Rendering Catenaries

[paraphrased from wikipedia]

A catenary is the shape that a chain or rope creates when it is hung at two fixed points. It looks like, but is not a parabola.

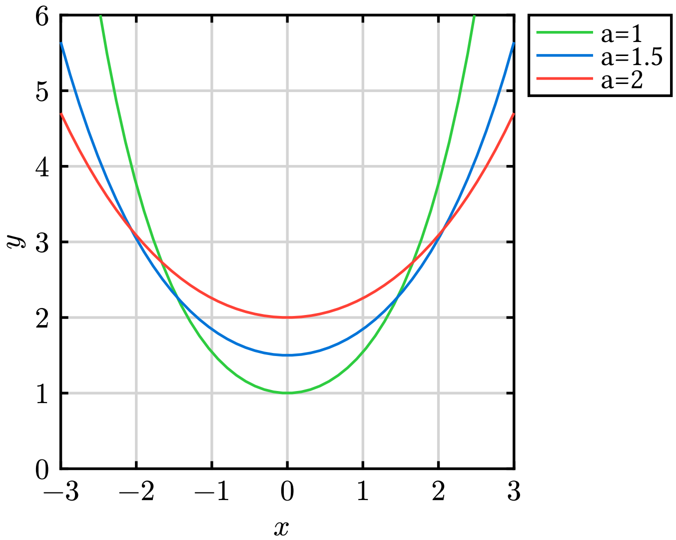
If there is added weight, as in the rope is supporting extra weight other than itself, the shape will not be a mix between catenary and parabola, depending on how impactful the weight of the rope is. As in: if the weight of the rope itself is negligeable compared to what it is supporting, then the shape will be a parabola.

Catenary equation

The basic equation for catenaries is as follows:

$$y = a \cosh\left(\frac{x}{a}\right) \tag{1}$$

Where $a > 0$.



The catenary is centered on the y axis, and is above the x axis. The minima is at $(0, a)$. a determines how "wide" the catenary is.

The reverse is also possible, getting x from y.

$$x = a \text{ acosh}\left(\frac{y}{a}\right) \tag{2}$$

where acosh is the inverse of cosh, and only returns the positive values of x.

The length of the curve between two points at x_1 and x_2 is:

$$L = a \sinh\left(\frac{x_2}{a}\right) - a \sinh\left(\frac{x_1}{a}\right) \tag{3}$$

Catenary from point displacement and arc length

Goal: to draw a catenary with anchor points spaced apart by the vector "disp", with a length of rope between them L.

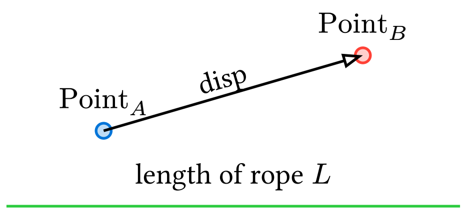
$$\text{disp} = \begin{pmatrix} h \\ v \end{pmatrix} \tag{4}$$

The catenary needs has anchor points: Point_A and Point_B, where

$$\text{Point}_B = \text{Point}_A + \text{disp} \tag{5}$$

In summary:

Knowns: disp, L
Unknowns: a, Point_A, Point_B



a is the only paramter needed for the catenary equation in Equation 1. However, the section of the catenary which has the two points on the curve with a difference of "disp" must be found.

Since there is a direct relation between x and y from Equation 1 and Point_A, Point_B from disp, finding either x_A, x_B, y_A or y_B is sufficient.

Keeping in mind $x_B = x_A + h$ and $y_B = y_A + v$, and the hyperbolic identities:
$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$
$\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$

Both Point_A and Point_B are on the catenary, so an equation for v by using Equation 1 is:

$$v = a\cosh\left(\frac{x_B}{a}\right) - a\cosh\left(\frac{x_A}{a}\right) \tag{6}$$

This can be simplified to:

$$\begin{aligned} \frac{v}{a} &= \cosh\left(\frac{x_A + h}{a}\right) - \cosh\left(\frac{x_A}{a}\right) \\ \frac{v}{2a} &= \cosh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \end{aligned} \tag{7}$$

$$v^2 = 4a^2\cosh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{8}$$

The equation for L in Equation 3 can be similarly simplified:

$$
\begin{aligned}
L &= a\sinh\left(\frac{x_B}{a}\right) - a\sinh\left(\frac{x_A}{a}\right) \\
\frac{L}{a} &= \sinh\left(\frac{x_A + h}{a}\right) - \sinh\left(\frac{x_A}{a}\right) \\
\frac{L}{2a} &= \sinh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right)
\end{aligned} \tag{9}
$$

$$
L^2 = 4a^2\sinh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{10}
$$

Finding a

Keeping in mind

$\cosh^2(x) - \sinh^2(x) = 1$

Substracting Equation 10 from Equation 8 gives:

$$
\begin{aligned}
v^2 - L^2 &= 4a^2\cosh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) - 4a^2\sinh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \\
&= 4a^2\sinh^2\left(\frac{h}{2a}\right)\left[\cosh^2\left(\frac{2x_A + h}{2a}\right) - \sinh^2\left(\frac{2x_A + h}{2a}\right)\right] \\
&= 4a^2\sinh^2\left(\frac{h}{2a}\right)
\end{aligned} \tag{11}
$$

Therefore:

$$
\sqrt{v^2 - L^2} = 2a\sinh\left(\frac{h}{2a}\right) \tag{12}
$$

Solving for a must be done numerically.

Finding x_A

x_A can be solved for in Equation 7:

$$
\begin{aligned}
\frac{v}{2a} &= \cosh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \\
\cosh\left(\frac{2x_A + h}{2a}\right) &= \frac{v}{2a\sinh\left(\frac{h}{2a}\right)} \\
\frac{2x_A + h}{2a} &= \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right) \\
2x_A &= 2a\ \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right) - h
\end{aligned} \tag{13}
$$

$$
x_A = a\ \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right) - \frac{h}{2} \tag{14}
$$

A similar equation can be derived from Equation 9