# Causal Mediation in Natural Experiments

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## Abstract

Natural experiments are a cornerstone of applied economics, providing settings for estimating causal effects of a treatment with a compelling argument for treatment ignorability. Economists are often interested in understanding the mechanisms through which causal effects operate, and mediation methods aim to estimate these components. However, conventional mediation methods rely on a selection-on-observables assumption, assuming the mediator is conditionally ignorable — in addition to the natural experiment for the original treatment. This paper shows that conventional estimates of mediation effects are contaminated by selection bias when the mediator is not ignorable. Using the case of a Roy model for a mediator, I show that individuals' selection based on expected gains and costs is inconsistent with mediator ignorability without implausible behavioural assumptions. I develop a control function approach, which correctly estimates mediation effects when selection into the mediator follows a selection model, using cost of mediator take-up as an instrument. Simulations confirm that this method corrects for selection bias in conventional mediation estimates, and performs comparably to a selection-on-observables approach when the mediator selection does not follow a selection model. I illustrate the approach by estimating the proportion of the causal effect of genes associated with education that operates via a direct genetic channel versus indirectly through extended schooling. Finally, I provide an implementation of this method in the *R* package *mediate-controlfun*, offering an accessible tool for robust mediation analysis in natural experiment settings.

**Keywords:** Causal Mediation.

**JEL Codes:** D31, D91, I24, J24, Z00.

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Conventional CM methods rely on a selection-on-observables assumption, which may not hold true in observational work. I explicitly connect the assumptions behind CM methods to those of selection into treatment in classic labour and observation economic research (Heckman & Vytlacil 2005). When a mediator, here education, is not randomly assigned then conventional CM methods for estimating direct and indirect effects are contaminated by selection bias. I write this as both a non-parametric non-identification result, and with a model-based regression framework with a correlated error term(e.g., as in the Imai et al. 2010 linear model approach). Structural assumptions could solve the identification problem, for example if selection into education follows a Roy model or errors terms have a known distribution (Heckman 1979).

Intro paragraph on the selection bias term. These are similar, in spirit, to selection bias of Heckman et al. (1998). Also give a result closely related to bad control bias (also known as M-bias, Ding 2015, Cinelli 2022). Also similar to the non-identification result of Bugni Canay McBride (2024).

This paper proceeds as follows. Section 1 introduces causal mediation, and develops selection bias in mediation estimates in natural experiments. Section 2 describes the selection bias in applied settings, a regression framework and a natural experiment with selection based on costs and benefits. Section 3 identifies and estimates average direct and indirect effects under a selection model, with simulation evidence that this approach purges bias in mediation estimates. Section 5 concludes.

# 1 Direct and Indirect Effects

Causal mediation decomposes causal effects into two channels, through a mediator (indirect effect) and through all other paths (direct effect).

To develop notation for direct and indirect effects, write $Z_i$ for an exogenous binary variable, $D_i$ an intermediary outcome (mediator), and $Y_i$ an outcome for individuals $i = 1, \dots, n$.[1] The outcomes are a sum of their potential outcomes.

$$\begin{aligned} D_i &= Z_i D_i(1) + (1 - Z_i) D_i(0), \\ Y_i &= Z_i Y_i(1, D_i(1)) + (1 - Z_i) Y_i(0, D_i(0)). \end{aligned}$$

Write $\boldsymbol{X}_i$ for a set of control variables, and assume $Z_i$ is ignorable — possibly conditional on $\boldsymbol{X}_i$.

$$Z_i \perp\!\!\!\perp D_i(z), Y_i(z', d), \text{ for } z, z', d = 0, 1$$

Then there are only two average effects which are identified.

The average first-stage refers to the effect of the treatment on mediator, $Z \to D$.

$$\mathbb{E}\left[D_i \mid Z_i = 1\right] - \mathbb{E}\left[D_i \mid Z_i = 0\right] = \mathbb{E}\left[D_i(1) - D_i(0)\right]$$

It common in the economics literature to assume that $Z$ influences $D$ in at most one direction, $\Pr\left(D_i(1) \geq D_i(0)\right) = 1$ — monotonicity (Imbens & Angrist 1994). I assume monotonicity (and its conditional variant) holds through-out to simplify notation.[2]

The reduced-form effect refers to the effect of the treatment on outcome, $Z \to Y$, and is also known as the intent-to-treat effect in experimental settings, or total effect in causal mediation literature.

$$\mathbb{E}\left[Y_i \mid Z_i = 1\right] - \mathbb{E}\left[Y_i \mid Z_i = 0\right] = \mathbb{E}\left[Y_i(1, D_i(1)) - Y_i(0, D_i(0))\right]$$

[1] Other literatures use different notation. For example, Imai et al. (2010) write $T_i, M_i, Y_i$ for the randomised treatment, mediator, and outcome, respectively. I use $Z_i, D_i, Y_i$ to stick to the instrumental variables notation Angrist et al. (1996), more familiar in empirical economics (Angrist & Pischke 2009).

[2] Assuming monotonicity also brings closer to the IV notation, and has other beneficial implications in this setting (see Subsection 2.5).

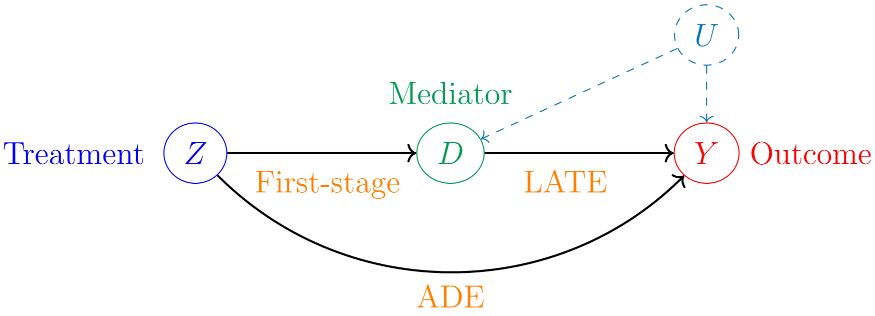
$Z_i$ affects outcome $Y_i$ directly, and indirectly via the $D_i(Z_i)$ channel, with no reverse causality. Figure 1 visualises the design, where the direction arrows denote the causal direction (and no reverse causality). On the other hand, mediation aims to decompose the reduced form effect of $Z \to Y$ into these two separate pathways.

$$\text{Average Indirect Effect (AIE), } D(Z) \to Y: \quad \mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right]$$

$$\text{Average Direct Effect (ADE), } Z \to Y: \quad \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right]$$

These effects are not separately identified without further assumptions.

**Figure 1:** Structural Causal Model for Causal Mediation.



**Note**: This figures shows the structural causal model behind causal mediation. LATE refers to the effect $D \to Y$ local to $Z$ compliers, so that AIE = average first-stage × LATE. Unobserved confounder $U$ represents this paper's focus on the case that $D_i$ is not ignorable, by showing an implied unobserved confounder. Section 2 formally defines $U$ in this set-up.

## 1.1 Causal Mediation (CM) Estimands

The conventional approach to estimating direct and indirect effects assumes both $Z_i$ and $D_i$ are ignorable, conditional on $\boldsymbol{X}_i$.

**Definition 1.** *Sequential Ignorability (Imai et al. 2010).*

$$Z_i \perp\!\!\!\perp D_i(z), Y_i(z', d) \;\mid\; \boldsymbol{X}_i, \qquad \text{for } z, z', d = 0, 1 \tag{1}$$

$$D_i \perp\!\!\!\perp Y_i(z', d) \;\mid\; \boldsymbol{X}_i, Z_i = z', \qquad \text{for } z', d = 0, 1 \tag{2}$$

If 1(1) and 1(2) hold, then the direct and indirect effects are identified by two-stage mean differences, after conditioning on $\boldsymbol{X}_i$.[3]

$$\mathbb{E}_{D_i, \boldsymbol{X}_i} \left[ \underbrace{\mathbb{E}\left[Y_i \mid Z_i = 1, D_i, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \mid Z_i = 0, D_i, \boldsymbol{X}_i\right]}_{\text{Second-stage regression, } Y_i \text{ on } Z_i \text{ holding } D_i \text{ constant}} \right] = \underbrace{\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right]}_{\text{Average Direct Effect (ADE)}}$$

$$\mathbb{E}_{Z_i, \boldsymbol{X}_i} \left[ \underbrace{\left( \mathbb{E}\left[D_i \mid Z_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[D_i \mid Z_i = 0, \boldsymbol{X}_i\right] \right)}_{\text{First-stage regression, } D_i \text{ on } Z_i} \times \underbrace{\left( \mathbb{E}\left[Y_i \mid Z_i, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \mid Z_i, D_i = 0, \boldsymbol{X}_i\right] \right)}_{\text{Second-stage regression, } Y_i \text{ on } D_i \text{ holding } Z_i \text{ constant}} \right]$$

$$= \underbrace{\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right]}_{\text{Average Indirect Effect (AIE)}}$$

I refer to the estimands on the left-hand side as Causal Mediation (CM) estimands in the following. These estimands are typically estimated with linear models (Imai et al. 2010):

$$D_i = \phi + \pi Z_i + \boldsymbol{\psi}_1' \boldsymbol{X}_i + \eta_i$$

$$Y_i = \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \boldsymbol{\psi}_2' \boldsymbol{X}_i + \varepsilon_i$$

And so the CM estimands are composed from OLS estimates, $\widehat{\gamma} + \widehat{\delta}\mathbb{E}\left[D_i\right]$ for the Average Direct Effect (ADE) and $\widehat{\pi}\left(\widehat{\beta} + \mathbb{E}\left[Z_i\right]\widehat{\delta}\right)$ for the average indirect effect (AIE). While this is the most common approach in the applied literature, I do not focus on the linear formulation of this problem as it assumes homogenous treatment effects and linear confounding. These assumptions are unnecessary to my analysis; it suffices to note that heterogeneous treatment effects and non-linear confounding would bias OLS estimates of CM estimands in the same manner that is well documented elsewhere (see e.g., Angrist 1998, Słoczyński 2022). I focus on fundamental problems that plague causal mediation methods in practice, regardless of estimation method.

[3] Imai et al. (2010) show a general identification statement; I show identification in terms of two-stage regression, which is more familiar in economics. This reasoning is in line with G-computation reasoning (Robins 1986); Subsection A.1 states the Imai et al. (2010) identification result, and then develops the two-stage regression notation which holds as a consequence of sequential ignorability.

## 1.2 Bias in Causal Mediation Estimates

Mediation methods are the main method that researchers then answer the following question: how did $Z$ lead to a causal effect on $Y$, and through which channels? In observational work this may include a natural experiment that quasi-randomly assigns $Z_i$ to individuals, regardless of their preferences or selection patterns — i.e., justifying assumption 1(1). Rarely does observational research employ an additional, overlapping identification design for $D_i$ as part of the analysis, and instead estimate CM estimands by assuming this $D_i$ is ignorable conditional on $\boldsymbol{X}_i$.[4] This approach leads to biased estimates, and contaminates inference regarding direct and indirect effects.

**Theorem 1.** *Absent an identification strategy for the mediator, causal mediation estimates are at risk of selection bias. Suppose 1(1) holds, but 1(2) does not. Then CM estimands are contaminated by selection bias and group difference terms.*

*Proof.* See Subsection A.4 for the extended proof. □

Below I present the relevant selection bias and group difference terms, omitting the conditional on $\boldsymbol{X}_i$ notation for brevity.

For the average direct effect: CM estimand = ADE + selection bias + group differences.

$$\mathbb{E}_{D_i}\Big[\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i\right]\Big]$$

$$= \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right]$$

$$+ \mathbb{E}_{D_i}\Big[\mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(0) = d\right]\Big]$$

$$+ \mathbb{E}_{D_i}\left[\Big(1 - \Pr\left(D_i(1) = d\right)\Big)\begin{pmatrix}\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d\right] \\ - \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(0) = 1 - d\right]\end{pmatrix}\right]$$

[4] Imai et al. (2013) call attention to the need for a separate research design to isolate causal effects of $D_i$ in randomised controlled trials; Subsection A.3 overviews literature, finding many papers that employ mediation methods with a research design for $Z_i$, but not for $D_i$.

For the average indirect effect: CM estimand = AIE + selection bias + group differences.

$$
\begin{aligned}
&\mathbb{E}_{Z_i}\left[\Big(\mathbb{E}\left[D_i \,|\, Z_i = 1\right] - \mathbb{E}\left[D_i \,|\, Z_i = 0\right]\Big) \times \Big(\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 1\right] - \mathbb{E}\left[Y_i \,|\, Z_i, D_i = 0\right]\Big)\right] \\
&= \mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right] \\
&\quad + \Pr\left(D_i(1) = 1, D_i(0) = 0\right)\Big(\mathbb{E}\left[Y_i(Z_i, 0) \,|\, D_i = 1\right] - \mathbb{E}\left[Y_i(Z_i, 0) \,|\, D_i = 0\right]\Big) \\
&\quad + \Pr\left(D_i(1) = 1, D_i(0) = 0\right) \times \\
&\quad \left[\Big(1 - \Pr\left(D_i = 1\right)\Big)\begin{pmatrix}\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, D_i = 1\right] \\ - \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, D_i = 0\right]\end{pmatrix} \right. \\
&\quad \left. + \left(\frac{1 - \Pr\left(D_i(1) = 1, D_i(0) = 0\right)}{\Pr\left(D_i(1) = 1, D_i(0) = 0\right)}\right)\begin{pmatrix}\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, D_i(1) = 0 \text{ or } D_i(0) = 1\right] \\ - \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0)\right]\end{pmatrix}\right]
\end{aligned}
$$

The selection bias terms come from systematic differences between the treated and untreated groups, differences not fully unexplained by $\boldsymbol{X}_i$. These selection bias terms would equal to zero if the mediator was ignorable (2), but do not necessarily average to zero if not. The group differences represent the fact that a matching estimator gives an average effect on the treated group and, when selection-on-observables does not hold, this is systematically different from the average effect (Heckman et al. 1998).[5,6]

## 2 Causal Mediation in Applied Settings

In this section, I further develop the issue of selection in causal mediation estimates. First, I show the non-parametric bias terms from above can be written as omitted variables bias in

[5] The selection-on-observables approach could, instead, focus on the average effect on treated populations (as do Keele et al. 2015). This runs into a problem of comparisons: CM estimates would give average effects on different treated groups. The CM estimand for the ADE on treated gives the ADE local to the $Z_i = 1$ treated group, and local to the $D_i = 1$ group for the AIE. In this way, these ADE and AIE on treated terms are not comparable to each other, so I focus on the true averages to avoid these misaligned comparisons.

[6] The group differences term is longer for the average indirect effect estimate, because the indirect effect is comprised from the effect of $D_i$ local to $Z_i$ compliers; a matching estimator gets the average effect on treated, and the longer term adjusts for differences with the complier average effect.

a regression framework. Second, I show how selection bias operates in an applied model for selection into a mediator based on costs and benefits.

## 2.1 Regression Framework

Inference for direct and direct effects can be written in a regression framework, showing how correlation between the error term and the mediator persistently biases estimates. Write $Y_i(Z, D)$ as a sum of observed factors $Z_i, \boldsymbol{X}_i$ and unobserved factors, $U_{1,i}, U_{0,i}$ (following the notation of Heckman & Vytlacil 2005). Put $\mu_D(Z; \boldsymbol{X}_i) = \mathbb{E}\left[Y_i(Z_i, 0) \,|\, \boldsymbol{X}\right]$, to give a representation of the average direct and indirect effects.

$$\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right] = \mathbb{E}\Big[\Big(D_i(1) - D_i(0)\Big) \times \Big(\mu_1(Z_i; \boldsymbol{X}_i) - \mu_0(Z_i; \boldsymbol{X}_i)\Big)\Big],$$
$$\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right] = \mathbb{E}\Big[\mu_{D_i}(1; \boldsymbol{X}_i) - \mu_{D_i}(0; \boldsymbol{X}_i)\Big].$$

Then define the error terms.

$$U_{0,i} = Y_i(Z_i, 0) - \mu_0(Z_i; \boldsymbol{X}_i), \quad U_{1,i} = Y_i(Z_i, 1) - \mu_1(Z_i; \boldsymbol{X}_i)$$

With this notation, observed data $Z_i, D_i, Y_i$ take the following representation, which characterises direct effects, indirect effects, and bias from selection.

$$D_i = \phi + \pi Z_i + \varphi(\boldsymbol{X}_i) + \eta_i \tag{3}$$
$$Y_i = \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \zeta(\boldsymbol{X}_i) + \underbrace{U_{0,i} + D_i\left(U_{1,i} - U_{0,i}\right)}_{\text{Correlated error term.}} \tag{4}$$

First-stage (3) is identified. $\alpha, \zeta(\boldsymbol{X}_i)$ are the intercept terms, and $\beta, \gamma, \delta$ are values that comprise mediation effects — all whose values may depend on $\boldsymbol{X}_i$, see Subsection A.6 for full definitions. The average direct and indirect effects are averages of these terms, across the distribution of $\boldsymbol{X}_i$.

$$\begin{aligned}
\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right] &= \mathbb{E}\left[\pi\left(\beta + Z_i\delta\right)\right], \\
\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right] &= \mathbb{E}\left[\gamma + \delta D_i\right].
\end{aligned}$$

By construction, $U_i = U_{1,i} - U_{0,i}$ is an unobserved confounder. The regression estimates of Equation (4) give unbiased estimates only if $D_i$ is also conditionally ignorable: $D_i \perp\!\!\!\perp U_i$. If not, then regression estimates suffer from omitted variables bias if they do not adjust for the unobserved confounder $U_i$.

## 2.2 Selection on Costs and Benefits

The key to noting that CM is at risk of bias is noting that $D_i \perp\!\!\!\perp U_i$ is unlikely to hold in applied settings. Without an identification strategy for $D_i$, in addition to that for $Z_i$, bias will persist, given how we conventionally think of selection into treatment.

Consider a model where individual $i$ selects into a mediator after $Z_i, \boldsymbol{X}_i$, based on costs and benefits. The model has $i$ taking the mediator if the benefits of doing so exceed the costs. Write $C_i$ for individual $i$'s costs of taking mediator $D_i$, and $\mathbb{1}\left\{.\right\}$ for the indicator function.

$$D_i\left(z'\right) = \mathbb{1}\left\{\underbrace{Y_i\left(z', 1\right) - Y_i\left(z', 0\right)}_{\text{Benefits}} \geq \underbrace{C_i}_{\text{Costs}}\right\}, \quad \text{for } z' = 0, 1$$

Paragraph here talking about why the Roy model is useful. (Roy 1951, Heckman & Honore 1990).

Write in here with the updated notation of the above, and lead to the $D_i \not\perp\!\!\!\perp U_i$ statement, unless $\boldsymbol{X}_i$ is incredibly rich.

$$C_i(Z_i) = \mu_C(Z_i; \boldsymbol{X}_i) + U_{C,i}$$

And so we can write the first-stage selection equation in full.

$$D_i(z) = \mathbb{1}\left\{ \underbrace{U_{C,i} + U_{0,i} - U_{1,i}}_{\text{Unobserved}} \leq \underbrace{\mu_1(z; \boldsymbol{X}_i) - \mu_0(z; \boldsymbol{X}_i) - \mu_C(z; \boldsymbol{X}_i)}_{\text{Observed}} \right\}$$

## 2.3 Applied Settings

Three parapgraphs on what goes on in empirical settings. Survey the papers, and speak about it heavily in one paragraph.

table:

name — $Z \to Y$ — design for $Z$ — Primary mediatory — controls — Possible $U$.

## 2.4 Older Writing

This section connects causal mediation, without assuming the mediator is randomly assigned (i.e., under selection), with classic labour economics models for selection into treatment.

## 2.5 Selection Model Representation

The IV literature assumes a first-stage monotonicity condition, where randomised $Z_i$ influences mediator $D_i$ in at most one direction.

**Definition 2.** *First-stage Monotonicity (Imbens & Angrist 1994).*

$$\Pr\left(D_i(1) \geq D_i(0)\right) = 1 \tag{5}$$

Assuming 2(5) in a mediation setting opens mediation to the wide literature on IV and selection models for identification in the presence of selection.

**Theorem 2.** *Under monotonicity, mediator $D_i$ can be represented by a selection model. Suppose 2(5) holds, then there is a function $\mu(.)$ and random variable $U_i$ such that $D_i$ takes the following form.*

$$D_i(z) = \mathbb{1}\left\{\mu(z) \geq U_i\right\}, \quad \forall z = 0, 1$$

*Proof.* Special case of the Vytlacil (2002) equivalence result; see Subsection A.5. □

Theorem 2 is a powerful result: it says that at the cost of assuming monotonicity (as is done in the IV literature), then selection into $D_i$ takes a latent index form, and opens up identification in a mediation context to the wide literature on identifying treatment effects in selection models.

## 2.6 A Regression Framework for Direct and Indirect Effects

## 2.7 Selection into Education

In the education context, point identifying direct and indirect effects requires the *researcher controls for all sources of selection-into-education.*

While this assumption may hold true in two-way randomised experiments (e.g., in a laboratory or two-way RCT), it is unlikely to hold in the case of quasi-experimental variation in $Z$, or when modelling education as a mediator — absent a separate identification strategy for education $D$. To expand this point in an econometric selection-into-treatment framework, suppose selection follows a Roy model, where individual $i$ weighs the costs and benefits of completing education.

$$D_i(Z_i) = \mathbb{1}\left\{ \underbrace{C_i(Z_i)}_{\text{Costs}} \leq \underbrace{Y_i(Z_i, 1) - Y_i(Z_i, 0)}_{\text{Gains}} \right\}$$

Education choice $D_i(z)$ is clearly related to $Y_i(z, d)$ in this model, so let's see what the equation looks like in terms of sequential ignorability. As above, decompose costs into observed and unobserved factors.

$$C_i(Z_i) = \mu_C(Z_i; \boldsymbol{X}_i) + U_{C,i}$$

And so we can write the first-stage selection equation in full.

$$D_i(z) = \mathbb{1}\left\{ \underbrace{U_{C,i} + U_{0,i} - U_{1,i}}_{\text{Unobserved}} \leq \underbrace{\mu_1(z; \boldsymbol{X}_i) - \mu_0(z; \boldsymbol{X}_i) - \mu_C(z; \boldsymbol{X}_i)}_{\text{Observed}} \right\}$$

Sequential ignorability, where $Y_i(z, d) \perp\!\!\!\perp D_i(z') \mid \boldsymbol{X}_i$, would then require that $\mathbb{E}\left[U_{0,i} - U_{1,i} \mid D_i\right] = 0$. In the Roy model above, this would assume every single contribution for returns to education is contained in $\boldsymbol{X}_i$; if there are any unobserved sources by which people have systematically different returns to education, then they would select into education based on this, and bias naïve mediation estimates. This is unlikely to hold true, unless there is another identification strategy for $D_i$ — in addition to the one used for $Z_i$.

### 2.8 Estimating Direct and Indirect Effects

Quasi-experimental work does not take the assumption of "selection-on-observables" at face value without an explicit research design (Angrist & Pischke 2009) or modelling approach to address this issue.

A classical approach to modelling this issue, is a selection model approach (Heckman 1974, 1979). The approach assume $U_0, U_1$ follow a known distribution (e.g, bivariate normal), and estimates the regression via maximum likelihood. Alternatively, a control function approach estimates the system in two stages, avoiding (some) distributional assumptions if an instrument is used, at the cost of efficiency. In the following, I estimate direct and indirect effects first by OLS (assuming sequential ignorability), and then via both variants of the sample selection models, to compare estimates. Future work will consider estimates by using an alternative instrument for education, in the framework of Frölich & Huber (2017) to avoid the modelling assumptions inherent to sample selection models.

## 3 Control Function Estimates

If you could control for $U_i$, then you would. Laffers et al, for example, tests sequential ignoability.

# 4 Discussion and Future Work

This project aims to achieve two main goals: first, to test the claims that genetics (specifically Ed PGI) is associated with labour market outcomes independently, and secondly to connect the mediation literature to classical labour economic methods for adjusting for selection bias. Adjusting conventional mediation methods via a structural selection model for education reduces estimates for the direct channel in the genetic association from 50% to 0%. These results bring into question previous claims, in the context of Ed PGI, that genetics affect outcomes independently of education.

This work so far has focused on genetic association, and not causal effects, because the HRS data have no clear research design for random variation in Ed PGI. This means that the above estimates are only correlational because the EA Score is heritable, and not randomized. However, there is opportunity to analyse random genetic variation, thanks to Mendelian independent assortment. If a father has an Ed PGI of $X$ and a mother $Y$, then genetic mixing at conception means their child is expected to have an EA Score of $\frac{X+Y}{2}$. Thanks to genetic mixing, their child may have EA Score above or below the expected value, as they randomly inherited more/fewer genes in the EA Score from the parent with a higher score (a.k.a. random Mendelian segregation, ?). The HRS has no data on parental genetic information, so the estimates above did not control for parents' scores and are thus not causal (Young et al. 2022). In-progress work is expanding on the above, using UK Biobank data on genetic data after controlling for parents genes, expanding these results from genetic associations to genetic effects.

Secondly, this project has so far connected causal mediation to classical approaches to selection into treatment, using a Roy model as a key structural example for which selection models can overcome selection bias in mediation analyses. However, an explicit research design for years of education (in addition to Ed PGI) is necessary for realistic estimates — in the sense of a causally identified analysis (Angrist & Pischke 2009). An overlapping instrument for years of education is necessary to compare to the results of classical sample selection models.

## 5 Summary and Concluding Remarks

This paper studies the returns to higher education, using IV methods from the epidemiology literature and adjustments from the causal mediation literature to tackle violations of the exclusion restriction. First, I derive identification of the average mechanism effect under a selection-on-observables type assumption, and partial identification when unobserved selection confounding. I apply these methods to a sample of retirement age Americans in the years 1990–2021, using genetic information to instrument for higher education, estimating that higher education leads to roughly 40% higher earnings (point estimates), or between 8–44% higher earnings (partial bounds). Additionally, women had significantly higher returns to higher education over this time period.

The methods here provide alternatives to assuming the exclusion restriction in empirical applications of IV models, so can be useful in sensitivity analyses for any application of IV methods. Mendelian randomisation is a particularly useful application of IV methods, though the exclusion restriction is particularly problematic in practice. The approach allows researchers to use MR to study effects of both health conditions and behaviours with significant selection-into-treatment concerns, such as higher education.

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# A Appendix

This project used computational tools which are fully open-source. Any comments or suggestions may be sent to me at seh325@cornell.edu, or raised as an issue on the Github project.

A number of statistical packages, for the R language (R Core Team 2023), made the empirical analysis for this paper possible.

- *Tidyverse* (Wickham et al. 2019) collected tools for data analysis in the R language.
- *DoubleML* (Bach et al. 2024) implemented doubly robust methods used in the empirical analysis.
- *GRF* (Athey et al. 2019, Tibshirani et al. 2023) compiled forest computational tools for the R language.
- *Stargazer* (Hlavac 2018) provided methods to efficiently convert empirical results into presentable output in LaTeX.

## A.1 Identification in Causal Mediation

Imai et al. (2010, Theorem 1) states that the direct and indirect effects are identified under sequential ignorability, at each level of $Z_i = 0, 1$. For $z' = 0, 1$:

$$\mathbb{E}\left[Y_i(1, D_i(z')) - Y_i(0, D_i(z'))\right] = \int\int \Big(\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i, \boldsymbol{X}_i\right]\Big) dF_{D_i \,|\, Z_i = z', \boldsymbol{X}_i} dF_{\boldsymbol{X}_i},$$

$$\mathbb{E}\left[Y_i(z', D_i(1)) - Y_i(z', D_i(0))\right] = \int\int \mathbb{E}\left[Y_i \,|\, Z_i = z', D_i, \boldsymbol{X}_i\right] \Big(dF_{D_i \,|\, Z_i = 1, \boldsymbol{X}_i} - dF_{D_i \,|\, Z_i = 0, \boldsymbol{X}_i}\Big) dF_{\boldsymbol{X}_i}.$$

I focus on the averages, which are identified by consequence of the above.

$$\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right] = \mathbb{E}_{Z_i}\left[\mathbb{E}\left[Y_i(1, D_i(z')) - Y_i(0, D_i(z')) \,|\, Z_i = z'\right]\right]$$
$$\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right] = \mathbb{E}_{Z_i}\left[\mathbb{E}\left[Y_i(z', D_i(1)) - Y_i(z', D_i(0)) \,|\, Z_i = z'\right]\right]$$

My estimand for the average direct effect is a simple rearrangement of the above. The estimand for the average indirect effect relies on a different sequence, relying on (1) sequential ignorability, (2) conditional monotonicity. These give (1) identification of, and equivalence between, LADE conditional on $\boldsymbol{X}_i$ and ADE conditional on $\boldsymbol{X}_i$, (2) identification of the complier score.

$$
\begin{aligned}
&\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0)) \,|\, \boldsymbol{X}_i\right] \\
&= \Pr\left(D_i(1) = 1, D_i(0) = 0 \,|\, \boldsymbol{X}_i\right) \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, D_i(1) = 1, D_i(0) = 0, \boldsymbol{X}_i\right] \\
&= \Pr\left(D_i(1) = 1, D_i(0) = 0 \,|\, \boldsymbol{X}_i\right) \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, \boldsymbol{X}_i\right] \\
&= \Big(\mathbb{E}\left[D_i \,|\, Z_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[D_i \,|\, Z_i = 0, \boldsymbol{X}_i\right]\Big) \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \,|\, \boldsymbol{X}_i\right] \\
&= \Big(\mathbb{E}\left[D_i \,|\, Z_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[D_i \,|\, Z_i = 0, \boldsymbol{X}_i\right]\Big)\Big(\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i, D_i = 0, \boldsymbol{X}_i\right]\Big)
\end{aligned}
$$

Monotonicity is not technically required for the above. Breaking monotonicity would not change the identification of any of the above; it would be the same except replacing the complier score with a complier or defier score, $\Pr\left(D_i(1) \neq D_i(0) \,|\, \boldsymbol{X}_i\right) = \mathbb{E}\left[D_i \,|\, Z_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[D_i \,|\, Z_i = 0, \boldsymbol{X}_i\right]$.

## A.2 Continuous Average Causal Responses

Section here relating the approach to the average causal response function (see e.g., Angrist Imbens JASA 1996, Andrew Bacon for DiD 2023).

## A.3 Previous Literature

Create a table in this section that surveys previous research which employs mediation methods while having a clear causal design for $Z_i$, but not $D_i$.

| Paper | Field | Research Design for $Z_i$ | Research Design for $D_i$ | Selection bias? |
|---|---|---|---|---|
| Paper name 1. | | | | |

## A.4 Bias in Mediation Estimates

Suppose that $Z_i$ is ignorable conditional on $\boldsymbol{X}_i$, but $D_i$ is not.

### A.4.1 Bias in Direct Effect Estimates

To show that the conventional approach to mediation gives an estimate for the ADE with selection and non-complier bias, start with the components of the conventional estimands. This proof starts with the relevant expectations, conditional on a specific value of $\boldsymbol{X}_i$. For each $d' = 0, 1$.

$$
\begin{aligned}
\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = d', \boldsymbol{X}_i\right] =& \mathbb{E}\left[Y_i(1, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right], \\
\mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = d', \boldsymbol{X}_i\right] =& \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(0) = d', \boldsymbol{X}_i\right]
\end{aligned}
$$

And so

$$\begin{aligned}
&\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = d', \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = d', \boldsymbol{X}_i\right] \\
=&\mathbb{E}\left[Y_i(1, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(0) = d', \boldsymbol{X}_i\right] \\
=&\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] \\
&+ \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(0) = d', \boldsymbol{X}_i\right]
\end{aligned}$$

The final term is a sum of the ADE, conditional on $D_i(1) = d'$, and a selection bias term — difference in baseline terms between the (partially overlapping) groups for whom $D_i(1) = d'$ and $D_i(0) = d'$.

To reach the final term, note the following.

$$\begin{aligned}
&\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, \boldsymbol{X}_i\right] \\
=&\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] \\
&+ \Big(1 - \Pr\left(D_i(1) = d' \,|\, \boldsymbol{X}_i\right)\Big) \begin{pmatrix} \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = 1 - d', \boldsymbol{X}_i\right] \end{pmatrix}
\end{aligned}$$

The second term is a difference term between the average and the average for relevant complier groups.

Collect everything together, as follows.

$$\begin{aligned}
&\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = d', \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = d', \boldsymbol{X}_i\right] \\
=&\underbrace{\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, \boldsymbol{X}_i\right]}_{\text{ADE, conditional on } \boldsymbol{X}_i} \\
&+ \underbrace{\mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,|\, D_i(0) = d', \boldsymbol{X}_i\right]}_{\text{Selection bias}} \\
&+ \underbrace{\Big(1 - \Pr\left(D_i(1) = d' \,|\, \boldsymbol{X}_i\right)\Big) \begin{pmatrix} \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = d', \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,|\, D_i(1) = 1 - d', \boldsymbol{X}_i\right] \end{pmatrix}}_{\text{Non-complier bias}}
\end{aligned}$$

The proof is achieved by applying the expectation across $D_i = 0, 1$, and $\boldsymbol{X}_i$.

### A.4.2 Bias in Indirect Effect Estimates

To show that the conventional approach to mediation gives an estimate for the AIE with selection and non-complier bias, start with the definition of the ADE — the direct effect among compliers times the size of the complier group.

This proof starts with the relevant expectations, conditional on a specific value of $\boldsymbol{X}_i$.

$$\begin{aligned} &\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0)) \mid \boldsymbol{X}_i\right] \\ =& \Pr\left(D_i(1) = 1, D_i(0) = 0 \mid \boldsymbol{X}_i\right) \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i(1) = 1, D_i(0) = 0, \boldsymbol{X}_i\right] \end{aligned}$$

When $D_i$ is not ignorable, the bias comes from estimating the second term, $\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i(1) = 1\right.$

For each $z' = 0, 1$.

$$\begin{aligned} \mathbb{E}\left[Y_i \mid Z_i = z', D_i = 1, \boldsymbol{X}_i\right] =& \mathbb{E}\left[Y_i(z', 1) \mid D_i = 1, \boldsymbol{X}_i\right], \\ \mathbb{E}\left[Y_i \mid Z_i = z', D_i = 0, \boldsymbol{X}_i\right] =& \mathbb{E}\left[Y_i(z', 0) \mid D_i = 0, \boldsymbol{X}_i\right] \end{aligned}$$

So compose the CM estimand, as follows.

$$\begin{aligned} &\mathbb{E}\left[Y_i \mid Z_i = z', D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \mid Z_i = z', D_i = 0, \boldsymbol{X}_i\right] \\ =&\mathbb{E}\left[Y_i(z', 1) \mid D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(z', 0) \mid D_i = 0, \boldsymbol{X}_i\right] \\ =&\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i = 1, \boldsymbol{X}_i\right] + \mathbb{E}\left[Y_i(z', 0) \mid D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(z', 0) \mid D_i = 0, \boldsymbol{X}_i\right] \end{aligned}$$

The final term is a sum of the AIE, among the treated group $D_i = 1$, and a selection bias term — difference in baseline terms between the groups $D_i = 1$ and $D_i = 0$.

The AIE is the direct effect among compliers times the size of the complier group, so we need to compensate for the difference between the treated group $D_i = 1$ and complier group $D_i(1) = 1, D_i(0) = 0$.

Start with the difference between treated group's average and overall average.

$$\begin{aligned} &\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i = 1, \boldsymbol{X}_i\right] \\ =&\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid \boldsymbol{X}_i\right] \\ &+ \Big(1 - \Pr\left(D_i = 1 \mid \boldsymbol{X}_i\right)\Big) \begin{pmatrix} \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i = 1, \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i = 0, \boldsymbol{X}_i\right] \end{pmatrix} \end{aligned}$$

Then the difference between the compliers' average and the overall average.

$$\begin{aligned} &\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i(1) = 1, D_i(0) = 0, \boldsymbol{X}_i\right] \\ =&\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid \boldsymbol{X}_i\right] \\ &+ \frac{1 - \Pr\left(D_i(1) = 1, D_i(0) = 0 \mid \boldsymbol{X}_i\right)}{\Pr\left(D_i(1) = 1, D_i(0) = 0 \mid \boldsymbol{X}_i\right)} \begin{pmatrix} \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid D_i(1) = 0 \text{ or } D_i(0) = 1, \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \mid \boldsymbol{X}_i\right] \end{pmatrix} \end{aligned}$$

Collect everything together, as follows.

$$
\begin{aligned}
&\mathbb{E}\left[Y_i \,|\, Z_i = z', D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = z', D_i = 0, \boldsymbol{X}_i\right] \\
=& \underbrace{\mathbb{E}\left[Y_i(z', D_i(1)) - Y_i(z', D_i(0)) \,|\, \boldsymbol{X}_i\right]}_{\text{AIE, conditional on } \boldsymbol{X}_i, Z_i = z'} \\
&+ \underbrace{\mathbb{E}\left[Y_i(z', 0) \,|\, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i(z', 0) \,|\, D_i = 0, \boldsymbol{X}_i\right]}_{\text{Selection bias}} \\
&+ \underbrace{\left[\begin{array}{l}
\Big(1 - \Pr\left(D_i = 1 \,|\, \boldsymbol{X}_i\right)\Big) \begin{pmatrix} \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \,|\, D_i = 1, \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \,|\, D_i = 0, \boldsymbol{X}_i\right] \end{pmatrix} \\
+ \dfrac{1 - \Pr\left(D_i(1) = 1, D_i(0) = 0 \,|\, \boldsymbol{X}_i\right)}{\Pr\left(D_i(1) = 1, D_i(0) = 0 \,|\, \boldsymbol{X}_i\right)} \begin{pmatrix} \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \,|\, D_i(1) = 0 \text{ or } D_i(0) = 1, \boldsymbol{X}_i\right] \\ - \mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \,|\, \boldsymbol{X}_i\right] \end{pmatrix}
\end{array}\right]}_{\text{Non-complier bias}}
\end{aligned}
$$

The proof is finally achieved by multiplying by the complier score, $\Pr\left(D_i(1) = 1, D_i(0) = 0 \,|\, \boldsymbol{X}_i\right)$ $= \mathbb{E}\left[D_i \,|\, Z_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[D_i \,|\, Z_i = 0, \boldsymbol{X}_i\right]$, then applying the expectation across $Z_i = 0, 1$, and $\boldsymbol{X}_i$.

## A.5 Proof of the Selection Model Representation

Write the proof in here, following Vytlacil (2002) construction in the forward direction. Note that the notation needs updating for no exclusion restriction.

## A.6 A Regression Framework for Direct and Indirect Effects

Put $\mu_D(Z; \boldsymbol{X}) = \mathbb{E}\left[Y_i(Z, D) \,|\, \boldsymbol{X}\right]$ and $U_{D,i} = Y_i(Z, D) - \mu_D(Z; \boldsymbol{X})$, so we have the following expressions.

$$
Y_i(Z_i, 0) = \mu_0(Z_i; \boldsymbol{X}_i) + U_{0,i}, \quad Y_i(Z_i, 1) = \mu_1(Z_i; \boldsymbol{X}_i) + U_{1,i}
$$

$U_{0,i}, U_{1,i}$ are error terms with unknown distributions, mean independent of $Z_i, \boldsymbol{X}_i$ by definition — but possibly correlated with $D_i$.

$Z_i$ is independent of potential outcomes, so that $U_{0,i}, U_{1,i} \perp\!\!\!\perp Z_i$. Thus, the first-stage regression of $Z \to Y$ has unbiased estimates.

$$\begin{aligned} D_i &= Z_i D_i(1) + (1 - Z_i) D_i(0) \\ &= D_i(0) + Z_i \left[ D_i(1) - D_i(0) \right] \\ &= \underbrace{\mathbb{E}\left[ D_i(0) \,|\, \boldsymbol{X}_i \right]}_{\text{Intercept}} + \underbrace{Z_i \mathbb{E}\left[ D_i(1) - D_i(0) \right]}_{\text{Regressor}} \\ &\quad + \underbrace{D_i(0) - \mathbb{E}\left[ D_i(0) \,|\, \boldsymbol{X}_i \right] + Z_i \big( D_i(1) - D_i(0) - \mathbb{E}\left[ D_i(1) - D_i(0) \,|\, \boldsymbol{X}_i \right] \big)}_{\text{Mean-zero independent error term, since } Z_i \perp\!\!\!\perp D_i \,|\, \boldsymbol{X}_i} \\ &=: \phi + \pi Z_i + \varphi(\boldsymbol{X}_i) + \eta_i \\ \implies \mathbb{E}\left[ D_i \,|\, Z_i, \boldsymbol{X}_i \right] &= \phi + \pi Z_i + \varphi(\boldsymbol{X}_i), \text{ and thus unbiased estimates since } Z_i \perp\!\!\!\perp \phi, \eta_i. \end{aligned}$$

$Z_i$ is also assumed independent of potential outcomes $Y_i(.,.)$, so that $U_{0,i}, U_{1,i} \perp\!\!\!\perp Z_i$. Thus, the reduced form regression $Z \to Y$ also leads to unbiased estimates.

The same cannot be said of the regression that estimates direct and indirect effects, without further assumptions.

$$\begin{aligned} Y_i &= Z_i Y_i(1, D_i(1)) + (1 - Z_i) Y_i(0, D_i(0)) \\ &= Z_i D_i Y_i(1,1) \\ &\quad + (1 - Z_i) D_i Y_i(0,1) \\ &\quad + Z_i (1 - D_i) Y_i(1,0) \\ &\quad + (1 - Z_i)(1 - D_i) Y_i(0,0) \\ &= Y_i(0,0) \\ &\quad + Z_i \left[ Y_i(1,0) - Y_i(0,0) \right] \\ &\quad + D_i \left[ Y_i(0,1) - Y_i(0,0) \right] \\ &\quad + Z_i D_i \left[ Y_i(1,1) - Y_i(1,0) - \left( Y_i(0,1) - Y_i(0,0) \right) \right] \end{aligned}$$

And so $Y_i$ can be written as a regression equation in terms of the observed factors and error terms.

$$\begin{aligned} Y_i &= \mu_0(0; \boldsymbol{X}_i) \\ &\quad + D_i \left[ \mu_1(0; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i) \right] \\ &\quad + Z_i \left[ \mu_0(1; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i) \right] \\ &\quad + Z_i D_i \left[ \mu_1(1; \boldsymbol{X}_i) - \mu_0(1; \boldsymbol{X}_i) - \left( \mu_1(0; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i) \right) \right] \\ &\quad + U_{0,i} + D_i \left( U_{1,i} - U_{0,i} \right) \\ &=: \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \zeta(\boldsymbol{X}_i) + U_{0,i} + D_i \left( U_{1,i} - U_{0,i} \right) \end{aligned}$$

With the following definitions:

- $\alpha = \mathbb{E}\left[ \mu_0(0; \boldsymbol{X}_i) \right]$ and $\zeta(\boldsymbol{X}_i) = \mu_0(0; \boldsymbol{X}_i) - \alpha$ are the intercept terms.
- $\beta = \mu_1(0; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i)$ is the indirect effect under $Z_i = 0$

- $\gamma = \mu_0(1; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i)$ is the direct effect under $D_i = 0$.
- $\gamma = \mu_1(1; \boldsymbol{X}_i) - \mu_0(1; \boldsymbol{X}_i) - (\mu_1(0; \boldsymbol{X}_i) - \mu_0(0; \boldsymbol{X}_i))$ is the interaction effect.
- $U_{0,i} + D_i\left(U_{1,i} - U_{0,i}\right)$ is the remaining error term.

This sequence gives us the resulting regression equation:

$$\mathbb{E}\left[Y_i \,|\, Z_i, D_i, \boldsymbol{X}_i\right] = \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \zeta(\boldsymbol{X}_i) + \mathbb{E}\left[D_i\left(U_{1,i} - U_{0,i}\right) \,|\, \boldsymbol{X}_i\right]$$

Taking the conditional expectation, and collecting for the expressions of the direct and indirect effects:[7]

$$\begin{aligned}\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right] &= \mathbb{E}\left[\pi\left(\beta + Z_i\delta\right)\right] \\ \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right] &= \mathbb{E}\left[\gamma + \delta D_i\right]\end{aligned}$$

These terms are conventionally estimated in a simultaneous regression (Imai et al. 2010).

If sequential ignorability does not hold, then the regression estimates from estimating the mediation equations (without adjusting for the contaminated bias term) suffer from omitted variables bias.

$$\begin{aligned}\mathbb{E}_{\boldsymbol{X}_i}\left[\mathbb{E}\left[Y_i \,|\, Z_i = D_i = 0, \boldsymbol{X}_i\right]\right] &= \alpha + \mathbb{E}\left[D_i\left(U_{1,i} - U_{0,i}\right)\right] \\ \mathbb{E}_{\boldsymbol{X}_i}\left[\mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = 0, \boldsymbol{X}_i\right]\right] &= \beta + \frac{\operatorname{Cov}\left(D_i,\ D_i\left(U_{1,i} - U_{0,i}\right)\right)}{\operatorname{Var}\left(D_i\right)} \\ \mathbb{E}_{\boldsymbol{X}_i}\left[\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = 0, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = 0, \boldsymbol{X}_i\right]\right] &= \gamma + \frac{\operatorname{Cov}\left(Z_i,\ D_i\left(U_{1,i} - U_{0,i}\right)\right)}{\operatorname{Var}\left(Z_i\right)} \\ \mathbb{E}_{\boldsymbol{X}_i}\begin{bmatrix}\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i = 0, \boldsymbol{X}_i\right] \\ -\left(\mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = 1, \boldsymbol{X}_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i = 0, \boldsymbol{X}_i\right]\right)\end{bmatrix} &= \delta + \frac{\operatorname{Cov}\left(Z_i D_i,\ D_i\left(U_{1,i} - U_{0,i}\right)\right)}{\operatorname{Var}\left(Z_i D_i\right)}\end{aligned}$$

And so the direct and indirect effect estimates are contaminated by these bias terms.

**Senan note:** write here OLS estimates of CM estimands, with the OVB terms.

Rearrange the above to be $\mathbb{E}\left[Y_i \,|\, Z_i = 1, D_i\right] - \mathbb{E}\left[Y_i \,|\, Z_i = 0, D_i\right]$ and such.

[7] These equations have simpler expressions after assuming constant treatment effects in a linear framework; I have avoided this as having compliers, and controlling for observed factors $\boldsymbol{X}_i$ only makes sense in the case of heterogeneous treatment effects.