End-Semester Exam: Math-I (Linear Algebra)

Duration: 120 minutes **Maximum Marks:** 80

Question 1.

(a) (5 marks) Find an LU-factorization of the following matrix:

$$A = \begin{bmatrix} 4 & 3 & -5 & 0 \\ 4 & 5 & -7 & -6 \\ 4 & 3 & -4 & 4 \end{bmatrix}$$

(b) (5 marks) Use the LU-factorization method to solve the linear system $A\mathbf{x} = \mathbf{b}$ where A is the matrix given in part (a) and $\mathbf{b}$ is the vector given below. Write down all the calculations involved.

$$\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Solution.

(a) We reduce A to an echelon form U:

$$\begin{bmatrix} 4 & 3 & -5 & 0 \\ 4 & 5 & -7 & -6 \\ 4 & 3 & -4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 4 & 5 & -7 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & 2 & -2 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

We perform the corresponding column operations on the identity matrix to obtain L:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{C_1 \to C_1 + C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{C_1 \to C_1 + C_2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 & 0 \\ 0 & 2 & -2 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(b) We first solve $L\mathbf{y} = \mathbf{b}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \\ 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus

$$\mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Next we solve $U\mathbf{x} = \mathbf{y}$.

$$\left[\begin{array}{rrrr|r} 4 & 3 & -5 & 0 & 1 \\ 0 & 2 & -2 & -6 & -3 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_3} \left[\begin{array}{rrrr|r} 4 & 3 & -5 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 5R_3} \left[\begin{array}{rrrr|r} 4 & 3 & 0 & 20 & 11 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \to 1/2 R_2} \left[\begin{array}{rrrr|r} 4 & 3 & 0 & 20 & 11 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \left[\begin{array}{rrrr|r} 4 & 0 & 0 & 17 & 19/2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to 1/4 R_1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 17/4 & 19/8 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

Therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 19/8 \\ 1/2 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -17/4 \\ -1 \\ -4 \\ 1 \end{bmatrix}$$

Rubric.

Part (a):

- 2 marks for reducing A to an echelon form U - please deduct 1/2 mark for each calculation error
- 2 marks for finding L - 1 mark for each operation on I
- 1 mark for writing the LU factorization

Part (b):

- 2 marks for solving Ly=b. Deduct 1/2 mark for each calculation error.
- 3 marks for solving Ux=y. Deduct 1/2 mark for each calculation error.

Question 2.

(a) (5 marks) Diagonalize

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

(b) (5 marks) Define $\langle .,. \rangle : \mathbb{R}^3 \to \mathbb{R}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T A \mathbf{y}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,$$

where A is the matrix given in part (a). Show that $\langle .,. \rangle$ is an inner product.

Solution.

(a)

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -4 & 4 \\ -4 & 5-\lambda & 0 \\ 4 & 0 & 9-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (7-\lambda)(\lambda^2 - 14\lambda + 45) - 16(9-\lambda) - 16(5-\lambda) \\ &= (7-\lambda)(1-\lambda)(13-\lambda) \end{aligned}$$

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 7, \lambda_3 = 13$

Clearly, the last two rows of $A - \lambda I$ are lineary independent for all values of λ. Therefore we can find an eigenvector corresponding to each λ by finding the cross product of the last two rows of $A - \lambda I$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & 5-\lambda & 0 \\ 4 & 0 & 9-\lambda \end{vmatrix} = (5-\lambda)(9-\lambda)\hat{\imath} + 4(9-\lambda)\hat{\jmath} - 4(5-\lambda)\hat{k} = \begin{bmatrix} (5-\lambda)(9-\lambda) \\ 4(9-\lambda) \\ -4(5-\lambda) \end{bmatrix}$$

Put $\lambda = 1$ to get $\mathbf{v}_1 = \begin{bmatrix} 32 \\ 32 \\ -16 \end{bmatrix}$

Put $\lambda = 7$ to get $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 8 \\ 8 \end{bmatrix}$

Put $\lambda = 13$ to get $\mathbf{v}_3 = \begin{bmatrix} 32 \\ -16 \\ 32 \end{bmatrix}$

Therefore $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 32 & -4 & 32 \\ 32 & 8 & -16 \\ -16 & 8 & 32 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

(b) Verification of inner product properties:

Linear in the first argument:

(i) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$.

$$\begin{aligned} \langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (\mathbf{x} + \mathbf{y})^T A\mathbf{z} \\ &= \mathbf{x}^T A\mathbf{z} + \mathbf{y}^T A\mathbf{z} \\ &= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle \end{aligned}$$

(ii) Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^3$, $c \in \mathbb{R}$.

$$\begin{aligned} \langle c\mathbf{x}, \mathbf{z} \rangle &= (c\mathbf{x})^T A\mathbf{z} \\ &= c\mathbf{x}^T A\mathbf{z} \\ &= c\langle \mathbf{x}, \mathbf{z} \rangle \end{aligned}$$

Symmetry:

Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^3$.

$$\begin{aligned} \langle \mathbf{x}, \mathbf{z} \rangle &= \mathbf{x}^T A\mathbf{z} \\ &= (\mathbf{x}^T A\mathbf{z})^T \quad (\because \mathbf{x}^T A\mathbf{z} \text{ is a } 1 \times 1 \text{ matrix}) \\ &= \mathbf{z}^T A^T \mathbf{x} \\ &= \mathbf{z}^T A\mathbf{x} \quad (\because A \text{ is symmetric}) \\ &= \langle \mathbf{z}, \mathbf{x} \rangle \end{aligned}$$

Positive-definite property:

Clearly,

$$\langle \mathbf{0}, \mathbf{0} \rangle = \mathbf{0}^T A\mathbf{0} = 0$$

Next, suppose $\mathbf{x} \in \mathbb{R}^3$. Since A has strictly positive eigenvalues, the quadratic form $\mathbf{x}^T A\mathbf{x} > 0$ if $\mathbf{x} \neq \mathbf{0}$.

Rubric.

(a)
- 1/2 mark for correct characteristic polynomial
- 1 mark for correct eigenvalues
- 1/2 mark for listing the eigenvector columns of P in the **same order** as the eigenvalues in D, even if they are not computed or computed incorrectly.
- 1 mark for each correctly computed eigenvector (using any method). Please note that eigenvectors are unique only up to scalar multiples.

(b)
- 1 mark for linearity, either in first or second argument
- 1 mark for symmetry
- 1 mark for verifying that $\langle \mathbf{0}, \mathbf{0} \rangle$ is zero
- 2 marks for correctly justifying that the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite

Question 3.

(a) (5 marks) Find a QR factorization of

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

(b) (5 marks) Let $W = \operatorname{Col} A$ (where A is the matrix in part (a)). and $\mathbf{v} = (1, 1, 1, 1)$. Find vectors $\hat{\mathbf{v}} \in W$ and $\mathbf{z} \in W^{\perp}$ such that $\mathbf{v} = \hat{\mathbf{v}} + \mathbf{z}$.

Solution.

(a) We apply the QR algorithm:

$$\|\mathbf{a}_1\| = \sqrt{\mathbf{a}_1 \cdot \mathbf{a}_1} = 6$$

Define

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = (5/6, 1/6, -1/2, 1/6)$$

Next

$$\begin{aligned} \mathbf{a}_2 - \operatorname{proj}_{\mathbf{a}_1} \mathbf{a_2} &= \mathbf{a}_2 - (\mathbf{q_1} \cdot \mathbf{a}_2)\mathbf{q_1} \\ &= (9, 7, -5, 5) - ((9, 7, -5, 5) \cdot (5/6, 1/6, -1/2, 1/6))(5/6, 1/6, -1/2, 1/6) \\ &= (9, 7, -5, 5) - (10, 2, -6, 2) \\ &= (-1, 5, 1, 3) \end{aligned}$$

Define

$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - (\mathbf{q_1} \cdot \mathbf{a}_2)\mathbf{q_1}}{\|\mathbf{a}_2 - (\mathbf{q_1} \cdot \mathbf{a}_2)\mathbf{q_1}\|} = (-1/6, 5/6, 1/6, 1/2)$$

Thus

$$Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -1/2 & 1/6 \\ 1/6 & 1/2 \end{bmatrix}$$

Next

$$R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -1/2 & 1/6 \\ -1/6 & 5/6 & 1/6 & 1/2 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

(b) Observe that the columns of Q are an orthonormal basis of W, as they were obtained using the Gram-Schmidt process on the columns of A and normalizing the vectors obtained.

Thus the projection of $\mathbf{v}$ onto W is

$$\begin{aligned} \hat{\mathbf{v}} &= (\mathbf{v} \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{v} \cdot \mathbf{q}_2)\mathbf{q}_2 \\ &= (5/6 + 1/6 - 1/2 + 1/6))(5/6, 1/6, -1/2, 1/6) + (-1/6 + 5/6 + 1/6 + 1/2)(-1/6, 5/6, 1/6, 1/2) \\ &= (5/9, 1/9, -1/3, 1/9) + (-2/9, 10/9, 2/9, 2/3) \\ &= (1/3, 11/9, -1/9, 7/9) \end{aligned}$$

and

$$\mathbf{z} = \mathbf{v} - \hat{\mathbf{v}} = (2/3, -2/9, 10/9, 2/9)$$

Rubric.

(a)
- 1 mark for finding $\mathbf{q}_1$
- 1 mark for computing orthogonal complement of $\mathbf{a}_2$ in direction of $\mathbf{a}_2$
- 1 mark for normalizing to find $\mathbf{q}_2$
- 1 mark for writing the correct formula for R
- 1 mark for computing R

(b)
- 1 mark for observing that the columns of Q are an orthonormal basis of W
- 1 mark for justifying that the columns of Q are an orthonormal basis of W correctly
- 1 mark for the correct formula for the orthogonal projection onto W
- 1 mark for substituting correct values into the formula and finding $\hat{\mathbf{v}}$
- 1 mark for finding $\mathbf{z}$

Question 4. (10 marks) Let W be a real vector space. Let V be a non-empty set and let $f : V \to W$ be a bijection (i.e. a 1-1 and onto function). Define

$$v_1 \oplus v_2 := f^{-1}(f(v_1) + f(v_2)), \quad \forall v_1, v_2 \in V$$

and

$$c \star v := f^{-1}(cf(v)), \quad \forall c \in \mathbb{R}, v \in V.$$

Show that V is a vector space under the operations $\oplus$ and $\star$.

Solution.

Verification of axioms:

(i) Closure under vector addition:

Let $v_1, v_2 \in V$. Since W is closed under vector addition, $f(v_1) + f(v_2) \in W$. Since f is a bijection, $f^{-1}(f(v_1) + f(v_2)) \in V$. Therefore V is closed under the operation $\oplus$.

(ii) Commutativity of vector addition in V:

Let $w_1, w_2 \in V$. Then

$$\begin{aligned} w_1 \oplus w_2 &= f^{-1}(f(w_1) + f(w_2)) \\ &= f^{-1}(f(w_2) + f(w_1)) \\ &= w_2 \oplus w_1 \end{aligned}$$

(iii) Associativity of vector addition in V:

Let $w_1, w_2, w_3 \in V$. Then

$$\begin{aligned} w_1 \oplus (w_2 \oplus w_3) &= f^{-1}(f(w_1) + f(w_2 \oplus w_3)) \\ &= f^{-1}(f(w_1) + f(f^{-1}(f(w_2) + f(w_3)))) \\ &= f^{-1}(f(w_1) + f(w_2) + f(w_3)) \end{aligned}$$

Since we have already established commutativity,

$$\begin{aligned} (w_1 \oplus w_2) \oplus w_3 &= w_3 \oplus (w_1 \oplus w_2) \\ &= f^{-1}(f(w_3) + f(w_1) + f(w_2)) \\ &= f^{-1}(f(w_1) + f(w_2) + f(w_3)) \end{aligned}$$

(iv) Existence of Additive Identity in V:

Define

$$0_V := f^{-1}(\mathbf{0}),$$

where $\mathbf{0}$ is a zero vector in W. Let $w \in V$.

$$\begin{aligned} 0_V \oplus w &= f^{-1}(f(0_V) + f(w)) \\ &= f^{-1}(\mathbf{0} + f(w)) \\ &= f^{-1}(f(w)) \\ &= w \end{aligned}$$

(v) Existence of Additive inverse in V:

Let $w \in V$. Consider $u = f^{-1}(-f(w))$.

Then

$$\begin{aligned} w \oplus u &= f^{-1}(f(w) + f(u)) \\ &= f^{-1}(f(w) + (-f(w))) \\ &= f^{-1}(\mathbf{0}) \\ &= 0_V \end{aligned}$$

(vi) Closure with respect to scalar multiplication:

Let $v \in V, c \in \mathbb{R}$. Since W is closed under scalar multiplication, $cf(v) \in W$. Since f is a bijection, $f^{-1}(cf(v)) \in V$. Therefore V is closed under the operation $\star$.

(vii) First Distributive Law:

Let $c \in \mathbb{R}, w_1, w_2 \in V$. Then

$$\begin{aligned} c \star (w_1 \oplus w_2) &= f^{-1}(cf(w_1 \oplus w_2)) \\ &= f^{-1}(c(f(w_1) + f(w_2))) \\ &= f^{-1}(cf(w_1) + cf(w_2)) \end{aligned}$$

$$\begin{aligned} (c \star w_1) \oplus (c \star w_2) &= f^{-1}(f(c \star w_1) + f(c \star w_2)) \\ &= f^{-1}(cf(w_1) + cf(w_2)) \end{aligned}$$

Hence

$$c \star (w_1 \oplus w_2) = (c \star w_1) \oplus (c \star w_2)$$

(viii) Second Distributive Law:

Let $c_1, c_2 \in \mathbb{R}, w \in V$. Then

$$\begin{aligned} (c_1 + c_2) \star w &= f^{-1}((c_1 + c_2)f(w)) \\ &= f^{-1}(c_1 f(w) + c_2 f(w)) \end{aligned}$$

$$\begin{aligned} (c_1 \star w) \oplus (c_2 \star w) &= f^{-1}(f(c_1 \star w) + f(c_2 \star w)) \\ &= f(c_1 f(w) + c_2 f(w)) \end{aligned}$$

Therefore $(c_1 + c_2) \star w = (c_1 \star w) \oplus (c_2 \star w)$.

(ix) Let $c_1, c_2 \in \mathbb{R}, w \in V$.

$$\begin{aligned} c_1 \star (c_2 \star w) &= f^{-1}(c_1 f(c_2 \star w)) \\ &= f^{-1}(c_1 c_2 f(w)) \\ &= c_1 c_2 \star w \end{aligned}$$

(x) Let $w \in V$.

$$\begin{aligned} 1 \star w &= f^{-1}(1.f(w)) \\ &= f^{-1}(f(w)) \\ &= w \end{aligned}$$

Rubric. 1 mark for each correctly verified axiom.

Question 5.

(a) (5 marks) We define the *trace* of an $n \times n$ matrix A to be

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii},$$

where a_{ij} are the entries of A. Show that if A and B are $n \times n$ matrices then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

(b) (2 marks) Show that if A and B are similar matrices then $\operatorname{tr}(A) = \operatorname{tr}(B)$

(c) (3 marks) Show that the trace of a diagonalizable matrix equals the sum of its eigenvalues.

Solution.

(a)

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ik} a_{ki} \\
&= \operatorname{tr}(BA)
\end{aligned}$$

(b) Let A and B be similar matrices. This means there exists an invertible matrix P such that $B = PAP^{-1}$. Using part (a), we get

$$\operatorname{tr}((P^{-1})(PA)) = \operatorname{tr}((PA)(P^{-1}))$$

Therefore

$$\operatorname{tr}(A) = \operatorname{tr}(B)$$

(c) Suppose A is diagonalizable. Then A is similar to a diagonal matrix D. By part (b),

$$\operatorname{tr}(A) = \operatorname{tr}(D)$$

We know that the eigenvalues of a diagonal matrix are its diagonal entries (done in class). Therefore the trace of D is the sum of the eigenvalues of D.

We also know that similar matrices have the same eigenvalues with the same multiplicities (done in class).

Therefore the sum of the eigenvalues of A equals the sum of the eigenvalues of D.

Therefore $\operatorname{tr} A$ equals the sum of the eigenvalues of A.

Rubric.

(a)

- 1 mark for the correct formula for the diagonal entries of AB
- 1 mark for expressing the trace of AB correctly as a double summation (as shown)
- 3 marks for correctly reversing the order of the summation so that the result matches with the trace of BA

(b)

- 1 mark for recognizing that A and B are products of the matrices AP and P^{-1} in reversed order
- 1 mark for applying the result from part (a) and reaching the desired conclusion

(c)

- 1 mark for writing that A is similar to a diagonal matrix, say D
- 1/2 mark for concluding that A and D have the same trace
- 1/2 mark for writing that the eigenvalues of a diagonal matrix are its eigenvalues
- 1/2 for writing that similar matrices have the same eigenvalues with same multiplicities
- 1/2 mark for concluding that the trace of A equals the sum of its eigenvalues as a consequence of the above

Question 6.

(a) (5 marks) Find an orthonormal basis for $\mathbb{P}_2$ with the inner product

$$\langle p(t), q(t)\rangle = \int_{-1}^{1} p(t)q(t)\,\mathrm{dt}\,.$$

(b) (5 marks) Define $T : \mathbb{R}^3 \to \mathbb{P}_2$ by

$$T(a, b, c) = ap_1 + bp_2 + cp_3$$

where $\{p_1, p_2, p_3\}$ is the orthonormal basis you found in part (a). Show that T is a linear transformation. Define

$$\mathbf{v} \star \mathbf{w} := \langle T(\mathbf{v}), T(\mathbf{w})\rangle \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^3.$$

Find an orthonormal basis of $\mathbb{R}^3$ with respect to the inner product $\star$.

Solution.

(a) We apply Gram-Schmidt to the basis $\{1, t, t^2\}$ and normalize the resulting polynomials. Let

$$q_1 = 1, \quad q_2 = t, \quad q_3 = t^2$$

$$\|q_1\|^2 = \int_{-1}^{1} \mathrm{dt} = 2$$

Let $p_1 = 1/\sqrt{2}$. Next

$$\begin{aligned} q_2 - \langle q_2, p_1\rangle p_1 &= t - \frac{1}{2}\int_{-1}^{1} t\,\mathrm{dt} \\ &= t \end{aligned}$$

Normalizing, we obtain

$$p_2 = \frac{q_2 - \langle q_2, p_1\rangle p_1}{\|q_2 - \langle q_2, p_1\rangle p_1\|} = \frac{t}{\sqrt{\int_{-1}^{1} t^2\,\mathrm{dt}}} = \sqrt{\frac{3}{2}}t$$

Next,

$$\begin{aligned} q_3 - \langle q_3, p_1\rangle p_1 - \langle q_3, p_2\rangle p_2 &= t^2 - \frac{1}{2}\int_{-1}^{1} t^2\,\mathrm{dt} - \frac{3}{2}\int_{-1}^{1} t^3\,\mathrm{dt} \\ &= t^2 - \frac{1}{3} \end{aligned}$$

Normalizing, we obtain

$$\begin{aligned} p_3 &= \frac{t^2 - 1/3}{\sqrt{\int_{-1}^{1}(t^2 - 1/3)^2\,\mathrm{dt}}} \\ &= \frac{\sqrt{45}}{2\sqrt{2}}\left(t^2 - \frac{1}{3}\right) \end{aligned}$$

(b) Let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T(\alpha\mathbf{x} + \beta\mathbf{y}) &= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\ &= (\alpha x_1 + \beta y_1)p_1 + (\alpha x_2 + \beta y_2)p_2 + (\alpha x_3 + \beta y_3)p_3 \\ &= \alpha(x_1 p_1 + x_2 p_2 + x_3 p_3) + \beta(y_1 p_1 + y_2 p_2 + y_3 p_3) \\ &= \alpha T(\mathbf{x}) + \beta T(\mathbf{y}) \end{aligned}$$

Therefore T is a linear transformation.

If $T(\mathbf{x}) = 0$ then $x_1 p_1 + x_2 p_2 + x_3 p_3 = 0$. Since $\{p_1, p_2, p_3\}$ is a linearly independent set, this can only happen if $\mathbf{x} = 0$. Therefore T is 1-1. By a result covered in Quiz 2, $\star$ is an inner product.

The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis with respect to $\star$. Indeed for $i, j \in \{1, 2, 3\}$,

$$\mathbf{e}_i \star \mathbf{e}_j = \langle T(\mathbf{e}_i), T(\mathbf{e}_j)\rangle = \langle p_i, p_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Rubric.

(a)
- 1/2 mark for identifying a basis of $\mathbb{P}_2$
- 1/2 mark for finding p_1
- 1 mark for step 2 of Gram-Schmidt
- 1/2 mark for normalizing to obtain p_2
- 1.5 marks for step 3 of Gram-Schmidt
- 1 mark for normalizing to obtain p_3

(b)
- 1 mark for showing that T is linear
- 1 mark for identifying the standard basis as an orthonormal basis with respect to $\star$
- 3 marks for correctly justifying why it is orthonormal **with respect to** $\star$ (please **do not award marks** if the student uses the usual Euclidean dot product instead of $\star$)

Question 7 (10 marks)**.** Let A be a diagonalizable $n \times n$ matrix having exactly three distinct nonzero eigenvalues α, β and γ, with multiplicities p, q and r respectively. Find the largest value of m for which $\{I, A, A^2, \ldots, A^m\}$ is a linearly independent subset of $M_{n\times n}(\mathbb{R})$.

Solution. Let $A = PDP^{-1}$ where D is diagonal and P is invertible.

Let $p(x) = (x-\alpha)(x-\beta)(x-\gamma)$. Then

$$p(A) = Pp(D)P^{-1} \quad \text{(covered in class)}$$

If d_{jj} is the j-th diagonal entry of D then $p(d_{jj})$ is the j-th diagonal entry of the diagonal matrix $p(D)$ (also covered in class).

Therefore the diagonal entries of $p(D)$ are either $p(\alpha), p(\beta)$ or $p(\gamma)$. Thus $p(D) = 0$. Hence $p(A) = 0$.

As $p(x)$ is a polynomial of degree 3, the set $\{I, A, A^2, A^3\}$ is linearly dependent.

We will show that the set $\{I, A, A^2\}$ is linearly independent, and conclude that the largest possible value of m is 2.

Suppose $c_1 I + c_2 A + c_3 A^3 = 0$. Put $q(x) = c_1 + c_2 x + c_3 x^2$. As argued earlier,

$$q(A) = 0 \implies q(D) = 0 \implies q(\alpha) = q(\beta) = q(\gamma) = 0.$$

However, q is at most quadratic, so cannot have 3 distinct roots, unless it is the zero polynomial, in which case $c_1 = c_2 = c_3 = 0$.

Rubric.

- 5 marks for showing that the set $\{I, A, A^2, A^3\}$ is linearly dependent
- 5 marks for showing that the set $\{I, A, A^2\}$ is linearly independent

Question 8. (10 marks) Define $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^2$, where

$$A = \begin{bmatrix} 1 & -4/5 \\ 4 & -11/5 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_\mathcal{B}$ is orthogonal.

Solution.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -4/5 \\ 4 & -11/5-\lambda \end{bmatrix}$$

$$\det A - \lambda I = \lambda^2 + \frac{6}{5}\lambda + 1$$

Eigenvalues: $-\dfrac{3}{5} \pm \dfrac{4}{5}i$

We select $\lambda = -\frac{3}{5} - \frac{4}{5}i$ and solve $(A - \lambda I)\mathbf{x} = 0$. Let us bring $A - \lambda I$ to echelon form:

$$\begin{bmatrix} \frac{8+4i}{5} & -\frac{4}{5} \\ 4 & \frac{-8+4i}{5} \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & \frac{-8+4i}{5} \\ \frac{8+4i}{5} & -\frac{4}{5} \end{bmatrix} \xrightarrow{R_2 \to R_2 - (2+i)/5R_1} \begin{bmatrix} 4 & \frac{-8+4i}{5} \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to 1/4R_1} \begin{bmatrix} 1 & \frac{-2+i}{5} \\ 0 & 0 \end{bmatrix}$$

Eigenvector:

$$\mathbf{v} = \begin{bmatrix} \frac{2-i}{5} \\ 1 \end{bmatrix}$$

Required basis:

$$\mathcal{B} = \{\operatorname{Re}\mathbf{v}, \operatorname{Im}\mathbf{v}\} = \left\{ \begin{bmatrix} 2/5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/5 \\ 0 \end{bmatrix} \right\}$$

Rubric.

- 1 mark for characteristic polynomial
- 1 mark for eigenvalues
- 4 marks for finding a complex eigenvector correctly, using either of the two eigenvalues
- 3 marks for identifying the required basis as the real and imaginary parts of the eigenvector
- 1 mark for the answer