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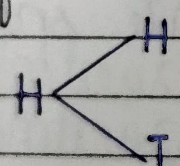
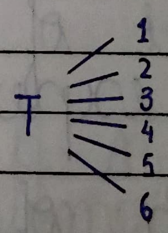
Ch-02
Probability

Experiment
The process from which we get a set of data.

Sample space
The set of all possible outcomes of an experiment.
Denoted as 'S'

Q. Write the sample space of an experiment consisting of flipping a coin and then flipping a coin once if head appears and if tail appears roll a die once.

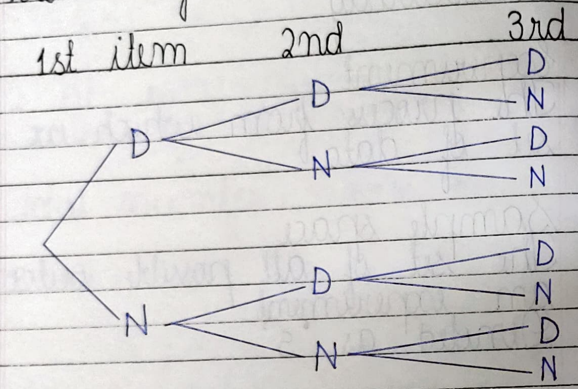




S = {HH, HT, T1, T2, T3, T4, T5, T6}

Tree diagram:
Technique to list the elements of sample space systematically.

Q. Suppose 3 items are selected at random from manufacturing process. Each item is inspected & classified as defective and non-defective. List elements of sample space using the tree diagram



Sample point:

DDD DDN DND DNN
NDD NDN NND NNN

Event:
Subset of the sample space of an experiment

Q. Which of the following events are equal?

A = {1, 3}
B = {x | x is number on die}
C = {x | $x^2 - 4x + 3 = 0$}
D = {x | x is no. of heads, six coins flipped simultaneously}

A and C are equal events
A = {1, 3} C = {1, 3}
B = {1, 2, 3, 4, 5, 6} D = {0, 1, 2, 3, 4, 5, 6}

Mutually exclusive / Disjoint events

Two events A and B are mutually exclusive event if there is no common element between A & B

$A \cap B = \phi$

eg: A: shows even number by rolling a die
B: shows odd number by rolling a die
A = {2, 4, 6} B = {1, 3, 5}
$A \cap B = \phi$

Union event
Union of two events A, B is an event in which element lies in A or B or both.

Intersection event
Intersection event of A and B is $A \cap B$ in which element lies in A and B both

Complement event (A'):
In which element lies in S but not in A

$A' = S - A$

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Probability of an event

The probability of an event A is sum of weights of all sample points in A

$$P(A) = \frac{\text{no. of favorable cases}}{\text{total no. of cases}}$$

eg: rolling a fair die

$S = \{1, 2, 3, 4, 5, 6\}$

A = die shows even number

$P(A) = \frac{3}{6} = \frac{1}{2}$

Q Find probability of atmost 2 heads by tossing 2 coins simultaneously

$S = \{HH, HT, TH, TT\}$

A: atmost two heads

$P(A) = \frac{4}{4} = 1 = P(S)$

Property:

1. For any event A of sample space S, $0 \le P(A) \le 1$
2. $P(\phi) = 0$
3. $P(S) = 1$

Additive rule

If A and B are two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Also written as

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Note:

If A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$

Q Prove that $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Let $B \cup C = D$

$P(A \cup B \cup C) = P(A \cup D)$

$= P(A) + P(D) - P(A \cap D)$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap (B \cup C))$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

Note:
If A, B, C are mutually exclusive then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

If $A_1, A_2, \ldots A_n$ are mutually exclusive event then
$P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Partitions of event

Suppose $A_1, A_2, \ldots A_n$ are partition of sample space S then
$P(A_1 \cup A_2 \ldots \cup A_n) = 1$

Compliment rule of probability

$A' = S - A$
$P(A') = 1 - P(A)$

$A \cup A' = S$
$P(A \cup A') = P(S) = 1$
$P(A) + P(A') - P(A \cap A') = 1$
$P(A) + P(A') = 1$

Q Prove that $P(A' \cap B') = 1 + P(A \cap B) - P(A) - P(B)$

$P(A' \cap B') = P(A \cup B)'$
$= 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$

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50. An experiment involving tossing a pair of fair dice one green & one red & recording the no turn up.
A is an event that sum is greater than 8. C is an event that number greater than 4 comes up on green die Find P(A). P(C), $P(A \cap C)$

A: sum > 8
C: no > 4 comes up on green die

$S = \{(1,1), (1,2) \ldots (1,6),$
$(2,1), (2,2) \ldots (2,6),$
$\vdots$
$(6,1), (6,2) \ldots (6,6)\}$

$A = \{(3,6), (4,5), (4,6), (5,4), (5,5), (5,6), (6,3), (6,4), (6,5), (6,6)\}$

$P(A) = \frac{10}{36}$

$C = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$P(C) = \frac{12}{36}$

$P(A \cap C) = \frac{7}{36}$

53 The probability that an American industry will locate in Shanghai is 0.7, probability that it will locate in Beijing is 0.4 and probability that it will locate in either Shanghai or Beijing or both is 0.8. What is the probability that industry will locate in i) both cities
ii) neither

A: industry in Shanghai

B: industry in Beijing

$P(A) = 0.7$ $\quad$ $P(B) = 0.4$

$P(A \cup B) = 0.8$

i) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
$= 0.7 + 0.4 - 0.8$
$= 0.3$

ii) $P(A' \cap B') = P(A \cup B)'$
$= 1 - P(A \cup B)$
$= 1 - 0.8$
$= 0.2$

59. In a poker hand consisting of 5 cards, find probability of holding
a) 3 aces
b) 4 hearts and 1 club

$S = {}^{52}C_5 = \dfrac{52!}{5!\,47!}$
$= \dfrac{48 \times 49 \times 50 \times 51 \times 52}{1 \times 2 \times 3 \times 4 \times 5}$
$= 2598960$

a) A = 3 aces
$= {}^{4}C_3 \times {}^{48}C_2 = \dfrac{4!}{3!\,1!} \times \dfrac{48!}{46!\,2!}$
$= 4 \times 1128 = 4512$

$P(A) = \dfrac{4512}{2598960} = 0.0017$

b) $B = {}^{13}C_4 \times {}^{13}C_1 = \dfrac{13!}{4!\,9!} \times \dfrac{13!}{1!\,12!}$
$= 715 \times 13$
$= 9295$

$P(B) = \dfrac{9295}{2598960} = 0.0035$

68. Consider the decisions made by 6 customers. Suppose probability is 0.4 that atmost 2 of these purchase electric oven. (a) What is the probability that atleast 3 purchase electric oven?
(b) Suppose probability all 6 purchase electric oven is 0.07 while 0.104 is all 6 purchase gas oven. What is the probability that atleast one of

each type is purchased?

A: atmost 2 purchase electric oven

A': atleast 3 purchase electric oven

$$P(A') = 1 - P(A)$$
$$= 1 - 0.4$$
$$= 0.6$$

Probability that atleast one of each type is purchased

$$1 - (0.07 + 0.104)$$
$$= 1 - 0.174$$
$$= 0.826$$

11-10-22

Conditional probability

The probability of an event B occuring when it is known that some event A has occured

Denoted as

$P(B|A)$ read as probability of B given A

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

eg: Suppose sample space is populat. of small town acc to gender & employment status is given. What is the probability that a person is randomly selected who is male given he is employed

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

Suppose, M is an event: person selected is male

E is person is employed

$$P(M|E) = \frac{P(M \cap E)}{P(E)}$$

$$P(M \cap E) = \frac{460}{900}$$

$$P(E) = \frac{600}{900}$$

$$P(M|E) = \frac{460}{600} = \frac{23}{30}$$

ii) Find probability: person is selected at random who is employed given he is male

$$P(E|M) = \frac{P(E \cap M)}{P(M)} = \frac{460/900}{500/900} = \frac{23}{25}$$

iii) Person is selected randomly is unemployed given person is female

U: unemployed

F: female is selected

$$P(U|F) = \frac{P(U \cap F)}{P(F)}$$

$$= \frac{260/900}{400/900} = \frac{13}{20}$$

iv) Person is selected at random is female and is employed

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

$$= \frac{140}{600} = \frac{7}{30}$$

Independent Event

Two events A, B are independent iff $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Q Suppose an experiment in which 2 cards are drawn in succession with replacement

A: first card is an ace

B: second card is club

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\frac{P(B \cap A)}{P(A)} = \frac{13}{52} = \frac{1}{4}$$

$$P(B) = \frac{1}{4}$$

$$P(B|A) = P(B)$$

Independent event

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Theorem-1

If two events A and B are independent iff

$$P(A \cap B) = P(A)P(B)$$

Theorem-2

If an experiment in which events $A_1, A_2 \ldots A_k$ occur then

$$P(A_1 \cap A_2 \cap \ldots A_k) = P(A_1)P(A_2|A_1)\, P(A_3|(A_1 \cap A_2)) \ldots P(A_k|(A_1 \cap A_2 \ldots \cap A_{k-1}))$$

Note:

If all the events $A_1, A_2 \ldots A_k$ are independent then

$$P(A_1 \cap A_2 \ldots A_k) = P(A_1)P(A_2) \ldots P(A_k)$$

eg 40 Three cards are drawn in succession without replacement. Find probability that event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is event that first card is red ace, A_2 is second card is 10 or jack, A_3 is third card is greater than 3 but less than 7.

A_1: 1st card is red ace
A_2: 2nd card is 10 or jack
A_3: 3rd card is greater than 3 but less than 7

$P(A_1) = \frac{2}{52} = \frac{1}{26}$

$P(A_2 | A_1) = \frac{8}{51}$

$P(A_3 | (A_1 \cap A_2)) = \frac{12}{50}$

$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2 | A_1) P(A_3 | (A_1 \cap A_2))$

$= \frac{2}{52} \times \frac{8}{51} \times \frac{12}{50}$

$= \frac{8}{5525}$

Q-75 A random sample of 200 adults are classified below by sex & their level of education attained

Education	Male	Female
Elementary	38	45
Secondary	28	50
College	22	17

a) person is male, given person has secondary education

b) person does not have college degree given person is female

E: person having elementary education
S: person having secondary education
C: person having college education
M: male
F: female

$P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{28/200}{78/200}$

$= \frac{28}{78}$

$P(C'|F) = \frac{P(C' \cap F)}{P(F)} = \frac{95/200}{112/200}$

$= \frac{95}{112}$

77. In senior year of high school graduating class of 100 students 42 studied math, 68 studied psychology, 54 studied history, 22 studied both math & history, 25 studied both math & psychology, 7 studied history but neither math nor psychology, 10 studied all three and 8 did not take any.

a) person enrolled in psychology takes all 3 subjects

b) person not taking psychology is taking both history & math

75, 76, 77, 80, 88, 91

M : mathematics
H : history
P : psychology

$M \cap H = 22$
$M \cap P = 25$
$H \cap M' \cap P' = 7$
$M \cap P \cap H = 10$
$M' \cap H' \cap P' = 8$

(a) $\frac{10}{68}$

(b) $P(M \cap H) - P(M \cap P \cap H)$
$= \frac{22-10}{100-68} = \frac{12}{32}$

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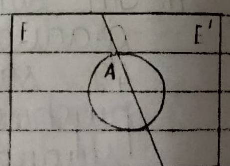
Baye's Theorem

Total probability

$P(A) = P(E \cap A) + P(E' \cap A)$

or

$P(A) = P(E)P(A|E) + P(E')P(A|E')$



Suppose S is sample space of experiment. E & E' are partitions of S

$E \cup E' = S$
$E \cap E' = \phi$

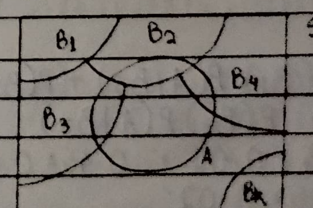
For any event A of S
$P(A) = P(E \cap A) + P(E' \cap A)$ is total probability

Proof:
$A = (E \cap A) \cup (E' \cap A)$
$P(A) = P((E \cap A) \cup (E' \cap A))$
$= P(E \cap A) + P(E' \cap A) - P((E \cap A) \cap (E' \cap A))$
$P(A) = P(E \cap A) + P(E' \cap A)$
$P(A) = P(E)P(A|E) + P(E')P(A|E')$

General formula

Suppose $B_1, B_2 \ldots B_k$ are the partitions of S where $P(B_i) \neq 0$, $i = 1, 2 \ldots k$. For any event A of S

$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + \ldots P(B_k \cap A)$
$= \sum_{i=1}^{k} P(B_i \cap A)$

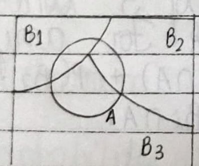


$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + \ldots + P(B_k)P(A|B_k)$

$= \sum_{i=1}^{k} P(B_i)P(A|B_i)$

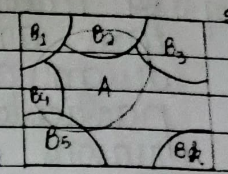
eg-41 In a certain assembly plant three machines B_1, B_2, B_3 make 30%, 45%, 25% of products. It is known from past experience that 2%, 3%, 2% of products made by each machine are defective. Now, suppose that a finished product is selected. What is the probability that it is defective?

A: product is defective
B_1: product made by machine B_1
B_2: product made by machine B_2
B_3: product made by machine B_3



$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$$
$$= 0.3 \times 0.02 + 0.45 \times 0.03 + 0.25 \times 0.02$$
$$= 0.024$$
$$= 2.4\%$$

Baye's Theorem



If the events $B_1, B_2, \dots B_k$ constitute a partition of sample space S such that $P(B_i) \neq 0$, $i = 1, 2, \dots k$ then for any event A of S such that $P(A) \neq 0$

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} \quad r = 1, 2, \dots k$$

where $P(A) = \sum_{i=1}^{k} P(B_i \cap A)$

or $P(A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$

eg-42 With reference to 41, if product was chosen randomly and found to be defective, probability that it was made by B_3?

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

$$= \frac{0.25 \times 0.02}{0.024}$$

$$= \frac{0.005}{0.024} = \frac{10}{49} = 0.$$

Eg 43 A manufacturing firm employs three analytical plans for the design and development of particular product. For cost reasons all three are used at varying time. In fact plans 1, 2, 3 are used for 30%, 20%, 50% of products. Defect rate is different for three procedures

$P(D|P_1) = 0.01$ $\quad P(D|P_2) = 0.03$

$P(D|P_3) = 0.02$

If random product was found to be defective, which plan was most likely used?

$P(P_1) = 0.3 \quad P(P_2) = 0.2 \quad P(P_3) = 0.5$

$$P(P_1|D) = \frac{P(P_1 \cap D)}{P(D)}$$

$$P(D) = \sum_{i=1}^{3} P(P_i)P(D|P_i)$$

$$= P(P_1)P(D|P_1) + P(P_2)P(D|P_2) + P(P_3)P(D|P_3)$$

$$= 0.019$$

$$P(P_1 \cap D) = P(P_1)P(D|P_1)$$
$$= 0.003$$

$$P(P_1|D) = 0.158$$

$$P(P_2|D) = \frac{P(P_2 \cap D)}{P(D)}$$

$$P(P_2 \cap D) = P(P_2)P(D|P_2)$$
$$= 0.03 \times 0.2$$

$$P(P_2|D) = 0.316$$

$$P(P_3|D) = \frac{P(P_3 \cap D)}{P(D)}$$

$$P(P_3 \cap D) = P(P_3)P(D|P_3)$$
$$= 0.02 \times 0.5$$

$$P(P_3|D) = 0.526$$

Conditional probability of defect given plan 3 is largest

Q-95 In a certain region of country it is known from past experience that probability of selecting an adult over 40 years with cancer is 0.05. If the probability of a doctor correctly diagnosing person with cancer is 0.78 and incorrect diagnosing is 0.06 what is the probability that an adult over 40 years is having cancer

C adult has cancer

D adult diagnosed having cancer

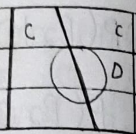
$P(C) = 0.05 \qquad P(C') = 0.95$

$P(D|C) = 0.78 \qquad P(D|C') = 0.06$

$P(D) = P(C)P(D|C) + P(C')P(D|C')$

$= 0.05 \times 0.78 + 0.95 \times 0.06$

$= 0.096$



Q-97 What is the probability that a person diagnosed having cancer actually has disease?

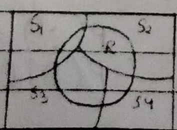
$P(C|D) = \dfrac{P(C \cap D)}{P(D)}$

$= \dfrac{P(C)\,P(D|C)}{P(D)}$

$= \dfrac{0.05 \times 0.78}{0.096}$

$= 0.46$

Q-96 Police plan to enforce speed limits using radar traps at four different locations. Radar traps at each of the locations L_1, L_2, L_3, L_4 will be operated 40%, 30%, 20%, 30% of time. If person who is speeding on her way to work has probabilities 0.2, 0.1, 0.5, 0.2 of passing through these locations what is the probability that she will receive a ticket?



95, 96, 97, 98

R : radar traps

S_1 : person speeding at L_1

$P(S_1) = 0.4$ $\quad$ $P(S_2) = 0.3$

$P(S_3) = 0.2$ $\quad$ $P(S_4) = 0.3$

$P(R|S_1) = 0.2$ $\quad$ $P(R|S_2) = 0.1$

$P(R|S_3) = 0.5$ $\quad$ $P(R|S_4) = 0.2$

$P(R) = P(S_1)P(R|S_1) + P(S_2)P(R|S_2) + P(S_3)P(R|S_3) + P(S_4)P(R|S_4)$

$= 0.08 + 0.03 + 0.1 + 0.06$

$= 0.27$

Q-98 What is the probability that she passed through radar trap at L_2?

$P(S_2|R) = \dfrac{P(S_2 \cap R)}{P(R)}$

$P(S_2 \cap R) = P(S_2)P(R|S_2)$

$= 0.3 \times 0.1$

$= 0.03$

$P(S_2|R) = \dfrac{0.03}{0.27}$

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Ch-03
Random Variable and Probability Distribution

Concept of random variable
A random variable is a fun" that associates real number with each element of a sample space

Denoted with capital letter

An experiment in which 2 balls are drawn in succession without replacement from bag having 4 red & 3 blue balls
Y represents no of red balls

S = {RR, RB, BR, BB}
Y = {2, 1, 0}

Discrete Sample Space
A sample space if it contain finite no. of possibilities or countable infinite no. of possibilities

Countable infinite
Every element of set is a 1-1 correspondance to natural number set

Continuous sample space
Sample space is said to be continuous if it contains infinite number of possibilities equal to number of point on line segment

Random variable are of two types
1 Discrete
2 Continuous

Discrete random variable
Random variable is discrete if its set of possible outcomes are countable

eg- tossing 2 coins simultaneously
X: no. of heads

X = {0, 1, 2}

Continuous random variable
Random variable takes some values on continuous scale

Bernoulli random variable
A random variable bernoulli if it assigns 0 or 1

eg- testing a device for working assign 1 and 0 for not working.

22.10.22

Discrete Probability Distribution

If x is a discrete random variable and f(x) is probability distribution function then for each x

1. $f(x) \geq 0$
2. $\sum f(x) = 1$
3. $P(X = x) = f(x)$

Cumulative distribution function (F(x)) of a discrete random variable x with probability distribution function f(x) is

$$F(x) = P(X = x) = \sum_{x \leq x} f(x) \quad -\infty < x < \infty$$

eq-9 A shipment of 20 similar laptop computers to a retail outlet contains 3 defective. If school makes random purchase of 2 of these computers find prob distribution for number of defectives

X = random variable
= no of defective laptops
X = {0, 1, 2}

$F(0) = P(X = 0) = \frac{{}^{17}C_2 \cdot {}^3C_0}{{}^{20}C_2}$

$= \frac{68}{95}$

$F(1) = P(X = 1) = \frac{{}^3C_1 \cdot {}^{17}C_1}{{}^{20}C_2}$

$= \frac{51}{190}$

$F(2) = P(X = 2) = \frac{{}^3C_2 \cdot {}^{17}C_0}{{}^{20}C_2}$

$= \frac{3}{190}$

x	0	1	2
F(x)	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

eq-10 Find the cumulative distribution function of random variable x from eq-9. Using F(x), verify that f(2) = 3/8

$f(x) = 1/16 ({}^4C_x)$

x : models with side air-bag
x = 0, 1, 2, 3, 4

$f(0) = \frac{1}{16} {}^4C_0 = \frac{1}{16}$

$f(1) = \frac{1}{16} {}^4C_1 = \frac{1}{4}$

$f(2) = \frac{1}{16} {}^4C_2 = \frac{3}{8}$

$f(3) = \frac{1}{16} {}^4C_3 = \frac{1}{4}$

$f(4) = \frac{1}{16}$

$F(0) = \frac{1}{16}$

$F(1) = \frac{5}{16}$

$F(2) = \frac{11}{16}$

$F(3) = \frac{15}{16}$

$F(4) = 1$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/16 & 0 \leq x < 1 \\ 5/16 & 1 \leq x < 2 \\ 11/16 & 2 \leq x < 3 \\ 15/16 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

$f(2) = F(2) - F(1)$
$= \frac{11}{16} - \frac{5}{16} = \frac{3}{8}$

29-10-22

Continuous Probability Distribution Function

X is a continuous random variable. The function $f(x)$ is probability density function if

1. $f(x) \geq 0$ for all $x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

Note
If X is cont^n random variable then $P(a < X < b)$, $P(a \leq X < b)$, $P(a < X \leq b)$, $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

Cumulative distribution function

The cumulative distribution fun^n $F(x)$ of cont^n random variable X with probability mass function $f(x)$ is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = P(X \leq x)$$

* $P(a < X < b) = F(b) - F(a)$

* $f(x) = \frac{d}{dx}(F(x))$

eg. 11 Suppose error in reaction temp. in °C. for controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} x^2/3, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

a) Verify that $f(x)$ is density func^n
b) Find $P(0 < X \leq 1)$

$f(x) \geq 0$ for $x \in R$

a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$

$= \int_{-1}^{2} \frac{x^2}{3}\,dx = \left.\frac{x^3}{9}\right]_{-1}^{2} = 1$

$f(x)$ is density function

b) $P(0 < X < 1)$

$= \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{3}\,dx$

$= \frac{1}{9}$

eg-12 For density function of eg-11 find $F(x)$ and its to evaluate $P(0 < x \leq 1)$

$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{x} f(t)\,dt$

$= \int_{-1}^{x} \frac{t^2}{3}\,dt = \left.\frac{t^3}{9}\right]_{-1}^{x} = \frac{x^3+1}{9}$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x^3+1}{9} & -1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$P(0 < X \leq 1) = F(1) - F(0)$
$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

ex-3.3 Let W be random variable giving number of heads minus no. of tails in three tosses of a coin. List elements of sample space S for three tosses of coin and to each sample point assign a value w of W

S = { HHH, HHT, HTH, THH, THT, TTH, HTT, TTT }

W = { 3, 1, -1, -3 }

W	3	1	-1	-3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$f(x) = \begin{cases} 1/8 & x = 3, -3 \\ 3/8 & x = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

31-10-22

ex-7 Total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a cont" random variable x that has density function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{Elsewhere} \end{cases}$$

Find probability that over period one year a family runs their vacuum cleaner

a) less than 120 hours

b) bet" 50 and 100 hours

$$P(X<1.2) = \int_{-\infty}^{1.2} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.2} f(x)\,dx$$

$$= \int_{0}^{1} x\,dx + \int_{1}^{1.2} 2-x\,dx$$

$$= \frac{x^2}{2}\Big]_0^1 + 2x - \frac{x^2}{2}\Big]_1^{1.2}$$

$$= \frac{1}{2} + 2.4 - \frac{1.44}{2} - 2 + \frac{1}{2}$$

$$= 1.4 - 0.72$$

$$= 0.68$$

b) $P\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} x\,dx$

$$= \frac{x^2}{2}\Big]_{1/2}^{1} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

ex-12 An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given cumulative distribution function of T. no. of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1 \\ 1/4, & 1 \le t < 3 \\ 1/2, & 3 \le t < 5 \\ 3/4, & 5 \le t < 7 \\ 1 & t \ge 7 \end{cases}$$

find (a) $P(T=5)$

(b) $P(T>3)$

(c) $P(1.4<T<6)$

(d) $P(T \le 5 \mid T \ge 2)$

(a) $P(T=5) = F(5) - F(4)$

$$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

(b) $P(T>3) = 1 - P(T \le 3)$

$$= 1 - F(3)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

(c) $P(1.4<T<6) = F(6) - F(1.4)$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

(d) $P(T \le 5 \mid T \ge 2) = \dfrac{P((T \le 5) \cap (T \ge 2))}{P(T \ge 2)}$

$$= \frac{P(2 \le T \le 5)}{P(T \ge 2)} = \frac{F(5) - F(2)}{1 - P(T<2)}$$

$$= \frac{\frac{3}{4} - \frac{1}{4}}{1 - 1/4} = \frac{2}{3}$$

Ex-21 Consider density function

$$f(x) = \begin{cases} k\sqrt{x} & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) Evaluate k

b) Find $F(x)$ and use it to evaluate $P(0.3 < X < 0.6)$

(a) $\int_{-\infty}^{\infty} f(x)dx = 1$

$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)dx$

$= \int_{0}^{1} f(x)\,dx = 1$

$= \int_{0}^{1} k\sqrt{x}\;dx = 1$

$k \frac{3x^{3/2}}{2} \Big]_0^1 = 1$

$k = \frac{3}{2}$

$$f(x) = \begin{cases} \frac{3}{2}\sqrt{x} & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

(b) $F(x) = \int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{0} F(t)\,dt + \int^{x} F(t)dt$

$= \int_0^x \frac{3}{2}\sqrt{t}\;dt = t^{3/2}\Big]_0^x$

$= x^{3/2}$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^{3/2} & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$P(0.3 < X < 0.6) = F(0.6) - F(0.3)$

$= (0.6)^{3/2} - (0.3)^{3/2}$

$= 0.3$

Joint Probability Distribution

Let us consider X, Y are two discrete random variable. The function $f(x,y)$ is a joint probability distribution of X & Y

1. $f(x,y) \ge 0 \quad \forall\ (x,y)$
2. $\sum_x \sum_y f(x,y) = 1$
3. $P(X = x, Y = y) = f(x,y)$

For any region A in xy plane

$P((X,Y) \in A) = \sum\sum_A f(x,y)$

For continuous case

X, Y are two continuous random variable

$f(x,y)$

1. $f(x,y) \ge 0 \quad \forall\ (x,y)$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
3. $P((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy$

where A is any region in xy plane.

Marginal distribution of random variable X alone & Y alone

X alone

$$g(x) = \sum_{y} f(x,y)$$

Y alone

$$h(y) = \sum_{x} f(x,y)$$

For continuous random variable

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

eg·14 Two ballpoint pens are selected at random from box that has 3 blue pens, 2 red, 3 green pens. If X is no of blue pens selected, Y is no. of red pens selected find

a) joint probability funⁿ $f(x,y)$

b) $P[(X,Y) \in A]$. A is the region $\{(x,y) \mid x+y \le 1\}$

$f(x,y)$ Y \ X	0	1	2
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$
1	$\frac{6}{28}$	$\frac{6}{28}$	0
2	$\frac{1}{28}$	0	0

$$f(x,y) = {}^{3}C_{x}\,{}^{2}C_{y}\,{}^{3}C_{(2-x-y)} / {}^{8}C_{2}$$

$$0 \le x+y \le 2$$

$x = 0,1,2$ $\quad y = 0,1,2$

(b) $P[(X,Y) \in A] = P(X+Y \le 1)$

$= f(0,0) + f(0,1) + f(1,0)$

$= \frac{3}{28} + \frac{9}{28} + \frac{6}{28} = \frac{18}{28}$

$= \frac{9}{14}$

(c) $P((X,Y) \in A) = P(X+Y \ge 1)$

$= f(0,1) + f(1,0) + f(2,0) + f(1,1)$
$+ f(1,2) + f(2,2)$

$= \frac{9}{28} + \frac{3}{28} + \frac{6}{28} + \frac{6}{28} + \frac{1}{28}$

$= \frac{25}{28}$

1·11·22

eg·16 Find marginal distributions of X alone and Y alone of the problem of eg·14

X alone

$$g(x) = \sum_{y} f(x,y)$$

$g(0) = \sum_{y} f(0,y) = f(0,0) + f(0,1) + f(0,2)$

$= \frac{3}{28} + \frac{6}{28} + \frac{1}{28}$

$= \frac{10}{28}$

$g(1) = \sum_y f(1,y) = f(1,0) + f(1,1) + f(1,2)$

$= \frac{15}{28}$

$g(2) = \sum_y f(2,y) = \frac{3}{28}$

x	0	1	2
$g(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Y alone

$h(y) = \sum_x f(x,y)$

$h(0) = \sum_x f(x,0) = \frac{15}{28}$

$h(1) = \sum_x f(x,1) = \frac{12}{28}$

$h(2) = \sum_x f(x,2) = \frac{1}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

eg.15 A privately owned business operates both a drive-in facility & walk in facility. On a randomly selected day, let X and Y resp. be proportions of time that the drive-in & walk in facilities are in use and suppose that joint density function of these random variables is

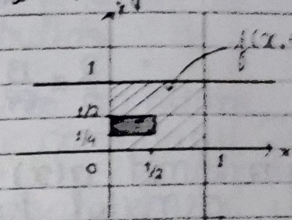
$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify condition 2

(b) Find $P[(X,Y) \in A]$ where

$A = \{(x,y) \mid 0 < x < \frac{1}{2},\ \frac{1}{4} < y < \frac{1}{2}\}$

(a) $f(x,y) \ge 0$ for all (x,y)



$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

$= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$

$= \frac{2}{5}\int_0^1 x^2 + 3xy\Big]_0^1 dy$

$= \frac{2}{5}\int_0^1 1 + 3y\,dy = \frac{2}{5}\left(y + \frac{3y^2}{2}\right)\Big]_0^1$

$= \frac{2}{5}\left(1 + \frac{3}{2}\right) = 1$

(b) $P((X,Y) \in A) = \int_{1/4}^{1/2}\int_0^{1/2} f(x,y)\,dx\,dy$

$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy = \frac{2}{5}\int_{1/4}^{1/2} x^2 + 3xy\Big]_0^{1/2} dy$

$= \frac{2}{5}\int_{1/4}^{1/2} \frac{1}{4} + \frac{3}{2}y\,dy$

$= \frac{2}{5}\left(\frac{1}{4}y + \frac{3}{4}y^2\right)_{1/4}^{1/2}$

$= \frac{2}{5}\left\{\frac{1}{8} + \frac{3}{16} - \frac{1}{16} - \frac{3}{64}\right\}$

$= \frac{2}{5}\left(\frac{8+12-4-3}{64}\right)$

$= \frac{2}{5} \times \frac{13}{64} = \frac{13}{160}$

eg 17 Find $g(x)$ & $h(y)$ for joint density function of eg-15

$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int \frac{2}{5}(2x+3y)\,dy$

$= \frac{4xy}{5} + \frac{3y^2}{5}\Big]_0^1$

$= \frac{4x+3}{5}$

$h(y) = \int_0^1 \frac{2}{5}(2x+3y)\,dx$

$= \frac{2x^2}{5} + \frac{2}{5}3xy\Big]_0^1$

$= \frac{2+6y}{5}$

$$g(x) = \begin{cases} \frac{4x+3}{5} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(y) = \begin{cases} \frac{2+6y}{5} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

5-11-22 Conditional distribution

Let X and Y be two random variables discrete & continuous. The conditional distribution of random variable Y given that X = x is

$f(y|x) = \frac{f(x,y)}{g(x)}$ provided $g(x) > 0$

Similarly conditional distribution of X given that Y = y is

$f(x|y) = \frac{f(x,y)}{h(y)}$ $h(y) > 0$

$P(a < X < b \mid Y = y) = \sum_{a<x<b} f(x|y)$ for dis random var

$P(a < X < b \mid Y = y) = \int_a^b f(x|y)\,dx$ contⁿ

$P(c < Y < d \mid X = x) = \sum_{c<y<d} f(y|x)$ random

$P(c < Y < d \mid \;\; = \;\;) = \int_c^d f(y|x)\,dy$ contⁿ

Independent random variable

$X, Y \quad f(x,y) \quad g(x) \quad h(y)$

If $f(x,y) = g(x)\,h(y)$

Ex-38 If joint probability distribution of X and Y is

$$f(x,y) = \frac{x+y}{30} \qquad x = 0,1,2,3;\quad y = 0,1,2$$

find

(a) $P(X \le 2, Y = 1)$ (c) $P(X > Y)$

(b) $P(X > 2, Y \le 1)$ (d) $P(X+Y = 4)$

(a) $P(X \le 2, Y = 1) = f(0,1) + f(1,1) + f(2,1)$
$= \frac{6}{30} = \frac{1}{5}$

(b) $P(X > 2, Y \le 1) = f(3,0) + f(3,1)$
$= \frac{3}{30} + \frac{4}{30} = \frac{7}{30}$

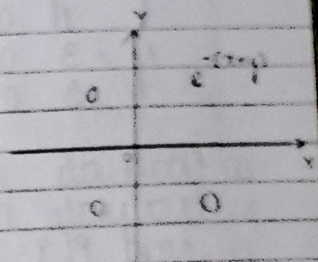
(c) $P(X > Y) = f(1,0) + f(2,0) + f(2,1) + f(3,0) + f(3,1) + f(3,2)$
$= \frac{1}{30} + \frac{2}{30} + \frac{3}{30} + \frac{3}{30} + \frac{4}{30} + \frac{5}{30}$
$= \frac{18}{30}$

(d) $P(X+Y = 4) = f(3,1) + f(2,2)$
$= \frac{8}{30}$

Ex-42 Let X and Y denote lengths of life, in years, of two components in an electronic system. If joint density function of these variables is

$$f(x,y) = \begin{cases} e^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $P(0 < X < 1 \mid Y = 2)$



$P(a < X < b \mid Y = 2)$
$= \int_a^b f(x|y)\,dx$

$f(x|y) = \frac{f(x,y)}{h(y)}$

$h(y) = \int_a^b f(x,y)\,dx = \int_{-\infty}^{\infty} f(x,y)\,dx$

$= \int_0^{\infty} e^{-(x+y)}\,dx = \int e^{-x}\cdot e^{-y}\,dx$

$= e^{-y}\int e^{-x}\,dx = e^{-y}\left[-e^{-x}\right]_0^{\infty}$

$= e^{-y}$

$f(x|y) = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x}$

$P(0 < X < 1 \mid Y = 2) = \int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1$

$= 1 - e^{-1}$

$= 0.63212$

ex.49 Let X denote no of times certain numerical control machine will malfunction : 1, 2 or 3 times on a day. Let Y denote no. of times a technician is called. Their joint probability distribution is

$f(x,y)$		$x=1$	$x=2$	$x=3$
y	1	0.05	0.05	0.1
	3	0.05	0.1	0.35
	5	0.00	0.2	0.1

a) Evaluate marginal distribution of X
b) Evaluate marginal distribution of Y
c) Find $P(Y=3 \mid X=2)$

a) $g(x) = \sum_y f(x,y)$

X	1	2	3
$g(x)$	0.1	0.35	0.55

b) $h(y) = \sum_x f(x,y)$

Y	1	3	5
$h(y)$	0.2	0.5	0.3

(c) $P(Y=3 \mid X=2) = \frac{f(x,y)}{g(2)} = \frac{f(2,3)}{g(2)}$

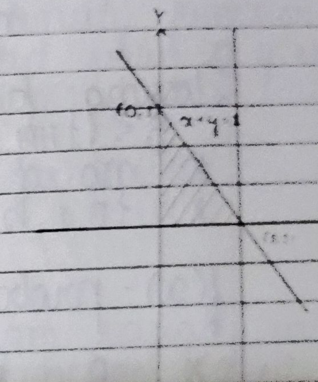
$= \frac{0.10}{0.35} = \frac{10}{35}$

$\frac{2}{7}$

ex.56 The joint density function of random variables X and Y is

$$f(x,y) = \begin{cases} 6x, & 0<x<1,\ 0<y<1-x \\ 0, & \text{elsewhere} \end{cases}$$

(a) Show X and Y are not independent
(b) Find $P(X>0.3 \mid Y=0.5)$



(a) $g(x) = \int_0^{1-x} f(x,y)\,dy$

$= \int_0^{1-x} 6x\,dy$

$= 6x(1-x)$
$= 6x - 6x^2$

$h(y) = \int_0^{1-y} 6x\,dx$

$= 3x]_0^{1-y}$

$= 3-3y$

$f(x,y) = g(x)\,h(y)$

Not independent

(b) $P(X>0.3 \mid Y=0.5) = \int_{0.3}^{0.5} f(x|y)\,dx$

$f(x|y) = \frac{f(x,y)}{h(0.5)}$

Date 7/11/22

Ch-04
Mathematical Expectation

Mean of a random variable

$x_1, x_2, \ldots x_n$

$$\text{Mean} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Tossing fair coin twice

$S = \{HH, HT, TH, TT\}$

X: no. of tails

$X = \{0, 1, 2\}$ Discrete

$f(x)$ - probability distribution funct.

X	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Mean of random variable X is the expected or avg value of X

Denoted as: $E(X) = \mu$

$$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4}$$

$$= 1$$

$$E(X) = \sum_x x f(x)$$

If X is cont^n random variable with probability density function $f(x)$ then

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$$

Theorem-1

Let X be a random variable with probability density function $f(x)$ then expected value of $g(x)$.

$$E(g(x)) = \mu_{g(x)} = \sum_x g(x) f(x)$$

$$E(g(x)) = \mu_{g(x)} = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Theorem-2

Let X, Y be random variables with prob distribution function $f(x,y)$

then $E(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y)$

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y) dx\, dy$$

Expected value of X alone

$$E(X) = \mu_x = \sum_x \sum_y x f(x,y) = \sum x g(x)$$

$$= \mu_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y) dx\, dy$$

$$= \int_{-\infty}^{\infty} x g(x) dx$$

expected value of Y alone

$E(Y) = \mu_Y = \sum_y y\,g(y)$

$E(Y) = \mu_Y = \int_{-\infty}^{\infty} y\,g(y)\,dy$

ex-4 A coin is biased such that head is three times as likely to occur as tail. Find expected no. of tails when coin is tossed twice

$P(T) = K$

$P(H) = 3K$

$P(H) + P(T) = 1$

$4K = 1$

$K = \frac{1}{4}$

$P(T) = \frac{1}{4} \quad P(H) = \frac{3}{4}$

$S = \{HH, HT, TH, TT\}$

X no. of tails

$= \{0, 1, 2\}$

X	0	1	2
f(x)	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

$E(X) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{6}{16} + 2 \cdot \frac{1}{16}$

$= \frac{1}{2}$

ex-12 If a dealer's profit, in units of $5000, on new automobile can be looked upon as random variable X having density function

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find average profit per automobile.

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{1} x f(x)\,dx + \int_{1}^{\infty} x f(x)\,dx$

$= \int_0^1 2x(1-x)\,dx = x^2 - \frac{2x^3}{3}\Big]_0^1$

$= \frac{1}{3}$

Total profit $= \$5000 \times \frac{1}{3}$

$= \$1666.67$

ex-10 Two tire-quality experts examine stacks of tires and assign quality rating to each tire on 3 point scale. Let X be rating given by expert A, Y by B. Find μ_x, μ_y

f(x,y)	y = 1	2	3
x = 1	0.1	0.05	0.02
2	0.1	0.35	0.05
3	0.03	0.1	0.20

Expected value of Y alone

$E(Y) = \mu_Y = \sum_y y g(y)$

$E(Y) = \mu_Y = \int_{-\infty}^{\infty} y g(y) dy$

ex-4 A coin is biased such that head is three times as likely to occur as tail. Find expected no. of tails when coin is tossed twice

$P(T) = K$
$P(H) = 3K$
$P(H) + P(T) = 1$
$4K = 1$
$K = \frac{1}{4}$

$P(T) = \frac{1}{4}$ $P(H) = \frac{3}{4}$

$S = \{HH, HT, TH, TT\}$
X : no. of tails
$= \{0, 1, 2\}$

X	0	1	2
f(x)	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

$E(X) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{6}{16} + 2 \cdot \frac{1}{16}$

$= \frac{1}{2}$

ex-12 If a dealer's profit, in units of \$5000, on new automobile can be looked upon as random variable X having density function

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find average profit per automobile.

$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{0} x f(x) dx + \int_{0}^{1} x f(x) dx + \int_{1}^{\infty} x f(x) dx$

$= \int_0^1 2x(1-x) dx = x^2 - \frac{2x^3}{3} \Big]_0^1$

$= \frac{1}{3}$

Total profit $= \$5000 * \frac{1}{3}$

$= \$1666.67$

ex-10 Two tire-quality experts examine stacks of tires and assign quality rating to each tire on 3 point scale. Let X be rating given by expert A, Y by B. Find μ_x, μ_y

	f(x,y)	y = 1	2	3
x	1	0.1	0.05	0.02
	2	0.1	0.35	0.05
	3	0.03	0.1	0.20

$g(x) = \sum_y f(x,y)$

$g(1) = \sum_y f(1,y)$

$g(2) = \sum_y f(2,y)$

$g(3) = \sum_y f(3,y)$

X	1	2	3
g(x)	0.17	0.50	0.33

$\mu_x = 0.17 + 2\times 0.5 + 3\times 0.33$
$= 0.17 + 1 + 0.99$
$= 2.16$

$g(y) = \sum_x f(x,y)$

Y	1	2	3
g(y)	0.23	0.5	0.27

$\mu_Y = 0.23 + 1 + 0.81$
$= 2.04$

8-11-22

20. A continuous random variable x has density function

$$f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & \text{elsewhere} \end{cases}$$

Find expected value of $g(x) = e^{2x/3}$

$E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

$= \int_{-\infty}^{0} g(x) f(x)\, dx + \int_{0}^{\infty} g(x) f(x)\, dx$

$= \int_0^\infty e^{2x/3} e^{-x}\, dx = \int_0^\infty e^{-x/3}\, dx$

$= \left.\frac{e^{-x/3}}{(-1/3)}\right]_0^\infty = -3e^{-x/3}\Big]_0^\infty = 3$

23. Suppose X and Y have following joint probability function:

f(x,y)		x = 2	x = 4
y	1	0.1	0.15
	3	0.2	0.3
	5	0.1	0.15

(a) Find expected value of $g(X,Y) = XY^2$

(b) Find μ_X and μ_Y

Here X, Y are two discrete random variable with probability distribution function $f(x,y)$

$E(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y)$

$= \sum_x \sum_y xy^2 f(x,y)$

$= 2(0.1) + 18(0.2) + 50(0.1) + 4(0.15)$
$+ 36(0.3) + 100(0.15)$
$= 35.2$

(b) $\mu_x = \sum_x x\, g(x)$

X	2	4
$g(x)$	0.4	0.6

$g(x) = \sum_y f(x,y)$

$\mu_x = 2\times 0.4 + 4 \times 0.6$
$= 0.8 + 2.4$
$= 3.2$

$\mu_y = \sum_y y\, g_1(y)$

$g_1(y) = \sum_y f(x,y)$

Y	1	3	5
$g_1(y)$	0.25	0.5	0.25

$\mu_y = 0.25 + 1.5 + 1.25$
$= 3$

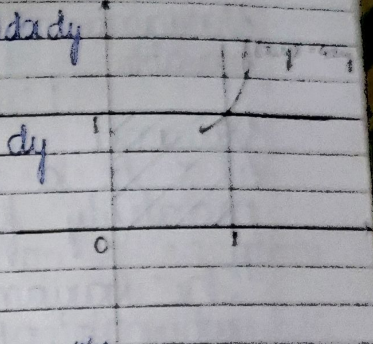
26. Let X and Y be random variables with joint density function

$$f(x,y) = \begin{cases} 4xy & 0<x,\ y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of $Z = \sqrt{X^2+Y^2}$

$$E(Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2}\, f(x,y)\, dx\, dy$$



$$= \int_0^1\int_0^1 \sqrt{x^2+y^2}\, 4xy\, dx\, dy$$

Let $x^2+y^2 = t$
$2x\,dx = dt$

$$y\int_{y^2}^{y^2+1} 2\sqrt{t}\, dt = 2y \frac{t^{3/2}}{(3/2)}\Big]_{y^2}^{y^2+1}$$

$$= \frac{4y}{3}\left[(y^2+1)^{3/2} - y^3\right]$$

$$= \int_0^1 \frac{4y}{3}\left[(y^2+1)^{3/2} - y^3\right] dy$$

$$= \frac{4}{3}\int_0^1 y(y^2+1)^{3/2} - y^4\ dy$$

$y^2+1 = w$
$2y\,dy = dw$

$$= \frac{4}{3}\int_1^2 w^{3/2}\frac{dw}{2} - \frac{4}{3}\int_0^1 y^4\, dy$$

$$= \frac{2}{3}\frac{w^{5/2}}{5/2}\Big]_1^2 - \frac{4}{3}\frac{y^5}{5}\Big]_0^1$$

$$= \frac{4}{15}(2^{5/2}-1) - \frac{4}{3}\times\frac{1}{5}$$

$$= \frac{4}{15}(2^{5/2}-2) = \frac{8}{15}(2^{3/2}-1)$$

Date 9 | 11 | 22

Variance and covariance of random variables

Variance

Let X be a random variable with probability function $f(x)$ and mean μ

The variance of X is σ^2

where $\sigma^2 = E((X-\mu)^2)$

For discrete

$$\sigma^2 = \sum_x (x-\mu)^2 f(x) , \quad \mu = \sum_x x f(x)$$

For continuous

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

Note

Let X be random variable with prob function $f(x)$,

$$\sigma^2 = E(X^2) - \mu^2$$

Prove that $\sigma^2 = E(X^2) - \mu^2$, μ is mean

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) f(x)$$

$$= \sum_x x^2 f(x) - \sum_x 2\mu x f(x) + \sum_x \mu^2 f(x)$$

$$= E(X^2) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x)$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

Standard deviation of random variable.

Suppose X is random variable +ve square root of σ^2 is standard deviation

$$\sigma = +\sqrt{\sigma^2}$$

Theorem

X be random variable, prob. funⁿ $f(x)$, mean μ_x then variance of any function $g(X)$ is $\sigma^2_{g(x)}$

For discrete

$$\sigma^2_{g(x)} = \sum (g(x) - \mu_x)^2 + f(x)$$

For continuous

$$\sigma^2_{g(x)} = \int_{-\infty}^{\infty} (g(x) - \mu_x)^2 f(x)\, dx$$

eg-9 Let random variable X represent no. of defective parts for machine when 3 parts are sampled.

x	0	1	2	3
$f(x)$	0.51	0.38	0.1	0.01

Calculate σ^2 a) using defⁿ

b) using $\sigma^2 = E(X^2) - \mu^2$

$\sigma^2 = \sum_x (x-\mu)^2 f(x)$ where $\mu = \sum_x x f(x)$

$\mu = \sum_x x f(x)$

$= 0\times 0.51 + 1\times 0.38 + 2\times 0.1 + 3\times 0.01$

$= 0.61$

$\sigma^2 = \sum_x (x-\mu)^2 f(x)$

$= (0-0.61)^2\times 0.51 + (1-0.61)^2\times 0.38 + (2-0.61)^2\times 0.1 + (3-0.61)^2\times 0.01$

$= 0.4979$

(b) $\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \sum_x x^2 f(x)$

$= 0 + 1^2\times 0.38 + 2^2\times 0.1 + 3^2\times 0.01$

$= 0.38 + 0.4 + 0.09$

$= 0.87$

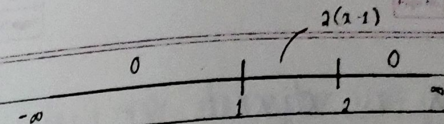
$\sigma^2 = 0.87 - (0.61)^2$

$= 0.4979$

eg-10 Weekly demand for drinking water in 1000s of L from local chain of efficiency stores is cont" random variable x having probability density

$$f(x) = \begin{cases} 2(x-1) & 1<x<2 \\ 0 & \text{elsewhere} \end{cases}$$

Find mean & variance of x



Mean of X is μ

$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_1^2 x f(x)\,dx = 2\int_1^2 x(x-1)\,dx$

$= \left[\frac{2x^3}{3} - x^2\right]_1^2 = \frac{5}{3}$

$\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_1^2 2x^2(x-1)\,dx$

$= \left[\frac{2x^4}{4} - \frac{2x^3}{3}\right]_1^2$

$= 8 - \frac{16}{3} - \frac{2}{4} + \frac{2}{3}$

$= \frac{17}{6}$

$\sigma^2 = \frac{17}{6} - \frac{25}{9} = \frac{1}{18}$

12-11-22

Theorem

Let X be random variable with probability function $f(x)$. Variance of random variable $g(x)$ is

$\sigma^2_{g(x)} = E((g(x) - \mu_{g(x)})^2)$

$\sigma^2_{g(x)} = \sum_x (g(x) - \mu_{g(x)})^2 f(x)$

If X is discrete
$$\sigma^2_{g(x)} = \int_{-\infty}^{\infty} (g(x)-\mu_{g(x)})^2 f(x)\,dx$$

If X is continuous

Covariance
Let X and Y are two random variables with probability distribution f(x, y) then covariance of X and Y is
$$\sigma_{XY} = E((X-\mu_X)(Y-\mu_Y))$$
$$= \sum_x \sum_y (x-\mu_x)(y-\mu_y) f(x,y)$$
$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_Y) f(x,y)\,dx\,dy$$

Note:
$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

Corelation coefficients
Let X and Y are two random variables with probability funⁿ f(x, y) The co-relation coefficient of X and Y is ρ_{XY}
$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, \quad -1 \le \rho_{XY} \le 1$$

* If $\rho_{XY} = 0$ then X & Y are indep and not related

* If ρ_{XY} is +ve X inc, Y inc or X ↓, Y ↓

* If ρ_{XY} is -ve X↑ Y↓ or X↓ Y↑

eg-11 Calculate variance of g(X) = 2X + 3. X is random varibable

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

$$\sigma^2_{g(X)} = E((g(X) - \mu_{g(X)})^2)$$
$$= \sum_x (g(X) - \mu_{g(X)})^2 f(x)$$
$$\mu_{g(X)} = \sum_x g(x) f(x)$$
$$= \sum_x (2X+3) f(x)$$
$$= \sum_x (2x+3) f(x)$$
$$= 3\cdot\frac{1}{4} + 5\cdot\frac{1}{8} + 7\cdot\frac{1}{2} + 9\cdot\frac{1}{8}$$
$$= \frac{5+28+9}{8} = 6$$
$$\sigma^2_{g(X)} = \sum_x (2x+3-6)^2 f(x)$$
$$= \sum_x (2x-3)^2 f(x)$$
$$= 9\times\frac{1}{4} + 1\times\frac{1}{8} + 1\times\frac{1}{2} + 9\times\frac{1}{8}$$
$$= 4$$

eg. 15 Find correlation coefficient between X and Y in 13.

$f(x,y)$		$x=0$	1	2	$h(y)$
y	0	3/28	9/28	3/28	15/28
	1	3/14	3/14	0	3/7
	2	1/28	0	0	1/28
	$g(x)$	5/14	15/28	3/28	1

σ_X : standard deviation of X alone
σ_Y : standard deviation of Y alone
σ_{XY} : covariance

$$\sigma_{XY} = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y)$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

$$\mu_X = \sum_x x g(x) = \frac{21}{28} = \frac{3}{4}$$

$$\mu_Y = \sum_y y h(y) = \frac{14}{28} = \frac{1}{2}$$

$$E(XY) = \sum_x \sum_y xy f(x,y)$$
$$= 1 \cdot \frac{3}{14} + 0 = \frac{3}{14}$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$
$$= \frac{3}{14} - \frac{3}{4} \cdot \frac{1}{2}$$
$$= \frac{3}{14} - \frac{3}{8} = \frac{-9}{56}$$

$$\sigma_X^2 = E(X)^2 - \mu_X^2$$

$$E(X^2) = \sum x^2 g(x)$$
$$= 0 + \frac{15}{28} + \frac{12}{28} = \frac{27}{28}$$

$$E(Y^2) = \sum y^2 h(y)$$
$$= 0 + \frac{3}{7} + \frac{4}{28} = \frac{16}{28}$$

$$\sigma_X^2 = \frac{27}{28} - \frac{9}{16} = \frac{45}{112}$$

$$\sigma_X = \sqrt{\frac{45}{112}}$$

$$\sigma_Y^2 = \frac{16}{28} - \frac{1}{4} = \frac{9}{28}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-9/56}{\sqrt{45/112} \times \sqrt{9/28}}$$

$$= \frac{-1}{\sqrt{5}} = -0.447$$

14-11-22

Mean and variance of linear combination of random variable

X is random variable with prob distribution function $f(x)$

$$E(X) = \mu = \sum_x x f(x)$$

$$E(X) = \mu = \int_{-\infty} x f(x)$$

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$E(X^2) = \sum_x x^2 f(x)$$

$$E(X^2) = \int_{\infty} x^2 f(x)\,dx$$

Theorem-1

If a, b are constants then

$$E(aX+b) = \mu_{ax+b} = aE(X) + b$$

eg- Determine mean of random variable $f(x) = 2X-1$, where X is random variable

X	4	5	6	7	8	9
$f(x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\mu_{2X-1} = E(2X-1) = 2E(X) - 1$$

$$E(X) = \sum_x x f(x) = \frac{4}{12} + \frac{5}{12} + \frac{6}{4} + \frac{7}{4} + \frac{8}{6} + \frac{9}{6}$$

$$= \frac{82}{12} = \frac{41}{6}$$

$$\mu_{2X-1} = 2\left(\frac{41}{6}\right) - 1 = \frac{41}{3} - 1$$

$$= \frac{38}{3} = 12.67$$

Theorem-2

Suppose X is random variable with two fun^n $g(x)$ and $h(x)$ then

$$E(g(X) \pm h(X)) = E(g(x)) \pm E(h(x))$$

Find mean of random variable $g(X) = 4X+3$ if X is random variable with prob density fun^n

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$E(g(x)) = E(4X+3)$$
$$= E(4X) + E(3)$$
$$= 4E(X) + 3$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{2} \frac{x^3}{3}\,dx = \left.\frac{x^4}{12}\right]_{-1}^{2}$$

$$= \frac{15}{12} = \frac{5}{4}$$

$$E(g(x)) = 8$$

Theorem-3

X and Y are two random variables with prob function $f(x,y)$

$$E(g(X,Y) \pm h(X,Y))$$
$$= E(g(X,Y)) \pm E(h(X,Y))$$

Note:
If X and Y are two independent random variable then

$$E(XY) = E(X)E(Y)$$

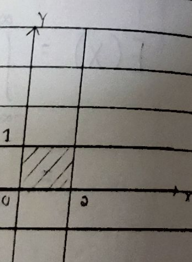
or $\sigma_{XY} = 0$

or $\rho_{XY} = 0$

eg-21 Ratio of gallium to arsenide does affect functioning of gallium-arsenide wafers. X denotes ratio of Ga to arsenide, Y is functional wafers retrieved during 1-hour period. X, Y are independent

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0<x<2,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$$



$$= \int_0^1\int_0^2 x^2 y \frac{(1+3y^2)}{4}\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3}\cdot\frac{y}{4}(1+3y^2)\right]_0^2 dy$$

$$= \int_0^1 \frac{2}{3}(y+3y^3)\,dy$$

$$= \frac{2}{3}\left(\frac{y^2}{2} + \frac{3y^3}{4}\right)_0^1 = \frac{2}{3}\left(\frac{1}{2}+\frac{3}{4}\right)$$

$$= \frac{5}{6}$$

$$E(X) = \int_{-\infty}^{\infty} x\,g(x)\,dx$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \frac{x}{4}(y+y^3)_0^1$$

$$= \frac{x}{2}$$

$$g(x) = \begin{cases} x/2, & 0<x<2 \\ 0 & \text{else} \end{cases}$$

$$E(X) = \int_0^2 x\cdot\frac{x}{2}\,dx = \left.\frac{x^3}{6}\right]_0^2 = \frac{4}{3}$$

$$h(Y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$= \frac{3y^2+1}{2}$$

$$E(Y) = \int_0^1 \frac{3y^3+y}{2}\,dy = \frac{1}{2}\left[\frac{3y^4}{4}+\frac{y^2}{2}\right]_0^1$$

$$= \frac{5}{8}$$

$$E(X)E(Y) = \frac{4}{3}\times\frac{5}{8} = \frac{5}{6} = E(XY)$$

Theorem-4

If X and Y are random variables with joint probability distribution f(x, y) and a, b, c are constants then $\sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}$

Corollary

If $b = 0$ $\sigma^2_{aX+b} = a^2\sigma_X^2 = a^2\sigma^2$

If $a = 1, b = 0,$ $\sigma^2_{X+c} = \sigma_X^2 = \sigma^2$

If $b = 0, c = 0,$ $\sigma^2_{aX} = a^2\sigma_X^2 = a^2\sigma^2$

Note

If X and Y are independent random variable then

$\sigma^2_{aX+bY+c} = a^2\sigma_X^2 + b^2\sigma_Y^2$

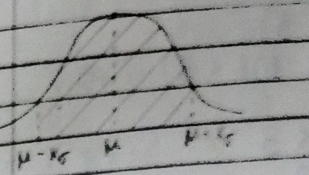
$\sigma^2_{aX-bY} = a^2\sigma_X^2 + b^2\sigma_Y^2$

If $X_1, X_2, \ldots X_n$ are independent then $\sigma^2_{a_1X_1+a_2X_2+\ldots a_nX_n} = a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2$

Theorem-5 - Chebyshev's Theorem

Probability that any random variable X will assume a value K standard deviation of mean is atleast $1 - \frac{1}{K^2}$

$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$



eg: 22 If X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 4$ & covariance $\sigma_{XY} = -2$. find variance of $Z = 3X - 4Y + 8$

$\sigma_Z^2 = \sigma^2_{3X-4Y+8} = \sigma^2_{3X-4Y}$

$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY}$

$= 9(2) + 16(4) - 24(-2)$

$= 130$

ex-77 X is random variable with mean $\mu = 10$, variance $\sigma^2 = 4$. Using Chebyshev's theorem, find

(a) $P(|X - 10| \geq 3)$

(b) $P(|X - 10| < 3)$

(c) $P(5 < X < 15)$

(d) value of constant c such that
$P(|X - 10| \geq c) \leq 0.04$

$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2}$

$\mu = 10, \quad \sigma^2 = 4 \Rightarrow \sigma = 2$

(a) $P(|X-10| \geq 3)$

$= 1 - P(|X-10| < 3)$

$-3 < X - 10 < 3$

$\Rightarrow 7 < X < 13$

$= 1 - P(7 < X < 13)$

$10 - 2K < X < 10 + 2K$

$10 - 2K = 7 \Rightarrow K = 3/2$

$10 + 2K = 13 \Rightarrow K = 3/2$

$P(7 < X < 13) \geq 1 - \frac{1}{K^2} = \frac{5}{9}$

$P(|X-10| \geq 3) \leq 1 - \frac{5}{9} = \frac{4}{9}$

(b) $P(|X-10| < 3) \geq \frac{5}{9}$

(c) $P(5 < X < 15)$

$10 - 2K = 5 \Rightarrow K = 5/2$

$10 + 2K = 15 \Rightarrow K = 5/2$

$P(5 < X < 15) \geq \frac{21}{25}$

(d) $P(|X-10| \geq C) \leq 0.04$

$1 - P(|X-10| < C) \leq 0.04$

$P(|X-10| < C) \geq 0.96$

$P(10 - C < X < 10 + C) \geq 0.96$

$1 - \frac{1}{K^2} = 0.96 \Rightarrow \frac{1}{K^2} = 0.04$

$\Rightarrow K^2 = \frac{1}{0.04} \Rightarrow K = 5$

$\mu - 5\sigma = 10 - C$

$10 - 10 = 10 - C$

$C = 10$

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Ch-05

Some Discrete Probability Distribution

Binomial Distribution

A bernouli's trial can result in success with probability p and a failure with probability $q = 1-p$. Then probability distribution of random variable x, the no. of success in n independent trials is

$$b(x; n, p) = {}^{n}C_{x}\, p^{x} q^{n-x} \qquad x = 0, 1, \ldots n$$

Mean and variance of binomial distribution is

$$\mu = np$$
$$\sigma^2 = npq$$

eg-1 The probability that certain kind of component will survive shock test is 3/4. Find probability that exact 2 of next 4 components tested survive

X: no of components survive

$n = 4 \qquad p = \frac{3}{4} \qquad q = \frac{1}{4}$

$$P(X=2) = b(2; n, p) = b\left(2; 4, \tfrac{3}{4}\right)$$

$$= {}^{4}C_{2}\left(\frac{3}{4}\right)^{2}\left(\frac{1}{4}\right)^{2} = \frac{4!}{2!\,2!} \times \frac{9}{16} \times \frac{9}{16}$$

$$= \frac{27}{128}$$

eg-2 Probability that a patient recovers from rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that (a) at least 10 survive (b) from 3 to 8 survive (c) exactly 5 survive?

X: no. of people survive

$n = 15 \qquad p = 0.4 \qquad q = 0.6$

(a) $P(X \geq 10) = 1 - P(X < 10)$

$$= \sum_{x=10}^{15} b(x; n, p)$$

$= b(10; 15, 0.4) + b(11; 15, 0.4) + b(12; 15, 0.4) + b(13; 15, 0.4) + b(14; 15, 0.4) + b(15; 15, 0.4)$

$= 0.0244 + 0.0074 + 0.0016 + 0.00025 + 0.000024 + 0.0000026$

$= 0.0338$

$P(X \geq 10) = 1 - P(X < 10)$

$= 1 - 0.9662$

$= 0.0338$

(b) $P(3 \leq X \leq 8) = \sum_{x=0}^{8} (x; n, p) - \sum_{x=0}^{2} (x; n, p)$

$= 0.9050 - 0.0271$

$= 0.8779$

(c) $P(X = 5) = \sum_{x=0}^{5} (x; n, p) - \sum_{x=0}^{4} (x; n, p)$

$= 0.4032 - 0.2173$

$= 0.1859$

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Multinomial Distribution

If a given trial results in k outcomes $E_1, E_2, \dots E_k$ with probabilities $P_1, P_2, \dots P_k$ then the probability distribution of random variables $X_1, X_2, \dots X_k$ denotes no. of occurances of $E_1, E_2, \dots E_k$ in m independent trial is

$$f(x_1, x_2, \dots x_k, P_1, P_2, \dots P_k, n)$$
$$= {}^mC_{x_1 x_2 \dots x_k} P_1^{x_1} P_2^{x_2} \dots P_k^{x_k}$$

where $\sum_{i=1}^{k} x_i = m$, $\sum_{i=1}^{k} P_i = 1$ and

$${}^mC_{x_1 x_2 \dots x_k} = \frac{m!}{x_1! x_2! \dots x_k!}$$

eg. 7 Complexity of arrivals & departures of planes at airport is such that computer simulation is often used to model ideal conditions. For certain airport with 3 runways, it is known that in ideal setting following are probabilities that individual runways are accessed by randomly arriving commerical jet

$R_1 : P_1 = 2/9$ $\quad R_2 : P_2 = 1/6$ $\quad R_3 : P_3 = 11/18$

What is the prob that 6 randomly arriving planes are distributed in following fashion?

$R_1 : 2$ $\quad R_2 : 1$ $\quad R_3 : 3$

$$P(X_1 = 2, X_2 = 1, X_3 = 3) = {}^nC_{x_1 x_2 x_3} P_1^{x_1} P_2^{x_2} P_3^{x_3}$$
$$= {}^6C_{2,1,3} \left(\frac{2}{9}\right)^2 \left(\frac{1}{6}\right) \left(\frac{11}{18}\right)^3$$
$$= \frac{6!}{2!\,1!\,3!} \left(\frac{4}{81}\right) \left(\frac{1}{6}\right)^1 \left(\frac{11}{18}\right)^3$$
$$= 0.1127$$

ex - 22 According to genetics theory, certain cross of guinea pigs will result in red, black, white offspring in ratio 8 : 4 : 4. Find probability that among 8 offspring 5 will be red, 2 black, 1 white

$m = 8$ $\quad x_1 = 5, x_2 = 2, x_3 = 1$

$P_1 = \frac{1}{2}$ $\quad P_2 = \frac{1}{4}$ $\quad P_3 = \frac{1}{4}$

$$P(X_1 = 5, X_2 = 2, X_3 = 1) = {}^nC_{x_1 x_2 x_3} P_1^{x_1} P_2^{x_2} P_3^{x_3}$$
$$= {}^8C_{5,2,1} \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^1$$
$$= \frac{8!}{5!\,2!\,1!} \times \frac{1}{32} \times \frac{1}{64}$$
$$= 0.082$$

15. It's known that 60% of mice inoculated with serum are protected from certain disease. If 5 mice are inoculated, find prob that
a) none contracts disease
b) fewer than 2 contract disease
c) more than 3 contract disease

9, 11, 15, 16, 17, 22

$n = 5$

X: no. of mice contracted the disease

$p = 40\% = 0.4 \qquad q = 0.6$

$P(X = x) = {}^{n}C_x\, p^x q^{n-x} \qquad x = 0, 1 \ldots 5$

(a) $P(X = 0) = {}^{5}C_0 (0.4)^0 (0.6)^5$

$= (0.6)^5$

$= 0.07776$

(b) $P(X < 2) = P(X = 0) + P(1)$

$= 0.07776 + {}^{5}C_1 (0.4)^1 (0.6)^4$

$= 0.07776 + 5(0.4)(0.6)^4$

$= 0.33696$

(c) $P(X > 3) = 1 - P(X \leq 3)$ or $P(X = 4) + P(X = 5)$

$= {}^{5}C_4 (0.4)^4 (0.6) + {}^{5}C_5 (0.4)^5 (0.6)^0$

$= 5(0.4)^4(0.6) + (0.4)^5$

$= 0.0768 + 0.01024$

$= 0.08704$

22-11-22

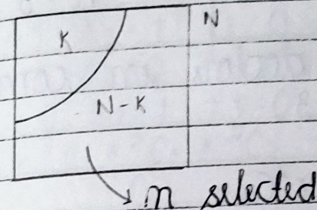
Hypergeometric Distribution

The probability distribution of the hypergeometric distribution X the no. of success in random sample size of n selected from N items of which K are success and $N-K$ are failure is

$$P(X = x) = h(x; N, n, K) = \frac{{}^{K}C_x\, {}^{N-K}C_{n-x}}{{}^{N}C_n}$$

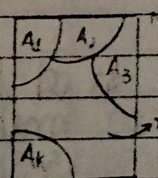
Mean $\mu = n \cdot \frac{K}{N}$

Variance $\sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{K}{N} \cdot \left(1 - \frac{K}{N}\right)$



Note:
Hypergeometric distribution is a process of without replacement

Multivariate hypergeometric distribution



If N items can be partitioned into $A_1 \ldots A_k$ with $a_1, a_2 \ldots a_k$ elements then probability distribution of random variables $X_1, X_2 \ldots X_k$ representing no. of elements selected from $A_1, A_2, \ldots A_k$ in random sample size n is

$P(X_1 = x_1, X_2 = x_2 \ldots X_k = x_k)$

$= f(x_1 \ldots x_k; a_1 \ldots a_k, N, n)$

$$= \frac{{}^{a_1}C_{x_1}\, {}^{a_2}C_{x_2} \cdots {}^{a_k}C_{x_k}}{{}^{N}C_n}, \qquad \sum_{i=1}^{k} a_i = N, \quad \sum_{i=1}^{k} x_i = n$$

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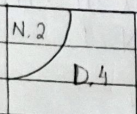
31. A random committee of size 3 is selected from 4 doctors, 2 nurses. Write formula for probability distribution of random variable x representing no. of doctors. Find $P(2 \le X \le 3)$



$N = 6$
$n = 3$

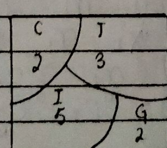
X: no. of doctors in committee
$x = 1, 2, 3$

$$P(X = x) = \frac{{}^4C_x \times {}^2C_{3-x}}{{}^6C_3}$$

$$P(2 \le X \le 3) = P(X = 2) + P(X = 3)$$
$$= \frac{{}^4C_2 \times {}^2C_1}{{}^6C_3} + \frac{{}^4C_3 \times {}^2C_0}{{}^6C_3}$$
$$= \frac{3}{5} + \frac{1}{5} = \frac{4}{5}$$

43. A foreign student club lists as its members 2 Canadians, 3 Japanese, 5 Italians, 2 Germans. If committee of 4 is selected, find prob. that
a) all nationalities are represented
b) all nationalities except Italian

$N = 12$
$n = 4$
$A_1 = C,\ A_2 = J$
$A_3 = I,\ A_4 = G$
$a_1 = 2,\ a_2 = 3,\ a_3 = 5,\ a_4 = 2$



31, 32, 43, 44, 47

X_1 = no. of persons represented Canada

a) $P(X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1)$
$$= \frac{{}^2C_1 \times {}^3C_1 \times {}^5C_1 \times {}^2C_1}{{}^{12}C_4}$$
$$= \frac{4}{33}$$

b) $P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2) +$
$P(X_1 = 1, X_2 = 2, X_3 = 0, X_4 = 1) +$
$P(X_1 = 2, X_2 = 1, X_3 = 0, X_4 = 1)$
$$= \frac{{}^2C_1 \times {}^3C_1 \times {}^5C_0 * {}^2C_2}{{}^{12}C_4} + \frac{{}^2C_1 \times {}^3C_2 \times {}^2C_1}{{}^{12}C_4}$$
$$+ \frac{{}^2C_2 \times {}^3C_1 \times {}^2C_1}{{}^{12}C_4}$$
$$= \frac{8}{165}$$

26-11-22

Negative binomial and geometric distribution

Negative binomial distribution

If repeated independent trials can result in a success probability p and failure probability is $q = 1 - p$ then probability distribution of random variable x which is no. of trial on which k^{th} success is

$$b^*(x; k, p) = {}^{x-1}C_{k-1}\, p^{k} q^{x-k}$$

$x = k, k+1, \ldots$

In particular $k = 1$ the -ve binomial is called geometric distribution and its probability funⁿ is

$$g(x, p) = p q^{x-1}, \quad x = 1, 2, 3 \ldots$$

Note:

Mean and variance of geometric distribution is $\mu = \frac{1}{p}$

$$\sigma^2 = \frac{1-p}{p^2}$$

49. Probability that person living in a certain city owns dog is estimated to be 0.3. Find probability that 10th person randomly interviewed in city is 5th one to own a dog

X : person interviewed owns a dog

$P = 0.3 \qquad q = 0.7$

$x = 10 \qquad K = 5$

$$b^*(x; k, p) = {}^{x-1}C_{k-1}\, p^k q^{x-k}$$

$$= {}^{9}C_4 (0.3)^5 (0.7)^5$$

$$= \frac{9!}{4!\,5!} (0.3)^5 (0.7)^5$$

$$= 0.0514$$

50. Find probability that a person flipping coin gets

a) third head on seventh flip

b) first head on fourth flip

X : no. of flips on which head comes

a) $P = 0.5 \qquad q = 0.5$

$x = 7 \qquad K = 3$

$$b^*(7; 3, 0.5) = {}^{6}C_2 (0.5)^3 (0.5)^4$$

$$= 0.117$$

b) $x = 4 \qquad K = 1$

$$b^*(4; 1, 0.5) = {}^{3}C_0 (0.5)^1 (0.5)^3$$

$$= (0.5)^4$$

$$= 0.0625$$

51. Three people toss a fair coin and odd one pays for coffee. If coins all turn up same they are tossed again. Find probability that fewer than 4 tosses are needed

X : no. of tosses when 1 pays for coffee

$p(x) = \frac{6}{8} = \frac{3}{4} \qquad q = \frac{2}{8} = \frac{1}{4}$

$k = 1 \qquad x = 1, 2, 3$

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$$\sum_{x=1}^{3} b^*(x;k,p) = {}^{0}C_0\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^0 + {}^{1}C_0\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^1 + {}^{2}C_0\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^2$$

$$= \frac{3}{4} + \frac{3}{16} + \frac{3}{64} = \frac{63}{64} = 0.984$$

(or)

$$\sum_{x=1}^{3} g(x,p)$$

70. A company purchases large lots of certain kind of electronic device. A method is used that rejects a lot if 2 or more defective units are found in random sample of 100 units.

a) What is the mean no. of defective units found in 100 units if lot is 1% defective

b) What is variance?

Mean $\mu = np$

Variance $\sigma^2 = npq$

$\mu = 1$

$\sigma^2 = 0.9$

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Poisson's distribution

Probability distribution of poisson's random variable x, representing no. of outcomes occuring in given interval of time or specified region denoted by t is

$$P(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0,1,2,\ldots$$

where λ is average no. of outcomes per unit time, distance, area, vol, etc

$\mu = \lambda t$

Mean & variance of poisson's distribution are λt or np

Note

Let X be a random variable with probability distribution $b(x;n,p)$

$$b(x;n,p) \xrightarrow{n\to\infty} P(x;\mu)$$

58. A certain area of eastern US is on average hit by 6 hurricanes a year. Find probability that in given year that area will be hit by

a) fewer than 4 hurricanes

b) from 6 to 8 hurricanes

X: no. of hurricanes hit particular area in U.S

$\lambda = 6 \quad t = 1$

(a) $P(X < 4) = P(X=0) + P(X=1) + P(X=2) + P(3)$

From Poisson's distribution

$$P(x;\mu) = \frac{e^{-\mu}\mu^{x}}{x!} \quad x = 0, 1, 2, 3$$

$$P(X<4) = \frac{e^{-6}6^{0}}{0!} + \frac{e^{-6}6^{1}}{1!} + \frac{e^{-6}6^{2}}{2!} + \frac{e^{-6}6^{3}}{3!}$$

$$= e^{-6}\left(1 + 6 + \frac{36}{2} + \frac{216}{6}\right)$$

$$= e^{-6}(61)$$

$$= 0.151$$

b) $P(6 \leq X \leq 8) = P(X \leq 8) - P(X < 6)$

$$= 0.8472 - 0.4457$$

$$= 0.4015$$

(or)

$$= e^{-6}\left(\frac{6^{6}}{6!} + \frac{6^{7}}{7!} + \frac{6^{8}}{8!}\right) = e^{-6}(162)$$

$$= 0.4015$$

69. Probability that person will die when he contracts virus infection is 0.001. Of next 4000 people infected, what is the mean no. who will die?

$$\mu = np$$

$$= 4000 \times 0.001$$

$$= 4$$

Ch - 06 Continuous Probability
Some Distributions

1. Uniform distribution
2. Normal distribution
3. Exponential distribution
4. Gamma Distribution

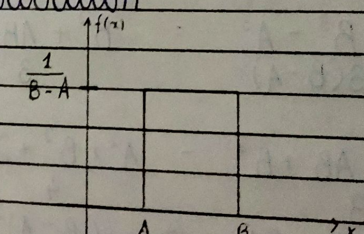
1. Uniform Distribution

Let X be continuous random variable, density function of the random variable X on the interval $[A, B]$ is

$$f(x) = \begin{cases} \frac{1}{B-A}, & A \leq X \leq B \\ 0, & \text{otherwise} \end{cases}$$

is called continuous uniform distribution

Otherwise called as rectangular distribution



$$\mu = \frac{A+B}{2}$$

$$\sigma^2 = \frac{(B-A)^2}{12}$$

$$\sigma = \frac{B-A}{...}$$

Find mean & variance of uniform distribution on interval [A, B]

$X, f(x), \mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$$f(x) = \begin{cases} \frac{1}{B-A}, & A \le X \le B \\ 0, & \text{else} \end{cases}$$

$-\infty \quad A \quad B \quad \infty$

$$\mu = \int_A^B x f(x)\,dx = \int_A^B \frac{x}{B-A}\,dx$$

$$= \frac{1}{B-A}\,\frac{x^2}{2} = \frac{B^2-A^2}{(B-A)2} = \frac{A+B}{2}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_A^B \frac{x^2}{B-A}\,dx$$

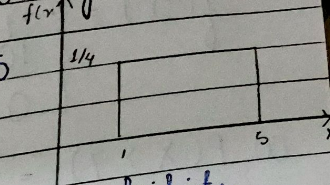
$$= \frac{B^3-A^3}{3(B-A)} = \frac{B^2+AB+A^2}{3}$$

$$\sigma^2 = \frac{A^2+AB+B^2}{3} - \frac{A^2+B^2+2AB}{4}$$

$$= \frac{A^2+B^2-2AB}{12} = \frac{(B-A)^2}{12}$$

$\sigma = \frac{B-A}{2\sqrt{3}}$ (only +ve value)

ex-2 Suppose X follows contⁿ uniform distribution from 1 to 5. Determine conditional probability $P(X>2.5 \mid X \le 4)$



$$f(x) = \begin{cases} \frac{1}{4}, & 1 \le X \le 5 \\ 0, & \text{else} \end{cases}$$

From conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(X>2.5 \mid X \le 4) = \frac{P((X>2.5) \cap (X \le 4))}{P(X \le 4)}$$

$$= \frac{P(2.5 \le X \le 4)}{P(X \le 4)} \qquad = \int_{2.5}^{4} f(x)\,dx$$

$$= \frac{1}{4}\,x\Big]_{2.5}^{4} = \frac{1}{4}(4-2.5) = \frac{1.5}{4}$$

$$= \frac{3}{8}$$

$$P(X \le 4) = \int_{-\infty}^{4} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$$

$$= \frac{1}{4}\,x\Big]_{-\infty}^{1} + \frac{1}{4}\,x\Big]_{1}^{4}$$

$$= \frac{3}{4}$$

$$P(X>2.5 \mid X \le 4) = \frac{1}{2}$$

4. A bus arrives every 10 min at bus stop. It's assumed that waiting time for particular individual is random variable

a) Probability that individual waits more than 7 min?
b) Between 2 and 7 minutes?

$$f(x) = \begin{cases} \frac{1}{10}, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

a) $P(X > 7) = \int_7^\infty f(x)\,dx = \int_7^{10} f(x)\,dx + \int_{10}^{\infty} f(x)\,dx$

$= \frac{1}{10} x \Big]_7^{10} = \frac{3}{10}$

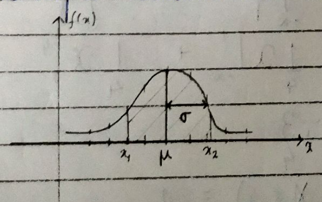
b) $P(2 < X < 7) = \int_2^7 f(x)\,dx = \frac{x}{10}\Big]_2^7$

$= \frac{5}{10} = \frac{1}{2}$

30-11-22

Normal Distributions

A cont^n random variable x having bell shape distribution

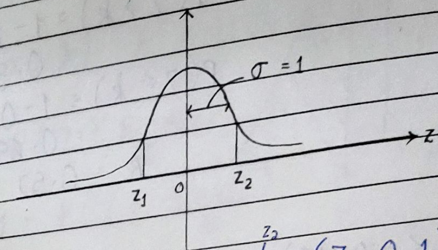


Density fun^n of normal distribution is $n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $-\infty < x < \infty$

$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma)\,dx$

Standard normal distribution (Z)

$Z = \frac{X-\mu}{\sigma}$ μ = mean
σ = standard deviation
$\mu = 0,\ \sigma = 1$

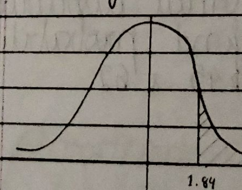


$P(Z_1 < Z < Z_2) = \int_{z_1}^{z_2} n(z;0,1)\,dz$

$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz$

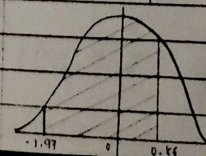
eg-2 Given a standard normal distribution, find area under curve that lies
a) to right of $z = 1.84$
b) bet^n $z = -1.97$, $z = 0.86$

(a)



$P(Z > 1.84) = 1 - P(Z \le 1.84)$
$= 1 - 0.9671$
$= 0.0329$

(b)



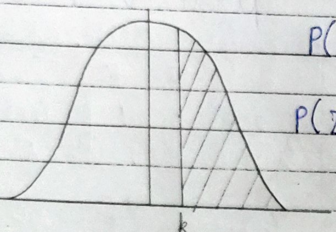
$P(-1.97 < Z < 0.86) =$
$P(Z < 0.86) - 1(Z < -1.97)$
$= 0.8051 - 0.0744$
$= 0.7807$

eg-3 Given standard normal distribution find k such that

a) $P(Z > k) = 0.3015$

b) $P(k < Z < -0.18) = 0.4197$

(a)



$P(Z > k) = 1 - P(Z \leq k)$

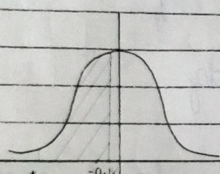
$= 0.3015$

$P(Z \leq k) = 1 - 0.3015$

$= 0.6985$

$k = 0.52$

(b) $P(k < Z < -0.18) = 0.4197$

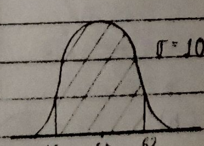


$0.4286 - P(Z < k) = 0.4197$

$P(Z < k) = 0.0089$

$k = -2.37$

eg-4 Given X with normal distribution with $\mu = 50$, $\sigma = 10$, find probability of x assumes value betn 45 & 62



$$Z = \frac{X - \mu}{\sigma}$$

$P(45 < X < 62) =$

$$P\left(\frac{45-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{62-\mu}{\sigma}\right)$$

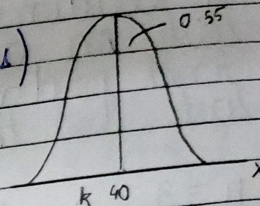
$= P(-0.5 < Z < 1.2)$

$= P(Z < 1.2) - P(Z < -0.5)$

$= 0.8849 - 0.3085$

$= 0.5764$

Find k: $P(X > k) = 0.55$, $\mu = 40$, $\sigma = 6$



$$P(X > k) = P\left(\frac{X-\mu}{\sigma} > \frac{K-\mu}{\sigma}\right)$$

$$= P\left(Z > \frac{K-40}{6}\right)$$

$$= 1 - P\left(Z \leq \frac{K-40}{6}\right)$$

$= 0.55$

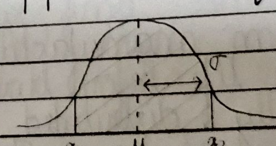
$$P\left(Z \leq \frac{K-40}{6}\right) = 0.45$$

$$\frac{K-40}{6} = -0.12$$

$K = 39.28$

3-12-22

Application of normal distribution



$$P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)\, dx$$

= ar under curve from x_1 to x_2

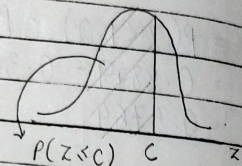
Standard normal distribution

$$Z = \frac{X - \mu}{\sigma}$$

$$P(z_1 \leq Z \leq z_2) = \int_{z_1}^{z_2} n(z, 0, 1)\, dz$$

$$n(z, 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$P(Z > c) = 1 - P(Z \leq c)$



eg-07 A certain type of storage battery lasts on avg. 3 years with $\sigma = 0.5$ years. Find probability that battery will last less than 2.3 years.

$\mu = 3 \qquad \sigma = 0.5$

X : life span of battery in year

$$Z = \frac{X-\mu}{\sigma} = \frac{2.3-3}{0.5} = \frac{-0.7}{0.5}$$

$$= -1.4$$

$$P(X < 2.3) = P\left(\frac{X-\mu}{\sigma} < \frac{2.3-\mu}{\sigma}\right)$$

$$= P(Z < -1.4) = 0.0808$$

eg-08 An electrical firm manufactures light bulbs that have life before burn-out that is normally distributed with mean 800 hours, $\sigma = 40$ hours. Find prob that bulb burns bet^n 778 & 834 hours.

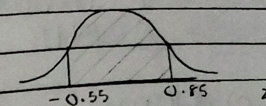
X : life span of bulb

$\mu = 800 \qquad \sigma = 40$

$P(778 < X < 834)$

$$= P\left(\frac{778-\mu}{\sigma} < Z < \frac{834-\mu}{\sigma}\right)$$

$$P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$



$= 0.8023 - 0.2912$

$= 0.5111$

15. A lawyer commutes daily from his suburban home to his midtown office. Avg time for one way trip is 24 min with $\sigma = 3.8$ min. Assume normal distribution.

a) What is probability that trip will take atleast 1/2 hour?

b) If office opens at 9 a.m and lawyer leaves house at 8:45 a.m what % of time is he late?

c) If he leaves house at 8:35 am, coffee is served from 8:50 - 9 a.m, probability that he misses it?

d) Find length of time above which we find slowest 15% of trips.

e) Prob that 2 of next 3 trips will take atleast 1/2 hour.

X : time for oneway trip

$\mu = 24 \qquad \sigma = 3.8$

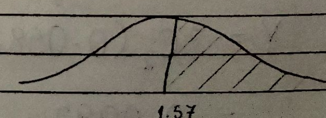
a) $P(X \geq 30)$

$= P(Z \geq 1.57)$

$= 1 - P(Z < 1.57)$

$= 1 - 0.9418$

$= 0.0582$



b) $P(X > 15) = P(Z > -2.37)$
$= 1 - P(Z < -2.37)$
$= 1 - 0.0089$
$= 0.9911$

c) Home 8:35 am   office 8:50 - 9 am   office 9 a.m

$P(X > 25) = P\left(Z > \frac{25-24}{3.8}\right)$
$= P(Z > 0.26)$
$= 1 - P(Z \leq 0.26)$
$= 1 - 0.6026$
$= 0.3974$

d) $P(X \geq c) = 0.15$
$= P\left(Z \geq \frac{c-24}{3.8}\right) = 0.15$
$\Rightarrow 1 - P\left(Z < \frac{c-24}{3.8}\right) = 0.15$
$P\left(Z < \frac{c-24}{3.8}\right) = 0.85$
$\frac{c-24}{3.8} = 1.04 \Rightarrow c = 27.952$

(e) $Y = {}^nC_y P^y q^{n-y}$
$n = 3 \quad y = 2 \quad P = 0.0582$
$Y = {}^3C_2 (0.0582)^2 (0.9418)$
$= 0.0092$

3.12.22

Normal approximation to binomial

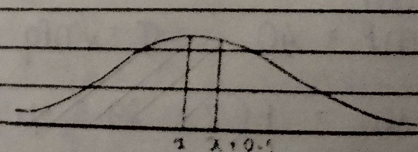
Q. Coin is tossed 400 times. Find the probability obtaining exactly 205 heads.

X: no. of heads $n = 400$ $x = 205$
$p = \frac{1}{2} \quad q = \frac{1}{2}$ $x = 0, 1, 2$

$b(x; n, p) = {}^nC_x p^x q^{n-x}$
$= {}^{400}C_{205} \left(\frac{1}{2}\right)^{205} \left(\frac{1}{2}\right)^{195}$
$= 0.035195$

Theorem: Let X be binomial random variable with μ, σ^2. For large n X has approximately normal distribution with $\mu = np$, $\sigma^2 = npq$ and

$P(X \leq x) = \sum_{k=0}^{x} b(k; n, p)$
$\approx$ area under curve to left of $x + 0.5$
$= P\left(Z \leq \frac{x + 0.5 - \mu}{\sigma}\right)$



Note :

1. $P(X \le x) = P\left(Z \le \frac{x+0.5-\mu}{\sigma}\right)$

2. $P(X < x) = P\left(Z \le \frac{x-0.5-\mu}{\sigma}\right)$

3. $P(x_1 \le X \le x_2) = P\left(\frac{x_1-0.5-\mu}{\sigma} \le Z \le \frac{x_2+0.5-\mu}{\sigma}\right)$

4. $P(x_1 < X < x_2) = P\left(\frac{x_1+0.5-\mu}{\sigma} < Z < \frac{x_2-0.5-\mu}{\sigma}\right)$

5. $P(X > x) = 1 - P(X \le x)$

6. $P(X \ge x) = 1 - P(X < x)$

n is large i.e $n \ge 30$

eg-15 Probability that patient recovers from blood disease is 0.4. If 100 people are known to have contracted this disease what is the prob that fewer than 30 survive?

X : no. of persons recover from blood disease

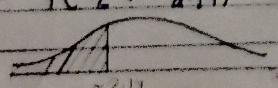
$p = 0.4 \quad q = 0.6 \quad n = 100$

$P(X < 30)$

$\mu = np = 40 \qquad \sigma = \sqrt{npq} = \sqrt{24}$

$P(X<30) = P\left(Z \le \frac{29.5-\mu}{\sigma}\right) = P\left(Z \le \frac{29.5-40}{\sqrt{24}}\right)$

$= P(Z < -2.14) = 0.0162$



24. Coin is tossed 400 times. Use normal curve approximation

(a) between 185 and 210 heads inclusive

(b) exactly 205 heads

(c) fewer than 176 or more than 227

$n = 400 \qquad p = 1/2 \qquad q = 1/2$

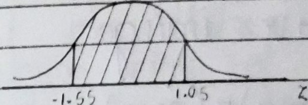
X : no. of heads

$\mu = np = 200$

$\sigma = 10$

(a) $P(185 \le X \le 210) = P\left(\frac{184.5-200}{10} \le Z \le \frac{210.5-200}{10}\right)$

$= P(-1.55 \le Z \le 1.05)$



$= P(Z \le 1.05) - P(Z \le -1.55)$

$= 0.8531 - 0.0606$

$= 0.7925$

(b) $P(X = 205) = P(204 < X < 206)$

$= P\left(\frac{204.5-200}{10} < Z < \frac{205.5-200}{10}\right)$

$= P(0.45 < Z < 0.55)$

$= P(Z < 0.55) - P(Z < 0.45)$

$= 0.7088 - 0.6736$

$= 0.0352$

(c) $P(X \leq 176) + P(X > 227)$

$= P(X \leq 176) + 1 - P(X \leq 227)$

$= P\left(Z \leq \frac{175.5 - 200}{10}\right) + 1 - P\left(Z \leq \frac{227.5 - 200}{10}\right)$

$= P(Z \leq -2.45) + 1 - P(Z \leq 2.75)$

$= 0.0071 + 1 - 0.9970$

$= 0.0101$

6-12-22

Gamma and Exponential distribution

Gamma function

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \qquad \alpha > 0$

Properties

1. $\Gamma(n) = (n-1)!$
   n is +ve integer
2. $\Gamma(n) = (n-1)\Gamma(n-1)$
   n is real number
3. $\Gamma(1) = 1$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Gamma Distribution

Continuous random variable x has gamma distribution with parameter α, β if density function is

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\alpha > 0, \quad \beta > 0$

Note:
If $\alpha = 1$ with parameter β then gamma distribution is exponential distribution

Exponential distribution

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \quad \beta > 0 \\ 0 & \text{else} \end{cases}$$

$\mu = \alpha\beta \qquad \sigma^2 = \alpha\beta^2$: Gamma

$\mu = \beta \qquad \sigma^2 = \beta^2$: Exponential

eg-19 In biomedical study with rats, a dose-response investigation is used to determine effect of dose of a toxicant on their survival time. For certain dose of toxicant, study determines survival time in weeks has gamma distribution with $\alpha = 5$, $\beta = 10$. Probability that rat survives no longer than 60 weeks?

X: survival time (in weeks)

$\alpha = 5 \qquad \beta = 10$

$P(X \leq 60)$

$$f(x) = \begin{cases} \frac{1}{10^5 \Gamma(5)} x^4 e^{-x/10} & x > 0 \\ 0 & \text{else} \end{cases}$$

(c) $P(X < 176) + P(X > 227)$

$= P(X < 176) + 1 - P(X \leq 227)$

$= P\left(Z < \frac{175.5 - 200}{10}\right) + 1 - P\left(Z \leq \frac{227.5 - 200}{10}\right)$

$= P(Z \leq -2.45) + 1 - P(Z \leq 2.75)$

$= 0.0071 + 1 - 0.9970$

$= 0.0101$

6-12-22

Gamma and Exponential distribution

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Continuous random variable x has gamma distribution with parameters α, β if density function is

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\alpha > 0$, $\beta > 0$

Note:
If $\alpha = 1$ with parameter β then gamma distribution is exponential distribution

Exponential distribution

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x > 0 \qquad \beta > 0 \\ 0 & \text{else} \end{cases}$$

$\mu = \alpha\beta \qquad \sigma^2 = \alpha\beta^2 \qquad$: Gamma

$\mu = \beta \qquad \sigma^2 = \beta^2 \qquad$: Exponential

eg-19 In biomedical study with rats a dose-response investigation is used to determine effect of dose of a toxicant on their survival time. For certain dose of toxicant, study determines survival time in weeks has gamma distribution with $\alpha = 5$, $\beta = 10$. Probability that rat survives no longer than 60 weeks?

X: survival time (in weeks)

$\alpha = 5 \qquad \beta = 10$

$P(X \leq 60)$

$$f(x) = \begin{cases} \frac{1}{10^5\Gamma(5)} x^4 e^{-x/10}, & x > 0 \\ 0, & \text{else} \end{cases}$$

$$P(X \leq 60) = \int_0^{60} f(x)\,dx$$

$$= \int_0^{60} \frac{1}{10^5 \Gamma(5)} x^4 e^{-\frac{x}{10}}\,dx$$

Incomplete gamma function

$$F(x;\alpha) = \int_0^x \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}\,dy$$

Let $y = \frac{x}{10} \Rightarrow x = 10y$

$dx = 10\,dy$

$x \to 0,\ y \to 0$

$x \to 60,\ y \to 6$

$$= \int_0^6 \frac{1}{10^5 \Gamma(5)} 10^4 y^4 e^{-y} \times 10\,dy$$

$$= \int_0^6 \frac{1}{\Gamma(5)} y^4 e^{-y}\,dy = F(6;5)$$

$= 0.7150$

eg-20 Length of time in months between customer complaints about certain product is gamma distribution with $\alpha = 2$, $\beta = 4$. Following changes, 20 months passed before first complaint. Does it appear as if quality control tightening was effective?

$\alpha = 2 \quad \beta = 4$

$$f(x) = \begin{cases} \frac{1}{4^2 \Gamma(2)} x^1 e^{-\frac{x}{4}}, & x > 0 \\ 0, & \text{else} \end{cases}$$

$P(X \geq 20) = 1 - P(X \leq 20)$

$$P(X \leq 20) = \int_0^{20} f(x)\,dx$$

$$= \int_0^{20} \frac{1}{4^2 \Gamma(2)} x e^{-\frac{x}{4}}\,dx$$

$y = \frac{x}{4} \Rightarrow x = 4y$

$dx = 4\,dy$

$x = 0,\ y = 0$

$x = 20,\ y = 5$

$$= \int_0^5 \frac{1}{4^2 \Gamma(2)} 4y\, e^{-y}\, 4\,dy$$

$$= \int_0^5 \frac{y e^{-y}}{\Gamma(2)}\,dy = F(5;2)$$

$= 0.9600$

$P(X \geq 20) = 1 - 0.9600$

$= 0.04$

Date 7 12 22

41. If a random variable x has gamma distribution, $\alpha = 2$, $\beta = 1$ find $P(1.8 < X < 2.4)$

$$f(x) = \begin{cases} \frac{1}{1^2 \Gamma(2)} x^1 e^{-x}, & x > 0 \\ 0, & \text{else} \end{cases}$$

$$P(1.8 < X < 2.4) = \int_{1.8}^{2.4} f(x)\,dx$$

$$= \int_{1.8}^{2.4} x e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}\Big]_{1.8}^{2.4}$$

$$= -2.4e^{-2.4} - e^{-2.4} + 1.8e^{-1.8} + e^{-1.8}$$

$$= -3.4e^{-2.4} + 2.8e^{-1.8}$$
$$= -0.3084 + 0.4628$$
$$= 0.1547$$

54. Lifetime in weeks of transistor is known to follow gamma distribution with $\mu = 10$ weeks, $\sigma = \sqrt{50}$ weeks
a) Probability that transistor will last atmost 50 weeks
b) Not survive first 10 weeks

X: lifetime of transistor
$\mu = 10 \qquad \sigma = \sqrt{50}$
$\mu = \alpha\beta \qquad \sigma^2 = \alpha\beta^2$
$= \mu \cdot \beta$

$\Rightarrow 50 = 10 \cdot \beta \Rightarrow \beta = 5, \ \alpha = 2$

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$$f(x) = \begin{cases} \frac{1}{5^2 \Gamma(2)} x e^{-x/5} & x > 0 \\ 0 & \text{else} \end{cases}$$

(a) $$P(X \le 50) = \int_0^{50} \frac{1}{25\Gamma(2)} \cdot xe^{-x/5}\,dx$$

$$= \int_0^{50} \frac{1}{25} x e^{-x/5}\,dx$$

$$= 24.9875 \times \frac{1}{25} = 0.9995$$

b) $$P(X \le 10) = \int_0^{10} \frac{1}{25} x e^{-x/5}\,dx$$

$$= \frac{1}{25} \times 14.8498$$

$$= 0.59399$$

46. Life in years of electrical switch has exponential distribution with avg life $\beta = 2$. If 100 switches are installed, prob that atmost 30 fail during first year?

X: life of battery
$\beta = 2 \qquad n = 100$

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & \text{else} \end{cases}$$

$$P(X < 1) = \int f(x)\,dx$$

$$= \int_0^1 \frac{1}{2} e^{-x/2}\, dx = 0.3934$$

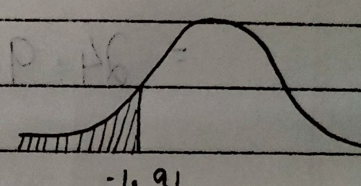
$$q = 0.6065$$

Y = no. of battery fail in a year

$$P(Y \leq 30) = P\left(\frac{Y-\mu}{\sigma} \leq \frac{30-\mu}{\sigma}\right)$$

$$= P\left(Z \leq \frac{30-39.34}{\sqrt{23.85971}}\right)$$

$$= P\left(Z \leq -\frac{9.34}{4.88}\right)$$



$$= P(Z \leq -1.91)$$

$$= 0.0281$$