Test 2: Math 1 (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

February 8th

Duration: 60 minutes **Maximum Marks:** 10

Question 1 (6 marks)**.**

(a) (2 marks) Let $(W, \langle ., . \rangle)$ be an inner product space (i.e. W is a vector space with an inner product $\langle ., . \rangle$ defined on it).

Let $T : V \to W$ be an injective linear transformation. Define

$$u \star v := \langle T(u), T(v) \rangle, \quad \forall u, v \in V.$$

Show that $\star$ is an inner product defined on V.

(b) (4 marks) Define $T : \mathbb{R}_2 \to \mathbb{P}_1$ by

$$T(a, b) = a + bt$$

Define

$$\mathbf{x} \star \mathbf{y} := \int_0^1 T(\mathbf{x})T(\mathbf{y})\,\mathrm{dt}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$$

Show that $\star$ is an inner product on $\mathbb{R}^2$. Compute $\mathrm{proj}_{(1,0)}(0, 1)$ with respect to $\star$.

Solution.

(a) Verification of properties:

Linearity in first argument:

(i) Let $v, w, z \in V$.

$$\begin{aligned}(v + w) \star z &= \langle T(v + w), T(z) \rangle \\ &= \langle T(v) + T(w), T(z) \rangle \quad (\because T \text{ is linear}) \\ &= \langle T(v), T(z) \rangle + \langle T(w), T(z) \rangle \quad (\because \langle ., . \rangle \text{ is bilinear}) \\ &= v \star z + w \star z\end{aligned}$$

(ii) Let $v, z \in V$, $c \in \mathbb{R}$.

$$\begin{aligned}(cv) \star z &= \langle T(cv), T(z) \rangle \\ &= \langle cT(v), T(z) \rangle \quad (\because T \text{ is linear}) \\ &= c\langle T(v), T(z) \rangle \quad (\because \langle ., . \rangle \text{ is bilinear}) \\ &= c(v \star z)\end{aligned}$$

Symmetry:

Let $v, z \in V$. As $\langle ., . \rangle$ is symmetric,

$$v \star z = \langle T(v), T(z) \rangle = \langle T(z), T(v) \rangle = z \star v$$

Positive-definite property:

Clearly,

$$0 \star 0 = \langle T(0), T(0) \rangle = \langle 0, 0 \rangle = 0$$

Next, if $v \star v = 0$, then

$$\langle T(v), T(v) \rangle = 0 \implies T(v) = 0,$$

because $\langle ., . \rangle$ is positive-definite. If $T(v) = 0$ then $v = 0$, because T is 1-1.

(b) We first show that $\star$ is an inner product.

First method:

We show that the given mapping T is a 1-1 linear transformation. The rest follows from part (a), because

$$\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t)\,\mathrm{dt}$$

is an inner product on $\mathbb{P}_2$.

Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$ be elements of $\mathbb{R}^2$. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha\mathbf{x} + \beta\mathbf{y}) &= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\ &= \alpha x_1 + \beta y_1 + (\alpha x_2 + \beta y_2)t \\ &= \alpha(x_1 + x_2 t) + \beta(y_1 + y_2 t) \\ &= \alpha T(\mathbf{x}) + \beta T(\mathbf{y}) \end{aligned}$$

Next $T(\mathbf{x}) = T((x_1, x_2)) = 0 \implies x_1 + x_2 t = 0 \implies x_1 = x_2 = 0 \implies \mathbf{x} = 0$. As the kernel of T is trivial, T is 1-1.

Second method:

We verify that $\star$ satisfies the properties of an inner product.

Linear in the first argument:

(i) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$.

$$\begin{aligned} (\mathbf{x} + \mathbf{y}) \star \mathbf{z} &= \int_0^1 (x_1 + x_2 t + y_1 + y_2 t)(z_1 + z_2 t)\,\mathrm{dt} \\ &= \int_0^1 (x_1 + x_2 t)(z_1 + z_2 t)\,\mathrm{dt} + \int_0^1 (y_1 + y_2 t)(z_1 + z_2 t)\,\mathrm{dt} \\ &= \mathbf{x} \star \mathbf{z} + \mathbf{y} \star \mathbf{z} \end{aligned}$$

(ii) Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^2$, $c \in \mathbb{R}$.

$$\begin{aligned}(c\mathbf{x}) \star \mathbf{z} &= \int_0^1 c(x_1 + x_2 t)(z_1 + z_2 t)\,\mathrm{dt} \\ &= c\int_0^1 (x_1 + x_2 t)(z_1 + z_2 t)\,\mathrm{dt} \\ &= c(\mathbf{x} \star \mathbf{z})\end{aligned}$$

Symmetry:

Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^2$.

$$\begin{aligned}\mathbf{x} \star \mathbf{z} &= \int_0^1 (x_1 + x_2 t)(z_1 + z_2 t)\,\mathrm{dt} \\ &= \int_0^1 (z_1 + z_2 t)(x_1 + x_2 t)\,\mathrm{dt} = \mathbf{z} \star \mathbf{x}\end{aligned}$$

Positive-definite property:

Clearly,

$$0 \star 0 = \int_0^1 (0 + 0t)^2\,\mathrm{dt} = 0$$

Next, suppose $\mathbf{x} = (x_1, x_2)$ is a vector such that

$$\int_0^1 (x_1 + x_2 t)^2\,\mathrm{dt} = 0$$

Now

$$\begin{aligned}\int_0^1 (x_1 + x_2 t)^2\,\mathrm{dt} &= \int_0^1 (x_1^2 + x_2^2 t^2 + 2x_1 x_2 t)\,\mathrm{dt} \\ &= x_1^2 + \frac{x_2^2}{3} + x_1 x_2 \\ &= x_1^2 + \frac{x_2^2}{3} + x_1 x_2 - \frac{x_2^2}{4} + \frac{x_2^2}{4} \\ &= \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{x_2^2}{12}\end{aligned}$$

The sum of the squares of two real numbers can only be zero if the numbers are zero. Therefore $x_1 = x_2 = 0$.

Next

$$\begin{aligned}\mathrm{proj}_{(1,0)}(0,1) &= \frac{(1,0) \star (0,1)}{(1,0) \star (1,0)}(1,0) \\ &= \frac{\int_0^1 (1 + 0t)(0 + 1t)\,\mathrm{dt}}{\int_0^1 (1 + 0t)(1 + 0t)\,\mathrm{dt}}(1,0) \\ &= (1/2, 0)\end{aligned}$$

Rubric.

(a)
- 1/2 mark for symmetry
- 1/2 mark for linearity in either first or second argument
- 1 mark for correct proof of positive-definite property (the fact that T is 1-1 must be mentioned)

(b)
- First method: 1/2 mark for showing T is linear, 1/2 mark for showing T is 1-1, 1 mark for citing part (a)
- Second method: 1.5 marks for showing the positive-definite property correctly (in case a student has cited the result stated in class about the definite integral of a non-negative continuous function, that should be clearly stated.), 1/2 mark for the other properties.
- 1 mark for writing the correct formula for the projection
- 1/2 mark for substituting the values correctly in the formula
- 1/2 mark for computing the final answer

Question 2 (4 marks)**.** Diagonalize if possible:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + \lambda$$

Thus the eigenvalues are: $\lambda_1 = 0, \lambda_2 = \frac{1+\sqrt{5}}{2}, \lambda_3 = \frac{1-\sqrt{5}}{2}$.

As A is a 3×3 matrix having 3 distinct eigenvalues, it is diagonalizable, i.e. $A = PDP^{-1}$, where the columns of P are eigenvectors corresponding to λ_1, λ_2 and λ_3 in that order, and D is a diagonal matrix whose diagonal entries are λ_1, λ_2 and λ_3 in the same order.

As the eigenvalues are distinct, each eigensapce is one-dimensional. A short way of finding an eigenvector basis is to find the orthogonal complement of the row space of $A - \lambda I$. This can be found by finding the cross-product of any two linearly independent rows. Observe that the second and third rows of

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 2 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}$$

are linearly independent for all three values of λ. Therefore we compute

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = \begin{bmatrix} \lambda^2 \\ 2\lambda + 1 \\ \lambda \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 0 & (3+\sqrt{5})/2 & (3-\sqrt{5})/2 \\ 1 & 2+\sqrt{5} & 2-\sqrt{5} \\ 0 & (1+\sqrt{5})/2 & (1-\sqrt{5})/2 \end{bmatrix}$$

Rubric.

- 1 mark for finding the characteristic polynomial
- 1/2 mark for finding the three eigenvalues correctly
- 1/2 mark for either stating that A is diagonalizable or explicitly writing the diagonal matrix D
- 1/2 mark for listing the eigenvector columns of P in the **same order** as the eigenvalues in D, even if they are not computed or computed incorrectly.
- 1/2 mark for each correctly computed eigenvector (Caution: each of these can differ from the given answer by a scalar.) Needless to say, any method can be used to find these eigenvectors, as long as it produces a valid eigenvector.