ASSIGNMENT-2

Q1. a) Rephrase the definitions for the reflexive, symmetric, transitive and antisymmetric properties of a relation R (on a set A), using quantifiers

b) use the results of part (a) to specify when a relation R (on a set A) is (i) not reflexive (ii) not symmetric; (iii) not transitive; and (iv) not antisymmetric.

Ans a) The Relation R on the set A is:

(i) reflexive if $\forall x \in A\ (x, x) \in R$

(ii) symmetric if $\forall x, y \in A\ [(x, y) \in R \Rightarrow (y, x) \in R]$

(iii) transitive if $\forall x, y, z \in A\ [(x, y)(y, z) \in R \Rightarrow (x, z) \in R]$

(iv) antisymmetric if $\forall x, y \in A\ [(x, y)(y, x) \in R \Rightarrow x = y]$

b) The Relation R on the set A is:

(i) not reflexive if $\exists x \in A\ (x, x) \notin R$

(ii) not symmetric if $\exists x, y \in A\ [(x, y) \in R \wedge (y, x) \notin R]$

(iii) not transitive if $\exists x, y, z \in A\ [(x, y), (y, z) \in R \wedge (x, z) \notin R]$

(iv) not antisymmetric if $\exists x, y \in A\ [(x, y), (y, x) \in R \wedge x \neq y]$

Q2. For each of the following statements about relations on a set A, where $|A| = n$, determine whether the statement is true or False. If it is false, give a counter example.

a) If R is a relation on A and $|R| \geq n$ then R is reflexive.

b) If R_1, R_2 are relations on A and $R_2 \supseteq R_1$, then R_1 reflexive (symmetric, anti symmetric, transitive) $\Rightarrow$ R_2 reflexive (symmetric, antisymmetric, transitive)

c) If R_1, R_2 are relations on A and $R_2 \supseteq R_1$, then R_2 reflexive (symmetric, antisymmetric, transitive) $\Rightarrow$ R_1 reflexive (symmetric, antisymmetric, transitive).

d) If R is an equivalence relation on A, then $n \leq |R| \leq n^2$

Ans a) False: Let $A = \{1, 2\}$ and $R = \{(1, 2), (2, 1)\}$

b) (i) Reflexive: True

(ii) Symmetric: False. Let $A = \{1, 2\}$,

$R_1 = \{(1, 1)\}$,

$R_2 = \{(1, 1)(1, 2)\}$

(iii) Antisymmetric & Transitive: False.

Let $A = \{1, 2\}$, $R_1 = \{(1, 2)\}$, $R_2 = \{(1, 2), (2, 1)\}$.

c) (i) Reflexive : False. Let $A = \{1, 2\}$, $R_1 = \{(1, 1)\}$,
$R_2 = \{(1, 1), (2, 2)\}$

(ii) Symmetric : False. Let $A = \{1, 2\}$, $R_1 = \{(1, 2)\}$,
$R_2 = \{(1, 2)(2, 1)\}$

(iii) Antisymmetric : True.

(iv) Transitive : False. Let $A = \{1, 2\}$, $R_1 = \{(1, 2), (2, 1)\}$
$R_2 = \{(1, 1), (1, 2), (2, 1)(2, 2)\}$.

d) True.

Q3) If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, & antisymmetric.

Ans (a) 2^{12}

(b) $(2^4)(2^6) = 2^{10}$

(c) 2^6

(d) 2^{11}

(e) $(2^4)(2^5) = 2^9$

(f) $2^4 \cdot 3^6$

(g) $2^4 \cdot 3^5$ (h) (2^4) (i) 1

Q4) Let A be a set with $|A| = n$, and let R be a relation on A that is antisymmetric. What is maximum value for $|R|$? How many antisymmetric relations can have this size?

Ans There are n ordered pairs of the form (x, x), $x \in A$. For each of the $\frac{(n^2 - n)}{2}$ sets $\{(x, y), (y, x)\}$ of ordered pairs where $x, y \in A$, $x \neq y$, one element is chosen. This result in a maximum value of $n + \frac{(n^2 - n)}{2} = \left(\frac{n^2 + n}{2}\right)$

The number of antisymmetric relations that can have this size is $2(n^2 - n)/2$.

Q5) With $A = \{1, 2, 3, 4\}$, let $R = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$ be a relation on A. Find two relations S, T on A where $S \neq T$ but $R \circ S = R \circ T = \{(1, 1), (1, 2), (1, 4)\}$

Ans Let $S = \{(1, 1), (1, 2) (1, 4)\}$.

$T = \{(2, 1), (2, 2), (1, 4)\}$.

Q6) Let A be a set with $|A| = n$, and let R be an equivalence relation on A with $|R| = r$. Why is $r-n$ always even?

Ans $r-n$ counts the elements in R of the form (a,b), $a \neq b$. Since R is symmetric, $r-n$ is even......

Q7) Determine how many integer solution there are to $x_1+x_2+x_3+x_4=19$ if

a) $0 \leq x_i$ for all $1 \leq i \leq 4$

b) $0 \leq x_i < 8$ for all $1 \leq i \leq 4$

c) $0 \leq x_1 \leq 5$, $0 \leq x_2 < 6$, $3 \leq x_3 \leq 7$, $3 \leq x_4 \leq 8$

Ans a) Given $x_1+x_2+x_3+x_4 = 19$

$0 \leq x_i ; 1 \leq i \leq 4 \Rightarrow \binom{19+4-1}{19} = \binom{22}{19}$

$$\therefore {}^{22}C_{19} = \frac{22!}{19!(22-19)!} = \frac{22!}{19! \cdot 3!}$$

$$= \frac{22 \times \cancel{21}^{7} \times \cancel{20}^{10} \times \cancel{19}}{\cancel{19} \times \cancel{3}_1 \times \cancel{2}_1 \times 1}$$

$N = 1540$

(b) For $1 \le i \le 4$:

Let C_i : $x_i \geq 8$

$N(C_1) = x_1 + x_2 + x_3 + x_4 = 19$

$= x_1 - 8 + x_2 + x_3 + x_4 = 19$

$= x_1 + x_2 + x_3 + x_4 = 19 - 8$

$= x_1 + x_2 + x_3 + x_4 = 11$

$: \binom{11+4-1}{11} = \binom{14}{11}$

$\therefore {}^{14}C_{11} = \frac{14!}{11!(14-11)!}$

$= \frac{14!}{11!\,3!}$

$= \frac{\cancel{14}^{7} \times 13 \times \cancel{12}^{4} \times \cancel{11}}{\cancel{11} \times \cancel{3}_{1} \times \cancel{2}_{1} \times 1} \Rightarrow 364$

$N(C_2) = 364$

$N(C_3) = 364$

$N(C_4) = 364$

Let $C_i C_j$: $x_i x_j \geq 8$

$N(C_1 C_2) = x_1 + x_2 + x_3 + x_4 = 19$

$= x_1 - 8 + x_2 - 8 + x_3 + x_4 = 19$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 19 - (8+8)$

$\Rightarrow x_1 + x_2 + x_3 + x_4 = 19 - 16$
$= 3$

$$\therefore \binom{3+4-1}{3} = \binom{6}{3}$$

$$6C_3 = \frac{6!}{3!\,3!} = \frac{\cancel{6}^{2} \times 5 \times \cancel{4}^{2} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{2} \times 1} = 20$$

$N(C_1\ C_3): 20$
$N(C_1\ 4): 20$
$N(C_2\ C_3): 20$
$N(C_2\ C_4): 20$
$N(C_3\ C_4): 20$

$$\therefore N(\overline{C_1}\ \overline{C_2}\ \overline{C_3}\ \overline{C_4}) = N - S_1 + S_2$$
$$= 1540 - 4 \times 364 + 6 \times 20$$
$$= 1540 - 1456 + 120$$
$$= \underline{204}$$

c) The number of solution for $x_1 + x_2 + x_3 + x_4 = 19$ where $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $3 \le x_3 \le 7$, $3 \le x_4 \le 8$ equals the no of soln for $x_1 + x_2 + x_3 + x_4 = 13$ with $0 \le x_1 \le 5$, $0 \le x_2 \le 6$, $0 \le x_3 \le 4$, $0 \le x_4 \le 5$

Define the condition C_i, $1 \le i \le 4$ as follows

Let $C_1 : x_1 \geq 6$

Let $C_2 : x_2 \geq 7$

Let $C_3 : x_3 \geq 5$

Let $C_4 : x_4 \geq 6$

$$x_1 + x_2 + x_3 + x_4 = 13$$

$$\therefore N = \binom{4+13-1}{13} = \binom{16}{13}$$

$$\therefore 16C_{13} = \frac{16!}{13!(16-13)!} = \frac{16!}{13!\,3!}$$

$$= \frac{16 \times \cancel{15}^{5} \times \cancel{14}^{7} \times \cancel{13}}{\cancel{13} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 560$$

$$N(C_1) = x_1 + x_2 + x_3 + x_4 = 13$$

$$= x_1 - 6 + x_2 + x_3 + x_4 = 13$$

$$= x_1 + x_2 + x_3 + x_4 = 13 - 6$$

$$= 7$$

$$N(C_1) = \binom{7+4-1}{7} = \binom{10}{7} = 10C_7$$

$$\frac{10!}{7!\,3!} = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4} \times \cancel{7}^{1}}{\cancel{7} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 120 = N(C_4)$$

$N(C_2) = x_1 + x_2 + x_3 + x_4 = 13$
$= x_1 + x_2 - 7 + x_3 + x_4 = 13$
$= x_1 + x_2 + x_3 + x_4 = 13 - 7$
$= x_1 + x_2 + x_3 + x_4 = 6$

$N(C_2) = \binom{6+4-1}{6} = \binom{9}{6}$

$9C_6 = \frac{9!}{6!(9-6)!} = \frac{9!}{6!\,3!}$

$\Rightarrow \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7 \times \cancel{6}}{\cancel{6} \times \cancel{3}_{1} \times \cancel{2}_{1} \times 1} = 84$

$N(C_3) = x_1 + x_2 + x_3 + x_4 = 13$
$= x_1 + x_2 + x_3 - 5 + x_4 = 13$
$= x_1 + x_2 + x_3 + x_4 = 13 - 5$
$\Rightarrow x_1 + x_2 + x_3 + x_4 = 8$

$N(C_3) : \binom{8+4-1}{8} = \binom{11}{8}$

$11C_8 = \frac{11!}{8!\,3!} = \frac{11 \times \cancel{10}^{5} \times \cancel{9}^{3} \times \cancel{8}}{\cancel{8}_{1} \times \cancel{3}_{1} \times \cancel{2}_{1} \times 1}$

$= 165$

Similarlly

$N(C_1\ C_2) = 1$

$N(C_1 C_3) = x_1 + x_2 + x_3 + x_4 = 13$

$= x_1 - 6 + x_2 + x_3 - 5 + x_4 = 13$

$= x_1 + x_2 + x_3 + x_4 = 13 - 6 + 5$

$= 13 - 11$

$= 2$

$N(C_1 C_3) = \binom{2+4-1}{2} = \binom{5}{2} = 5C_2$

$= \frac{5!}{2!\,3!} = \frac{5 \times \cancel{4}^{2} \times \cancel{4}^{1} \times \cancel{3} \times \cancel{2}}{\cancel{2} \times \cancel{3} \times \cancel{2} \times 1}$

$= 10$

$N(C_1 C_4) = x_1 + x_2 + x_3 + x_4 = 13$

$= x_1 - 6 + x_2 + x_3 + x_4 - 6 = 13$

$= x_1 + x_2 + x_3 + x_4 = 13 - 6 + 6$

$= 13 - 12$

$= 1$

$N(C_1 C_4) = \binom{1+4-1}{1} = \binom{4}{1} = 4C_1$

$\frac{4!}{1!\,3!} = \frac{4 \times 3 \times 2 \times 1}{1 \times 3 \times 2 \times 1} = 4$

$N(C_2 C_3) = \binom{4}{1} = 4C_1 = 4$

$N(C_2 C_4) = 1$

$N(C_3 C_4) = 5C_2 = 10$

$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - S_1 + S_2$

$= 560 - [2\times 120 + 84 + 165] + 2[1+4+10]$

$= 560 - 489 + 30$

$= \underline{\underline{101}}$ (Ans)

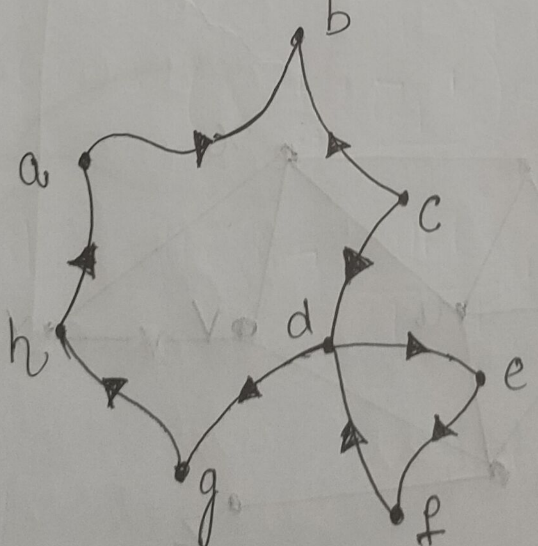
Q8) For the directed graph $G=(V,E)$ in Fig 7.12 classify each of the following statements as true or false.

a) Vertex c is the origin of two edges in G

b) Vertex g is adjacent to vertex h.

c) There is a directed path in G from d to b.

d) There are two directed cycles in G

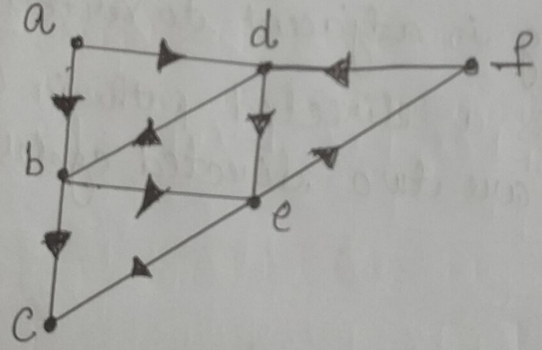


Ans (a) True (b) True (c) True (d) False

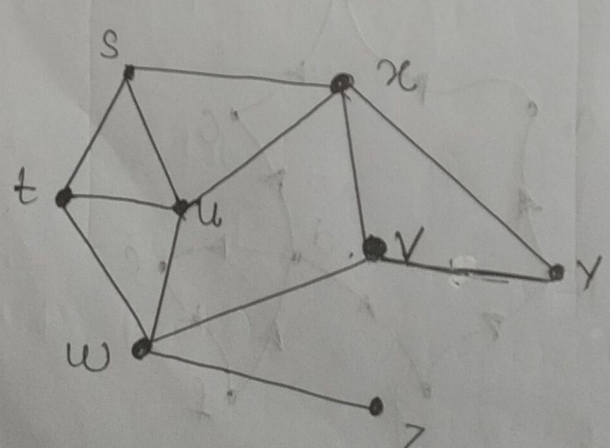
Q9) a) Draw the digraph $G_1 = (V_1, E_1)$ and $V_1 = \{a, b, c, d, e, f\}$ and $E_1 = \{(a,b), (a,d), (b,c), (b,e), (d,b), (d,e), (e,c), (e,f), (f,d)\}$

b) Draw the undirected graph $G_2 = (V_2, E_2)$ where $V_2 = \{s, t, u, v, w, x, y, z\}$ and $E_2 = \{\{s,t\}, \{s,u\}, \{s,x\}, \{t,u\}, \{t,w\}, \{u,w\}, \{u,x\}, \{v,w\}, \{v,x\}, \{v,y\}, \{w,z\}, \{x,y\}\}$.

Ans a)

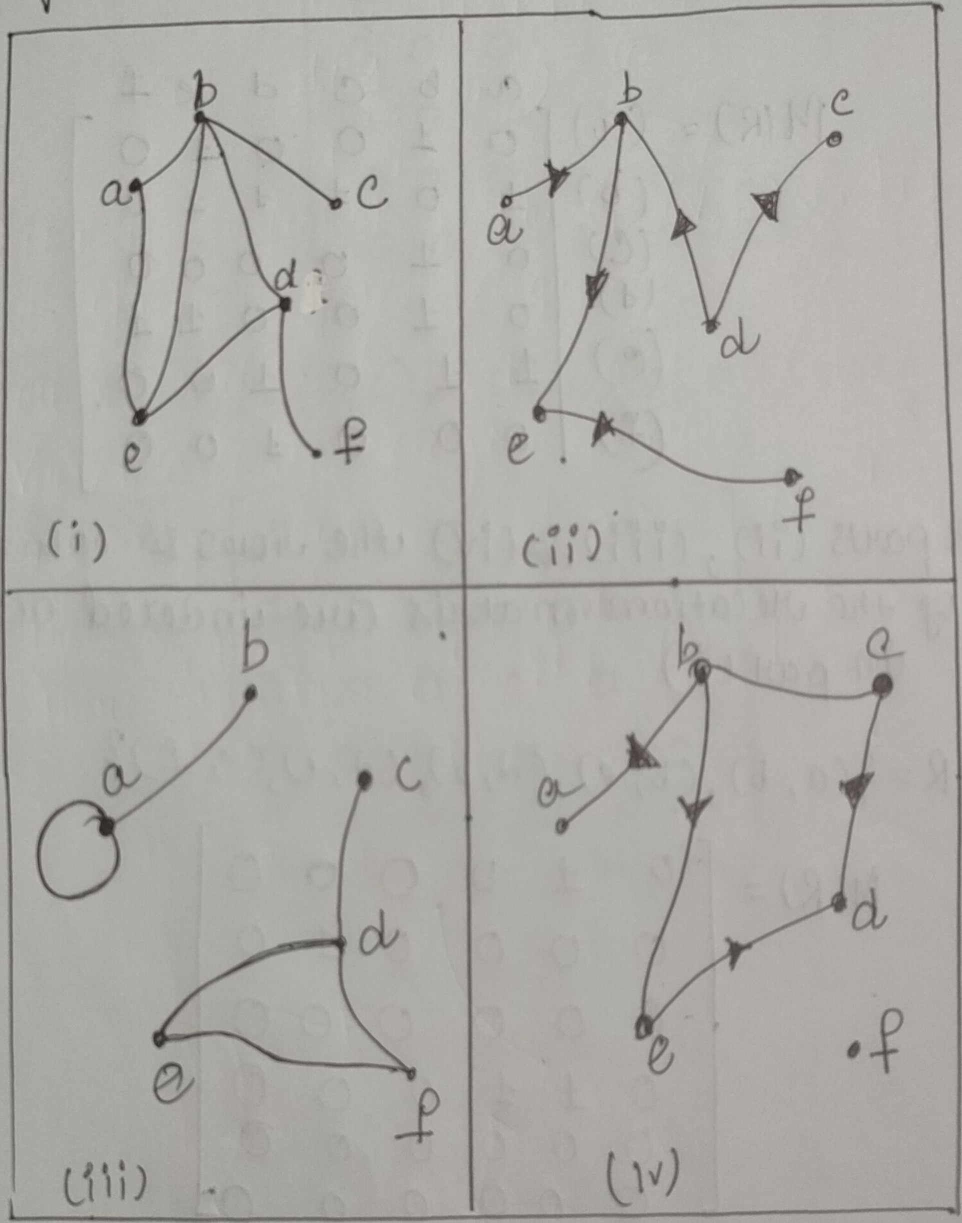


b)



Q10) Fo... in Fig...

Q.10) For $A = \{a, b, c, d, e, f\}$ each graph or digraph in Fig 7.13 represents a relation R on A



Ans

(i) $R = \{(a,b),(b,a),(a,e),(e,a),(b,c),(c,b),(b,d),(d,b),$
$(b,e),(e,b),(d,e),(e,d),(d,f),(f,d)\}$

$$M(R) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

For parts (ii), (iii), & (iv) the rows & columns of the relation matrix are indexed as in par (i)

(ii) $R = \{(a,b),(b,e),(d,b),(d,c),(e,f)\}$

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(iii) $R = \{(a,a),(a,b),(b,a),(c,d),(d,c),(d,e)$
$(e,d),(d,f),(f,d),(e,f),(f,e)\}$

$$M(R) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(iv) $R = \{(b,a), (b,c), (c,b), (b,e), (c,d), (e,d)\}$

$$M(R) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$