MC-208 Linear Algebra

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Assignment - 3

Ans 1.

$$\begin{bmatrix} 4 & 3 & 0 \\ 2 & 1 & 0 \\ 5 & 7 & 9 \end{bmatrix} = A$$

Characteristic polynomial

$$\Rightarrow \lambda^3 - tr(A)\lambda^2 + \begin{bmatrix} \text{sum of minors} \\ \text{along diagonals} \end{bmatrix} \lambda - det(A)$$

$tr(A) = 14$ Sum of minors along diagonal $= 43$

$det(A) = -18$

$$\Rightarrow \lambda^3 - 14\lambda^2 + 43\lambda + 18$$

$$\Rightarrow (\lambda - 9)(\lambda^2 - 5\lambda - 2)$$

$$\Rightarrow \boxed{(\lambda - 9)\left(\lambda + \frac{5 \mp \sqrt{33}}{2}\right)\left(\lambda - \frac{5 \mp \sqrt{33}}{2}\right)}$$

$$\begin{array}{r} \lambda^2 - 5\lambda - 2 \\ \lambda - 9 \overline{\smash{)}\lambda^3 - 14\lambda^2 + 43\lambda + 18} \\ \lambda^3 - 9\lambda^2 \\ \hline -5\lambda^2 + 43\lambda \\ -5\lambda^2 + 45\lambda \\ \hline -2\lambda + 18 \\ -2\lambda + 18 \\ \hline 0 \end{array}$$

In this case the characteristic & the minimal polynomial are the same as both of them do ~~not~~ have the same number of irreducible factors.

Ans 2. a) $T : R^2 \longrightarrow R^2$ $T(x, y) = (2x - 9y, 3x - 5y)$

Usual basis of $R^2 = \{ \underbrace{(1, 0)}_{v_1}, \underbrace{(0, 1)}_{v_2} \}$

$T(v_1) = (2, 3)$

$T(v_2) = (-9, -5)$

$a(1,0) + b(0,1) = (2,3)$

$(a,b) \equiv (2,3)$

$a(1,0) + b(0,1) = (-9,-5)$

$(a,b) \equiv (-9,-5)$

Matrix Representation of $T(x,y)$ $= \begin{bmatrix} 2, & 3 \\ -9 & -5 \end{bmatrix} =$

$\det(T) = -10 + 27 = \underline{\underline{17}}$

b) $T: R^3 \to R^3$ $\quad T(x,y,z) = (3x-2z, 5y+7z, x+y+z)$

Usual basis for $R^3 = \{ \underbrace{(1,0,0)}_{v_1} \underbrace{(0,1,0)}_{v_2} \underbrace{(0,0,1)}_{v_3} \}$

$T(v_1) = (3,0,1)$

$T(v_2) = (0,5,1)$

$T(v_3) = (-2,7,1)$

$a(1,0,0) + b(0,1,0) + c(0,0,1) = (3,0,1)$

$(a, b, c) \equiv (3,0,1)$

$a(1,0,0) + b(0,1,0) + c(0,0,1) = (0,5,1)$

$(a,b,c) \equiv (0,5,1)$

$a(1,0,0) + b(0,1,0) + c(0,0,1) = (-2,7,1)$

$(a,b,c) \equiv (-2,7,1)$

Matrix Rep: $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 1 \\ -2 & 7 & 1 \end{bmatrix}$

$\det(T) = -6 + 10 = \underline{\underline{4}}$

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Ans 3.

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} = A$$

$$chr(A) = \lambda^3 - tr(A)\,\lambda^2 + \begin{bmatrix} \text{Sum of minors} \\ \text{along the} \\ \text{diagonal} \end{bmatrix} \lambda - det(A)$$

$tr(A) = 0$

$det(A) = 1(-2) + 3(-6) + 3(12)$

$= -2 - 18 + 36 = 16$

$$\begin{bmatrix} \text{Sum of minors} \\ \text{along the diag} \end{bmatrix} = -2 - 14 + 4 = -12$$

$$\lambda^3 - 12\lambda - 16 = 0$$

$$\begin{array}{r} \lambda^2 - 2\lambda - 8 \\ \lambda + 2 \overline{\big)\ \lambda^3 - 12\lambda - 16} \\ \lambda^3 + 2\lambda^2 \\ \hline -2\lambda^2 - 12\lambda \\ -2\lambda^2 - 4\lambda \\ \hline -8\lambda - 16 \\ -8\lambda - 16 \\ \hline 0 \end{array}$$

$(\lambda+2)(\lambda^2 - 2\lambda - 8)$

$(\lambda+2)(\lambda+2)(\lambda-4)$

eigen values are $\lambda_1 = -2$, $\lambda_2 = 4$

There are only two linearly independent eigen vectors that exist

for $\lambda = -2$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} =$$

$3x - 3y + 3z = 0$

$3x - 3y + 3z = 0$

$6x - 6y + 6z = 0$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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for $\lambda = 4$

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$-3x - 3y + 3z = 0$ $\qquad$ $x + y - z = 0$

$3x - 9y + 3z = 0$ $\qquad$ $x - 2y + z = 0$

$6x - 6y = 0$ $\qquad$ $x = y$

~~Eigen Value~~ $\qquad$ ~~Eigenvector~~

~~4~~ $\qquad$ ~~(1, 1, 0)~~

~~-2~~ $\qquad$ ~~(1, 0, -1)~~

$x + y - z = 0$ $\qquad$ $2x = z$

$x - 2y + z = 0$

$x = y$

So, the eigenvalues are 4 & -2 and the eigen vectors are

$(1/2, 1/2, 1)$ & $\{(1,1,0), (1,0,-1)\}$

Ans 4. $T : R^2 \longrightarrow R^2$ $\qquad$ $T(x,y) = (3x+3y, x+5y)$

Usual basis of R^2 : $\{\underbrace{(1,0)}_{v_1}, \underbrace{(0,1)}_{v_2}\}$

$T(v_1) = (3,1)$

$T(v_2) = (3,5)$

Matrix Rep. $= \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$

$Chr(T) = \lambda^2 - 8\lambda + 12 = 0$

$\lambda^2 - 6\lambda - 2\lambda + 12 = 0$

$(\lambda - 6)(\lambda - 2) = 0$

Eigenvalues are 6 & 2

for $\lambda = 6$ $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-3x + y = 0$

$3x - y = 0$

Eigenvectors $= \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

for $\lambda = 2$ $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x + y = 0$

$3x + 3y = 0$

eigenvectors $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenspace $= \{ (1/3, 1), (1, -1) \}$

Ans 5. $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} = A$

$tr(A) = 7$

$det(A) = 12$

sum of minors $= 4 + 6 + 6$

$= 16$

$chr(A) = \lambda^3 - 7\lambda^2 + 16\lambda - 12$

$$\lambda^2 - 5\lambda + 6$$
$$\lambda - 2 \overline{\big) \lambda^3 - 7\lambda^2 + 16\lambda - 12}$$
$$\lambda^3 - 2\lambda^2$$
$$-5\lambda^2 + 16\lambda$$
$$-5\lambda^2 + 10\lambda$$
$$6\lambda - 12$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12$$

$$(\lambda-2)(\lambda^2-5\lambda+6)$$
$$(\lambda-2)(\lambda-2)(\lambda-3)$$

$(\lambda-2)(\lambda-3)$ $\qquad$ $(\lambda-2)^2(\lambda-3)$

$\Downarrow$ By cayley hamilton thm $\rightarrow$ True

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the minimal polynomial is
$$(\lambda-2)(\lambda-3) = \lambda^2 - 5\lambda + 6$$

Ans 6. $\begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}$

$tr(A) = 5$
sum $= 9 - 4 + 3 = 8$
$\det(A) = -3(9) - 3(-11) - 2(1)$
$= -27 + 33 - 2 = 4$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda^2 - 4\lambda + 4$$
$$\lambda - 1 \overline{\big) \lambda^3 - 5\lambda^2 + 8\lambda - 4}$$
$$\lambda^3 - \lambda^2$$
$$-4\lambda^2 + 8\lambda$$
$$-4\lambda^2 + 4\lambda$$
$$4\lambda - 4$$

$$(\lambda-1)(\lambda-2)^2$$

$$\begin{bmatrix} -4 & 3 & -2 \\ -7 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{bmatrix}$$

$= \Big[1 \quad -$

The matrix chr and minimal polynomial are the same.

$$J = \begin{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

To find $\Rightarrow$ Q where $J = Q^{-1} A Q$

for $\lambda = 2$ $\quad A - 2I$

$$\begin{bmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x + 3y - 2z = 0$ $\qquad x = y$

$-7x + 4y - 3z = 0$

$x - y = 0$ $\qquad x = y = -z$

$2x + 2z = 0$ $\qquad x = -z$

$3x + 3z = 0$

eigen vector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = 1$

$$\begin{bmatrix} -4 & 3 & -2 \\ -7 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x + 3y - 2z = 0$ $\qquad$ eigen Vector $= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$-7x + 5y - 3z = 0$

$x - y + z = 0$ $\qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Ans 7. Let f be the minimal polynomial of linear operator T, over a vector space S.

$\Rightarrow f(T) = 0$

Let $g(t)$ be ~~an~~ another polynomial such that $g(T) = 0$

then; $\deg(f) \leq \deg(g)$

[By the defn of minimal polynomial]

By division theorem of polynomial forms over field, there exists polynomials $q(t)$ & $r(t)$ such that

$$fq + r = g$$

where $\deg(r) < \deg(f)$

Putting T in the above eqn, we get

$$f(T).q(T) + r(T) = g(t)$$

$$\Rightarrow r(T) = 0$$

Contradicts the minimality

Hence the minimal polynomial is always unique.

Ans 8 $A = \begin{bmatrix} c & c & c \\ c & c & c \\ c & c & c \end{bmatrix}$

$tr(A) = 3c$

$det(A) = 0$

sum of minors $= 0$

$chr(A) = \lambda^3 - (3c)\lambda^2 = 0$

$\lambda^2(\lambda - 3c) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 3c$

$$\begin{bmatrix} c & c & c \\ c & c & c \\ c & c & c \end{bmatrix}\begin{bmatrix} -2c & c & c \\ c & -2c & c \\ c & c & -2c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

minimal polynomial $m(T) = \lambda(\lambda - 3c)$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3c \end{bmatrix}$$

Ans 9. $\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$ $tr(A) = 6$

$det(A) = -20 - 21 = -41$

$sum = 6 - 7 - 7 = -8$

$\lambda^3 - 6\lambda^2 - 8\lambda + 41$

By C.H thm

$A^3 - 6A^2 - 8A = -41$

Multiplying by A^{-1}

$$A^2 - 6A - 8 = -41A^{-1}$$

$$\begin{bmatrix} 20 & 3 & 8 \\ 27 & 13 & 18 \\ 20 & 5 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 12 \\ 54 & 12 & 0 \\ 30 & 0 & 18 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\boxed{\frac{-1}{41}\begin{bmatrix} 6 & -3 & -4 \\ -27 & -7 & 18 \\ -10 & 5 & -7 \end{bmatrix} = A^{-1}}$$

Ans16 $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

$\det(A) = 3(2) - 1(2) - 1(0)$
$= 6 - 2 = 4$

$tr(A) = 5$

sum of minors along diag $= 2 + 2 + 4$
$= 8$

$$chr(A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$8 = 20 + 160 - 4$

$$\begin{array}{r} \lambda^2 - 3\lambda + 2 \\ \lambda - 2 \overline{) \lambda^3 - 5\lambda^2 + 8\lambda - 4} \\ \underline{\lambda^3 - 2\lambda^2} \\ -3\lambda^2 + 8\lambda \\ \underline{-3\lambda^2 + 6\lambda} \\ 2\lambda - 4 \end{array}$$

$chr(A) = (\lambda - 2)(\lambda - 2)(\lambda - 1)$
$= (\lambda - 2)^2(\lambda - 1)$

for $\lambda = 2$ $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x + y - z = 0$
$2x = z$ $\qquad x = 1/2$
$2x + 2y - 2z = 0$ $\qquad z = 1$

eigenvector = $\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$

for $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + y - z = 0$

$2x + y - z = 0$

$2x + 2y - z = 0$

eigenvector = $\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

There are only 2 linearly independent eigenvectors.

$\therefore$ The given matrix is not similar to a diagonal matrix over R or C.