Ch-04
Mathematical Expectation

4. A coin is biased such that head is three times as likely to occur as tail. Find expected no. of tails when coin is tossed twice.

$P(T) = K \qquad P(H) = 3K$

$P(T) + P(H) = 1$

$4K = 1$

$K = \frac{1}{4}$

$P(T) = \frac{1}{4} \qquad P(H) = \frac{3}{4}$

$S = \{HH, HT, HT, TT\}$

X : no. of tails

$= \{0, 1, 2\}$

X	0	1	2
$f(x)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

$\mu = E(X) = 0 \cdot \frac{9}{16} + 1 \cdot \frac{6}{16} + 2 \cdot \frac{1}{16}$

$= \frac{1}{2}$

7. By investing in particular stock, a person can make profit in one year of \$4000 with probability 0.3 or take loss of \$1000 with 0.7 probability. What is this person's expected gain?

Expected gain $= E(X)$

$= 4000(0.3) + (-1000)(0.7)$

$= 1200 - 700$

$= \$500$

10. Two tire-quality experts examine stacks of tires & assign quality rating on 3-point scale. X is rating given by expert A and Y denote rating by B. Following is the joint distribution of X and Y

$f(x,y)$		y = 1	2	3
x	1	0.10	0.05	0.02
	2	0.10	0.35	0.05
	3	0.03	0.10	0.20

Find μ_x and μ_y

$g(x) = \sum_y f(x,y)$

$g(1) = \sum_y f(1,y)$

$g(2) = \sum_y f(2,y)$

$g(3) = \sum_y f(3,y)$

X	1	2	3
g(x)	0.17	0.50	0.33

$\mu_x = 1(0.17) + 2(0.5) + 3(0.33)$

$= 0.17 + 1 + 0.99$

$= 2.16$

$g(y) = \sum_x f(x, y)$

Y	1	2	3
g(y)	0.23	0.5	0.27

$\mu_Y = 0.23 + 1 + 0.81$
$= 2.04$

12. If a dealer's profit, in units of $5000 on new automobile can be looked upon as random variable having density function

$$f(x) = \begin{cases} 2(1-x) & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

find average profit per automobile

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} x \cancel{f(x)}^{\to 0} dx + \int_{0}^{1} x f(x)\, dx + \int_{1}^{\infty} x \cancel{f(x)}^{\to 0} dx$$

$$= \int_0^1 2(1-x).x\, dx = \left[x^2 - \frac{2x^3}{3} \right]_0^1$$

$$= \frac{1}{3}$$

Total profit $= \$5000 * \frac{1}{3}$

$= \$1666.67$

Date

Page No.

15 Assume (X, Y) are uniformly distributed on circle with radius a. Then joint probability density function is

$$f(x,y) = \begin{cases} 1/\pi a^2 & , x^2 + y^2 \leq a^2 \\ 0 & , \text{otherwise} \end{cases}$$

Find μ_X

$$\mu_X = \sum_x \sum_y x\, f(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$= \frac{1}{\pi a^2} \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x\, dx\, dy$$

$$= \frac{1}{\pi a^2} \int_{-a}^{a} \left(\frac{a^2-y^2}{2}\right) - \left(\frac{a^2-y^2}{2}\right) dy$$

$$= 0$$

20 A continuous random variable has density function

$$f(x) = \begin{cases} e^{-x} & , x > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

Find expected value of $g(X) = e^{2X/3}$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$= \int_{-\infty}^{0} g(x) \cancel{f(x)}^{0}\, dx + \int_{0}^{\infty} g(x) f(x)\, dx$$

$$= \int_0^\infty e^{2x/3} e^{-x} \, dx = \int_0^\infty e^{-x/3} \, dx$$

$$= -3e^{-x/3} \Big]_0^\infty = 3$$

23. Let X, Y have following joint probability function:

$f(x,y)$		x	
		2	4
y	1	0.1	0.15
	3	0.2	0.3
	5	0.1	0.15

(a) Find expected value of $g(X,Y) = XY^2$

(b) Find μ_X and μ_Y

(a) $E(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y)$

$$= \sum_x \sum_y xy^2 f(x,y)$$

$= 2(0.1) + 18(0.2) + 50(0.1) + 4(0.15) + 36(0.3) + 100(0.15)$

$= 35.2$

(b) $\mu_X = \sum_x x g(x)$

X	2	4
$g(x)$	0.4	0.6

$g(x) = \sum_y f(x,y)$

$\mu_X = 2 \times 0.4 + 4 \times 0.6$

$= 3.2$

$$\mu_Y = \sum_y y\, h(y)$$

$$h(y) = \sum_x f(x, y)$$

Y	1	3	5
h(y)	0.25	0.5	0.25

$$\mu_Y = 0.25 + 1.5 + 1.25$$
$$= 3$$

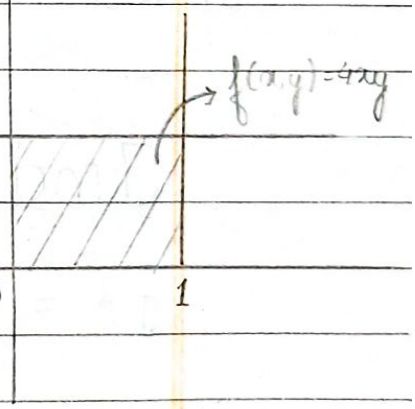
26. Let X and Y be random variable with joint density function

$$f(x, y) = \begin{cases} 4xy & 0 < x, y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find expected value of

$$Z = \sqrt{X^2 + Y^2}$$

$$E(Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z\, f(x, y)\, dx\, dy$$



$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2 + y^2}\, f(x, y)\, dx\, dy$$

$$= \int_0^1\int_0^1 \sqrt{x^2 + y^2}\; 4xy\, dx\, dy$$

Let $x^2 + y^2 = t \Rightarrow 2x\,dx = dt$

$$y \int_{y^2}^{y^2+1} 2\sqrt{t}\, dt = 2y \left. \frac{t^{3/2}}{3/2} \right]_{y^2}^{y^2+1}$$

$$= \frac{4y}{3}\left[(y^2+1)^{3/2} - y^3 \right]$$

$$\int_0^1 \frac{4}{3} y \left[(y^2+1)^{3/2} - y^3 \right] dy$$

Let $y^2 + 1 = w \Rightarrow 2y\,dy = dw$

$$= \frac{4}{3} \int_1^2 \frac{w^{3/2}}{2} dw - \frac{4}{3} \int_0^1 y^4 dy$$

$$= \frac{2}{3} \left. \frac{w^{5/2}}{5/2} \right]_1^2 - \frac{4}{3} \left. \frac{y^5}{5} \right]_0^1$$

$$= \frac{4}{15} (2^{5/2} - 1) - \frac{4}{15} = \frac{4}{15} (2^{5/2} - 2)$$

$$= \frac{8}{15} (2^{3/2} - 1)$$

34. Let X be random variable

x	-2	3	5
$f(x)$	0.3	0.2	0.5

Find standard deviation of X

$$\sigma^2 = E(X^2) - \mu^2$$

$$\mu = \sum_x x f(x)$$

$$= -0.6 + 0.6 + 2.5 = 2.5$$

$$E(X^2) = \sum_x x^2 f(x)$$

$$= 1.2 + 1.8 + 12.5 = 15.5$$

Date
Page No.

$\sigma^2 = 15.5 - (2.5)^2$
$= 9.25$

$\sigma = 3.04$

35. Random variable X, representing the number of errors per 100 lines of software code has probability distribution :

x	2	3	4	5	6
$f(x)$	0.01	0.25	0.4	0.3	0.04

Find variance of X

$\mu = 0.02 + 0.75 + 1.6 + 1.5 + 0.24$
$= 4.11$

$E(X^2) = \sum_x x^2 f(x)$

$= 0.04 + 2.25 + 6.4 + 7.5 + 1.44$
$= 17.63$

$\sigma^2 = E(X^2) - \mu^2$
$= 17.63 - 16.8921$
$= 0.7379$

50. For laboratory assignment, if equipment is working, density function of the observed outcome is

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find variance & standard deviation of X

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{1} x f(x)\,dx = 2\int_{0}^{1} x - x^2\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{\infty} 2x^2(1-x)\,dx$$

$$= 2\int_{0}^{1} x^2 - x^3\,dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{6}$$

$$\sigma^2 = E(X^2) - \mu^2$$
$$= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\sigma = 0.2357$$

57. X is random variable with the following probability distribution

x	-3	6	9
$f(x)$	1/6	1/2	1/3

Find $E(X)$ and $E(X^2)$ and then evaluate $E[(2X+1)^2]$

$$E(X) = -3\left(\frac{1}{6}\right) + 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right)$$

$$= \frac{11}{2}$$

$$E(X^2) = 9\left(\frac{1}{6}\right) + 36\left(\frac{1}{2}\right) + 81\left(\frac{1}{3}\right)$$

$$= \frac{93}{2}$$

$$E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1$$

$$= 4\left(\frac{93}{2}\right) + 4\left(\frac{11}{2}\right) + 1$$

$$= 209$$

58. Total time measured in units of 100 hours that teenager runs hair dryer over one year is continuous random variable x with density function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate mean of $Y = 60X^2 + 39X$, where Y is equal to no. of kilowatt hours

$$E(X) = \int_0^1 x^2\,dx + \int_1^2 2x - x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 + \left.\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - \frac{2}{2} + \frac{1}{3} = 1$$

$$E(Y) = 60E(X^2) + 39E(X)$$

$$E(X^2) = \int_0^1 x^3\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \left.\frac{x^4}{4}\right]_0^1 + \left.\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}$$

$$= \frac{7}{6}$$

$$E(Y) = 60\left(\frac{7}{6}\right) + 39(1)$$

$$= 70 + 39 = 109 \text{ kW}$$

60. X, Y are independent random variable

$f(x,y)$		x	
		2	4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

Find (a) $E(2X - 3Y)$

(b) $E(XY)$

$$E(X) = 2(0.4) + 4(0.6)$$
$$= 3.2$$

$$E(Y) = 1(0.25) + 3(0.5) + 5(0.25)$$
$$= 3$$

Date

Page No.

a) $E(2X-3Y) = 2E(X) - 3E(Y)$
$= 2(3.2) - 3(3)$
$= 6.4 - 9 = -2.6$

b) $E(XY) = E(X)E(Y)$
$= 3.2 \times 3$
$= 9.6$

64. X and Y are independent random variables

$$g(x) = \begin{cases} 8/x^3, & x > 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find expected value of $Z = XY$

$E(Z) = E(XY) = E(X)E(Y)$

$$= \int_0^1 \int_2^\infty \frac{8xy}{x^3} \cdot 2y \, dx \, dy$$

$$= \int_0^1 16y^2 \cdot \frac{x^{-1}}{-1} \Big]_2^\infty dy$$

$$= \int_0^1 8y^2 \, dy$$

$$= \frac{8y^3}{3} \Big]_0^1 = \frac{8}{3}$$

67 Joint density function of X, Y

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y) & , 0 < x < 1 , 1 < y < \\ 0 & , \text{elsewhere} \end{cases}$$

Find expected value of $g(X,Y) = \frac{X}{Y^3} + X^2Y$

$$E(g(X,Y)) = E\left[\frac{X}{Y^3} + X^2Y\right]$$

$$= E\left(\frac{X}{Y^3}\right) + E(X^2Y)$$

$$E\left(\frac{X}{Y^3}\right) = \int_1^2\int_0^1 \frac{2}{7}(x+2y)\cdot\frac{x}{y^3}\,dx\,dy$$

$$= \frac{2}{7}\int_1^2 \frac{1}{3y^3} + \frac{1}{y^2}\,dy$$

$$= \frac{2}{7}\left(\frac{y^{-2}}{-6} + \frac{y^{-1}}{-1}\right)_1^2$$

$$= \frac{2}{7}\left(\frac{-1}{6.4} - \frac{1}{2} + \frac{1}{6} + 1\right)$$

$$= \frac{15}{84}$$

$$E(X^2Y) = \int_1^2\int_0^1 \frac{2}{7}(x+2y)\,x^2y\,dx\,dy$$

$$= \frac{2}{7}\int_1^2 \frac{y}{4} + \frac{2y^2}{3}\,dy = \frac{139}{252}$$

$$E[g(X,Y)] = \frac{15}{84} + \frac{139}{252} = \frac{46}{63}$$

75. An electrical firm manufactures 100 w bulb, which according to specifications written on package, has mean life of 900 hours with standard deviation 50 hours. At most, what % of bulbs fail to last even 700 hours? Assume distribution is symmetric about mean

$\mu = 900$ hours $\quad \sigma = 50$ hours

$\mu - K\sigma = 700 \Rightarrow 900 - 50K = 700$

$200 = 50K$

$K = 4$

Using Chebyshev's theorem

$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$

$\Rightarrow P(\mu - 4\sigma < X < \mu + 4\sigma) \geq \frac{15}{16}$

$P(700 < X < 1100) \geq 0.9375$

$P(X \leq 700) \leq 0.03125$

77. A random variable X has mean $\mu = 10$ and $\sigma^2 = 4$. Using Chebyshev's theorem, find

a) $P(|X - 10| \geq 3)$ b) $P(|X - 10| < 3)$

c) $P(5 < X < 15)$

d) value of c such that

$P(|X - 10| \geq c) \leq 0.04$

$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$

(a) $P(|X - 10| \geq 3) = 1 - P(|X - 10| < 3)$

$-3 < X - 10 < 3$

$\Rightarrow 7 < X < 13$

$1 - P(7 < X < 13)$

$10 - 2K < X < 10 + 2K$

$10 - 2K = 7 \Rightarrow K = 3/2$

$10 + 2K = 13 \Rightarrow K = 3/2$

$P(7 < X < 13) \geq 1 - \frac{1}{K^2} = \frac{5}{9}$

$P(|X - 10| \geq 3) \leq 1 - \frac{5}{9} = \frac{4}{9}$

Date

Page No.

(b) $P(|X-10| < 3) \geq \frac{5}{9}$

(c) $P(5 < X < 15)$

$10 - 2K = 5 \Rightarrow K = 5/2$

$10 + 2K = 5 \Rightarrow K = 5/2$

$P(5 < X < 15) \geq \frac{21}{25}$

(d) $P(|X-10| \geq C) \leq 0.04$

$1 - P(|X-10| < C) \leq 0.04$

$P(|X-10| < C) \geq 0.96$

$P(10 - C < X < 10 + C) \geq 0.96$

$1 - \frac{1}{K^2} = 0.96 \Rightarrow \frac{1}{K^2} = 0.04$

$K = 5$

$\mu - 5\sigma = 10 - C$

$10 - 10 = 10 - C$

$C = 10$