Extending the Limits of Tractability

Rangaballav Pradhan

ITER, SOADU

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-complete problem. What should I do?

A. Theory says you're unlikely to find poly-time algorithm.

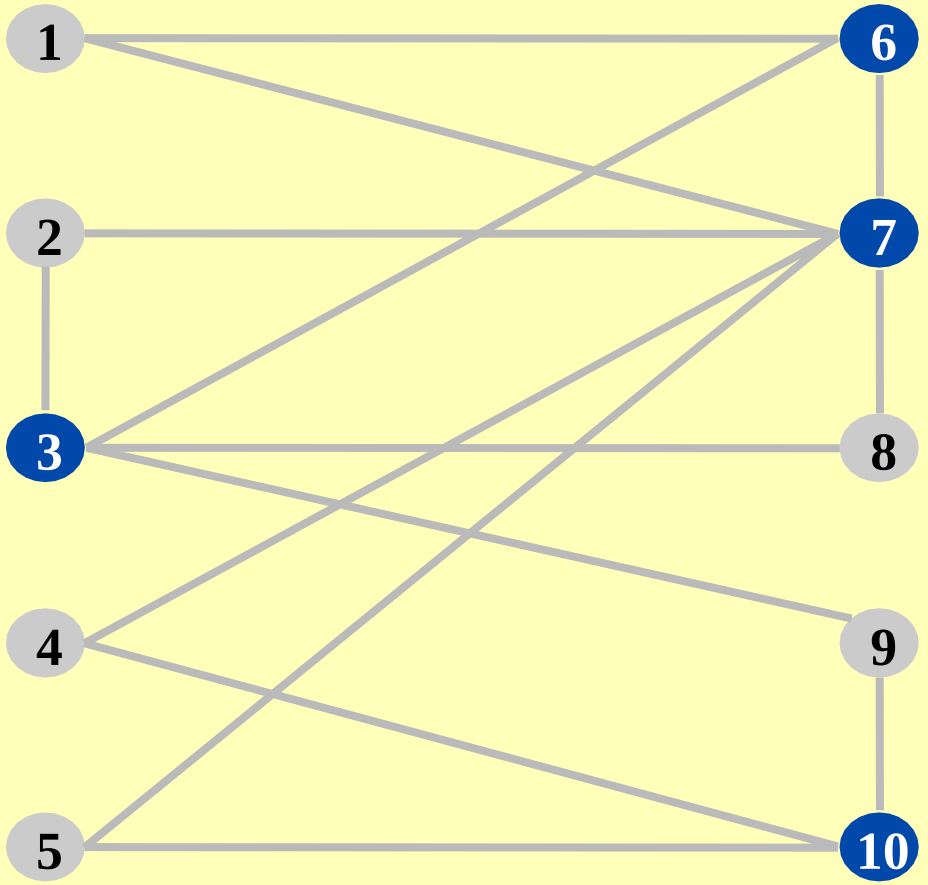
Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

Our goal. Solve some special cases of **NP**-complete problems.

Vertex cover

Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$ or $v \in S$ or both?



S = {3, 6, 7, 10} a vertex cover of size k = 4

Finding small vertex covers

Q. VERTEX-COVER is **NP**-complete. But what if k is small?

Brute force. $O(k\, n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k.
- Takes $O(k\, n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k, say to $O(2^k\, k\, n)$.

Ex. $n = 1{,}000$, $k = 10$.

Brute. $k\, n^{k+1} = 10^{34}\ 2^k$ • infeasible.

Better. $k\, n = 10^7$ • feasible.

Remark. If k is a constant, then the algorithm is poly-time; if k is a small constant, then it's also practical.

Finding small vertex covers

Claim. Let (u, v) be an edge of G. G has a vertex cover of size $\leq k$ iff at least one of $G - \{ u \}$ and $G - \{ v \}$ has a vertex cover of size $\leq k - 1$.

delete v and all incident edges

Pf. $\Rightarrow$

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u.
- $S - \{ u \}$ is a vertex cover of $G - \{ u \}$.

Pf. $\Leftarrow$

- Suppose S is a vertex cover of $G - \{ u \}$ of size $\leq k - 1$.
- Then $S \cup \{ u \}$ is a vertex cover of G. ▪

Claim. If G has a vertex cover of size k, it has $\leq k\,(n - 1)$ edges.

Pf. Each vertex covers at most $n - 1$ edges. ▪

Finding small vertex covers: algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

```
Vertex-Cover(G, k) {
   if (G contains no edges)     return true
   if (G contains ≥ kn edges) return false
   let (u, v) be any edge of G
   a = Vertex-Cover(G - {u}, k-1)
   b = Vertex-Cover(G - {v}, k-1)
   return a or b
}
```

Pf.

- Correctness follows from previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time

Finding small vertex covers: recursion tree

$$T(n,k) \le \begin{cases} c & \text{if } k = 0 \\ cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k ckn$$