG payment) in surrogate valuation domain. We will utilize this transformation in the design of our mechanism.

2.2 Sink mechanism

Faltings et al [FFRA04] proposed a budget balanced, truthful and individually rational mechanism for social choice problems. Since budget balance, incentive compatibility and pareto-efficiency can not all be achieved by the same mechanism, the proposed mechanism has to compromise on efficiency but is applicable to domains where budget balance is an utmost requirement. VCG can not be used in those settings since it requires an external mediator who can handle the surplus money which is not preferable in a lot of cases. Let us have a look at the proposed mechanism:

- Each agent $A_i \in A, i = 1..k$ is asked to state its utilities for different value combinations
- Choose an excluded coalition E of one or more agents using a method that does not depend on the utility stated by the agents.
- Compute the assignment:

$$S_E = V*_{RR_E} \tag{5}$$

  where $R_E = \cup_{A_L \in E} R_l$. Optionally, if there are several equally optimal assignments, choose the one with the best utility according to the relations in R_E.
- Make each agent A_i pay to agents in E the VCGtax for the solution S_E:

$$pay(A_i \rightarrow E) = VCGtax_{-E}(A_i) = \sum_{r_m \in R(R_i \cup R_E)} r_m(V*_{R(R_i \cup R_E)}) - r_m(V*_{RR_E}) \tag{6}$$

  and distribute the tax among the agents in E according to some predetermined scheme.

This is the sink mechanism as applied in monetary settings. When cast in our setting, we can think how to achieve budget balance in terms of resource wastage by applying this mechanism.

2.3 Our Proposed Mechanism

2.3.1 Mechanism

Combining the two mechanisms discussed above, we now come up with the following mechanism for divisible resource allocation which does not involve monetary transfers, does not waste any resource and is incentive-compatible: The takeaways from the two mechanisms discussed above were that the partial allocation mechanism is equivalent to running the VCG scheme for surrogate valuations (log of valuations) and sink mechanism involves running the VCG scheme by excluding a coalition of agents who are paid the surplus tax. We combine these two mechanisms to have the desirable properties of both of these mechanisms by running the sink mechanism on surrogate valuations of the agents. This mechanism is truthful since the sink mechanism (which basically runs VCG scheme) is truthful even in the surrogate valuation domain (since PA mechanism is also truthful and that is how we obtained the connection between PA mechanism and VCG). We do not waste any resource by the construction of the mechanism directly, which basically allocates any left over resource to the sink agents.

Nath and Sandholm [NS17] showed that a truthful budget-balanced mechanism must have one or more *sink* agents and that the resulting inefficiency is minimized when we have a single sink agent. Therefore, we use the Faltings mechanism with one sink agent to minimize inefficiency.

2.3.2 Bound

We now provide the social welfare bound for our proposed mechanism and start by discussing two results from literature that will be useful in our derivation:

- Conitzer et al [GNC+11] showed that in case of non-negative valuations, the sink mechanism proposed by Faltings [FFRA04] has a social welfare ratio lower bound = $\frac{n-1}{n}$.
- Another result we use in our proof is that, for 2 agents and any number of resources, the social welfare ratio lower bound for proportionally fair solution with utilitarian welfare is $\frac{2\sqrt{3}+3}{4\sqrt{3}}$ [CGG13b].

Let there be n agents and their valuation functions be $v_1, v_2, ..., v_n$. Their corresponding surrogate valuations will be $v_i' = \log v_i$.

Using the first result above, we have

$$\frac{\sum_i v_i'(x_{sink})}{\sum_i v_i'(x_{opt})} \geq \frac{n-1}{n} \tag{7}$$

Here, x_{opt} will be equal to x_{PF}(proportionally fair allocation) since the product of $v_i s$ (which becomes sum of $v_i' s$ on transformation) is to be maximized.

Using the second result above, we have

$$\frac{\prod_i v_i(x_{PF})}{\sum_i v_i(x_{optimal})} \geq \frac{2\sqrt{3}+3}{4\sqrt{3}} \tag{8}$$

Rearranging Equation 8 and taking log on both sides:

$$\log \prod_i v_i(x_{PF}) \geq \log \frac{2\sqrt{3}+3}{4\sqrt{3}} \sum_i v_i(x_{optimal}) \tag{9}$$

Dividing the whole equation by $\sum_i v_i(x_{optimal})$ and replacing $\frac{2\sqrt{3}+3}{4\sqrt{3}}$ by k for a while, we have:

$$\frac{\log \prod_i v_i(x_{PF})}{\sum_i v_i(x_{optimal})} \geq \frac{\log k \sum_i v_i(x_{optimal})}{\sum_i v_i(x_{optimal})} \tag{10}$$

The right-hand side of the equation ($\frac{\log kt}{t}$) is a convex function maximized at t = $\frac{e^{1/k}}{k}$. For this value of t, the inequality 10 becomes:

$$\frac{\log \prod_i v_i(x_{PF})}{\sum_i v_i(x_{optimal})} \geq e^{-1/k} \tag{11}$$

Using the fact that $\sum_i v_i'(x_{opt})$ is essentially $\log \prod_i v_i(x_{PF})$, we can multiply inequalities 7 and 11 to get the following bound on the ratio of social welfare of our applied mechanism and the optimal one:

$$\frac{\sum_i v_i'(x_{sink})}{\sum_i v_i(x_{optimal})} \geq \frac{n-1}{n} e^{-1/k} \tag{12}$$

3 Single-peaked valuations

In this section, we move to single-peaked valuations to come up with a truthful scheme which does not waste any resource. As it turns out, the uniform rule given by Sprumont [Spr91] satisfies these conditions well. Ruben Juarez and Jung S. You [JY19] further characterize this rule from the domain of ordinal utility to cardinal utility.

3.1 Uniform Rule

Let N denote the set of agents and R denote the set of peaks. In this setting, the uniform rule is characterized as follows:

$$\phi_i^*(R) = \begin{cases} \min\{x^*(R_i), \lambda(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \geqslant 1, \\ \max\{x^*(R_i), \mu(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \leqslant 1, \end{cases}$$

where $\lambda(R)$ solves the equation $\Sigma_{i \in N} \min\{x^*(R_i), \ \lambda(R)\} = 1$ and $\mu(R)$ solves the equation $\Sigma_{i \in N} \max\{x^*(R_i), \ \mu(R)\} = 1$.

The uniform rule is strategy-proof as proved rigorously by Sprumont [Spr91]. It, by its construction, does not waste any resource by either giving agents greater than or equal to their peaks in case of under-demanded resource, and less than or equal to their peaks in case of over-demanded resource. This fits our requirements from the mechanism making it suitable to be employed in our use-case. We now need to estimate the inefficiency of this mechanism which is calculated for the case of cardinal utilities (as is our use case) by Ruben Juarez and Jung S. You [JY19].

The uniform rule, however, is not optimal in both the over-demanded and under-demanded cases. Next, we look at examples to see the sub-optimality of this rule in the two cases:

- **Over-demanded** - Let us consider a scenario with N players with their utility functions being $u_i(x) = a_i x$ for all $x \in [0, C]$ such that $a_i < a_j$ if $i < j$. Now, since the peak of all agents lies at x=C, this is an over-demanded case and uniform rule makes an allocation $\frac{C}{N}$ to all agents. The optimal solution here would be to give all of the resource to agent N since that maximizes the utility sum.
- **Under-demanded** - Let us consider a scenario with N players with their utility functions for agent 1 to N-1 being $u_i(x) = (C - x)$ for all $x \in [0, C]$ and the utility function for agent N being $u_N(x) = \alpha(C - x)$ such that $0 < \alpha < 1$. The peak for agents in this case is equal to 0 making it an under-demanded case. The uniform rule allocates $\frac{C}{N}$ to all agents. The optimal solution here is to give all of the resource to agent N since that would maximize the overall utility sum for all agents.

We observe that the sub-optimality arises because the mechanism only looks at the peaks, not the complete utility profiles while allocating the resource. Also, The considered valuation range is unrestricted which eases construction of examples such that there is a mismatch between utilitarian output and uniform rule allocation. A potential future work is to analyze the mechanisms under a restricted range and proposing a modification to the uniform rule to take the whole valuation profiles into consideration. For N players, the social welfare ratio for the under-demanded and over-demanded cases for the uniform rule are proved to be $\frac{N-1}{N}$ and $\frac{1}{N}$ respectively. [JY19].

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