Time: 30 minutes Max marks: ~~10~~ 11

Instructions:
- Do not plagiarize. Do not assist your classmates in plagiarism.
- Show your full solution for the questions to get full credit.
- Attempt all questions that you can.
- In the unlikely case a question is not clear, discuss it with an invigilating TA. Please ensure that you clearly include any assumptions you make, even after clarification from the invigilator.

1. (3 points) Using the homogeneous representation of lines and points, it can be shown that parallel lines intersect at a *point at infinity*.
   (a) (1 point) Let $\ell = [a, b, c]^\top$ and $\ell' = [a', b', c']^\top$ be homogeneous representation of two lines. Given these, how would you compute the slope and intercept in the (image) plane to write the equation of the line?
   (b) (1 point) If ℓ and ℓ' are parallel, write the relationship between the elements (a, b, c) and (a', b', c').
   (c) (1 point) Use the above relationship to show that the point at intersection of ℓ and ℓ' is a point at infinity and has the form $[x_1, x_2, 0]$.
   (d) (1 point) In the previous part, write x_1 and x_2 in terms of the elements of ℓ or ℓ'.

Solution:

(a) (1 point) The equation of the line in homogeneous representation is $\ell^\top \mathbf{x} = 0$, where $\ell = [a, b, c]^\top$ is the line and $\mathbf{x} = [x_1, x_2, x_3]^\top$ is the homogeneous representation of the point $\mathbf{x}$. The equation of the line turns out to be:

$$ax_1 + bx_2 + cx_3 = 0$$

In the canonical form, $x_3 = 1$, therefore, we have the equation of the line as:

$$ax_1 + bx_2 + c = 0$$
$$\Rightarrow x_2 = \frac{-a}{b}x_1 + \frac{-c}{b}$$

The slope is then given by $\frac{-a}{b}$ and the intercept of the line is given by $\frac{-c}{b}$. Similarly for ℓ', the slope and intercept will be $\frac{-a'}{b'}$ and $\frac{-c'}{b'}$ respectively.

(b) (1 point) Parallel lines will have the same slope, therefore we will have the following relationship:

$$\frac{a}{b} = \frac{a'}{b'}$$
$$\Rightarrow \frac{a}{a'} = \frac{b}{b'}$$
$$\Rightarrow ab' = ba'$$

(c) (1 point) The cross product $\ell \times \ell'$ is computed as follows:

$$\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} -cb' + bc' \\ ca' - ac' \\ -ba' + ab' \end{bmatrix}$$

From the previous part, we see that for parallel lines, $ab' = ba' \Rightarrow ab' - ba' = 0$. The resulting vector, which is the point of intersection of ℓ and ℓ' will always have its last element zero. In other words, the point will have the form $[x_1, x_2, 0]$.

(d) (1 point) $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \cong \begin{bmatrix} -b(c\frac{b'}{b} - c') \\ a(c\frac{a'}{a} - c') \\ 0 \end{bmatrix} \cong \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} \cong \begin{bmatrix} -b' \\ a' \\ 0 \end{bmatrix}$

2. (3 points) The geometry of two cameras imposes the Epipolar constraint.
   (a) (½ point) Write the epipolar constraint using the Essential Matrix $\mathbf{E}$ and the corresponding points ($\mathbf{x}$ and $\mathbf{x}'$) in appropriate coordinates. Given the pixel coordinates of point correspondences ($\mathbf{p}$ and $\mathbf{p}'$) and the intrinsic matrices of the two cameras K and K', write the relationship between ($\mathbf{p}, \mathbf{p}'$) and ($\mathbf{x}$, $\mathbf{x}'$).
   (b) (½ point) Write the Essential Matrix in terms of relative rotation and translation between the two cameras.
   (c) (2 points) What is the rank of the Essential matrix? Why is it so? If $\mathbf{E}$ is full-rank, prove it, else, find at least one vector that lies in the null-space of the Essential matrix.

Solution:

(a) (½ point) The epipolar constraint is given as $\mathbf{x}^\top \mathbf{E}\mathbf{x}' = 0$, where $\mathbf{x}, \mathbf{x}'$ are point correspondences in the normalized camera coordinate frame. They are obtained using the intrinsic camera parameter matrix and the pixel coordinates as $\mathbf{x} = K^{-1}\mathbf{p}$ and similarly $\mathbf{x}' = K'^{-1}\mathbf{p}'$.

(b) (½ point) Given the translation vector $\mathbf{t} = [t_1, t_2, t_3]^\top$ and the rotation matrix $\mathbf{R}$, the Essential matrix is given as $\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$, where $[\mathbf{t}]_\times$ represents the skew-symmetric cross-product matrix of $\mathbf{t}$ given by

$$\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$$

(c) (2 points) (1/2 points) The rank of the Essential matrix is 2.
(1/2 points) Since $\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$, and $\mathbf{R}$ is orthogonal ($\mathbf{R}\mathbf{R}^\top = \mathbf{I}$) and full-rank, the rank of E will be the same as $[\mathbf{t}]_\times$, which for a general vector $\mathbf{t}$ is 2.
(1 points) This can be confirmed by Gaussian elimination too, however, a simpler way is to make the geometric observation, which trivially tells us that the cross-product ($\mathbf{a} \times \boldsymbol{b}$) is zero only when the two vectors are pointing in the same (or exactly opposite) direction. Therefore the null space (or the left null space) has dimensionality 1, which implies for a 3×3 matrix that its rank is 2. The vector in the left null space is the vector with which we are taking the cross-product, i.e., $\mathbf{t}$ and the vector in the null space is $\mathbf{R}^\top \mathbf{t}$ and it can be verified by multiplying $[\mathbf{t}]_\times \mathbf{R}\mathbf{R}^\top \mathbf{t} = [\mathbf{t}]_\times \mathbf{t} = \mathbf{t} \times \mathbf{t} = 0$.

3. (4 points) State True / False with justification or select the answer(s) for the MCQs as the case may be. **Note**: No partial grading for MCQs: any incorrect answer, or any missing correct answers will not get any credit.
   (a) (1 point) Suppose you're calibrating a camera using multiple views of a known planar checkerboard. After estimating the intrinsic matrix $\mathbf{K}$, you find that the skew parameter is non-zero. What could this indicate about your camera or calibration process?
      A. The camera has rectangular pixels.
      B. The checkerboard used was not planar.
      C. The image sensor has non-rectangular pixels and the axes are not orthogonal.
      D. The radial distortion model used is incorrect.

(b) (1 point) If the essential matrix has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t \\ 0 & t & 0 \end{bmatrix}$$

, then the motion between the two cameras is given by

A. a horizontal translation.
B. a vertical translation.
C. no rotation.
D. a 90° rotation about the X-axis.
E. None of the above

(c) (1 point) State True / False with justification. The solution for the Fundamental matrix obtained by solving the $\mathbf{Af} = 0$ system of equations always has rank 2.

(d) (1 point) State True / False with justification. If the camera has a pure translation along the Z-axis of the camera's coordinate frame, the epipoles coincide with the principal point (the projection of the camera centre on the image plane).

Solution:

(a) (1 point) For a non-zero skew parameter, the correct choice is the following:

A. The camera has rectangular pixels.
B. The checkerboard used was not planar.
C. The image sensor has non-rectangular pixels and the axes are not orthogonal.
D. The radial distortion model used is incorrect.

(b) (1 point) For the given Essential Matrix,

A. a horizontal translation.
B. a vertical translation.
C. no rotation.
D. a 90° rotation about the X-axis.
E. None of the above

(c) (1 point) **False**. The solution is obtained via SVD and there is no way to guarantee that the resulting vector when reshaped to a 3×3 matrix will have rank 2.

(d) (1 point) **True**. The principal point is the camera centre's projection on the image plane is along the Z-axis. If the motion is a pure translation along the Z-axis, the baseline will be along the Z-axis and therefore the intersection of the baseline (the epipole) with the image plane will be exactly coinciding with the principal point.