Q1) $P(X \mid \mu_i, \Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(X-\mu_i)^T \Sigma^{-1} (X-\mu_i)\right)$

$\Sigma = I$

Let the discriminate function be

$$g_i(X) = \log(P(X \mid \mu_i, \Sigma))$$

$$g_i(X) = \log\left[\frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(X-\mu_i)^T \Sigma^{-1} (X-\mu_i)\right)\right]$$

$$= -\frac{1}{2}(X-\mu_i)^T \Sigma^{-1} (X-\mu_i) - \frac{d}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma|)$$

$$g_1(X) = -\frac{1}{2}\left[\begin{bmatrix}0.5\\0.25\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right]^T . I . \left[\begin{bmatrix}0.5\\0.25\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}\right] -$$

this is same for all discriminant so can be ignored

1

$$\frac{3}{2}\log(2\pi) \underline{- \frac{1}{2}\log(1)}$$

$$= -\frac{1}{2}\begin{bmatrix}-0.5 & 0.25 & 1\end{bmatrix}\begin{bmatrix}-0.5\\0.25\\1\end{bmatrix} - \frac{3}{2}\log(2\pi)$$

$$= -0.65625 - 2.75681 = -3.413$$

$$g_2(X) = -\frac{1}{2}\begin{bmatrix}0.5 & -0.75 & 1\end{bmatrix}\begin{bmatrix}0.5\\-0.75\\1\end{bmatrix} - \frac{3}{2}\log(2\pi)$$

1

$$= -0.90625 - 2.75681 = -3.663$$

$g_1(X) > g_2(X)$, Hence X belongs to class 1

$\underline{Sol^n}$ (2) The likelihood ratio test is given by

$$\frac{p(x|\omega_1)}{p(x|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{p(\omega_2)}{p(\omega_1)}$$ — 0.25

If LHS > RHS, then $x \in \omega_1$ else $x \in \omega_2$

Now, $p(x|\omega_1) \sim N(x|\mu_1, I)$

$p(x|\omega_2) \sim N(x|\mu_2, I)$

MVG, $N(x|\mu, \Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\}$

RHS : $\frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{p(\omega_2)}{p(\omega_1)}$

$= \frac{2-0}{1-0} \frac{\cancel{1/2}}{\cancel{1/2}}$ ($\because$ priors are equal)

$= 2$

LHS : $\frac{p(x|\omega_1)}{p(x|\omega_2)} = \frac{\cancel{\frac{1}{(2\pi)^{d/2}|I|^{1/2}}} \exp\left\{-\frac{1}{2}(x-\mu_1)^T I^{-1}(x-\mu_1)\right\}}{\cancel{\frac{1}{(2\pi)^{d/2}|I|^{1/2}}} \exp\left\{-\frac{1}{2}(x-\mu_2)^T I^{-1}(x-\mu_2)\right\}}$

$$= \exp\left\{-\frac{1}{2}\left[(x-\mu_1)^T(x-\mu_1) - (x-\mu_2)^T(x-\mu_2)\right]\right\}$$

where $x-\mu_1 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \end{bmatrix}$

$$x-\mu_2 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.75 \\ -0.75 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -3/4 \\ -1/2 \end{bmatrix}$$

LHS then becomes

$$LHS = \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2+\cancel{\left(\frac{1}{2}\right)^2} - \left(\left(\frac{-3}{4}\right)^2+\left(\frac{-3}{4}\right)^2+\cancel{\left(\frac{-1}{2}\right)^2}\right)\right]\right\}$$

$$= \exp\left\{-\frac{1}{\cancel{2}} \times \cancel{2}\left[\left(\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]\right\}$$

$$= \exp\left\{\left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}$$

$$= \exp\left\{\left(\frac{3}{4}+\frac{1}{4}\right)\left(\frac{3}{4}-\frac{1}{4}\right)\right\}$$

$$= \exp\left(\frac{1}{2}\right)$$ — 0.75

$\because \exp\left(\frac{1}{2}\right) < 2$ i.e. LHS < RHS — 1

$\Rightarrow x \in \omega_2$

Q.3) Discriminant for class i is

$$g_i(x) = \ln P(x \mid \mu_i, \Sigma) + \ln P(\mu_i \mid \mu_0, \Sigma_0).$$

1/2

Given, $p(x \mid \mu_i, \Sigma) \sim N(\mu_i, I)$

$$P(\mu_i \mid \mu_0, \Sigma_0) \sim N(\mu_0, \Sigma_0).$$

So, $$g_i(x) = -\frac{1}{2}\{(x-\mu_i)^T \Sigma^{-1} (x-\mu_i)\} - \frac{1}{2}\{(\mu_i-\mu_0)^T \Sigma_0^{-1} (\mu_i-\mu_0)\}$$

1/2

$$= -\frac{1}{2}\cdot\{x^T \Sigma^{-1} x - 2\mu_i^T \Sigma^{-1} x + \mu_i^T \Sigma^{-1} \mu_i\} - \frac{1}{2}\cdot\{\mu_i^T \Sigma_0^{-1} \mu_i - 2\mu_0^T \Sigma_0^{-1} \mu_i + \mu_0^T \Sigma_0^{-1} \mu_0\}$$

Removing terms that do not impact discriminant for different classes

$$= \mu_i^T \Sigma^{-1} x - \frac{1}{2}\mu_i^T \Sigma^{-1} \mu_i - \frac{1}{2}\mu_i^T \Sigma_0^{-1} \mu_i + \mu_0^T \Sigma_0^{-1} \mu_i.$$

1/2

Given, $\Sigma = I, \Rightarrow \Sigma^{-1} = I$

$$= \mu_i^T I x - \frac{1}{2}\mu_i^T I \cdot \mu_i - \frac{1}{2}\mu_i^T \Sigma_0^{-1} \mu_i + \mu_0^T \Sigma_0^{-1} \mu_i.$$

$\therefore \; w = I\mu_i = \mu_i.$

$$b = -\frac{1}{2}\mu_i^T \mu_i - \frac{1}{2}\mu_i^T \Sigma_0^{-1} \mu_i + \mu_0^T \Sigma_0^{-1} \mu_i.$$

$\therefore \; g_i(x) = w^T x + b.$

1/2

Q4.) $p(x|\omega_1) = a\,p(x|\mu_1, \sigma^2) + b\,p(x|\mu_2, \sigma^2)$

$p(x|\omega_2) = b\,p(x|\mu_1, \sigma^2) + a\,p(x|\mu_2, \sigma^2)$

To find D.B →

We equate the posteriors.

$p(\omega_1|x) = p(\omega_2|x)$

$\Rightarrow p(\omega_1)(p(x|\omega_1)) = p(\omega_2)\,p(x|\omega_2)$

$\Rightarrow p(x|\omega_1) = p(x|\omega_2)$

$\Rightarrow a\,p(x|\mu_1, \sigma^2) + b\,p(x|\mu_2, \sigma^2) = b\,p(x|\mu_1, \sigma^2) + a\,p(x|\mu_2, \sigma^2)$

$\Rightarrow p(x|\mu_1, \sigma^2)[a-b] = p(x|\mu_2, \sigma^2)[a-b]$

$\Rightarrow p(x|\mu_1, \sigma^2) = p(x|\mu_2, \sigma^2)$

Given $\mu_1 = 1$, $\mu_2 = -1$ & $\sigma^2 = 1$

$\Rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{1}{2}(x-\mu_1)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu_2)^2}$

$\Rightarrow e^{-\frac{1}{2}(x-1)^2} = e^{-\frac{1}{2}(x+1)^2}$

$\Rightarrow (x-1)^2 = (x+1)^2$

$\Rightarrow x^2 - 2x + 1 = x^2 + 2x + 1$

$\therefore x = 0$

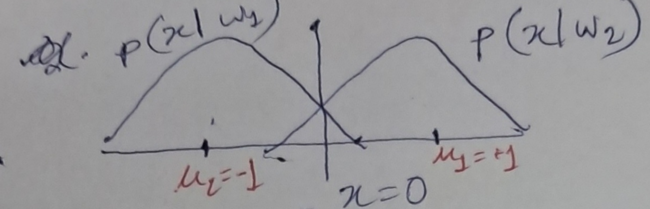
The D.B lies at $x = 0$

1

$$P(\text{error}|x) = \min\left(P(\omega_1|x),\ P(\omega_2|x)\right)$$

$$p(x|\omega_1) \approx b\, p(x|\mu_2, \sigma^2) \qquad b << a$$

$$p(x|\omega_2) \approx a\, p(x|\mu_1, \sigma^2)$$



0.5

$$P(\text{error}) = \frac{1}{2}\int_{-\infty}^{0} p(x|\omega_2)\,dx + \frac{1}{2}\int_{0}^{\infty} p(x|\omega_1)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} a\, p(x|\mu_1, \sigma^2) + b\, P(x|\mu_2, \sigma^2) + \frac{1}{2}\int_{0}^{\infty} a\, p(x|\mu_2, \sigma^2) + b\, P(x|\mu_1, \sigma^2)$$

$$= \frac{1}{2}\left[a\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}dx + b\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}}dx\right]$$

0.5

$$+ \frac{1}{2}\left[a\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}}dx + b\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}dx\right]$$

$$= \frac{1}{2}\left(1 + \text{erf}\left(\frac{1}{\sqrt{2}}\right)\right)$$