Data Science (CSE558)

End-Sem

<u>Time: 2 Hours</u> <u>Maximum Marks: 50</u>

Instructions

- 1. Please read all questions carefully before writing your answers. The 1-mark questions need to answered with a single word. An incorrect answer to these questions fetches negative marks.
- 2. Please write neat, short and meaningful answers. Untidy answers will not be evaluated.
- 3. All parts of a single question should be answered collectively, i.e., all at one place.

All The Best!

- 1. Consider a weighted coin such that probability of getting head is 0.2.
 - a. What is the probability of getting at least 40 heads from 100 random tosses of this coin? (3 Marks)
 - b. Use this coin to define a random variable that can be used to design a family of random matrices of size k x d such that a matrix randomly sampled from the family ensure preservation of distance between 2^k pairs of points. (5 Marks)
- 2. Let A and B be two events in a sample space such that Pr(A) = 6/11 and Pr(B) = 6/13.
 - a. Can the events be disjoint?

(+1/-0.5 Marks)

b. Prove or disprove your answer.

(2 Marks)

- 3. Recall the hypothesis tests for population means, i.e., z-test and t-test. For a sample of size n, after computing the test statistics why degree of freedom is taken into account in t-test whereas not in z-test?

 (2 Marks)
- 4. Consider a bag with four unbiased dice as {d1, d2, d3, d4}. For every 1 ≤ i ≤ 4, di has i-faces numbered 1 to i.
 - a. Let one die be selected uniformly at random from the bag and it has been rolled 3 times. The
 outcome is {1,3,1}. Use maximum likelihood estimation to determine the most likely dice selected
 from the bag.
 (2 Marks)
 - b. Use Bayes' rule to compute the probability of the most probable die for the given outcome after every roll. Compute Pr(die|{1}) after 1st roll, compute Pr(die|{3}) after 2nd roll and so on. (4 Marks)
 - c. Let a die be selected from the bag to play snakes and ladder between three friends. Each player can start her game only when she rolls 1. Let player-1, Player-2, and Player-3 take 3, 7, and 2 rolls to start their respective games. Using maximum likelihood estimation determine, the most likely die selected from the bag.

 (3 Marks)
- 5. A random point v on the surface of a unit sphere in a d-dimensional space, centered at origin is generated as follow: for every i ϵ [1, ..., d], v(i) is either +(d^{-0.5}) or -(d^{-0.5}) with equal probability. Let x and y be two such points.
 - a. Prove that as d tends to infinity, the points tend to be orthogonal.

(2 Marks)

b. For a fixed d, use Chebyshev's to bound the probability of the event $|x^Ty| \ge 0.1$.

(3 Marks)

c. For a fixed d, use Bernstein to prove that there are at most $O(2^{0.01d})$ points such the following event E is true with probability at least 0.9.

E: For all pair of points (x, y) from the set, $|x^Ty| < 0.1$.

(5 Marks)

- 6. Let A be a matrix of size n x m such that its rank is 10, where 10 < min(n,m). Let A = $U\Sigma V^T$ where U and V are two orthonormal square matrices of size n x n and m x m respectively.
 - a. How many non-zero singular values are there in Σ ?

(+1/-0.5 Marks)

- b. Let, k < 10 and Z = $[U_{11-k}, U_{12-k}, ..., U_9, U_{10}]$ consisting of k singular vectors from U. B = ZZ^TA be the low rank representation of A. Then calculate the exact Frobenius norm of the difference matrix between the original data and the projected data i.e., $||A B||_F^2$ (3 Marks)
- 7. Continuing the above question, consider another matrix B of size m x n. The running time of AB is $O(n^2m)$, which worries you because it has a quadratic dependence on n. So, your friend proposed the following steps to improve the running time of the matrix-matrix product.
 - Sample a random vector 'q' in R^m such that, every index of q is -1 or +1 with equal probability.
 - Compute C = Aq and $D = q^TB$.
 - Return X = CD.
 - a. What is the running time of the above algorithm?

(+1/-0.5 Marks)

- b. You noticed that, $E[qq^T] = I$ (m x m identity matrix). Hence, E[X] = AB and it is an unbiased estimator. If the rank(A) = rank(B) = rank(AB) = 10 then what is the rank of X? Why? (2 Marks)
- 8. Let A be a matrix of size n x m. Let C be a set of 2k columns in A and R be set of k rows in A. With the selected C and R design an efficient algorithm to compute U for the CUR decomposition. State its running time? (5 Marks)
- 9. Let palyer1 and player2 are playing for car1 and car2 in two different Monty Hall game separately. Let player1 did not change her preference upon given a choice and she lost the car1. In a separate game let player2 decides to change her preference upon given the option. With this strategy is the player2 guaranteed to win the car2 in her game? (+1/-0.5 Marks)
- guaranteed to win the car2 in her game? (+1/-0.5 Marks)

 10. Let H_m be a n x n Hadamard matrix, where n = 2^m for some positive integer m. Let $H_m = \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix}$ such that $H_0 = [1]$. Let v be a n-dimensional vector. Write an efficient pseudocode for the matrix vector product H_m v. What is its running time? (5 Marks)
- 11. (Bonus) Recall that sparsity in JL matrix has an advantage in the running time. Is this always useful? Explain your reason with an analysis. (3 Marks)

Rubric

Q = (a) Random Vañable
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Unbiased coin

$$Ps(\vec{H}) = 0.16 = 0.5$$
 $Ps_{\Lambda}(\vec{T}) = 0.5$
 1 Mark
 $Ps_{\Lambda}(\vec{T}) = 0.5$

$$f_{mue:-}$$
 $E[x] = 0$
 $Var(x) = 1(0.16) + 1(0.16)$
 $K - (E(x))$
 $We need 0$

Prove:
$$\frac{0.5}{k} + \frac{0.5}{K}$$

$$Var = 1$$

$$K$$

Que.3

- 1.For Z-test No degree of freedom because population variance is known and fixed.
- 2.For T-test DoF is critical because the test relies on sample variance and smaller sample size increases uncertainty. **2 Marks if the above points are mentioned in the answer.**

Que.7

Time complexity = $O(n^2 + nm)$ ———— **1 M** / **-0.5M**

Rank of X = 1 ———— **1 M**

Reason(C, D only 1,1 vectors) —----1 M

Que.8

SVD method:

Running time --1M Running time = O(K³)

ntersection ---- 2M

Pseudo Inverse — 2M

Other method:

Calculation of U — 1M

C+, R+ calculation —-- 2M

Efficient Algorithm —--- 1M

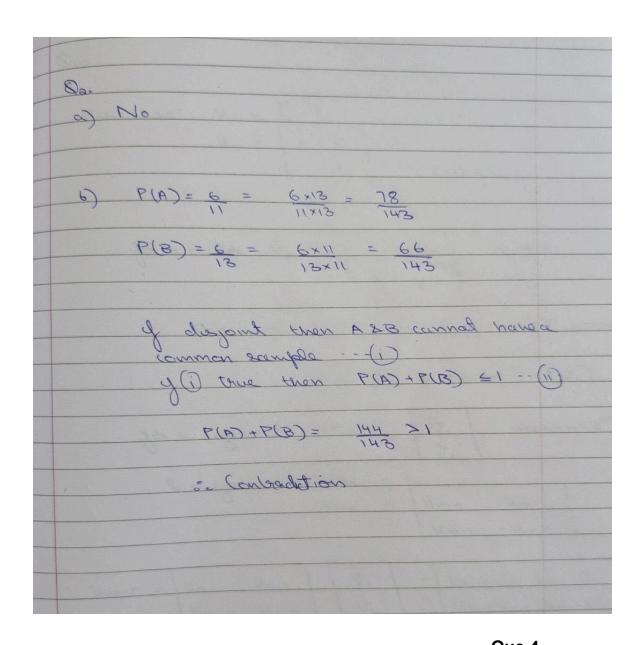
Time Complexity —--- **1M** Running time = $O(k^2n + kmn + k^2m)$

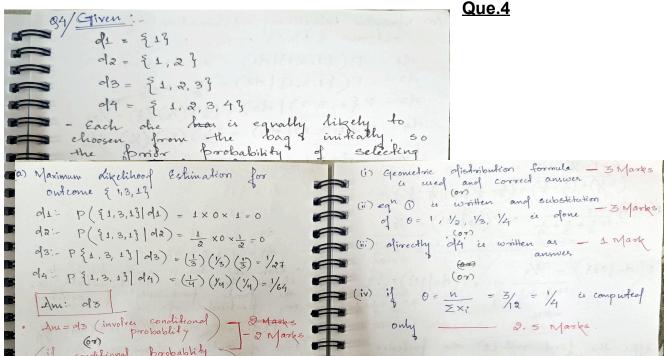
Marks awarded for both methods

Que.9

Answer: No, 1 Mark for correct, -0.5 for incorrect.

Que-2



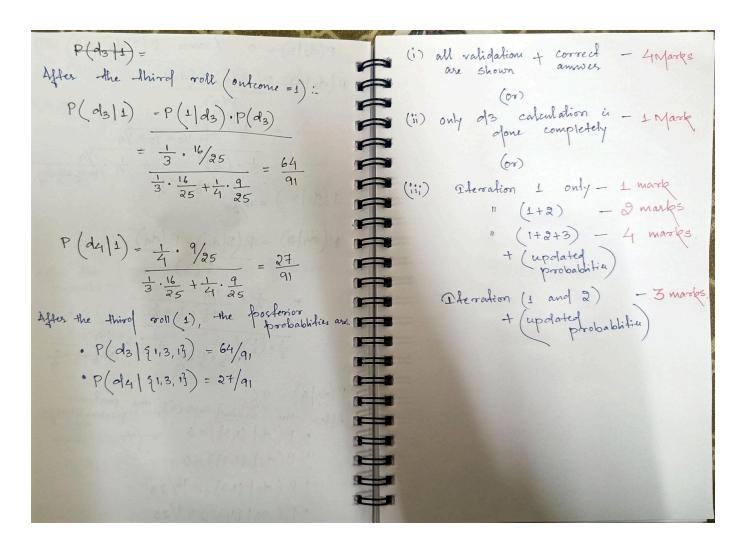


$$P(d_{2}|3) = 0 \quad (\text{since } P(3|d_{3}) = 0)$$

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$$P(d_{2}|3) = \frac{1}{1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{4}} \quad P(d_{3}|3) = \frac{1}{4}$$

$$P(d_{3}|3) = \frac{1}{4} \quad P(d_{3}|3) =$$



Que.5

Q5.
$$? \in \{ \frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}} \}$$
 $x, y \in \mathbb{R}^d$
 $X, y \in \mathbb{R}^d$
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$$E[x^*y] = E[x^*y^*] = \sum_{i=1}^d E[x^*y^*] = \sum_{i=1}^d E[x^*y^*] = 0.5$$
 $E[x^*y^*] = E[x^*y] \cdot E[y^*]$
 $x^*, y^* = \begin{cases} \sqrt{d} & h^{-1/2} & y \in E[x^*] = 0 \end{cases} 0.5$
 $E[x^*y^*] = \begin{cases} \sqrt{d} & h^{-1/2} & y \in E[x^*] = 0 \end{cases} 0.5$
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(b) Let
$$\pm$$
 be the $R.v$ for representing $x^T.y$

$$\mu = E \left[x^T.y \right] = 0$$

$$Var (7) = Var (x^T.y) = 1/d = v^2$$
Chebyshev's Inequality:
$$P(|7 - 4|), kv) \leq \frac{1}{k^2}$$

$$P(|7 - 4|), kv) \leq \frac{1}{k^2}$$

$$= \frac{100}{d^2} \left[0.5 \right] \left(k = \frac{0.1}{v} \right)$$

() Bernstein's Inequality:

$$X_b = X_b = \begin{cases} 1/49 & b = 1/2 \\ -1/4 & b = 1/2 \end{cases} 0.5$$

$$S = \sum_{i=1}^{d} X_i^2, \quad \left\{ X_i^2 - E \left[X_i^2 \right] = \frac{1}{d} \right\}$$

$$= \left[S \right] = E \left[X_i^2 X_i^2 \right] = 0, \quad \left\{ X_i^2 - E \left[X_i^2 \right] = \frac{1}{d} \right\}$$

$$\Rightarrow b = 1/d$$

Using the inequality

=)
$$P(15-01,0.1) \leq exp(-\frac{(0.1)^2}{\sqrt{1+\frac{0.1}{a}}})$$

$$\Rightarrow P (|x^{T}y^{T}|, 0.1) = 1 - P (|x^{T}y^{T}| < 0.1)$$
 $= ex \cdot P (-\frac{ct}{210})$

If there are m points which form the set,

no d pairs (21, y) =
$$m(m^{-1})$$

P (P ($1 \times 7 \times 1 \times 0.1$) for all pairs)

= $\left(1 - e^{d/10}\right) \frac{m(m^{-1})}{a} > 0.9$

10 $e^{-d/10} = d/10 > 0.9$

=) $m(m^{-1}) = d/10 > 0.9$

Que.6

6.a: 10, 1 Mark for correct, -0.5 for incorrect.

6.b: 1 Mark for expression B, 2 marks for exact frobenius norm.

ZZ is projection Materix which is Spanned July 'Z', Z = (UII-K, UIZ-K, ---, UIO).

and, All the vectors in 'U' matrix is Orthonor

- mal i.e. every frain wise Vector is Zero.

that, efforively

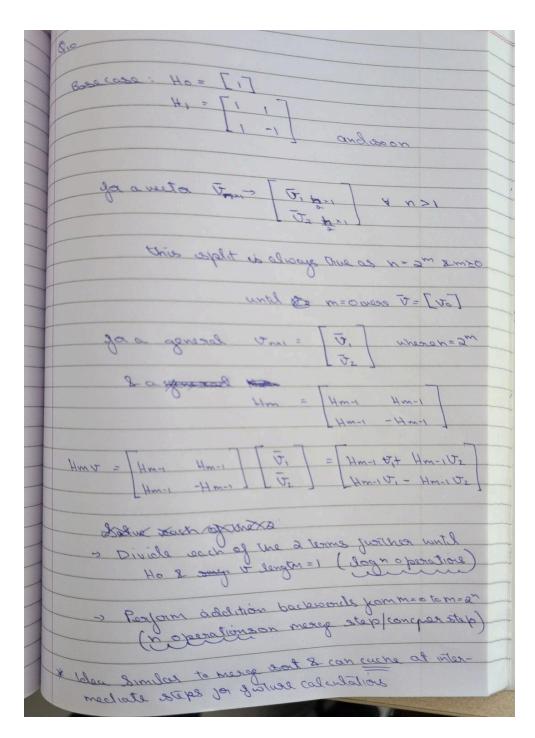
B = 68 M8 V8 + 63 My V9 + CIOMIO VIO — 1 TROM

Assuming K=3.

After 11A-B112 =

11 \(\subseteq \text{C.M.VT} - \frac{2}{2} \text{C.V.J.M. 112} \) 11 5 6, M, V, - 2 6; V, M, 1/2 3 Cancel and to Zero from to to 8 get : 11 = 6; M, V, 112 F > \$ ₹ 62 -> 2 Mark.

Que-10



Time Complexity: nLog(n)

Note: In Q10. of the endsem. If you have used a recursive algorithm. Partial marks have been awarded for correct divide and merge steps. And another mark for the correct time complexity mentioned. There are no marks for a rank approximation or any other randomized prediction of the result, an exact computation was required.