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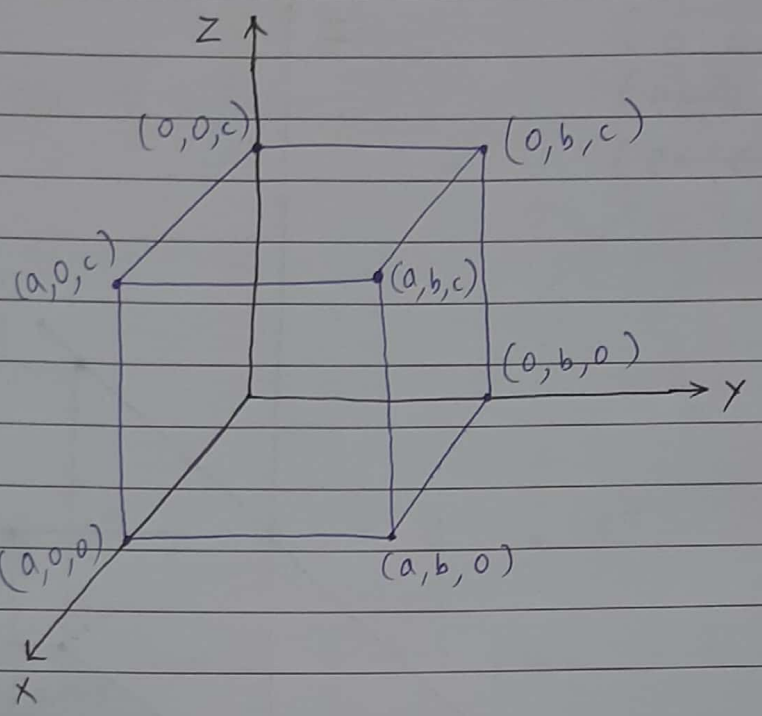
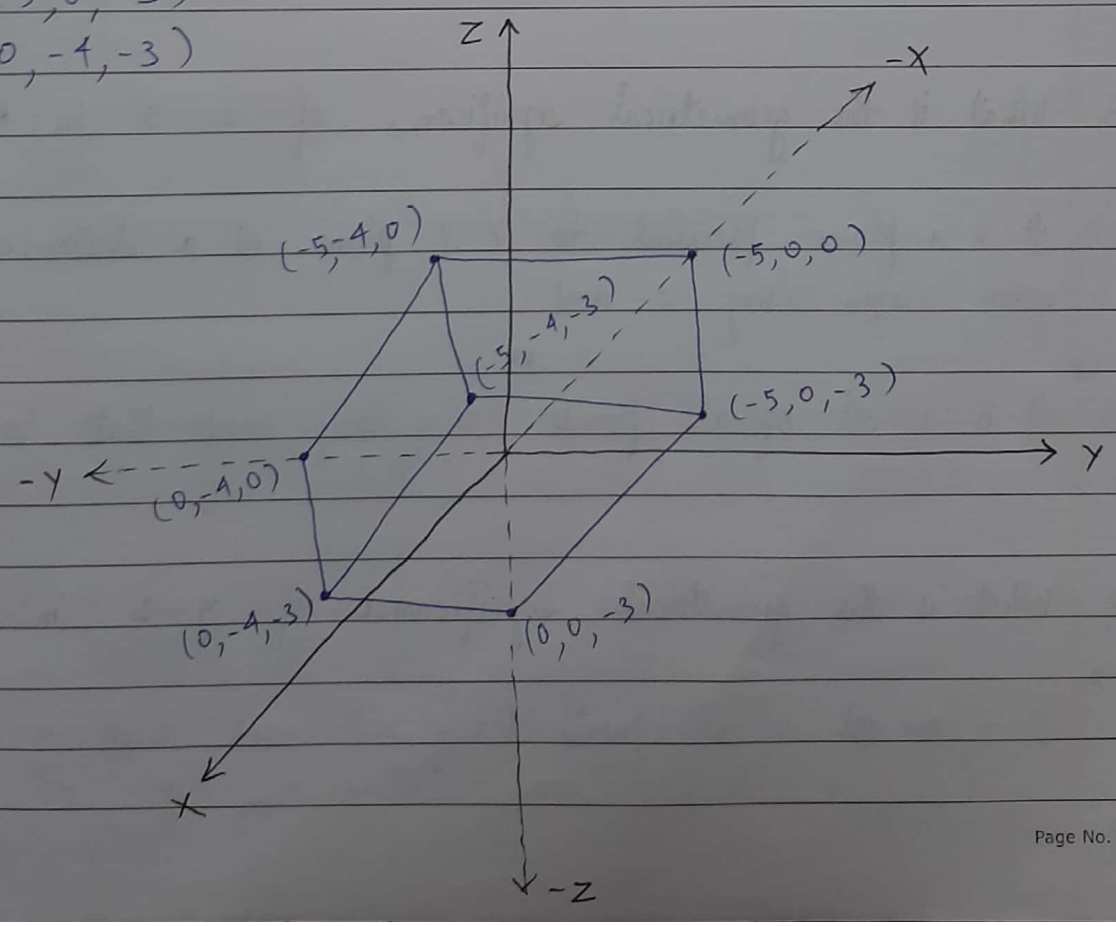
Chap. 12 Vectors and Geometry in Space



(a, b, c)
(a, b, 0)
(a, 0, c)
(0, b, c)

projections

Q. (-5, -4, -3)
(-5, -4, 0)
(-5, 0, -3)
(0, -4, -3)



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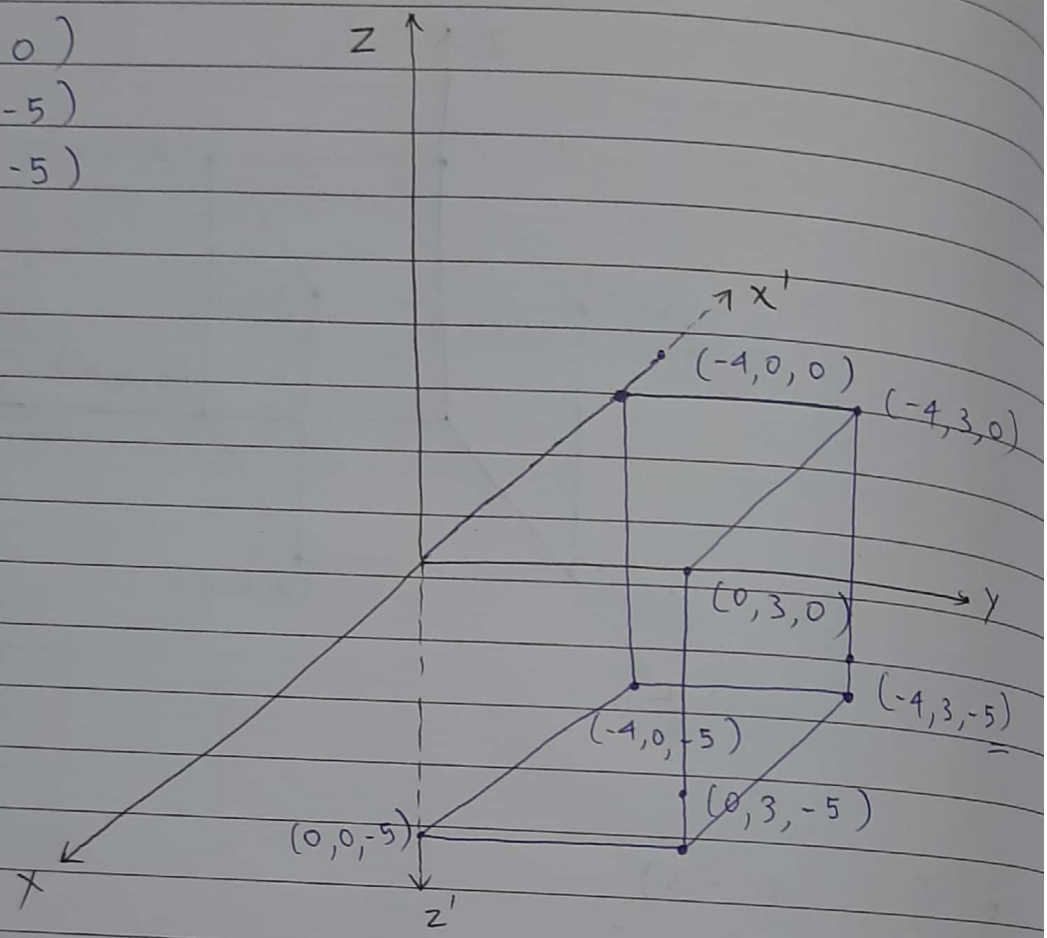
Q. Locate the point $(-4, 3, -5)$ as the corner point of a box.

$(-4, 3, 0)$
$(-4, 0, -5)$
$(0, 3, -5)$



Q. What is the geometrical significance of $z = 5$ in R^3 ? $\rightarrow (R \times R \times R)$

$\Rightarrow$ It is a plane parallel to x-y plane at a distance of 5 units from origin along z-axis.

or

It is a set of all points (x, y, z) such that $z = 5$.

Q. What is the geometrical significance of $y = 1$ in R^3.

$\Rightarrow$ It is a set of all points (x, y, z) such that $y = 1$.

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* Vector → locus of a point

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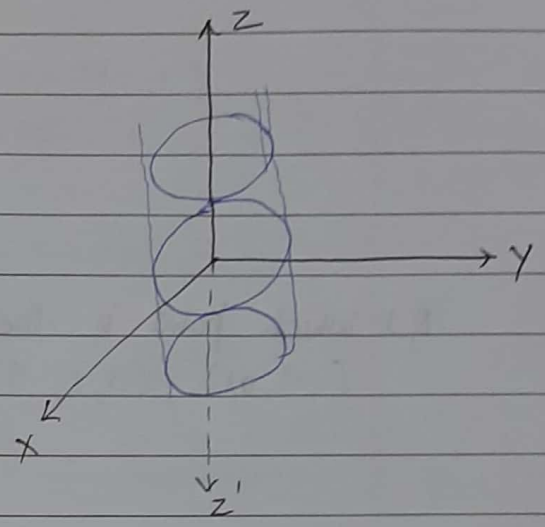
Q. Significance of $x^2 + y^2 = 1$ in R^3.

⇒ $x^2 + y^2 = 1$, $z = K \in R$.

It represents a ⊗ cylinder.

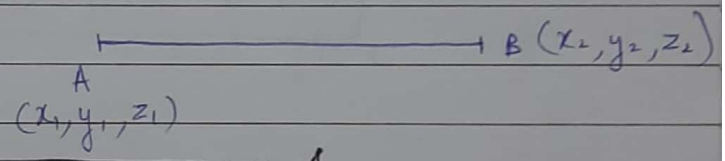


Q. Significance of $x^2 + y^2 = 1$, $z = 1$ in R^3.

⇒ It is a ⊗ plance parallel to x-y plane at a distance of 1 unit from origin along z-axis.

Note :-

① Distance formula :-



$$\boxed{\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}.$$

Q.4) Find an equ^n of the set of all points which is equidistant from the point $A(-1, 5, 3)$ and $B(6, 2, -2)$.

⇒ Let $P(x, y, z)$ be a point equidistant from A & B such that $\overline{PA} = \overline{PB}$.

$$\Rightarrow (x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2.$$

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$\Rightarrow x^2 + 1 + 2x + y^2 + 25 - 10y + z^2 + 9 - 6z = x^2 + 36 - 12x + y^2 + 1 - 4y + z^2 + 4 + 4z$.

$\Rightarrow$

Q.3 Which point is closest to the Y-Z plane ?
$(-4, 0, 1)$; $(3, 1, -5)$, $(2, 4, 6)$.

Q. $(3, 7, 5) \rightarrow$ Dist. of the point from X-axis ?
$(3, 0, 0)$

$= \sqrt{49 + 25} = \sqrt{74}$ Ans.

Note: ② Equation of sphere :-

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}.$$

Q.11 What is the intersection of the sphere with centre at $(1, -4, 3)$ and radius 5 with X-Z plane ?

$$(x-1)^2 + (0+4)^2 + (z-3)^2 = 5^2.$$

$\Rightarrow x^2 + 1 - 2x + 16 + z^2 + 9 - 6z = 25 - y^2 - 8y$.

$\Rightarrow x^2 + z^2 - 2x - 6z + y^2 + 8y = 25$.

$\because y = 0$.

$\Rightarrow (x-1)^2 + 16 + (z-3)^2 = 25$.

$\Rightarrow (x-1)^2 + (z-3)^2 = 9$. Ans.

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Q.17. Find the centre & radius of the given sphere :-

$$2x^2 + 2y^2 + 2z^2 = 8x - 24z + 1$$

$$2x^2 - 8x + 2y^2 + 2z^2 + 24z = 1$$

$$= \cancel{\sqrt{2}x +}$$

$$\Rightarrow x^2 + y^2 + z^2 - 4x + 12z = 1/2$$

$$\Rightarrow x^2 - 2\cdot 2x + 2^2 + y^2 + z^2 + 2\cdot 6\cdot z + 6^2 = \frac{1}{2} + 36 + 4$$

$$\Rightarrow (x-2)^2 + (y-0)^2 + (z+6)^2 = \frac{81}{2} = \left(\frac{9}{\sqrt{2}}\right)^2$$

$\therefore$ Centre $\equiv (2, 0, -6)$

Radius $\equiv \frac{9}{\sqrt{2}}$ Ans.

Q.35 Write inequalities to describe the region :-

The region b/w Y-Z plane & vertical plane $X = 5$.

$\Rightarrow$ $\cancel{(y-b)^2 + (z-c)^2 + 25 = r^2}$.

$0 \le x \le 5$ Ans.

Q. Significance of $y < 5$ in $\mathbb{R}^3$.

$\Rightarrow$ Half - space left to the plane $y = 5$.

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→ **Vectors** :-

→ **Components of a vector** :-

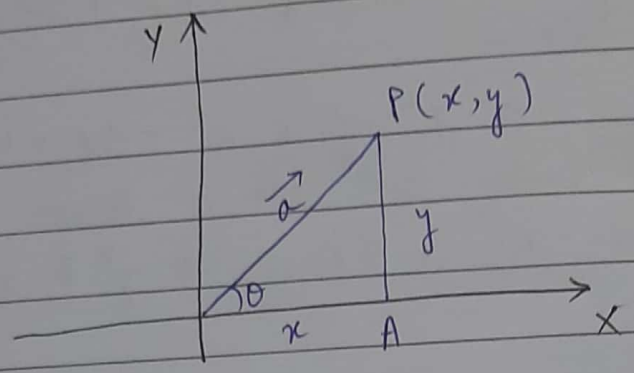
$\vec{OP} = \vec{a} = \langle x, y \rangle$

$(x, y) \rightarrow$ Point

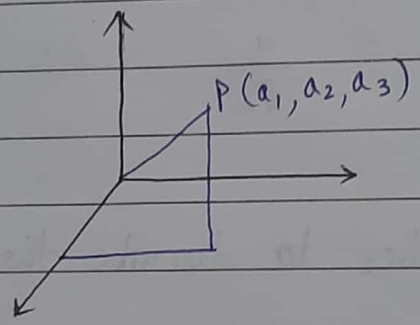
$\langle x, y \rangle$

↓

Vector representation.



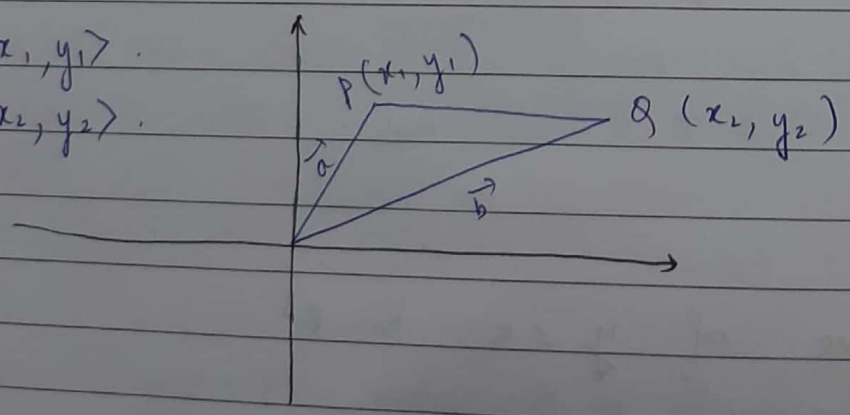
In 3-D $\vec{a} = \langle a_1, a_2, a_3 \rangle$ = P.V of a point.



→ **P.V of a line segment** :-

$\vec{a} = \langle x_1, y_1 \rangle$.

$\vec{b} = \langle x_2, y_2 \rangle$.



$$\boxed{\text{P.V of } \vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle}.$$

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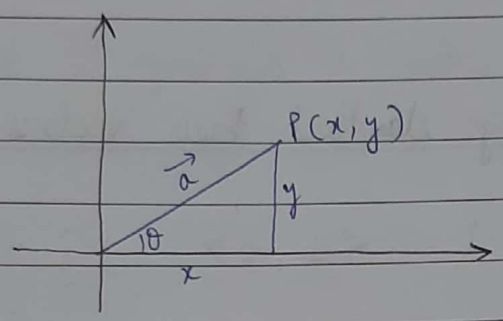
P.V of a line segment joining $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is

$$\boxed{P.V = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle}.$$

→ <u>Magnitude of a vector</u> :-



$$|\vec{a}| = \sqrt{x^2 + y^2}.$$

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}.$$

If $|\vec{a}| = 1$ → Unit vector.

$$\cos\theta = \frac{x}{|\vec{a}|} \Rightarrow x = |\vec{a}| \cos\theta.$$

$$\sin\theta = \frac{y}{|\vec{a}|} \Rightarrow y = |\vec{a}| \sin\theta.$$

$$\vec{a} = \langle x, y \rangle = \langle |\vec{a}| \cos\theta, |\vec{a}| \sin\theta \rangle.$$

→ <u>Unit vector in a dir^n of non-zero vector</u> :-

Let $\vec{a} = \langle a_1, a_2 \rangle$, $|\vec{a}| \neq 1$.

$$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle a_1, a_2 \rangle}{|\vec{a}|} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|} \right\rangle.$$

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eg:- $\vec{a} = \langle 3, 4 \rangle$.

$\Rightarrow |\vec{a}| = 5$.

$\Rightarrow \hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$.

$|\hat{a}| = \sqrt{\dfrac{9}{25} + \dfrac{16}{25}} = 1$.

→ <u>Vectors using standard basis vectors</u> :-

In 2-D, $\hat{i} = \langle 1, 0 \rangle$; $\hat{j} = \langle 0, 1 \rangle$.

In 3-D, $\hat{i} = \langle 1, 0, 0 \rangle$; $\hat{j} = \langle 0, 1, 0 \rangle$
; $\hat{k} = \langle 0, 0, 1 \rangle$.

$$\vec{a} = \langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle$$
$$= a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle.$$
$$= a_1 \hat{i} + a_2 \hat{j}.$$

Q.26 Find a vector that has same dir^n as $P\langle -2, 4, 2 \rangle$ and has length 6.

→

$|\vec{P}| = \sqrt{24}$

$\hat{P} = \left\langle \dfrac{-2}{\sqrt{24}}, \dfrac{4}{\sqrt{24}}, \dfrac{2}{\sqrt{24}} \right\rangle$.

$\vec{Q} = 6\hat{P} = \left\langle \dfrac{-12}{\sqrt{24}}, \dfrac{24}{\sqrt{24}}, \dfrac{12}{\sqrt{24}} \right\rangle$ Ans.

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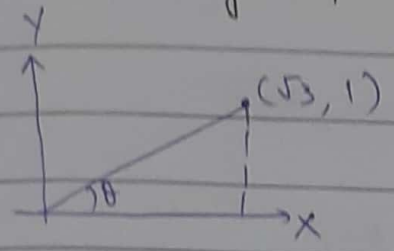
Q.27 Find the angle b/w the vector & +x-axis. $(\sqrt{3}\hat{i} + \hat{j})$.

→



~~sinθ~~ → $\tan\theta = \frac{1}{\sqrt{3}}$.

$\Rightarrow \theta = 30^\circ$ Ans.

Q.29 If $\vec{v}$ lies in the 1st quadrant & makes an $\angle\ \pi/3$ with +x-axis & magnitude of $\vec{v} = 4$. Find components of $\vec{v}$.

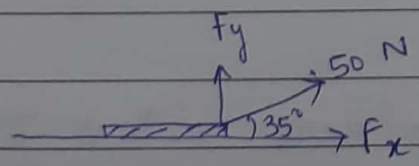
$\Rightarrow \sin\frac{\pi}{3} = \frac{y}{4} \Rightarrow \frac{\sqrt{3}}{2} = \frac{y}{4} \rightarrow y = 2\sqrt{3}$.

$\cos\frac{\pi}{3} = \frac{x}{4} \rightarrow \frac{1}{2} = \frac{x}{4} \rightarrow x = 2$.

$\Rightarrow \vec{v} = 2\hat{i} + 2\sqrt{3}\hat{j}$. Ans.

Q.30 If a child pulls a sled through the snow on a level surface ~~on~~ with a force of 50 N and exerted at an $\angle\ 35^\circ$ above the horizontal. Find the vertical & horizontal component of this force

⇒



$F_y = 50 \sin 35^\circ$.

$F_x = 50 \cos 35^\circ$.

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→ **Dot Product** (Scalar / Inner product) :-

Let $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$.

$$\vec{a} \cdot \vec{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle$$

$$= \underline{a_1 b_1 + a_2 b_2} .$$

$$\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= \underline{a_1 b_1 + a_2 b_2 + a_3 b_3} .$$

- **Properties** :-

i) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

iii) $c(\vec{a} + \vec{b}) = c \cdot \vec{a} + c \cdot \vec{b}$.

iv) $0 \cdot \vec{a} = 0$.

Def^n: If 'θ' be the angle b/w 2 non-zero vectors $\vec{a}$ & $\vec{b}$,

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta} .$$

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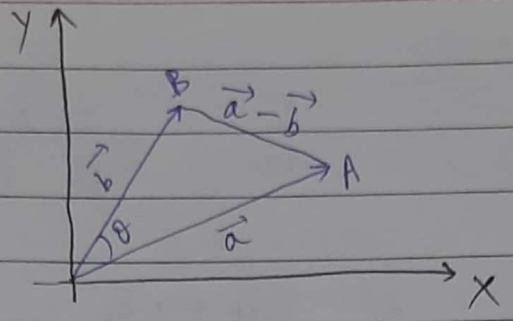

$$|\vec{AB}|^2 = |\vec{OA}|^2 + |\vec{OB}|^2 - 2\,|OA|\,|OB|\cos\theta .$$

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,|\vec{a}|\,|\vec{b}|\cos\theta .$$

a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \boxed{\theta = \cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]}$.

b) $\theta = 90^\circ$
$\vec{a}\cdot\vec{b} = 0$.

Two vectors are orthogonal means $\vec{a}\cdot\vec{b} = 0$.

c) Let $\theta = 0$, $\vec{a} = \vec{b}$ $\Rightarrow 1 = \dfrac{\vec{a}\cdot\vec{a}}{|\vec{a}||\vec{a}|}$.

If $a = \hat{i} \Rightarrow \hat{i}\cdot\hat{i} = 1$.
$a = \hat{j} \Rightarrow \hat{j}\cdot\hat{j} = 1$.
$a = \hat{k} \Rightarrow \hat{k}\cdot\hat{k} = 0$.

Q.26 Find the value of x such that the $\angle$ b/w the vector $\langle 2, 1, -1\rangle$ & $\langle 1, x, 0\rangle$ is 45°.

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$\Rightarrow 2 + x = 0$

$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{2+x}{\sqrt{6} \cdot \sqrt{1+x^2}}$

$\Rightarrow 2\sqrt{2} + x\sqrt{2} = \sqrt{6} \cdot \sqrt{1+x^2}$.

$\Rightarrow 8 + 2x^2 + 8x = 6 \cdot 1 + x^2$.

$\Rightarrow 4 + x^2 + 4x = 6 + x^2$

$\Rightarrow 2x^2 = 4x - 2$

$\Rightarrow 2x^2 = 4x + 1$.

$\Rightarrow x^2 + 2x + 1 = 0$.

$\Rightarrow x = \frac{+2 \pm \sqrt{4-4}}{2}$

$\Rightarrow x =$

Q.27 Find a unit vector which is orthogonal to both $\hat{i}+\hat{j}$ & $\hat{i}+\hat{k}$.

$\Rightarrow$

$\frac{\hat{i}+\hat{j}}{\sqrt{\hat{i}^2+\hat{j}^2}} \cdot \frac{\hat{i}+\hat{k}}{\sqrt{\hat{i}^2+\hat{k}^2}} = 0$.

$\Rightarrow \hat{i}+\hat{j}\sqrt{\hat{i}^2+\hat{k}^2} \cdot \hat{i}+\hat{k}\sqrt{\hat{i}^2+\hat{j}^2} = 0$.

$\Rightarrow 1 + 0 + 0 + 0$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be a unit vector $\perp^r$ to $\vec{b} = \hat{i}+\hat{j}$ & $\vec{c} = \hat{i}+\hat{k}$.

$\vec{a} \cdot \vec{b} = 0$.

$\Rightarrow a_1 + a_2 = 0$ — (i).

$\vec{a} \cdot \vec{c} = 0 \Rightarrow a_1 + a_3 = 0$ — (ii).

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$\Rightarrow a_2 = a_3 = -a_1$

$|\vec{a}| = 1$.

$a_1 = \frac{1}{\sqrt{3}}$, $a_2 = -\frac{1}{\sqrt{3}}$, $a_3 = -\frac{1}{\sqrt{3}}$.

$a_1 = -\frac{1}{\sqrt{3}}$, $a_2 = \frac{1}{\sqrt{3}}$, $a_3 = \frac{1}{\sqrt{3}}$.

Q. Find 2 unit vectors that makes an $\angle 60^\circ$ with vector $\vec{v}$.
$\vec{v} = \langle 3, 4 \rangle$.

$\Rightarrow \hat{v} = \frac{3\hat{i} + 4\hat{j}}{5}$

Let $\vec{a} = \langle a_1, a_2 \rangle$.
$\vec{b} = \langle b_1, b_2 \rangle$.

$\Rightarrow \because |\vec{a}| = 1$.

$\Rightarrow a_1^2 + a_2^2 = 1$ — (I).

$\cos 60^\circ = \frac{\langle a_1, a_2 \rangle \cdot \langle 3, 4 \rangle}{1 \cdot 5}$

$\Rightarrow \frac{1}{2} = \frac{3a_1 + 4a_2}{5}$.

$\Rightarrow 6a_1 + 8a_2 = 5$ — (II).

$\Rightarrow a_2 = \frac{5 - 6a_1}{8}$.

$\Rightarrow a_1^2 + \frac{(5-6a_1)^2}{64} = 1$.

$\Rightarrow 64a_1^2 + (5-6a_1)^2 = 64$.

$\Rightarrow a_1, a_2 =$

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Q.24 Determine whether the given vectors are $\perp^r$, $\parallel^{el}$ or neither.

a) $u = \langle -3, 9, 6 \rangle$, $v = \langle 4, -12, -8 \rangle$

$\Rightarrow$

$\vec{u} \cdot \vec{v} \neq 0$ $\therefore$ ~~Neither~~

$\vec{v} = -\frac{4}{3} \langle -3, 9, 6 \rangle$

$\because \vec{v} = \alpha \vec{u}$

$\therefore \vec{v} \parallel^{el} \vec{u}$ Ans.

b) $\vec{u} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{v} = 2\hat{i} - \hat{j} + \hat{k}$.

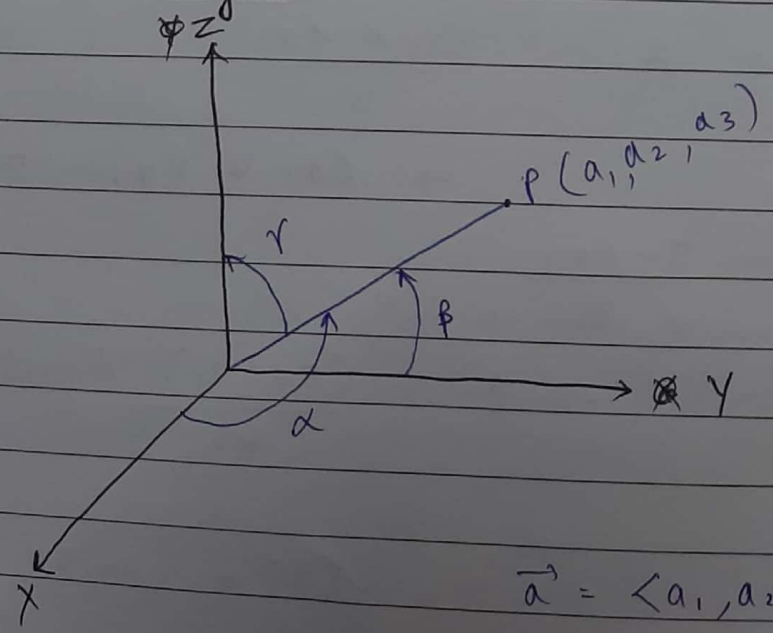
$\Rightarrow$

$\vec{u} \cdot \vec{v} \neq 0$. $\therefore$ Neither Ans.

c) $\vec{u} = \langle a, b, c \rangle$
$\vec{v} = \langle -b, a, 0 \rangle$

$\Rightarrow \because \vec{u} \cdot \vec{v} = 0$ $\therefore$ Orthogonal Ans.

→ <u>Dir^n cosines & angles</u> :-



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

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$$\begin{cases} \cos\alpha = \dfrac{\vec{a}\cdot\hat{i}}{|\vec{a}||i|} = \dfrac{a_1}{|\vec{a}|} \\ \cos\beta = \dfrac{a_2}{|\vec{a}|} \; ; \; \cos\gamma = \dfrac{a_3}{|\vec{a}|} \end{cases}$$

→ <u>Direction cosines</u> :- D.C of $\vec{a} = \langle a_1, a_2, a_3 \rangle$.

Q.34 Find dc's and ∠s of $\langle 6, 3, -2 \rangle$.

⇒

Let $\vec{a} = \langle 6, 3, -2 \rangle$.

$\cos\alpha = \frac{6}{7}$; $\cos\beta = \frac{3}{7}$; $\cos\gamma = -\frac{2}{7}$.

$\alpha = \cos^{-1}\left(\frac{6}{7}\right)$; $\beta = \cos^{-1}\left(\frac{3}{7}\right)$; $\gamma = \cos^{-1}\left(\frac{2}{7}\right)$.

Q.35 If a vector has dir^n angles $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$. Find γ.

⇒

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

$\Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$.

$\Rightarrow \cos^2\gamma = \frac{1}{4}$. $\rightarrow$ $\cos\gamma = \pm\frac{1}{2}$.

$\rightarrow \gamma = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ Ans.

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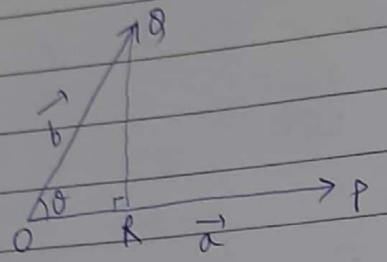
→ **Projection** (Scalar, vector) :-



$$\cos\theta = \frac{OR}{|\vec{b}|}.$$

$$OR = |\vec{b}|\cos\theta.$$

Scalar projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

$$\boxed{\text{Comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}.$$

Vector projection of $\vec{b}$ on '$\vec{a}$',

$$\boxed{\text{Proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right)}.$$

Q.40 Find both scalar & vector proj^n of $\vec{b}$ on $\vec{a}$.
$\vec{a} = \langle 1, 4\rangle$, $\vec{b} = \langle 2, 3\rangle$.

→

$$\text{Comp}_{\vec{a}}\,\vec{b} = \frac{14}{\sqrt{17}}. \qquad \text{Proj}_{\vec{a}}\,\vec{b} = \frac{14}{\sqrt{17}} \times \left(\frac{\hat{i}+4\hat{j}}{\sqrt{17}}\right).$$

Ans.

Q.29 Find the acute ∠ b/w the lines $2x - y = 3$ & $3x + y = 7$.

→

$2x - y = 3$

$\rightarrow y = 2x - 3$ — (I).

$3x + y = 7$.

$\rightarrow y = 7 - 3x$ — (II).

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(I), $m_1 = \frac{2}{1} = \frac{y_2 - y_1}{x_2 - x_1}$.

$\Rightarrow \vec{v_1} = \langle 1, 2 \rangle$.

(II) $m_2 = \frac{-3}{1} \Rightarrow \vec{v_2} = \langle 1, -3 \rangle$.

$$\Rightarrow \cos\theta = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}||\vec{v_2}|} = \frac{1-6}{\sqrt{5}\sqrt{10}} = \frac{-5}{\sqrt{5}\sqrt{10}} = -\frac{1}{\sqrt{2}}.$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right).$$

$$\Rightarrow \theta = 180^\circ - 135^\circ = 45^\circ \text{ Ans.}$$

Q.3) Find the acute $\angle$ b/w the curve at their pt. of intersec^n
$y = x^2$, $y = x^3$.

$\Rightarrow$ Pt. of intersection $= (1, 1)$; $(0, 0)$.

$m_1 = 2x$. $\Rightarrow \vec{v_1} = \langle 1, 2x \rangle$. ; $\vec{v_1} = \langle 0, 0 \rangle$.

$m_2 = 3x^2$. $\Rightarrow \vec{v_2} = \langle 1, 3x^2 \rangle$. $\vec{v_2} = \langle 0, 0 \rangle$.

$$\cos\theta = \frac{1 + 6x^2}{\sqrt{5}\sqrt{10}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{5}\sqrt{10}}\right). \Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right).$$

& $\theta = 0^\circ$ Ans.

→ <u>Cross Product</u> (Vector Product) :-

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$; $\vec{b} = \langle b_1, b_2, b_3 \rangle$.
let $\vec{c} = \langle c_1, c_2, c_3 \rangle$.

$\vec{a} \cdot \vec{c} = 0$.

$a_1 c_1 + a_2 c_2 + a_3 c_3 = 0$ —— (1).

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$\vec{b} \cdot \vec{c} = 0$.

$\Rightarrow b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$ —— (II).

(I) $\times b_3$ − (II) $\times a_3$.

$\Rightarrow b_3 (a_1 c_1 + a_2 c_2) - a_3 (b_1 c_1 + b_2 c_2) = 0$.

$\Rightarrow \underbrace{(b_3 a_1 - a_3 b_1)}_{p} c_1 + c_2 \underbrace{(b_3 a_2 - a_3 b_2)}_{q} = 0$

$\Rightarrow p c_1 + q c_2 = 0$.

If $c_1 = q$ & $c_2 = -p$,

$c_1 = b_3 a_2 - a_3 b_2$.

$c_2 = b_3 a_1 - a_3 b_1$.

<u>Def^n</u> :- If $\vec{a}$ & $\vec{b}$ are 2 non-zero vectors then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

<u>Def^n</u> :- If $\vec{a}$ & $\vec{b}$ are 2 non-zero vectors, θ is the angle b/w them and $\hat{n}$ is a unit normal vector,

then,

$$\boxed{\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}}.$$

<u>Note</u> :-

(I) $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.

$$\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}}.$$

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(ii) $|\vec{a} \times \vec{b}|$ = area of $\|^{ogm}$.

(iii) $\frac{1}{2}|\vec{a} \times \vec{b}|$ = area of $\triangle$.

Q.19 Find 2 unit vectors which are orthogonal to both the vector $\langle 3, 2, 1 \rangle$ and $\langle -1, 1, 0 \rangle$.

→

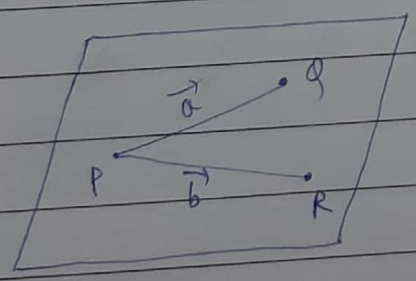
$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = i(-1) - j(1) + k(5).$$

$$= -\hat{i} - \hat{j} + 5\hat{k}.$$

Let $\vec{c} = -\hat{i} - \hat{j} + 5\hat{k}$.

$$\hat{c} = \frac{-\hat{i} - \hat{j} + 5\hat{k}}{\sqrt{27}}$$ Ans.

Q. Find a vector orthogonal to the plane passes through the pt $P(1, 4, 6)$; $Q(-2, 5, -1)$; $R(1, -1, 1)$.



$\vec{a} = \vec{PQ} = \langle -3, 1, -7 \rangle$.

$\vec{b} = \vec{PR} = \langle 0, -5, -5 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}.$$

$$\Rightarrow -40\hat{i} - 15\hat{j} + 15\hat{k}$$

Ans

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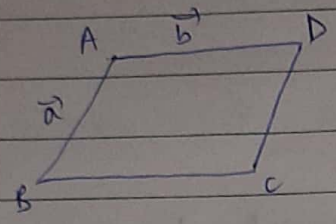
Q.27 Find the area of $\|^{gm}$ with vertices $A(-2,1)$; $B(0,4)$; $C(4,2)$; $D(2,-1)$.



$\vec{b} = 4\hat{i} - 2\hat{j}$

$\vec{a} = +2\hat{i} - 3\hat{j}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 0 \\ 2 & -3 & 0 \end{vmatrix}.$$

$$= -16\hat{k}$$

$\therefore |\vec{a} \times \vec{b}| = 16$ Ans.

Q.43 If $\vec{a} \cdot \vec{b} = \sqrt{3}$.

$\vec{a} \times \vec{b} = \langle 1, 2, 2 \rangle$.

Find $\angle$ b/w $\vec{a}$ & $\vec{b}$.

$\rightarrow$

$$\sin\theta \; \cancel{\vec{a} \times \vec{b}} = \frac{|\hat{i} + 2\hat{j} + 2\hat{k}|}{|\vec{a}|\,|\vec{b}|} = \frac{3}{\vec{a} \cdot \vec{b}} \cos\theta.$$

$\Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ Ans

$\rightarrow$ <u>Properties</u> :-

a) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. i.e $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

b) $\vec{a} \times \vec{a} = 0$.

c) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$. (Scalar Triple product)

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d) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

e) $|\vec{a}\cdot(\vec{b}\times\vec{c})|$ = Vol. of a parallopiped.

f) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$.

Q.48 If $\vec{a}+\vec{b}+\vec{c}=0$ S.T $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

$\Rightarrow$

$\vec{a}+\vec{b}+\vec{c} = 0$. $\quad \vec{b} = -(\vec{a}+\vec{c})$

$\Rightarrow \vec{a} = -(\vec{b}+\vec{c})$.

$\Rightarrow \vec{a}\times\vec{b} = \cancel{-(\vec{b}+\vec{c})\times(}$

$= -(\vec{b}+\vec{c})\times-(\vec{a}+\vec{c})$.

$= -(\vec{a}\times\vec{b}) + \vec{a}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}\times\vec{c}$.

$= -[(\vec{a}\times\vec{b}) + (\vec{b}\times\vec{c}) + \vec{c}\times\vec{a}]$.

$\vec{b}\times\vec{c} = -(\vec{a}+\vec{c})\times-(\vec{a}+\vec{b})$

$= -(\vec{a}\times\vec{a}) + (\vec{a}\times\vec{b}) + (\vec{c}\times\vec{a}) + \vec{c}\times\vec{b})$

$= -[(\vec{a}\times\vec{b}) + (\vec{c}\times\vec{a}) + (\vec{c}\times\vec{b})]$.

proved.

Q.44 i) Find all vectors v s.t $\langle 1,2,1\rangle \times v = \langle 3,1,-5\rangle$.

ii) Explain why there is no v such that $\langle 1,2,1\rangle \times v = \langle 3,1,5\rangle$

$\Rightarrow$ ~~v × ξ~~ a) let $v = a\hat{i} + b\hat{j} + c\hat{k}$.

$\rightarrow a\hat{i} + b\hat{j} + c\hat{k} \times (\hat{i} + 2\hat{j} + \hat{k}) =$

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$$\Rightarrow \vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (b-2c)\hat{i} + (a-c)\hat{j} + (2a-b)\hat{k}.$$
$$= 3\hat{i} + \hat{j} - 5\hat{k}.$$

$\Rightarrow$ $b - 2c = 3 \quad \Rightarrow b = 3 + 2c.$

$a - c = 1 \quad \Rightarrow a = 1 + c.$

$2a - b = -5. \quad \Rightarrow \cancel{b} = 2 + 2c - b = -5.$

$\Rightarrow 2c - b = -3.$

$\Rightarrow \cancel{2a - 3 - 2c = -3}.$

$\Rightarrow$

$\Rightarrow v = \langle x, 2a-5, a-1 \rangle, \; x \in R.$

$x = 0 \quad \Rightarrow v \cdot \langle 3, 1, -5 \rangle = 0.$

$\Rightarrow 0 - 5 + 5 = 0.$

$x = 3 \quad \Rightarrow 9 + 1 - 10 = 0.$

b) $\langle 1, 2, 1 \rangle \cdot \langle 3, 1, 5 \rangle = 10 \neq 0.$

$\therefore$ There is no vector v that $\langle 1, 2, 1 \rangle \times v = \langle 3, 1, 5 \rangle$

Ans

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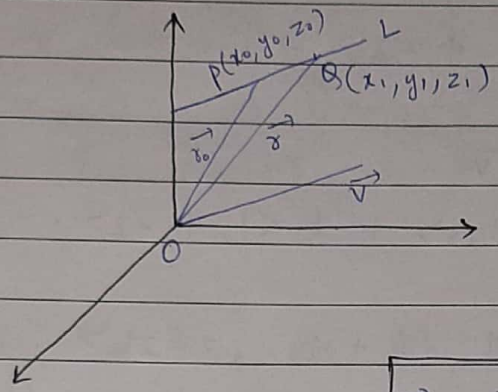
→ Equation of Lines and Planes :-

→ Equation of a line passing through a pt. and ||el to a vector :-



$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$.

$\vec{r} = \langle x_1, y_1, z_1 \rangle$.

$\vec{PQ} = \vec{r} - \vec{r_0}$.

$\vec{r} - \vec{r_0} = t\vec{v}$,

$\Rightarrow \boxed{\vec{r} = \vec{r_0} + t\vec{v}} \rightarrow$ vector equ^n of line.

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle.$$

$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.$$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \rightarrow \text{Parametric equ}^n \text{ of a line.}$$

$$t = \frac{x - x_0}{a} \; ; \; \frac{y - y_0}{b} = t \; ; \; \frac{z - z_0}{c} = t.$$

$$\Rightarrow \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \rightarrow \text{Symmetric equ}^n \text{ of line.}$$

Note :- The two lines are ||el to each other if their dir^n nos. are proportionate. ~~ie~~

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Q. Find the vector equ^n of a line that is passing through a pt. $(5,1,3)$ & $\parallel^{el}$ to $\hat{i} + 4\hat{j} - 2\hat{k}$. Find 2 other pts. on the line.

$\Rightarrow$ a) Let $r_0 = \langle 5, 1, 3 \rangle$.

$\vec{v} = \langle 1, 4, -2 \rangle$.

$\vec{r} = \vec{r_0} + t\vec{v}$

$= \langle 5,1,3 \rangle + t\langle 1, 4, -2 \rangle$.

$\Rightarrow \vec{r} = \langle 5+t, 1+4t, 3-2t \rangle$.

b) $t = 1 \Rightarrow \vec{r_1} = \langle 6, 5, 1 \rangle$

$t = 2 \Rightarrow \vec{r_2} = \langle 7, 9, -1 \rangle$. Ans.

Q. i) Find the parametric & symmetric equ^n of line passing through $A(2,4,-3)$ & $B(3,-1,1)$.

ii) At what pt. it intersects x-y plane?

$\Rightarrow \vec{AB} = \hat{i} - 5\hat{j} + 4\hat{k}$.

Let $\vec{v} = \langle \underset{a}{1}, \underset{b}{-5}, \underset{c}{4} \rangle$. $\rightarrow$ (dir^n vector of $\vec{AB}$).

$x = x_0 + at \Rightarrow x = 2 + t$.

$y = y_0 + bt \Rightarrow y = 4 - 5t$.

$z = z_0 + ct \Rightarrow z = -3 + 4t$.

& $\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$.

In x-y plane, $z = 0$.

$\Rightarrow x - 2 = \dfrac{y-4}{-5} = \dfrac{3}{4}$ Ans.

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Q. Find the pt. at which the line $x = 2+3t$, $y = -4t$, $z = 5+t$ intersect the plane $4x + 5y - 2z = 18$.

$\Rightarrow 8 + 12t - 20t - 10 - 2t = 18$.

$\Rightarrow -2 - 10t = 18$.

$\Rightarrow t = -2$.

$\Rightarrow x = -4$, $y = +8$, $z = 3$. Ans.

Q.22 Determine the two lines are $\parallel^{el}$, skew or intersecting each other. Find pt. of intersection.

$$L_1 = \frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3}.$$

$$L_2 = \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}.$$

$\Rightarrow$

$\vec{v_1} = \langle 1, -1, 3 \rangle = \langle a_1, b_1, c_1 \rangle$.

$\vec{v_2} = \langle 2, -2, 7 \rangle = \langle a_2, -b_2, c_2 \rangle$.

$\Rightarrow \parallel^{el} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{1}{2} = \frac{+1}{+2} \neq \frac{3}{7}$.

$\therefore$ Not $\parallel^{el}$.

From L_1,

$x = t$, $y = -t+1$, $z = 3t+2$.

From L_2,

$x = 2s+2$, $y = -2s+3$, $z = 7s$.

$t = 2s+2$ — (I).

$-t+1 = -2s+3$ — (II).

$3t+2 = 7s$ — (III).

$5 = s+9 \Rightarrow s = -4$.

$\therefore t = -10$.

$-10 \neq -6$.

$\therefore$ No pt. of intersection.

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Q.21 $L_1 = \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$.

$L_2 = \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$.

If intersecting, find $\angle$ b/w them.

$\rightarrow$ $\vec{v_1} = \langle 1, -2, -3 \rangle = \langle a_1, b_1, c_1 \rangle$.

$\vec{v_2} = \langle 1, 3, -7 \rangle = \langle a_2, b_2, c_2 \rangle$.

$\parallel^{el} \Rightarrow \frac{1}{1} \neq \frac{-2}{3} \neq \frac{+3}{7}$ $\therefore$ Not $\parallel^{el}$.

From L_1,

$x = t+2, \quad y = -2t+3, \quad z = -3t+1$.

L_2,

$x = s+3, \quad y = 3s-4, \quad z = -7s+2$.

$\Rightarrow$ Let

$t+2 = s+3$ — (I)

$-2t+3 = 3s-4$ — (II)

$-3t+1 = -7s+2$ — (III).

$2 \times$ (I) + (II),

$\Rightarrow 7 = 5s+2$

$\Rightarrow 5s = 5 \Rightarrow s = 1$.

$\therefore t = 2$.

$\Rightarrow \cos\theta = \cancel{\frac{(0\hat{i} + 5\hat{j} - 4\hat{k})}{\sqrt{14}\sqrt{59}}} = \frac{(1-6+21)}{\sqrt{14}\sqrt{59}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{16}{\sqrt{14}\sqrt{59}}\right)$ Ans.

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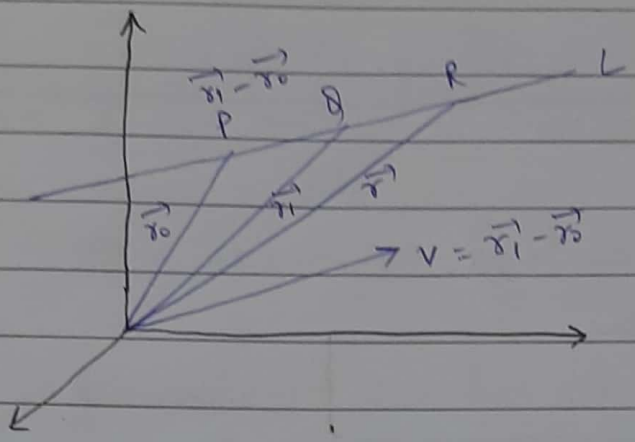
Note :- ① Equ^n of line passing through 2 pts. (x_0, y_0, z_0) & (x_1, y_1, z_1):-



$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$.

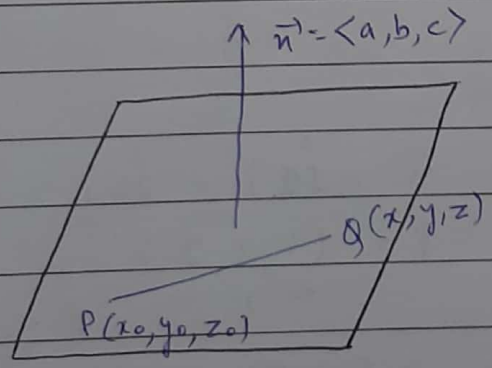
$\vec{r_1} = \langle x_1, y_1, z_1 \rangle$.

$\vec{r} = \langle x, y, z \rangle$.

$\vec{PQ} = \vec{r_1} - \vec{r_0}$.

$\therefore$ Equ^n of line, $\boxed{\vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0})}$.

→ Equation of Planes :-



$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$.

$\vec{r} = \langle x, y, z \rangle$.

$\vec{PQ} = \vec{r} - \vec{r_0}$.

Equ^n of plane passing through a pt. (x_0, y_0, z_0) with a normal vector $\vec{n}$ is given by, $n \cdot (\vec{r} - r_0) = 0$.

$$\rightarrow \boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}.$$

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Note: ① $\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$.

components of normal vector

$\Rightarrow \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \rightarrow$ algebraic equ^n of plane.

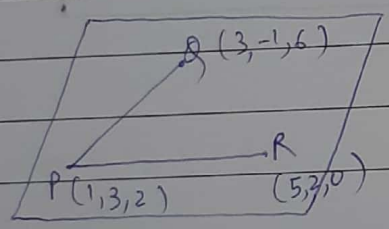
② $ax + by + cz - ax_0 - by_0 - cz_0 = 0$

$\Rightarrow \boxed{ax + by + cz + d = 0}$ where $d = -ax_0 - by_0 - cz_0$.

$\rightarrow$ Standard equ^n of plane.

Q. Find the equ^n of the plane that is passing through $P(1,3,2)$; $Q(3,-1,6)$; $R(5,2,0)$.

$\Rightarrow$



$\vec{a} = \vec{PQ} = \langle 2, -4, 4 \rangle$.

$\vec{b} = \vec{PR} = \langle 4, -1, -2 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$\vec{n} = \overset{a}{12}\,\hat{i} + \overset{b}{20}\,\hat{j} + \overset{c}{14}\,\hat{k}$ (normal vector)

$P = (\underset{x_0}{1}, \underset{y_0}{3}, \underset{z_0}{2})$

$\therefore$ Equ^n of plane $= 12(x-1) + 20(y-3) + 14(z-2) = 0$

$\rightarrow 12x - 12 + 20y - 60 + 14z - 28 = 0$.

$\rightarrow 12x + 20y + 14z - 100 = 0$.

$\rightarrow 6x + 10y + 7z - 50 = 0$ Ans.

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Note: (i) Angle b/w 2 planes is same as angle b/w their normal vectors.

(ii) The 2 planes are $\parallel^{el}$ if one of the normal vector is a scalar multiple of other normal vector.

Q. i) Find 'θ' b/w 2 planes $x+y+z=1$ & $x-2y+3z=1$.
* ii) Find symmetric equ^n for line of intersection of these 2 planes.

$\Rightarrow$ i) $x+y+z=1$
let $\vec{n_1} = \langle 1, 1, 1 \rangle$.

$x-2y+3z=1$.
Let $\vec{n_2} = \langle 1, -2, 3 \rangle$.

$$\Rightarrow \cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{1-2+3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}.$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \text{ Ans.}$$

ii) $x+y+z=1$.
$x-2y+3z=1$.
If $z=0$,
$\Rightarrow x+y=1$
$x-2y=1$.
$\Rightarrow 3y=0 \Rightarrow y=0$ & $x=1$ & $z=0$.

$\therefore (1,0,0)$ is a pt. lies over the line.

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$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 2\hat{j} - 3\hat{k}.$$

$$= \langle 5, -2, -3 \rangle.$$

$\therefore$ Parametric equ^n $\Rightarrow$

$$x = 1 + 5t.$$
$$y = -2t$$
$$z = -3t.$$

Ans.

Q.54 Determine 2 given planes are $\parallel^{el}$, $\perp^r$ or neither. If neither then find the angle b/w them.

$x + y + z = 1$ & $x - y + z = 1$.

$\Rightarrow$

$\vec{n_1} = \langle 1, 1, 1 \rangle$.

$\vec{n_2} = \langle 1, -1, 1 \rangle$.

$$\cos\theta = \frac{1 - 1 + 1}{\sqrt{3}\,\sqrt{3}} = \frac{1}{3}. \quad \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right).$$

* Q.57 Find the ~~param~~ symmetric equ^n for the line of intersection of the planes $x + y + z = 1$ & $x + 2y + 2z = 1$.

$\Rightarrow$

$x + y + z = 1$

$x + 2y + 2z = 1$.

for $(z = 0)$,

$x + y = 1$.

$x + 2y = 1$.

$\Rightarrow -y = 0 \Rightarrow y = 0$, $x = 1$, $z = 0$.

$\therefore (1, 0, 0)$ lies on ~~plane~~ line.

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$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \hat{i}(2-2) + \hat{j}(2-1) + \hat{k}(2-1)$$

$$= 0\hat{i} - \hat{j} + \hat{k}.$$

$$= \langle 0, -1, 1 \rangle.$$

$\therefore$ Symmetric equ^n of line.

$$= \frac{x-1}{0} = \frac{y-0}{-1} = \frac{z-0}{1}.$$

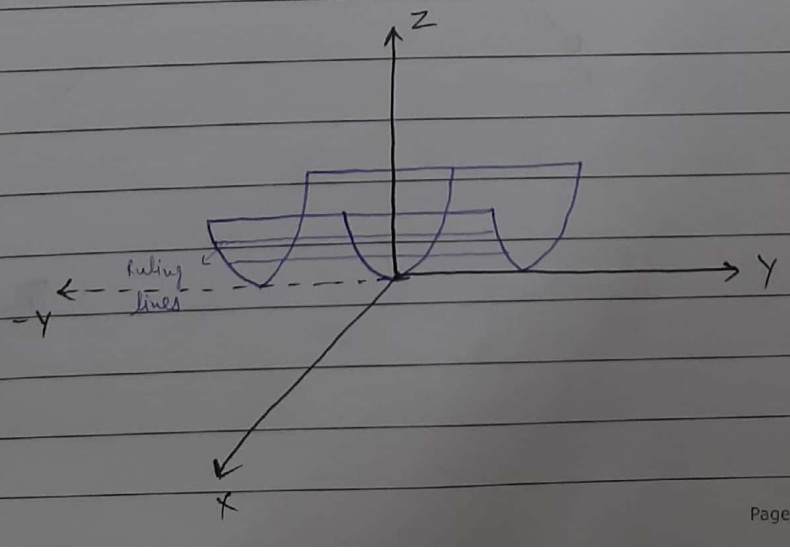
$$\Rightarrow \frac{-y}{1} = \frac{z}{1}, \quad x = 1 \quad \text{Ans.}$$

→ <u>Cylinder and Quadratic Surface</u> :-

Cylinder is a surface that consists of the no. of rolling lines $\parallel^{el}$ to a given line and moves in the dir^n of a plane curve.

Q. $z = x^2 \quad \in R^2$

$z = x^2 \quad \in R^3.$



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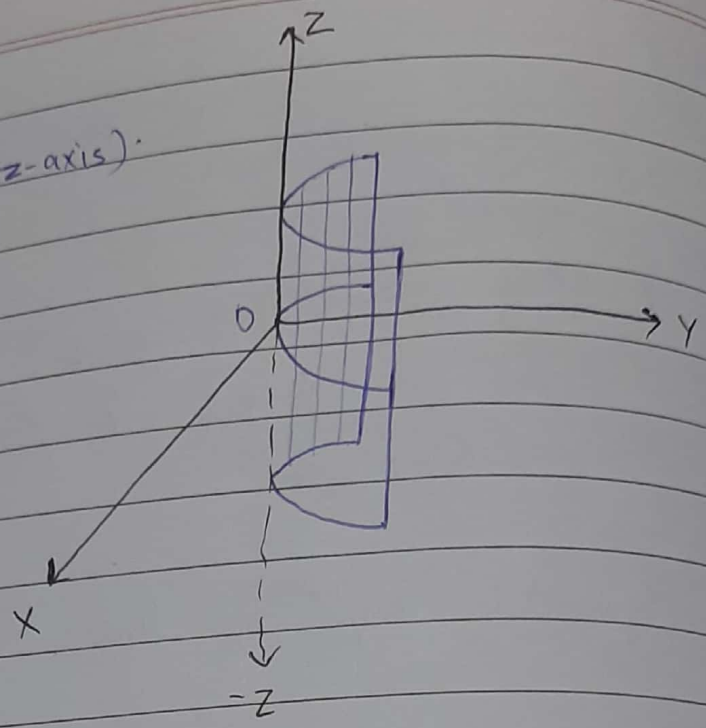
Q. $y = x^2$ in R^3

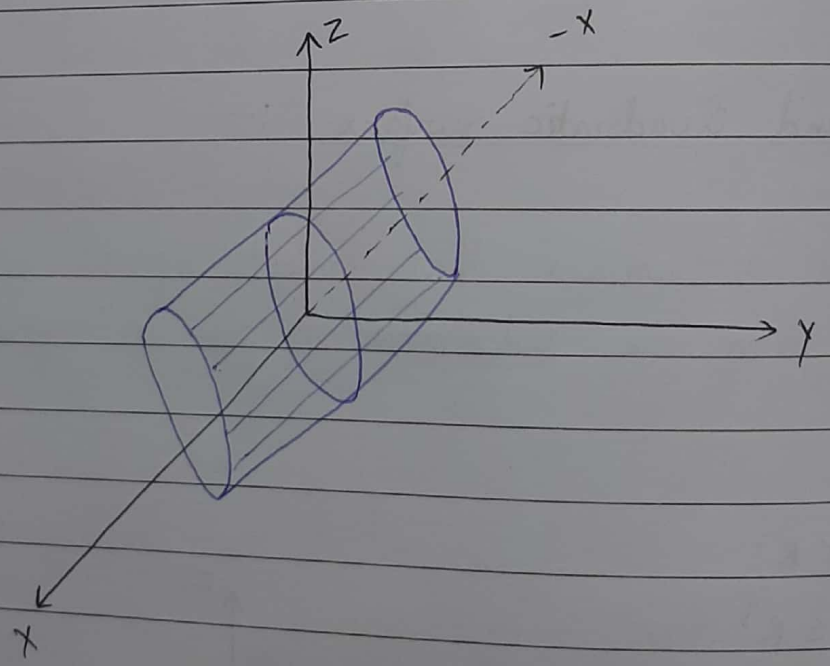
(z - missing → move in z-axis).



Q.3 Describe and sketch $y^2 + 4z^2 = 4$ in R^3.

→

$$\frac{y^2}{2^2} + \frac{z^2}{1^2} = 1 .$$ (Elliptical cylinder).



Note : ① $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ (Ellipsoid).

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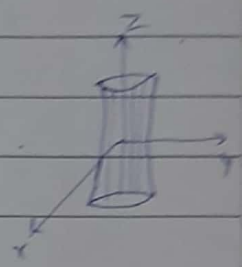
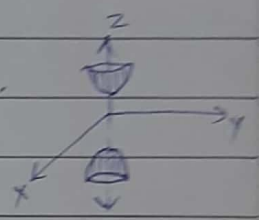
② $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (Paraboloid).

③ $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (Cone).

④ $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ (Hyperboloid of 1-sheet).



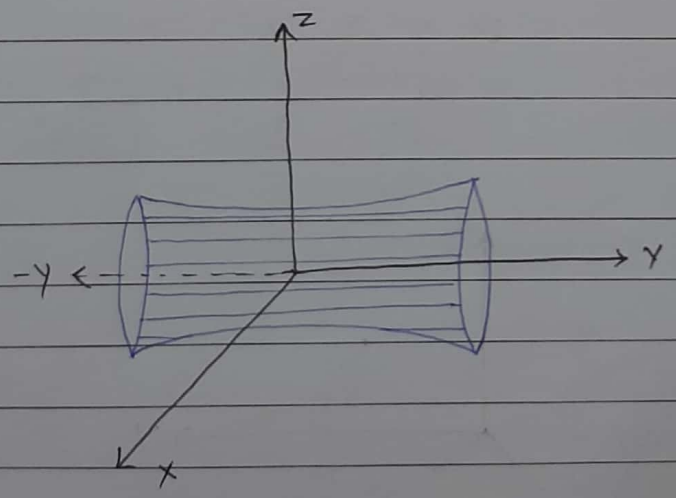
⑤ $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Hyperboloid of 2-sheet).



8.18 Sketch & identify the surface. $4x^2 - 16y^2 + z^2 = 16$.

$\cancel{\frac{x^2}{1^2} - 4y^2}$

$$\frac{x^2}{2^2} - \frac{y^2}{1^2} + \frac{z^2}{4^2} = 1.$$



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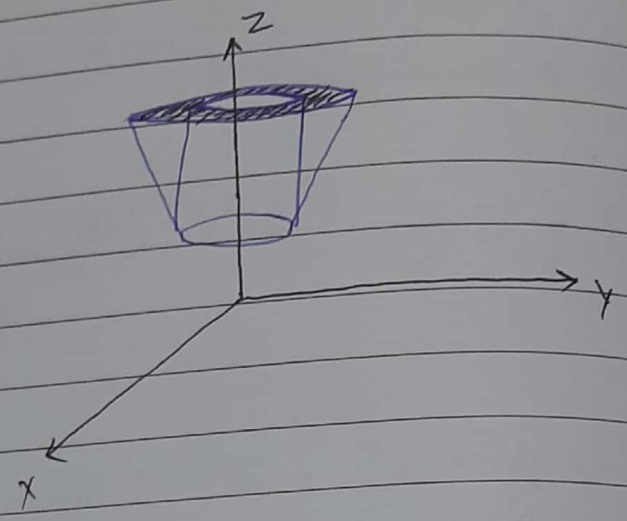
Q.41 Sketch the region bounded by the 2 surface $z = \sqrt{x^2+y^2}$ & $x^2+y^2 = 1$ where $1 \le z \le 2$.

$\Rightarrow$

$z^2 = x^2 + y^2$

$x^2 + y^2 = 1$.

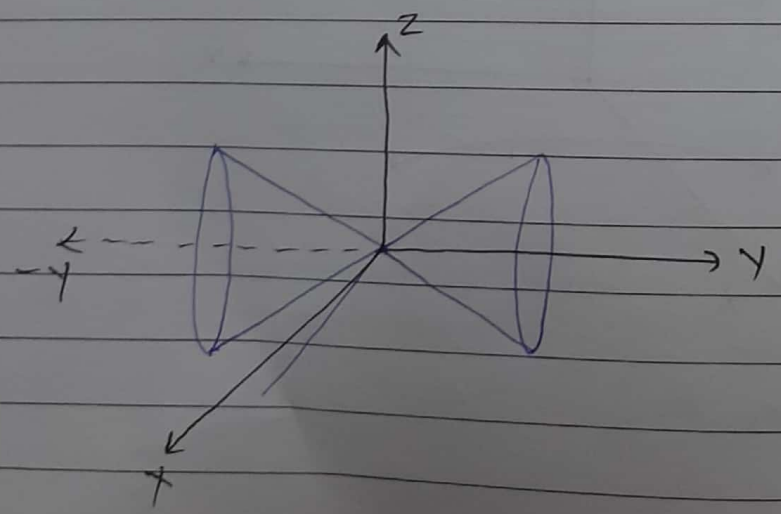


Q. Reduce the equ^n to one of it's standard form, classify & sketch it. $4x^2 - y^2 + 2z^2 = 0$.

$$\cancel{\frac{x^2}{1^2} + -\frac{y^2}{2^2} + \frac{z^2}{(\sqrt{2})^2} = 0}.$$

$\Rightarrow y^2 = 4x^2 + 2z^2$.

$$\Rightarrow \frac{y^2}{1^2} = \frac{x^2}{(1/2)^2} + \frac{z^2}{(1/\sqrt{2})^2}.$$



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