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Q1. Maintenance p, Load h → both independent.
$p \in \{0,1\}$, $h \in \{0,1\}$.

If machine has both $=1$, chances of good functioning is high.

Data: S_i : iid responses.

a) θ_1 : prob $(p=1)$, θ_2 : prob $(h=1)$.

We model the trials as ~~n~~ Bernoulli trials.
Suppose ~~S_{ip}~~ S_{ip} is the value of p for ~~any~~ an S_i.
Similarly, suppose S_{ih} is the value of h for ~~any~~ an S_i.

For θ_1: Likelihood of observed data is:

$$\mathcal{L}_{\theta_1} = \prod_{i=1}^{n} \theta_1^{S_{ip}} (1-\theta_1)^{1-S_{ip}}$$

0.5

$$\log \mathcal{L}_{\theta_1} = \sum_{i=1}^{n} S_{i,p} \log \theta_1 + (1-S_{ip}) \log(1-\theta_1)$$

$$\hat{\theta}_1 = \underset{\theta_i}{\operatorname{argmax}} \log \mathcal{L}_{\theta_1}.$$

Take derivative of $\log \mathcal{L}_{\theta_1}$ and equate to 0:

$$\sum_{i=1}^{n} \left(\frac{S_{i,p}}{\theta_1} - \frac{1-S_{ip}}{1-\theta_1} \right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{S_{ip}(1-\theta_1) - (1-S_{i,p})\theta_1}{\cancel{\theta_1(1-\theta_1)}} = 0$$

$$\Rightarrow \sum_{i=1}^{n} S_{i,p} - \theta = 0$$

$$n\theta_1 = \sum_{i=1}^{n} S_{i,p}$$

$$\boxed{\theta_1 = \sum_{i=1}^{n} \frac{S_{i,p}}{n}}$$

Similarly for θ_2, we have $\boxed{\theta_2 = \sum_{i=1}^{n} \frac{S_{i,h}}{n}}$ 1.5

If we view S_i as a vector in $\mathbb{R}^2$, we can get:

$$\boxed{(\theta_1, \theta_2) = \sum_{i=1}^{n} \frac{S_i}{n}}$$

b) Value of $(\theta_1, \theta_2) = \sum_{i=1}^{2} \frac{S_i}{n} = \frac{(1,0)+(1,1)}{2}$

$= 0.5 \; \underline{(1, 0.5)}$ 0.5

Prob. that machine is high functioning for given (p, h):

$$P(\text{high functioning}) = \theta_1^{p}(1-\theta_1)^{1-p} \cdot (\theta_2)^{h}(1-\theta_2)^{1-h}$$

Given $p=0, h=1$;

$$P(\text{high functioning} \mid p=0, h=1) = 1^{0} \cdot (1-1)^{1-0} \; 0.5^{1} (0.5)^{0}$$

$= 1 (0)^1 (0.5) \cdot 1$

$= \underline{0}$ 0.5

The machine has 0 probability to be high functioning.

3.) Given $\mu_1 = \mu_2$, $\Sigma_1 = 1$, $\Sigma_2 = 2$

(a) At β^* for chernoff bound $\frac{\partial K(\beta)}{\partial \beta} = 0$.

$$\Rightarrow \frac{\partial}{\partial \beta}\left[\frac{1}{2} \ln\left(\frac{\beta + 2(1-\beta)}{2^{(1-\beta)}}\right)\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial \beta}\left[\frac{1}{2} \ln\left(\frac{2-\beta}{2^{(1-\beta)}}\right)\right] = 0.$$

$$\Rightarrow \frac{\partial}{\partial \beta}\left[\frac{1}{2}\left\{\ln(2-\beta) - \ln\left(2^{(1-\beta)}\right)\right\}\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial \beta}\left[\frac{1}{2}\left\{\ln(2-\beta) - (1-\beta)\ln(2)\right\}\right] = 0$$

$$\Rightarrow \frac{-1}{2-\beta} + \ln(2) = 0 \Rightarrow \frac{1}{2-\beta} = \ln(2)$$

$$\Rightarrow \boxed{\beta^* = 2 - \frac{1}{\ln(2)}}$$

(b) β^* for Bhattacharya bound., $\beta^* = \frac{1}{2}$.

$$e^{-K\left(\frac{1}{2}\right)} = \exp\left\{-\frac{1}{2} \ln\left(\frac{\frac{1}{2} + 2\left(1-\frac{1}{2}\right)}{2^{\left(1-\frac{1}{2}\right)}}\right)\right\}$$

$$= \exp\left\{-\frac{1}{2} \ln\left(\frac{\frac{1}{2} + 1}{2^{1/2}}\right)\right\}$$

$$= \exp\left\{-\frac{1}{2} \ln\left(\frac{3}{2\sqrt{2}}\right)\right\} = \exp\left\{\ln\left(\left(\frac{3}{2\sqrt{2}}\right)^{-\frac{1}{2}}\right)\right\}$$

$$= \left(\frac{2\sqrt{2}}{3}\right)^{\frac{1}{2}}$$