Mid-Semester Exam: Math 1 (Linear Algebra)

Indraprastha Institute of Information Technology, Delhi

December 18th, 2022

Duration: 60 minutes **Maximum Marks:** 30

Question 1 (4 marks).

(a) (2 marks) Find the RREF of the matrix

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{bmatrix}$$

(b) (2 marks) Without doing any computations, find the RREF of the matrix

$$\begin{bmatrix} 4 & -3 & 1 & 0 \\ -5 & 4 & 0 & 1 \end{bmatrix}$$

Justify your answer.

Solution.

(a) One possible sequence of row operations (this sequence is NOT unique):

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{5}{4}R1} \begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & 1/4 & -5/4 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & 3/4 & 1/4 & 0 \\ 0 & 1/4 & -5/4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1/4 & -5/4 & 1 \end{bmatrix} \xrightarrow{R_2 \to 4R_2} \begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -5 & 4 \end{bmatrix}$$

(b) Using part (a), the inverse of the matrix

$$\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

is

$$\begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$$

Hence the inverse of the matrix

$$\begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$$

is

$$\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

Therefore the RREF of

$$\begin{bmatrix} 4 & -3 & 1 & 0 \\ -5 & 4 & 0 & 1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 5 & 4 \end{bmatrix}$$

Rubric.

(a) 2 marks for finding the correct RREF. You may deduct half a mark for every calculation error.

(b) 1 mark for recognizing that the first two columns of part (a) and part (b) form matrices that are inverses of each other. 1 mark for deducing the RREF using this idea.

Question 2 (6 marks)**.**

(a) (3 marks) Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 4 & 10 & 2 & 1 \end{bmatrix}$$

Solve the homogeneous system $A\mathbf{x} = 0$ (where $\mathbf{x} = (x_1, x_2, x_3, x_4)$) and write the general solution in parametric vector form.

(b) (3 marks) Let $\mathbf{v}$ be the solution of the above system which corresponds to $x_3 = 0$ and $x_4 = 1$. Let $\mathbf{w}$ be the solution that corresponds to $x_3 = 1$ and $x_4 = 0$. Show that $\{\mathbf{v}, \mathbf{w}\}$ is a basis of $\operatorname{Nul} A$.

Solution.

(a) We reduce this to RREF:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 4 & 10 & 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 4 & 10 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & -2 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the general solution in parametric vector form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$\mathbf{v} = \begin{bmatrix} -3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

We know that $\operatorname{Nul} A$ is the space of all solutions of the homogeneous equation $A\mathbf{x} = 0$. Every solution of this equation is described in parametric vector form as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ and every such linear combination is a solution of the homogeneous equation. Hence

$$\operatorname{Span}\{\mathbf{v}, \mathbf{w}\} = \operatorname{Nul} A$$

Therefore we only need to show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. Consider the equation

$$c_1\mathbf{v} + c_2\mathbf{w} = 0$$

where c_1 and c_2 are real unknowns. Substituting the values of $\mathbf{v}$ and $\mathbf{w}$ we get

$$\begin{bmatrix} -3/2c_1 + 2c_2 \\ 1/2c_1 - c_2 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $c_1 = c_2 = 0$.

Rubric.

(a) 2 marks for bringing to RREF and 1 mark for writing the solution in parametric vector form. Alternatively, 1 mark for echelon form, 1 mark for back substitution and 1 mark for writing the solution in parametric vector form.

(b) 1 mark for stating that $\operatorname{Nul} A = \operatorname{Span}\{\mathbf{v}, \mathbf{w}\}$, 1 mark for setting up the linear independent equation, and 1 mark for showing that it has only the trivial solution.

Question 3 (5 marks)**.**

(a) (2 marks) Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ 4 & 10 & 2 & 1 \\ -1 & -3 & 1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 17 \\ 4 \end{bmatrix}$$

Show that

$$\begin{bmatrix} x_1 + 1 \\ x_2 + 1 \\ x_3 + 1 \\ x_4 + 1 \end{bmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is a solution of $A\mathbf{x} = 0$.

(b) (3 marks) Solve the system of equations $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ -1 & -3 & 1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}$$

Solution.

(a) As the row sums of matrix A are the entries of the vector $\mathbf{b}$ it is obvious that the vector

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is a solution of the system $A\mathbf{x} = \mathbf{b}$. So by Theorem 6, Chapter 1 of the David C Lay text,

$$\begin{bmatrix} x_1 + 1 \\ x_2 + 1 \\ x_3 + 1 \\ x_4 + 1 \end{bmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is a solution of $A\mathbf{x} = 0$.

(b) Clearly the vector $\mathbf{p} = (1, 1, 1, 1)$ is a particular solution of $A\mathbf{x} = \mathbf{b}$. So as a consequence of Theorem 6 mentioned above, we need only solve the homogeneous system $A\mathbf{x} = 0$. We reduce A to RREF:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \\ -1 & -3 & 1 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ -1 & -3 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 2 & 7 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 2 & 7 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 2 & 7 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 7/2 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 & 17/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 7/2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 17/2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7/2 \end{bmatrix}$$

The solution of the homogeneous system in parametric form is

$$\mathbf{x} = x_4 \begin{bmatrix} -17/2 \\ 4 \\ -7/2 \\ 1 \end{bmatrix}$$

Thus the general soution of the system is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -17/2 \\ 4 \\ -7/2 \\ 1 \end{bmatrix}$$

Rubric.

(a) 1 mark for spotting that (1,1,1,1) is a solution and 1 mark for applying Theorem 6.

(b) 1 mark for using Theorem 6 to solve the homogeneous system, 1 mark for solving the homogeneous system and 1 mark for writing the general solution of the nonhomogeneous system. Alternatively, 2 marks for solving the nonhomogeneous system Ax=b and 1 mark for writing the general solution.

Question 4 (5 marks)**.**

(a) (2 marks) Let A be an $m \times n$ matrix, where $n > 3$. Let B be the $m \times 3$ matrix constructed using the last three columns of A. Show that $\operatorname{col} B$ is a subspace of $\operatorname{col} A$. Does a similar relation hold between $\operatorname{nul} B$ and $\operatorname{nul} A$? Justify your answer.

(b) (3 marks) Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 7 \\ 1 & 0 & 1 & 0 & 7 \\ 2 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Let

$$S = \left\{ \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} \right\}$$

Extend S to a basis of $\operatorname{col} A$.

Solution.

(a) **First Solution:**

Let

$$A = [\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n],$$

where $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$. Then

$$B = [\mathbf{a}_{n-2}\ \mathbf{a}_{n-1}\ \mathbf{a}_n]$$

Let $\mathbf{v} \in \operatorname{col} B$. Then there exist real numbers c_1, c_2, c_3 such that

$$\mathbf{v} = c_1\mathbf{a}_n + c_2\mathbf{a}_{n-1} + c_3\mathbf{a}_{n-2}$$

Put $c_k = 0$ for $k = 1, \dots, n-3$. Then

$$\mathbf{v} = \sum_{k=1}^{n} c_{n+1-k}\mathbf{a}_k \in \operatorname{col} A.$$

Thus

$$\operatorname{col} B \subset \operatorname{col} A.$$

As both $\operatorname{col} A$ and $\operatorname{col} B$ are vector spaces under the same vector addition and scalar multiplication operations, $\operatorname{col} B$ is a subspace of $\operatorname{col} A$.

No similar relationship can hold between $\operatorname{Nul} A$ and $\operatorname{Nul} B$ because $\operatorname{Nul} A \subset \mathbb{R}^n$ and $\operatorname{Nul} B \subset \mathbb{R}^3$ and $n \neq 3$.

Second Solution:

Since $\{\mathbf{a}_{n-2}, \mathbf{a}_{n-1}, \mathbf{a}_n\} \subset \operatorname{col} A$, it follows that $\operatorname{Span}\{\mathbf{a}_{n-2}, \mathbf{a}_{n-1}, \mathbf{a}_n\}$ is a subspace of $\operatorname{col} A$, by Theorem 1, Chapter 4, DCL.

The remaining part of the solution is the same as solution 1.

(b) **First Solution:**

Let $\mathbf{a}_1, \ldots, \mathbf{a}_5$ denote the columns of A, ordered from left to right. It is obvious that

$$\operatorname{Span}\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \mathbb{R}^3$$

Hence $\mathbb{R}^3 \subset \operatorname{col} A \subset \mathbb{R}^3$. Therefore

$$\operatorname{col} A = \mathbb{R}^3$$

If we add any two of the four remaining four columns of A to the set S, then the resultant set is a basis of $\mathbb{R}^3$. I will demonstrate this using the first two columns of A, but the same argument will hold for any other choice.

Consider $S_1 = \{\mathbf{a}_5, \mathbf{a}_1, \mathbf{a}_2\}$. Let $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ be any vector. We will show that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. From this it follows that S_1 spans $\mathbb{R}^3$. It also follows from this that S_1 is linearly independent because the choice $\mathbf{b} = 0$ has only the trivial solution.

We reduce the matrix $[\mathbf{a}_5 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}]$ to echelon form:

$$\begin{bmatrix} 7 & 2 & 1 & b_1 \\ 7 & 1 & 0 & b_2 \\ 7 & 2 & 0 & b_3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 7 & 2 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 7 & 2 & 0 & b_3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 7 & 2 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & 0 & -1 & b_3 - b_1 \end{bmatrix}$$

As this echelon form has a pivot in every column except for the augmented column, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Second Solution: We use the Spanning Set theorem.

Clearly, $\operatorname{col} A = \operatorname{Span} S_0$ where

$$S_0 = \{\mathbf{a}_1, \ldots, \mathbf{a}_5\}$$

Since $\mathbf{a}_1 = (2,1,2) = 2(1,0,0) + 1(0,1,0) + 2(0,0,1) = 2\mathbf{a}_2 + 1.\mathbf{a}_3 + 2\mathbf{a}_4 + 0.\mathbf{a}_5$, we can eliminate $\mathbf{a}_1$ from S_0 to obtain

$$S_1 = \{\mathbf{a}_2, \ldots, \mathbf{a}_5\}$$

Next we observe that $7\mathbf{a}_2 + 7\mathbf{a}_3 + 7\mathbf{a}_4 = \mathbf{a}_5$. Hence

$$\mathbf{a}_2 = \frac{1}{7}\mathbf{a}_5 - \mathbf{a}_3 - \mathbf{a}_4 \in \operatorname{Span} S_1.$$

So we eliminate $\mathbf{a}_2$ from S_1 to obtain

$$S_2 = \{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$$

If we show that S_2 is a linearly independent set it follows from the Spanning Set theorem that is a basis for $\operatorname{col} A$. So let us consider the equation

$$c_1\mathbf{a}_3 + c_2\mathbf{a}_4 + c_3\mathbf{a}_5 = 0$$

where c_1, c_2 and c_3 are real unknowns. Substituting the values of $\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ we obtain

$$\begin{bmatrix} 7c_3 \\ c_1 + 7c_3 \\ c_2 + 7c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, the only possible solution is $c_1 = c_2 = c_3 = 0$.

Rubric.

(a) 1 mark for showing that $\operatorname{col} B$ is a subspace of $\operatorname{col} A$ using any correct argument. 1 mark for justifying that no subset comparison can hold between $\operatorname{Nul} A$ and $\operatorname{Nul} B$ because they are subsets of Euclidean spaces of different dimensions.

(b) 1 mark for finding a correct basis (it should be a basis that contains the vector (7,7,7)). 2 marks for a logically sound justification.

Question 5 (10 marks)**.**

(a) (7 marks) For each $j \in \{0, \ldots, n\}$, let $p_j(x)$ be a nonzero polynomial of degree j. Show that the $\{p_0(x), \ldots, p_n(x)\}$ is a linearly independent subset of $\mathbb{P}_n$ (the vector space of all polynomials with real coefficients which have degree at most n).

(b) (3 marks) Let $S = \{1, x, x^2, (x+1)^3, x^3, 2x^4, (x-1)^5, x^5+3, 3x^4+6x^3\}$ (a subset of $\mathbb{P}_5$). Find a basis for $\operatorname{Span} S$.

Solution.

(a) We use induction on n. For $n = 0$, p_0 is a nonzero constant, and hence $\{p_0\}$ is a linearly independent set.

Assume that the result is true for $n \in \{0, \ldots, k-1\}$ where $k \in \mathbb{N}$. We show that it holds for $n = k$.

Consider the equation

$$c_0p_0 + \cdots + c_kp_k = 0$$

in real unknowns $c_0, \ldots, c_k$. Let $p_k = a_kx^k + q_k$ where a_k is a nonzero real number and q_k is a polynomial of degree less than k. Then

$$c_0p_0 + \cdots + c_k(a_kx^k + q_k) = 0$$

As the coefficient of x^k on the LHS is c_ka_k it follows that

$$c_ka_k = 0$$

Since $a_k \neq 0$ we must have $c_k = 0$. By the induction hypothesis the polynomials $p_0, \ldots, p_{k-1}$ are linearly independent, so

$$c_0 = \cdots = c_{k-1} = 0$$

(b) By part (a), the set $\{1, x, x^2, x^3, 2x^4, x^5 + 3\}$ is a basis of $\mathbb{P}_5$. Since

$$\{1, x, x^2, x^3, 2x^4, x^5 + 3\} \subset \operatorname{Span} S,$$

it follows from Theorem 1, Chapter 4, DCL that $\mathbb{P}_5$ is a subspace of $\operatorname{Span} S$. Clearly $\operatorname{Span} S$ is a subspace of $\mathbb{P}_5$. Hence

$$\mathbb{P}_5 = \operatorname{Span} S$$

Thus $\{1, x, x^2, x^3, 2x^4, x^5 + 3\}$ is a basis of $\operatorname{Span} S$.

Rubric.

(a)
- 1 mark for using induction
- 1 mark for checking base case
- 1 mark for setting up linear dependence equation
- 3 marks for identifying the highest degree coefficient ($c_k a_k$) and arguing that it must be zero and concluding that $c_k = 0$
- 1 mark for showing that the remaining coefficients are zero.

(b) 2 marks for using part (a) to find a basis for $\mathbb{P}_5$, 1 mark for showing that $\mathbb{P}_5$ is the same as Span S.