SHOBHIT RAJ
2022482

SML ASSIGNMENT - 1

Q1.

$$P(x|\omega_1) = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}$$

$$P(x|\omega_2) = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$$

$P(\omega_1) = P(\omega_2) = 1/2$ (as $P(\omega_1) + P(\omega_2) = 1$) ($a_2 > a_1 \rightarrow$ given).

Decision boundary at $P(\omega_1|x) = P(\omega_2|x) \rightarrow$

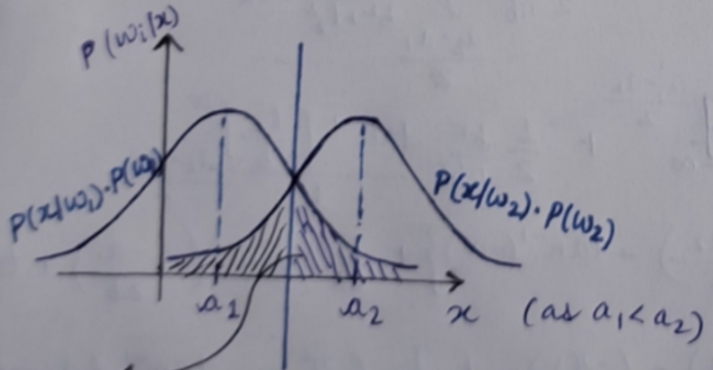
$$\frac{P(x|\omega_1)\cdot \cancel{P(\omega_1)}}{\cancel{P(x)}} = \frac{P(x|\omega_2)\cdot \cancel{P(\omega_2)}}{\cancel{P(x)}} \quad \left(\text{Posterior} = \frac{\text{Likelihood} \times \text{prior}}{\text{evidence}}\right)$$

$$\Rightarrow \cancel{\frac{1}{\pi b}} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \cancel{\frac{1}{\pi b}} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$$

$$\Rightarrow (x-a_1)^2 = (x-a_2)^2 \Rightarrow \cancel{x^2} + a_1^2 - 2a_1x = \cancel{x^2} + a_2^2 - 2a_2x$$

$$\Rightarrow 2a_2x - 2a_1x = a_2^2 - a_1^2$$

$$\Rightarrow x = \frac{a_1+a_2}{x} \rightarrow \text{decision boundary.}$$



error will happen in these regions

$x = \frac{a_1+a_2}{2} \rightarrow$ decision boundary

As $P(\text{error}|x) = \min(P(\omega_1|x), P(\omega_2|x))$

$$\rightarrow P(\text{error}) = \int_x P(\text{error}|x)\, p(x)\, dx$$

$$\Rightarrow P(\text{error}) = \int_x \min(P(\omega_1|x), P(\omega_2|x)) \cdot p(x)\, dx$$

same (take common)

$$= \int_x \min\left(\frac{P(x|\omega_1)\cdot P(\omega_1)}{P(x)}, \frac{P(x|\omega_2)\, P(\omega_2)}{P(x)}\right) p(x)\, dx$$

$$\Rightarrow P(\text{error}) = \int_x \frac{\overset{\to \text{ as } p(\omega_1)=p(\omega_2)=1/2}{P(\omega_1)}}{\cancel{P(x)}} \cdot \min \cdot (P(x|\omega_1), P(x|\omega_2)) \cdot \cancel{p(x)}\, dx$$

$$= \frac{1}{2} \int_x \min \cdot (P(x|\omega_1), P(x|\omega_2))\, dx$$

Now, using decision boundary $x = \frac{a_1+a_2}{2}$.

when $x < \frac{a_1+a_2}{2} \rightarrow P(x|\omega_2)$ is min. (from graph)

$x > \frac{a_1+a_2}{2} \rightarrow P(x|\omega_1)$ is min.

$$\therefore P(\text{error}) = \frac{1}{2} \int_{-\infty}^{\frac{a_1+a_2}{2}} P(x|\omega_2)\, dx + \frac{1}{2} \int_{\frac{a_1+a_2}{2}}^{\infty} P(x|\omega_1)\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{\pi b} \cdot \frac{1}{1+\frac{(x-a_2)^2}{b^2}}\, dx + \frac{1}{2} \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}\, dx$$

$$= \frac{1}{2} \frac{b^2}{\pi b} \int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{b^2+(x-a_2)^2}\, dx + \frac{1}{2} \frac{b^2}{\pi b} \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{b^2+(x-a_1)^2}\, dx$$

Let $x - a_2 = t_1 \rightarrow dx = dt_1$, $x - a_1 = t_2 \rightarrow dx = dt_2$

$$\Rightarrow \frac{1}{2} \frac{b^2}{\pi b} \int_{-\infty}^{\frac{a_1-a_2}{2}} \frac{1}{b^2+t_1^2}\, dt_1 + \frac{1}{2} \frac{b^2}{\pi b} \int_{\frac{a_2-a_1}{2}}^{\infty} \frac{1}{b^2+t_2^2}\, dt_2$$

$$\Rightarrow \frac{1}{2} \frac{\cancel{b^2}}{\pi \cancel{b}} \frac{\tan^{-1}\left[\frac{t_1}{b}\right]_{-\infty}^{\frac{a_1-a_2}{2}}}{\cancel{b}} + \frac{1}{2} \frac{\cancel{b^2}}{\pi \cancel{b}} \frac{\tan^{-1}\left[\frac{t_2}{b}\right]_{\frac{a_2-a_1}{2}}^{\infty}}{\cancel{b}}$$

$$\Rightarrow \frac{1}{2\pi} \left\{ \tan^{-1}\left(\frac{a_1-a_2}{2b}\right) - \tan^{-1}(-\infty) \right\} + \frac{1}{2\pi} \left\{ -\tan^{-1}\left(\frac{a_2-a_1}{2b}\right) + \tan^{-1}(\infty) \right\}$$

$$\Rightarrow \frac{1}{2\pi} \left(\tan^{-1}\left(\frac{a_1-a_2}{2b}\right) - \left(\frac{-\pi}{2}\right) \right) + \frac{1}{2\pi} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a_2-a_1}{2b}\right) \right)$$

$$\Rightarrow \frac{1}{2\pi} \left(\frac{2\pi}{2} + \left(-\tan^{-1}\left(\frac{a_2-a_1}{2b}\right) - \tan^{-1}\left(\frac{a_2-a_1}{2b}\right) \right) \right)$$ (as $\tan^{-1}(x) = -\tan^{-1}(-x)$)

$$\Rightarrow \frac{1}{2} + \frac{-2\tan^{-1}\left(\frac{a_2-a_1}{2b}\right)}{2\pi} = \underbrace{\frac{1}{2} - \frac{1}{\pi} \tan^{-1}\left(\frac{a_2-a_1}{2b}\right)}_{P(\text{error})}$$

Q2. For unbiased estimate,

$$\mathrm{cov}(\vec{x}) \approx \frac{1}{N-1}(\vec{x}-\vec{\mu}_x)(\vec{x}-\vec{\mu}_x)^T$$

covariance matrix of $\vec{x}$ ($d \times d = 3 \times 3$ matrix)

$$X = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad ; X \in \mathbb{R}^{d \times N} \rightarrow \text{Here, } d=3,\ N=3$$

3 samples in 3-dimensions.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{3}\begin{bmatrix} 1+0+0 \\ -1+0+1 \\ 0+1+1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \end{bmatrix}$$

$$\therefore \vec{x}-\vec{\mu}_x = [x_1-\mu_x \quad x_2-\mu_x \quad x_3-\mu_x]$$

$$= \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & \frac{1}{3} & 1/3 \end{bmatrix}$$

$$\Rightarrow (\vec{x}-\vec{\mu}_x)^T = \begin{bmatrix} 2/3 & -1 & -2/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & 1 & 1/3 \end{bmatrix}$$

$$\therefore \mathrm{cov}(\vec{x}) = \frac{1}{3-1}\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1 & 0 & 1 \\ -2/3 & 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2/3 & -1 & -2/3 \\ -1/3 & 0 & 1/3 \\ -1/3 & 1 & 1/3 \end{bmatrix}$$

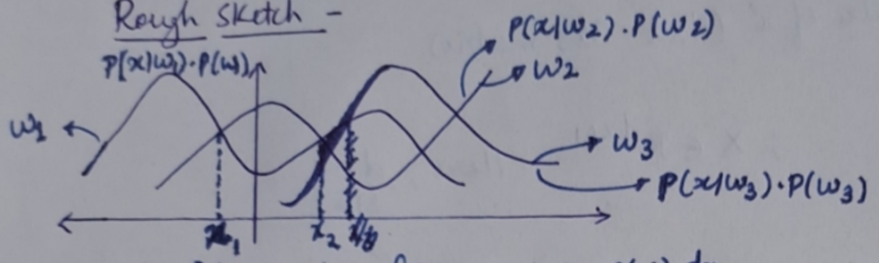
$$= \frac{1}{2}\cdot\begin{bmatrix} 2/3 & -1 & -2/3 \\ -1 & 2 & 1 \\ -2/3 & 1 & 2/3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/3 & -1/2 & -1/3 \\ -1/2 & 1 & 1/2 \\ -1/3 & 1/2 & 1/3 \end{bmatrix}}_{} \rightarrow \text{unbiased covariance matrix of } \vec{x}.$$

Q3. a) Let 3 classes $\rightarrow \omega_1, \omega_2, \omega_3$

$P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$

$P(\text{error}) = 1 - P(\text{correct})$

Rough Sketch -



As $P(\text{error}) = \int_x P(\text{error}|x) \cdot p(x)\,dx$

$= \int \min\left(P(\omega_1|x), P(\omega_2|x), P(\omega_3|x)\right) p(x)\,dx$

$= 1 - \int \max\left(P(\omega_1|x), P(\omega_2|x), P(\omega_3|x)\right) \cdot p(x)\,dx$

$\underline{(1 - P(\text{correct}))}$

$$\therefore P(\text{error}) = 1 - \int_{-\infty}^{\infty} \max\left(P(\omega_1|x), P(\omega_2|x), P(\omega_3|x)\right) \cdot p(x)\,dx$$

Now, as the distribution is not known, can't be deduced further.

Here, assuming ω_1 & ω_2 intersect at x_1

ω_2 & ω_3 intersect at x_2

ω_1 & ω_3 intersect at x_2 ~~too~~.

(as $P(\omega_1) = P(\omega_2) = P(\omega_3)$)

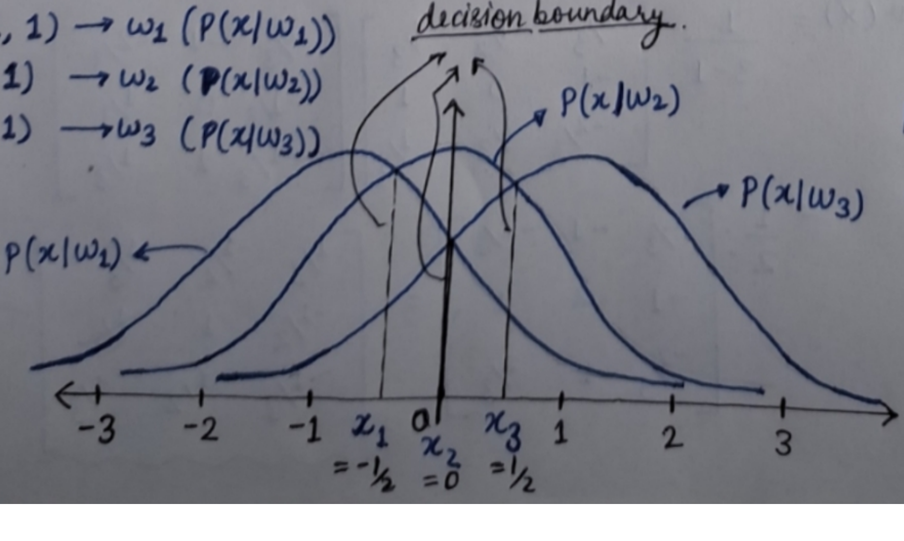
$$\therefore P(\text{error}) = 1 - \int_{-\infty}^{\infty} \max\cdot\left(P(x|\omega_1), P(x|\omega_2), P(x, \omega_3)\right) \cdot \underbrace{\frac{\overset{1/3}{\cancel{P(\omega_1)}}\cdot \cancel{P(x)}}{\cancel{P(x)}}}_{\text{common (taken out)}}\,dx \quad (1)$$

$$\Rightarrow P(\text{error}) = 1 - \frac{1}{3}\left[\int_{-\infty}^{x_1} P(x|\omega_1)\,dx + \int_{x_1}^{x_2} P(x|\omega_2)\,dx + \int_{x_2}^{\infty} P(x|\omega_3)\,dx.\right]$$

b) $N(-1, 1) \rightarrow \omega_1 \;(P(x|\omega_1))$

$N(0, 1) \rightarrow \omega_2 \;(P(x|\omega_2))$

$N(1, 1) \rightarrow \omega_3 \;(P(x|\omega_3))$



Here, assuming $P(\omega_1) = P(\omega_2) = P(\omega_3) = \frac{1}{3}$.

(from part (a)).

→ w_1 & w_2 intersect at x_1 → $P(x|\omega_1) = P(x|\omega_2) \Rightarrow N(-1,1) = N(0,1)$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} e^{\frac{-(x+1)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Rightarrow x^2+1+2x = x^2 \Rightarrow \underline{x_1 = \frac{-1}{2}}.$$

→ similarly, w_1 & ω_3 intersect at x_2 → $N(-1,1) = N(1,1)$

$$\Rightarrow x^2+1+2x = x^2+1-2x \Rightarrow \underline{x_2 = 0}.$$

→ similarly, w_2 & ω_3 intersect at x_3 → $N(0,1) = N(1,1)$

$$\Rightarrow x^2 = x^2+1-2x \Rightarrow \underline{x_3 = \frac{1}{2}}.$$

$$P(\text{error}) = 1 - P(\text{correct})$$

$$= 1 - \int_{-\infty}^{\infty} \max\cdot\left(P(x|\omega_1), P(x|\omega_2), P(x|\omega_3)\right) \cdot \frac{\cancel{P(\omega_1)}^{1/3}}{\cancel{P(x)}} \cdot \cancel{P(x)}\, dx$$

(from part (a)) 1

$$= 1 - \frac{1}{3}\left[\int_{-\infty}^{x_1} P(x|\omega_1)\,dx + \int_{x_1}^{x_3} P(x|\omega_2)\,dx + \int_{x_3}^{\infty} P(x|\omega_3)\,dx\right]$$

$$= 1 - \frac{1}{3}\left[\int_{-\infty}^{-1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} dx + \int_{-1/2}^{1/2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx + \int_{1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-1)^2}{2}} dx\right]$$

Convert to standard Gaussian ← Let $x+1 = u_1$ Let $x-1 = u_2$

$$= 1 - \frac{1}{3}\left[\int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-u_1^2/2} du_1 + \int_{-1/2}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_{-1/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u_2^2/2} du_2\right]$$

$$= 1 - \frac{1}{3}\left[\phi\left(\tfrac{1}{2}\right) + \left(\phi\left(\tfrac{1}{2}\right) - \phi\left(\tfrac{-1}{2}\right)\right) + \left(1 - \phi\left(\tfrac{-1}{2}\right)\right)\right]$$

as $\phi(z) = 1 - \phi(-z)$

$$= 1 - \frac{1}{3}\left[3\phi\left(\tfrac{1}{2}\right) - \left(1 - \phi\left(\tfrac{1}{2}\right)\right)\right] = 1 - \frac{1}{3}\left[4\phi\left(\tfrac{1}{2}\right) - 1\right]$$

$$= \frac{4}{3} - \frac{4}{3}\phi\left(\tfrac{1}{2}\right) = \frac{4}{3}\left(1 - \phi\left(\tfrac{1}{2}\right)\right) = \frac{4}{3}\phi\left(\tfrac{-1}{2}\right).$$

$$\therefore P(\text{error}) = \frac{4}{3}\phi\left(\tfrac{-1}{2}\right)$$

Q4. $P(x|\omega_1) = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}$

$P(x|\omega_2) = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$

$P(\omega_1) = P(\omega_2) = 1/2$ (as $P(\omega_1) + P(\omega_2) = 1$) ($a_2 > a_1 \rightarrow$ given)

0-1 loss $\Rightarrow$ $\lambda_{ij} \rightarrow$ loss in taking action α_i given $x \in \omega_j$

$\therefore \lambda_{11} = 0, \lambda_{22} = 0$
$\lambda_{12} = 1, \lambda_{21} = 1$ } as correct decision = 0, Incorrect decision = 1

$$\Rightarrow \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Here, $R(\alpha_1|x) = \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x)$

$R(\alpha_2|x) = \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_2|x)$

If $R(\alpha_1|x) < R(\alpha_2|x) \rightarrow$ take action α_1

$\rightarrow$ Reducing after subitituting:-

$$\frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(\omega_2)}{P(\omega_1)} \longrightarrow \text{Likelihood Ratio Test}$$

$$\Rightarrow \frac{\frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}}{\frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}} > \cancel{\frac{1-0}{1-0}} \cdot \frac{1/2}{1/2} \rightarrow 1$$

$\Rightarrow 1+\left(\frac{x-a_2}{b}\right)^2 > 1+\left(\frac{x-a_1}{b}\right)^2 \Rightarrow \cancel{x^2}+a_2^2-2a_2x > \cancel{x^2}+a_1^2-2a_1x$

$\Rightarrow a_2^2 - a_1^2 > 2x(a_2-a_1) \Rightarrow \frac{a_1+a_2}{2} > x$

$\therefore$ When $x < \frac{a_1+a_2}{2} \rightarrow$ take action α_1 (decide in favour of ω_1)

Given $x \rightarrow$

& when $x > \frac{a_1+a_2}{2} \rightarrow$ take action α_2 (decide in favour of ω_2)

This is the likelihood ratio test.

——X——

Please Turn Upside Down!!