1) Rendering Catenaries

[paraphrased from wikipedia]

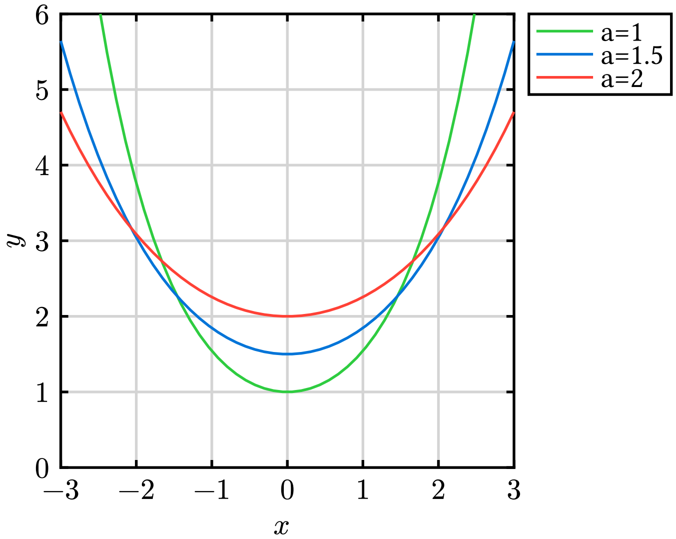
A catenary is the shape that a chain or rope creates when it is hung at two fixed points. It looks like, but is not a parabola.

If there is added weight, as in the rope is supporting extra weight other than itself, the shape will be a mix between catenary and parabola, depending on how impactful the weight of the rope is. As in: if the weight of the rope itself is negligeable compared to what it is supporting, then the shape will be a parabola.

2) Catenary equation

$$y = a \cosh\left(\frac{x}{a}\right) \tag{1}$$

Where $a > 0$.



Any two points (and the segment between) chosen on any of these curves are valid catenaries.

The catenary is centered on the y axis, and is above the x axis. The minima is at $(0, a)$. a determines how "wide" the catenary is.

The reverse is also possible, getting x from y.

$$x = a \text{ acosh}\left(\frac{y}{a}\right) \tag{2}$$

where acosh is the inverse of cosh. This will and only returns the positive values of x.

The length of the curve between two points at x_1 and x_2, where $x_2 > x_1$, is:

$$L = a \sinh\left(\frac{x_2}{a}\right) - a \sinh\left(\frac{x_1}{a}\right) \tag{3}$$

3) Algorithm for rendering catenaries in pixel art

Pixel art catenaries can be drawn if the parameter a is known. An x and y displacement is necessary to render the wanted section.

First iterate through the x coordinates and use Equation 1 to get the corresponding y coordinate, draw all paris. Once done, there will be gaps when the y displacement from one x coord to the next is greater than 1.

Fix this by iterating through the y coordinates, using Equation 2 to get x, draw all pairs. This will overwrite some of the pixels drawn during the first loop, but will get rid of the gaps.

```
x_disp = ...
y_disp = ...
fn cat_y(x) = a * cosh(x/a)
fn cat_x(y) = a * acosh(y/a)

for x_im in 0..w
  x = x_im + x_disp
  y = cat_y(x)
  y = y.round()
  y_im = y - y_disp
  # FLIPPING: (0,0) is usually top-left for images
  # flip and make (0,h-1) the top-left
  y_im = h - 1 - y_im
  draw(x_im,y_im)

for y_im in 0..h
  # See FLIPPING note
  y_im = h - 1 - y_im
  y = y_im + y_disp
  x = cat_x(y)
  x = x.round()
  x_im = x - x_disp
  draw(x,y)
```

4) Catenary from anchor displacement and arc length

Goal: to draw a catenary with anchor points spaced apart by the vector $\mathrm{disp}_{\mathrm{AB}}$, with a length of rope between them L. h is the displacement in the x direction and v in the y.

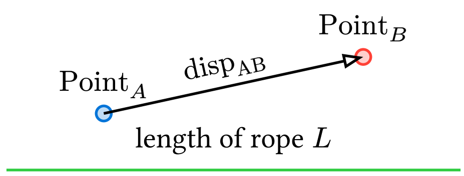
$$\mathrm{disp}_{\mathrm{AB}} = \begin{pmatrix} h \\ v \end{pmatrix}$$

The catenary has two anchor points: Point_A and Point_B, where

$$\mathrm{Point}_B = \mathrm{Point}_A + \mathrm{disp}_{\mathrm{AB}}$$

In summary:

Knowns: $\mathrm{disp}_{\mathrm{AB}}$, L
Unknowns: a, Point_A, Point_B



a is the only paramter needed for the catenary equation in Equation 1. However, the section of the catenary which has the two points on the curve with a difference of disp_{AB} must also be found.

Since there is a direct relation between x and y from Equation 1 and Point_A and Point_B from disp_{AB}, finding either x_A, x_B, y_A or y_B is sufficient.

Keeping in mind $x_B = x_A + h$ and $y_B = y_A + v$, and the hyperbolic identities:

$$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

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Both Point_A and Point_B are on the catenary, so an equation for v by using Equation 1 is:

$$v = a\cosh\left(\frac{x_B}{a}\right) - a\cosh\left(\frac{x_A}{a}\right)$$

This can be worked to:

$$\frac{v}{a} = \cosh\left(\frac{x_A+h}{a}\right) - \cosh\left(\frac{x_A}{a}\right)$$

$$\frac{v}{2a} = \sinh\left(\frac{2x_A+h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \tag{4}$$

$$v^2 = 4a^2\sinh^2\left(\frac{2x_A+h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{5}$$

The equation for L in Equation 3 can be similarly worked:

$$\frac{L}{a} = \sinh\left(\frac{x_A+h}{a}\right) - \sinh\left(\frac{x_A}{a}\right)$$

$$\frac{L}{2a} = \cosh\left(\frac{2x_A+h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \tag{6}$$

$$L^2 = 4a^2\cosh^2\left(\frac{2x_A+h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{7}$$

$\sinh^2$ is an even function, so this function is also valid

4.a) Finding a

Keeping in mind

$$\cosh^2(x) - \sinh^2(x) = 1$$

Substracting Equation 5 from Equation 7 gives:

$$\begin{aligned} L^2 - v^2 &= 4a^2\cosh^2\left(\frac{2x_A+h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) - 4a^2\sinh^2\left(\frac{2x_A+h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \\ &= 4a^2\sinh^2\left(\frac{h}{2a}\right)\left[\cosh^2\left(\frac{2x_A+h}{2a}\right) - \sinh^2\left(\frac{2x_A+h}{2a}\right)\right] \\ &= 4a^2\sinh^2\left(\frac{h}{2a}\right) \end{aligned}$$

Therefore for $a \geq 0$:

$$\sqrt{L^2 - v^2} = 2a\sinh\left(\frac{h}{2a}\right)$$

Solving for a must be done numerically.

$$f(a) = 0 = 2a\sinh\left(\frac{h}{2a}\right) - \sqrt{L^2 - v^2}$$

$$\frac{df(a)}{da} = 2\sinh\left(\frac{h}{2a}\right) - \frac{h}{a}\cosh\left(\frac{h}{2a}\right)$$

4.b) Finding x_A

x_A can be solved for in Equation 4:

$$\frac{v}{2a} = \cosh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right)$$

$$\cosh\left(\frac{2x_A + h}{2a}\right) = \frac{v}{2a\sinh\left(\frac{h}{2a}\right)}$$

$$\frac{2x_A + h}{2a} = \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right)$$

$$2x_A = 2a\ \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right) - h$$

$$x_A = a\ \operatorname{acosh}\left(\frac{v}{2a\sinh\left(\frac{h}{2a}\right)}\right) - \frac{h}{2}$$

it can be similarly solved for in Equation 6

5) Avoiding having to solve numerically for a

a can be solved for algebraically if there is no y displacement between Point_A, Point_B, and the amount of "sag" H is known from the y level of the points down to the minima of the catenary.

In summary:

Knowns: L, H, $v = 0$
Unknowns: a, Point_A, Point_B, h

if $v = 0$ then the minima is centered between Point_A and Point_B.

$$x_B = -x_A = \frac{h}{2}$$

Keeping in mind

$$\sinh(-x) = -\sinh(x)$$

Equation 3 can be worked down to:

$$
\begin{aligned}
L &= a\sinh\left(\frac{x_B}{a}\right) - a\sinh\left(\frac{x_A}{a}\right) \\
&= a\sinh\left(\frac{x_B}{a}\right) + a\sinh\left(\frac{x_B}{a}\right) \\
L &= 2a\sinh\left(\frac{h}{2a}\right) \\
L^2 &= 4a^2\sinh^2\left(\frac{h}{2a}\right) \qquad (8)
\end{aligned}
$$

Reworking Equation 1:

$$
\begin{aligned}
y_B &= a\cosh\left(\frac{x_B}{a}\right) \\
&= a\cosh\left(\frac{h}{2a}\right) \\
y_B^2 &= a^2\cosh^2\left(\frac{h}{2a}\right) \qquad (9)
\end{aligned}
$$

Substracting Equation 8 from Equation 9:

$$
\begin{aligned}
y_B^2 - \frac{L^2}{4} &= a^2\cosh^2\left(\frac{h}{2a}\right) - a^2\sinh^2\left(\frac{h}{2a}\right) \\
&= a^2\left(\cosh^2\left(\frac{h}{2a}\right) - \sinh^2\left(\frac{h}{2a}\right)\right) \\
&= a^2 \\
y_B^2 - \frac{L^2}{4} &= a^2 \qquad (10)
\end{aligned}
$$

Since a is the minima, and H is from the minima to the y of the points

$$y_A = y_B = H + a$$

Plug into Equation 10:

$$
\begin{aligned}
a^2 &= (H+a)^2 - \frac{L^2}{4} \\
&= H^2 + 2Ha + a^2 - \frac{L^2}{4} \\
2Ha &= \frac{L^2}{4} - H^2 \\
a &= \frac{L^2}{8H} - \frac{H}{2} \qquad (11)
\end{aligned}
$$

To get a non-symmetrical catenary using this method, an arbitrary amount of curve can be ignored. For example, Point_A can be moved by 20 to the right:

$$x_A \text{ new} = x_A + 20$$

$$y_A \text{ new} = a \cosh\left(\frac{x_A \text{ new}}{a}\right)$$

Could also move it up or down instead