Test 1: Math 1 (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

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Duration: 60 minutes **Maximum Marks:** 10

Question 1 (5 marks)**.**

(a) (2.5 marks) Let $c \in \mathbb{R}$ be a fixed scalar. Define $T : \mathbb{P}_2 \to \mathbb{P}_3$ by

$$T(a_0 + a_1 x + a_2 x^2) = \int_c^x (a_0 + a_1 t + a_0 t^2)\,\mathrm{dt}$$

Show that T is a 1-1 linear transformation.

(b) (2.5 marks) Find the matrix of T with respect to the ordered bases $\mathcal{B} = \{1, x - c, (x - c)^2\}$ of $\mathbb{P}_2$, and $\mathcal{C} = \{1, x - c, x^2 - c^2, x^3 - c^3\}$ of $\mathbb{P}_3$.

Solution.

(a) Let $p(x), q(x) \in \mathbb{P}_2$. We know (from high school) that

$$\int_c^x (p(t) + q(t))\,\mathrm{dt} = \int_c^x p(t)\,\mathrm{dt} + \int_c^x q(t)\,\mathrm{dt}$$

Therefore $T(p(x) + q(x)) = T(p(x)) + T(q(x))$. Similarly if $p(x) \in \mathbb{P}_2$ and $\alpha \in \mathbb{R}$ then

$$\int_c^x \alpha p(t)\,\mathrm{dt} = \alpha \int_c^x p(t)\,\mathrm{dt}$$

Therefore $T(\alpha p(x)) = \alpha T(p(x))$.

As T respects addition and scalar multiplication, it is a linear transformation. Next we show that T is 1-1.

First method:

T is 1-1 if and only if $\ker T = \{0\}$. Therefore we will show that $\ker T = \{0\}$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 \in \ker T$. Then

$$\int_c^x (a_0 + a_1 t + a_2 t^2)\,\mathrm{dt} = 0.$$

Therefore

$$a_0t\Big|_c^x + \frac{1}{2}a_1t^2\Big|_c^x + \frac{1}{3}a_2t^3\Big|_c^x = 0 \implies a_0(x-c) + \frac{1}{2}a_1(x^2-c^2) + \frac{1}{3}a_2(x^3-c^3) = 0$$

A polynomial is zero only if all its coefficients are zero. Therefore

$$a_0 = \frac{1}{2}a_1 = \frac{1}{3}a_2 = 0 \implies a_0 = a_1 = a_2 = 0 \implies p(x) = 0$$

As the choice of $p(x)$ was arbitrary, it follows that $\ker T = \{0\}$.

Second method:

We show that $T(p(x)) = T(q(x)) \implies p(x) = q(x)$.

Let $p(x) = a_0 + a_1x + a_2x^2, q(x) = b_0 + b_1x + b_2x^2$ be polynomials such that $T(p(x)) = T(q(x))$. Then

$$\int_c^x (a_0 + a_1t + a_2t^2)\,\mathrm{dt} = \int_c^x (b_0 + b_1t + b_2t^2)\,\mathrm{dt}$$

Therefore

$$a_0t\Big|_c^x + \frac{1}{2}a_1t^2\Big|_c^x + \frac{1}{3}a_2t^3\Big|_c^x = b_0t\Big|_c^x + \frac{1}{2}b_1t^2\Big|_c^x + \frac{1}{3}b_2t^3\Big|_c^x$$

Hence

$$(a_0-b_0)(x-c) + \frac{1}{2}(a_1-b_1)(x^2-c^2) + \frac{1}{3}(a_2-b_2)(x^3-c^3) = 0.$$

A polynomial is zero only if all its coefficients are zero. Therefore

$$a_0 - b_0 = \frac{1}{2}(a_1-b_1) = \frac{1}{3}(a_2-b_2) = 0 \implies a_0 = b_0, a_1 = b_1, a_2 = b_2 \implies p(x) = q(x).$$

(b)

$$T(1) = \int_c^x \mathrm{dt} = x - c \implies [T(1)]_{\mathcal{C}} = (0, 1, 0, 0)$$

$$T(x-c) = \int_c^x (t-c)\,\mathrm{dt} = \frac{1}{2}(x^2-c^2) - c(x-c) \implies [T(x-c)]_{\mathcal{C}} = (0, -c, 1/2, 0)$$

$$\begin{aligned} T((x-c)^2) &= \int_c^x (t^2 - 2ct + c^2)\,\mathrm{dt} \\ &= \frac{1}{3}(x^3-c^3) - c(x^2-c^2) + c^2(x-c) \end{aligned}$$

Therefore

$$[T((x-c)^2)]_{\mathcal{C}} = (0, c^2, -c, 1/3)$$

Thus the matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is

$$[T]_{\mathcal{B},\mathcal{C}} = [[T(1)]_{\mathcal{C}} \quad [T(x-c)]_{\mathcal{C}} \quad [T((x-c)^2)]_{\mathcal{C}}] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -c & c^2 \\ 0 & 1/2 & -c \\ 0 & 0 & 1/3 \end{bmatrix}$$

Rubric.

(a)
- 1/2 mark for correctly showing that T respects vector addition
- 1/2 mark for correctly showing that T respect scalar multiplication
- First method of 1-1: 1 mark for choosing to show that the kernel of T is trivial, 1/2 mark for finishing the argument correctly
- Second method: 1/2 mark for choosing to show that $T(p(x)) = T(q(x))$ implies $p(x) = q(x)$, 1/2 mark for correctly calculating the required integrals, 1/2 mark for concluding that the coefficients of the polynomials must be equal

(b)
- 1 mark for stating the correct formula or for substituting correct values in the formula, i.e. the step
$$[T]_{\mathcal{B},\mathcal{C}} = [[T(1)]_{\mathcal{C}} \quad [T(x-c)]_{\mathcal{C}} \quad [T((x-c)^2)]_{\mathcal{C}}]$$
- 1/2 mark for computing the required integrals correctly
- 1 mark for correctly identifying the coefficients of $T(1), T(x-c)$ and $T((x-c)^2)$ with respect to basis $\mathcal{C}$.

Question 2 (5 marks)**.**

(a) (2.5 marks) Let Q be an invertible 2×2 matrix. Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by
$$T(A) = QAQ^{-1} - A$$
Show that T is a linear transformation and find the kernel of T. Is T a 1-1 transformation? Justify your answer.

(b) (2.5 marks) Find the dimension of the kernel of T for the following choices of Q:

(i) $Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Solution.

(a) Let $A, B \in M_{2\times 2}(\mathbb{R})$.
$$\begin{aligned} T(A+B) &= Q(A+B)Q^{-1} - (A+B) \\ &= (QA+QB)Q^{-1} - A - B \\ &= QAQ^{-1} + QBQ^{-1} - A - B \\ &= T(A) + T(B) \end{aligned}$$

Let $A \in M_{2\times 2}(\mathbb{R}), c \in \mathbb{R}$.
$$\begin{aligned} T(cA) &= QcAQ^{-1} - cA \\ &= c(QAQ^{-1} - A) \\ &= cT(A) \end{aligned}$$

As T respects addition and scalar multiplication, it is a linear transformation.

$$\begin{aligned}\ker T &= \{A \in M_{2\times 2}(\mathbb{R}) \mid T(A) = 0\} \\ &= \{A \in M_{2\times 2}(\mathbb{R}) \mid QAQ^{-1} - A = 0\} \\ &= \{A \in M_{2\times 2}(\mathbb{R}) \mid QA = AQ\}\end{aligned}$$

Clearly $\ker T \neq \{0\}$, because $T(I) = QIQ^{-1} - I = 0$, so $I \in \ker T$. Hence T is not 1-1.

(b) (i) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{aligned}T(A) &= QAQ^{-1} - A \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Hence T is the zero transformation, and $\ker T = M_{2\times 2}(\mathbb{R})$. Therefore $\dim \ker T = 4$.

(ii) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$\begin{aligned}T(A) &= QAQ^{-1} - A \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & 2b/3 \\ 3c/2 & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} 0 & -b/3 \\ c/2 & 0 \end{bmatrix}\end{aligned}$$

Thus $A \in \ker T \iff b = c = 0$. Hence

$$\ker T = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Therefore $\dim \ker T = 2$.

Rubric.

(a)
- 1/2 mark for correctly showing that T respects vector addition
- 1/2 mark for correctly showing that T respect scalar multiplication

- 1/2 mark for using the correct definition of the kernel and substituting the appropriate terms into the definition
- 1/2 mark for answering that T is not 1-1
- 1/2 mark for correct justification.

(b)

- 1 mark for finding the correct dimension in (i).
- 1 mark for finding the correct dimension in (ii)
- 1/2 mark for correct justification in part (ii)