DSc Assignment-1

Manan, Aggarwal
2022273

Shobhit, Raj
2022482

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1 Question 3

(a) Since the dice is unbiased therefore probability of the dice to land on any of its faces would be $\frac{1}{k}$. $\left\lfloor\sqrt{k}\right\rfloor$ is also a number on the dice again with the probability $\frac{1}{k}$.

The expected number of rolls until we see this specific number follows a geometric distribution, and the expected value of a geometric distribution with success probability p is $\frac{1}{p}$.

So, the expected number of rolls $E[\text{rolls}]$ is:

$$E[\text{rolls}] = \frac{1}{\frac{1}{k}} = k$$

Thus, the expected number of rolls until we see the number $\left\lfloor\sqrt{\lfloor k \rfloor}\right\rfloor$ on its upward face is:

$$E[\text{rolls}] = k$$

(b) Given a dice with k faces we can find the probability of finding r^{th} unique value. When finding the r^{th} value we would have already realized $(r-1)^{\text{th}}$ unique values. Therefore for a roll after finding $r-1$ unique values we can find a unique value with probability $\frac{k-(r-1)}{k}$.

$$E[\text{rolls for finding } r^{\text{th}} \text{ unique value} \mid r{-}1 \text{ unique values}] = \frac{1}{\frac{k-(r-1)}{k}} = \frac{k}{k-(r-1)}$$

To find number of rolls to get all the values,

$$E[\text{rolls}] = \sum_{i=1}^{k} \frac{k}{k-(r-1)} = k \sum_{i=1}^{k} \frac{1}{k-(r-1)}$$

(c) We have a 3-faced dice with

$$P(1) = P(3) = \frac{1}{4}, \quad P(2) = \frac{1}{2}$$

We need to find the expected number of rolls to see every number from 1 to 3 at least once. The strategy is again similar to the coupon collector problem, but with varying probabilities.

Let the expected number of rolls be E_3. Start by seeing one number. On the next roll, the probability of seeing a new number is $1 - P(\text{already seen})$. Thus, the expected number of rolls for each step is:

1. To get the first number, we need 1 roll.

2. After seeing 1 number, the probability of seeing another new number will be:

   - if 1 was seen in the first roll P(2 or 3) $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
   - if 2 was seen in the first roll P(1 or 3) $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
   - if 3 was seen in the first roll P(1 or 2) $= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

   $$E_2 = \frac{1}{4} * \frac{1}{\frac{3}{4}} + \frac{1}{2} * \frac{1}{\frac{1}{2}} + \frac{1}{4} * \frac{1}{\frac{3}{4}} = \frac{1}{4} * \frac{4}{3} + \frac{1}{2} * 2 + \frac{1}{4} * \frac{4}{3} = \frac{5}{3}$$

3. After seeing 2 numbers, the probability of seeing 3^{rd} number will be:

   - if 1 was not seen then P(2 and 3 seen):
     - P(1^{st} 2 seen then 3) $= \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$
     - P(1^{st} 3 seen then 2) $= \frac{1}{4} * \frac{2}{3} = \frac{1}{6}$
   - if 2 was not seen then P(1 and 3 seen):
     - P(1^{st} 1 seen then 3) $= \frac{1}{4} * \frac{1}{3} = \frac{1}{12}$
     - P(1^{st} 3 seen then 1) $= \frac{1}{4} * \frac{1}{3} = \frac{1}{12}$
   - if 3 was not seen then P(1 and 2 seen):
     - P(1^{st} 1 seen then 2) $= \frac{1}{4} * \frac{2}{3} = \frac{1}{6}$
     - P(1^{st} 2 seen then 1) $= \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$

   $$E_3 = (\frac{1}{4} + \frac{1}{6}) * \frac{1}{\frac{1}{4}} + (\frac{1}{12} + \frac{1}{12}) * \frac{1}{\frac{1}{2}} + (\frac{1}{6} + \frac{1}{4}) * \frac{1}{\frac{1}{4}} = \frac{5}{12} * 4 + \frac{1}{6} * 2 + \frac{5}{12} * 4 = \frac{11}{3}$$

   Expected number of rolls before we get all the numbers is:

   $$E[\text{rolls}] = E_1 + E_2 + E_3 = 1 + \frac{5}{3} + \frac{11}{3} = \frac{19}{3} \approx 6.33$$