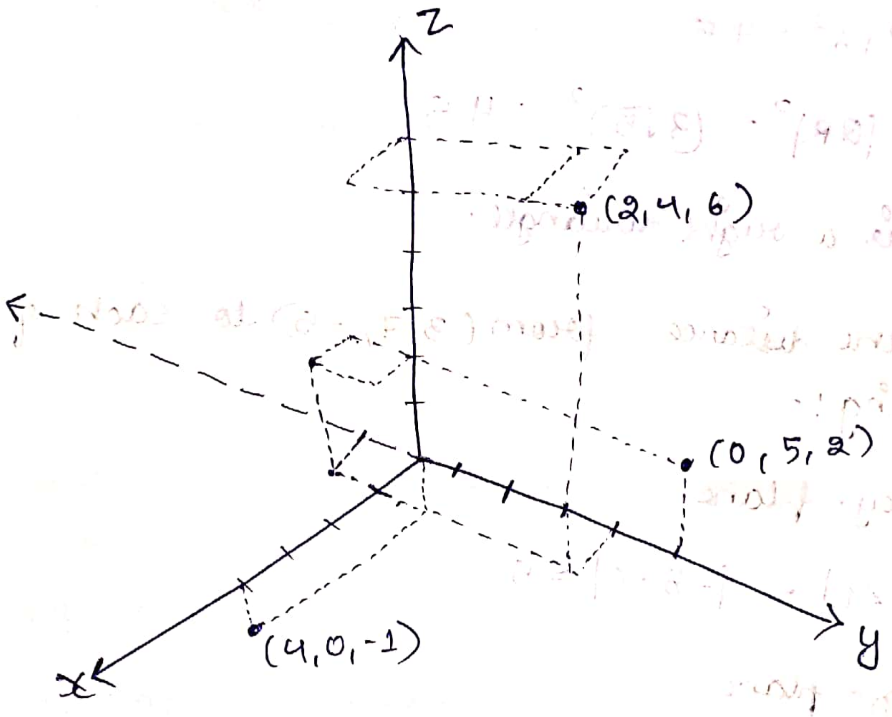
VECTORS AND THE GEOMETRY OF SPACE

C.W.

② sketch the points $(0,5,2)$, $(4,0,-1)$, $(2,4,6)$ & $(1,-1,2)$ on a single set of coordinates axes.

(Ans)



③ which of the points A $(-4,0,-1)$, B $(3,1,-5)$ and C $(2,4,6)$ is closest to the y-z plane? which point lies in the x-z plane?

(Ans) C is with the smallest x is closest to the plane y-z

A with $y=0$ is in the plane $x-z$

⑧ find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle.

$P(2,-1,0)$, $Q(4,1,1)$, $R(4,-5,4)$

(Ans) $|PQ| = \sqrt{(4-2)^2 + (1+1)^2 + (1-0)^2}$

$= \sqrt{4+4+1} = 3$

$|QR| = \sqrt{(4-4)^2 + (-5-1)^2 + (4-1)^2}$

$= \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$

$|PR| = \sqrt{(4-2)^2 + (-5+1)^2 + (4-0)^2}$

$= \sqrt{4+16+16} = \sqrt{36} = 6$

since, $|PQ|^2 + |PR|^2 = |QR|^2$

$= 3^2 + 6^2 = 45$

and $|QR|^2 = (3\sqrt{5})^2 = 45$

so, PQR is a right triangle.

(10) Find the distance from $(3, 7, -5)$ to each of the following:-

(a) The xy-plane

(Ans) $|z_2 - z_1| = |-5-0| = 5$

(b) the yz-plane

(Ans) $|x_2 - x_1| = |3-0| = 3$

(c) The xz-plane

(Ans) $|y_2 - y_1| = |7-0| = 7$

(d) The x-axis

(Ans) $(3,0,0)$ is close to $(3,7,-5)$

so, $\sqrt{(3-3)^2 + (7-0)^2 + (-5-0)^2} = \sqrt{104} = 10.19$

(e) The y-axis

(Ans) $(0,7,0)$ is close to $(3,7,-5)$

so $\sqrt{(3-0)^2 + (7-7)^2 + (-5-0)^2} = \sqrt{34} = 5.83$

(f) The z-axis

(Ans) $(0,0,-5)$ is close to $(3,7,-5)$

so, $\sqrt{(3-0)^2 + (7-0)^2 + (-5-(-5))^2} = \sqrt{58} = 7.61$

(11) Find an equation of the sphere with center $(1,-4,3)$ & radius 5. What is the intersection of this sphere with the xz-plane?

(Ans) $(x-1)^2 + [y-(-4)]^2 + (z-3)^2 = (5)^2$

$\Rightarrow (x-1)^2 + (y+4)^2 + (z-3)^2 = 25$

To find the intersection of this sphere with the xz-plane set $y=0$

so $(x-1)^2 + [0+4]^2 + (z-3)^2 = 25$

$\Rightarrow (x-1)^2 + (z-3)^2 = 25-16$

$\Rightarrow (x-1)^2 + (z-3)^2 = 9$

centers are $(1,0,3)$ with radius $r = \sqrt{9} = 3$

(16) Show that the equation represents a sphere, and find its centre & ~~sphere~~ radius

$$x^2+y^2+z^2+8x-6y+2z+17=0$$

(Ans) $x^2+y^2+z^2+8x-6y+2z+17=0$

$\Rightarrow x^2+8x+y^2-6y+z^2+2z+17=0$

$\Rightarrow x^2+8x+16+y^2-6y+9+z^2+2z+1 = -17+16+9+1$

$\Rightarrow (x+4)^2 + (y-3)^2 + (z+1)^2 = 9$

so, the centre is $(-4,3,-1)$ and radius is $r=\sqrt{9}=3$

(17) $2x^2+2y^2+2z^2 = 8x-24z+1$

(Ans) $2x^2+2y^2+2z^2-8x+24z = 1$

$\Rightarrow 2x^2-8x+2y^2+2z^2+24z = 1$

$\Rightarrow 2(x^2-4x+y^2+z^2+12z) = 1$

$\Rightarrow x^2-4x+y^2+z^2+12z = 1/2$

$\Rightarrow x^2-4x+4+y^2+z^2+12z+36 = 1/2+4+36$

$\Rightarrow (x-2)^2+y^2+(z+6)^2 = 81/2$

so, the centre is $(2,0,-6)$ & radius is $r = \sqrt{\frac{81}{2}} = \frac{9}{\sqrt{2}}$

(25) Describe in words the region of R^3 represented by the equations or inequalities.

$y < 8$

(Ans) $y < 8$ covers all regions that would lie to the left of the plane parallel to the xz-plane that intersected at $y = 8$

(34) $x^2 + y^2 + z^2 > 2z$

(Ans) $x^2 + y^2 + z^2 - 2z > 0$

$\Rightarrow x^2 + y^2 + z^2 - 2z + 1 > 0 + 1$

$\Rightarrow x^2 + y^2 + (z-1)^2 > 1$

All the points are outside & not on the sphere of radius 1 centered at $(0, 0, 1)$

(35) write inequalities to describe the region;
the region between the yz-plane & the vertical plane $x = 5$

(Ans) In the yz-plane, we have $x = 0$. This means that the region b/w the yz-plane & the plane $x = 5$ is given by
$0 < x < 5$

(41) Find an equation of the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$. Describe the set

(Ans) $A(-1, 5, 3)$ and $B(6, 2, -2)$

(x, y, z) is a point that is equidistant from $A(-1, 5, 3)$ and $B(6, 2, -2)$

so, $\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}$

$\Rightarrow (x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 =$
$x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4$

$$\Rightarrow 2x - 10y - 6z + 35 = -12x - 4y + 4z + 44$$

$$\Rightarrow 14x - 6y - 10z = 9$$

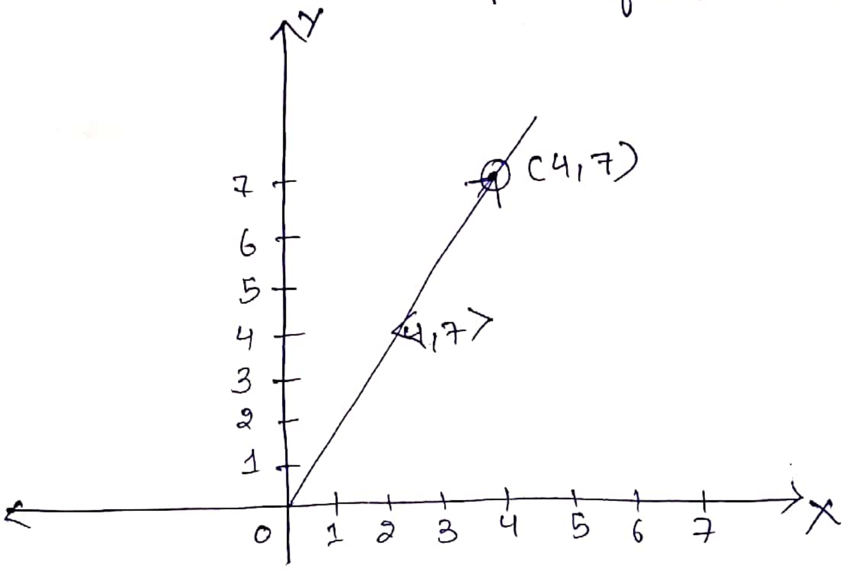
$$\Rightarrow 2(7x - 3y - 5z) = 9$$

$$\Rightarrow 7x - 3y - 5z = 9/2$$

Exercise 12.2

2) what is the relationship between the point (4, 7) and the vector ⟨4, 7⟩? Illustrate with a sketch.

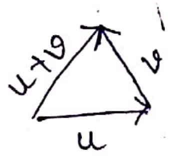
(Ans) The point (4, 7) is the terminal point of the vector ⟨4, 7⟩.



5) copy the vectors in the figure & use them to draw the following vectors.

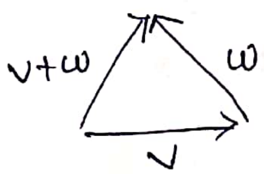
a) u + v

(Ans)

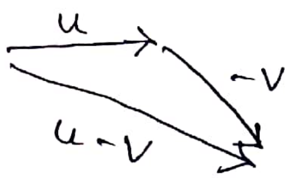


b) u + w

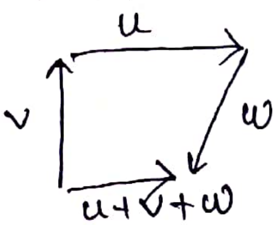
c) v + w



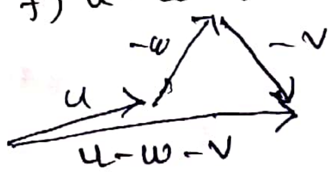
d) u − v



e) v + u + w

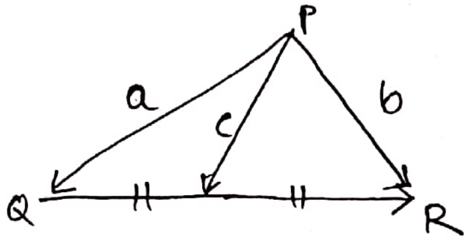


f) u − w − v

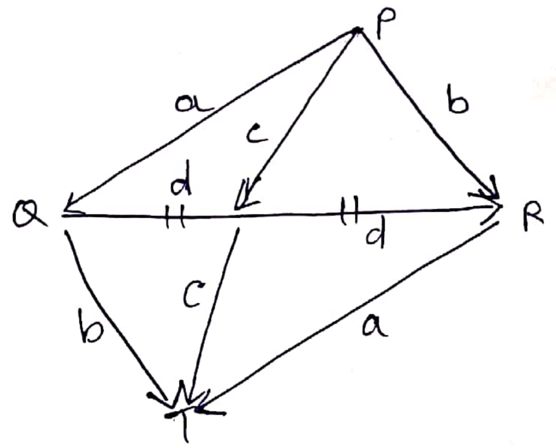


Scanned with CamScanner

7) In the fig. the tip of 'c' & the tail of 'd' are both the midpoint of QR. Express 'c' and 'd' in terms of 'a' and 'b'



(Ans)



To get from P to T we can travel on a and b.

$$2c = a + b$$

$$\Rightarrow c = \frac{1}{2}a + \frac{1}{2}b$$

To get from the center point to point R

$$d + c = b$$

$$\Rightarrow d = -c + b$$

Put the value of 'c'

$$\Rightarrow d = -\left(\frac{1}{2}a + \frac{1}{2}b\right) + b$$

$$\Rightarrow d = -\frac{1}{2}a - \frac{1}{2}b + b$$

$$\Rightarrow d = -\frac{1}{2}a + \left(1 - \frac{1}{2}\right)b$$

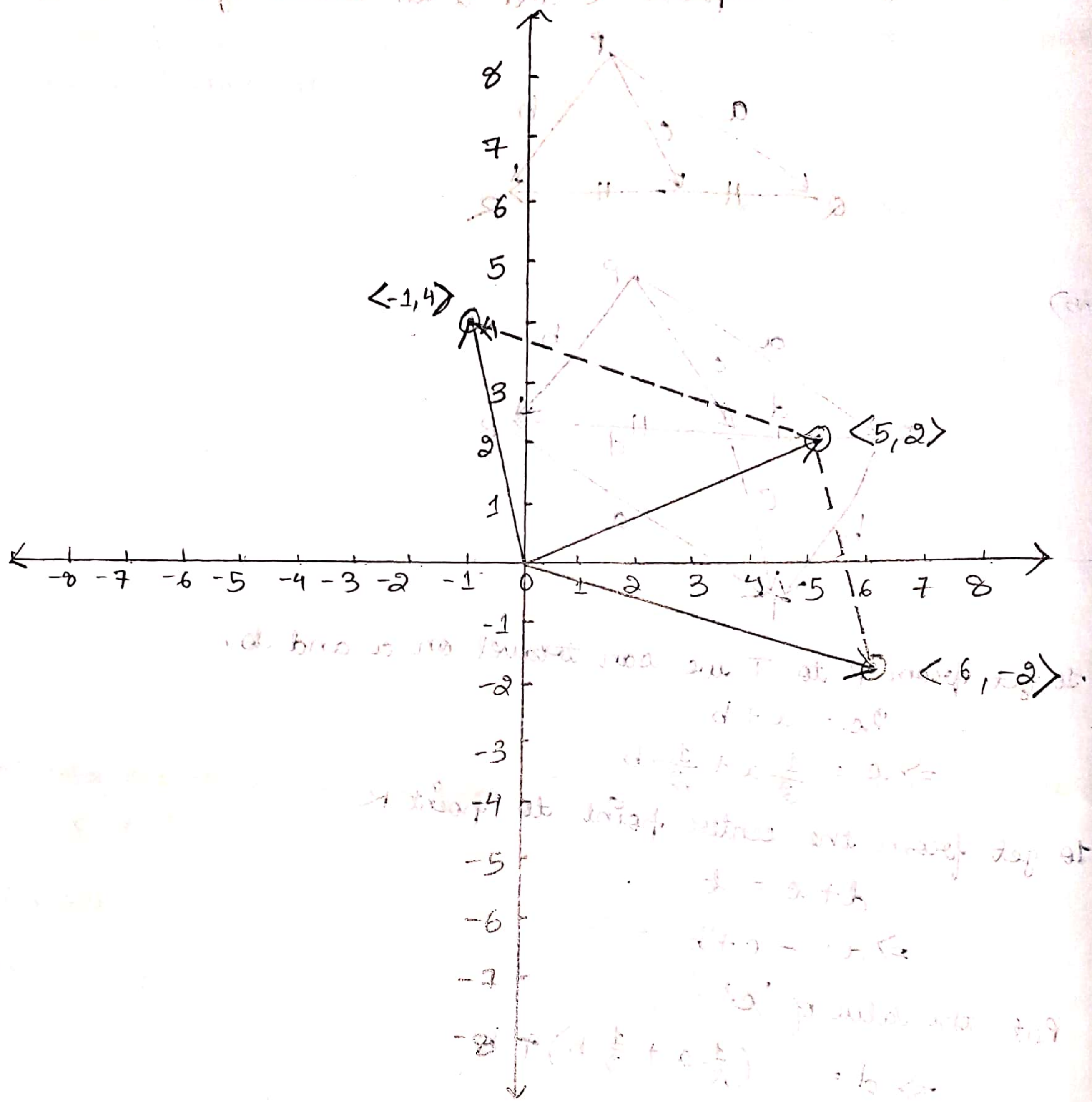
$$\Rightarrow d = -\frac{1}{2}a + \frac{1}{2}b$$

15) $\langle -1, 4\rangle$, $\langle 6, -2\rangle$

(Ans) $\langle -1, 4\rangle + \langle 6, -2\rangle$

$= \langle -1 + 6, 4 + (-2)\rangle$

$= \langle 5, 2\rangle$



20) Find $a+b$, $2a+3b$, $|a|$ and $|a-b|$

$a = 4i + j$, $b = i - 2j$

(Ans) $a+b = 4i+j+i-2j$

$= 5i - j$

$2a+3b = 2(4i+j)+3(i-2j)$

$= 8i + 2j + 3i - 6j$

$= 11i - 4j$

$|a| = \sqrt{(4)^2 + (1)^2} = \sqrt{17}$

$|a-b| = |4i+j-i+2j|$

$= |3i+3j|$

$= \sqrt{(3)^2+(3)^2} = \sqrt{18} = 3\sqrt{2}$

21) $a = i+2j-3k$, $b = -2i-j+5k$

(Ans) $a+b = i+2j-3k - 2i-j+5k$

$= -i+j+2k$

$2a+3b = 2(i+2j-3k)+3(-2i-j+5k)$

$= 2i+4j-6k-6i-3j+15k$

$= -4i+j+9k$

$|a| = \sqrt{(1)^2+(2)^2+(-3)^2} = \sqrt{14}$

$|a-b| = |3i+3j-8k|$ $[= |i+2j-3k+2i+j-5k|]$

$= \sqrt{(3)^2+(3)^2+(-8)^2}$

$= \sqrt{82}$

25) Find a unit vector that has same direction as the given vector.

$8i-j+4k$

(Ans) Let $V = 8i-j+4k$

$|V| = \sqrt{8^2+(-1)^2+(4)^2}$

$= \sqrt{81} = 9$

so, unit vector $u = \frac{V}{|V|} = \frac{8i-j+4k}{9}$

$= \frac{8}{9}i - \frac{1}{9}j + \frac{4}{9}k$

26) Find a vector that has the same magnitude & direction as $\langle -2,4,2 \rangle$ but has length 6

(Ans) Let $\vec{v}$ be the vector with same magnitude & direction as $\langle -2,4,2 \rangle$ but has length 6

then $\vec{c} = \langle -2,4,2 \rangle$

$$\hat{c} = \frac{\langle -2,4,2 \rangle}{\sqrt{(-2)^2+(4)^2+(2)^2}} = \frac{\langle -2,4,2 \rangle}{\sqrt{24}} = \frac{\langle -2,4,2 \rangle}{2\sqrt{6}}$$

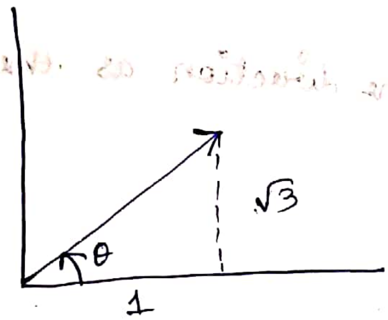
so $\vec{v} = 6 \times \hat{c}$

$$= \frac{\cancel{6}^{3} \langle -2,4,2 \rangle}{\cancel{2}\cancel{\sqrt{6}}\,\sqrt{2}} = \frac{\langle -2,4,2 \rangle}{\sqrt{2}}$$

27) What is the angle between the given vector & the positive direction of the x-axis?

$i + \sqrt{3} j$

(Ans)



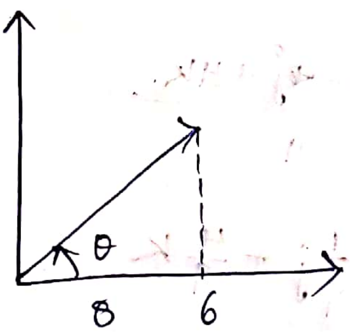
$$\tan\theta = \frac{\sqrt{3}}{1}$$

$$\Rightarrow \theta = \tan^{-1}(\sqrt{3})$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{3}}$$

28) $8i + 6j$

(Ans)



$$\tan\theta = \frac{6}{8}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{6}{8}\right)$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\Rightarrow \theta \approx 36.87^{\circ}$$

Scanned with CamScanner

29) If v lies in the first quadrant & makes an angle $\pi/3$ with the positive x-axis and $|v| = 4$, find v in component form.

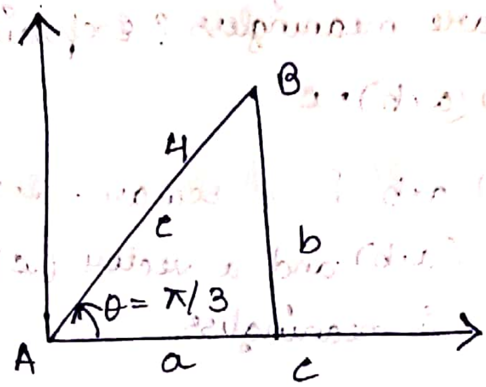
(Ans) $a = r\cos\theta$

$b = r\sin\theta$

$a = 4\cos\pi/3 = 4 \times 1/2 = 2$

$b = 4\sin\pi/3 = 4 \cdot \sqrt{3}/2 = 2\sqrt{3}$

$v = \langle 2, 2\sqrt{3} \rangle$



30) If a child pulls a sledge through the snow on a level path with a force of 50 N exerted at angle 38° above the horizontal, find the horizontal & vertical components of the force

(Ans) $r = |F| = 50N$

$a = r\cos\theta = 50\cos 38^\circ \approx 39.4\,N$

$b = r\sin\theta = 50\sin 38^\circ \approx 30.8\,N$

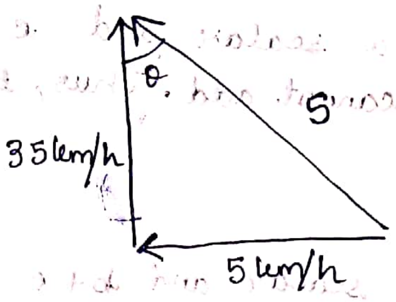
horizontal $\approx 39.4\,N$ and vertical $\approx 30.8\,N$

35) A women walks due west on the deck of a ship at 5 km/h. The ship is moving north at a speed of 35 km/h. Find the speed & direction of the women relative to the surface of the water.

(Ans) so, $s = \sqrt{(5)^2 + (35)^2}$

$= \sqrt{1250}$

≈ 35.35 km/h



so, $\theta = \text{arc}\tan^{-1}\left(\frac{5}{35}\right) \approx 8^\circ$

$\therefore$ she is moving approx. 35.35 km/h in the direction 8 degrees west of north relative to the water surface.

Exercise 12.3

1) which of the following expressions are meaningful? which are meaningless? Explain

a) $(a \cdot b) \cdot c$

(Ans) $a \cdot b$ is a scalar. The dot product cannot take a scalar $(a \cdot b)$ and a vector (c) as arguments. Thus the expressions is meaningless.

b) $(a \cdot b) c$

(Ans) $a \cdot b$ is a scalar. Thus, it can scale c, leading to a meaningful expression.

c) $|a|(b \cdot c)$

(Ans) $|a|$ and $b \cdot c$ are both scalars. The multiplication of two scalars, meaning the expression is meaningful.

d) $a \cdot (b + c)$

(Ans): a and $(b+c)$ are both vectors. Thus, the dot product of the two vectors is a meaningful.

e) $a \cdot b + c$

(Ans) $a \cdot b$ is a scalar and c is a vector. A scalar & a vector cannot add. Thus, the expression is meaningless.

f) $|a| \cdot (b + c)$

(Ans) $|a|$ is a scalar and $b + c$ is a vector. The dot product between a scalar and vector cannot be taken. Thus, the expression is meaningless.

9) Find $a \cdot b$

$|a| = 6$, $|b| = 5$, the angle between a and b is $2\pi/3$

(Ans) $|a| = 6$, $|b| = 5$, $\theta = 2\pi/3$

so, $|a| \cdot |b| \cos\theta$

$= 6 \cdot 5 \cdot \cos(2\pi/3)$

$= 6 \cdot 5 \cdot (-0.5) = -15$

10) $|a| = 3$, $|b| = \sqrt{6}$, the angle between a and b is 45°

(Ans) $|a| = 3$, $|b| = \sqrt{6}$, $\theta = 45^\circ$

so, $|a||b| \cos\theta$

$= 3 \cdot \sqrt{6} \cdot \cos(45^\circ)$

$= 3 \cdot \sqrt{6} \cdot 1/\sqrt{2}$

$= 3 \cdot \sqrt{6} \cdot \sqrt{2}/\cancel{5} = 3\sqrt{3}$

17) Find the angle between the vectors

$a = \langle 3, -1, 5 \rangle$, $b = \langle -2, 4, 3 \rangle$

(Ans) $a \cdot b = |a||b| \cos\theta$

$\Rightarrow \theta = \cos^{-1} \left(\frac{a \cdot b}{|a||b|} \right)$

so, $|a| = \sqrt{(3)^2 + (-1)^2 + (5)^2}$

$= \sqrt{33}$

$|b| = \sqrt{(-2)^2 + (4)^2 + (3)^2}$

$= \sqrt{29}$

$a \cdot b = \langle 3, -1, 5 \rangle \cdot \langle -2, 4, 3 \rangle$

$= (-6) - 4 + 15 = 5$

so, $\theta = \cos^{-1} \left(\frac{5}{|\sqrt{35}||\sqrt{29}|} \right)$

$= 80.97^\circ = 1.413$ radians

$\Rightarrow \boxed{\theta = 81^\circ}$

Scanned with CamScanner

18) $a = \langle 4,0,2 \rangle$, $b = \langle 2,-1,0 \rangle$

(Ans) $a \cdot b = |a|\,|b| \cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\frac{a \cdot b}{|a|\,|b|}\right)$

so, $|a| = \sqrt{(4)^2 + (0)^2 + (2)^2}$

$= \sqrt{20}$

$|b| = \sqrt{(2)^2 + (-1)^2 + (0)^2}$

$= \sqrt{5}$

$a \cdot b = \langle 4,0,2 \rangle \cdot \langle 2,-1,0 \rangle$

$= 8 + 0 + 0$

$= 8$

$\theta = \cos^{-1}\left(\frac{8}{|\sqrt{20}|\,|\sqrt{5}|}\right)$

$= \cos^{-1}\left(\frac{8}{10}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \boxed{\theta = 37^0}$

24) Determine whether the given vectors are orthogonal, parallel or neither

(a) $u = \langle -3,9,6 \rangle$, $v = \langle 4,-12,-8 \rangle$

(Ans) $u \cdot v = \langle -3,9,6 \rangle \cdot \langle 4,-12,-8 \rangle$

$= (-3*4) + (9*-12) + (6*-8) = -168$

$|u| = \sqrt{(-3)^2 + (9)^2 + (6)^2} = \sqrt{126}$

$|v| = \sqrt{(4)^2 + (-12)^2 + (-8)^2} = \sqrt{224}$

$|u| * |v| = \sqrt{126} * \sqrt{224} = 168$

since, $u \cdot v = -|u| * |v|$ the vectors are parallel.

(b) $u = \hat{i} - \hat{j} + 2\hat{k}$, $v = 2\hat{i} - \hat{j} + \hat{k}$

(Ans) $u \cdot v = \langle 1,-1,2\rangle \cdot \langle 2,-1,1\rangle$

$= 2+1+2 = 5 \neq 0 \Rightarrow$ neither parallel or perpendicular.

(c) $u = \langle a,b,c\rangle$, $v = \langle -b,a,0\rangle$

(Ans) $\vec{u} \cdot \vec{v} = a*(-b) + b*a + c*0$

$= -ab + ab + 0 = 0$

The vectors are orthogonal.

26) Find the values of x such that the angle between the vectors $\langle 2,1,-1\rangle$ and $\langle 1,x,0\rangle$ is 45°

(Ans) $\langle 2,1,-1\rangle \cdot \langle 1,x,0\rangle$

$= \sqrt{2^2+1^2+(-1)^2} \times \sqrt{(1)^2+(x)^2} \times \cos 45^\circ$

$\Rightarrow 2+x = \sqrt{6}\sqrt{1+x^2} \cdot \frac{\sqrt{3}}{2}$

$\Rightarrow 2+x = \frac{\sqrt{12}}{2}\sqrt{1+x^2}$

$\Rightarrow 2+x = \sqrt{3}\sqrt{1+x^2}$

$\Rightarrow (2+x)^2 = (\sqrt{3}\sqrt{1+x^2})^2$ [squaring both sides]

$\Rightarrow x^2+4x+4 = 3(1+x^2)$

$\Rightarrow x^2+4x+4 = 3+3x^2$

$\Rightarrow 2x^2-4x-1=0$

$\Rightarrow x = \frac{4 \pm \sqrt{16-4 \times 2 \times (-1)}}{2 \times 2}$

$\Rightarrow x = \frac{4 \pm \sqrt{24}}{4}$

$\Rightarrow x = \frac{4 \pm 2 \times \sqrt{6}}{4}$

$\Rightarrow \boxed{x = 1 \pm \frac{\sqrt{6}}{2}}$

27) Find a unit vector that is orthogonal to both $\hat{i}+\hat{j}$ & $\hat{i}+\hat{k}$

(Ans) $a = x\hat{i} + y\hat{j} + z\hat{k}$

Scanned with CamScanner

so, $x\cdot 1 + y\cdot 1 = 0$ (as $\hat{i}+\hat{j}$)

$x\cdot 1 + z\cdot 1 = 0$ (as $\hat{i}+\hat{k}$)

substituting we get $x=-y$ and $x=-z$

so, $a = x\hat{i} - x\hat{j} - xk$

so, unit vector $= a = \frac{1}{\sqrt{3}}(\hat{i}-\hat{j}-k)$

(28) Find two unit vectors that make an angle of 60° with $v=\langle 3,4\rangle$

(Ans) $V = \langle 3,4\rangle$

$|a| = \sqrt{x^2+y^2} = 1$

so, $a\cdot v = |a||v|\cos\theta$

$\Rightarrow \langle x,y\rangle\cdot\langle 3,4\rangle = 1\cdot\sqrt{3^2+4^2}\cos 60^\circ$

$\Rightarrow 3x+4y = 5(1/2)$

$\Rightarrow 6x+8y = 5$

$\Rightarrow y = \frac{1}{8}(5-6x)$

so, $x^2+y^2=1$

$\Rightarrow x^2 + \left(\frac{1}{8}(5-6x)\right)^2 = 1$

$\Rightarrow x^2 + \frac{1}{64}(25-60x+36x^2) = 1$

$\Rightarrow 64x^2+25-60x+36x^2 = 1$

$\Rightarrow 100x^2-60x-39=0$

$\Rightarrow x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-60)\pm\sqrt{(60)^2-4(100)(-39)}}{2(100)}$

$= \frac{60\pm\sqrt{19200}}{200}$

$= \frac{60\pm\sqrt{6400\times 3}}{200}$

$$= \frac{60 \pm 80\sqrt{3}}{200}$$

$$= \frac{3 \pm 4\sqrt{3}}{10} \approx -0.39282, 0.99282$$

so, $y = \frac{1}{8}(5-6(-0.39282)) \approx 0.919615$

and $y = \frac{1}{8}(5-6(0.99382)) \approx -0.119615$

so, the unit vectors are

$\langle -0.39282, 0.919615 \rangle$ and $\langle 0.99282, -0.119615 \rangle$

29) Find the acute angle between the lines

$2x - y = 3, \quad 3x + y = 7$

(Ans) $2x - y = 3 \Rightarrow y = 2x - 3$ ------- (i)

$3x + y = 7 \Rightarrow y = -3x + 7$ ------- (ii)

$$\left.\begin{array}{l} a = \langle 1, 2 \rangle \\ b = \langle 1, -3 \rangle \end{array}\right\} \left(\frac{\Delta y}{\Delta x}\right)$$

$a \cdot b = |a||b| \cos\theta$

$$\Rightarrow \theta = \cos^{-1} \frac{a \cdot b}{|a||b|}$$

$$= \cos^{-1} \frac{1(1) + 2(-3)}{\sqrt{1^2 + 2^2}\sqrt{1^2 + (-3)^2}}$$

$$= \cos^{-1} \frac{-5}{\sqrt{5}\sqrt{10}}$$

$$= \cos^{-1} \frac{-5}{\sqrt{50}}$$

$$= \cos^{-1} \frac{-5}{5\sqrt{2}} = \cos^{-1} \frac{-1}{\sqrt{2}} = \frac{3\pi}{4} = 135^\circ$$

so, the angle (acute) $= 180^\circ - 135^\circ = 45^\circ$

30) $x + 2y = 7, \quad 5x - y = 2$

(Ans) $x + 2y = 7$

$\Rightarrow y = -\frac{1}{2}x + \frac{7}{2}$ --------- (i)

$5x - y = 2$

$\Rightarrow y = 5x - 2 \text{ .... (ii)}$

$a = \langle 2, -1 \rangle$

$b = \langle 1, 5 \rangle$

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

$$\Rightarrow \theta = \cos^{-1} \frac{(2)(1) + (-1)(5)}{\sqrt{2^2 + (-1)^2}\sqrt{1^2 + 5^2}}$$

$$\Rightarrow \theta = \cos^{-1} \frac{-3.}{\sqrt{5} \cdot \sqrt{26}}$$

$$\Rightarrow \theta = 105.3^\circ$$

so, the accute angle $= 180^\circ - 105.3^\circ = 74.7^\circ$

31) Find the accute angles between the curves at their points of intersection.

$y = x^2, y = x^3$

(Ans) Let $f(x) = x^2$

& $g(x) = x^3$

so, $x^2 = x^3$

$\Rightarrow x^2 - x^3 = 0$

$\Rightarrow x^2(1 - x) = 0$

$\Rightarrow \boxed{x = 0}$ and $\boxed{x = 1}$

Again, $f'(x) = 2x$

$g'(x) = 3x^2$

when $x = 0$,

$f'(0) = 2 \times 0 = 0$

$g'(0) = 3(0)^2 = 0$

so, $\theta = 0^\circ$ at $x = 0$

when $x = 1$,

$f'(1) = 2\times1 = 2$

$g'(x) = 3\,(1)^2 = 3$

$a = \langle 1, 2\rangle$

$b = \langle 1, 3\rangle$

so, $\cos\theta = \dfrac{a\cdot b}{|a|\,|b|}$

$\Rightarrow \theta = \cos^{-1} \dfrac{1\times1 + 2\times3}{\sqrt{5}\cdot\sqrt{10}}$

$= \cos^{-1} \dfrac{7}{\sqrt{50}} = 8.13^\circ$

so, $\tan^{-1} \dfrac{3}{1} = 71.57^\circ$

$\tan^{-1} \dfrac{2}{1} = 63.43^\circ$

so, $71.57^\circ - 63.43^\circ = 8.13^\circ$

34) Find the direction cosines & direction angles of the vector.
$\langle 6, 3, -2\rangle$

(Ans) Let $v = \langle 6, 3, -2\rangle$

$|v| = \sqrt{6^2 + 3^2 + (-2)^2}$

$= 7$

$\cos\alpha = \dfrac{6}{7}$, $\cos\beta = \dfrac{3}{7}$, $\cos\gamma = \dfrac{-2}{7}$

$\Rightarrow \alpha = \cos^{-1}\left(\dfrac{6}{7}\right) = 31^\circ$

$\Rightarrow \beta = \cos^{-1}(3/7) = 65^\circ$

$\Rightarrow \gamma = \cos^{-1}(-2/7) = 107^\circ$

36) If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$ find the third direction angle γ.

(Ans) we know that,

$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$

so, $\alpha = \pi/4$ and $\beta = \pi/3$

so, $\dfrac{1}{2} + \dfrac{1}{4} + \cos^2(\gamma) = 1$

$\Rightarrow \cos^2(\gamma) = 1/4 \Rightarrow \gamma = \cos^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \boxed{\gamma = \dfrac{\pi}{3}}$

40) Find the scalar & vector projections of b onto a

$a = \langle 1,4 \rangle$, $b = \langle 2,3 \rangle$

(Ans) $a = \langle 1,4 \rangle$ and $b = \langle 2,3 \rangle$

$|a| = \sqrt{1^2+4^2}$

$= \sqrt{17}$

scalar projections b onto a $= \dfrac{a \cdot b}{|a|^2} \cdot a$

$= \dfrac{14}{17} (1,4)$

$= \langle 14/17 , 56/17 \rangle$

(vector projection)

52) Find the workdone by a force of 100N acting in the direction N50°W in moving an object 5m due west.

(Ans) workdone $= W = |F||D| \cos\theta$

$= 100 \times 5 \times \cos 50°$

$= 321.39$ N-m

Exercise 12.4

3) Find the cross product $a \times b$ & verify that it is orthogonal to both a & b.

$$a = i + 3j - 2k, \quad b = -i + 5k$$

(Ans) $a = \langle 1, 3, -2 \rangle$ and $b = \langle -1, 0, 5 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix}$$

$$= i(15-0) - j(5-2) + k(0+3)$$

$$= 15i + 3j + 3k$$

$$\Rightarrow a \times b = \langle 15, -3, 3 \rangle$$

4) $a = j + 7k \qquad b = 2i - j + 4k$

(Ans) $a = \langle 0, 1, 7 \rangle$ and $b = \langle 2, -1, 4 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 0 & 1 & 7 \\ 2 & -1 & 4 \end{vmatrix}$$

$$= i(4+7) - j(0-14) + k(0-2)$$

$$= 11i + 14j - 2k$$

$$= \langle 1, 14, -2 \rangle$$

(19) Find two unit vectors orthogonal to both $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$

(Ans) $a = \langle 3, 2, 1 \rangle$ and $b = \langle -1, 1, 0 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= (0-1)i - (0+1)j + (3+2)k$$

$$= \langle -1, -1, 5 \rangle$$

Let $\vec{v} = \langle -1, -1, 5 \rangle$

$$|\vec{v}| = \sqrt{(-1)^2 + (-1)^2 + (5)^2} = \sqrt{27} = 3\sqrt{3}$$

First unit vector $= \dfrac{\langle -1, -1, 5\rangle}{3\sqrt{3}}$

$$= \left\langle \frac{-1}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}} \right\rangle$$

second unit vector $= (-1)\ \dfrac{\langle -1, -1, 5\rangle}{3\sqrt{3}}$

$$= \left\langle \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{-5}{3\sqrt{3}} \right\rangle$$

27) Find the area of the parallelogram with vertices $A(-2, 1)$, $B(0,4)$, $C(4,2)$, and $D(2,-1)$.

(Ans) $\vec{AD} = (2,1) - (2,-1) = (-4, 2)$

$\vec{CD} = (4,2) - (2,-1) = (2,3)$

Area $= |\vec{AD} \times \vec{CD}|$

$$= \begin{vmatrix} i & j & k \\ -4 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= \left| k \begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix} \right|$$

$$= |k(-12-4)| = |-16k| = 16$$

29) Find a non zero vector orthogonal to the plane through the points P, Q and R and (b), find the area of triangle PQR

$P(1,0,1)$, $Q(-2,1,3)$ and $R(4,2,5)$

(Ans) a) $\vec{PQ} = \langle -3, 1, 2\rangle$

$\vec{PR} = \langle 3, 2, 4\rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= \langle (4-4) - (-12-6) + (-6-3) \rangle$$

$$= \langle 0, 18, -9 \rangle$$

b) area of triangle PQR $= \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

$$= \frac{1}{2} \quad i(4-4) - j(-12-6) + k(-6-3)$$

$$= \frac{18j - 9k}{2} = \frac{\sqrt{(18)^2 + (9)^2}}{2}$$

$$= \frac{9\sqrt{5}}{2}$$

35) Find the volume of the parallelopiped with adjacent edges PQ, PR and PS

P(−2, 1, 0), Q(2, 3, 2), R(1, 4, −1), S(3, 6, 1)

(Ans) $\vec{PQ} = \langle 4, 2, 2 \rangle$

$\vec{PS} = \langle 5, 5, 1 \rangle$

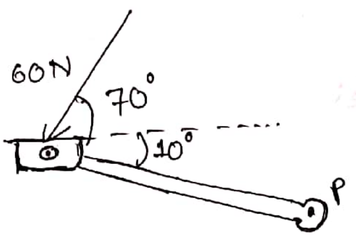
$\vec{PR} = \langle 3, 3, -1 \rangle$

$$\vec{PS} \cdot (\vec{PQ} \times \vec{PR}) = \begin{vmatrix} 5 & 5 & 1 \\ 4 & 2 & 2 \\ 3 & 3 & -1 \end{vmatrix}$$

$$= 5(-2-6) - 5(-4-6) + 1(12-6)$$

$$= -40 + 50 + 6 = 16$$

39) A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18cm long. Find the magnitude of the torque about P.



(Ans) $|T| = |r \times F| = |r||F| \sin\theta$

$|r| = 18$, $|F| = 60\,N$, $\theta = 80°$

so, $|T| = 18 \cdot 60 \cdot \sin(80°) = 10.6$ Nm

43) If $a \cdot b = \sqrt{3}$ and $a \times b = \langle 1, 2, 2 \rangle$ find the angle between a&b

(Ans) $|a||b| = \frac{\sqrt{3}}{\cos\theta}$ (As, $a \cdot b = |a||b| \cos\theta$)

$\Rightarrow |a \times b| = \frac{\sqrt{3}}{\cos\theta} \sin\theta = \sqrt{3} \tan\theta$

so, $\tan\theta = \frac{1}{\sqrt{3}} |\langle 1, 2, 2 \rangle|$

$= \frac{\sqrt{1+4+4}}{\sqrt{3}} = \frac{3}{\sqrt{3}}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{\sqrt{3}}\right)$

$= \tan^{-1}(\sqrt{3})$

$\Rightarrow \boxed{\theta = 60^\circ}$

44) a) Find all vectors v such that
$\langle 1, 2, 1 \rangle \times v = \langle 3, 1, -5 \rangle$

b) Explain why there is no vector v such that
$\langle 1, 2, 1 \rangle \times v = \langle 3, 1, 5 \rangle$

(Ans) a) Let $v = \langle x, y, z \rangle$

so, $\langle 1, 2, 1 \rangle \times \langle x, y, z \rangle = \langle 3, 1, -5 \rangle$

$\Rightarrow \langle 2z - y, x - z, y - 2x \rangle = \langle 3, 1, -5 \rangle$

so, $2z - y = 3$ ------- (i)

$x - z = 1$ ------- (ii)

$y - 2x = -5$ ------- (iii)

so, $z = x - 1$ (from eq^n(ii))

Put the value of z in eq^n(i) $\Rightarrow$ we get

$2(x-1) - y = 3$

$\Rightarrow 2x - 2 - y = 3$

$\Rightarrow 2x - y = 5$

$\Rightarrow y = 2x - 5$

Using x as a free variable we can get the following vectors
$v = \langle x, 2x-5, x-1 \rangle$

b) From the above equations the first two equations obtains $y-2x=-1$ which contradicts the 3rd equation.

So, there is no solution.

(14) If $a+b+c=0$, show that
$$a\times b = b\times c = c\times a$$

Ans) $a\times b = b\times c = c\times a$

As, $a+b+c=0$
$\Rightarrow b = -a-c$

So, $a\times b = a\times(-a-c)$
$= -a\times c = c\times a$

$a\times b = (-b-c)\times b = -c\times b = b\times c$
(as, $a+b+c=0$
$\Rightarrow a = -b-c$)

(OR)

$a\times(a+b+c) = a\times 0$
$\cancel{a\times a} + a\times b + a\times c = 0$
$a\times b = c\times a$

$b\times(a+b+c) = b\times 0$
$b\times a + \cancel{b\times b} + b\times c = 0$
$b\times c = a\times b$

$\boxed{a\times b = b\times c = c\times a}$

Exercise 12.5

2) Find a vector equation & parametric equations for the line

The line through the point $(6, -5, 2)$ & parallel to the vector $\langle 1, 3, -2/3 \rangle$

(Ans) Here, $r = \langle 6, -5, 2 \rangle$ and $v = \langle 1, 3, -2/3 \rangle$

so, $r = \langle 6, -5, 2 \rangle + \langle t, 3t, -2t/3 \rangle$

$\Rightarrow r = \langle 6+t, -5+3t, 2-\frac{2}{3}t \rangle$

$\Rightarrow x = 6+t,\ y = -5+3t,\ z = 2-\frac{2}{3}t$

3) The line through the point $(2, 2.4, 3.5)$ and parallel to the vector $3\hat{i}+2\hat{j}-k$

(Ans) Here, $r = \langle 2, 2.4, 3.5 \rangle$ and $v = \langle 3, 2, -1 \rangle$

so, $r = \langle 2, 2.4, 3.5 \rangle + t\langle 3, 2, -1 \rangle$

$\Rightarrow r = \langle 2+3t, 2.4+2t, 3.5-t \rangle$

$\Rightarrow x = 2+3t,\ y = 2.4+2t,\ z = 3.5-t$

22) Determine whether the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection

$$L_1 :- \frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3}$$

$$L_2 :- \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}$$

(Ans) The direction vectors are the numbers in the denominators of the equations

$L_1 :- \langle 1, -1, 3 \rangle$

$L_2 :- \langle 2, -2, 7 \rangle$

Parametric form of L_1 :-

$$\frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3} = t$$

$\Rightarrow x = t,\ y = 1-t,\ z = 2+3t$

Parametric form of L_2 :-

$$\frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7} = s$$

$\Rightarrow x = 2+2s, \; y = 3-2s, \; z = 7s$

$\Rightarrow t = 2+2s$ ---------- (i)

$1-t = 3-2s$ ---------- (ii)

$2+3t = 7s$ ---------- (iii)

Put the value of 't' in eq^n (ii) we get

$$1-(2+2s) = 3-2s$$

$$\Rightarrow 1-2-2s = 3-2s$$

$$\Rightarrow -1 = 3$$

this is a contradiction, so there is no solution. The lines donot intersect, so they are skew.

23) Find an equation of the plane

the plane through the point $(6,3,2)$ and perpendicular to the vector $\langle -2,1,5 \rangle$

(Ans) Let $P = (x_0, y_0, z_0)$

and $n = \langle a,b,c \rangle$

so, $P = (6,3,2)$

and $n = \langle -2,1,5 \rangle$

so, $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\Rightarrow ax+by+cz = a(x_0)+by_0+cz_0$

$\Rightarrow -2x+y+5z = -2(6)+1(3)+5(2)$

$\Rightarrow -2x+y+5z = -12+3+10$

$\Rightarrow -2x+y+5z = 1$

23) The plane through the point $(4,0,-3)$ & with normal vector $j+2k$

(Ans) Let $P = (x_0, y_0, z_0)$

$n = \langle a,b,c \rangle$

so, $P = (4,0,-3)$

$n = \langle 0,1,2 \rangle$

so, $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\Rightarrow ax+by+cz = ax_0+by_0+cz_0$

$\Rightarrow y+2z = 0(4)+1(0)+2(-3) \Rightarrow y+2z = -6$

54) Determine whether the planes are parallel, perpendicular or neither. If neither find the angle between them.

$2x-3y+4z=5, \quad x+6y+4z=3$

(Ans) $2x-3y+4z=5$

and $x+6y+4z=3$

Let $n=\langle 2,-3,4\rangle$ and $m=\langle 1,6,4\rangle$

$$\cos\theta = \frac{m \cdot n}{|m||n|}$$

$$= \frac{2-18+16}{\sqrt{4+9+16}\ \sqrt{1+36+16}} = 0$$

so, $\theta = 90^\circ$

Then the two vectors are perpendicular.

55) $x=4y-2z, \quad 8y=1+2x+4z$

(Ans) $-x+4y-2z=0$

and $-2x+8y-4z=1$

so, $\langle -1,4,-2\rangle \langle -2,8,-4\rangle$

$\Rightarrow \frac{-1}{-2} = \frac{4}{8} = \frac{-2}{-4} \Rightarrow \frac{1}{2} = \frac{1}{2} = \frac{1}{2}$

57) a) Find parametric equations for the line of intersection of the planes and

b) find the angle between the planes

$x+y+z=1, \quad x+2y+2z=1$

(Ans) a) $x+y+z=1$

$x+2y+2z=1$

set $z=0$,

$x+y+0=1$

$\Rightarrow y=1-x$

so, $x+2y+0=1$

$\Rightarrow x+2(1-x)=1$

$\Rightarrow -x+2=1$

$\Rightarrow \boxed{x=1}$

so, $\boxed{y=0}$

so, point $(x_0,y_0,z_0)=(1,0,0)$

vectors are :- $\langle 1,1,1\rangle$ and $\langle 1,2,2\rangle$

so, cross products are $= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$

$= (2-2)i - (2-1)j + (2-1)k$

$= \langle 0, 1, 1\rangle$

so, $x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

$x = 1 + 0(t)$

$y = 0 + (-1)t$

$z = 0 + (1)t$

$\Rightarrow \boxed{x = 1}$

$\boxed{y = -t}$

$\boxed{z = t}$

b) $n_1 = \langle 1,1,1\rangle$

$n_2 = \langle 1,2,2\rangle$

$\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

$\Rightarrow \theta = \cos^{-1} \dfrac{1(1) + 1(2) + 1(2)}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+2^2+2^2}}$

$\Rightarrow \theta = \cos^{-1} \dfrac{5}{3\sqrt{3}} \approx 15.8^0$

(59) Find symmetric equations for the line of intersection of the planes.

$5x - 2y - 2z = 1$, $4x + y + z = 6$

(Ans) $5x - 2y - 2z = 1 \rightarrow \langle 5,-2,-2\rangle$

$4x + y + z = 6 \longrightarrow \langle 4,1,1\rangle$

so, $\langle 5,-2,-2\rangle \times \langle 4,1,1\rangle$ (cross product)

$= \langle -2(1) - (-2)(1), -2(4) - 5(1), 5(1) - (-2)4\rangle$

$= \langle 0, -13, 13\rangle$

we can ~~value~~ reduce the vector magnitude by dividing by 13

$\langle 0,-1,1\rangle$

at $z=0$,

so, $5x-2y=1$ ---------- ①

$4x+y=6$ ------- ②

Multiplying '2' in equation ②, we get

$$8x+2y=12$$

Adding equation ① and ②, we get.

$$5x-2y+8x+2y=13$$

$$\Rightarrow 13x=13$$

$$\Rightarrow x=1$$

Put the value of 'x' in eqⁿ ②, we get

$$4(1)+y=6$$

$$\Rightarrow \boxed{y=2}$$

First on the line :- $(x_0, y_0, z_0) = (1,2,0)$

so, $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

$$\Rightarrow \quad \cancel{x} = \frac{y-2}{-1} = \frac{z-0}{1}$$

$$\Rightarrow 2-y=z$$

$$\Rightarrow \boxed{y-2=-z}$$

Exercise 12.6

1) a) what does the equation $y = x^2$ represent as a curve in R^2?

b) what does it represent as a surface in R^3?

c) what does the equation $z = y^2$ represent?

(Ans) a) $y = x^2$ represents parabola because it satisfies quadratic equation $y = ax^2 + bx + c$ where $b = 0$, $c = 0$.

b) In R^3, $y = x^2$ represents a parabolic cylinder. It is the 2-D parabola on the xy plane that is stretched in the directions parallel to the z-axis

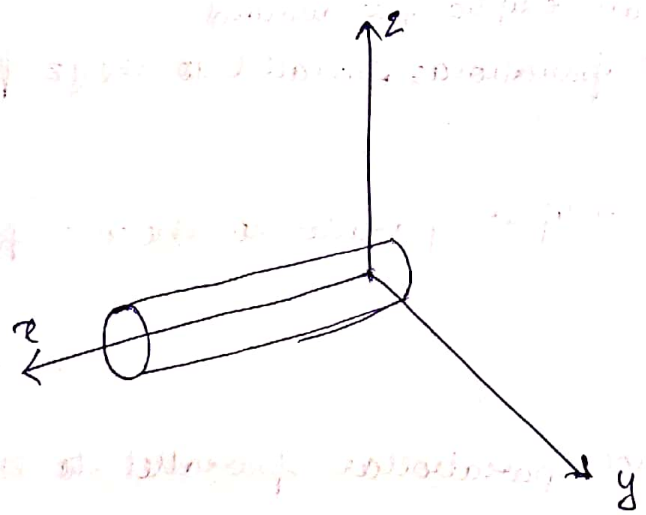
c) In R^3, $z = y^2$ represents a parabolic cylinder. It is the 2-D parabola on the yz-plane that is stretched in the directions parallel to the x-axis

8) Describe & sketch the surface

$$y^2 + 4z^2 = 4$$

(Ans) $y^2 + 4z^2 = 4$

$$\Rightarrow \frac{y^2 + 4z^2}{4} = 1 \Rightarrow \frac{y^2}{4} + z^2 = 1$$



The result is a infinite elliptic cylinder centered around the x-axis.

18) Use traces to sketch and identify the surface

$$4x^2 - 16y^2 + z^2 = 16$$

(Ans) $4x^2 - 16y^2 + z^2 = 16$

To find traces, we set each variable to a constant K.

Let $x = k$

$4k^2 - 16y^2 + z^2 = 16$

$\Rightarrow z^2 - 16y^2 = 16 - 4k^2$

The traces parallel to the y-z plane are hyperbolas,

Let $y = k$

$4x^2 - 16k^2 + z^2 = 16$

$\Rightarrow 4x^2 + z^2 = 16 + 6k^2$

The traces parallel to the x-z plane are obliques.

Let $z = k$

$4x^2 - 16y^2 + k^2 = 16$

$\Rightarrow 4x^2 - 16y^2 = 16 - k^2$

The traces parallel to the xy- plane are hyperbolas.

(30) Reduce the equation to one of the standard forms, classify the surfaces & sketch it.

$4x^2 - y + 2z^2 = 0$

(Ans) $4x^2 - y + 2z^2 = 0$

$\Rightarrow y = 4x^2 + 2z^2$

$\Rightarrow \frac{y}{4} = x^2 + \frac{z^2}{2}$

This is in the form of an elliptic paraboloid

Traces with $y = k$ produce parabolas parallel to the yz-plane.

$y = 4k^2 + 2z^2$

Traces with $y = k$ produce ellipses parallel to the xz-plane.

$4x^2 - k + 2z^2 = 0$

$\Rightarrow \frac{x^2}{k/4} + \frac{z^2}{k/2} = 1$

Traces with $z = k$ produce parabolas parallel to the xy-plane

$y = 4x^2 + 2k^2$

(31) $x^2 + 2y - 2z^2 = 0$

(Ans) $x^2 + 2y - 2z^2 = 0$

$\Rightarrow 2y = 2z^2 - x^2$

$\Rightarrow y = z^2 - \frac{x^2}{2}$ (dividing '2' on both sides)

If $z = k$, then

$$y = k^2 - \frac{x^2}{2}$$

traces are parabolas opening in xy- direction

if $y = k$, then

$$k = z^2 - \frac{x^2}{2}$$

traces are hyperbolas

If $x = k$, then

$$y = z^2 - \frac{k^2}{2}$$

traces are parabolas in +y direction.

(11) sketch the region bounded by the surfaces.
$z = \sqrt{x^2+y^2}$ and $x^2+y^2 = 1$ for $1 \le z \le 2$

(Ans) $z = \sqrt{x^2+y^2}$ is a circular cone

$x^2+y^2 = 1$ is a cylinder