Missile Target Defender Game

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June 29, 2021

1 Introduction

I have looked a solution to the Missile Target Defender game: A spin on the 2 player pursuit evasion game, this 3 player game includes an additional player i.e the defender that tries to stop the pursuer from intercepting the target. On inspecting the literature, most methods split the timeline into 2 parts. The first is a game between the missile, defender and the target and the second part is a game solely between the pursuer and evader. This is effected by considering two different cost functions or considering an impulse function. Based on the terminal weights, game has been classified as between decisive players, bad defender, bad pursuer and solution to each case has been obtained analytically.

All of the solutions I've looked at considers the reduced order problem through terminal projection.[1] [2]. The problem has also been solved for the case when the controls of each player is constrained. [4]. Sufficiency conditions for the existence of a solution have been derived[5]. This problem has been solved analytically by Rusnak et al. [1]. This has been solved as the lady, the bandit and the body guard game in literature [3].

However, I've adopted the solution obtained from the Differential Riccati equation of the modified problem. As in literature, the optimisation problem has been modified and expressed using a state vector constructed from projected states derived through the state transition matrices. The control for each player has been obtained by integrating backwards in time based on terminal boundary conditions twice for two sepaprate time periods. A sufficient condition to ensure saddle point solution to exist has been stated and the system has been simulated on MATLAB. The defender is assumed to stop acting after the first duration of time. Assumption has been made that the missile, target and defender are decisive and their control actions are unbounded and simulation has been designed as such.

2 Problem Formulation

3 player game has been set up. Let y_M, y_T, y_D denote the positions of the missile, target and the defender respectively as shownn in Fig. 1

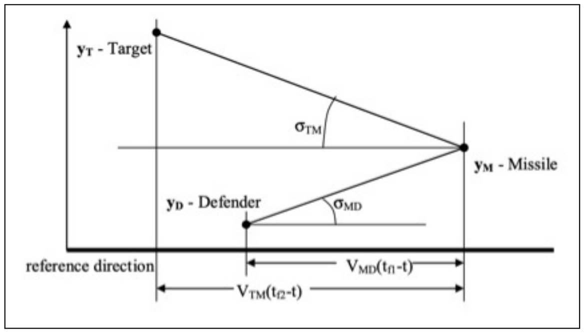

Figure 1: Engagement Geometry

V_{MD}, V_{TM} are the closing velocities of the Missile-Target and Missile-Defender respectively and σ_{MD}, σ_{TM} are the Defender-to-Missile and Target-to-Missile line-of-sight angles respectively.

The missile, target and defenders are assumed to have acceleration control perpendicular to the reference direction and are given by a_{Mc}, a_{Tc}, a_{Dc} respectively.

The dynamics of this system is given as

$$\dot{x} = Ax + Ba_{Mc} + Ca_{Tc} + Da_{Dc}$$

Here x is the state vector defined as

$$x = \begin{bmatrix} y_M - y_T \\ y_D - y_M \end{bmatrix}$$

Let t_{f1} be the the expected interception moment of the Missile by the Defender and t_{f2} be the the expected interception moment of the Target by the Missile. In the initial optimisation problem, the missile tries to minimise it's relative position w.r.t the target at the expected moment of interception while the target tries to maximise it. The defender tries to minimise the relative position of the it w.r.t the missile at the expected moment of interception with the missile while the missile maximises this. All 3 players individually also wish to achieve their objective by using the least amount of control effort as possible. This can be modelled as

At t_{f1},

$$\min_{a_{Mc}} \max_{a_{Dc}} -[y_M(t_{f1}) - y_D(t_{f1})]^2$$

At t_{f2},

$$\min_{a_{Mc}} \max_{a_{Tc}} [y_M(t_{f1}) - y_T(t_{f1})]^2$$

Control effort is minimised as follows

$$\min_{a_{Mc}}\{\int_0^{t_{f2}} a_{Mc}^T R_M a_{Mc} dt\}$$
$$\min_{a_{Tc}}\{\int_0^{t_{f2}} a_{Tc}^T R_T a_{Tc} dt\}$$
$$\min_{a_{Dc}}\{\int_0^{t_{f1}} a_{Dc}^T R_D a_{Dc} dt\}$$

This multi objective optimisation problem has been modified to a reduced order optimisation problem with new projected states for the state vector x_β given by $\hat{x}_\beta$. Let $\phi_\beta(t_f, t_0)$ be the State transition matrix for the missile, target or defender given by

$$\phi_\beta(t_f, t) = e^{A_\beta(t_f - t)}$$

Here β can be the Missile, Target or Defender. The projected states are now given by

$$\hat{x}_\beta = \phi_\beta(t_f, t) x_\beta$$

Using these projected states, the new state vector can be constructed in terms of the zero effort miss as

$$Z = \begin{bmatrix} Z_{MT} \\ Z_{DM} \end{bmatrix} = \begin{bmatrix} \hat{x}_M - \hat{x}_T \\ \hat{x}_D - \hat{x}_M \end{bmatrix} \tag{1}$$

The dynamics of the zero effort miss are given as

$$\dot{Z} = B a_{Mc} + C a_{Tc} + D(t) a_{Dc} \tag{2}$$

Here D(t) is time varying and given by

$$D(t) = \begin{cases} D, & \text{if } 0 \leq t \leq t_{f1} \\ 0, & t_{f1} < t \leq t_{f2} \end{cases}$$

Cost function J defined as

$$J = \frac{1}{2}\{Z^T(t_{f2}) G_{MT} Z(t_{f2}) + \int_0^{t_{f2}} [-Z(t) G_{DM} \delta(t - t_{f1}) Z(t) + a_{Mc}^T R_M a_{Mc} - a_{Tc}^T R_T a_{Tc} - a_{Dc}^T R_D a_{Dc}] dt\} \tag{3}$$

Here $\delta(.)$ is the unit impulse function defined as

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

3 Control Design Method

Now that the optimisation problem has been posed, the solution can be obtained by constructing the Hamiltonian as

$$H = \frac{1}{2}\left\{-Z(t)G_{DM}\delta(t-t_{f1})Z(t) + a_{Mc}^T R_M a_{Mc} - a_{Tc}^T R_T a_{Tc} - a_{Dc}^T R_D a_{Dc}\right\} + \lambda^T[Ba_{Mc} + Ca_{Tc} + D(t)a_{Dc}]$$

Based on the Hamiltonian, the co-state equation is obtained from the condition

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

Using above equation,

$$\dot{\lambda}(t) = G_{DM}\delta(t-t_{f1})Z(t)$$

Subject to the boundary condition

$$\lambda(t_{f2}) = G_{MT}Z(t_{f2})$$

Let the co-state be expressed as

$$\lambda(t) = P(t)Z(t)$$

We obtain the optimal control terms from the Hamiltonian using the condition

$$[\frac{\partial H}{\partial u}]_{u=a_\beta^*} = 0$$

Here β can be acceleration of the Missile, Target or Defender.

We obtain the optimal control terms as

$$a_{Mc}^* = -R_M^{-1}B^T P(t)Z(t) \tag{4}$$
$$a_{Tc}^* = R_T^{-1}C^T P(t)Z(t) \tag{5}$$
$$a_{Dc}^* = R_D^{-1}D(t)^T P(t)Z(t) \tag{6}$$

From the co-state dynamics equation we obtain an equation for P(t) as

$$-\dot{P}(t) = -G_{DM}\delta(t-t_{f1}) - P[BR_M^{-1}B^T - CR_T^{-1}C^T - D(t)R_D^{-1}D(t)^T]P \tag{7}$$

Boundary conditions obtained are:

$$P(t_{f2}) = G_{MT}$$
$$P(t_{f1}^-) = P(t_{f1}^+) - G_{DM}$$

The control terms are obtained by integrating Eq. 7 backwards in time. To effect this we assume a psedovariable $\tau = (t_{f2} - t)$ and propagate Eq. 7 forwards with the terminal conditions as initial states

$$\dot{P}(\tau) = -G_{DM}\delta(\tau-t_{f1}) - P[BR_M^{-1}B^T - CR_T^{-1}C^T - D(\tau)R_D^{-1}D(\tau)^T]P \tag{8}$$

3.1 Sufficiency Conditions

It is important to examine the conditions for a solution to the game to exist.Following closely along [3] Assuming a solution to Eq. 7 to exists with boundary condition $P(t_f) = Q_f$,

Define M as

$$M = Z_0^T P Z_0 - Z(t_f)^T Q_f Z(t_f) + \int_0^{t_f} \frac{d}{dt}(Z^T P Z)dt \geq 0$$

The latter term in the equation above

$$\int_0^{t_f} \frac{d}{dt}(Z^T P Z)dt = \int_0^{t_f} (\dot{Z}^T P Z + Z^T \dot{P} Z + Z^T P \dot{Z})dt)$$

Substituting optimal control terms from Eq. 6 and using the optimal solution we obtain

$$J = Z_0 P_0 Z_0 + \int_0^{t_f} (a_M^* - a_M)^T R_M (a_M^* - a_M) - (a_T^* - a_T)^T R_T (a_T^* - a_T) - (a_D^* - a_D)^T R_D (a_D^* - a_D) \quad (9)$$

From the above equation we obtain that the necessary condition for optimality is the positive indices $R_M, R_T, R_D > 0$

Another condition from [3] is on the terminal weights $G_{MT} \to \infty$, $G_{DM} \to \infty$

3.2 Nash Equilibrium Solution

From the expression for the cost function in Eq. 9 we notice that for optimum cost

$$\begin{aligned}
J(a_M^*, a_T^*, a_D^*) &\leq J(a_M, a_T^*, a_D^*) \\
J(a_M^*, a_T, a_D^*) &\leq J(a_M^*, a_T^*, a_D^*) \\
J(a_M^*, a_T^*, a_D) &\leq J(a_M^*, a_T^*, a_D^*)
\end{aligned}$$

From the above we observe that the Nash equilibrium solution would require the Target and Defender to be coordinated. If there is a lack of coordination the Missile can leverage that benefit

Remark. *If we examine D(t) we notice the control weighting for the Defender goes to 0 after t_{f1} and the game continues despite the defender intercepting the target. This is to model the scenario where due to unforeseen circumstances, the defender fails to incapacitate the missile in which case it would be impossible for the defender to manoeuvre and re capture the missile hence the latter half of the game is only between the Missile and Target*

4 Simulation

Simulation has been set up on MATLAB. All players are assumed to have first order actuators

$$H(s) = \frac{a_\beta(s)}{a_{\beta c}(s)} = \frac{1}{\tau_\beta}$$

Here β can be the Missile, Target or Defender The values are given as

$$\tau_M = 0.1, \tau_T = 0.2, \tau_D = 0.05,$$

The initial conditions are given as

$$\begin{aligned} y_M(0) &= 0m, \quad \dot{y_M}(0) = 100m/s \\ y_T(0) &= 0m, \quad \dot{y_M}(0) = 50m/s \\ y_D(0) &= 0m, \quad \dot{y_D}(0) = 48m/s \end{aligned}$$

Cost function parameters are:

$$G_{MT} = \begin{bmatrix} 1000 & 0 \\ 0 & 0 \end{bmatrix} \quad G_{DM} = \begin{bmatrix} 0 & 0 \\ 0 & 1000 \end{bmatrix}$$

$$R_M = 1 \quad R_D = 3/5 \quad R_T = 5/3$$

The control terms are synthesized using the projected states and the zero effort miss shown below

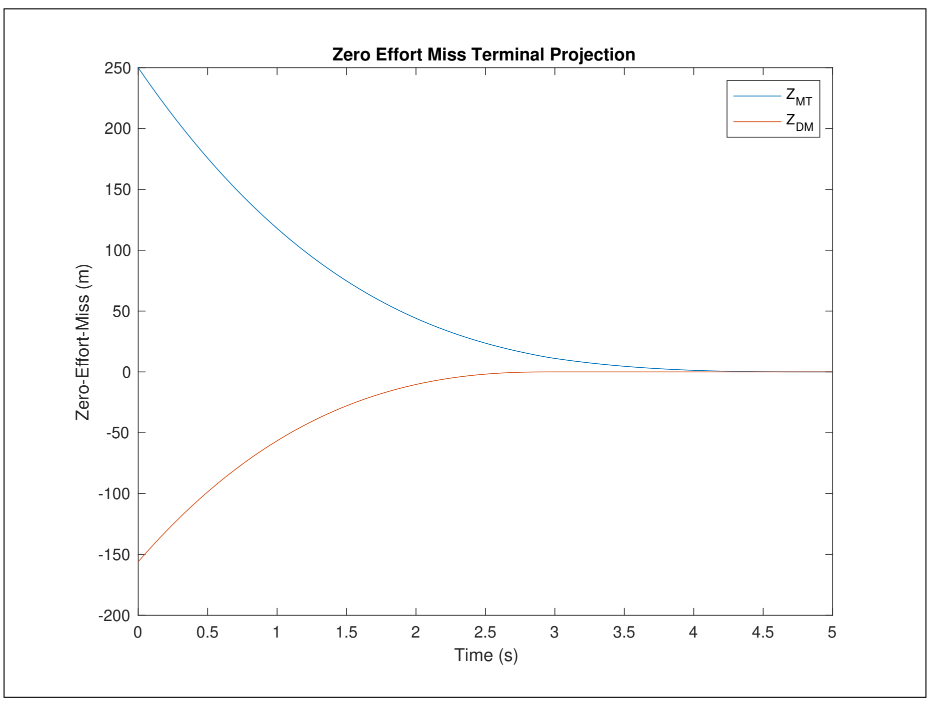

Figure 2: Zero effort Miss vs Time

Convergence to 0 is observed in both Z.E.M states

Next we look at the control effort

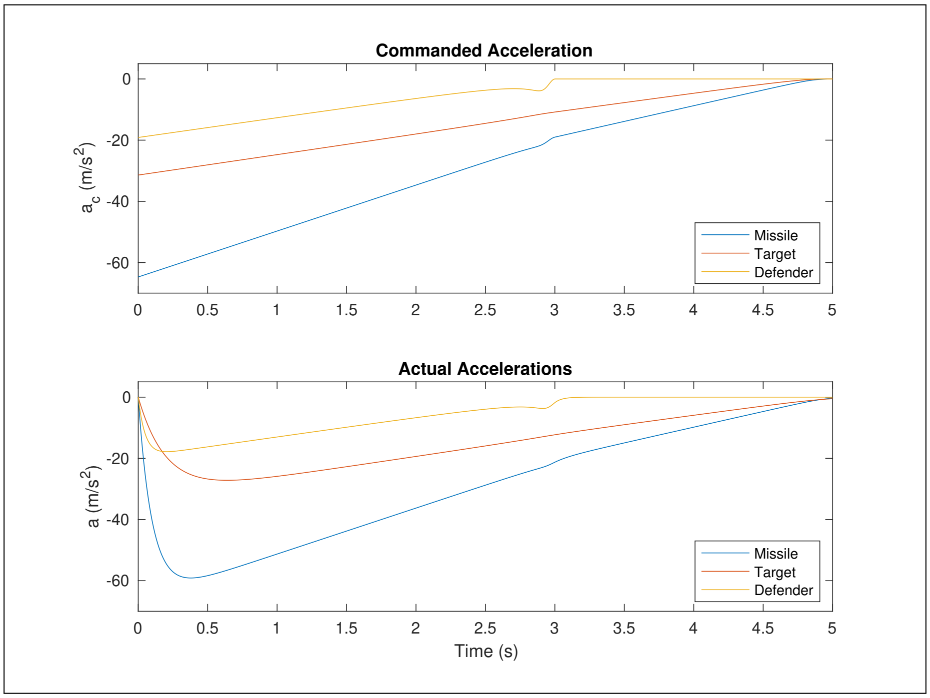

Figure 3: Control vs Time

The plot above has a small jump at t = 3 sec which is the t_{f1} for this simulation. The Defender's control effort goes to 0 at t_{f1}

Next we look at the plot of the vertical displacements

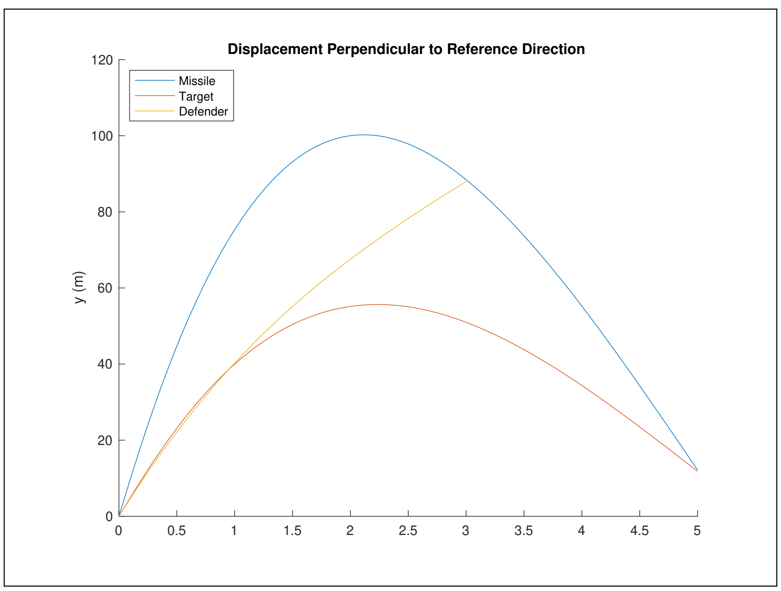

Figure 4: Vertical displacements

All 3 player started from initial vertical displacements and the convergence is shown

It is important to study the trajectories shown below

Figure 5: Trajectories

The trajectories at the end of t_{f1} are shown below

Figure 6: Trajectories at t = t_{f1}

Although it appears that the defender has indeed intercepted the missile, the game goes on in the off chance that the defender fails to invalidate the missile.

Next we look at the plots of the Navigation gains These gains are the effective navigation gains for

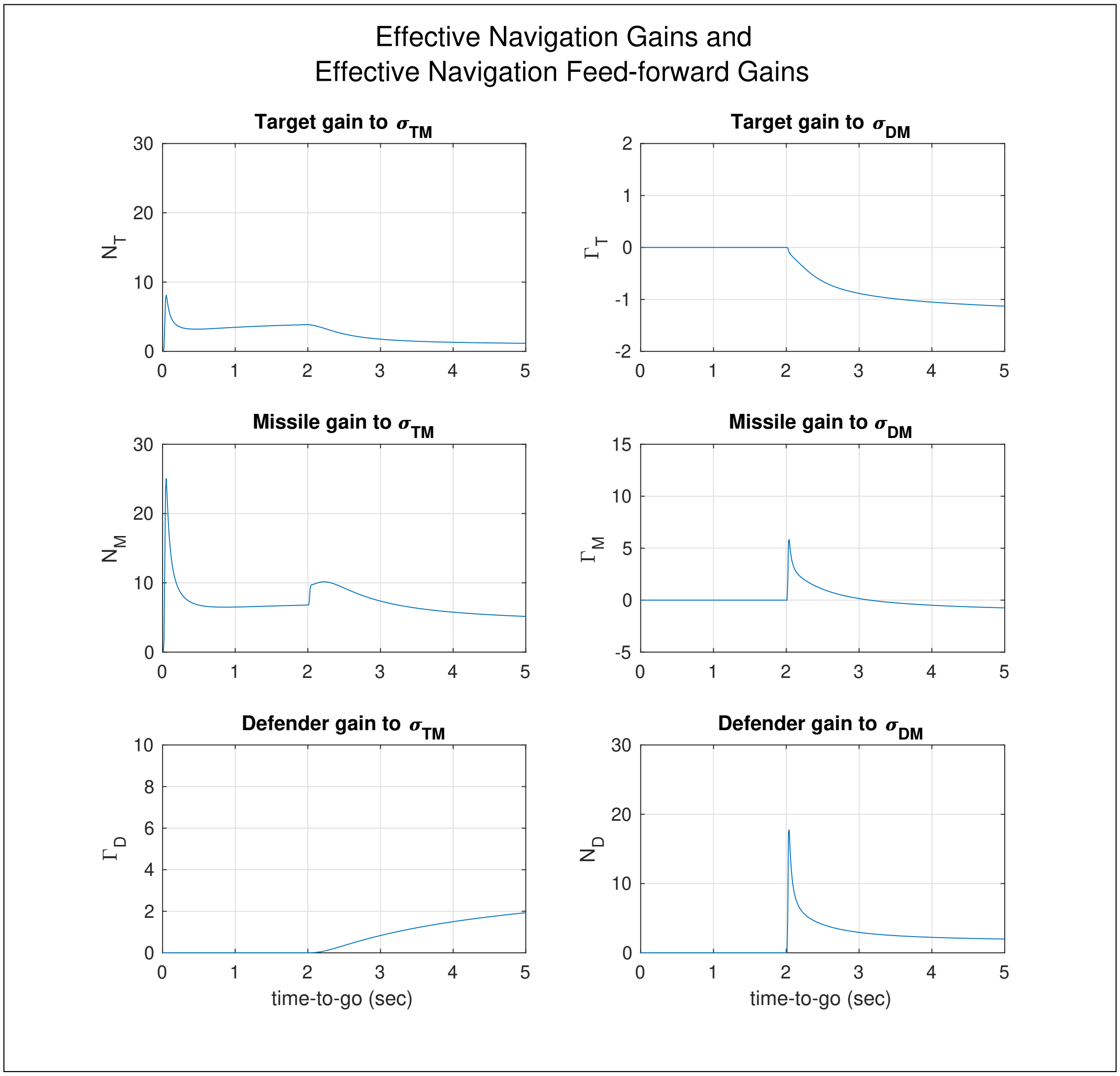

Figure 7: Navigation gains

various scenarios obtained from the solution of the DRE N_β are the Effective navigation gains and Γ_β are the Effective feed-forward Navigation gains defined as

$$N_\beta = (t_f - t)^2 K_\beta$$
$$\Gamma_\beta = (t_f - t)^2 L_\beta$$

Since control accelerations (a_β) can be written as

$$a_\beta = K_\beta Z_{MT} + L_\beta Z_{DM} \tag{10}$$

for β being the Missile, Target or Defender

This would be useful in examining the navigational gains for the unmodified system

Next are the plots of the difference in the vertical position between the two player

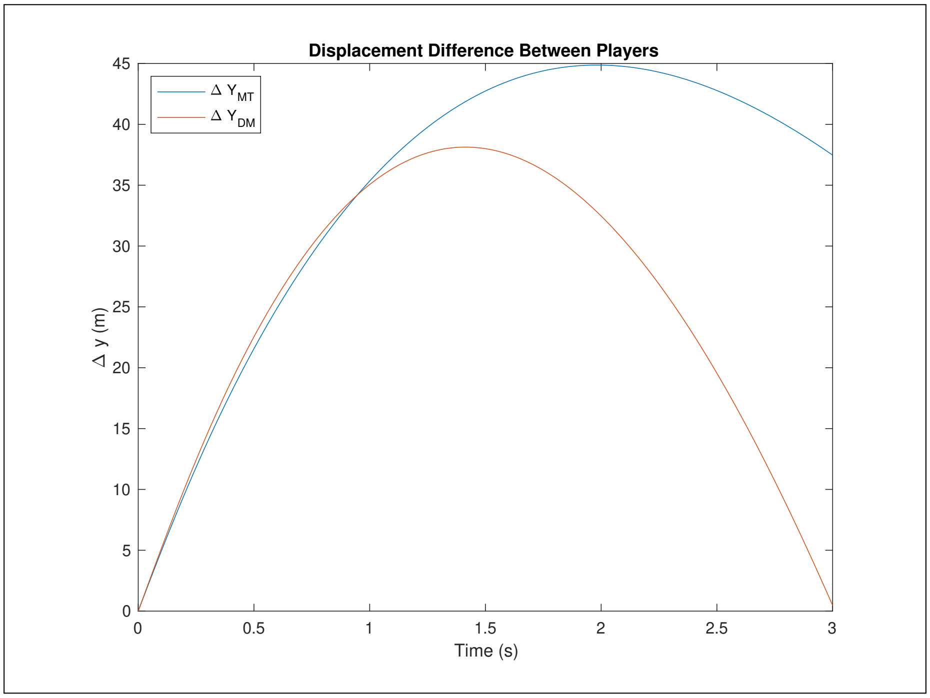

Figure 8: Trajectories at t = t_{f1}

5 Conclusion

The missile, target and defender problem has been formulated and an approach to a solution has been proposed. Sufficiency conditions for the existence of a solution have been stated and derived. The entire engagement has been simulated and the results have been discussed. We see that the defender not only tries to minimise the distance between itself and the missile but also helps increase the distance between the target and the missile since the missile tries to simultaneously maximise its distance from the defender while also minimises it's distance from the target which is indicative of cooperation between the Target and Defender.

References

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