Causal Mediation in Natural Experiments

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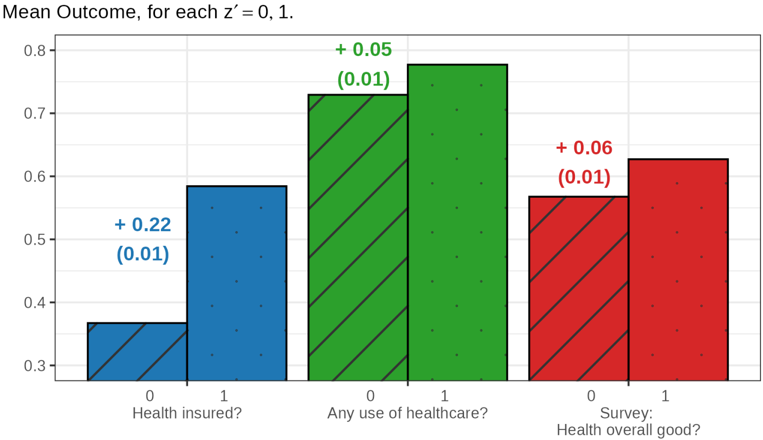
Cornell, Labour Economics Seminar
17 November 2025

Notes for long version

1. Translate the wording for everyone (mechanism, quasi-random), and be clearer about suggestive. Use words like necessary but not sufficient.
2. Needs a clearer introduction, which accurately overviews and previews the approach findings
3. Novelty needs to be loud, so put it first, Write ATE = ADE + AIE for Levon, and enumerates folks theorems for why CM did not take off in econ but did in medicine epi psych)
4. Evan Riehl recommends a slide with quotes from top 5s that investigates mechanisms (note the approach is necessary but not sufficient for mechanism analysis)
5. Mention Kwon Roth result on my data, reject null then move on....
6. Longer presentation needs clear reasoning on the IV.

Intro: Oregon Health Insurance Experiment

In 2008, Oregon gave access to socialised health insurance by wait-list lottery (Finkelstein et al, 2012).



Applied practice:

⇒ Suggestive evidence for healthcare as mechanism for wait-list lottery. . . .

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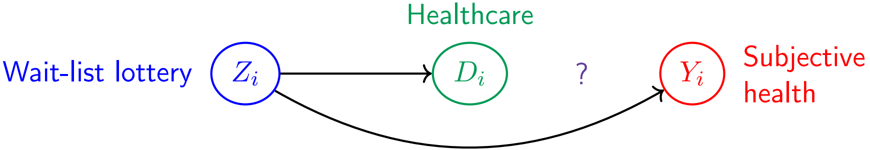
Figure: Model for Suggestive Evidence of a Mechanism.



Unanswered questions:

- Is $D_i \to Y_i$ small, large, or even existent?
- Where else do we accept assumed causal effects without evidence?

Plausible direct effects, which are overlooked:

- Psychological gains, less worry for financial consequences of health
- income effect

Introduction — Road map

Causal Mediation (CM) is an additional framework, with clear identification and assumptions to add quantitative analysis to studying mechanisms.
Introduce CM longer here, and the big picture.

Introduction — Contributions

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1. Problems with conventional approach to CM in observational settings. **[Negative result]**
2. Recovering valid CM effects, via MTE + control function modelling. **[Positive result]**

New insights from intersection of two fields:

- **Causal Mediation (CM).**

  Imai Keele Yamamoto (2010), Frölich Huber (2017), Deuchert Huber Schelker (2019), Huber (2020), Kwon Roth (2024).
- **Labour theory, Selection-into-treatment, MTEs.**

  Roy (1951), Heckman (1979), Heckman Honoré (1990), Vycatil (2002), Heckman Vycatil (2005), Brinch Mogstad Wiswall (2017), Kline Walters (2019).

Introduction – CM

Consider ignorable treatment $Z_i = 0, 1$, binary mediator $D_i = 0, 1$, and continuous outcome Y_i.

Assumption: **Mediator Ignorability** (**MI**, Imai Keele Yamamoto 2010)
mediator D_i is *also* ignorable, conditional on $\boldsymbol{X}_i$ and Z_i realisation.

Average Direct Effect (ADE) and Average Indirect Effect (AIE) are identified by two-stage regression

- ADE is causal effect $Z_i \to Y_i$, blocking the indirect D_i path
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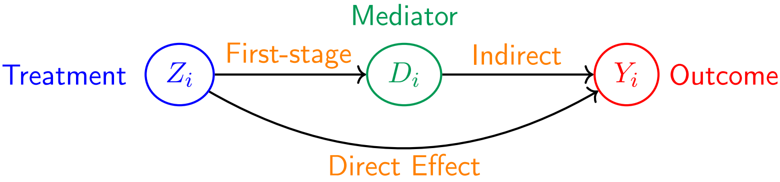

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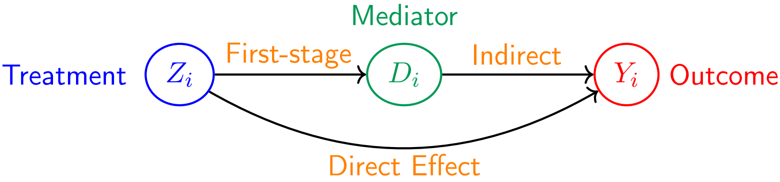

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1. Selection Bias

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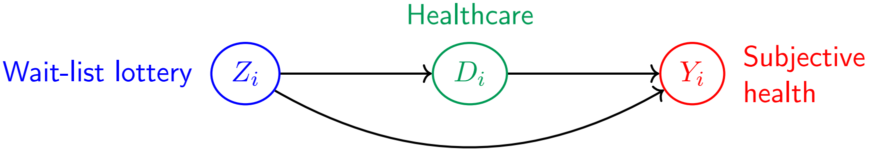
Would this assumption hold true in settings economists study?

E.g., Oregon Health Insurance Experiment.



1. Treatment is as-good-as random (2008 Oregon wait-list lottery).
2. Healthcare is quasi-random, conditional on lottery realisation Z_i and demographic controls $\boldsymbol{X}_i$.

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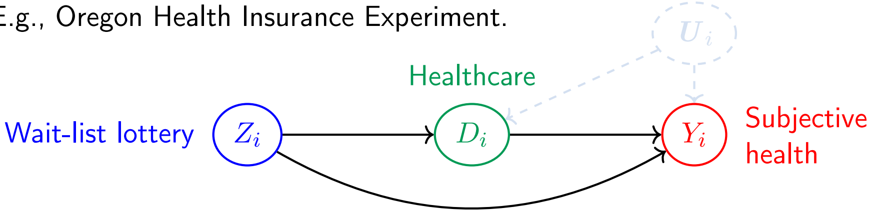

Theorem: If choice to attend healthcare is unconstrained, based on costs and benefits (Roy model) and demographics do not explain all benefits $\Longrightarrow$ **MI** does not hold, there is unobserved confounding.

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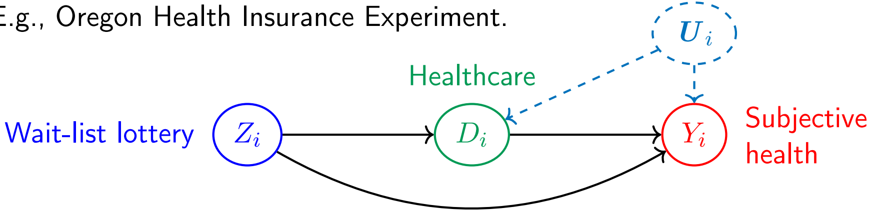

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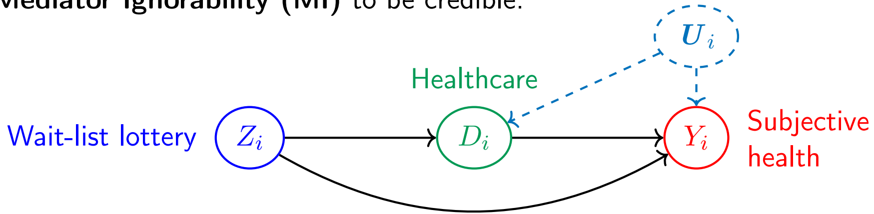

If not, then CM effects are contaminated by bias terms, similar to classical selection bias (e.g., Heckman Ichimura Smith Todd 1998).

- ADE: $\text{CM Estimand} = \text{ADE} + \left(\text{Selection Bias} + \text{Group difference bias}\right)$
- AIE: $\text{CM Estimand} = \text{AIE} + \left(\text{Selection Bias} + \text{Group difference bias}\right)$

ADE biases AIE biases

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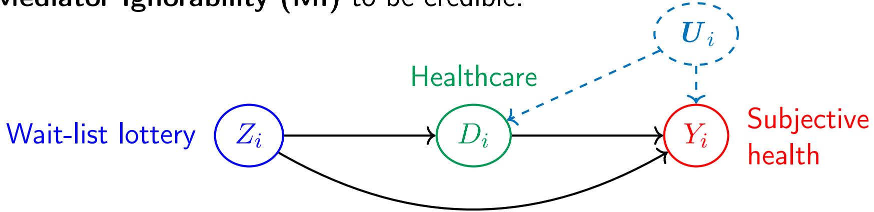

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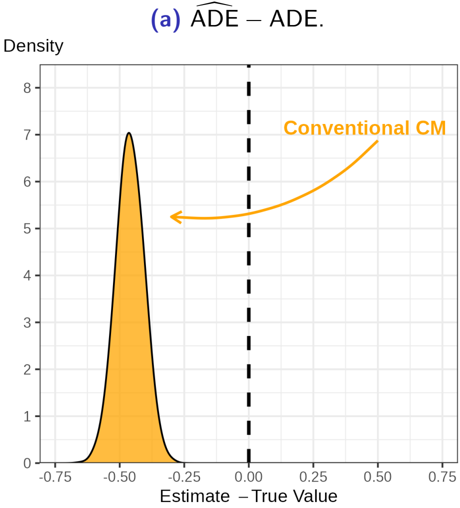
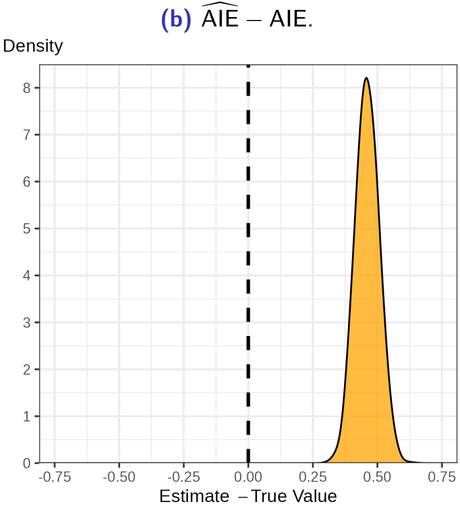
1. Selection Bias

In a simulation with Roy selection-into-D_i, CM estimates are biased.

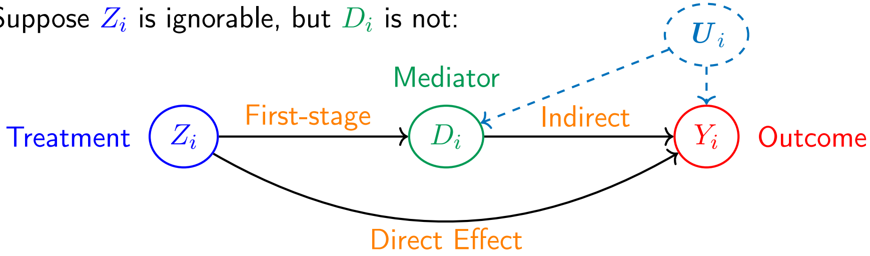




2. CM with Selection

Conventional CM methods do not identify ADE + AIE in a natural experiment setting, but can we build a credible structural model?

Suppose Z_i is ignorable, but D_i is not:



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1. Average first-stage, $Z_i \rightarrow D_i$, is identified
2. Average second-stage, $Z_i, D_i \rightarrow Y_i$, is not — represented by $\boldsymbol{U}_i$.

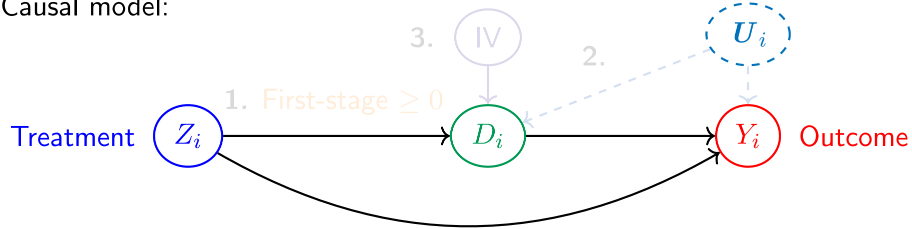
Intuition: model $\boldsymbol{U}_i$ via mediator MTE to identify ADE + AIE.

2. CM with Selection — Identification

MTE assumptions:

1. Mediator monotonicity
2. Selection on mediator benefits
3. IV for mediator take-up cost.

Causal model:



Proposition: Under MTE assumptions, the mediator MTE is identified.

Theorem: Mediation second-stage effects, $Z_i, D_i \to Y_i$, are identified by the MTE associated Control Functions (CFs).

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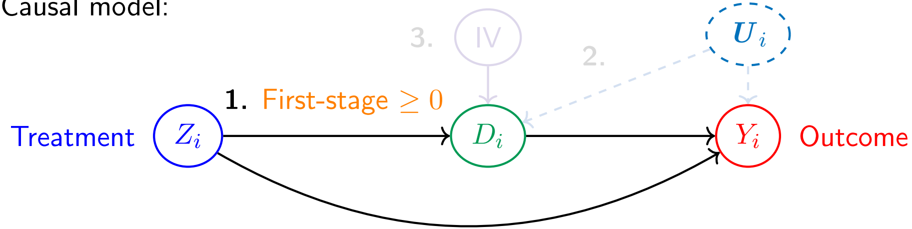

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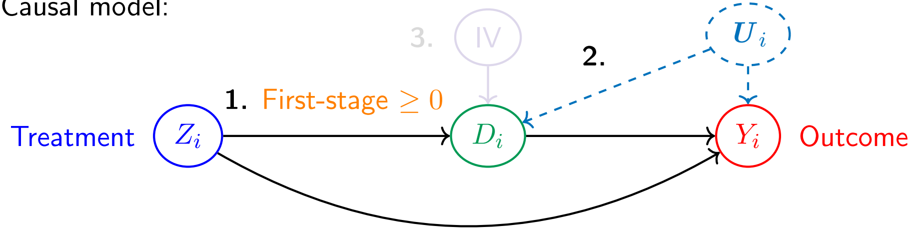

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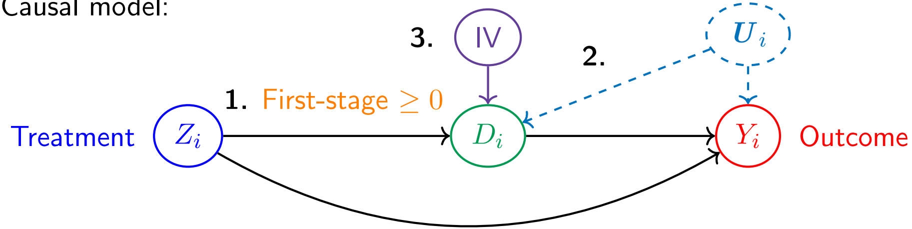

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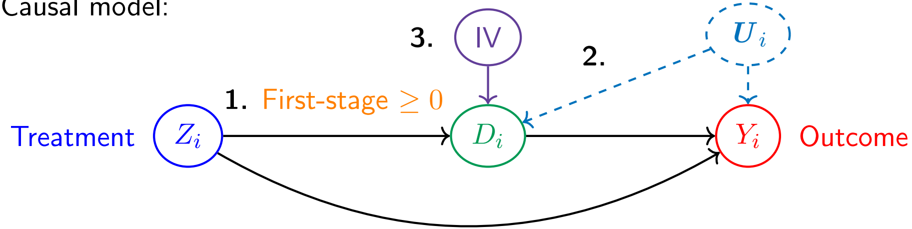

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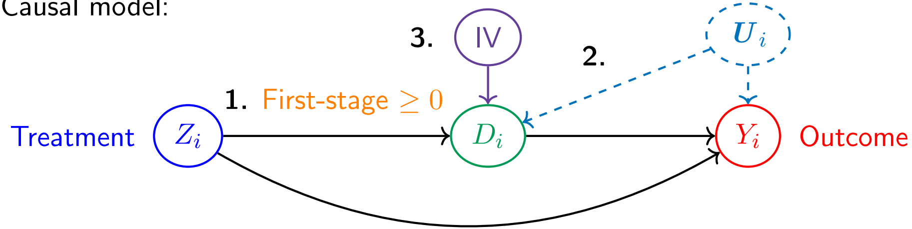

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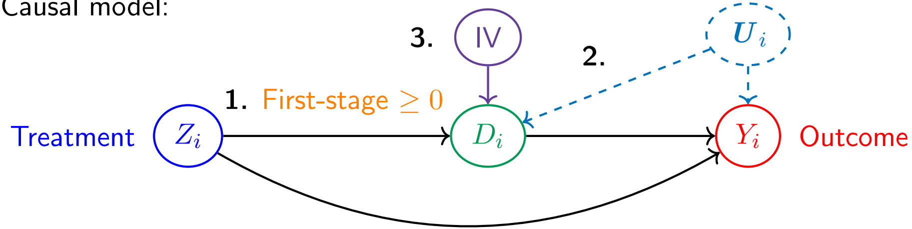

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2. CM with Selection — Estimation

In practice, this means two-stage CM estimation, with CF in second-stage.

Parametric CF Estimation Recipe:

1. Estimate mediation first-stage with probit, including the IV.
2. Estimate mediation second-stage by OLS, with Mills ratio CF terms (Heckman 1979).
3. Compose CM estimates from two-stage plug-in estimates (same as parametric MTEs, Björklund Moffitt 1987).

Semi-parametric CF Estimation Recipe:

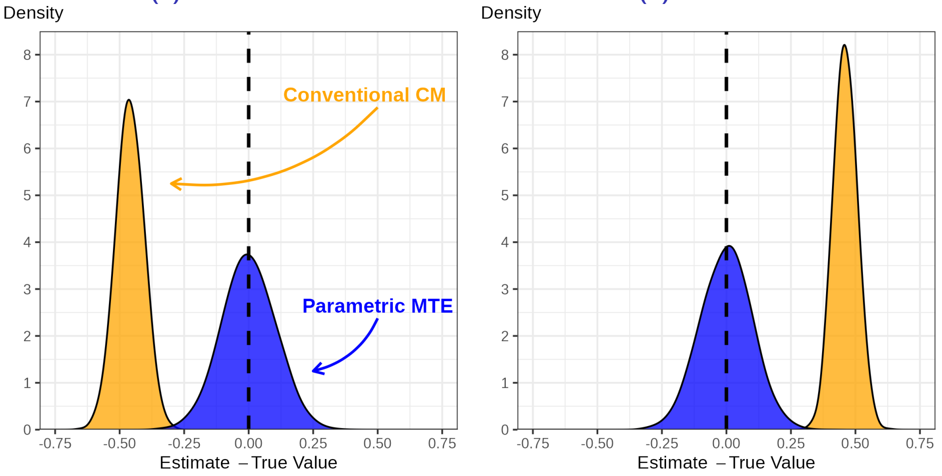
Replace **2.** with semi-parametric CFs (same estimation as MTEs).

$\Longrightarrow$ Conventional CM estimates (two-stages) + IV-guided CF adjustment.

2. CM with Selection — Estimation

Figure: CM Estimates from 10,000 DGPs with **Normal** Errors.

(a) $\widehat{\text{ADE}} - \text{ADE}$. (b) $\widehat{\text{AIE}} - \text{AIE}$.



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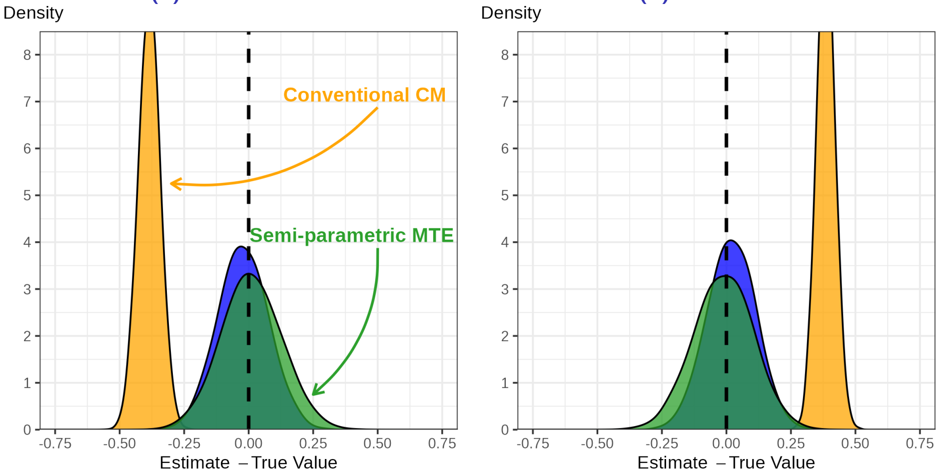
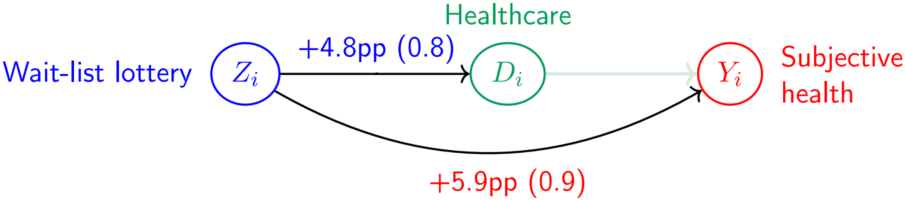


3. Returning to Oregon

Winning access to Medicaid increases healthcare usage, and subjective health:



CM is quantitatively estimating the entire system:

- Use correlational estimate of $D_i \to Y_i$
- Does visiting healthcare at least once increase subjective health 12 months later?
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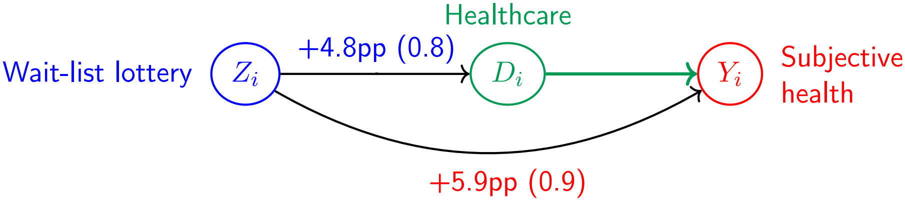

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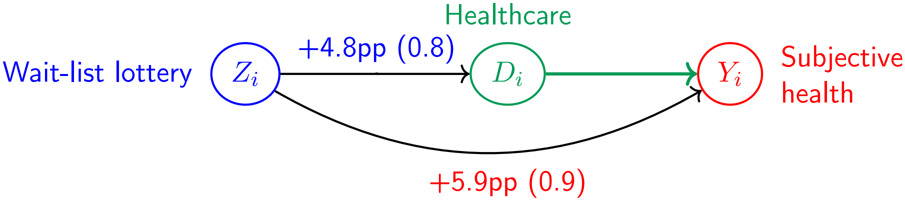

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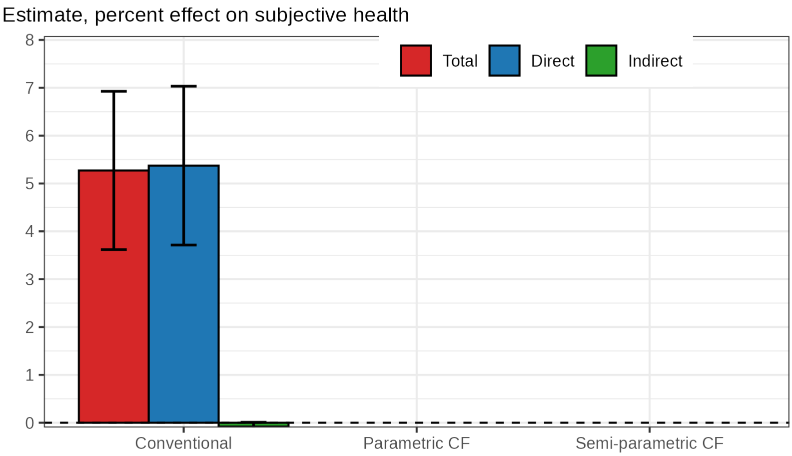
Conventional CM estimates lottery effects as mostly direct, ≈ 0 healthcare.



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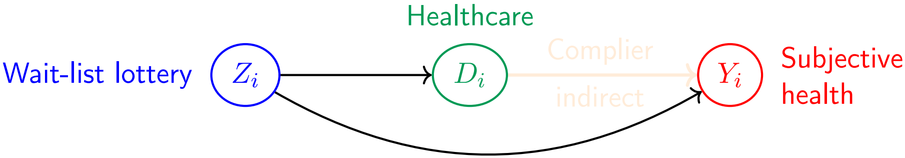

My approach to **CM** is modelling selection-into-D_i via mediator MTE:

- Uses an estimate of $D_i \to Y_i$ (plus complier extrapolation)
- Regular healthcare location pre-lottery serves as first-stage IV ▸ IV.
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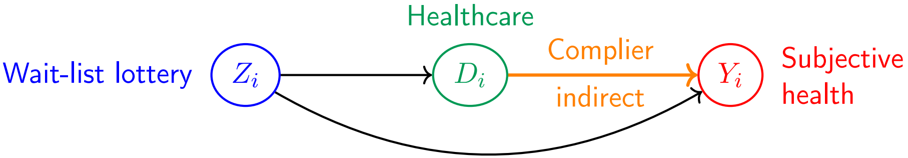

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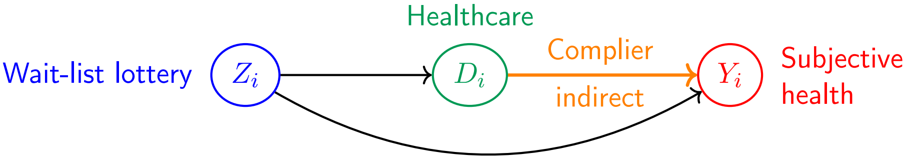

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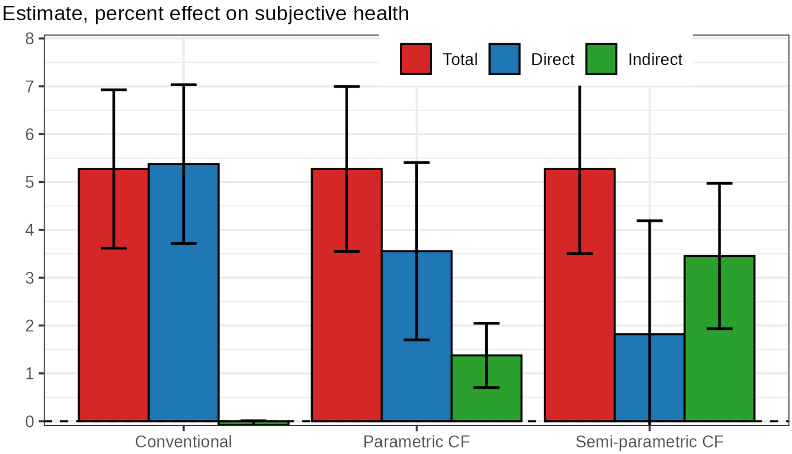
Using my approach, with regular healthcare location as an excluded IV, restores indirect effect through increasing healthcare visitation.



Conclusion

Overview:

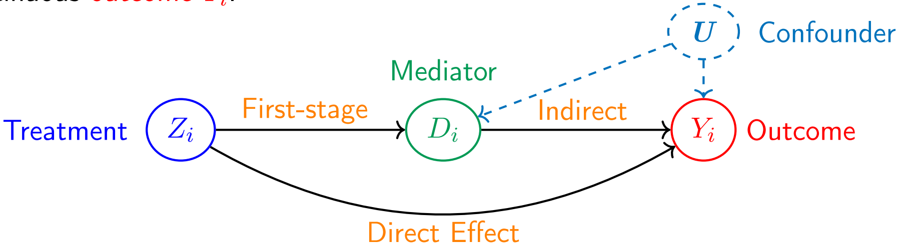
1. CM as alternative to "suggestive evidence for mechanisms."
2. Selection bias in conventional CM analyses with no case for mediator ignorability.
3. Connect CM with labour theory + selection-into-treatment + MTEs.

Caveats and points to remember:

- Structural assumptions and IV for identification + estimation (not ideal).
- Application to Oregon Health Insurance Experiment, showing subjective health + well-being effects mediated by healthcare.
- **Credible** analyses of mechanisms are hard in practice, wide confidence intervals show true uncertainty.

Appendix: CM Guiding Model

Consider binary treatment $Z_i = 0, 1$, binary mediator $D_i = 0, 1$, and continuous outcome Y_i.



Average Direct Effect (ADE) : $\mathbb{E}\left[Y_i\left(1, D_i(Z_i)\right) - Y_i\left(0, D_i(Z_i)\right)\right]$

- ADE is causal effect $Z \to Y$, blocking the indirect D_i path.

Average Indirect Effect (AIE): $\mathbb{E}\left[Y_i\left(Z_i, D_i(1)\right) - Y_i\left(Z_i, D_i(0)\right)\right]$

- AIE is causal effect of $D_i(Z_i) \to Y_i$, blocking the direct Z_i path.

Group Difference — ADE

CM effects contaminated by (less interpretable) bias terms.

CM Estimand = ADEM + Selection Bias

$$\underbrace{\mathbb{E}_{D_i}\Big[\mathbb{E}\left[Y_i \mid Z_i = 1, D_i\right] - \mathbb{E}\left[Y_i \mid Z_i = 0, D_i\right]\Big]}_{\text{Estimand, Direct Effect}}$$

$$= \underbrace{\mathbb{E}_{D_i = d'}\left[\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \,\middle|\, D_i(1) = d'\right]\right]}_{\text{Average Direct Effect on Mediator (ADEM) take-up — i.e., } D_i(1) \text{ weighted}}$$

$$+ \underbrace{\mathbb{E}_{D_i}\Big[\mathbb{E}\left[Y_i(0, D_i(Z_i)) \,\middle|\, D_i(1) = d'\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \,\middle|\, D_i(0) = d'\right]\Big]}_{\text{Selection Bias}}$$

The weighted ADE you get here is a positive weighted sum of local ADEs, but with policy irrelevant weights $D_i(1) = d'$.

$\implies$ consider this group bias, noting difference from true ADE. Back

Selection Bias — Direct Effect

CM Effects + contaminating bias.

$$\text{CM Estimand} = \text{ADE} + \Big(\text{Selection Bias} + \text{Group difference bias}\Big)$$

Model

$$\underbrace{\mathbb{E}_{D_i=d'}\left[\mathbb{E}\left[Y_i \mid Z_i = 1, D_i = d'\right] - \mathbb{E}\left[Y_i \mid Z_i = 0, D_i = d'\right]\right]}_{\text{Estimand, Direct Effect}}$$

$$= \underbrace{\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))\right]}_{\text{Average Direct Effect}}$$

$$+ \underbrace{\mathbb{E}_{D_i=d'}\left[\mathbb{E}\left[Y_i(0, D_i(Z_i)) \mid D_i(1) = d'\right] - \mathbb{E}\left[Y_i(0, D_i(Z_i)) \mid D_i(0) = d'\right]\right]}_{\text{Selection Bias}}$$

$$+ \underbrace{\mathbb{E}_{D_i=d'}\left[\begin{array}{l}\Big(1 - \Pr\big(D_i(1) = d'\big)\Big) \\ \times \left(\begin{array}{l}\mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \mid D_i(1) = 1 - d'\right] \\ - \mathbb{E}\left[Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \mid D_i(0) = d'\right]\end{array}\right)\end{array}\right]}_{\text{Group difference bias}}$$

Group-diff

Group Difference — AIE

CM effects contaminated by (less interpretable) bias terms.

$$\text{CM Estimand} = \text{AIEM} + \Big(\text{Selection Bias} + \text{Group difference bias}\Big)$$

$$\underbrace{\mathbb{E}_{Z_i}\left[\Big(\mathbb{E}\left[D_i \mid Z_i = 1\right] - \mathbb{E}\left[D_i \mid Z_i = 0\right]\Big) \times \Big(\mathbb{E}\left[Y_i \mid Z_i, D_i = 1\right] - \mathbb{E}\left[Y_i \mid Z_i, D_i = 0\right]\Big)\right]}_{\text{Estimand, Indirect Effect}}$$

$$= \underbrace{\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0)) \mid D_i = 1\right]}_{\text{Average Indirect Effect on Mediated (AIEM) — i.e., } D_i = 1 \text{ weighted}}$$

$$+ \underbrace{\overline{\pi}\Big(\mathbb{E}\left[Y_i(Z_i, 0) \mid D_i = 1\right] - \mathbb{E}\left[Y_i(Z_i, 0) \mid D_i = 0\right]\Big)}_{\text{Selection Bias}}$$

$$+ \underbrace{\overline{\pi}\left[\left(\frac{1 - \Pr\left(D_i(1) = 1, D_i(0) = 0\right)}{\Pr\left(D_i(1) = 1, D_i(0) = 0\right)}\right)\Big(\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i(1) = 0 \text{ or } D_i(0) \right. \\ \left. - \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0)\right]\right.}_{\text{Groups difference Bias}}$$

The weighted AIE you get here is not a positive weighted sum of local AIEs, because the AIE is only about $D(Z)$ compliers. ▸ Model.

$\implies$ consider this group bias, noting difference from true AIE. ▸ Back

Selection Bias — Indirect Effect

CM Effects + contaminating bias, where $\overline{\pi} = \Pr\left(D_i(0) \neq D_i(1)\right)$.

CM Estimand = AIE + (Selection Bias + Group difference bias) ▸ Model

$$\underbrace{\mathbb{E}_{Z_i}\left[\Big(\mathbb{E}\left[D_i \mid Z_i = 1\right] - \mathbb{E}\left[D_i \mid Z_i = 0\right]\Big) \times \Big(\mathbb{E}\left[Y_i \mid Z_i, D_i = 1\right] - \mathbb{E}\left[Y_i \mid Z_i, D_i = 0\right]\Big)\right]}_{\text{Estimand, Indirect Effect}}$$

$$= \underbrace{\mathbb{E}\left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))\right]}_{\text{Average Indirect Effect}}$$

$$+ \underbrace{\overline{\pi}\Big(\mathbb{E}\left[Y_i(Z_i, 0) \mid D_i = 1\right] - \mathbb{E}\left[Y_i(Z_i, 0) \mid D_i = 0\right]\Big)}_{\text{Selection Bias}}$$

$$+ \underbrace{\overline{\pi}\left[\Big(1 - \Pr\left(D_i = 1\right)\Big)\begin{pmatrix}\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i = 1\right] \\ - \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i = 0\right]\end{pmatrix} + \left(\frac{1 - \Pr\left(D_i(1) = 1, D_i(0) = 0\right)}{\Pr\left(D_i(1) = 1, D_i(0) = 0\right)}\right)\begin{pmatrix}\mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i(Z_i) \neq Z_i\right] \\ - \mathbb{E}\left[Y_i(Z_i, 1) - Y_i(Z_i, 0)\right]\end{pmatrix}\right]}_{\text{Groups difference Bias}}$$

▸ Group-diff

Semi-parametric Control Functions

Semi-parametric specifications for the CFs λ_0, λ_1 bring some complications to estimating the AIE.

$$\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 0, \boldsymbol{X}_i\right] = \alpha + \gamma Z_i + \varphi\big(\boldsymbol{X}_i\big) + \boxed{\rho_0 \lambda_0\big(\pi(Z_i; \boldsymbol{X}_i)\big)},$$

$$\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 1, \boldsymbol{X}_i\right] = (\alpha + \beta) + (\gamma + \delta) Z_i + \varphi\big(\boldsymbol{X}_i\big) + \boxed{\rho_1 \lambda_1\big(\pi(Z_i; \boldsymbol{X}_i)\big)}$$

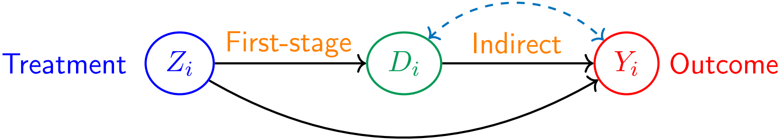
Intercepts, $\alpha, (\alpha + \beta)$, and relevance parameters ρ_0, ρ_1 are not separately identified from the CFs $\lambda_0(.), \lambda_1(.)$ so CF extrapolation term $(\rho_1 - \rho_0)\Gamma\big(\pi(0; \boldsymbol{X}_i), \pi(1; \boldsymbol{X}_i)\big)$ is not directly identified or estimable.

These problems can be avoided by estimating the AIE using its relation to the ATE, $\widehat{\text{AIE}}^{\text{CF}} =$

$$\widehat{\text{ATE}} - (1 - \overline{Z}) \underbrace{\left(\frac{1}{N}\sum_{i=1}^{N} \widehat{\gamma} + \widehat{\delta}\, \widehat{\pi}(1; \boldsymbol{X}_i)\right)}_{\widehat{\text{ADE}} \text{ given } Z_i = 1} - \overline{Z} \underbrace{\left(\frac{1}{N}\sum_{i=1}^{N} \widehat{\gamma} + \widehat{\delta}\, \widehat{\pi}(0; \boldsymbol{X}_i)\right)}_{\widehat{\text{ADE}} \text{ given } Z_i = 0}.$$

Appenidx: CM with Selection

Suppose Z_i is ignorable, D_i is not, so we have the following causal model.



Then this system has the following random coefficient equations:

$$D_i = \phi + \overline{\pi} Z_i + \varphi(\boldsymbol{X}_i) + U_i$$

$$Y_i = \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \zeta(\boldsymbol{X}_i) + \underbrace{\left(1 - D_i\right) U_{0,i} + D_i U_{1,i}}_{\text{Correlated error term}}$$

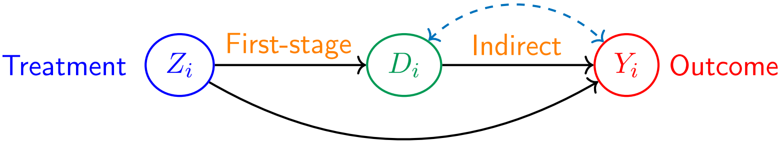
where β, γ, δ are functions of $\mu_{d'}(z'; \boldsymbol{X}_i)$.

$$\text{ADE} = \mathbb{E}\left[\gamma + \delta D_i\right], \quad \text{AIE} = \mathbb{E}\left[\overline{\pi}\left(\beta + \delta Z_i + \widetilde{U}_i\right)\right]$$

with $\widetilde{U}_i = \mathbb{E}\left[U_{1,i} - U_{0,i} \mid \boldsymbol{X}_i, D_i(0) \neq D_i(1)\right]$ unobserved complier gains.

Appenidx: CM with Selection

Suppose Z_i is ignorable, D_i is not, so we have the following causal model.



Main problem, second-stage is not identified:

$$
\begin{aligned}
\mathbb{E}\left[Y_i \,|\, Z_i, D_i, \boldsymbol{X}_i\right] \;=\; & \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \varphi(\boldsymbol{X}_i) \\
& + \underbrace{(1 - D_i)\,\mathbb{E}\left[U_{0,i} \,|\, D_i = 0, \boldsymbol{X}_i\right] + D_i \mathbb{E}\left[U_{1,i} \,|\, D_i = 1, \boldsymbol{X}_i\right]}_{\text{Unobserved } D_i \text{ confounding.}}
\end{aligned}
$$

Identification intuition: Identify second-stage via MTE control function.

Appenidx: CM with Selection — Identification

Assume:

1. Mediator monotonicity, $\Pr\left(D_i(0) \leq D_i(1) \,|\, \boldsymbol{X}_i\right) = 1$

$$\implies D_i(z') = \mathbb{1}\left\{U_i \leq \pi(z'; \boldsymbol{X}_i)\right\}, \quad \text{for } z' = 0, 1 \text{ (Vycatil 2002).}$$

2. Selection on mediator benefits, Cov$\left(U_i,\, U_{0,i}\right)$, Cov$\left(U_i,\, U_{1,i}\right) \neq 0$

$\implies$ First-stage take-up informs second-stage confounding.

3. There is an IV for the mediator, $\boldsymbol{X}_i^{\text{IV}}$ among control variables $\boldsymbol{X}_i$.

$$\implies \pi(Z_i; \boldsymbol{X}_i) = \Pr\left(D_i = 1 \,|\, Z_i, \boldsymbol{X}_i\right) \text{ is separately identified.}$$

Proposition:

$$\mathbb{E}\left[Y_i(z', 1) - Y_i(z', 0) \,\middle|\, Z_i = z', \boldsymbol{X}_i, U_i = p'\right]$$
$$= \beta + \delta z' + \mathbb{E}\left[U_{1,i} - U_{0,i} \,\middle|\, \boldsymbol{X}_i, U_i = p'\right], \qquad \text{for } p' \in (0, 1).$$

Appenidx: CM with Selection — Identification

The marginal effect has corresponding Control Functions (CFs), describing unobserved selection-into-D_i,

$$\rho_0 \lambda_0(p') = \mathbb{E}\left[U_{0,i} \,\middle|\, p' \leq U_i\right], \quad \rho_1 \lambda_1(p') = \mathbb{E}\left[U_{1,i} \,\middle|\, U_i \leq p'\right].$$

These CFs restore second-stage identification, by extrapolating from $\boldsymbol{X}_i^{\text{IV}}$ compliers to $D_i(Z_i)$ mediator compliers,

$$\begin{aligned} \mathbb{E}\left[Y_i \,|\, Z_i, D_i, \boldsymbol{X}_i\right] \; &= \; \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \varphi(\boldsymbol{X}_i) \\ &+ \underbrace{\rho_0 \left(1 - D_i\right) \lambda_0\big(\pi(Z_i; \boldsymbol{X}_i)\big) + \rho_1 D_i \lambda_1\big(\pi(Z_i; \boldsymbol{X}_i)\big)}_{\text{CF adjustment.}} \end{aligned}$$

This adjusted second-stage re-identifies the ADE and AIE,

$$\text{ADE} = \mathbb{E}\left[\gamma + \delta D_i\right], \;\; \text{AIE} = \mathbb{E}\left[\overline{\pi}\left(\beta + \delta Z_i + \underbrace{\left(\rho_1 - \rho_0\right) \Gamma\big(\pi(0; \boldsymbol{X}_i), \, \pi(1; \boldsymbol{X}_i)\big)}_{\text{Mediator compliers extrapolation.}}\right)\right]$$

Appenidx: CM with Selection — Estimation

Will explain how estimation works, with simulation evidence.

1. Random treatment $Z_i \sim \text{Binom}(0.5)$, for $n = 5,000$.
2. $(U_{0,i}, U_{1,i}) \sim \text{BivariateNormal}(0, 0, \sigma_0, \sigma_1, \rho)$, Costs $C_i \sim N(0, 0.5)$.

Roy selection-into-D_i, with constant partial effects + interaction term.

$$D_i(z') = \mathbb{1}\left\{C_i \leq Y_i(z', 1) - Y_i(z', 0)\right\},$$
$$Y_i(z', d') = \left(z' + d' + z'd'\right) + U_{d'} \qquad \text{for } z', d' = 0, 1.$$

Following the previous, these data have the following first and second-stage equations, where $\boldsymbol{X}_i^{\text{IV}}$ is an additive cost IV:

$$D_i = \mathbb{1}\left\{C_i - \left(U_{1,i} - U_{0,i}\right) \leq Z_i - \boldsymbol{X}_i^{\text{IV}}\right\}$$
$$Y_i = Z_i + D_i + Z_i D_i + (1 - D_i) U_{0,i} + D_i U_{1,i}.$$

$\implies$ unobserved confounding by BivariateNormal $\left(U_{0,i}, U_{1,i}\right)$.

Appenidx: CM with Selection — Estimation

Errors are normal, so system is Heckman (1979) selection model.

CFs are the inverse Mills ratio, with $\phi(.)$ normal pdf and $\Phi(.)$ normal cdf,

$$\lambda_0(p') = \frac{\phi(-\Phi^{-1}(p'))}{\Phi(-\Phi^{-1}(p'))}, \quad \lambda_1(p') = \frac{\phi(\Phi^{-1}(p'))}{\Phi(\Phi^{-1}(p'))}, \quad \text{for } p' \in (0,1).$$

Parametric Estimation Recipe:

1. Estimate first-stage $\pi(Z_i; \boldsymbol{X}_i)$ with probit, including $\boldsymbol{X}_i^{\text{IV}}$.
2. Include λ_0, λ_1 CFs in second-stage OLS estimation.
3. Compose CM estimates from two-stage plug-in estimates.

$\rightarrow$ Same as conventional CM estimates (two-stages), with CFs added.

$$\widehat{\text{ADE}} = \mathbb{E}\left[\widehat{\gamma} + \widehat{\delta} D_i\right], \quad \widehat{\text{AIE}} = \mathbb{E}\left[\widehat{\overline{\pi}}\left(\widehat{\beta} + \widehat{\delta} Z_i + \underbrace{(\widehat{\rho}_1 - \widehat{\rho}_0)\,\Gamma\big(\widehat{\pi}(0; \boldsymbol{X}_i),\, \widehat{\pi}(1; \boldsymbol{X}_i)\big)}_{\text{Mediator compliers extrapolation.}}\right)\right]$$

Appenidx: CM with Selection — Estimation

If errors are not normal, then CFs do not have a known form, so semi-parametrically estimate them (e.g., splines).

$$\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 0, \boldsymbol{X}_i\right] = \alpha + \gamma Z_i + \varphi\big(\boldsymbol{X}_i\big) + \rho_0 \lambda_0\big(\pi(Z_i; \boldsymbol{X}_i)\big),$$

$$\mathbb{E}\left[Y_i \,|\, Z_i, D_i = 1, \boldsymbol{X}_i\right] = (\alpha + \beta) + (\gamma + \delta) Z_i + \varphi\big(\boldsymbol{X}_i\big) + \rho_1 \lambda_1\big(\pi(Z_i; \boldsymbol{X}_i)\big)$$

Semi-parametric Estimation Recipe:

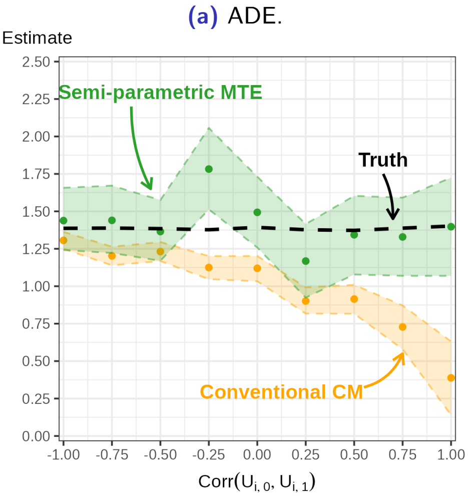
1. Estimate first-stage $\pi(Z_i; \boldsymbol{X}_i)$, including $\boldsymbol{X}_i^{\text{IV}}$.
2. Estimate second-stage separately for $D_i = 0$ and $D_i = 1$, with regressors $\lambda_0(p'), \lambda_1(p')$, semi-parametric in $\widehat{\pi}(Z_i; \boldsymbol{X}_i)$.
3. Compose CM estimates from two-stage plug-in estimates.

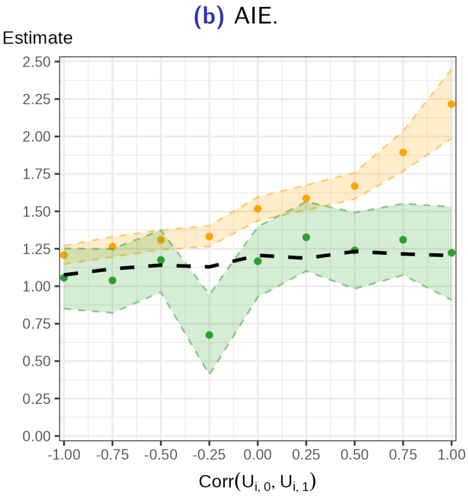
$\rightarrow$ Same as conventional CM estimates, with semi-parametric CFs. CFs.

$$\widehat{\text{ADE}} = \mathbb{E}\left[\widehat{\gamma} + \widehat{\delta} D_i\right], \quad \widehat{\text{AIE}} = \mathbb{E}\left[\widehat{\overline{\pi}}\left(\widehat{\beta} + \widehat{\delta} Z_i + \left(\widehat{\rho}_1 - \widehat{\rho}_0\right) \Gamma\big(\widehat{\pi}(0; \boldsymbol{X}_i), \widehat{\pi}(1; \boldsymbol{X}_i)\big)\right)\right]$$

Appenidx: CM with Selection — Estimation

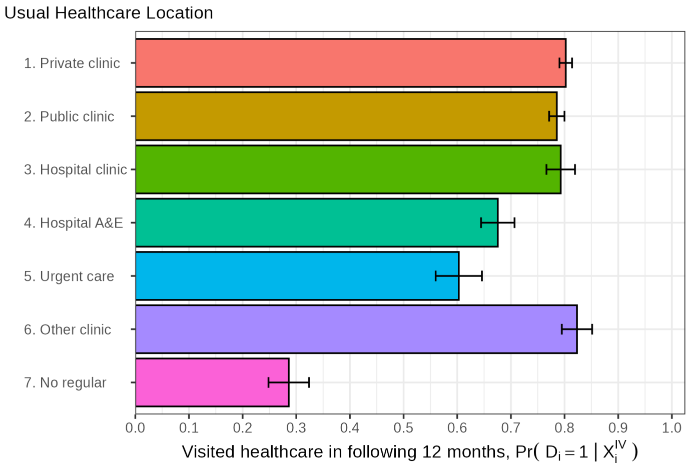
Figure: CF Adjusted Estimates Work with Different Error Term Parameters.





Appenidx: OHIE IV

IV first-stage F stat. is 124, for all categories (minus base).



Structural estimate of mediator compliers' $D_i \to Y_i$ is +32.9pp (4.4).