Intermediate Discrete Mathematics

9/9/23

CH-5 Language Finite State Machines. Vending Machines.

Defn:- A non empty finite set of symbols is said to be alphabet. It is denoted by Σ → collection of well defined distinct elements.

Ex:- $\Sigma = \{0, 1\}$, $\Sigma = \{a, b, c, d\}$

Defn:- If Σ is alphabet and $n \in Z^+$ we define the power of Σ is as follows:-

(I) $\Sigma^1 = \Sigma$

(II) $\Sigma^{n+1} = \{xy : x \in \Sigma, y \in \Sigma^n\}$, where xy denotes the juxtaposition of x and y.

Ex:- $\Sigma = \{0, 1\}$ Total element = 2

$\Sigma^3 = \Sigma^{2+1} = \{xy : x \in \Sigma, y \in \Sigma^2\}$ $x: \{0, 1\}$

2^2 elements $\Sigma^2 = \Sigma^{1+1} = \{xy : x \in \Sigma, y \in \Sigma\}$ $y: \{0, 1\}$

$= \{00, 01, 10, 11\}$ → Juxtaposition of x & y.

2^3 elements $\Sigma^3 = \Sigma^{2+1} = \{000, 001, 010, 011, 100, 101, 110, 111\}$

→ If $\Sigma = \{a, b, c, d, e\}$ Total elements = 5 So, $\Sigma^5 = 5^5$

For all $n \in Z$, total number of element in Σ^n is $|\Sigma|^n$

Note:- For an alphabet Σ, we define $\Sigma^0 = \{\lambda\}$, where λ denotes empty string.

Defn:- If Σ is an alphabet then

i) $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n = \bigcup_{n \in Z^+} \Sigma^n$ (Range same)

ii) $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$

Note:- $\Sigma^* = \Sigma^0 \cup \Sigma^+$

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \cdots\cdots$

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \cdots\cdots$

$= \Sigma^0 \cup \Sigma^+$

Ex:- If $\Sigma = \{0,1\}$. Find Σ^+ & Σ^* → Infinite Set.

$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \dots = \{0,1\} \cup \{00,01,10,11\} \cup \dots$

$\Sigma^1 = \{0,1\}$ $\quad \Sigma^2 = \Sigma^{1+1} = \{00,01,10,11\}$

$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2$

$\Sigma^0 = \{\lambda\} \cup \{0,1,00,01,10,11\} \cup \dots$

$\left.\begin{array}{l} \Sigma^* = \{\lambda, 0, 1, 00, 01, 10, 11, \dots\} \\ \Sigma^+ = \{0,1,00,01,10,11,\dots\} \end{array}\right\}$ Ans.

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Defn:- If $w_1, w_2 \in \Sigma^+$, then we may write $w_1 = x_1 x_2 \dots x_n$ and $w_2 = y_1 y_2 \dots y_m$ for $m, n \in Z^+$ and $x_i, y_j \in \Sigma$ for all $1 \le i \le n$, $1 \le j \le m$. We say that strings w_1 and w_2 are equal if $m = n$ and $x_i = y_i$ for all $1 \le i \le m$.

$\Sigma = \{0,1\} \Rightarrow \Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n = \{0,1,00,01,10,11\}$

Let $w_1 = 0$ then $n = 1$, $x_1 = 0$, $w_1 = x_1$

Let $w_2 = 01$ then $n = 2$, $x_1 = 0$, $x_2 = 1$ $\quad w_2 = x_1 x_2$

Defn:- Let $w = x_1 x_2 \dots x_n \in \Sigma^+$ where $x_i \in \Sigma$ for all $1 \le i \le n$. We define the length of w as the value of n and is denoted by $\|w\|$ (norm of w)

$w_1 = 0$, $\|w_1\| = 1$ $\quad w_3 = 000$, $\|w_3\| = 3$

$w_2 = 01$, $\|w_2\| = 2$

Defn:- If $x, y \in \Sigma^+$, with $x = x_1 x_2 \dots x_m$ and $y = y_1 y_2 \dots y_n$ so that each $x_i, y_j \in \Sigma$ for $1 \le i \le m$, $1 \le j \le n$. Then the concatenation of x & y is xy → string.

0 & 1 is 01

Note:- Concatenation of $x\lambda = \lambda x = x$ and $\lambda . \lambda = \lambda$

Defn:- For each $x \in \Sigma^*$, we define the power of x by
$x^0 = \lambda$, $x^1 = x$, $x^2 = x.x$, $x^3 = x.x^2$, $x^{n+1} = x.x^n$

Ex:- $\Sigma = \{0,1\}$ $x = 01$ length of $x = ||x|| = 2$

$x^2 = x.x = 0101$ (concatenation) $||x^2|| = 4$

$x^5 = x.x^4 = 01x.x^3 = 0101.xx^2 = 010101x.x$
$= 0101010101$ $||x^5|| = 10$

$||x|| = 2$, $||x^2|| = 2||x||$, $||x^3|| = 3||x||$, $||x^4|| = 4||x||$, $||x^5|| = 5||x||$

$\therefore \boxed{||x^n|| = n||x||}$

Defn:- If $x, y \in \Sigma^*$ and $w = xy$, then the string x is called a prefix of w and y is called a suffix of w.
If $x \neq \lambda$ then y is proper suffix of w, and if $y \neq \lambda$ then x is proper prefix of w.

$w = \underbrace{x}_{\text{prefix}}\underbrace{y}_{\text{suffix}}$

Ex:- $\Sigma = \{0,1\}$ $w = 0011$

→ $x = 0$; $y = 011$; x is proper prefix of w as $y \neq \lambda$
→ $x = 00$; $y = 11$;
→ $x = 001$; $y = 1$;
→ $x = 0011$; $y = \lambda$; x is prefix
→ $y = 1$; $x = 001$; y is proper suffix of w as $x \neq \lambda$
→ $y = 11$; $x = 00$;
→ $y = 011$; $x = 0$;
→ $y = 0011$; $x = \lambda$; y is suffix

(*) w is prefix / suffix of w itself.

Defn:- If $x, y, z \in \Sigma^*$ and $w = xyz$, then y is called a substring of w. When atleast one of x and z is different from λ, we call y is a proper substring

Ex:- $\Sigma = \{0, 1\}$, $W = 00101110$
Identify whether 1011 is a proper substring or not.

Note:- Case 1 :- $x = \lambda$, $y \neq \lambda$, y is proper substring
Case 2 : $x \neq \lambda$, $y = \lambda$, y is proper substring
Case 3 :- $x \neq \lambda$, $y \neq \lambda$, y is proper substring
Case 4 : $x = \lambda$, $y = \lambda$, y is substring.

Soln:- $x = 00 \neq \lambda$
$y = 1011$ $\therefore$ y is a proper substring of w.
$z = 10 \neq \lambda$

Defn:- For a given Σ, any subset of Σ^* is called a language over Σ. This includes the subset ϕ, which is empty language.

Ex:- $\Sigma = \{0,1\}$; $\Sigma^* = \{\lambda, 0, 1, 00, 01, 10, 11 \ldots\}$
Consider $A = \{00, 10, 11, 000, 100\}$ $\rightarrow$ A is a languge over Σ
$A \subset \Sigma^*$ (subset)

Ex:- If Σ includes all the alphabets, numbers, special symbols then any programming lang ie. python, C, C# is known as language over Σ.

Exercise - 1

Let $\Sigma = \{a, b, c, d, e\}$ Then compute :

i) $\|\Sigma^2\| = \cancel{2\|\Sigma\| = 2 * 5 = 10}$

ii) $\|\Sigma^3\| = \cancel{3\|\Sigma\| = 3 * 5 = 15}$

iii) How many strings in Σ^* have length at most 5.

i) $\Sigma^2 = \{aa, ab, ac, ad, \ldots, ee\}$

$|\Sigma^2| = 5^2 = 25$

ii) $\Sigma^3 = \{aaa, aab, \ldots, eee\}$

$= |\Sigma^3| = 5^3 = 125$

iii) $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots$

$= \{\lambda, a, b, c, d, e, \underbrace{aa, ab, ac, ad, ae, \ldots ee}_{\Sigma^2},$
$\underbrace{aaa, aab, \ldots, eee}_{\Sigma^3}, \underbrace{aaaa, aaab, \ldots eeee}_{\Sigma^4},$
$\underbrace{aaaaa, aaaab, \ldots, eeee}_{\Sigma^5}, \ldots\}$

$= 1 + 5 + 5^2 + 5^3 + 5^4 + 5^5$

$= 1 + 5 + 25 + 125 + 625 + 3125$

$= 3906$ Ans.

$\therefore \Sigma^+ = 3906 - 1 = 3905$

Exercise - 3

If $x \in \Sigma^*$ and $\|x^3\| = 36$, then what is $\|x\| = ?$

Soln: $\|x^n\| = n\|x\|$

$\Rightarrow \|x^3\| = 3\|x\| = 36$

$\therefore \|x\| = 12$

Defn:- A finite state machine is a five-tuple $M = (S, i, C, V, W)$ where S = the set of internal states for M. i = the input alphabet, C = the output alphabet for M. $V: S \times i \to S$ is a next state function and $W = S \times i \to C$ is the output function.

we need to get / will be given

Ex:- Consider the finite state machine $M = (S, i, C, V, W)$ where $S = \{S_0, S_1, S_2\}$, $i = C = \{0, 1\}$, V and W are given table.

internal states, input alphabet, initial state, next state, last state, output alphabet in terms of 0,1 only.

	V		W	
	0	1	0	1
S_0	S_0	S_1	0	0
S_1	S_2	S_1	0	0
S_2	S_0	S_1	0	1

$S = \{S_0, S_1, S_2\}$ $\quad i = \{0, 1\}$

$V = S \times i \to S$

$= \{(S_0, 0), (S_0, 1), (S_1, 0), (S_1, 1), (S_2, 0), (S_2, 1)\}$

$W = S \times i \to C$

$= \{(S_0, 0), (S_0, 1), (S_1, 0), (S_1, 1), (S_2, 0), (S_2, 1)\}$

$\therefore V(S_0, 0) = S_0$	$W(S_0, 0) = 0$
$V(S_0, 1) = S_1$	$W(S_0, 1) = 0$
$V(S_1, 0) = S_2$	$W(S_1, 0) = 0$
$V(S_1, 1) = S_1$	$W(S_1, 1) = 0$
$V(S_2, 0) = S_0$	$W(S_2, 0) = 0$
$V(S_2, 1) = S_1$	$W(S_2, 1) = 1$

Find the out string of 1010

State	s_0	$v(s_0,1) = s_1$	$v(s_1,0) = s_2$	$v(s_2,1) = s_1$	$v(s_1,0) = s_2$
Input	1	0	1	0	—
Output	$w(s_0,1) = 0$	$w(s_1,0) = 0$	$w(s_2,1) = 1$	$w(s_1,0) = 0$	—

∴ Output is :- 0010 Ans.

Exercise 6.2

1) Let M be the finite state machine defined by $M = (s, i, c, v, w)$ where $s = \{s_0, s_1, s_2\}$, $i = c = \{0,1\}$ and v, w are defined by the following table.

	v 0	v 1	w 0	w 1
s_0	s_0	s_1	0	0
s_1	s_2	s_1	0	0
s_2	s_0	s_1	0	1

Find the output strings for each input strings.

a) 1010101 ⇒ 0010101

b) 1001001 ⇒ 0000000

c) 101001000 ⇒ 001000000

State	s_0	s_1	s_2	s_1	s_2	s_1	s_2	s_1
Input	1	0	1	0	1	0	1	—
Output	$w(s_0,1) = 0$	0	1	0	1	0	1	—

State	s_0	s_1	s_2	s_0	s_1	s_2	s_0	s_1
Input	1	0	0	1	0	0	1	—
Output	0	0	0	0	0	0	0	—

State	s_0	s_1	s_2	s_1	s_2	s_0	s_1	s_2	s_0	s_0
Input	1	0	1	0	0	1	0	0	0	—
Output	0	0	1	0	0	0	0	0	0	—

HW Example 6.7, 6.8, 6.19

Exercise 6.1 → Q4, 8, 10

Exercise 6.2 → Q3

CHAP-7 Relation

Defn:- For set A, B any subset of $A \times B$ is called a relation from A to B.

Note:- $|A| = n$, $|B| = m$ $\therefore |A \times B| = mn$

Total no. of relation from A to B is 2^{mn}

Ex1:- $A = \{1, 2, 3, 4\}$ $A \times A = \{(1,1), (1,2) \ldots (4,4)\}$

$R = \{(1,1), (1,2), (1,3)\} \cancel{(4,4)\}}$

→ We can say R is a subset of $A \times A$ $\Rightarrow R \subseteq A \times A$

Ex2:- Define a relation R over $\mathbb{Z}$

ex:- $R = \{(1,1), (1,2), (1,3)\} \subseteq Z \times Z$

ex:- $R_1 = \{(a,b) : a \leq b, \ a, b \in Z\}$

Relation

Reflexive	Symmetric	Transitive
Let A be the set, R be the relation over A, $\forall x \in A$ $\Rightarrow (x,x) \in R$, then we say R is reflexive relation. Ex:- $A = \{1,2,3\}$ $R = \{(1,1), (2,2), (3,3)\}$ R is reflexive relation.	$(x,y) \in R \Leftrightarrow xRy$ Ex:- $R_3 = \{(1,1), (2,2), (3,3)\}$ is reflexive & symmetric both.	For a set A, a relation R on A is called transitive if xRy hold, yRz hold which implies xRz hold for all $x, y, z \in A$ Ex:- Define a relation R on Z^+ by aRb hold if a divides b

$R_1 = \{(1,1),(2,2),(1,3)\}$
R_1 is not reflexive as $(3,3)$ is not present

let xRy is true $\Rightarrow x|y \Rightarrow$
$y = ex$ for some $e \in Z^+$
yRz is true $\Rightarrow y|z \Rightarrow z = dy$
$xRz \Rightarrow x|z \Rightarrow z =$

that is $b = ac$ for some $c \in Z^+$
xRx hold $\to$ reflexive
xRy hold $\Rightarrow x|y$ $2|4$ but $4 \nmid 2$
not symmetric

Q1.) Let $A = \{1,2,3,4,5\}$
Define a relation R such that $R = \{(x,y) \in A \times A ; x \le y\}$
check R is reflexive or not.

$R = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$

So, we can say R is reflexive.

Note:- If x relate to y then we can write $x \overset{R}{\sim} y$, where R is the relation.

Defn:- Relation R on a set A is called symmetric if xRy hold then yRx also holds for $x, y \in A$

Ex:- Let $A = \{1,2,3\}$
$R_1 = \{(1,2),(2,1),(1,3),(3,1)\} \to$ symmetric
$R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\} \to$ Reflexive & Symmetric

Transitive.

Let xRy is true $\Rightarrow x|y \Rightarrow y = ex$ for some $e \in Z^+$
yRz $\quad y|z \Rightarrow z = dy$ for some $d \in Z^+$

$xRz = x|z$?
$\Rightarrow z = d_1 x$?

$z = dy$
$= d \times (ex)$
$= (de)x$

$\Rightarrow z = d_1 x$, for some $d_1 = de \in Z^+$
$\Rightarrow x|z$
$\Rightarrow xRz$ is true.

$\therefore R$ is transitive.

JMC

Ex:- consider a relation R on $\mathbb{Z}$ such that aRb hold when $ab \geq 0$

$\forall x \in \mathbb{Z}$, xRx True ; $x \times x \geq 0 \Rightarrow x^2 \geq 0$ true.

$\therefore$ R is reflexive.

if xRy is true $\Rightarrow$ yRx is also true.

since xRy is true $\Rightarrow xy \geq 0 \Rightarrow yx \geq 0 \Rightarrow$ yRx holds.

$\therefore$ R is symmetric.

counter ex for transitivity:-

xRy true $\Rightarrow (-1) \times 0 \geq 0$

yRz true $\Rightarrow 0 \times 1 \geq 0$

$xRz \Rightarrow x \times z \geq 0$

$\Rightarrow (-1) \times 1 \geq 0$ not possible.

$\therefore$ R is not transitive.

Defn:- Given a relation R on set A, R is called anti-symmetric if aRb hold & bRa hold which imply a=b.

Ex:- Let U be the set. Define the relation R on P(U) by ARB is true if $A \subseteq B$ (A, B $\rightarrow$ are 2 subsets)

consider ARB hold $\Rightarrow A \subseteq B$
BRA hold $\Rightarrow B \subseteq A$ } $A = B$

So, R is anti symmetric.

Defn:- A relation R is said to be partial order relation if it is reflexive, anti-symmetric and transitive.

Defn:- A relation R is said to be equivalence relation if it is reflexive, symmetric and transitive.

Note:- Let $A = \{1, 2, 3, \ldots, n\}$ then total no. of anti-symmetric relation is $2^n . 3^{\left(\frac{n^2-n}{2}\right)}$

Exercise 7.1

1.) $A = \{1, 2, 3, 4\}$, given an example R on A that is

i) reflexive, symmetric but not transitive.

$R = \{\underbrace{(1,1), (2,2), (3,3), (4,4)}_{\text{Reflexive}}, \underbrace{(1,2), (2,1)}_{\text{Symmetric}}, (2,3), (3,2)\}$

not transitive

(1,2), (2,3) but (1,3) is not there

5.) (a.) $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ where aRb hold if a | b.

(b.) R is the relation on $\mathbb{Z}$ where aRb hold if a | b.

(c.) For a given universe U and a fixed set $C \subseteq U$. Define R on P(U) where ~~aRb~~ ARB holds if $A \cap C = B \cap C$

(d.) A be the set of all lines in $\mathbb{R}^2$, define the relation R on A where $l_1 R l_2$ hold if l_1 is perpendicular to l_2

(e.) R is the relation on $\mathbb{Z}$ where xRy if x-y is odd

(f.) let T be the set of all triangles in $\mathbb{R}^2$. Define a relation R on T where $t_1 R t_2$ is true if t_1 and t_2 have an angle of same measure.

g.) R is relation on $\mathbb{Z} \times \mathbb{Z}$, where (a, b) R (c, d) hold if $a \leq c$.

(a.) $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ $\quad aRb = a|b$

→ R is reflexive as $a|a$, $b|b$ is true.

→ R is not symmetric as $a|b$ True $\not\Rightarrow$ $b|a$ True

→ R is transitive : if $a|b$ True & $b|c$ True

$\Rightarrow a|c$ True

→ R is anti symmetric : if $a|b$ is True & $b|a$ is True

$\Rightarrow a = b$ is True

(b.) $R \subseteq \mathbb{Z}$

→ R is not reflexive : $a|a$, $b|b$, $c|c$ not always True

exception $(0|0)$

→ R is not symmetric : if $a|b \not\Rightarrow b|a$ eg:- $2|4 (T) \Rightarrow 4|2 (F)$

→ R is ~~not~~ transitive : if $a|b$ is True & $b|c$ is True

$\Rightarrow a|c$ True

→ R is anti symmetric : if $a|b$ is True & $b|a$ is True

$\Rightarrow a = b$ True.

c.) $C \subseteq U$ $\quad$ ARB hold if $A \cap C = B \cap C$

→ R is reflexive : ARA $\Rightarrow A \cap C = A \cap C$ True.

→ R is symmetric : if ARB is True $\Rightarrow A \cap C = B \cap C$

check BRA is True $\Rightarrow B \cap C = A \cap C$

→ R is transitive check ARB hold $\Rightarrow A \cap C = B \cap C$

check BRD hold $\Rightarrow B \cap C = D \cap C$

Now check ARD is true ? $\Rightarrow A \cap C = D \cap C$

True

→ R is not anti symmetric : ARB is True : $A \cap C = B \cap C$
& BRA is True : $B \cap C = A \cap C$
check $A = B$ → True.

(*) Very few relation are both symmetric & anti-symmetric

d) $l_1 R l_2$ if l_1 is $\perp$ l_2

→ R is not reflexive $\parallel$

→ R is symmetric

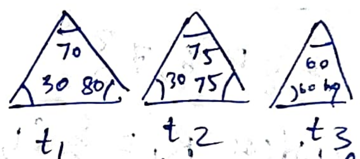
→ R is not transitive $\perp \; \perp \; \parallel$

→ R is not antisymmetric.

f.) $t_1 R t_2$ hold t_1 & t_2 same measure.

$T = \{t_1, t_2, t_3, t_4\}$ each & every element is a triangle.

$T \times T = \{(t_1, t_1), (t_1, t_2), (t_1, t_3) ...\}$



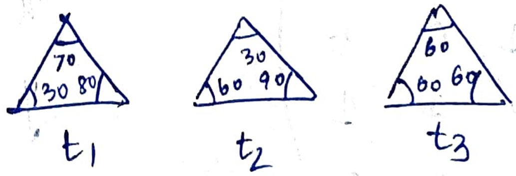
$R \subseteq T \times T$

→ R is reflexive $t_1 R t_1$ is True ; an angle of a single triangle is same.

→ R is symmetric : $t_1 R t_2$ is True : one angle of both triangle is same.
$t_2 R t_1$ is True.

→ R is not transitive : $t_1 R t_2$ is True & $t_2 R t_3$ is True
but $t_1 R t_3$ is not true.

Ex:-



4.) $R \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$

$(a,b) R (c,d)$ hold if $a \leq c$

→ R is reflexive : there is no condition for b and d so we can take $a \leq c$. by taking (a,a)

Ex:- $(1,2), (1,2)$

Defn:- An $m \times n$ matrix $(e_{ij})_{m\times n}$ is said to be <u>zero one matrix</u> if all the entries of $(e_{ij})_{m \times n}$ are either zero or one.

Ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$: Zero One Matrix

<u>Ex 7.21</u> Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$

Consider $R_1 = \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B$

$R_2 = \{(w, 5), (x, 6)\} \subseteq B \times C$

compute $R_1 \circ R_2$ by using matrix representation;

Soln:- R_1 : A to B → columns, ↓ rows

R_2 : B to C → columns, ↓ rows

$M(R_1)$: $\begin{array}{c|c} & \begin{matrix} w & x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times4} \end{array}$

$M(R_2)$: $\begin{array}{c|c} & \begin{matrix} 5 & 6 & 7 \end{matrix} \\ \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{4\times3} \end{array}$

Cont Next →

7.2 Defn.:- If A, B, C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, then the composite relation $R_1 \underset{\text{compose}}{\circ} R_2$ is a relation from A to C, defined by

$R_1 \circ R_2 = \{(x, z) : x \in A, z \in C\}$ and $\exists$ a $y \in B$ such that $(x, y) \in R_1$ and $(y, z) \in R_2$

Ex:- consider previous ex :- Straight Forward Method.

$R_1 = \{(1, x), (2, x), (3, y), (3, z)\}$

$R_2 = \{(w, 5), (x, 6)\}$

$R_1 \circ R_2 = \{(1, 6), (2, 6)\}$

$$\begin{matrix} x & y \\ \downarrow & \downarrow \\ (1, & x) \in R_1 \\ (x, & 6) \in R_2 \\ \uparrow & \uparrow \\ y & z \end{matrix}$$

$\therefore R_1 \circ R_2 = \{x, z\}$

Defn:- Given a set A and a relation R on A. We define the power of R as follows:-

i) $R^1 = R$

ii) $R^{n+1} = R \circ R^n$

Ex:- let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$ $\checkmark R \subseteq A \times A$

compute R^2 and R^3.

Soln:- $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$

$R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$

$\therefore R^2 = R \circ R \Rightarrow \{(1, 4), (1, 2), (3, 4)\}$

$R^3 = R \circ R^2$

$= \{(1, 4)\}$

$R^4 = R \circ R^3$

$= \{\} = \phi$

Ex 7.21

$$M(R_1) \times M(R_2) = \begin{array}{c|ccc} & 5 & 6 & 7 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array}_{4\times 3}$$

$R_1 : A \to B$

$R_2 : B \to C$

$R_1 \circ R_2 : A \to C \to$ columns

$\downarrow$ rows

$\therefore R_1 \circ R_2 = \{(1,6), (2,6)\}$

Exercise 7.2

1) For $A = \{1,2,3,4\}$. Let R and I be the relation is defined by $R = \{(1,2), (1,3), (2,4), (4,4)\}$

$I = \{(1,1), (1,2), (1,3), (2,3), (2,4)\}$

Find $R \circ I$, $I \circ R$, I^2, I^3, R^2 and R^3

$R \circ I = \{(1,3), (1,4)\}$

$I \circ R = \{(1,2), (1,3), (1,4), (2,4)\}$

$I^2 = I \circ I = \{(1,2), (1,3), (2,4)$

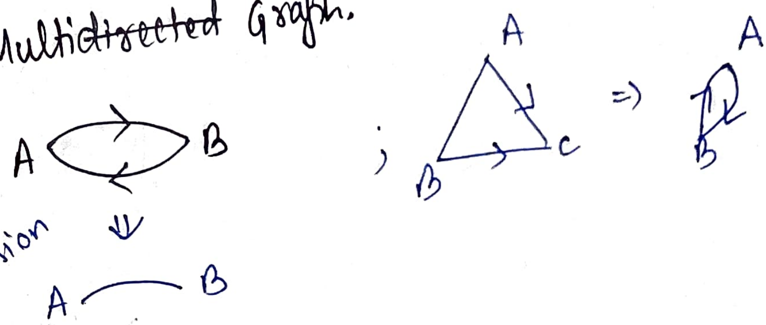
Def^n:- Directed Graph :

Weakly Connected Digraph

Strongly connected Digraph

(*) Every strongly connected digraph is a weakly connected digraph. But the converse is not true.

Note:- Multi~~directed~~ Graph.



edge compression

Multigraph:

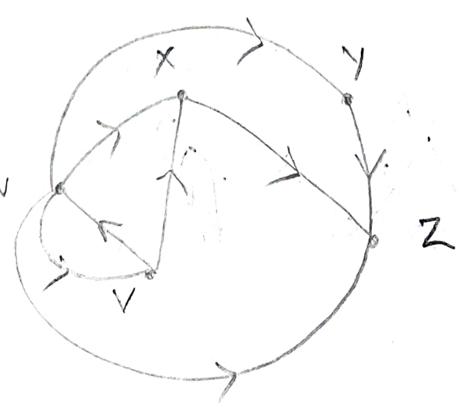
Q18.) For $A = \{v, w, x, y, z\}$ each of the following is (0,1) matrix for a relation R over A. Determine the relation $R \subseteq A \times A$ in each case and draw a digraph.

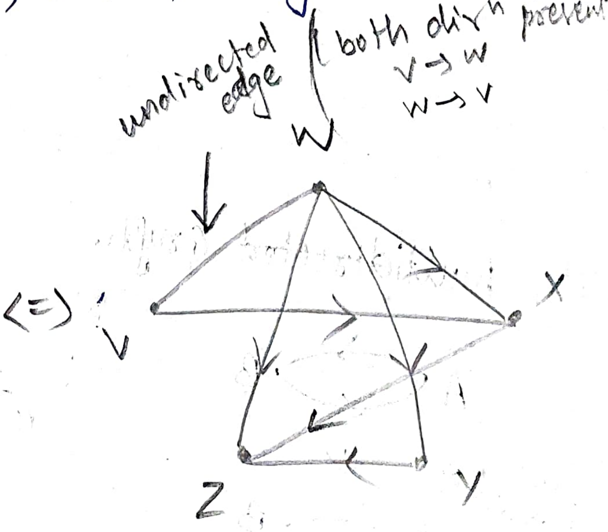
a.) $M(R) = \begin{array}{c|ccccc} & v & w & x & y & z \\ \hline v & 0 & 1 & 1 & 0 & 0 \\ w & 1 & 0 & 1 & 1 & 1 \\ x & 0 & 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & 0 & 0 \end{array}$

b) $M(R) = \begin{array}{c|ccccc} & v & w & x & y & z \\ \hline v & 0 & 1 & 1 & 1 & 0 \\ w & 1 & 0 & 1 & 0 & 0 \\ x & 1 & 1 & 0 & 0 & 1 \\ y & 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 1 & 1 & 0 \end{array}$

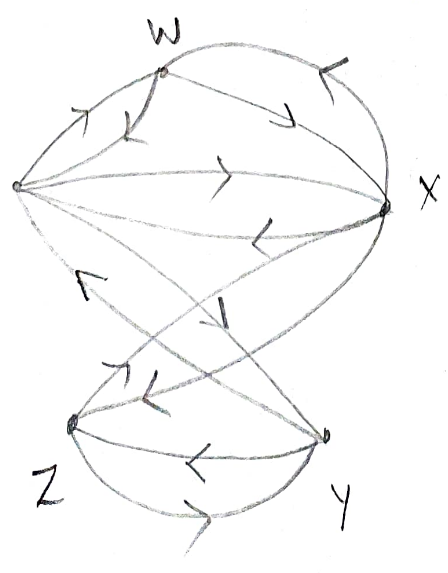
(*) $\boxed{R \text{ over } A \iff R \subseteq A \times A}$ $\quad R: A \to A$

a.) $R = \{(v,w), (v,x), (w,v), (w,x), (w,y), (w,z), (x,z), (y,z)\}$





b.) $R = \{(v,w), (v,x), (v,y), (w,v), (w,x), (x,v), (x,w), (x,z), (y,v), (y,z), (z,x), (z,y)\}$



CHAP-8

Remember statement

Theorem 8.1 : The principle of Inclusion and Exclusion.

Consider a set S, with $|S| = N$ and conditions c_i for $1 \le i \le n$, each of which may be satisfied by some of the elements of S. The no. of elements of S that satisfy none of the conditions c_i, is denoted by

$$\bar{N} = N(\bar{c}_1 \bar{c}_2 \ldots \bar{c}_n) \text{ and is defined by:}$$

total no. of elements satisfying the condition c_i

$$\bar{N} = N - [N(c_1) + N(c_2) + \cdots + N(c_n)]$$
$$+ [N(c_1 c_2) + N(c_2 c_3) + \cdots + N(c_{n-1} c_n)] \quad \frac{n(n-1)}{2} \text{ terms}$$
$$- [N(c_1 c_2 c_3) + N(c_2 c_3 c_4) + \cdots + N(c_{n-2} c_{n-1} c_n)]$$
$$+ \cdots + (-1)^n N(c_1 c_2 \cdots c_n), \text{ where}$$

$N(c_i)$ be the total no. of elements satisfying condition c_i.

Ex:- c_1, c_2 $\quad \bar{N} = N - [N(c_1) + N(c_2)] + [N(c_1 c_2)]$

c_1, c_2, c_3 $\quad \bar{N} = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1 c_2) + N(c_2 c_3) + N(c_2 c_3)] - [N(c_1 c_2 c_3)]$

Ex 8.1 Let $S = 100$ be the total no. of students enrolled in a central college. Consider c_1 and c_2 such that

c_1 : Students enrolled in History

c_2 : Students enrolled in Economics.

If 35 out of 100 students enrolled in History and 30 out of 100 students enrolled in Economics and 9 students enrolled in both, then how many students are not taking any of the subjects ?

soln:- $N(\bar{C_1} \times \bar{C_2}) = N - [N(C_1) + N(C_2)] + N(C_1 C_2)$

$= 100 - [35 + 30] + 9$

$= 44$

~~HW~~

Example 8.2, Ex-8.1 Q3

Example 8.5 Find total no. of integer solution of $x_1 + x_2 + x_3 + x_4 = 18$ such that $0 \le x_i \le 7$, for $1 \le i \le 4$.

$S = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18, \; x_i \ge 0, \; x_i \in Z\}$

$|S| = \underbrace{18}_{\text{Total}} + \underbrace{4}_{\text{no. of variables}} - 1 \; \underbrace{C_{18}}_{\text{Position}} = {}^{21}C_{18}$

whatever may be the Qs here - it will be +1

Let C_1 be the condition such that

C_1: x_1 be the element where $x_1 \ge 8$

C_i: x_i be the element where $x_i \ge 8$, $i = 2, 3, 4$

$\bar{C_1} \cap \bar{C_2} \cap \bar{C_3} \cap \bar{C_4} = x_1 \le 7, x_2 \le 7, x_3 \le 7, x_4 \le 7$

$N[\bar{C_1} \times \bar{C_2} \times \bar{C_3} \times \bar{C_4}) = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4)]$

$+ [N(C_1 C_2) + N(C_1 C_3) + N(C_2 C_4) + N(C_1 C_4)$

$+ N(C_2 C_3) + N(C_3 C_4)]$.

$- [N(C_1 C_2 C_3) + \cdots + N(C_3 C_4 C_2)]$.

$- N(C_1 C_2 C_3 C_4)$.

$S_1 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18$ and

x_1 satisfy C_1, $x_2 \ge 0, x_3 \ge 0, x_4 \ge 0\}$

$|S_1| = N(C_1)$

$S_1 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18,\ x_1 \geq 8,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0\ ;\ x_i \in \mathbb{Z}\}$

Let $y_1 = x_1 - 8 \geq 0$

$S_1 = \{(y_1, x_2, x_3, x_4) : y_1 + x_2 + x_3 + x_4 = \underset{(18-8)}{10},\ y_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0\ ;\ y_1, x_2, x_3, x_4 \in \mathbb{Z}\}$

$= {}^{10+4-1}C_{10} = {}^{13}C_{10}$

$N(C_1) = {}^{13}C_{10}$, $N(C_2) = {}^{13}C_{10}$, $N(C_3) = {}^{13}C_{10}$, $N(C_4) = {}^{1}$

$S_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18,\ x_1 \geq 0,\ \underbrace{x_2 \geq 8}_{x_2 \text{ satisfy } C_2},\ x_3 \geq 0,\ x_4 \geq 0\}$

$S_1 \cap S_2 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18,\ \underbrace{x_1 \geq 8}_{C_1},\ \underbrace{x_2 \geq 8}_{C_2},\ x_3 \geq 0,\ x_4 \geq 0\}$

Let $y_1 = x_1 - 8 \geq 0$

$z_1 = x_2 - 8 \geq 0$

$S_1 \cap S_2 = \{(y_1, z_1, x_3, x_4) : y_1 + z_1 + x_3 + x_4 = 18 - 8 - 8 = 2,\ y_1 \geq 8,\ z_1 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0\}$

$|S_1 \cap S_2| = {}^{2+4-1}C_2 = {}^5C_2$

$N(C_2 C_3) = {}^5C_2$ ......

$N(C_1 C_2 C_3) \Rightarrow$

$S_3 = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 18,\ x_1 \geq 8,\ x_2 \geq 8,\ x_3 \geq 8,\ x_4 \geq 0\}$

$|S_3| = N(C_1 C_2 C_3) = 0$ [As $8 + 8 + 8 = 24 \neq 18$ not possible]

$N(C_1 C_2 C_3 C_4) = 0$ - - -

$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = {}^{21}C_{18} - \underbrace{{}^4C_1 \times {}^{13}C_{10}}_{\downarrow} \times {}^4C_2 \times {}^5C_2 + 0 + 0$

$= 246$

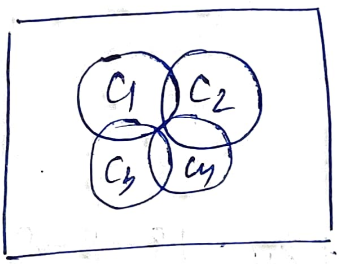
$4 \times {}^{13}C_{18}$ $\quad N(C_1), N(C_2), N(C_3), N(C_4)$

HW What is the total no. of integers soln of $x_1 + x_2 + x_3 + x_4 = 18$ such that $0 \le x_1 \le 5$, $0 \le x_2 \le 4$, $0 \le x_3 \le 9$, $0 \le x_4 \le 5$

Exercise 8.1

Let S be the non empty set such that $|S| = N$. Consider C_1, C_2, C_3, C_4 conditions where each of them may be satisfied by some of the elements of S. Then prove that

$N(\bar{C_2}\bar{C_3}\bar{C_4}) = N(C_1\bar{C_2}\bar{C_3}\bar{C_4}) + N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4})$



CHAP-9

9.2

Defn:- Let $a_0, a_1, a_2, a_3 \ldots$ be the sequence of real nos. The function $f(x)$ is called generating function if we can write $f(x)$ as follows:

Sequence :- 1, 1, 1, 1

$1+x+x^2+\ldots$

$= \frac{1}{1-x}$: generating function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

Ex-2

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$$
$$= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + 0 + 0$$
$$\left(a_0 = {}^nC_0,\ a_1 = {}^nC_1,\ a_n = {}^nC_n,\ a_{n+1} = 0,\ a_{n+2} = 0\right)$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

$(1+x)^n$ is the generating function for sequence of real numbers, $a_0, a_1, a_2, \ldots$

$$(1+x+x^2+\ldots+x^n)(1-x) = 1-x^{n+1}$$
$$= \frac{1-x^{n+1}}{1-x} = 1+x+x^2+\ldots+x^n$$
$$= \sum_{j=0}^{n} x^j$$

$$\lim_{n\to\infty} x^{n+1} = \begin{cases} 1, & x = 1 \\ 0, & 0 \le x < 1 \\ \infty, & x > 1 \end{cases}$$

$$\lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \lim_{n\to\infty} \sum_{j=0}^{n} x^j$$
$$= \lim_{n\to\infty} \frac{1}{1-x} - \lim_{n\to\infty} \frac{x^{n+1}}{1-x} = \sum_{j=0}^{\infty} x^j$$
$$= \frac{1}{1-x} - 0 = \sum_{j=0}^{\infty} x^j;\ |x|<1 \Rightarrow 1+x+x^2+\ldots;\ |x|<1$$

$\therefore \frac{1}{1-x}$ is the generating function for the sequence of real nos $1, 1, 1, \ldots\ldots$

Ex-3 Find the generating function in each of the following cases :

i) where sequence of real nos are $\underset{a_0}{1}, \underset{a_1}{2}, \underset{a_2}{3}, \underset{a_3}{4}, \ldots\ldots$

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$

$f(x) = 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 + \ldots\ldots$

$= \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \ldots\ldots\right)$

$= \left(\frac{1}{(1-x)^2}\right) = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

$\therefore f(x) = \frac{1}{(1-x)^2}$ is the generating function for the sequence of real no. $1, 2, 3, 4, \ldots\ldots$

Find the generating function for each cases :

Ex-1 Find the generating function for each cases :

i) Sequence of real nos $0, 1, 2, 3, \ldots\ldots$

ii) " " " " $1^2, 2^2, 3^2, \ldots\ldots$

iii) " " " " $0, 1^2, 2^2, 3^2, \ldots\ldots$

iv) " " " " $1, 2, 2^2, 2^3, 2^4, \ldots\ldots$

v) " " " " $1, 1, 1, 3, 1, 1, 1 \ldots\ldots$

vi) " " " " $0, 2, 6, 12, 20, 30, 42, \ldots\ldots$

(i) $f(x) = 0 + 1 \cdot x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$

$= x + 2x^2 + 3x^3 + \ldots\ldots$

$= x(1 + 2x + 3x^2 + \ldots\ldots)$

$\Rightarrow x \times \frac{1}{(1-x)^2}$

$0, 0, 1, 2, 3 \Rightarrow x^2 \times \frac{1}{(1-x)^2}$

$0, 0, 0, 1, 2, 3 \Rightarrow x^3 \times \frac{1}{(1-x)^2}$

(ii.) $f(x) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$$

$$\Rightarrow \frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^3 + 4^2x^4$$

(iii.) $f(x) = x\left(\frac{x+1}{(1-x)^3}\right) = 0 + 1^2x + 2^2x^2 + \dots$

$= x + 2^2x^2 + \dots$

$= x(1 + 2^2x + \dots$

$= x \times \left(\frac{x+1}{(1-x)^3}\right)$

(iv.) $f(x) = 1 + 2x + 2^2x^2 + 2^3x^3 + \dots$

$\frac{1}{1-x} = 1 + x + \cancel{2}x^2 + x^3 + \dots$

↑replace x by $2x$

$$\Rightarrow \frac{1}{(1-2x)} = 1 + 2x + 2^2x^2 + 2^3x^3 + \dots$$

(v.) $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \underbrace{x^3} + \dots \quad + \underbrace{2x^3}$

$f(x) = 1 + x + x^2 + \underline{\underline{3x^3}} + x^4 + \dots$

$\therefore f(x) = \frac{1}{1-x} + 2x^3$

(vi) $f(x) = 0 + 2x + 6x^2 + 12x^3 + \dots$

$a_0 = 0,\ a_1 = 2,\ a_2 = 6,\ a_3 = 12$

$a_0 = 0^2 + 0,\ a_1 = 1^2 + 1,\ a_2 = 2^2 + 2,\ a_3 = 3^2 + 3 \dots a_n = n^2 + n$

$; n \geq 0$

$f(x) = (0^2 + 0) + (1^2 + 1)x + (2^2 + 2)x^2 + (3^2 + 3)x^3 + \dots$

$= (0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \dots) + (0 + x + 2x^2 + 3x^3 + \dots)$

$$f(x) = x \times \left(\frac{x+1}{(1-x)^3}\right) + x \times \left(\frac{1}{(1-x)^2}\right)$$

Exercise 9.2

Q1.) (f.) Find the g.f where $0, 0, 1, a, a^2$

Q2.) (f) $f(x) = \frac{1}{1-x} + 3x^7 - 11$, compute sequence of real no.

Soln:- Q1.) $f(x) = 0 + 0.x + 1.x^2 + a.x^3 + a^2x^4 + \ldots$

$$= x^2(1 + ax + a^2x^2 + \ldots)$$

$$\text{g.f. } f(x) = x^2 \times \frac{1}{(1-ax)}$$

Q2.) $f(x) = \frac{1}{1-x} + 3x^7 - 11$

$$= (1 + x + x^2 + \ldots) + 3x^7 - 11$$

$$= x + x^2 + \ldots + 4x^7 + x^8 + \ldots - 10$$

$$= -10 + x + x^2 + \ldots + x^6 + 4x^7 + x^8 + \ldots$$

Q6.) Determine the constant coefficient of the polynomial function

$$\left(3x^2 - \frac{2}{x}\right)^{15}$$

$$= (a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + \ldots + {}^nC_n\, a^0 b^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

A/Q

$n = 15$, $a = 3x^2$, $b = -\frac{2}{x}$

$$= \sum_{r=0}^{15} {}^{15}C_r\, (3x^2)^{15-r} \left(-\frac{2}{x}\right)^r$$

$$= \sum_{r=0}^{15} {}^{15}C_r\, 3^{15-r} (-2)^r\, x^{30-2r} \cdot x^{-r}$$

constant term

$$= \sum_{r=0}^{15} {}^{15}C_r \, 3^{15-r} (-2)^r x^{30-3r}$$

$$= {}^{15}C_0 \, 3^{15} (-2)^0 x^{30} + {}^{15}C_1 \, 3^{14} (-2)^1 x^{27} + \dots + {}^{15}C_{15} \, 3^0 (-2)^{15} x^{-15}$$

To find the constant term, we equate $30-3r = 0 \Rightarrow r = 10$

Constant term $= {}^{15}C_{10} \, 3^5 (-2)^{10}$

<u>Ex2</u> $f(x) = \left(3x^2 - \frac{2}{x}\right)^{1002}$

$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$ $\qquad n = 1002, \; a = 3x^2, \; b = -\frac{2}{x}$

$$\Rightarrow \sum_{r=0}^{1002} {}^{1002}C_r \, (3x^2)^{1002-r} \left(-\frac{2}{x}\right)^r$$

$$= \sum_{r=0}^{1002} {}^{1002}C_r \, 3^{1002-r} (-2)^r x^{2004-3r}$$

Constant term, $2004 - 3r = 0 \Rightarrow r = 668$

$\Rightarrow {}^{1002}C_{668} \, 3^{334} (-2)^{668}$

668
$3)\overline{2004}$

668
$\times 3$
2004

1002
-668
334

Section 9.1

<u>Ex9.1</u> Mildred ~~buys~~ buys 12 oranges for her children, Grace, Mary & Frank. How many ways we can distribute the oranges so that Grace gets at least 4, Mary & Frank get atleast 2 but Frank gets no more than 5. Also find generating function.

<u>Soln</u>:- Let Grace gets c_1 oranges.
Mary gets c_2 oranges.
Frank gets c_3 oranges.

$c_1 + c_2 + c_3 = 12$
$c_1 \geq 4, \; c_2 \geq 2, \; 2 \leq c_3 \leq 5$

Discrete Distribution Table.

G	M	T
4	3	5
4	4	4
4	5	3
4	6	2

G	M	T
5	2	5
5	3	4
5	4	3
5	5	2

G	M	T
6	2	4
6	3	3
6	4	2

G	M	T
7	2	3
7	3	2

G	M	T
8	2	2

Total 14 terms.

$f(x)$ 9 8 4

$f_G(x) = (x^4 + x^5 + x^6 + x^7 + x^8)$

$f_M(x) = (x^2 + x^3 + x^4 + x^5 + x^6)$.

$f_F(x) = (x^2 + x^3 + x^4 + x^5)$

$f(x) = (x^4 + x^5 + x^6 + x^7 + x^8).(x^2 + x^3 + x^4 + x^5 + x^6).$
$(x^2 + x^3 + x^4 + x^5)$

Generating function for this function.

Total 100 no. of terms identify these 14 terms.

Ex 9.2

HW Q1, Q2

CHAP-10 Recurrence Relation.

Consider the infinite sequence $1, a, a^2, a^3, \ldots\ldots$ geometric progression (common ratio = a)

$b_0\ b_1\ b_2\ b_3 \qquad b_n\ b_{n+1}\ b_{n+2}$

$\frac{b_1}{b_0} = a = \frac{b_2}{b_1} = \frac{b_3}{b_2} = \; = \frac{b_{n+1}}{b_n} = \ldots$

common ratio

$\frac{b_{n+1}}{b_n} = a \Rightarrow b_{n+1} = a.b_n \; ; \; n \geq 0 \leftarrow$ Recurrence Reln.

$\boxed{b_0 = 1}$

Linear Recurrence Relation.

Consider the linear recurrence relation in general form

$b_{n+1} = d\,b_n$, $n \geq 0$ and $b_0 = A$; Here d is any constant.

$n = 1$, $b_1 = db_0 \quad = A.d$

$b_2 = db_1 \quad = (Ad)d = Ad^2$

$b_3 = db_2 \quad = (Ad^2)d = A.d^3$

So, the solution of the recurrence relation is :-

$\boxed{b_n = A.d^n} \; ; \; n \geq 0$

Ex:- Solve the recurrence relation $b_{n+1} = 7b_n,\ n \geq 0,\ b_2 = 98$

$n=1,\ b_1 = 7b_0 \Rightarrow b_0 = 2$

$b_2 = 7b_1 = 98 \Rightarrow b_1 = \frac{98}{7} = 14 \Rightarrow b_2 = b_0.(7)^2$

$b_3 = 7b_2 \Rightarrow b_0(7)^3$

$\vdots$

$b_n = b_0(7)^n;\ n > 0$

$b_n = 2.(7)^n;\ n \geq 0$

Ex 10.2 A bank pays 6% (annual) interest on savings, compounding the interest monthly. If some one deposits 1000 dollar on the first day of May, how much will this deposit be worth, a year later?

Soln:- $P_0 = 1000$ dollar

Annual interest $= 6\%$

So, monthly interest $= \frac{6}{12} = 0.5\% = 0.005$

1st day of June $P_1 = P_0 + P_0 \times 0.005 \rightarrow$ 1st June

$P_2 = P_1 + 0.005 \times P_1 \rightarrow$ 1st July

$\vdots$

P_n be the total amount after completing n months.

$P_n = P_{n-1} + P_{n-1} \times 0.005$

$= (1.005)P_{n-1},\ n \geq 0,\ P_0 = \$1000$

Solution is $P_n = (1.005)^n \times 1000$ dollar

$\therefore P_{12} = (1.005)^{12} \times 1000$ dollar $= \$1061.678$

Ex 10.4 Find a_{12} where $a_{n+1}^2 = 5a_n^2,\ a_n > 0$ for $n \geq 0$ and $a_0 = 2$

Consider $b_n = a_n^2$

So, $b_{n+1} = 5b_n,\ b_0 = 4 = (a_0)^2 = 2^2$

$\therefore$ Soln $b_n = (5)^n \times 4 \Rightarrow (a_n)^2 = (5)^n \times 4 \Rightarrow a_n = (5)^{n/2} \times 2,\ n \geq 0$

So, $a_{12} = (5)^6 \times 2 = 31250$ Ans.

Ex 10.7 Solve $a_n = n a_{n-1}$, $n \geq 1$ and $a_0 = 1$

$a_1 = n a_0 = \cancel{n} = 1$

$a_2 = \cancel{n a_1 = n.n = n^2} = 2a_1 = 2.1$

$a_3 = \cancel{n a_2 = n.n^2 = n^3} = 3a_2 = 3.2.1$

$\vdots$

$a_n = \cancel{n a_{n-1}} = n(n-1)(n-2)\cdots 1 = n!$

Exercise 10.1

1) Find the recurrence relation with initial condition for the sequence

(i) 2, 10, 50, 250

(ii) 6, −18, 54, −162

i) $b_0 = 2$, $b_1 = 10$, $b_2 = 50$, $b_3 = 250$

$\frac{b_{n+1}}{b_n} = 5 \Rightarrow b_{n+1} = 5b_n$, $n \geq 0$; $b_0 = 2$

ii) $b_0 = 6$, $b_1 = -18$, $b_2 = -3a_1$
$= -3a_0$

$b_n = -3a_{n-1}$

Q3) If $a_{n+1} = d a_n$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{240}$, find d
$n \geq 0$

$a_5 = d a_4$ $\quad a_5 = d(d a_3) = d^2 a_3$

$a_4 = d a_3$

$a_4 = d \times \frac{153}{49}$ $\quad a_5 = d^2 \times \frac{153}{49} = \frac{1377}{240} = d^2 \times \frac{153}{49}$

Section 10.2

Let $K \in \mathbb{Z}^+$ and $C_0 (\neq 0), C_1, C_2, \ldots\ldots C_K (\neq 0)$ be real nos. If a_n is a discrete function then
$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots\cdots + C_K a_{n-K} = f(n)$ for $n \geq K$
is a linear recurrence relation with constant coefficient of order K.

Note:-
① If $f(n) = 0$, then eqⁿ ① is called homogenous linear recurrence relation with constant coefficient of order k.
② If $f(n) \neq 0$, then eqⁿ ① is non-homogenous.

ex:- $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = f(n)$ is linear recurrence relation of order 2.

Consider the linear homogenous recurrence relation of order 2 is $C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$ ——— ①
Let $a_n = A \cdot d^n$ be the trial solution of ①

$a_{n-1} = A d^{n-1}$

$a_{n-2} = A d^{n-2}$, $A \neq 0$

So, we have $C_0 A d^n + C_1 A d^{n-1} + C_2 A d^{n-2} = 0$

$\Rightarrow C_0 d^n + C_1 d^{n-1} + C_2 d^{n-2} = 0$

$\Rightarrow C_0 d^2 + C_1 d + C_2 = 0$ ——— ②

Let $d = d_1$ and $d = d_2$ be two values satisfying ②
Solution is $a_n = A d_1^n, A d_2^n$
General solⁿ of eqⁿ ① is written as

$a_n = b_1 d_1^n + b_2 d_2^n$; b_1 & b_2 are constant.

Case A d_1 & d_2 are real and distinct.

Case B d_1 & d_2 are real but equal. } Not in syllabus.

Case C d_1 & d_2 are complex numbers. }

Ex10.9 Solve, $a_n + a_{n-1} - 6a_{n-2} = 0$, ——— (1)
where $n \geq 2$, $a_0 = -1$, $a_1 = 8$

Let $a_n = Ad^n$ be the trial soln of (1)

From (1), $A.d^n + Ad^{n-1} - 6Ad^{n-2} = 0$

$\Rightarrow d^2 + d - 6 = 0$

$\Rightarrow$ ~~$d=1$, $d=5$~~ $\Rightarrow d = 2$, $d = -3$

$\underset{d_1}{\downarrow}$ $\underset{d_2}{\downarrow}$

Since d_1 & d_2 both are real and distinct. So the general solution is:

$a_n = b_1 2^n + b_2 (-3)^n$; $n \geq 0$

$n=0$ $a_0 = b_1 2^0 + b_2(-3)^0$

$\Rightarrow -1 = b_1 + b_2$ ——— (3)

$n=1$ $a_1 = b_1 2 - 3b_2$

$\Rightarrow 8 = 2b_1 - 3b_2$ ——— (4)

Solving eqn (3) & (4), we have $b_1 = 1$, $b_2 = -2$;

So, the solution is

$a_n = 2^n - 2 \times (-3)^n$, $n \geq 2$

Q2.) Find the solution of Fibonacci Series.

$F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, $F_0 = 0$, $F_1 = 1$.

Let $F_n = Ad^n$ be the trial solution.

$Ad^{n+2} = Ad^{n+1} + Ad^n$

$= Ad^{n+2} - Ad^{n+1} - Ad^n = 0$

$= Ad^n(d^2 - d - 1) = 0$

$\Rightarrow d^2 - d - 1 = 0$