Q.9 $b=4$ at $t=-2$
$b=3$ at $t=-1$
$b=1$ at $t=0$
$b=0$ at $t=2$

Line : $b = C + Dt$

$$\begin{aligned} C - 2D &= 4 \\ C - D &= 3 \\ C &= 1 \\ C + 2D &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Ax = b$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{35}\begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix}$$

The best line $61/35 - (36/35)^t$

The projection of b onto the column space of A is :- $p = A\hat{x}$

$$= \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 61/35 \\ -36/35 \end{bmatrix} = \begin{bmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{bmatrix}$$

08/05/23

4.2 :- Properties of Determinants.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $|A| = ad - bc$
or, $\det(A) = ad - bc$

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, Then :-

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Properties :-

1. The determinant of the identity matrix is 1.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(I) = 1$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(I) = 1$$

2. The determinant changes sign when two rows are exchanged.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

3. The determinant depends linearly on the first row.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ tc & td \end{vmatrix} = t^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Let A be the matrix of order n. then
$|tA| = t^n \det(A)$

4. If two rows of A are equal, then $|A| = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0 \quad (\because R_1 = R_3)$$

5. Subtracting a multiple of one row from another row leaves the same determinant

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$LHS = \begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} lc & ld \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

6. If A has a row of zeros, then $|A| = 0$

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{vmatrix} = 0$$

7. If A is triangular, then $|A|$ is the product of its diagonal elements.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11}\ a_{12}\ a_{13}$$

(upper triangular)

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11}\ a_{12}\ a_{13}$$

(Lower triangular)

If A is a diagonal matrix then $|A|$ is the product of the diagonal elements.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \Rightarrow |A| = a_{11}\ a_{22}\ a_{33}$$

All singular matrices have a zero determinant.

8. If A is singular, then $|A| = 0$. If A is invertible then $|A| \neq 0$.

9. $|AB| = |A| \cdot |B|$

$$A A^{-1} = I$$
$$\Rightarrow |A A^{-1}| = |I|$$
$$\Rightarrow |A|\,|A^{-1}| = |I| = 1$$
$$\Rightarrow \boxed{|A^{-1}| = \frac{1}{|A|}}$$

10. The transpose of A has the same determinant as A itself. $\Rightarrow \det(A^T) = \det(A)$
$\Rightarrow |A^T| = |A|$

Problem Set 4.2

Q2. Let A be a 4x4 matrix with $|A| = \frac{1}{2}$

Then, $|2A| = 2^4 |A| = 2^4 \times \frac{1}{2} = 2^3 = 8$

$$|-A| = (-1)^4 |A| = |A| = \frac{1}{2}$$

$$|A^2| = |A \times A| = |A| \times |A| = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$|A^{-1}| = \frac{1}{|A|} = 2$$

Q3. Let A be a 3 by 3 matrix with $|A| = -1$
Then

$$\left|\frac{1}{2} A\right| = \left(\frac{1}{2}\right)^3 \times |A| = \frac{1}{8} \times -1 = -\frac{1}{8}$$

$$|-A| = (-1)^3 \times |A| = 1$$

$$|A^2| = (|A|)^2 = (-1)^2 = 1$$

$$|A^{-1}| = \frac{1}{|A|} = -1$$

Q4.

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \quad R_4 \leftarrow R_4 + 2R_2$$

$$= \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{5}{2} R_3$$

↳ upper triangular form.

$$|A| = 1(-1)(-2)\,10 = \boxed{20}$$

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2} R_1$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \quad R_3 \leftarrow R_3 + \frac{2}{3} R_2$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & -11/4 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{3}{4} R_3$$

↳ upper triangular Form

$$|A| = 2 \times \frac{3}{2} \times \frac{4}{3} \times \frac{-11}{4} = \boxed{-11}$$

Interchanging the 3rd and 4th rows of the matrix B, we have

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \quad R_2 \leftarrow R_2 + \frac{1}{2} R_1$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4/3 & -1 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{2}{3} R_2$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -11/3 \end{bmatrix} \quad R_4 \leftarrow R_4 + \frac{4}{3} R_3$$

$$|C| = 2 \times \frac{3}{2} \times (-1) \times \left(\frac{-11}{3}\right) = 11$$

Q.5 a) $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$ (a rank one matrix)

$$= \begin{bmatrix} 2 & -1 & 2 \\ 8 & -4 & 8 \\ 4 & -2 & 4 \end{bmatrix}$$

$|A| = 0$ [two columns of A are identical

b) $U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (upper triangular matrix)

$|U| = 4 \times 1 \times 2 \times 2 = 16$

c) $U^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 8 & 2 & 2 & 0 \\ 8 & 2 & 6 & 2 \end{bmatrix}$ (lower triangular matrix)

$|U^T| = 4 \times 1 \times 2 \times 2 = 16$

d) $|U^{-1}| = \frac{1}{|U|} = \frac{1}{16}$

e) $M = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}$ (Reverse triangular)

$= \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_4$

$= \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $R_2 \leftrightarrow R_3$

$|M| = 4 \times 1 \times 2 \times 2 = \boxed{16}$

Q.6 $\begin{vmatrix} a-mc & b-md \\ c-la & d-lb \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} - \begin{vmatrix} mc & md \\ c-la & d-lb \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ la & lb \end{vmatrix} - \begin{vmatrix} mc & md \\ c & d \end{vmatrix} + \begin{vmatrix} mc & md \\ la & lb \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} - m \begin{vmatrix} c & d \\ c & d \end{vmatrix} + lm \begin{vmatrix} c & d \\ a & b \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - lm \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= (1 - lm) \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Q.13

i) $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow |A| = 12 - 2 = 10$

ii) $A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3/10 & -1/5 \\ -1/10 & 2/5 \end{bmatrix}$

$|A^{-1}| = \frac{6}{50} - \frac{1}{50} = \frac{5}{50} = \frac{1}{10}$

iii) $A - \lambda I = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$

$|A - \lambda I| = (4-\lambda)(3-\lambda) - 2$
$= (\lambda - 4)(\lambda - 3) - 2$
$= \lambda^2 - 7\lambda + 12 - 2$
$= \lambda^2 - 7\lambda + 10$

$\lambda^2 - 7\lambda + 10 = 0$
$\Rightarrow \lambda^2 - 5\lambda - 2\lambda + 10 = 0$
$\Rightarrow \lambda(\lambda - 5) - 2(\lambda - 5) = 0$
$\Rightarrow \boxed{\lambda = 2, 5}$

For $\lambda = 2, 5$, $A - \lambda I$ is singular.

Q.20 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$|A^{-1}| = \frac{ad}{(ad-bc)^2} - \frac{bc}{(ad-bc)^2}$$

$$= \frac{ad-bc}{(ad-bc)^2} = \boxed{\frac{1}{ad-bc}}$$

But $|A^{-1}| = \frac{1}{ad-bc}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$

$= \frac{1}{ad-bc}(ad-bc) = \underline{\underline{1}}$ is wrong

4.4 : Application of Determinants

Given : Square matrix A

$|A| \neq 0 \Rightarrow A^{-1}$ will exist

Cofactor of $A = C$

Adjoint of $A = C^T$

$$A^{-1} = \frac{Adj.A}{|A|} = \frac{C^T}{|A|}$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $|A| = 1$

$C = \begin{bmatrix} 1 & -0 & 1 \\ 1 & 1 & 0 \\ -0 & 0 & 1 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$A^{-1} = \frac{C^T}{|A|} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $|A| = 4-6 = -2 \neq 0$

$\Rightarrow A^{-1}$ will exist

Cofactor of $A = C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adjoint of $A = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$A^{-1} = \frac{C^T}{|A|} = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |A| = ad - bc \neq 0$

A^{-1} will exist

Cofactor of $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Adjoint of $A = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$|A| = 1(20-0) - 2(0-0) + 3(0-0) = 20 \neq 0$

$\Rightarrow A^{-1}$ will exist

cofactor of $A = \begin{bmatrix} 20 & 0 & 0 \\ -10 & 5 & 0 \\ -12 & 0 & 4 \end{bmatrix}$

Adj. A = Transpose of cofactor $= \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$A^{-1} = \frac{\text{Adj.} A}{|A|} = \frac{1}{20} \begin{bmatrix} 20 & -10 & -12 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution of $Ax = b$: (Cramer's Rule)

Given : $a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

$x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$

If $D \neq 0$, then the system has unique solution

If $D = 0$ and atleast one of D_1, D_2 and D_3 is non-zero, then the system has no solution

If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system has infinite number of solution.

Ex:- $x_1 + 3x_2 = 0$
$2x_1 + 4x_2 = 6$

$D = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$, $D_1 = \begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix} = 0 - 18 = -18$, $D_2 = \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix} = 6 - 0 = 6$

$$x_1 = \frac{D_1}{D} = \frac{-18}{-2} = 9$$

$$x_2 = \frac{D_2}{D} = \frac{6}{-2} = -3$$

Ex:- $x + 4y - z = 1$
$x + y + z = 0.$
$2x + 3z = 0$

$$D = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(3-0) - 4(3-2) - 1(0-2)$$
$$= 3 - 4 + 2 = 1$$

$$D_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 1(3-0) = 3$$

$$D_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(0-0) - 1(3-2) - 1(0-0)$$
$$= -1$$

$$D_3 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2(0-1) = -2$$

$$x = \frac{D_1}{D} = \frac{3}{1} = 3$$

$$z = \frac{D_3}{D} = \frac{-2}{1} = -2$$

$$y = \frac{D_2}{D} = \frac{-1}{1} = 1$$

To find the pivots :-

Let A be the matrix of order n with full set of pivots. Let $A_1, A_2, A_3 \ldots A_n$ be the leading submatrices of A. Let $d_1, d_2, d_3 \ldots, d_n$ be the pivots of A. Then,

$$\boxed{d_k = \frac{\det(A_k)}{\det(A_{k-1})}}$$

where $k = 1, 2, 3 \ldots, n$ and $\det(A_0) = 1$

Ex:- $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

The leading submatrices of A are

$A_1 = [2]$ $\quad A_2 = \begin{bmatrix} 2 & 1 \\ 4 & -6 \end{bmatrix}$ $\quad A_3 = A$

$|A_1| = 2$ $\quad |A_2| = -12 - 4 = -16$ $\quad |A_3| = 2(-12) - 1(8) + 1(16) = -24 - 8 + 16 = -16$

The pivots of A are :-

$$d_1 = \frac{|A_1|}{|A_0|} = \frac{2}{1} = 2$$

$$d_2 = \frac{|A_2|}{|A_1|} = \frac{-16}{2} = -8$$

$$d_3 = \frac{|A_3|}{|A_2|} = \frac{-16}{-16} = 1$$

The volume of a box:-

Let l_1, l_2 and l_3 be three adjacent sides of a box. Then the volume of the box is

$$V = l_1 l_2 l_3$$

~~Let l_1, l_2, l_3 and l_4 be four adjacent~~

~~Chapter - 5~~

~~Eigenvalues & Eigenvectors~~

~~1. Sum of the~~

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = 0 \qquad x = 2, 3$$

$$|A - 2I| x = 0$$

$$\Rightarrow \begin{bmatrix} 1-2 & -1-0 \\ 2-0 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Let l_1, l_2, l_3, l_4 be four adjacent sides of a box. Then the volume of the box is:-

$$V = l_1 l_2 l_3 l_4$$

Similarly, for n edges,

$$V = l_1 l_2 l_3 \cdots l_n$$

Let A be a matrix whose rows are the edges $l_1, l_2, l_3 \cdots l_n$. Then:-

$$A = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}, \quad A^T = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}$$

$$AA^T = \begin{bmatrix} l_1^2 & l_1 l_2 & \cdots & l_1 l_n \\ l_2 l_1 & l_2^2 & \cdots & l_2 l_n \\ \vdots & & & \vdots \\ l_n l_1 & l_n l_2 & & l_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2 & 0 & \cdots & 0 \\ 0 & l_2^2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & l_n^2 \end{bmatrix} \Rightarrow \det(AA^T) = l_1^2 l_2^2 \cdots l_n^2$$

$$\Rightarrow \det(A) \cdot \det(A^T) = l_1^2 l_2^2 \cdots l_n^2$$

$$\Rightarrow (\det(A))^2 = l_1^2 l_2^2 \cdots l_n^2$$

$$\Rightarrow \det(A) = l_1 l_2 l_3 \cdots l_n$$

$$\Rightarrow |A| = \text{volume of the box.}$$

Ex:- Find the volume of the box whose three adjacent edges are $(-1,2,2)$, $(2,-1,2)$ & $(2,2,-1)$

Solⁿ:- The edge vectors are:-

$l_1 = (-1,2,2)$, $l_2 = (2,-1,2)$ and $l_3 = (2,2,-1)$

Volume of box is:-

$$V = \begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= -1(1-4) - 2(-2-4) + 2(4+2)$$
$$= 3+12+12$$

$\therefore V = 27$ cube unit

Ex:- Find area of the triangle with vertices $(2,2)$, $(-1,3)$ and $(0,0)$

Solⁿ Given: vertices $(2,2)$, $(-1,3)$, $(0,0)$

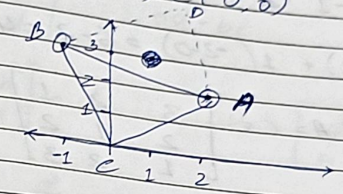
Area of the triangle is:-

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2} \times 2(6+2) = 8$$

Problem Set 4.4:-

Q.2 a) $A(2,2)$, $B(-1,3)$, $C(0,0)$



Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = \frac{1}{2} \times 8 = 4$

Area of ||gm $CADB = \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix} = 8$

b) $A = (2,2)$ $B = (-1,3)$, $C = (1,-4)$

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

New coordinates with origin $= (1,-4)$ [any vertex]

$A = (2-1, 2+4) = (1,6)$

$B = (-1-1, 3+4) = (-2,7)$

Area of $\Delta = \frac{1}{2}\begin{vmatrix} 1 & 6 \\ -2 & 7 \end{vmatrix}$

$= \frac{1}{2}(7+12)$

$= \frac{1}{2}(19) = \frac{19}{2}$

$$= \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ -1 & 3 & 1 \\ 1 & -4 & 1 \end{vmatrix} = \frac{1}{2}\left[2(3+4) - 2(-1-1) + 1(4-3)\right]$$

$$= \frac{1}{2}(14+4+1)$$

$$= \frac{19}{2}$$

Q.6

a) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A| = 2(4-1) + 1(-2-0) = 6-2 = 4 \neq 0$

Cofactor of $A = C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$A^{-1} = \frac{C^T}{|A|} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$|B| = 1(6-4) - 1(3-2) + 1(2-2) = 2-1 = 1 \neq 0$

cofactor of $B = C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$B^{-1} = \frac{C^T}{|B|} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

Q.7 i) $ax + by = 1$
$cx + dy = 0$

$\Rightarrow Ax = b$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$D_1 = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} = d$, $D_2 = \begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix} = c$

$x = \frac{D_1}{D} = \frac{d}{ad-bc}$, $y = \frac{D_2}{D} = \frac{c}{ad-bc}$

ii) $x + 4y - z = 1$
$x + y + z = 0$
$2x + 3z = 0$

$\Rightarrow Ax = b$

where $A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$D = \begin{vmatrix} 1 & 4 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(3-0) - 4(3-2) - 1(0-2)$
$= 3 - 4 + 2 = 1$

$D_1 = \begin{vmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3$

$D_2 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 1(0-0) - 1(3-2) - 1(0-0) = -1$

$$D_3 = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 1(0-0) - 4(0) + 1(0-2) = -2$$

$$x = \frac{D_1}{D} = 3, \quad y = \frac{D_2}{D} = -1, \quad z = \frac{D_3}{D} = -2$$

Q.27 Given: parallelogram with sides $(2,1)$ and $(2,3)$.

Area of the parallelogram $= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6 - 2 = 4$

Another parallelogram with sides $(2,2)$ and $(1,3)$

Area $= \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6 - 2 = 4$

$\therefore$ Area of both parallelograms are equal.

Q.29 1st edge of the box $= l_1 = (3,1,1) - (0,0,0)$
$= (3,1,1)$

2nd edge of the box $= l_2 = (1,3,1) - (0,0,0)$
$= (1,3,1)$

3rd edge of the box $= l_3 = (1,1,3) - (0,0,0)$
$= (1,1,3)$

Volume of the box $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3(9-1) - 1(3-1) + 1(1-3)$

$= 24 - 2 - 2 = 20$

Chapter-5

: Eigenvalues and Eigenvectors :

5.1 Introduction

Definition :- Let A be a square matrix of order n. Then the solutions of the equation $|A - \lambda I| = 0$ are called eigenvalues of the matrix A, where the unknown λ is a scalar.

Definition : Let A be a square matrix of order n. Then the nonzero vectors x such that $(A - \lambda I)x = 0$ are called the eigenvectors of A.

Notes :-

1. Sum of the eigenvalues of A = Trace of A.
2. Product of the eigenvalues of $A = |A|$.
3. The eigenvalues of A = The eigenvalues of A^T.
4. $|A - \lambda I| = 0$ is known as characteristic equation of A and $P(\lambda) = |A - \lambda I|$ is known as characteristic polynomial of A.

Ex. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Sol^n Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$