Analysis of Pendulum with Friction

Shobhit Voleti 130010042 [*]

October 21, 2016

Abstract

Github Repository: **Friction Pendulum**
This document gives brief introduction to the not so simple
pendulum with friction. A comprehensive analysis of the system
substantiated by appropriate simulation results have been presented.

1   Introduction [1]

An extension to the analysis of a simple pendulum, analysis of a pendulum
with friction involves dealing with terms that arise due to friction encoun-
tered during the angular motion of the pendulum.

This consists of a point mass attached to an inextensible string. A force
acting on the point mass is its weight. The weight can be split into two
components: the tangential force and the force acting away from the center
of rotation. The latter force, shown in brown, is balanced by the reaction
force acting towards the center and so can be ignored.
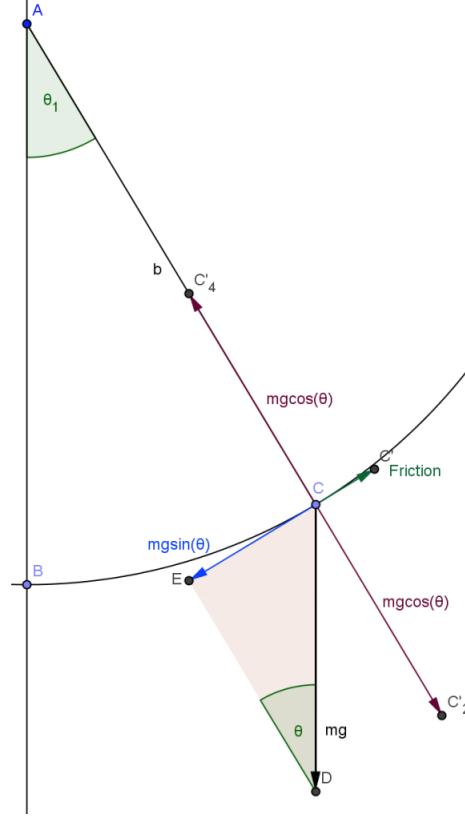
---

2 Dynamics [2]

Figure 1: Inverted Pendulum with sliding rod

The force due to friction acts tangentially on the bob. Since it acts in the opposite direction to the motion, the coefficient has to be negative.

Keeping the friction coefficient constant allows the differential equation to be linear and so, easy to solve. Making the frictional force proportional to the tangential velocity v rather than to the angular velocity ω prevents the pendulums length from affecting the frictional force.

Frictional force is given by the term,

$$F_s = \lambda \frac{dy}{dt} \tag{1}$$

The equations that govern the motion of the system are as follows:

$$m\ddot{y} + \lambda\dot{y} + \frac{mgy}{r} \tag{2}$$

3 Friction pendulum as generalized second order system

General second-order system is given by :

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{3}$$

where ζ is called damping ratio and ω_0 is called natural frequency.
From 2 we can get,

$$\ddot{y} + \frac{\lambda}{m}\dot{y} + \frac{g}{r}y = 0 \tag{4}$$

Comparing coefficient with 3, we get

$$\zeta = \frac{\lambda}{2\sqrt{km}}, \tag{5}$$

$$\omega = \sqrt{\frac{g}{r}} \tag{6}$$

where,

$$k = \frac{mg}{r} \tag{7}$$

4 State space model of the system[1]

We can describe the system only by its position and velocity. Therefore the state-space model of the system is given by :

$$\widetilde{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Therefore, equation of motion can be expressed as :

$$\dot{\widetilde{y}} = \begin{bmatrix} y_2 \\ -\frac{\lambda}{m}x_2 - \frac{k}{m}x_1 + F(y_2, y_1, t) \end{bmatrix}$$

5 Classification of system based on damping ratio

- Under-damped system ($\zeta < 1$)

- Critically damped system ($\zeta = 1$)

- Over-damped system ($\zeta > 1$)

Under-damped system :

In under-damped system ζ is less than 1.

This case best describes friction encountered on most surfaces The free response of the system is shown in the figure below.
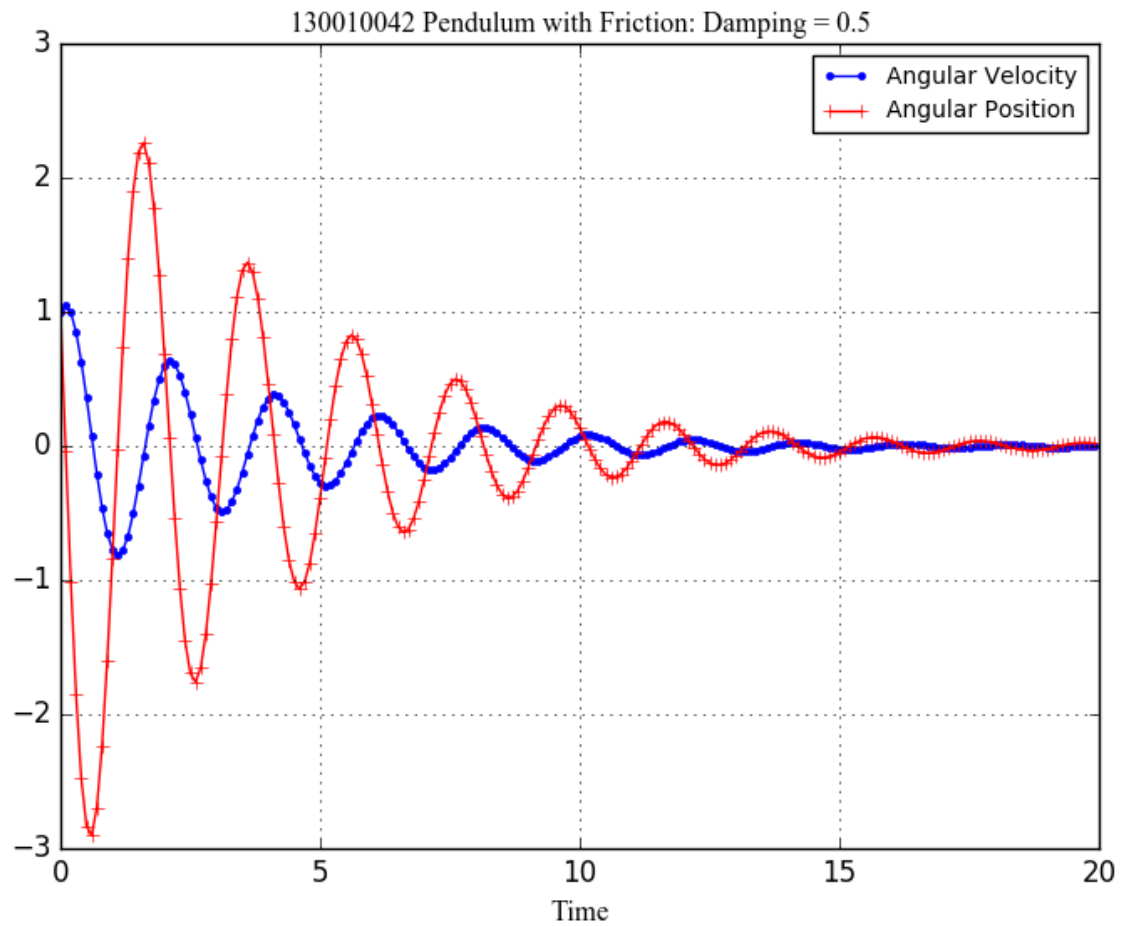

Figure 2: 130010042 Under-damped response

Critically-damped system :

In critically-damped system ζ is 1. The free response of the system is shown
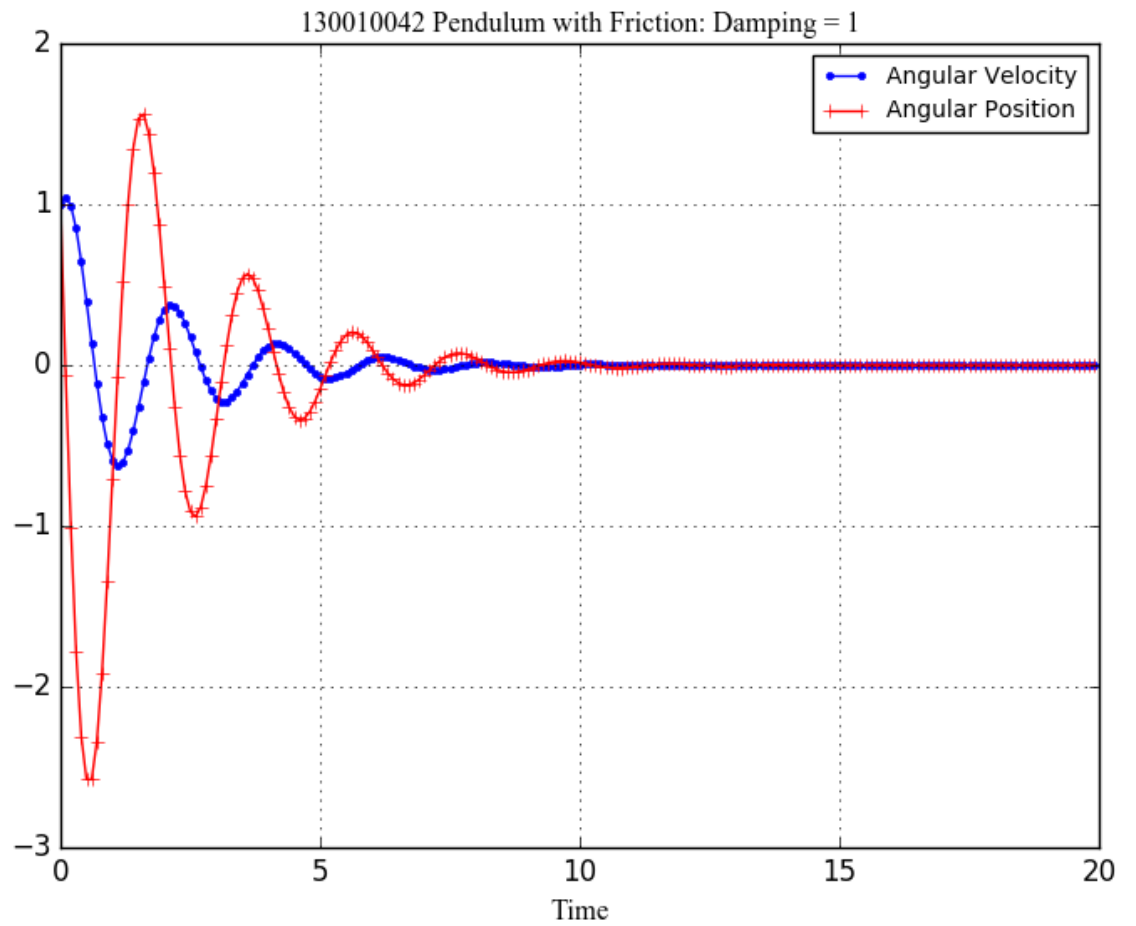in the figure below.

Figure 3: 130010042 Critically-damped response

Special case :
A special case of the system arises when there is zero damping i.e $\lambda = 0$ or
ζ is zero.
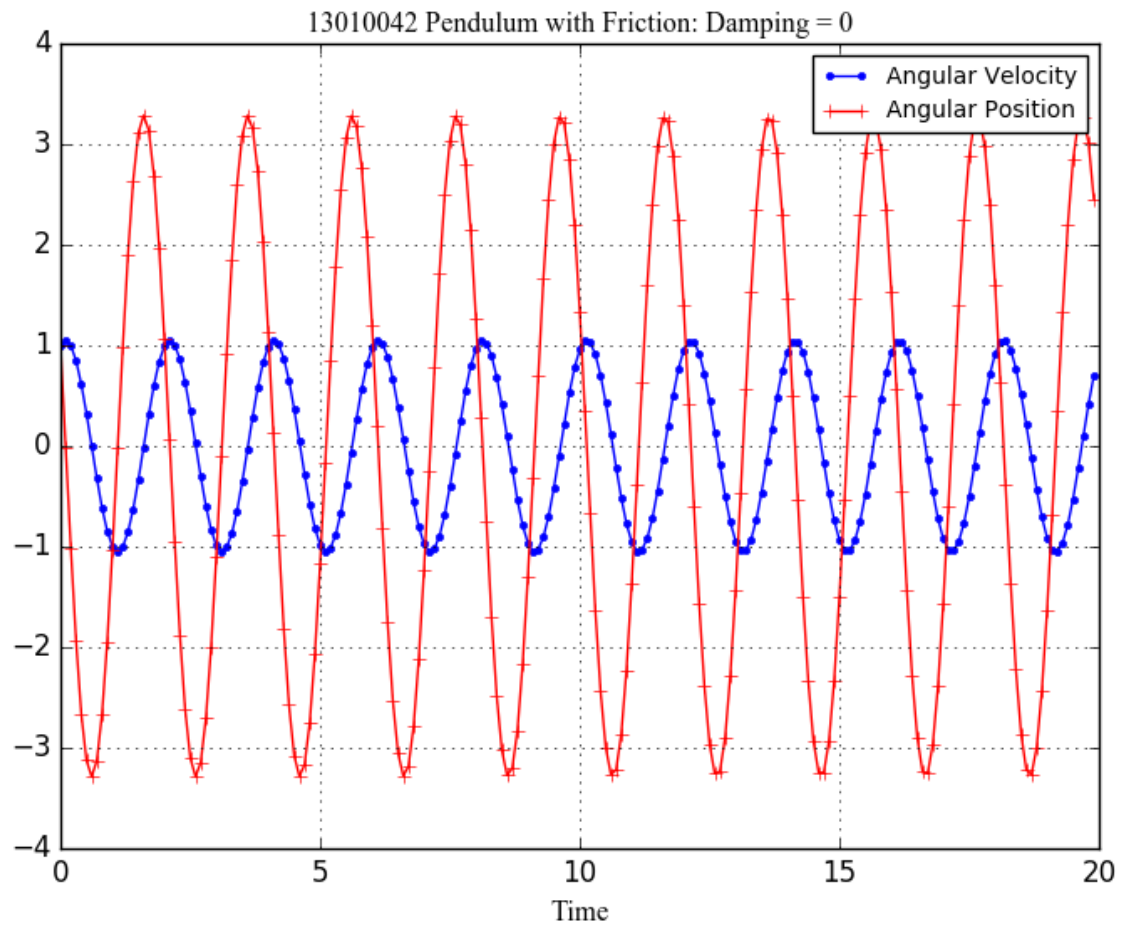The free response of the system is shown in the figure below.

Figure 4: 130010042 Friction-less response

References

[1] Open forum. www.google.com.

[2] Nrich Math. Not so simple pendulum.