Chapter 7:- Computation with matrices

7.2 Matrix Norm and condition Number:-

The Norm of a Matrix:-

The norm of a matrix A is the number $\|A\|$ defined by:-

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

In other words, $\|A\|$ bounds the "amplifying power" of the matrix.

$$\|Ax\| \leq \|A\| \, \|x\| \quad \text{for all vectors } x.$$

The norm of A measures the largest amount by which any vector (eigenvector or not) is amplified by matrix multiplication

$$\|A\| = \max \left(\frac{\|Ax\|}{\|x\|} \right)$$

The norm of a matrix A is the square root of the longest eigenvalue of A^TA i.e.

$$\boxed{\|A\| = \sqrt{\lambda_{max}(A^TA)}}$$

The condition Number of a Matrix:-

The condition number of 'c' of a matrix A is defined by:-

$$\boxed{C = \sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}}$$

$$||A^{-1}|| = \frac{1}{\sqrt{\lambda_{min}(A^TA)}}$$

$$||A||^2 = \lambda_{max}(A^TA)$$

$$||A^{-1}||^2 = \frac{1}{\lambda_{min}(A^TA)}$$

$$||A||^2 = \lambda_{max}(A^TA)$$

$$||A^{-1}||^2 = \frac{1}{\lambda_{min}(A^TA)}$$

$$C = \sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}$$

$$\Rightarrow \boxed{C = ||A||\,||A^{-1}||}$$

Eg:- Find the norm and condition number of matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Sol^n Given :- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$= (\lambda-1)(\lambda-2)-1=0$$

$$\Rightarrow \lambda^2-3\lambda+1=0$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$= \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

The norm of the matrix A is

$$\|A\| = \sqrt{\lambda_{max}(A^TA)}$$

$$= \sqrt{\frac{3+\sqrt{5}}{2}} = 1.618$$

$$\|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min}(A^TA)}} = \frac{1}{\sqrt{\frac{3-\sqrt{5}}{2}}} = \sqrt{\frac{2}{3-\sqrt{5}}}$$

$$= 1.618$$

The condition number of the matrix is :-

$$C = \|A\| \, \|A^{-1}\| = (1.618)^2 = 2.618$$

<u>The Norm and condition number of a positive definite matrix</u> :-

Let A be a positive definite matrix

$\Rightarrow$ A is symmetric

$\Rightarrow A^T = A$

Now, $\|A\| = \sqrt{\lambda_{max}(A^TA)}$

$$= \sqrt{\lambda_{max}(A^2)} = \sqrt{(\lambda_{max}(A))^2}$$

$= \lambda_{max}(A)$

$\Rightarrow \boxed{\|A\| = \lambda_{max}(A)}$, which is the formula for the ~~\$~~ norm of a positive definite matrix A.

$$\|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min}(A^TA)}}$$

$$= \frac{1}{\sqrt{\lambda_{min}(A^2)}} = \frac{1}{\sqrt{(\lambda_{min}(A))^2}}$$

$$\Rightarrow \boxed{\|A^{-1}\| = \frac{1}{\lambda_{min}(A)}}$$

$$C = \sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}$$

$$= \sqrt{\frac{\lambda_{max}(A^2)}{\lambda_{min}(A^2)}} = \sqrt{\frac{(\lambda_{max}(A))^2}{(\lambda_{min}(A))^2}}$$

$\Rightarrow \boxed{C = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}}$, which is the formula for the condition number of a positive definite matrix.

Ex:- Find the norm and condition number of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

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$\underline{\underline{Sol^n}}$ Given :- $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$A^T = A$

$\Rightarrow$ A is symmetric.

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a = 2, \quad b = -1, \quad c = 2$

$ac - b^2 = 4 - 1 = 3$

Here $a = 2 > 0$ and $ac - b^2 = 3 > 0$, so A is a positive definite matrix.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 2)^2 - 1 = 0$

$\Rightarrow \lambda - 2 = \pm 1$

$\Rightarrow \lambda = 2 \pm 1$

$= 1, 3$ are the eigenvalues of A.

The norm of the matrix A is :-

$$\|A\| = \lambda_{max}(A) = 3$$

The condition number c of the matrix A is

$$c = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = \frac{3}{1} = 3$$

Q- Find the norm and condition number of the matrix $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Solⁿ Given : $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

~~Given~~ $A^T A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$|A^T A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)^2 = 0$

$\Rightarrow \lambda = 4, 4$ are the eigenvalues of $A^T A$

The norm of the matrix A is :-

$||A|| = \sqrt{\lambda_{max}(A^T A)} = \sqrt{4} = 2$

$||A^{-1}|| = \dfrac{1}{\sqrt{\lambda_{min}(A^T A)}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

The norm of the matrix is :-

$||A|| = \sqrt{\lambda_{max}(A^T A)} = \sqrt{2}$

$||A^{-1}|| = \dfrac{1}{\sqrt{\lambda_{min}(A^T A)}} = \dfrac{1}{\sqrt{2}}$

The condition number of the matrix A is:-

$$C = \|A\| \, \|A^{-1}\| = 2 \times \frac{1}{2} = 1$$

Q. Find the norm and condition number of the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Soln Given : $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$|A^T A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$\lambda = 2, 2$ are the eigenvalues of $A^T A$

The norm of the matrix is:-

$$\|A\| = \sqrt{\lambda_{max}(A^T A)} = \sqrt{2}$$

$$\|A^{-1}\| = \frac{1}{\sqrt{\lambda_{min}(A^T A)}} = \frac{1}{\sqrt{2}}$$

The condition number of the matrix A is:-

$$C = \|A\| \, \|A^{-1}\| = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1$$

7.4. Iterative Methods for $Ax=b$ are :-

1. Jacobi's method
2. Gauss-Seidel method
3. Successive overrelaxation (SOR) method.

Given : A
Splitting of A : $L+D+U$
where L is a lower-triangular mat
D is a diagonal matrix and U is an upper triangular matrix.

Jacobi matrix :-

$$D^{-1}(-L-U)$$

Gauss-Seidel matrix :-

$$(D+L)^{-1}(-U)$$

SOR matrix :-

$$(D+\omega L)^{-1}[(1-\omega)D-\omega U]$$

Eg :- Find the Jacobi's matrix, Gauss-Seidel matrix and SOR matrix of the matrix $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$.

Soln Given $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$

Splitting of $A = L+D+U$ where,

$L=\begin{bmatrix}0 & 0\\ -1 & 0\end{bmatrix}$, $D=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$, $U=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$

Jacobi's Matrix :-

$$D^{-1}(-L-U)$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

Gauss-Seidel matrix :-

$$(D+L)^{-1}(-U)$$

$$D+L = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow (D+L)^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$(D+L)^{-1}(-U) = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$$

SOR matrix :-

$$(D+\omega L)^{-1}[(1-\omega)D - \omega U]$$

$$D+\omega L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\omega & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -\omega & 2 \end{bmatrix}$$

$$(D+\omega L)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ \omega & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ \omega/4 & 1/2 \end{bmatrix}$$

$$(1-\omega)D - \omega U = \begin{bmatrix} 2(1-\omega) & 0 \\ 0 & 2(1-\omega) \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix}$$

$$(D+\omega L)^{-1} = [(1-\omega)D - \omega U]$$

$$= \begin{bmatrix} 1/2 & 0 \\ \omega/4 & 1/2 \end{bmatrix} \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix}$$

$$= \begin{bmatrix} 1-\omega & \omega/2 \\ \omega/2\,(1-\omega) & \frac{\omega^2}{4} + (1-\omega) \end{bmatrix}$$

<u>Alternate method for Jacobi and Gauss-Seidel matrix:</u>

Given: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$Ax = b$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 = b_1$$
$$-x_1 + 2x_2 = b_2$$

$$\Rightarrow x_1 = \frac{1}{2}x_2 + \frac{b_1}{2}, \quad x_2 = \frac{1}{2}x_1 + \frac{b_2}{2}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1/2 \\ b_2/2 \end{bmatrix}$$

$\downarrow$

(Jacobi's matrix)

Again, $Ax = b$.

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2x_1 - x_2 &= b_1 \\ -x_1 + 2x_2 &= b_2 \end{aligned}$$

$$\Rightarrow x_1 = \frac{1}{2} x_2 + \frac{b_1}{2}$$

$$x_2 = \frac{1}{2} x_1 + \frac{b_2}{2}$$

$$\Rightarrow x_1 = 1/2\, x_2 + b_1/2$$

$$x_2 = 1/2 \left(\frac{1}{2} x_2 + \frac{b_1}{2} \right) + \frac{b_2}{2}$$

$$= \frac{1}{4} x_2 + \frac{b_1}{4} + \frac{b_2}{2}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_2/2 \\ b_1/4 / b_2/4 \end{bmatrix}$$

$\downarrow$

(Gauss-Seidel Matrix)

Q. Find the Jacobi's and Gauss-Seidel matrix of the matrix $A = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$

Solⁿ Given $A = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$

Splitting : $A = \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{U}$

Jacobi matrix :-

$$D^{-1}(-L-U) = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \end{bmatrix}$$

Gauss-Seidel Matrix :-

$(D+L)^{-1}(-U)$

$$D+L = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} \Rightarrow (D+L)^{-1} = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 1/9 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Alternate method :-

$Ax = b$.

$$\Rightarrow \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

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$\Rightarrow \quad 3x_1 = b_1$

$-x_1 + 3x_2 = b_2$

$\Rightarrow x_1 = b_1/3$

$x_2 = \frac{1}{3}x_1 + \frac{b_2}{3}$

~~Alternate~~ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1/3 \\ b_2/3 \end{bmatrix}$

Jacobi matrix

Again, $Ax = b$

$$\Rightarrow \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\Rightarrow 3x_1 = b_1$

$-x_1 + 3x_2 = b_2$

$\Rightarrow x_1 = b_1/3$

$x_2 = \frac{1}{3}x_1 + \frac{b_2}{3}$

$= \frac{1}{3}\left(\frac{b_1}{3}\right) + \frac{b_2}{3}$

$= \frac{b_1}{9} + \frac{b_2}{3}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} + \begin{bmatrix} b_1/3 \\ b_1/9 + b_2/9 \end{bmatrix}$$

gauss seidel matrix