Rendering Catenaries

[paraphrased from wikipedia]

A catenary is the shape that a chain or rope creates when it is hung at two fixed points. It looks like, but is not a parabola.

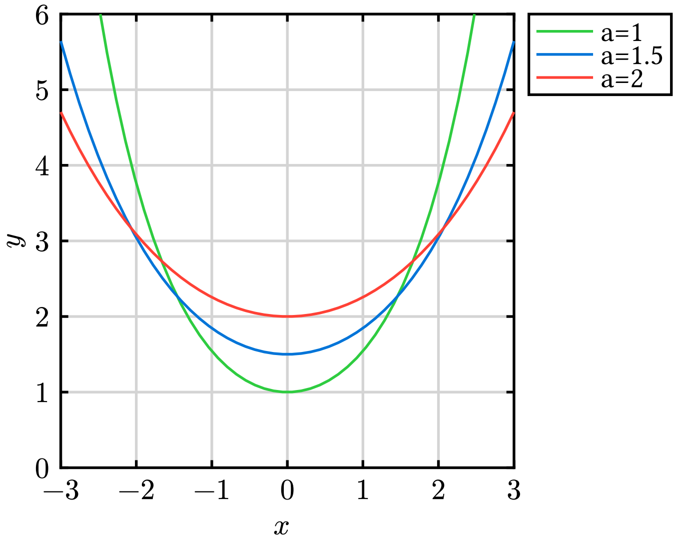
If there is added weight, as in the rope is supporting extra weight other than itself, the shape will not be a mix between catenary and parabola, depending on how impactful the weight of the rope is. As in: if the weight of the rope itself is negligeable compared to what it is supporting, then the shape will be a parabola.

Catenary equation

The basic equation for catenaries is as follows:

$$y = a \cosh\left(\frac{x}{a}\right) \tag{1}$$

Where $a > 0$.



The catenary is centered on the y axis, and is above the x axis. The minima is at $(0, a)$. a determines how "wide" the catenary is.

The reverse is also possible, getting x from y.

$$x = a \text{ acosh}\left(\frac{y}{a}\right) \tag{2}$$

where acosh is the inverse of cosh, and only returns the positive values of x.

The length of the curve between two points at x_1 and x_2 is:

$$L = a \sinh\left(\frac{x_2}{a}\right) - a \sinh\left(\frac{x_1}{a}\right) \tag{3}$$

Algorithm for rendering catenaries in pixel art

Pixel art catenaries can be drawn if the parameter a is known. An x and y displacement is necessary render the wanted section.

First iterate through the x coordinates and use Equation 1 to get the corresponding y coordinate, draw this pair. Once done, there will be gaps when the y displacement from one x-coord to the next is greater than 2.

Fix this by iterating through the y coordinates, using Equation 2 to get and draw coord pairs. This will overwrite some of the pixels drawn during the first loop, but will get rid of the gaps.

Pseudocode:

```
x_disp = ...
y_disp = ...
fn cat_y(x) = a * cosh(x/a)
fn cat_x(y) = a * acosh(y/a)

for x_im in 0..w
  x = x_im + x_disp
  y = cat_y(x)
  y = y.round()
  y_im = y - y_disp
  # (0,0) is usually top-left for images
  # flip and make (0,h-1) the top-left
  y_im = h - 1 - y_im
  draw(x_im,y_im)

for y_im in 0..h
  # (0,0) is usually top-left for images
  # flip and make (0,h-1) the top-left
  y_im = h - 1 - y_im
  y = y_im + y_disp
  x = cat_x(y)
  x = x.round()
  x_im = x - x_disp
  draw(x,y)
```

Catenary from anchor displacement and arc length

Goal: to draw a catenary with anchor points spaced apart by the vector disp_{AB}, with a length of rope between them L.

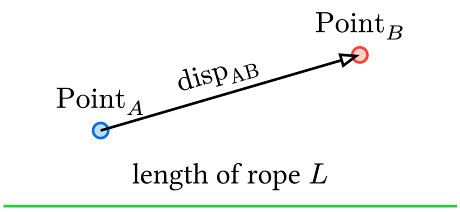
$$\text{disp}_{\text{AB}} = \begin{pmatrix} h \\ v \end{pmatrix}$$

The catenary needs has anchor points: Point_A and Point_B, where

$$\text{Point}_B = \text{Point}_A + \text{disp}_{\text{AB}}$$

In summary:

$$\text{Knowns: } \text{disp}_{\text{AB}}, L$$
$$\text{Unknowns: } a, \text{Point}_A, \text{Point}_B$$



a is the only paramter needed for the catenary equation in Equation 1. However, the section of the catenary which has the two points on the curve with a difference of disp_{AB} must be found.

Since there is a direct relation between x and y from Equation 1 and Point_A, Point_B from disp, finding either x_A, x_B, y_A or y_B is sufficient.

Keeping in mind $x_B = x_A + h$ and $y_B = y_A + v$, and the hyperbolic identities:

$$\sinh(x) - \sinh(y) = 2\cosh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

$$\cosh(x) - \cosh(y) = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right)$$

Both Point_A and Point_B are on the catenary, so an equation for v by using Equation 1 is:

$$v = a\cosh\left(\frac{x_B}{a}\right) - a\cosh\left(\frac{x_A}{a}\right)$$

This can be simplified to:

$$\frac{v}{a} = \cosh\left(\frac{x_A + h}{a}\right) - \cosh\left(\frac{x_A}{a}\right)$$

$$\frac{v}{2a} = \cosh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \tag{4}$$

$$v^2 = 4a^2\cosh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{5}$$

The equation for L in Equation 3 can be similarly simplified:

$$L = a\sinh\left(\frac{x_B}{a}\right) - a\sinh\left(\frac{x_A}{a}\right)$$

$$\frac{L}{a} = \sinh\left(\frac{x_A + h}{a}\right) - \sinh\left(\frac{x_A}{a}\right)$$

$$\frac{L}{2a} = \sinh\left(\frac{2x_A + h}{2a}\right)\sinh\left(\frac{h}{2a}\right) \tag{6}$$

$$L^2 = 4a^2\sinh^2\left(\frac{2x_A + h}{2a}\right)\sinh^2\left(\frac{h}{2a}\right) \tag{7}$$

Finding a
Keeping in mind

$$\cosh^2(x) - \sinh^2(x) = 1$$

Substracting Equation 7 from Equation 5 gives:

$$
\begin{aligned}
v^2 - L^2 &= 4a^2 \cosh^2\left(\frac{2x_A + h}{2a}\right) \sinh^2\left(\frac{h}{2a}\right) - 4a^2 \sinh^2\left(\frac{2x_A + h}{2a}\right) \sinh^2\left(\frac{h}{2a}\right) \\
&= 4a^2 \sinh^2\left(\frac{h}{2a}\right) \left[\cosh^2\left(\frac{2x_A + h}{2a}\right) - \sinh^2\left(\frac{2x_A + h}{2a}\right)\right] \\
&= 4a^2 \sinh^2\left(\frac{h}{2a}\right)
\end{aligned}
$$

Therefore:

$$
\sqrt{v^2 - L^2} = 2a \sinh\left(\frac{h}{2a}\right)
$$

Solving for a must be done numerically.

Finding x_A

x_A can be solved for in Equation 4:

$$
\begin{aligned}
\frac{v}{2a} &= \cosh\left(\frac{2x_A + h}{2a}\right) \sinh\left(\frac{h}{2a}\right) \\
\cosh\left(\frac{2x_A + h}{2a}\right) &= \frac{v}{2a \sinh\left(\frac{h}{2a}\right)} \\
\frac{2x_A + h}{2a} &= \operatorname{acosh}\left(\frac{v}{2a \sinh\left(\frac{h}{2a}\right)}\right) \\
2x_A &= 2a \ \operatorname{acosh}\left(\frac{v}{2a \sinh\left(\frac{h}{2a}\right)}\right) - h \\
x_A &= a \ \operatorname{acosh}\left(\frac{v}{2a \sinh\left(\frac{h}{2a}\right)}\right) - \frac{h}{2}
\end{aligned}
$$

it can be similarly solved for in Equation 6