

① equating posteriors de-B can be found

$$P(\omega_1|x) = P(\omega_2|x)$$

$$\Rightarrow P(x|\omega_1) \cdot P(\omega_1) = P(x|\omega_2) \cdot P(\omega_2)$$

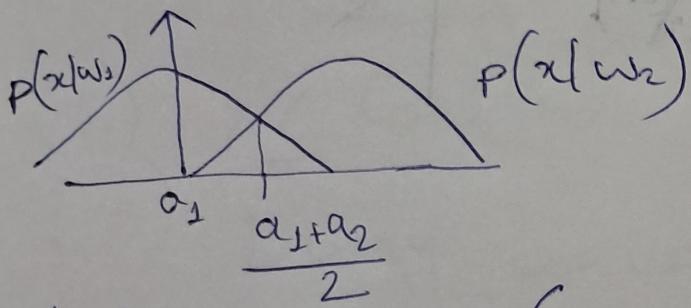
$$P(\omega_1) = P(\omega_2) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\pi b \left(1 + \frac{x-a_1}{b}\right)^2} = \frac{1}{\pi b \left(1 + \frac{x-a_2}{b}\right)^2}$$

$$\Rightarrow \pm \frac{x-a_1}{b} = \pm \frac{x-a_2}{b}$$

$$x = \frac{a_1 + a_2}{2}$$

1/2



$$\text{total error } P(\text{error}) = \int P(\text{error}|x) p(x) dx$$

$$P[\text{error}|x] = \min [P(\omega_1|x), P(\omega_2|x)]$$

$$1/4 \quad P(\text{error})_{a_1+a_2/2} = \int P(x|\omega_2) P(\omega_2) dx + \int_{-\infty}^{+\infty} P(x|\omega_1) P(\omega_1) dx$$

$$= \frac{1}{2} \left[ \int_{-\infty}^{a_1+a_2/2} \frac{1}{\pi b \left(1 + \frac{x-a_2}{b}\right)^2} dx + \int_{a_1+a_2/2}^{+\infty} \frac{1}{\pi b \left(1 + \frac{x-a_1}{b}\right)^2} dx \right]$$

$$= \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \left( \frac{a_2 - a_1}{2b} \right)$$

1/4

$$Q.2) \quad X = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \times \mathbb{R}^{d \times N} \Rightarrow d=3 \\ N=3$$

$$\mu = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \end{bmatrix} \quad \text{so } \sum_i X_i - \mu = \begin{bmatrix} 1 - \frac{1}{3} & 0 - 1/3 & 0 - 1/3 \\ -1 - 0 & 0 - 0 & 1 - 0 \\ 0 - 2/3 & 1 - 2/3 & 1 - 2/3 \end{bmatrix}$$

$$\Rightarrow \sum_i X_i - \mu = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix}.$$

Unbiased covariance matrix is  $\text{Cov}(X) = \frac{1}{N-1} \sum_i (X_i - \mu)^T$

$$\Rightarrow \text{Cov}(X) = \frac{1}{3-1} \cdot \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -1 & -2/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & 1 & 1/3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2/3 & -1 & -2/3 \\ -1 & 2 & 1 \\ -2/3 & 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 & -1/3 \\ -1/2 & 1 & 1/2 \\ -1/3 & 1/2 & 1/3 \end{bmatrix}$$

$$\therefore \text{Cov}(X) = \begin{pmatrix} 1/3 & -1/2 & -1/3 \\ -1/2 & 1 & 1/2 \\ -1/3 & 1/2 & 1/3 \end{pmatrix} \quad 0.5$$

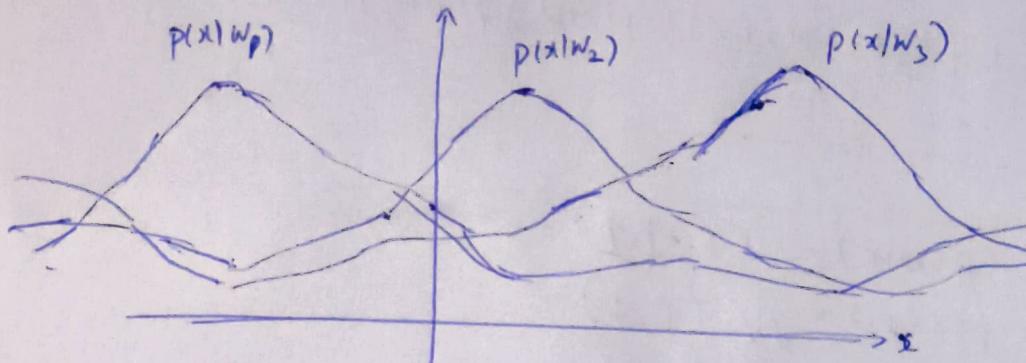
3 categories :  $w_1, w_2, w_3$

→ Constraint :  $p(w_1|x) + p(w_2|x) + p(w_3|x) = 1$ .

$$\rightarrow p(w_i|x) = \frac{p(x|w_i)p(w_i)}{p(x)} \Rightarrow [p(x|w_i)p(w_i) = p(w_i|x) \cdot p(x)].$$

[Bayes' theorem]  ~~$p(x|w_i)$  &  $p(w_i)$~~

$$[p(x|w_i) p(w_i) \propto p(w_i|x)]$$



1/4

$$p(\text{error}) = \int_x p(\text{error}|x)p(x)dx.$$

$$\begin{aligned} p(\text{error}|x) &= 1 - p(\text{correct}|x) \\ &= 1 - \max(p(w_1|x), p(w_2|x), p(w_3|x)) \end{aligned}$$

Since we assume equiprobable priors, we also have

$$\begin{aligned} p(w_1) &= p(w_2) = p(w_3) \\ \text{and } p(w_i) &= \frac{1}{3} \quad \forall i \in \{1, 2, 3\}. \end{aligned}$$

$$\begin{aligned} p(\text{error}) &= \int_x \left[ 1 - \max(p(w_1|x), p(w_2|x), p(w_3|x)) \right] p(x) dx \\ &= \int_x p(x) dx \Big|_0^1 - \int_x \max(p(w_1|x), p(w_2|x), p(w_3|x)) p(x) dx \\ &= 1 - \int_x \max(p(w_1|x), p(w_2|x), p(w_3|x)) p(x) dx. \end{aligned}$$

$$[\text{Bayes' theorem}] = 1 - \int_x \max \left( \frac{p(x|w_1)p(w_1)}{p(x)}, \frac{p(x|w_2)p(w_2)}{p(x)}, \frac{p(x|w_3)p(w_3)}{p(x)} \right) dx$$

$$= 1 - \int_x \max [p(x|w_1)p(w_1), p(x|w_2)p(w_2), p(x|w_3)p(w_3)] dx$$

$\therefore$  equiprobable priors:

$$P(\text{error}) = 1 - \int_x \max \left( p(x|w_1) \cdot \frac{1}{3}, p(x|w_2) \cdot \frac{1}{3}, p(x|w_3) \cdot \frac{1}{3} \right) dx$$

$$\boxed{P(\text{error}) = 1 - \frac{1}{3} \int_x \max(p(x|w_1), p(x|w_2), p(x|w_3)) dx.} \quad 1/2$$

(ii) We now have  
 $p(x|w_1) \sim \mathcal{N}(-1, 1)$   
 $p(x|w_2) \sim \mathcal{N}(0, 1)$   
 $p(x|w_3) \sim \mathcal{N}(1, 1)$

Find points of intersection between  ~~$p(x|w_i)$~~ ,  $p(x|w_{i+1})$  &  $i \in \{1, 2\}$ .

a)  $p(x|w_1), p(x|w_2)$

$$p(x|w_1) = p(x|w_2)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Rightarrow (x+1)^2 = x^2$$

$$\Rightarrow \boxed{x = -\frac{1}{2}} \quad 1/4$$

b)  $p(x|w_2), p(x|w_3)$

$$p(x|w_2) = p(x|w_3)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$$

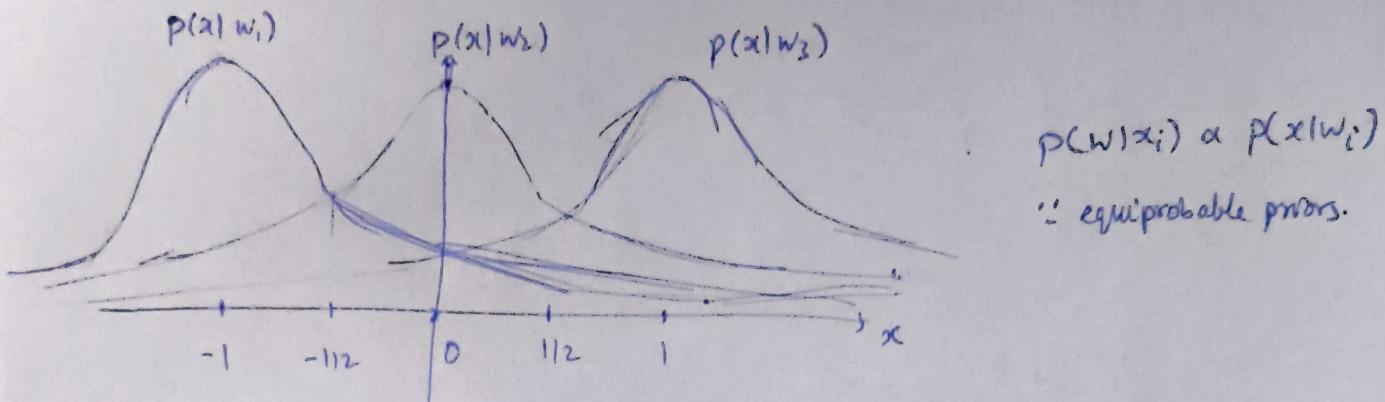
$$\Rightarrow x^2 = (x-1)^2$$

$$\Rightarrow \boxed{x = \frac{1}{2}} \quad 1/4$$

$\therefore$  for  $x \in (-\infty, -\frac{1}{2}]$ ,  $p(x|w_1)$  is max

for  $x \in (-\frac{1}{2}, \frac{1}{2}]$ ,  $p(x|w_2)$  is max

for  $x \in (\frac{1}{2}, \infty)$ ,  $p(x|w_3)$  is max.



$$P(\text{error}) = 1 - \frac{1}{3} \int_{-\infty}^{\infty} \max(p(x|w_1), p(x|w_2), p(x|w_3)) dx$$

$$= 1 - \frac{1}{3} \left[ \int_{-\infty}^{-1/2} p(x|w_1) dx + \int_{-1/2}^{1/2} p(x|w_2) dx + \int_{1/2}^{\infty} p(x|w_3) dx \right]$$

$$\text{a) } \int_{-\infty}^{-1/2} p(x|w_1) dx = \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} dx \quad 1/4$$

$$= \int_{-\infty}^{-1/2} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = \Phi\left(-\frac{1}{2}\right). \quad \text{--- (1)}$$

$$\text{b) } \int_{-1/2}^{1/2} p(x|w_2) dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right) \quad \text{--- (2)}$$

$$\text{c) } \int_{1/2}^{\infty} p(x|w_3) dx = \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx - \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx$$

$$= 1 - \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} dx$$

$$( \text{let } (x-1) = t ), \quad 1 - \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 1 - \Phi\left(-\frac{1}{2}\right) \quad \text{--- (3)}$$

1/4

$$\begin{aligned}
 P(\text{error}) &= 1 - \frac{1}{3} ((1) + (2) + (3)) \\
 &= 1 - \frac{1}{3} \left[ \Phi\left(\frac{1}{2}\right) + \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right) + 1 - \Phi\left(-\frac{1}{2}\right) \right] \\
 &= \frac{2}{3} - \frac{2}{3} \Phi\left(\frac{1}{2}\right) + \frac{2}{3} \Phi\left(-\frac{1}{2}\right) \\
 &= \frac{2}{3} \left[ 1 - \Phi\left(\frac{1}{2}\right) + \Phi\left(-\frac{1}{2}\right) \right] \\
 &= \frac{2}{3} \left[ 1 - \Phi\left(\frac{1}{2}\right) + 1 - \Phi\left(\frac{1}{2}\right) \right] : [\Phi(-x) = 1 - \Phi(x)] \\
 &\leftarrow \frac{4}{3} \left[ 1 - \Phi\left(\frac{1}{2}\right) \right] = \frac{4}{3} \Phi\left(-\frac{1}{2}\right)
 \end{aligned}$$

Q4)

$$\frac{f(x|w_1)}{f(x|w_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \cdot \frac{f(w_2)}{f(w_1)}$$

Given  $\Rightarrow$  0-1 loss,  $f(w_1) = f(w_2)$

$$\frac{f(x|w_1)}{f(x|w_2)} > 1$$

$$\frac{\frac{1}{\pi b \left[ 1 + \left( \frac{x-a_1}{b} \right)^2 \right]}}{\frac{1}{\pi b \left[ 1 + \left( \frac{x-a_2}{b} \right)^2 \right]}} > 1$$

$$\frac{1 + \left( \frac{x-a_2}{b} \right)^2}{1 + \left( \frac{x-a_1}{b} \right)^2} > 1$$

$$1 + \left( \frac{x-a_2}{b} \right)^2 > 1 + \left( \frac{x-a_1}{b} \right)^2$$

$$x^2 + a_2^2 - 2a_2 x > x^2 + a_1^2 - 2a_1 x$$

$$a_2^2 - a_1^2 > 2(a_2 - a_1)x$$

$$\boxed{x < \frac{a_1 + a_2}{2}}$$

1/2

if  $x$  satisfies this, it will mapped to class 1