VECTOR FUNCTIONS

Excercise 13.1

1) Find the domain of the vector function

$$r(t) = \langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \rangle$$

(Ans) The domain of the vector function is the domain of its component functions

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

$$= (\sqrt{4+t^2})\hat{i} + (e^{-3t})\hat{j} + (\ln(t+1))\hat{k}$$

so, $\sqrt{4-t^2}$ is defined when $4-t^2 \geq 0$

$$\Rightarrow t^2 \leq 4$$

$$\Rightarrow -2 \leq t \leq 2$$

e^{-3t} is defined for any $t \in R$

$\ln(t+1)$ is defined when $t+1 > 0 \Rightarrow t > -1$

$\Rightarrow$ domain is $(-1, 2]$

3) Find the limit

$$\lim_{t \to 0} \left(e^{-3t} i + \frac{t^2}{\sin^2 t} j + \cos 2t\, k\right)$$

(Ans) Let $x = \lim_{t \to 0} e^{-3t}$

$$= e^{-3 \times 0} = 1$$

$$y = \lim_{t \to 0} \frac{t^2}{\sin^2 t}$$

$$= \lim_{t \to 0} \frac{2t}{2\sin t \cos t} \quad \text{(L'Hospital rule)}$$

$$= \lim_{t \to 0} \frac{2t}{\sin 2t}$$

$$= \lim_{t \to 0} \frac{2}{2\cos 2t}$$

$$= \lim_{t \to 0} \frac{1}{\cos 2t}$$

$$= \frac{1}{\cos 2t} = 1$$

$$z = \lim_{t \to 0} \cos 2t$$

$$= \cos 2 \times 0 = 1$$

so, $\hat{i} + \hat{j} + k$

4) $\lim_{t \to 1} \left(\frac{t^2 - t}{t-1} \hat{i} + \sqrt{t+8}\, \hat{j} + \frac{\sin \pi t}{\ln t} k \right)$

Ans) $x = \lim_{t \to 1} \frac{t^2 - t}{t-1}$

$$= \lim_{t \to 1} \frac{t(t-1)}{t-1}$$

$$= \lim_{t \to 1} t = 1$$

$$y = \lim_{t \to 1} \sqrt{t+8}$$

$$= \sqrt{1+8} = 3$$

$$z = \lim_{t \to 1} \frac{\sin \pi t}{\ln t}$$

$$= \lim_{t \to 1} \frac{\pi \cos \pi t}{1/t}$$ (L' Hospital rule)

$$= \frac{\pi \cos \pi}{1} = -\pi$$

so, $\hat{i} + 3\hat{j} - \pi k$

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(7) sketch the curve with the given vector equation Indicate with an arrow the direction in which t increases

$r(t) = \langle \sin t, t \rangle$

(Ans) $r(t) = \langle \sin t, t \rangle$

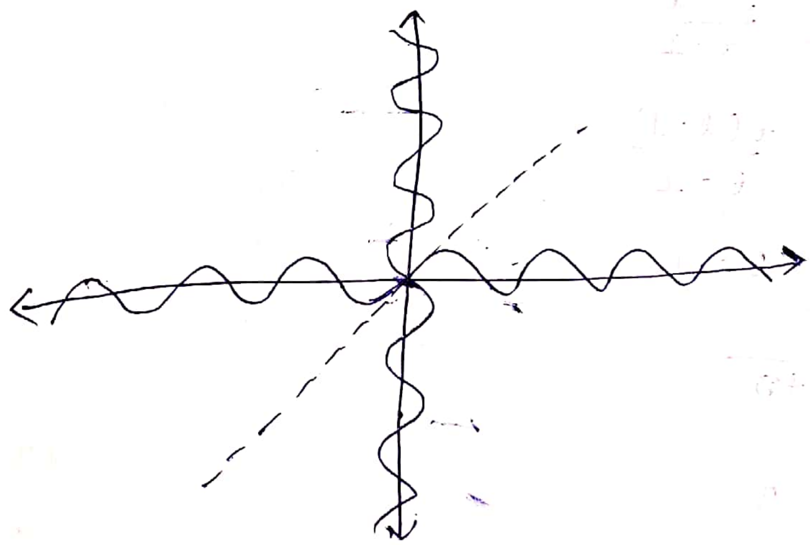
$x = \sin t$

and $y = t$

so, $\boxed{x = \sin y}$

so $y = \sin x$

As graph of $x = \sin y$ is the reflection of the graph of $y = \sin x$ in the line $x = y$



ii) $r(t) = \langle 1, \cos t, 2\sin t \rangle$

(Ans) $r(t) = \langle 1, \cos t, 2\sin t \rangle$

Here, $x = 1$

$y = \cos t$ -------- (1)

$z = 2\sin t$

$\Rightarrow \frac{z}{2} = \sin t$ ------- (2)

square both sides of eq^n (1), we get

$y^2 = \cos^2 t$ ------ (3)

squaring both sides of eq^n (2), we get

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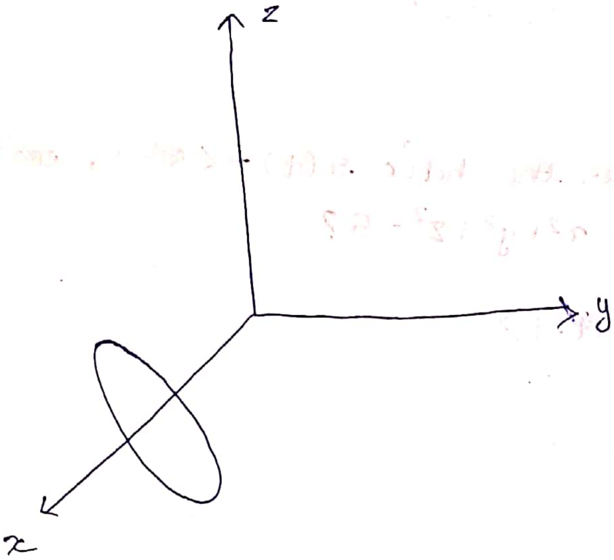
$\frac{z^2}{4} = \sin^2 t$ ------ (4)

Adding equation (3) and (4), we get

$y^2 + \frac{z^2}{4} = 1$ (ellipse)



18) Find a vector equation & parametric equations for the line segment that joins P to Q

P(1,0,1) and Q(2,3,1)

Ans) $r_0 = \langle -1, 2, -2 \rangle$ and $r_1 = \langle -3, 5, 1 \rangle$

$r(t) = (1-t)\langle -1, 2, -2 \rangle + t\langle -3, 5, 1 \rangle$

$\Rightarrow r(t) = \langle -1+t, 2-2t, -2+2t \rangle + \langle -3t, 5t, 1 \rangle$

$\Rightarrow r(t) = \langle -1-2t, 2+3t, -2+3t \rangle$

So, $x = -1-2t$, $y = 2t+3t$, $z = -2+3t$

29) At what points does the curve $r(t) = t\hat{i} + (2t-t^2)k$ intersect the paraboloid $z = x^2+y^2$?

Ans) $r(t) = t\hat{i} + (2t-t^2)k$

$x = t, y = 0, z = 2t - t^2$

Paraboloid equation :-

$z = x^2 + y^2$

$\Rightarrow 2t - t^2 = t^2 + 0^2$

$\Rightarrow 2t - 2t^2 = 0$

$\Rightarrow 2t(1-t) = 0 \Rightarrow \boxed{t = 0, 1}$

when $t=0$,

$x=0, y=0$ and $z=2(0)-0^2=0$

so point $A(0,0,0)$

when $t=1$,

$x=1, y=0, z=2(1)-1^2=1$

point $B(1,0,1)$

30) At what points does the helix $r(t)=\langle \sin t, \cos t, t\rangle$ intersect the sphere $x^2+y^2+z^2=5$?

(Ans) $r(t)=\langle \sin t, \cos t, t\rangle$

so, $x^2+y^2+z^2=5$

$\Rightarrow \sin^2 t+\cos^2 t+t^2=5$

$\Rightarrow 1+t^2=5$

$\Rightarrow t^2=4$

$\Rightarrow \boxed{t=\pm 2}$

so, $r(2)=\langle \sin(2), \cos(2), 2\rangle$

$=\langle 0.909, -0.416, 2\rangle$

and $r(-2)=\langle \sin(-2), \cos(-2), -2\rangle$

$=\langle 0.909, -0.416, -2\rangle$

40) The cylinder $x^2+y^2=4$ and the surface $z=xy$

(Ans) we know that, $x=r\cos t$

$y=r\sin t$

so, $x^2+y^2=4$

$\Rightarrow r^2\cos^2 t+r^2\sin^2 t=4$

$\Rightarrow r^2(\cos^2 t+\sin^2 t)=4$

$\Rightarrow \boxed{r=2}$

so, $x=2\cos t$

$y=2\sin t$

so, $z=[2\cos t][2\sin t]$

$\Rightarrow z=4\sin t\cos t$

so, $\langle 2\cos t, 2\sin t, 4\sin t\cos t\rangle$

41) The cone $z = \sqrt{x^2+y^2}$ and the plane $z = 1+y$

(Ans) $z = \sqrt{x^2+y^2}$ and $z = 1+y$

$\Rightarrow 1+y = \sqrt{x^2+y^2}$

$\Rightarrow x^2+y^2 = 1+2y+y^2$

$\Rightarrow x^2 = 1+2y$

$\Rightarrow y = \frac{x^2-1}{2}$

Let $x = t$

$\Rightarrow y = \frac{t^2-1}{2}$

so, $z = 1 + \frac{1}{2}(t^2-1)$

$= \frac{1}{2}(t^2+1)$

so, $r(t) = \langle t, \frac{1}{2}(t^2+1), \frac{1}{2}(t^2+1) \rangle$

48) Two particles travel along the space curves.

$r_1(t) = \langle t, t^2, t^3 \rangle$, $r_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$

Do the particles collide? Do their paths intersect?

(Ans) $t = 1+2s$, $t^2 = 1+6s$, $t^3 = 1+14s$.

$\quad$ ---- (1) $\quad$ ---- (2) $\quad$ ---- (3)

As $r_1(t) = r_2(t)$ (particles collide)

Multiply '-3' in equation (1), we get

$-3t = -3-6s$ ———— (4)

Adding equation (2) and (4) we get

$t^2 = 1+6s + (-3t = -3-6s)$

$\Rightarrow (t^2 - 3t = -2)$

$\Rightarrow (t-1)(t-2) = 0$

$\Rightarrow \boxed{t = 1, 2}$

Then, $\boxed{s = 0, 1/2}$

when $t = 1$, $s = 0$

point $A(1, 1, 1)$

when $t = 2$; $s = 1/2$

point $B(2, 4, 8)$.

$\therefore$ Particles donot collide but path intersect

Exercise 13.2

3) a) sketch the plane curve with the given vector equation
b) find $r'(t)$
c) sketch the position vector $r(t)$ and the tangent vector $r'(t)$ for the given value of t.

$$r(t) = \langle t-2, t^2+1 \rangle, \ t = -1$$

(Ans) a) $r(t) = \langle t-2, t^2+1 \rangle$

Here, $x = t-2$

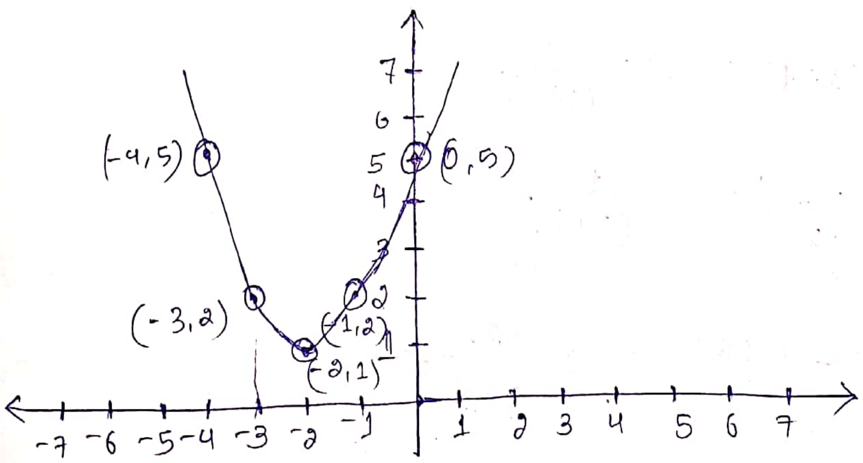
$\Rightarrow t = x+2$

$y = t^2+1$

$= (x+2)^2 + 1$

It is the form of $(h,k) = (-2,1)$

points
$t=-2$; $x=-4$; $y=5$
$t=-1$; $x=-3$; $y=2$
$t=0$; $x=-2$; $y=1$
$t=1$; $x=-1$; $y=2$
$t=2$; $x=0$; $y=5$



b) $r(t) = \langle t-2, t^2+1 \rangle$

$$r'(t) = \left\langle \frac{d[t-2]}{dt}, \frac{d[t^2+1]}{dt} \right\rangle$$

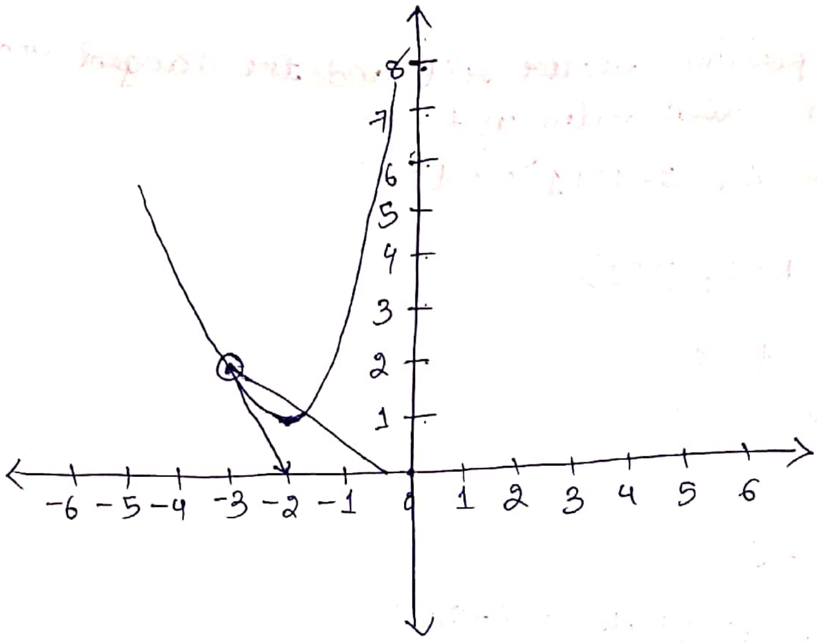
$$= \langle 1, 2t \rangle$$

c) $r(t) = \langle t-2, t^2+1 \rangle$, $t = -1$

$r'(t) = \langle 1, 2t \rangle$

$r(-1) = \langle (-1)-2, (-1)^2+1 \rangle = \langle -3, 2 \rangle$

$r'(t) = \langle 1, 2(-1) \rangle = \langle 1, -2 \rangle$



12) Determine the derivative of the vector function

$$r(t) = \frac{1}{1+t}\hat{i} + \frac{t}{1+t}\hat{j} + \frac{t^2}{1+t}\hat{k}$$

(Ans) Take the derivative of each component separately

$$\left[\frac{1}{1+t}\right]' = \left[(1+t)^{-1}\right]'$$

$$= (-1)(1+t)^{-2}(1) = \frac{-1}{(1+t)^2}$$

$$\left[\frac{t}{1+t}\right]' = \frac{1(1+t) - t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$\left[\frac{t^2}{1+t}\right]' = \frac{2t(1+t) - t^2(1)}{(1+t)^2} = \frac{2t + 2t^2 - t^2}{(1+t)^2}$$

$$= \frac{t^2+2t}{(1+t)^2}$$

$$r'(t) = \frac{-1}{(1+t)^2}\hat{i} + \frac{1}{(1+t)^2}\hat{j} + \frac{t^2+2t}{(1+t)^2}\hat{k}$$

17) Find the tangent vector $T(t)$ at the point with the given value of the parameter t

$$r(t) = \langle te^{-t}, 2\arctan t, 2e^t \rangle, t = 0$$

Ans) $r(t) = \langle te^{-t}, 2\arctan t, 2e^{t} \rangle, \ t=0$

$r'(t) = \langle e^{-t} - te^{-t}, \frac{2}{1+t^2}, 2e^{t} \rangle$

$r'(0) = \langle 1-0, \frac{2}{1+0}, 2 \rangle$

$= \langle 1, 2, 2 \rangle$

$|r'(0)| = \sqrt{(1)^2 + (2)^2 + (2)^2}$

$= \sqrt{1+4+4} = 3$

Tangent vector, $T(0) = \frac{r'(0)}{|r'(0)|}$

$= \frac{\langle 1, 2, 2 \rangle}{3}$

$= \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

20) $r(t) = \sin^2 t\,\hat{i} + \cos^2 t\,\hat{j} + \tan^2 t\,\hat{k}, \ t = \pi/4$

Ans) $r'(t) = 2\sin t\cos t\,\hat{i} - 2\cos t\sin t\,\hat{j} + 2\tan t\sec^2 t\,\hat{k}$

$r'(\pi/4) = 2\sin(\frac{\pi}{4})\cos(\frac{\pi}{4})\hat{i} - 2\cos(\frac{\pi}{4})\sin(\frac{\pi}{4})\hat{j} + 2\tan(\frac{\pi}{4})\sec^2(\frac{\pi}{4})\hat{k}$

$= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\hat{i} - 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\hat{j} + 2(1)(2)\hat{k}$

$= \hat{i} - \hat{j} + 4\hat{k}$

$|r'(\pi/4)| = \sqrt{(1)^2 + (-1)^2 + (4)^2} = \sqrt{18} = 3\sqrt{2}$

$T(\pi/4) = \left\langle \frac{1\hat{i} - 1\hat{j} + 4\hat{k}}{3\sqrt{2}} \right\rangle$

$= \frac{1}{3\sqrt{2}}\hat{i} - \frac{1}{3\sqrt{2}}\hat{j} + \frac{4}{3\sqrt{2}}\hat{k}$

$= \frac{\sqrt{2}}{6}\hat{i} - \frac{\sqrt{2}}{6}\hat{j} + \frac{2\sqrt{2}}{3}\hat{k}$

23) Find parametric equations for the tangent line to the curves with the given parametric equations at the specified point
$x = 1 + 2\sqrt{t}, \ y = t^3 - t, \ z = t^3 + t \ ; \ (3, 0, 2)$

(Ans) $r(t) = (1+2\sqrt{t})\hat{i} + (t^3 - t)\hat{j} + (t^3 + t)\hat{k}$

$r'(t) = \frac{1}{\sqrt{t}} i + (3t^2 - 1) j + (3t^2 + 1) k$

point $(3,0,2)$ corresponds to $t = 1$

$r'(1) = \frac{1}{\sqrt{1}} i + (3(1)^2 - 1) j + (3(1)^2 + 1) k$

$= i + 2j + 4k$

$r(t) = \langle 3, 0, 2 \rangle + t\langle 1, 2, 4 \rangle$

$r(t) = \langle 3+t, 2t, 2+4t \rangle$

Parametric equations,

$x = 3+t, \quad y = 2t, \quad z = 2+4t$

25) $x = e^{-t} \cos t, \quad y = e^{-t} \sin t, \quad z = e^{-t} \quad (1, 0, 1)$

(Ans) $r(t) = e^{-t} \cos t \, i + e^{-t} \sin t \, j + e^{-t} k$

$r'(t) = e^{-t}(-\cos t - \sin t) i + e^{-t}(-\sin t + \cos t) j - e^{-t} k$

point $(1, 0, 1)$ corresponds to $t = 0$

$r'(0) = e^{-0} (-\cos(0) - \sin(0)) i + e^{-0} (-\sin(0) + \cos(0)) \hat{j} - e^{-0} k$

$= -i + j - k$

$r(t) = \langle 1, 0, 1 \rangle + t\langle -1, 1, -1 \rangle$

$r(t) = \langle 1-t, t, 1-t \rangle$

Parametric equations

$x = 1 - t, \quad y = t, \quad z = 1 - t$

36) Evaluate the integral.

$$\int_0^1 \left(\frac{4}{1+t^2} j + \frac{2t}{1+t^2} k \right) dt$$

(Ans) $$\int_0^1 \left(\frac{4}{1+t^2} j + \frac{2t}{1+t^2} k \right) dt$$

$$= \int_0^1 \frac{4}{1+t^2} dt \, \hat{j} + \int_0^1 \frac{2t}{1+t^2} dt \, k$$

$$= \left[4 \tan^{-1} t\right]_0^1 j + \left[\ln[1+t^2]\right]_0^1 k$$

$$= 0i + \pi j + \ln 2k$$

44) Prove formula 3 of theorem 3

Ans) $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$

Let $u(t) = \langle f_1(t); f_2(t), f_3(t)\rangle$

then,

$f(t)u(t) = \langle f(t)f_1(t); f(t)f_2(t), f(t)f_3(t)\rangle$

so, $\frac{d}{dt}[f(t)u(t)] = \frac{d}{dt}\langle f(t)f_1(t), f(t)f_2(t), f(t)f_3(t)\rangle$

$$= \langle f'(t)f_1(t) + f(t)f_1'(t); f'(t)f_2(t) + f(t)f_2'(t); f'(t)f_3(t) + f(t)f_3'(t)\rangle$$

$$= \langle f'(t)f_1(t) + f'(t)f_2(t) + f'(t)f_3(t), f(t)f_1'(t) + f(t)f_2'(t) + f(t)f_3'(t)\rangle$$

$$= f'(t)\langle f_1(t) + f_2(t) + f_3(t)\rangle + f(t)\langle f_1'(t) + f_2'(t) + f_3'(t)\rangle$$

$$= f'(t)u(t) + f(t)u'(t)$$

(hence proved)

49) Find $f'(2)$ where $f(t) = u(t)\cdot v(t)$; $u(2) = \langle 1, 2, -1\rangle$ $u'(2) = \langle 3, 0, 4\rangle$ and $v(t) = \langle t, t^2, t^3\rangle$

Ans) $f(t) = v(t)u(t)$

$f'(t) = v'(t)\cdot u(t) + v(t)\cdot u'(t)$

$f'(2) = v'(2)\cdot u(2) + v(2)\cdot u'(2)$

$f'(2) = (1, 4, 12)\cdot(1, 2, -1) + (2, 4, 8)\cdot(3, 0, 4)$

$= 35$

where, $v(t) = (t, t^2, t^3)$

$v'(t) = (1, 2t, 3t^2)$

$v(2) = (2, 4, 8)$

$v'(2) = (1, 4, 12)$

$u(2) = (1, 2, -1)$

$u'(2) = (3, 0, 4)$

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Exercise 13.3

3) Find the length of the curve

$r(t) = \sqrt{2}t\,\hat{i} + e^{t}\hat{j} + e^{-t}\hat{k}\ ;\ 0 \le t \le 1$

(Ans) $r'(t) = \sqrt{2}\,\hat{i} + e^{t}\hat{j} + (-e^{-t})\hat{k}$

$|r'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}$

$\Rightarrow |r'(t)| = \sqrt{(e^t + e^{-t})^2}$

$\Rightarrow |r'(t)| = e^t + e^{-t}$

so, $L = \int_0^1 |r'(t)|\,dt$

$= \int_0^1 (e^t + e^{-t})\,dt$

$= (e^t - e^{-t})\Big|_0^1 = e - e^{-1}$

4) $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + \ln \cos t\,\hat{k}\quad , 0 \le t \le \pi/4$

(Ans) $r(t) = \langle \cos t, \sin t, \ln \cos t \rangle$

$r'(t) = \langle -\sin t, \cos t, -\dfrac{\sin t}{\cos t} \rangle$

$= \langle -\sin t, \cos t, -\tan t \rangle$

$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$

$= \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$

$= \sqrt{1 + \tan^2 t} = \sec t$

so, $L = \int_0^{\pi/4} \sec t\,dt$

$\Rightarrow L = \ln(\sec t + \tan t)\Big|_0^{\pi/4}$

$= \ln(\sqrt{2} + 1) - \ln(1)$

$= \ln(\sqrt{2} + 1)$

17) a) Find the unit tangent & unit normal vectors $T(t)$ & $N(t)$

b) Use formula 9 to find the curvature

$$r(t) = \langle t, 3\cos t, 3\sin t\rangle$$

(Ans) a) $r(t) = \langle t, 3\cos t, 3\sin t\rangle$

$$r'(t) = \langle 1, -3\sin t, 3\cos t\rangle$$

$$|r'(t)| = \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{10}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{10}} \langle 1, -3\sin t, 3\cos t\rangle$$

so, $T'(t) = \frac{3}{\sqrt{10}} \langle 0, -\cos t, -\sin t\rangle$

$$|T'(t)| = \sqrt{\left(\frac{3}{\sqrt{10}}\right)^2 [(-\cos t)^2 + (-\sin t)^2}$$

$$= \frac{3}{\sqrt{10}} \sqrt{\sin^2 t + \cos^2 t}$$

$$= \frac{3}{\sqrt{10}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$= \frac{\frac{3}{\sqrt{10}} (\langle 0, -\cos t, -\sin t\rangle)}{\frac{3}{\sqrt{10}}}$$

$$= \langle 0, -\cos t, -\sin t\rangle$$

b) $|r'(t)| = \sqrt{10}$

$$|T'(t)| = \frac{3}{\sqrt{10}}$$

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

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20) $r(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

(Ans) $r'(t) = \langle 1, t, 2t \rangle$

$$|r'(t)| = \sqrt{1^2 + t^2 + 4t^2} = \sqrt{5t^2+1}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \left\langle \frac{1}{\sqrt{5t^2+1}}, \frac{t}{\sqrt{5t^2+1}}, \frac{2t}{\sqrt{5t^2+1}} \right\rangle$$

$$T(t) = \langle (1+5t^2)^{-1/2}, t(1+5t^2)^{-1/2}, 2t(1+5t^2)^{-1/2} \rangle$$

$$T'(t) = \frac{-5t}{(1+5t^2)^{3/2}}, \frac{1}{(1+5t^2)^{3/2}}, \frac{2}{(1+5t^2)^{3/2}}$$

$$|T'(t)| = \frac{\sqrt{5}}{1+5t^2}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \left\langle \frac{-5t}{\sqrt{5}(1+5t^2)^{1/2}}, \frac{1}{\sqrt{5}(1+5t^2)^{1/2}}, \frac{2}{\sqrt{5}(1+5t^2)^{1/2}} \right\rangle$$

b) $$k = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{\sqrt{5}}{1+5t^2}}{(1+5t^2)^{1/5}} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

21) use theorem 10 to find the curvatures

$r(t) = t^3\hat{j} + t^2\hat{k}$

(Ans) $r(t) = t^3\hat{j} + t^2\hat{k}$

$r'(t) = 3t^2\hat{j} + 2t\hat{k}$

$r''(t) = 6t\hat{j} + 2\hat{k}$

$r'(t) \times r''(t) = (-6t^2)\hat{i}$

$|r'(t) \times r''(t)| = \sqrt{36t^4} = 6t^2$

$|r'(t)|^3 = (9t^4 + 4t^2)^{3/2}$

so, $$\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{6t^2}{(9t^4+4t^2)^{3/2}}$$

23) $r(t) = 3t\,\hat{i} + 4\sin t\,\hat{j} + 4\cos t\,\hat{k}$

(Ans) $r'(t) = 3\hat{i} + 4\cos t\,\hat{j} - 4\sin t\,k$

$r''(t) = -4\sin t\,\hat{j} - 4\cos t\,k$

$$|r'(t) \times r''(t)| = |-16\hat{i} + 12\cos t\,\hat{j} - 12\sin t\,k|$$

$$= \sqrt{256 + 144\cos^2 t + 144\sin^2 t}$$

$$= \sqrt{256 + 144} = \sqrt{400} = 20$$

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{20}{5^3} = \frac{4}{25}$$

25) Find the curvature of $r(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$

(Ans) $r(t) = \langle t, t^2, t^3 \rangle$

$P = \langle 1, 1, 1 \rangle$

Here $t = 1$

so, $r'(t) = \langle 1, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$$|r'(t) \times r''(t)| = |\langle 6t^2, -6t, 2 \rangle|$$

$$= \sqrt{36t^4 + 36t^2 + 4}$$

$$\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

Here $t = 1$, then

$$\frac{\sqrt{36 + 36 + 4}}{(\sqrt{1 + 4 + 9})^3} = \frac{\sqrt{76}}{(14)^{3/2}}$$

43) Use the formula in Exercise 42 to find the curvature $x = t^2, y = t^3$

(Ans) $x = t^2, y = t^3$

so, $x' = 2t$, $y' = 3t^2$

Again, $x'' = 2$, $y'' = 6t$

$$k = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$

$$= \frac{(2t \times 6t) - (3t^2 \times 2)}{[4t^2 + (3t^2)^2]^{3/2}}$$

$$= \frac{6t^2}{(4t^2 + 9t^4)^{3/2}}$$

(49) Find equations of the normal plane & osculating plane of the curve at the given point.

$x = 2\sin 3t$, $y = t$, $z = 2\cos 3t$ $\quad (0, \pi, -2)$

(Ans) $r(t) = \langle 2\sin 3t, t, 2\cos 3t \rangle = \langle 0, \pi, -2 \rangle$

so, $t = \pi$

$r'(t) = \langle 6\cos 3t, 1, -6\sin 3t \rangle$

$r''(t) = \langle -18\sin 3t, 0, -18\cos 3t \rangle$

$|r'(t)| = \sqrt{37}$

$|r''(t)| = \sqrt{324} = 18$

$r'(\pi) = \langle -6, 1, 0 \rangle$

$r''(\pi) = \langle 0, 0, 18 \rangle$

$r'(\pi) \times r''(\pi) = \langle 18, 108, 0 \rangle$

$\Rightarrow |r'(\pi) \times r''(\pi)| = \sqrt{11988} = 18\sqrt{37}$

$$B(\pi) = \frac{r'(\pi) \times r''(\pi)}{|r'(\pi) \times r''(\pi)|} = \frac{\cancel{18}\langle 1, 6, 0 \rangle}{\cancel{18}\sqrt{37}}$$

$$= \frac{1}{\sqrt{37}} \langle 1, 6, 0 \rangle$$

so, normal plane has normal vector T

$-6(x - 0) + 1(y - \pi) + 0(z + 2) = 0$

$\Rightarrow y = 6x + \pi$

Normal plane has normal vector B

$$1(x-0)+6(y-\pi)+0(z+2)=0$$

$$\Rightarrow x+6y=6\pi$$

(50) $x=t,\ y=t^2,\ z=t^3\ ;\ (1,1,1)$

(Ans) $r(t)=\langle t,t^2,t^3\rangle=\langle 1,1,1\rangle$

Here $t=1$

$$r'(t)=\langle 1,2t,3t^2\rangle$$

$$r''(t)=\langle 0,2,6t\rangle$$

$$|r'(t)|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$$

$$|r''(t)|=\sqrt{0^2+2^2+6^2}=\sqrt{40}=2\sqrt{10}$$

$$r'(1)=\langle 1,2,3\rangle$$

$$r''(1)=\langle 0,2,6\rangle$$

$$r'(1)\times r''(1)=\langle 6,-6,2\rangle$$

$$|r'(1)\times r''(1)|=\sqrt{(6)^2+(-6)^2+(2)^2}$$

$$=\sqrt{76}=2\sqrt{19}$$

$$B(v)=\frac{r'(1)\times r''(1)}{|r'(1)\times r''(1)|}=\frac{\langle 6,-6,2\rangle}{2\sqrt{19}}$$

$$T(1)=\frac{r'(1)}{|r'(1)|}=\frac{\langle 1,2,3\rangle}{\sqrt{14}}$$

Normal plane on vector T

$$1(x-1)+2(y-1)+3(z-1)=0$$

$$\Rightarrow x-1+2y-2+3z-3=0$$

$$\Rightarrow x+2y+3z=6$$

Normal plane on vector B

$$6(x-1)-6(y-1)+2(z-1)=0$$

$$\Rightarrow 6x-6y+2z=2$$

$$\Rightarrow 3x-3y+z=1$$

(53) At what point on the curve $x = t^3,\ y = 3t,\ z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

(Ans) $x = t^3,\ y = 3t,\ z = t^4$

$\Rightarrow r(t) = t^3\hat{i} + 3t\hat{j} + t^4\hat{k}$

$\Rightarrow r'(t) = 3t^2\hat{i} + 3\hat{j} + 4t^3\hat{k}$

Normal plane will be parallel to the given plane,

$\Rightarrow r'(t)$ is parallel to $6i + 6j - 8k$

$\Rightarrow r'(t) = k(6i + 6j - 8k)\ (k \neq 0)$

$\Rightarrow 3t^2\hat{i} + 3\hat{j} + 4t^3\hat{k} = k(6\hat{i} + 6\hat{j} + 8\hat{k})$

$\Rightarrow 3t^2 = 6k$ —— (1)

$3 = 6k$ —— (2)

$4t^3 = -8k$ —— (3)

Using (1) and (2),

$3t^2 = 3$

$\Rightarrow t = \pm 1$

From (2), we get

$k = 1/2$

Put the value of 't' in eq^n (3)

$\Rightarrow t = -1$ is satisfied

$x = -1,\ y = -3,\ z = 1$

so, point $(-1, -3, 1)$

Exercise 13.4

5) Find the velocity, acceleration & speed of a particle with the given position function. sketch the path of the particle & draw ~~by~~ the velocity & acceleration vectors for the specified value of t

$$r(t) = e^t\hat{i} + e^{2t}\hat{j} \;;\; t=0$$

(Ans) $r'(t) = \langle e^t, 2e^{2t} \rangle = v(t)$

$r''(t) = \langle e^t, 4e^{2t} \rangle = a(t)$

$v(0) = r'(0) = \langle e^0, 2e^{2\times 0} \rangle = \langle 1,2 \rangle$

$|v(0)| = |r'(0)| = \sqrt{1+4} = \sqrt{5}$

$a(0) = r''(0) = \langle e^0, 4e^{2\times 0} \rangle = \langle 1,4 \rangle$

$|a(0)| = |r''(0)| = \sqrt{1+16} = \sqrt{17}$

11) Find the velocity, acceleration & speed of a particle with the given position function

$$r(t) = \sqrt{2}t\hat{i} + e^t\hat{j} + e^{-t}\hat{k}$$

(Ans) $r(t) = \sqrt{2}t\hat{i} + e^t\hat{j} + e^{-t}\hat{k}$

$r'(t) = v(t) = \sqrt{2}\hat{i} + e^t\hat{j} - e^{-t}\hat{k}$

$r''(t) = a(t) = e^t\hat{j} + e^{-t}\hat{k}$

$|v(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}$

$= \sqrt{2 + e^{2t} + e^{-2t}}$

$= \sqrt{e^{2t} + 2\cdot e^t \cdot e^{-t} + e^{-2t}}$

$= \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$

13) $r(t) = e^t(\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k})$

(Ans) $r'(t) = v(t) = e^t(\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}) + e^t(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k})$

$= e^t ((\cos t - \sin t) i + (\sin t + \cos t) j + (t+1) k)$

$r''(t) = a(t) =$

$= e^t ((\cos t - \sin t) i + (\sin t + \cos t) j + (t+1) k) +$

$e^t ((-\sin t - \cos t) i + (\cos t - \sin t) j + k)$

$= e^t (-2\sin t\, i + 2\cos t\, j + (t+2) k)$

$|v(t)| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + (t+1)^2}$

$= e^t \sqrt{\cos^2 t - 2\cos t \sin t + \sin^2 t + 2 \sin t \cos t + \cos^2 t + (t+1)^2}$

$= e^t \sqrt{2 + (t+1)^2} = e^t \sqrt{t^2 + 2t + 3}$

20) what force is required so that a particle of mass m has the position function $r(t) = t^3 i + t^2 j + t^3 k$?

(Ans) $r(t) = t^3 \hat{i} + t^2 \hat{j} + t^3 \hat{k}$

$r'(t) = v(t) = 3t^2 \hat{i} + 2t \hat{j} + 3t^2 \hat{k}$

$r''(t) = a(t) = 6t \hat{i} + 2\hat{j} + 6t \hat{k}$

$F = ma$

$= m(6t \hat{i} + 2\hat{j} + 6t\hat{k})$

25) A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

(Ans) Let initial speed be v_0

Initial upward speed is $v_0 \sin\theta = v_0 \sin 45° = \frac{v_0}{\sqrt{2}}$

Initial vertical velocity is $+\frac{v_0}{\sqrt{2}}$

Final vertical velocity is $-\frac{v_0}{\sqrt{2}}$

Acceleration $= -a = -10\ m/s^2$

$v = u + at$

$\Rightarrow t = \frac{v-u}{a} = \frac{-v_0 - v_0}{-\sqrt{2}(10)}$

$\Rightarrow t = \frac{+2v_0}{\sqrt{2}(10)}$

$\Rightarrow t = \frac{v_0}{5\sqrt{2}}$

Horizontal speed $= v_0 \cos\theta = v_0 \cos 45^\circ = \frac{v_0}{\sqrt{2}}$

Therefore, $90 = \frac{v_0}{5\sqrt{2}} \times \frac{v_0}{\sqrt{2}}$

$\Rightarrow 90 = \frac{v_0^2}{10} \Rightarrow v_0^2 = 900$

$\Rightarrow v_0 = 30 m/s$

26) A gun is fired with angle of elevation 30°, what is the muzzle speed if the maximum height of the shaft is 500m?

(Ans) Let muzzle speed be u

Initial vertical velocity $= u \sin\theta = u \sin 30^\circ = \frac{u}{2}$

Acceleration $= -a = -10 m/s^2$

$v^2 - u^2 = 2as$

$\Rightarrow 0 - \left(\frac{u}{2}\right)^2 = 2(-10)(500)$

$\Rightarrow \frac{u^2}{4} = 40000$

$\Rightarrow u^2 = 40000 \Rightarrow \boxed{u = 200 m/s}$

39) Find the tangential & normal components of the acceleration vector.

$r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

(Ans) $r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$

$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}$

$= \sqrt{\sin^2 t + \cos^2 t + 1}$

$= \sqrt{2}$

Tangential component,

$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} = \frac{(-\sin t)(-\cos t) + (\cos t)(-\sin t) + (1)(0)}{\sqrt{2}}$

$$= \frac{\sin t \cos t - \sin t \cos t}{\sqrt{2}} = 0$$

Normal component,

$$a_N = \frac{r'(t) \times r''(t)}{|r'(t)|}$$

so, $r'(t) \times r''(t) = \langle \cos t\,(0) - (1)(-\sin t),\ (1)(-\cos t) - (-\sin t)(0),$
$(-\sin t)(-\sin t) - (\cos t)(-\cos t) \rangle$

$$= \langle \sin t, -\cos t, 1 \rangle$$

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$= \frac{\sqrt{(\sin t)^2 + (-\cos t)^2 + (1)^2}}{\sqrt{2}} = \frac{\sqrt{1+1}}{\sqrt{2}} = 1$$

42) $r(t) = t\hat{i} + \cos^2 t\,\hat{j} + \sin^2 t\,\hat{k}$

(Ans) $r'(t) = \langle 1, 2\cos t(-\sin t), 2\sin t \cos t \rangle$

$$= \langle 1, -\sin 2t, \sin 2t \rangle$$

$$r''(t) = \langle 0, -2\cos 2t, 2\cos 2t \rangle$$

$$|r'(t)| = \sqrt{(1)^2 + (-\sin 2t)^2 + (\sin 2t)^2}$$

$$= \sqrt{1 + \sin^2 2t + \sin^2 2t} = \sqrt{1 + 2\sin^2 2t}$$

Tangential component

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

$$= \frac{1(0) + (-\sin 2t)(-2\cos 2t) + (\sin 2t)(2\cos 2t)}{\sqrt{1 + 2\sin^2 2t}}$$

$$= \frac{2\sin 2t \cos 2t + 2\sin 2t \cos 2t}{\sqrt{1 + 2\sin^2 2t}}$$

$$= \frac{2\cdot 2\sin 2t \cos 2t}{\sqrt{1+2\sin^2 2t}} = \frac{2\sin 4t}{\sqrt{1+2\sin^2 2t}}$$

$$r'(t) \times r''(t) = \langle -\sin 2t\,(2\cos 2t) - \sin 2t\,(-2\cos 2t),\ \sin 2t\,(0) - 1(2\cos 2t),\ 1(-2\cos 2t) - (-\sin 2t)(0) \rangle$$

$$= \langle 0, -2\cos 2t, -2\cos 2t \rangle$$

Normal components,

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$= \frac{\sqrt{0^2 + (-2\cos 2t)^2 + (-2\cos 2t)^2}}{\sqrt{1+2\sin^2 2t}}$$

$$= \frac{\sqrt{2\cdot 4\cos^2 2t}}{\sqrt{1+2\sin^2 2t}} = \frac{2\sqrt{2\cos^2 2t}}{\sqrt{1+2\sin^2 2t}}$$