$$D_3 = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 1(0-0) - 4(0) + 1(0-2) = -2$$

$$x = \frac{D_1}{D} = 3, \quad y = \frac{D_2}{D} = -1, \quad z = \frac{D_3}{D} = -2$$

Q.27 Given: parallelogram with sides (2,1) and (2,3).

Area of the parallelogram $= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2 = 4$

Another parallelogram with sides (2,2) and (1,3)

$$\text{Area} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2 = 4$$

$\therefore$ Area of both parallelograms are equal.

Q.29 1st edge of the box $= l_1 = (3,1,1) - (0,0,0) = (3,1,1)$

2nd edge of the box $= l_2 = (1,3,1) - (0,0,0) = (1,3,1)$

3rd edge of the box $= l_3 = (1,1,3) - (0,0,0) = (1,1,3)$

Volume of the box $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3(9-1) - 1(3-1) + 1(1-3)$

$= 24 - 2 - 2 = 20$

Chapter-5

: Eigenvalues and Eigenvectors :

5.1 Introduction

Definition :- Let A be a square matrix of order n. Then the solutions of the equation $|A - \lambda I| = 0$ are called eigenvalues of the matrix A, where the unknown λ is a scalar.

Definition : Let A be a square matrix of order n. Then the nonzero vectors x such that $(A - \lambda I)x = 0$ are called the eigenvectors of A.

Notes :-

1. Sum of the eigenvalues of A = Trace of A.
2. Product of the eigenvalues of $A = |A|$.
3. The eigenvalues of A = The eigenvalues of A^T.
4. $|A - \lambda I| = 0$ is known as characteristic equation of A and $P(\lambda) = |A - \lambda I|$ is known as characteristic polynomial of A.

Ex. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

Sol^n Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(4-\lambda)+2=0$

$\Rightarrow (\lambda-1)(\lambda-4)+2=0$

$\Rightarrow \lambda^2-5\lambda+6=0$

$\Rightarrow \lambda^2-2\lambda-3\lambda+6=0$

$\Rightarrow (\lambda-2)(\lambda-3)=0$

$\Rightarrow \lambda=2,3$ are the eigenvalues of A.

For $\lambda=2$

$(A-2I)x=0$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1=1,\ x_2=-1$

$\Rightarrow \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$ is the eigenvector for $\lambda=2$

For $\lambda=3$

$(A-3I)x=0$

$$\Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} -2 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1=1,\ x_2-2$

So, $\begin{bmatrix} -1/2\,x_2 \\ x_2 \end{bmatrix}$ is the eigenvector for $\lambda=3$

Ex-2 Find eigenvalues and eigenvectors of the matrix $A=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

Solⁿ Given $A=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)(2-\lambda)=0$

$\Rightarrow \lambda=3,2$ are the eigenvalues of A

For $\lambda=3$

$(A-3I)x=0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pivot variable $= x_2$

Free variable $= x_1$

Nullspace :- $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ is the eigenvector for $\lambda=3$

For $\lambda=2$

$(A-2I)x=0$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pivot variable = x_1
Free variable = x_2

Nullspace = $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ is the eigenvector for $\lambda = 2$

Ex-3 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol^n $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)(1-\lambda)(-\lambda) = 0$

$\Rightarrow \lambda = 3, 1, 0$ are the eigenvalues of A

For $\lambda = 3$

$(A - 3I)x = 0$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot variable = x_2, x_3
free variable = x_1

Nullspace = $\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$ is the eigenvector for $\lambda = 3$

For $\lambda = 1$

$(A - \lambda I)x = 0$

$\Rightarrow (A - I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ~~$x_1 = x_2, x_3$~~ Pivot variables = x_1, x_3
Free variables = x_2

$2x_1 + 4x_2 + 2x_3 = 0$
$2x_3 = 0 \Rightarrow x_3 = 0$
$-1x_3 = 0$

$2x_1 + 4x_2 = 0$
$\Rightarrow x_1 = -2x_2$

Nullspace = $\begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} \Rightarrow$ is the eigenvector for $\lambda = 1$

For $\lambda = 0$

$(A - \lambda I)x = 0$

$\Rightarrow Ax = 0$

$\Rightarrow \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 0$

$\Rightarrow 3x_1 + 4x_2 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

$0 = 0$

Pivot variable = x_1, x_2

Free variable = x_3

$\Rightarrow x_2 = -2x_3$

$3x_1 + 4(-2x_3) + 2x_3 = 0$

$3x_1 + -8x_3 + 2x_3 = 0$

$\Rightarrow 3x_1 = 6x_3$

$\Rightarrow x_1 = 2x_3$

Nullspace = $\begin{bmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$ is the eigenvector of $\lambda = 0$

15/05/23

$\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ Eigenvalue of $A_{n \times n}$

$\text{Trace}(A) = \sum_{i=1}^{n} \lambda_i$

$\det(A) = \prod_{i=1}^{n} \lambda_i$

Eigenvalue of $A^k = \lambda_1^k, \lambda_2^k \dots \lambda_n^k$

$A \pm KI = \lambda_i \pm KI$

$P(A) = P(\lambda_i)$

Example:-

$A^2 + 2A + I \quad (\lambda_1, \lambda_2)$

$\Rightarrow \lambda_1^2 + 2\lambda_1 + 1$

$\Rightarrow \lambda_2^2 + 2\lambda_2 + 1$

$\boxed{\lambda^2 - \text{trace}(A)\lambda + |A| = 0} \leftarrow$ only for 2x2 matrix

19/05/23

Q.7. $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ $\quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

$\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$

Since it is an upper triangular matrix

For $\lambda_1 = 3$

$(A - 3I)x_1 = 0$

$= \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$= x_1\begin{bmatrix}0\\0\\0\end{bmatrix} + x_2\begin{bmatrix}4\\-2\\0\end{bmatrix} + x_3\begin{bmatrix}2\\2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

pivot variable $= x_2, x_3$
free variable $= x_1$

$$\therefore X_1 = \begin{bmatrix}x_1\\0\\0\end{bmatrix} \quad \forall\ x_1 \in R$$

For $\lambda_2 = 1$

$(A - I)\,X_2 = 0$

$$= \begin{bmatrix}2 & 4 & 2\\0 & 0 & 2\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= x_1\begin{bmatrix}2\\0\\0\end{bmatrix} + x_2\begin{bmatrix}4\\0\\0\end{bmatrix} + x_3\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$=$ pivot variable

Q.6 $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$

$\rightarrow$ choose $\lambda = 0$ for $\det(A) = (\lambda_1 - 0)(\lambda_2 - 0) \cdots (\lambda_n - 0)$

$$= \prod_{i=1}^{n} \lambda_i$$

Q.2 Solve $\dfrac{du}{dt} = Au$, $u(0) = \begin{bmatrix}0\\6\end{bmatrix}$

Q $\rightarrow$ Find eigenvalues of A^{-1}, where $A = \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$

Sol^n $A = \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$

Eigenvalues of A are $2, 3$ (Reference Ex-1)
Eigenvalues of A^{-1} are $\frac{1}{2}$ and $\frac{1}{3}$

Q :→ Find eigenvalues of A^2 and $A - 7I$, where

$$A = \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$$

Sol^n $A = \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$

Eigenvalues of A are 2 and 3
Eigenvalues of A^2 are 2^2 and $3^2 \Rightarrow 4$ & 9
Eigenvalues of $A - 7I$ are $2-7$ and $3-7$
i.e. -5 and -4

Q. Find the eigenvalues and eigenvectors of A^T, where
$$A = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}$$

$$\Rightarrow A^T = \begin{bmatrix} -6 & 2 \\ -1 & -3 \end{bmatrix}$$

$$A^T - \lambda I = \begin{bmatrix} -6-\lambda & 2 \\ -1 & -3-\lambda \end{bmatrix}$$

$$|A^T - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -6-\lambda & 2 \\ -1 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+6)(\lambda+3) + 2 = 0$$

$$\Rightarrow \lambda^2 + 9\lambda + 20 = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 5\lambda + 20 = 0$$
$$\Rightarrow \lambda(\lambda+4) + 5(\lambda+4) = 0$$
$$\Rightarrow (\lambda+5)(\lambda+4) = 0$$
$\Rightarrow \lambda = -4, -5$ are eigenvalues of A.

for $\lambda = -4$

$$\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} -2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = 1$$

So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = -4$

For $\lambda = -5$:
$$\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 2, \quad x_2 = 1$$

So, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = -5$

Q. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Solⁿ Given: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$= (2-\lambda)(3-\lambda)(4-\lambda) = 0$$
$\Rightarrow \lambda = 2, 3, 4$ are eigenvalues of A.

for $\lambda = 2$:-

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$

So, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen~~value~~vector for $\lambda = 2$.

for $\lambda = 3$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0, \; x_2 = 1, \; x_3 = 0$

So, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is the eigenvector for $\lambda = 3$

for $\lambda = 4$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1$

So, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = 4$

Q. Find the roots of the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

Sol^n Given : $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0$

$\Rightarrow (\lambda - 1)(\lambda - 4) + 2 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \rightarrow$ characteristic polynomial.

$\rightarrow \lambda^2 - 2\lambda - 3\lambda + 6 = 0$

$\Rightarrow \lambda(\lambda - 2) - 3(\lambda - 2) = 0$

$\Rightarrow \lambda = 3, \; \lambda = 2$ are the roots of charactristic polynomial.

Q. Find eigenvalues and eigenvector of $A - 7I$, where $A = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}$

Sol^n Given $A = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}$

Eigenvalues of A are 2 and 3
Eigenvectors of A are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively

Eigenvalues of $A - 7I$ are $2-7$ and $3-7$
$\Rightarrow -5$ & -4.

for $\lambda = -5$, the eigenvector of $A - 7I$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

for $\lambda = -4$, the eigenvector of $A - 7I$ is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

5.2 Diagonalization of a Matrix :-

Diagonalization :- Let A be an n by n matrix with n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $\Lambda = S^{-1} A S$ is a diagonal matrix.

$A \rightarrow$ diagonalized matrix
$S \rightarrow$ diagonalizing matrix.

Note :-

1. Any matrix with distinct eigenvalues can be diagonalized.

2. The diagonalizing matrix S is not unique.

$\Lambda = S^{-1} A S$ (Diagonalization)

$\Rightarrow S\Lambda = S(S^{-1}AS)$
$= SS^{-1}(AS)$
$= I\,(AS)$
$= AS$

$\Rightarrow S \Lambda S^{-1} = (AS) S^{-1}$
$= A(SS^{-1})$
$= AI$
$= A$

$\Rightarrow \boxed{A = S \Lambda S^{-1}} \Rightarrow$ Factorization.

Ex:- Diagonalize the matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

Sol^n $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda - 4)(\lambda - 2) - 3 = 0$
$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$
$\Rightarrow \lambda^2 - \lambda - 5\lambda + 5 = 0$
$\Rightarrow \lambda(\lambda - 1) - 5(\lambda - 1) = 0$
$\Rightarrow (\lambda - 5)(\lambda - 1) = 0$
$\Rightarrow \lambda = 5, 1$ are the eigenvalues of A

For $\lambda = 1$:-
$(A - \lambda I)x = 0$
$\Rightarrow (A - I)x = 0$
$\Rightarrow \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1\begin{bmatrix}3\\1\end{bmatrix} + x_2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = -1$$

So, $\begin{bmatrix}1\\-1\end{bmatrix}$ is the eigenvector for $\lambda = 1$

For $\lambda = 5$:-

$$(A-5I)x = 0$$

$$\Rightarrow \begin{bmatrix}-1 & 3\\1 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}-1\\1\end{bmatrix} + x_2\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow x_1 = 3, \quad x_2 = 1$$

So, $\begin{bmatrix}3\\1\end{bmatrix}$ is the eigenvector for $\lambda = 5$

$$S = \begin{bmatrix}1 & 3\\-1 & 1\end{bmatrix}$$

$$S^{-1} = \frac{1}{4}\begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}$$

$$\Lambda = S^{-1}AS$$

$$= \frac{1}{4}\begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}4 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\-1 & 1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}1 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 15\\-1 & 5\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$$

Ex:- Find the matrix A whose eigenvalues are 1 and 4 and whose eigenvectors are $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ respectively.

Sol^n Given: Eigenvalues are 1 and 4

Eigenvectors are $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$ respectively

$$\Lambda = \begin{bmatrix}1 & 0\\0 & 4\end{bmatrix} \qquad S^{-1} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$$

$$S = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$$

$$A = S\Lambda S^{-1}$$

$$= \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\-4 & 12\end{bmatrix} = \begin{bmatrix}-5 & 18\\-3 & 10\end{bmatrix}$$

Q.1 Diagonalize the matrix $A = \begin{bmatrix}2 & 0\\2 & -2\end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix}2-\lambda & 0\\2 & -2-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)(\lambda+2) = 0$$

$\Rightarrow \lambda = -2, 2$ are the eigenvalues of A

For $\lambda = -2$

$$\begin{bmatrix}4 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow \quad x_1 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 0 \quad x_2 = 1$

So, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = -2$

For $\lambda = 2$.

$\begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x_1 = 2 \quad , \quad x_2 = 1$

So, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda = 2$

$S = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$S^{-1} = \frac{-1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

$\Lambda = S^{-1} A S$

$= \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

$S \Lambda S^{-1}$

Q.2- Find the matrix A whose eigenvalues are 2 & 5 ~~respectively~~ and whose eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

Sol^n Given : Eigenvalues of A are 2 and 5
Eigenvectors of A are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively

$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$A = S \Lambda S^{-1}$ (factorization)

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

5.5 Complex Matrices :-

Complex matrix : A matrix is said to be a complex matrix if atleast one of its elements is a complex number.

Eg :- $A = \begin{bmatrix} 2 & 3+i \\ i & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & i & 2+i \\ 0 & 2i & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3+i \\ i & 4 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2 & 3-i \\ -i & 4 \end{bmatrix} \text{ (conjugate of A)}$$

$$B = \begin{bmatrix} 1 & i & 2+i \\ 0 & 2i & 3 \end{bmatrix} \Rightarrow \bar{B} = \begin{bmatrix} 1 & -i & 2-i \\ 0 & -2i & 3 \end{bmatrix} \text{ (conjugate of B)}$$

<u>Complex vector</u> :- A vector is said to be a <u>complex vector</u> if at least one of its components is a complex number.

Let $x = (x_1, x_2, x_3 \dots x_n)$ be a complex vector, then:-

length of x is :-

$$\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \dots |x_n|^2}$$

Length of square of x is:-

$$\|x\|^2 = |x_1|^2 + |x_2|^2 + \dots |x_n|^2$$

<u>Inner product</u> :- Let x and y be two complex vectors. Inner product of x with y is denoted by $\bar{x}^T y$ and is defined by:-

$$\bar{x}^T y = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots \bar{x}_n y_n .$$

where $x = (x_1, x_2 \dots x_n)$ and
$y = (y_1, y_2 \dots y_n)$.

Ex:- Let $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $y = \begin{bmatrix} 2+i \\ 2-4i \end{bmatrix}$

$$\|x\| = \sqrt{|1|^2 + |i|^2} = \sqrt{1+1} = \sqrt{2}$$

$$\|y\| = \sqrt{|2+i|^2 + |2-4i|^2} = \sqrt{5+20} = \sqrt{25} = 5$$

Inner product of x with y is:-

$$\bar{x}^T y = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 2+i \\ 2-4i \end{bmatrix}$$

$$= 2+i - i(2-4i)$$
$$= 2+i - 2i + 4i^2$$
$$= 2+0 - 2i - 4 \quad (\because i^2 = -1)$$
$$= -2 - i$$

<u>Hermitian Matrix</u> :-

A complex square matrix 'A' is said to be a Hermetian matrix if $\boxed{\bar{A}^T = A}$

If A is a Hermitian matrix, then it is denoted by A^H

Ex:- $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & i \\ -i & 5 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ $\quad \bar{B} = \begin{bmatrix} 4 & -i \\ i & 5 \end{bmatrix}$

$\bar{A}^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} = A$ $\quad \bar{B}^T = \begin{bmatrix} 4 & i \\ -i & 5 \end{bmatrix} = B$

Skew - Hermitian : A complex square matrix 'A' is said to be skew-hermitian if $\boxed{\bar{A}^{T} = -A}$

Eg:- $A = \begin{bmatrix} 3i & 2+i \\ -2+i & i \end{bmatrix}$ $B = \begin{bmatrix} i & 1+i \\ -1+i & 2i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & -i \end{bmatrix}$ $\bar{B} = \begin{bmatrix} -i & 1-i \\ -1-i & -2i \end{bmatrix}$

$\bar{A}^{T} = \begin{bmatrix} -3i & -2-i \\ 2-i & -i \end{bmatrix}$ $\bar{B}^{T} = \begin{bmatrix} -i & -1-i \\ 1-i & -2i \end{bmatrix}$

Unitary Matrix :-

A complex square matrix 'A' is said to be a unitary matrix if

$\boxed{\bar{A}^{T} = A^{-1}}$

or

$\boxed{\bar{A}^{T} A = I}$

Ex:- $A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$

$\bar{A}^{T} = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/ & -i/2 \end{bmatrix}$

$\bar{A}^{T} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So, A is a unitary matrix.

Note:- 1. A complex matrix with orthonormal columns is called a unitary matrix.

2. $(AB)^{H} = B^{H} A^{T}$ and $(A+B)^{H} = A^{H} + B^{H}$

Let A be any square matrix. Then $A + A^{H}$ is always Hermitian and $A - A^{H}$ is always skew Hermitian.

Let $B = A + A^{H}$

$B^{H} = (A+A^{H})^{H} = A^{H} + (A^{H})^{H} = A^{H} + A = B$

$\Rightarrow$ B is hermitian

$\Rightarrow$ $A + A^{H}$ is hermitian.

Again, let $C = A - A^{H}$

$C^{H} = (A - A^{H})^{H} = A^{H} - (A^{H})^{H} = A^{H} - A$

$= -(A - A^{H})$

$= -C$

$\Rightarrow$ C is skew hermitian

$\Rightarrow$ $A = A^{H}$ is skew hermitian.

$A = \underbrace{\dfrac{A + A^{H}}{2}}_{\text{Hermitian}} + \underbrace{\dfrac{A - A^{H}}{2}}_{\text{Skew-Hermitian}}$

$\Rightarrow$ Any square matrix can be expressed as a sum of Hermitian and skew-Hermitian matrix.

Notes:-

1. $(A^{H})^{H} = A$
2. $(\lambda A)^{H} = \bar{\lambda} A^{H}$

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Property -1:- The eigenvalues of a Hermitian matrix are real.

Proof:- Let A be a hermitian matrix. Then $\bar{A}^T = A$.

Let λ be an eigenvalue of A and x be a corresponding eigenvector. Then:-

$$Ax = \lambda x$$
$$\Rightarrow \bar{x}^T Ax = \bar{x}^T \lambda x$$
$$= \lambda \bar{x}^T x$$

$$\Rightarrow \lambda = \frac{\bar{x}^T Ax}{\bar{x}^T x}$$
$$= \frac{\bar{x}^T Ax}{\|x\|^2}$$ is reall as both numerator and denominator are real.

$\because \|x\|^2$ is real as it length square and $\bar{x}^T Ax$ is real as it conjugate equals with itself.

Property 2:- The eigenvalues of a skew-Hermitian matrix are purely imaginary or zero.

Proof:- Let A be a skew-Hermitian matrix. Then $\bar{A}^T = -A$.

Let λ be an eigenvalue of A and x be a corresponding eigenvector. Then:-
$$Ax = \lambda x$$

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$$\Rightarrow \bar{x}^T Ax = \bar{x}^T \lambda x$$
$$= \lambda \bar{x}^T x$$

$$\Rightarrow \lambda = \frac{\bar{x}^T Ax}{\bar{x}^T x} = \frac{\bar{x}^T Ax}{\|x\|^2}$$ is purely imaginary or zero as the denominator is real and the numerator is purely imaginary of zero.

$\because \|x\|^2$ is real as it length square and $\bar{x}^T Ax$ is pure imaginary or zero as $\bar{x}^T Ax = -(\bar{x}^T Ax)$

Property 3 :- The eigenvalues of a unitary matrix have absolute value 1.

Proof :- Let A be a unitary matrix. Then $\bar{A}^T = A^{-1}$ or $\bar{A}^T A = I$

Let λ be an eigenvalue of A and x be a corresponding eigenvector. Then:-

$$Ax = \lambda x \quad \text{and} \quad (\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T = \bar{\lambda}\bar{x}^T$$

Now $(\bar{A}\bar{x})^T Ax = \bar{\lambda} \bar{x}^T \lambda x$
$$= \bar{\lambda}\lambda \bar{x}^T x$$
$$= |\lambda|^2 \bar{x}^T x$$

$$\Rightarrow |\lambda|^2 = \frac{(\bar{A}\bar{x})^T Ax}{\bar{x}^T x}$$

$$= \frac{(\bar{x}^T \bar{A}^T)(Ax)}{\bar{x}^T x}$$

$$\Rightarrow |\lambda|^2 = \frac{\bar{x}^T (\bar{A}^T A) x}{\bar{x}^T x}$$

$$= \frac{\bar{x}^T I x}{\bar{x}^T x} = \frac{\bar{x}^T x}{\bar{x}^T x} = 1$$

$$\Rightarrow |\lambda|^2 = 1$$
$$\Rightarrow |\lambda| = 1$$

Problem set 5.5 :-

Q 1- Let $z_1 = 3+4i$, $z_2 = 1-i$

a) $z_1 = 3+4i = (3,4)$ lies in the 1st quadrant
$z_2 = 1-i = (1,-1)$ lies in the 4th quadrant.

b) $z_1 + z_2 = (3+4i) + (1-i) = 4+3i$
$z_1 z_2 = (3+4i)(1-i) = 3-3i+4i-4i^2$
$= 3+i+4 \quad (\because i^2=-1)$
$= 7+i$

c) $\bar{z}_1 = 3-4i$, $\bar{z}_2 = 1+i$

$|z_1| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$
$|z_2| = \sqrt{1^2+1^2} = \sqrt{2}$

unit circle :- A circle with center origin and radius 1.
Both the numbers lie outside the unit circle.

Q.2 $A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}$, $A^H = \bar{A}^T = \begin{bmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{bmatrix}$

$C = A^H A = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$

$C^H = \begin{bmatrix} 2 & i & -i \\ -i & 1 & 0 \\ i & 0 & 1 \end{bmatrix} = C$

So, C is Hermitian

Q.10 $x = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}$, $y = \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}$

$\|x\| = \sqrt{|2-4i|^2 + |4i|^2} = \sqrt{20+16} = \sqrt{36} = 6$

$\|y\| = \sqrt{|2+4i|^2 + |4i|^2} = \sqrt{20+16} = \sqrt{36} = 6$

$\bar{x}^T y = \begin{bmatrix} 2+4i & -4i \end{bmatrix} \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = (2+4i)^2 - 16i^2$
$= 4+16i+16i^2-16i^2$

Q.15 Let U and V be unitary then
$U^H U = I$ and $V^H V = I$

Now $(UV)^H (UV) = (V^H U^H)(UV)$
$= V^H (U^H UV)$
$= V^H (IV)$
$= V^H V = I$

$\Rightarrow UV$ is unitary

Q.22:- $(A^H A)^H = A^H (A^H)^H = A^H A$

$\Rightarrow A^H A$ is always Hermitian matrix.

$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}$, $A^H = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$

$$A^H A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$$

$$AA^H = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Q8:- Let $x = 2+i$ and $y = 1+3i$

$\bar{x} = 2-i$

$x\bar{x} = (2+i)(2-i) = 4-i^2 = 4+1 = 5$ $(\because i^2 = -1)$

$$xy = (2+i)(1+3i) = 2+7i+3i^2 = 2+7i-3 = -1+7i$$

$$\frac{1}{x} = \frac{1}{2+i} = \frac{2-i}{(2+i)(2-i)} = \frac{2-i}{4-i^2} = \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5}$$

$$\frac{x}{y} = \frac{2+i}{1+3i} = \frac{(2+i)(2-i)}{(1+3i)(2-i)} = \frac{4-i^2}{2+5i-3i^2}$$

$$= \frac{5}{5+5i} = \frac{1}{1+i}$$

$$= \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$$

$|x| = |2+i| = \sqrt{2^2+1^2} = \sqrt{5}$

$|y| = |1+3i| = \sqrt{1^2+3^2} = \sqrt{10}$

$|xy| = |(2+i)(1+3i)| = |2+7i+3i^2|$

$= |-1+7i|$

$= \sqrt{(-1)^2+7^2}$

$= \sqrt{50}$

$|xy| = \sqrt{50} = \sqrt{5}\sqrt{10} = |x|\,|y|$

$$\left|\frac{1}{x}\right| = \left|\frac{1}{2+i}\right| = \frac{1}{|2+i|} = \frac{1}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}} = \frac{1}{|x|}$$

Q. If A and B are hermitian, show that $AB - BA$ is skew-hermitian.

Proof:- Let A and B be Hermitian.

Then $A^H = A$ and $B^H = B$.

$(AB - BA)^H = (AB)^H - (BA)^H$

$= B^H A^H - A^H B^H$

$= BA - AB$

$= -(AB - BA)$

$\Rightarrow AB - BA$ is skew hermitian

Q. If A is a hermitian matrix, then show that iA is a skew-hermitian matrix :-

Proof:- Let A be a Hermitian matrix.

Then $A^H = A$

$(iA)^H = \bar{i} A^H = -iA$

$\Rightarrow iA$ is a skew-Hermitian matrix.

Q. Express the following matrix as a sum of hermitian and skew-hermitian matrix.

$$A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$$

Solⁿ:- Given : $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$

$$A^H = \bar{A}^T = \begin{bmatrix} 2-3i & 5 \\ 7i & 1+i \end{bmatrix}$$

$$A = \frac{A+A^H}{2} + \frac{A-A^H}{2}$$

$$\Rightarrow \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{2} - \frac{7}{5}i \\ \frac{5}{2} + \frac{7}{5}i & 1 \end{bmatrix} \rightarrow \text{Hermitian}$$

$$+ \begin{bmatrix} 3i & \frac{-5}{2} - \frac{7}{2}i \\ \frac{5}{2} + \frac{7}{2}i & -i \end{bmatrix} \rightarrow \text{Skew-Hermitian.}$$

CH-6

POSITIVE DEFINITE MATRICES

6.2 Test for Positive Definiteness :-

Positive Definite :-

Each of the following test is a necessary & sufficient condition for the real symmetric matrix A to be positive definite :-

Test-1 :- $x^T A x > 0$ for all nonzero real vector x

Test 2 :- All the eigenvalues of A are positive

Test III :- All the principal submatrices have +ve determinant

Test IV :- All the pivots (without row exchange) are positives

Eg:- Check the positive definiteness of the following matrix. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solⁿ Given $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow$ It is real symmetric

The principal submatrices are:-

$A_1 = [2]$, $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A_1| = 2$, $|A_2| = 4-1 = 3$, $|A_3| = |A|$
$= 2(4-1) + 1(-2-0)$
$= 6-2 = 4$

Since the determinants of all the principal submatrices are positive, so the given matrix A is positive definite.