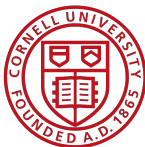


Causal Mediation in Natural Experiments

Senan Hogan-Hennessy
Economics Department, Cornell University
seh325@cornell.edu

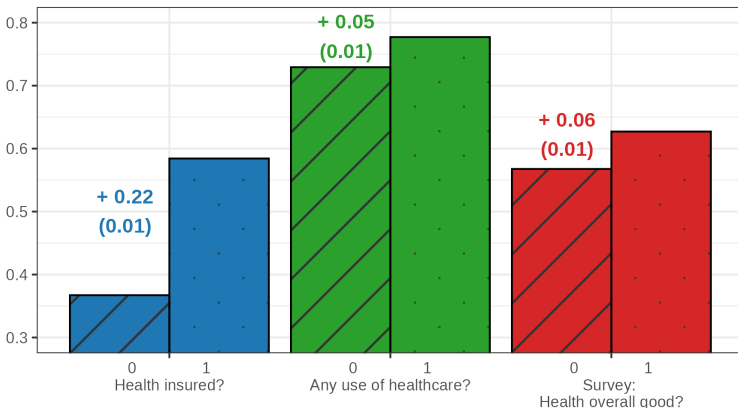


Cornell Placement Week
29 September 2025

Intro: Oregon Health Insurance Experiment

In 2008, Oregon gave access to socialised health insurance by wait-list lottery.

Mean Outcome, for each $z' = 0, 1$.



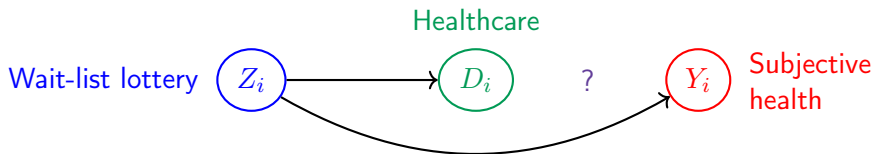
Applied practice:

⇒ Suggestive evidence for healthcare as mechanism for wait-list lottery. . . .

Intro: Oregon Health Insurance Experiment

In 2008, Oregon gave access to socialised health insurance by wait-list lottery (Finkelstein et al, 2012).

Figure: Model for Suggestive Evidence of a Mechanism.



Inconsistencies in suggestive evidence of mechanisms:

- Is $D_i \rightarrow Y_i$ small, large, or even existent?
- Where else do we accept assumed causal effects without evidence?

Introduction — Contributions

Causal Mediation (CM) is an alternative framework to studying mechanisms, with clear identification and assumptions required.

- ① Problems with conventional approach to CM in observational settings.
[Negative result]
 - ② Recovering valid CM effects, via MTE + control function modelling.
[Positive result]
-

New insights from intersection of two fields:

- **Causal Mediation (CM).**

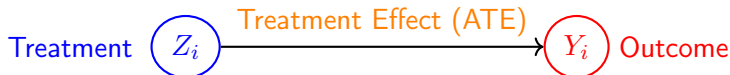
Imai Keele Yamamoto (2010), Frölich Huber (2017), Deuchert Huber Schelker (2019), Huber (2020), Kwon Roth (2024).

- **Labour theory, Selection-into-treatment, MTEs.**

Roy (1951), Heckman (1979), Heckman Honoré (1990), Vycatil (2002), Heckman Vycatil (2005), Brinch Mogstad Wiswall (2017), Kline Walters (2019).

Introduction – CM

Consider ignorable **treatment** $Z_i = 0, 1$, binary **mediator** $D_i = 0, 1$, and continuous **outcome** Y_i .



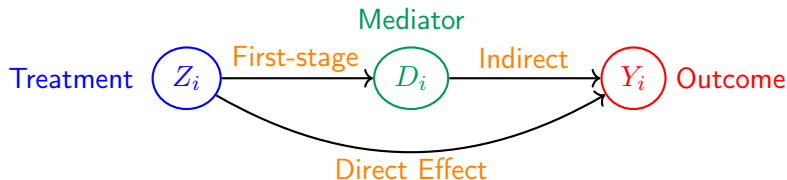
Assumption: **Mediator Ignorability** (MI, Imai Keele Yamamoto 2010)
mediator D_i is *also* ignorable, conditional on X_i and Z_i realisation.

Average Direct Effect (ADE) and Average Indirect Effect (AIE) are identified by two-stage regression

- ADE is causal effect $Z_i \rightarrow Y_i$, blocking the indirect D_i path
- AIE is causal effect of $D_i(Z_i) \rightarrow Y_i$, blocking the direct Z_i path.

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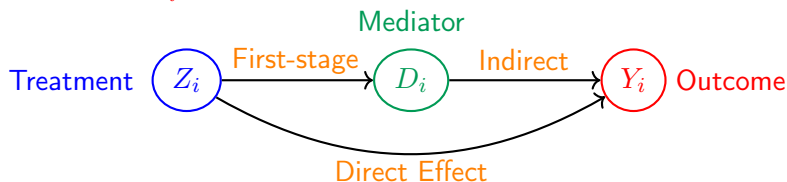
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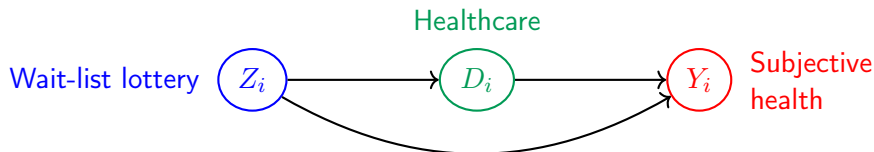
1. Selection Bias

Assumption: **Mediator ignorability** (MI, Imai Keele Yamamoto 2010)

mediator D_i is *also* ignorable, conditional on X_i , Z_i realisation

Would this assumption hold true in settings economists study?

E.g., Oregon Health Insurance Experiment.



- 1 Treatment is as-good-as random (2008 Oregon wait-list lottery).
- 2 Healthcare is quasi-random, conditional on lottery realisation Z_i and demographic controls X_i .

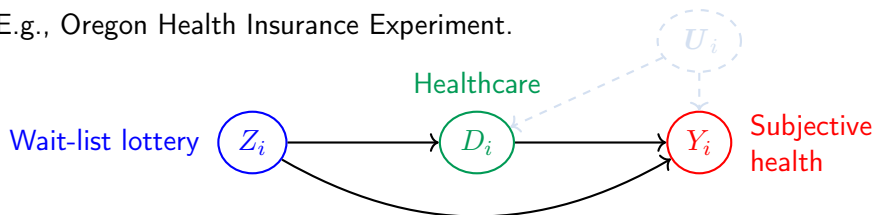
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Theorem: If choice to attend healthcare is unconstrained, based on costs and benefits (Roy model) and demographics do not explain all benefits \implies **MI** does not hold, there is unobserved confounding.

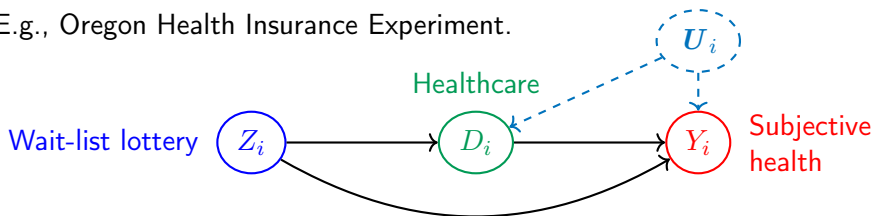
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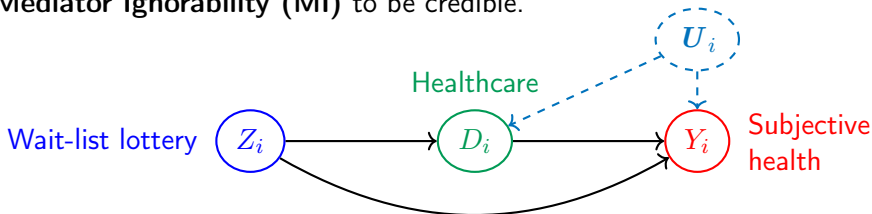
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1. Selection Bias

In an observational setting, need an additional credible research design for **Mediator Ignorability (MI)** to be credible.



If not, then CM effects are contaminated by bias terms, similar to classical selection bias (e.g., Heckman Ichimura Smith Todd 1998).

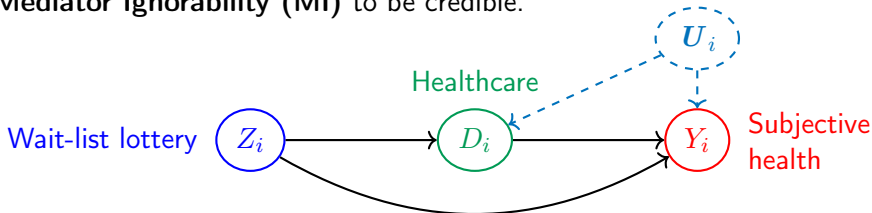
- ADE: CM Estimand = ADE + (Selection Bias + Group difference bias)
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► ADE biases

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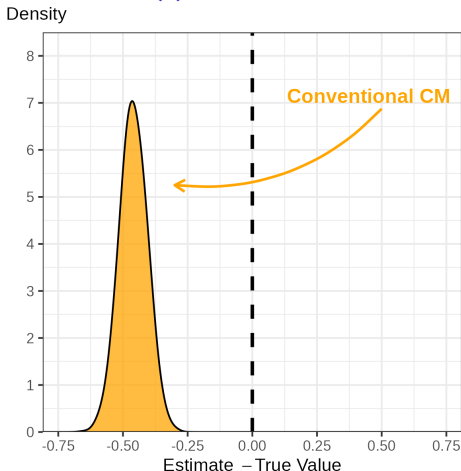
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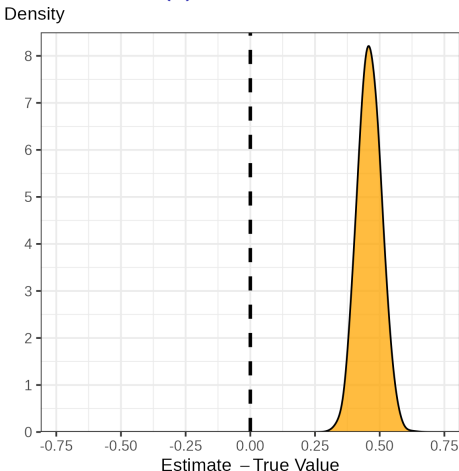
1. Selection Bias

In a simulation with Roy selection-into- D_i , CM estimates are biased.

(a) $\widehat{ADE} - ADE$.



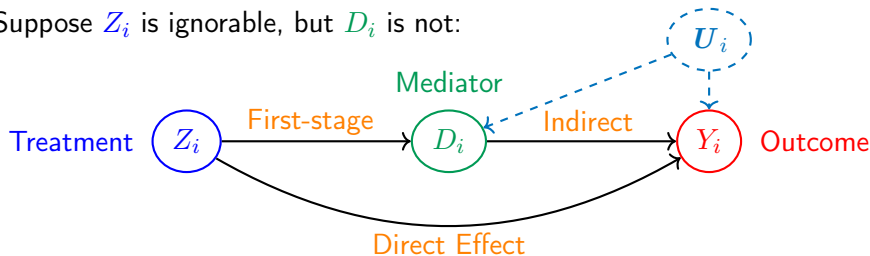
(b) $\widehat{AIE} - AIE$.



2. CM with Selection

Conventional CM methods do not identify $ADE + AIE$ in a natural experiment setting, but can we build a credible structural model?

Suppose Z_i is ignorable, but D_i is not:



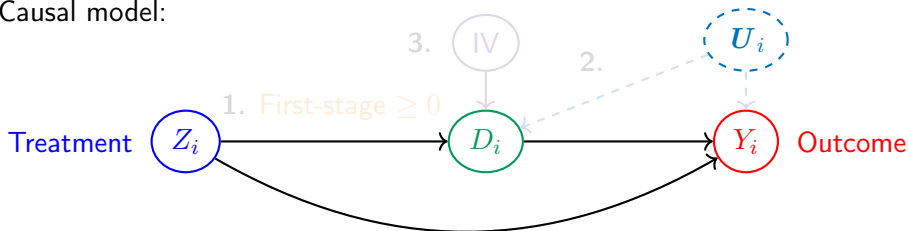
- 1 Average first-stage, $Z_i \rightarrow D_i$, is identified
- 2 Average second-stage, $Z_i, D_i \rightarrow Y_i$, is not — represented by U_i .

Intuition: model U_i via mediator MTE to identify $ADE + AIE$.

2. CM with Selection — Identification

- MTE assumptions:
- 1 Mediator monotonicity
 - 2 Selection on mediator benefits
 - 3 IV for mediator take-up cost.

Causal model:



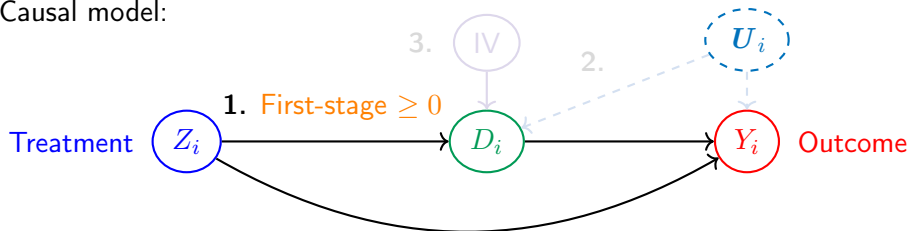
Proposition: Under MTE assumptions, the mediator MTE is identified.

Theorem: Mediation second-stage effects, $Z_i, D_i \rightarrow Y_i$, are identified by the MTE associated Control Functions (CFs).

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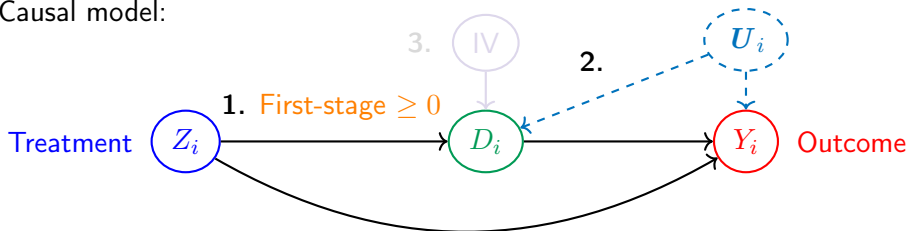
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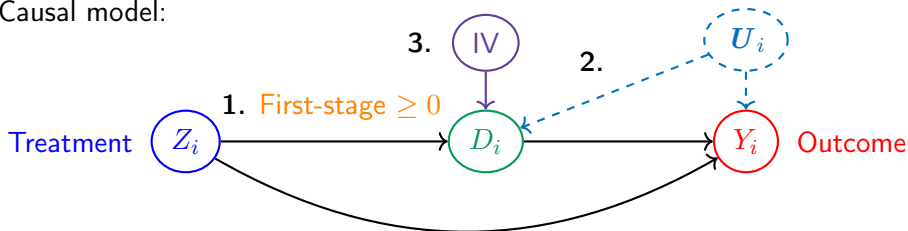
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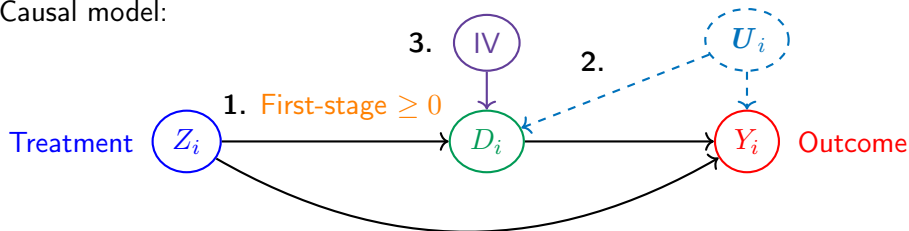
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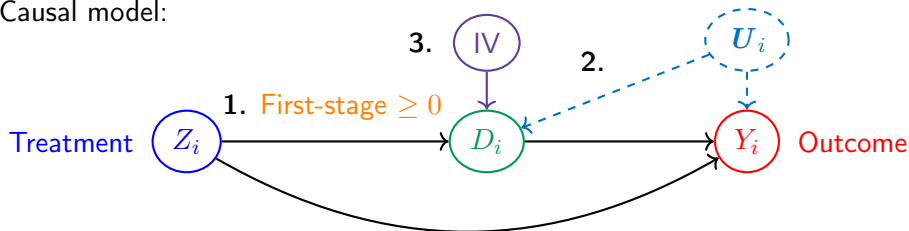
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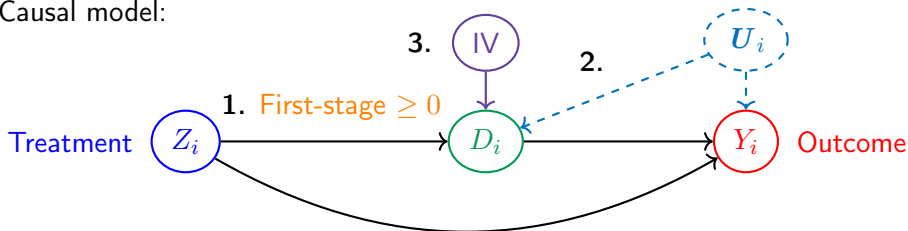
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Intuition: Identifies ADE + AIE by extrapolating from IV compliers to mediator compliers (MTE extrapolation e.g., Mogstad Torgovitsky 2018).

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2. CM with Selection — Estimation

In practice, this means two-stage CM estimation, with CF in second-stage.

Parametric CF Estimation Recipe:

- 1 Estimate mediation first-stage with probit, including the IV.
- 2 Estimate mediation second-stage by OLS, with Mills ratio CF terms (Heckman 1979).
- 3 Compose CM estimates from two-stage plug-in estimates (same as parametric MTEs, Björklund Moffitt 1987).

Semi-parametric CF Estimation Recipe:

Replace 2. with semi-parametric CFs (same estimation as MTEs).

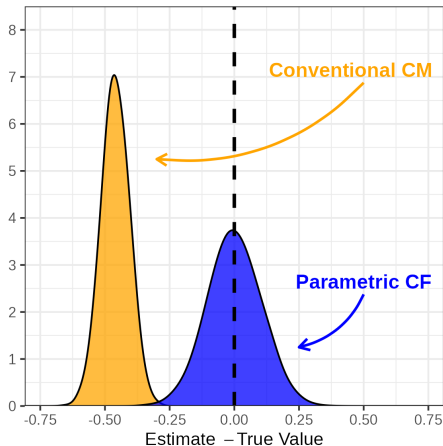
⇒ Conventional CM estimates (two-stages) + IV-guided CF adjustment.

2. CM with Selection — Estimation

Figure: CM Estimates from 10,000 DGPs with **Normal** Errors.

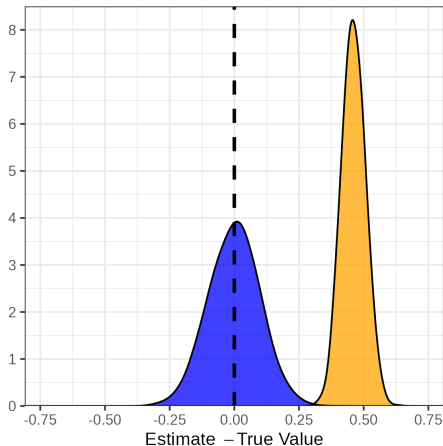
(a) $\widehat{ADE} - ADE$.

Density



(b) $\widehat{AIE} - AIE$.

Density



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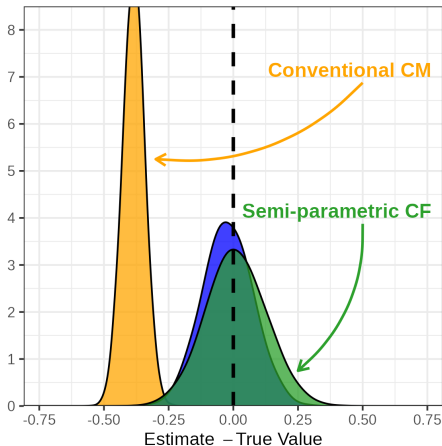
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Figure: CM Estimates from 10,000 DGPs with **Uniform** Errors.

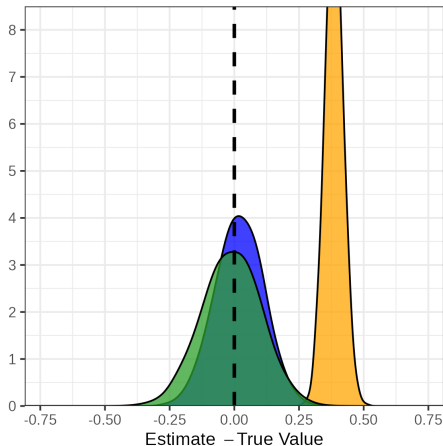
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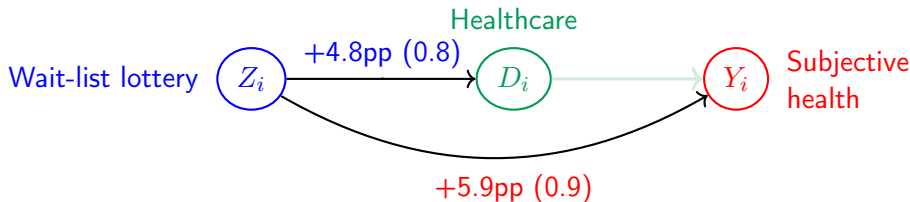
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Density



3. Returning to Oregon

Winning access to Medicaid increases healthcare usage, and subjective health:

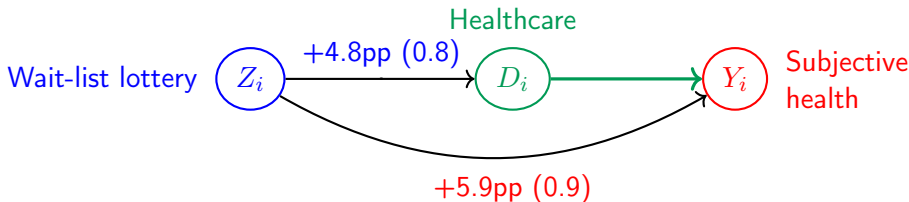


CM is quantitatively estimating the entire system:

- Use correlational estimate of $D_i \rightarrow Y_i$
- Does visiting healthcare at least once increase subjective health 12 months later?
- OLS for $D_i \rightarrow Y_i$ is ≈ 0 (not significant).

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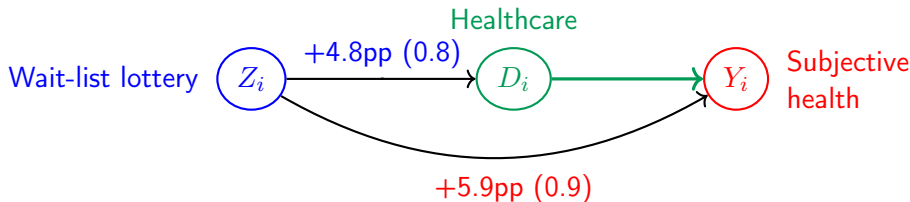


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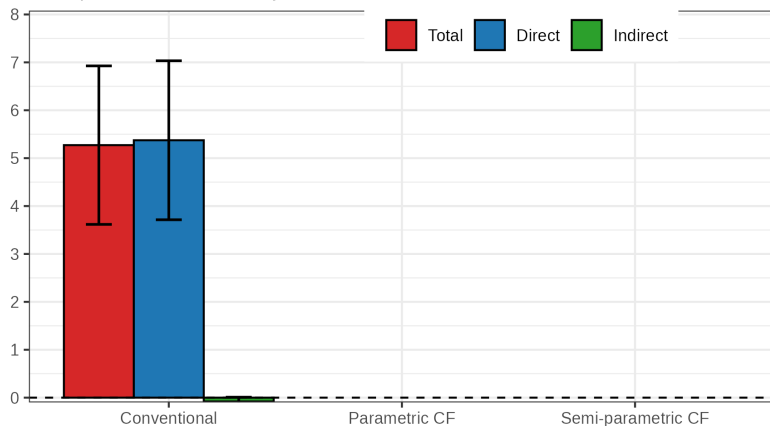
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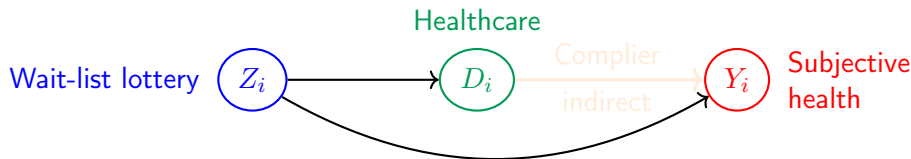
Conventional CM estimates lottery effects as mostly direct, ≈ 0 healthcare.

Estimate, percent effect on subjective health



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Winning access to Medicaid increases healthcare usage, and subjective health:

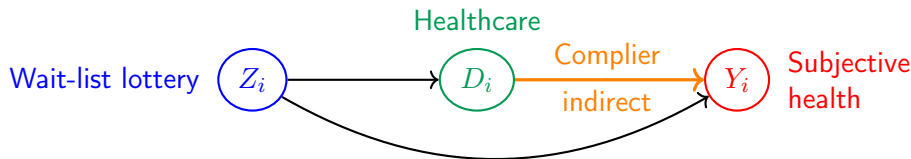


My approach to CM is modelling selection-into- D_i via mediator MTE:

- Uses an estimate of $D_i \rightarrow Y_i$ (plus complier extrapolation)
- Regular healthcare location pre-lottery serves as first-stage IV ▶ IV.
- IV + CF extrapolation estimates of $D_i \rightarrow Y_i$ are larger
⇒ smaller ADE estimates.

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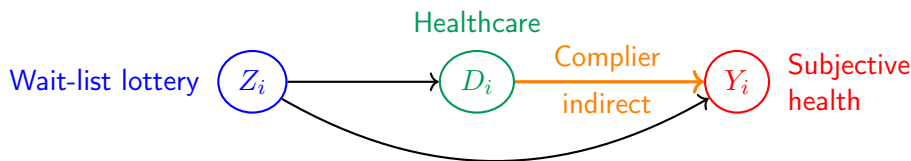


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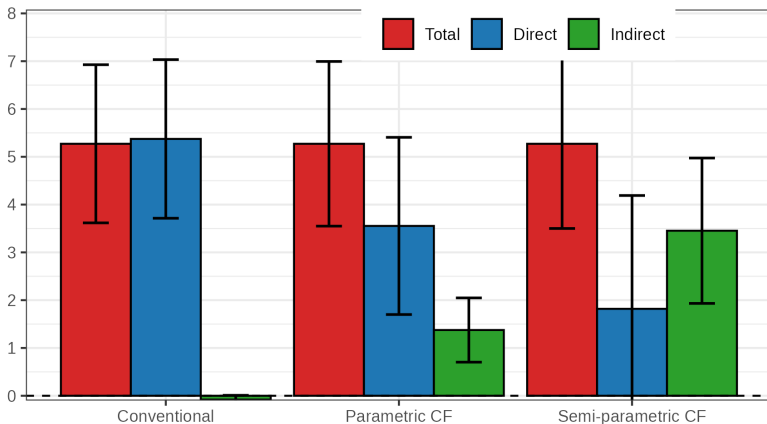
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Using my approach, with regular healthcare location as an excluded IV, restores indirect effect through increasing healthcare visitation.

Estimate, percent effect on subjective health



Conclusion

Overview:

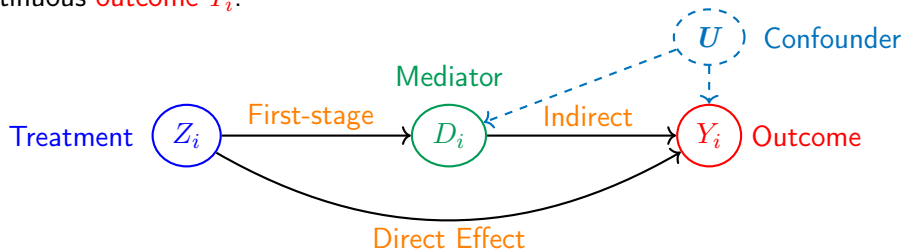
- ① CM as alternative to “suggestive evidence for mechanisms.”
- ② Selection bias in conventional CM analyses with no case for mediator ignorability.
- ③ Connect CM with labour theory + selection-into-treatment + MTEs.

Caveats and points to remember:

- Structural assumptions and IV for identification + estimation (not ideal).
- Application to Oregon Health Insurance Experiment, showing subjective health + well-being effects mediated by healthcare.
- **Credible** analyses of mechanisms are hard in practice, wide confidence intervals show true uncertainty.

Appendix: CM Guiding Model

Consider binary **treatment** $Z_i = 0, 1$, binary **mediator** $D_i = 0, 1$, and continuous **outcome** Y_i .



Average Direct Effect (ADE): $\mathbb{E} \left[Y_i \left(\mathbf{1}, D_i(Z_i) \right) - Y_i \left(\mathbf{0}, D_i(Z_i) \right) \right]$

- ADE is causal effect $Z \rightarrow Y$, blocking the indirect D_i path.

Average Indirect Effect (AIE): $\mathbb{E} \left[Y_i \left(Z_i, \mathbf{D_i(1)} \right) - Y_i \left(Z_i, \mathbf{D_i(0)} \right) \right]$

- AIE is causal effect of $D_i(Z_i) \rightarrow Y_i$, blocking the direct Z_i path.

Group Difference — ADE

CM effects contaminated by (less interpretable) bias terms.

$$\text{CM Estimand} = \text{ADEM} + \text{Selection Bias}$$

$$\begin{aligned} & \underbrace{\mathbb{E}_{D_i} \left[\mathbb{E} [Y_i \mid Z_i = 1, D_i] - \mathbb{E} [Y_i \mid Z_i = 0, D_i] \right]}_{\text{Estimand, Direct Effect}} \\ &= \underbrace{\mathbb{E}_{D_i=d'} \left[\mathbb{E} [Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \mid D_i(1) = d'] \right]}_{\text{Average Direct Effect on Mediator (ADEM) take-up — i.e., } D_i(1) \text{ weighted}} \\ & \quad + \underbrace{\mathbb{E}_{D_i} \left[\mathbb{E} [Y_i(0, D_i(Z_i)) \mid D_i(1) = d'] - \mathbb{E} [Y_i(0, D_i(Z_i)) \mid D_i(0) = d'] \right]}_{\text{Selection Bias}} \end{aligned}$$

The weighted ADE you get here is a positive weighted sum of local ADEs, but with policy irrelevant weights $D_i(1) = d'$.

⇒ consider this group bias, noting difference from true ADE. [▶ Back](#)

Selection Bias — Direct Effect

CM Effects + contaminating bias.

$$\text{CM Estimand} = \text{ADE} + \left(\text{Selection Bias} + \text{Group difference bias} \right)$$

► Model

$$\begin{aligned} & \underbrace{\mathbb{E}_{D_i=d'} \left[\mathbb{E} [Y_i \mid Z_i = 1, D_i = d'] - \mathbb{E} [Y_i \mid Z_i = 0, D_i = d'] \right]}_{\text{Estimand, Direct Effect}} \\ &= \underbrace{\mathbb{E} [Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i))]}_{\text{Average Direct Effect}} \\ &+ \underbrace{\mathbb{E}_{D_i=d'} \left[\mathbb{E} [Y_i(0, D_i(Z_i)) \mid D_i(1) = d'] - \mathbb{E} [Y_i(0, D_i(Z_i)) \mid D_i(0) = d'] \right]}_{\text{Selection Bias}} \\ &+ \underbrace{\mathbb{E}_{D_i=d'} \left[\left(1 - \Pr(D_i(1) = d') \right) \right.}_{\text{Group difference bias}} \\ &\quad \times \left. \left(\mathbb{E} [Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \mid D_i(1) = 1 - d'] \right. \right. \\ &\quad \left. \left. - \mathbb{E} [Y_i(1, D_i(Z_i)) - Y_i(0, D_i(Z_i)) \mid D_i(0) = d'] \right) \right]}_{\text{Group-diff}} \end{aligned}$$

Group Difference — AIE

CM effects contaminated by (less interpretable) bias terms.

$$\text{CM Estimand} = \text{AIEM} + \left(\text{Selection Bias} + \text{Group difference bias} \right)$$

$$\begin{aligned} & \underbrace{\mathbb{E}_{Z_i} \left[\left(\mathbb{E}[D_i | Z_i = 1] - \mathbb{E}[D_i | Z_i = 0] \right) \times \left(\mathbb{E}[Y_i | Z_i, D_i = 1] - \mathbb{E}[Y_i | Z_i, D_i = 0] \right) \right]}_{\text{Estimand, Indirect Effect}} \\ &= \underbrace{\mathbb{E} \left[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0)) \mid D_i = 1 \right]}_{\text{Average Indirect Effect on Mediated (AIEM) — i.e., } D_i = 1 \text{ weighted}} \\ &+ \underbrace{\pi \left(\mathbb{E}[Y_i(Z_i, 0) \mid D_i = 1] - \mathbb{E}[Y_i(Z_i, 0) \mid D_i = 0] \right)}_{\text{Selection Bias}} \\ &+ \underbrace{\pi \left[\left(\frac{1 - \Pr(D_i(1) = 1, D_i(0) = 0)}{\Pr(D_i(1) = 1, D_i(0) = 0)} \right) \left(\mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0) \mid D_i(1) = 0 \text{ or } D_i(0) = 1] - \mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0)] \right) \right]}_{\text{Groups difference Bias}} \end{aligned}$$

The weighted AIE you get here is not a positive weighted sum of local AIEs, because the AIE is only about $D(Z)$ compliers. [► Model](#).

⇒ consider this group bias, noting difference from true AIE. [► Back](#)

Selection Bias — Indirect Effect

CM Effects + contaminating bias, where $\bar{\pi} = \Pr(D_i(0) \neq D_i(1))$.

$$\text{CM Estimand} = \text{AIE} + \left(\text{Selection Bias} + \text{Group difference bias} \right) \quad \text{► Model}$$

$$\begin{aligned} & \underbrace{\mathbb{E}_{Z_i} \left[\left(\mathbb{E}[D_i | Z_i = 1] - \mathbb{E}[D_i | Z_i = 0] \right) \times \left(\mathbb{E}[Y_i | Z_i, D_i = 1] - \mathbb{E}[Y_i | Z_i, D_i = 0] \right) \right]}_{\text{Estimand, Indirect Effect}} \\ &= \underbrace{\mathbb{E}[Y_i(Z_i, D_i(1)) - Y_i(Z_i, D_i(0))]}_{\text{Average Indirect Effect}} \\ &+ \underbrace{\bar{\pi} \left(\mathbb{E}[Y_i(Z_i, 0) | D_i = 1] - \mathbb{E}[Y_i(Z_i, 0) | D_i = 0] \right)}_{\text{Selection Bias}} \\ &+ \bar{\pi} \left[\begin{aligned} & \left(1 - \Pr(D_i = 1) \right) \left(\mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0) | D_i = 1] \right. \\ & \quad \left. - \mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0) | D_i = 0] \right) \\ & + \left(\frac{1 - \Pr(D_i(1) = 1, D_i(0) = 0)}{\Pr(D_i(1) = 1, D_i(0) = 0)} \right) \left(\mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0) | D_i(Z_i) \neq Z_i] \right. \\ & \quad \left. - \mathbb{E}[Y_i(Z_i, 1) - Y_i(Z_i, 0)] \right) \end{aligned} \right] \end{aligned}$$

Groups difference Bias ► Group-diff

Semi-parametric Control Functions

Semi-parametric specifications for the CFs λ_0, λ_1 bring some complications to estimating the AIE.

$$\mathbb{E}[Y_i | Z_i, D_i = 0, \mathbf{X}_i] = \alpha + \gamma Z_i + \varphi(\mathbf{X}_i) + \rho_0 \lambda_0(\pi(Z_i; \mathbf{X}_i)),$$

$$\mathbb{E}[Y_i | Z_i, D_i = 1, \mathbf{X}_i] = (\alpha + \beta) + (\gamma + \delta) Z_i + \varphi(\mathbf{X}_i) + \rho_1 \lambda_1(\pi(Z_i; \mathbf{X}_i))$$

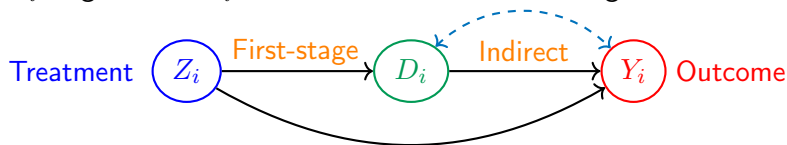
Intercepts, $\alpha, (\alpha + \beta)$, and relevance parameters ρ_0, ρ_1 are not separately identified from the CFs $\lambda_0(\cdot), \lambda_1(\cdot)$ so CF extrapolation term $(\rho_1 - \rho_0)\Gamma(\pi(0; \mathbf{X}_i), \pi(1; \mathbf{X}_i))$ is not directly identified or estimable.

These problems can be avoided by estimating the AIE using its relation to the ATE, $\widehat{\text{AIE}}^{\text{CF}} =$

$$\widehat{\text{ATE}} - (1 - \bar{Z}) \underbrace{\left(\frac{1}{N} \sum_{i=1}^N \hat{\gamma} + \hat{\delta} \hat{\pi}(1; \mathbf{X}_i) \right)}_{\widehat{\text{ADE}} \text{ given } Z_i=1} - \bar{Z} \underbrace{\left(\frac{1}{N} \sum_{i=1}^N \hat{\gamma} + \hat{\delta} \hat{\pi}(0; \mathbf{X}_i) \right)}_{\widehat{\text{ADE}} \text{ given } Z_i=0}.$$

Appendix: CM with Selection

Suppose Z_i is ignorable, D_i is not, so we have the following causal model.



Then this system has the following random coefficient equations:

$$D_i = \phi + \pi Z_i + \varphi(\mathbf{X}_i) + U_i$$

$$Y_i = \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \zeta(\mathbf{X}_i) + \underbrace{(1 - D_i) U_{0,i} + D_i U_{1,i}}_{\text{Correlated error term}}$$

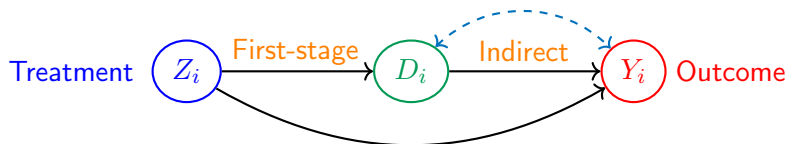
where β, γ, δ are functions of $\mu_{d'}(z'; \mathbf{X}_i)$.

$$\text{ADE} = \mathbb{E}[\gamma + \delta D_i], \quad \text{AIE} = \mathbb{E}\left[\pi(\beta + \delta Z_i + \tilde{U}_i)\right]$$

with $\tilde{U}_i = \mathbb{E}[U_{1,i} - U_{0,i} | \mathbf{X}_i, D_i(0) \neq D_i(1)]$ unobserved complier gains.

Appendix: CM with Selection

Suppose Z_i is ignorable, D_i is not, so we have the following causal model.



Main problem, second-stage is not identified:

$$\begin{aligned}\mathbb{E}[Y_i | Z_i, D_i, \mathbf{X}_i] &= \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \varphi(\mathbf{X}_i) \\ &+ (1 - D_i) \mathbb{E}[U_{0,i} | D_i = 0, \mathbf{X}_i] \\ &+ \underbrace{D_i \mathbb{E}[U_{1,i} | D_i = 1, \mathbf{X}_i]}_{\text{Unobserved } D_i \text{ confounding.}}\end{aligned}$$

Identification intuition: Identify second-stage via MTE control function.

Appendix: CM with Selection — Identification

Assume:

- ① Mediator monotonicity, $\Pr(D_i(0) \leq D_i(1) \mid \mathbf{X}_i) = 1$
 $\implies D_i(z') = \mathbb{1}\{U_i \leq \pi(z'; \mathbf{X}_i)\}, \text{ for } z' = 0, 1 \text{ (Vycatil 2002).}$
- ② Selection on mediator benefits, $\text{Cov}(U_i, U_{0,i}), \text{Cov}(U_i, U_{1,i}) \neq 0$
 \implies First-stage take-up informs second-stage confounding.
- ③ There is an IV for the mediator, \mathbf{X}_i^{IV} among control variables \mathbf{X}_i .
 $\implies \pi(Z_i; \mathbf{X}_i) = \Pr(D_i = 1 \mid Z_i, \mathbf{X}_i)$ is separately identified.

Proposition:

$$\begin{aligned} & \mathbb{E}[Y_i(z', 1) - Y_i(z', 0) \mid Z_i = z', \mathbf{X}_i, U_i = p'] \\ &= \beta + \delta z' + \mathbb{E}[U_{1,i} - U_{0,i} \mid \mathbf{X}_i, U_i = p'], \quad \text{for } p' \in (0, 1). \end{aligned}$$

Appendix: CM with Selection — Identification

The marginal effect has corresponding Control Functions (CFs), describing unobserved selection-into- D_i ,

$$\rho_0 \lambda_0(p') = \mathbb{E} [U_{0,i} \mid p' \leq U_i], \quad \rho_1 \lambda_1(p') = \mathbb{E} [U_{1,i} \mid U_i \leq p'] .$$

These CFs restore second-stage identification, by extrapolating from \mathbf{X}_i^{IV} compliers to $D_i(Z_i)$ mediator compliers,

$$\begin{aligned} \mathbb{E} [Y_i \mid Z_i, D_i, \mathbf{X}_i] &= \alpha + \beta D_i + \gamma Z_i + \delta Z_i D_i + \varphi(\mathbf{X}_i) \\ &\quad + \underbrace{\rho_0 (1 - D_i) \lambda_0(\pi(Z_i; \mathbf{X}_i)) + \rho_1 D_i \lambda_1(\pi(Z_i; \mathbf{X}_i))}_{\text{CF adjustment.}} \end{aligned}$$

This adjusted second-stage re-identifies the ADE and AIE,

$$\text{ADE} = \mathbb{E} [\gamma + \delta D_i], \quad \text{AIE} = \mathbb{E} \left[\bar{\pi} \left(\beta + \delta Z_i + \underbrace{(\rho_1 - \rho_0) \Gamma(\pi(0; \mathbf{X}_i), \pi(1; \mathbf{X}_i))}_{\text{Mediator compliers extrapolation.}} \right) \right]$$

Appendix: CM with Selection — Estimation

Will explain how estimation works, with simulation evidence.

- ① Random treatment $Z_i \sim \text{Binom}(0.5)$, for $n = 5,000$.
- ② $(U_{0,i}, U_{1,i}) \sim \text{BivariateNormal}(0, 0, \sigma_0, \sigma_1, \rho)$, Costs $C_i \sim N(0, 0.5)$.

Roy **selection-into- D_i** , with constant partial effects + interaction term.

$$D_i(z') = \mathbb{1} \{ C_i \leq Y_i(z', 1) - Y_i(z', 0) \},$$

$$Y_i(z', d') = (z' + d' + z'd') + U_{d'} \quad \text{for } z', d' = 0, 1.$$

Following the previous, these data have the following first and second-stage equations, where \mathbf{X}_i^{IV} is an additive cost IV:

$$D_i = \mathbb{1} \left\{ C_i - (U_{1,i} - U_{0,i}) \leq Z_i - \mathbf{X}_i^{\text{IV}} \right\}$$

$$Y_i = Z_i + D_i + Z_i D_i + (1 - D_i) U_{0,i} + D_i U_{1,i}.$$

\implies unobserved confounding by BivariateNormal $(U_{0,i}, U_{1,i})$.

Appendix: CM with Selection — Estimation

Errors are normal, so system is Heckman (1979) selection model.

CFs are the inverse Mills ratio, with $\phi(\cdot)$ normal pdf and $\Phi(\cdot)$ normal cdf,

$$\lambda_0(p') = \frac{\phi(-\Phi^{-1}(p'))}{\Phi(-\Phi^{-1}(p'))}, \quad \lambda_1(p') = \frac{\phi(\Phi^{-1}(p'))}{\Phi(\Phi^{-1}(p'))}, \quad \text{for } p' \in (0, 1).$$

Parametric Estimation Recipe:

- ① Estimate first-stage $\pi(Z_i; \mathbf{X}_i)$ with probit, including \mathbf{X}_i^{IV} .
 - ② Include λ_0, λ_1 CFs in second-stage OLS estimation.
 - ③ Compose CM estimates from two-stage plug-in estimates.
-

→ Same as conventional CM estimates (two-stages), with CFs added.

$$\widehat{\text{ADE}} = \mathbb{E} \left[\widehat{\gamma} + \widehat{\delta} D_i \right], \quad \widehat{\text{AIE}} = \mathbb{E} \left[\widehat{\pi} \left(\widehat{\beta} + \widehat{\delta} Z_i + \underbrace{(\widehat{\rho}_1 - \widehat{\rho}_0) \Gamma(\widehat{\pi}(0; \mathbf{X}_i), \widehat{\pi}(1; \mathbf{X}_i))}_{\text{Mediator compliers extrapolation.}} \right) \right]$$

Appendix: CM with Selection — Estimation

If errors are not normal, then CFs do not have a known form, so semi-parametrically estimate them (e.g., splines).

$$\mathbb{E}[Y_i | Z_i, D_i = 0, \mathbf{X}_i] = \alpha + \gamma Z_i + \varphi(\mathbf{X}_i) + \rho_0 \lambda_0(\pi(Z_i; \mathbf{X}_i)),$$

$$\mathbb{E}[Y_i | Z_i, D_i = 1, \mathbf{X}_i] = (\alpha + \beta) + (\gamma + \delta) Z_i + \varphi(\mathbf{X}_i) + \rho_1 \lambda_1(\pi(Z_i; \mathbf{X}_i))$$

Semi-parametric Estimation Recipe:

- 1 Estimate first-stage $\pi(Z_i; \mathbf{X}_i)$, including \mathbf{X}_i^{IV} .
- 2 Estimate second-stage separately for $D_i = 0$ and $D_i = 1$, with regressors $\lambda_0(p')$, $\lambda_1(p')$, semi-parametric in $\hat{\pi}(Z_i; \mathbf{X}_i)$.
- 3 Compose CM estimates from two-stage plug-in estimates.

→ Same as conventional CM estimates, with semi-parametric CFs. ▶ CFs.

$$\widehat{\text{ADE}} = \mathbb{E}[\hat{\gamma} + \hat{\delta} D_i], \quad \widehat{\text{AIE}} = \mathbb{E}\left[\hat{\pi}\left(\hat{\beta} + \hat{\delta} Z_i + (\hat{\rho}_1 - \hat{\rho}_0) \Gamma(\hat{\pi}(0; \mathbf{X}_i), \hat{\pi}(1; \mathbf{X}_i))\right)\right]$$

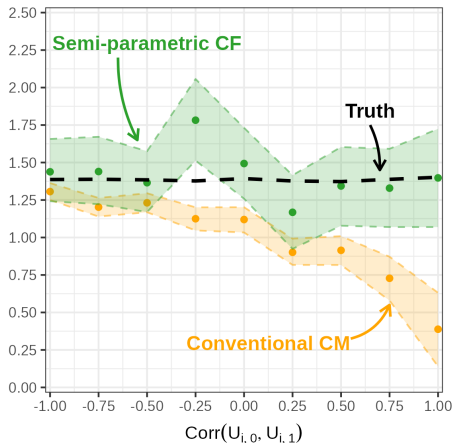
Appendix: CM with Selection — Estimation

Figure: CF Adjusted Estimates Work with Different Error Term Parameters.

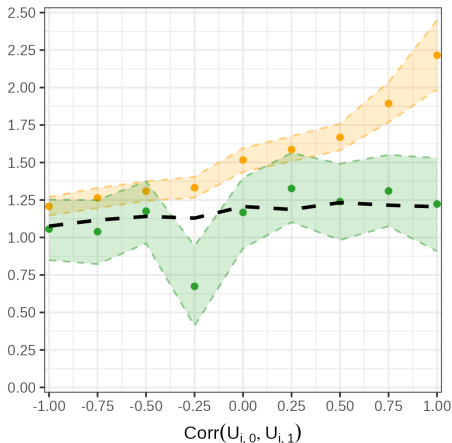
(a) ADE.

(b) AIE.

Estimate



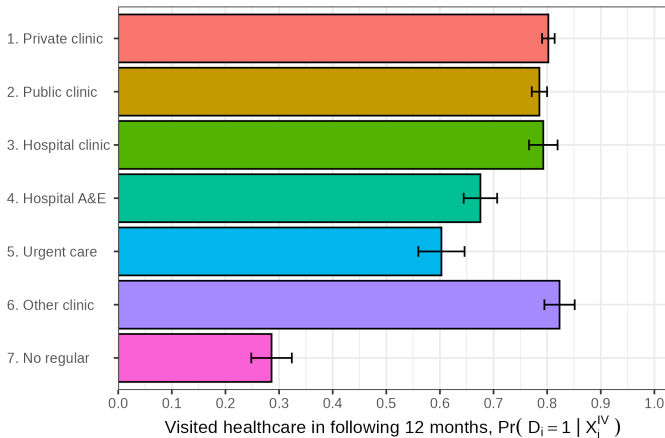
Estimate



Appendix: OHIE IV

IV first-stage F stat. is 124, for all categories (minus base).

Usual Healthcare Location



Structural estimate of mediator compliers' $D_i \rightarrow Y_i$ is +32.9pp (4.4).