Solutions

1) a) $$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 10.$$

$$x_i \geq 0 \quad \text{for} \quad 1 \leq i \leq 6.$$

$\therefore$ Generating function is $f(x) = (1 + x + x^2 + \dots + x^{10})^6$.

$\therefore$ Required number of ways = Coefficient of x^{10} in $f(x)$.

(Ans).

b) $$x_1 + x_2 + \dots + x_n = r$$

$$x_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n.$$

$\therefore$ Generating function is $f(x) = (1 + x + x^2 + \dots + x^r)^n$

$\therefore$ Required number of ways = Coefficient of x^r in $f(x)$.

(Ans).

2) $n \in \mathbb{Z}^+$

$$f(x) = (1 + x + x^2)(1+x)^n$$

$$= (1 + x + x^2)\left(1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n\right)$$

a) Coefficient of $x^7 = {}^nC_7 + {}^nC_6 + {}^nC_5$

b) 〃 〃 $x^8 = {}^nC_8 + {}^nC_7 + {}^nC_6$

c) $0 \leq r \leq n+2$, $r \in \mathbb{Z}$

Coefficient of $x^r = {}^nC_r + {}^nC_{r-1} + {}^nC_{r-2}$.

3) a) $f(x) = x^3 (1-2x)^{10}$

$= x^3 \left[1 + {}^{10}C_1 (-2x) + {}^{10}C_2 (-2x)^2 + \dots + {}^{10}C_r (-2x)^r + \dots + {}^{10}C_{10} (-2x)^{10} \right]$

$\therefore$ Coefficient of x^{15} in $f(x)$

$=$ Coefficient of x^{12} in $(1-2x)^{10}$

$= 0$

b) $g(x) = \dfrac{x^3 - 5x}{(1-x)^3}$

$= (x^3 - 5x)(1-x)^{-3}$

$= (x^3 - 5x) \left[1 + 3x + \dfrac{(-3)(-3-1)}{2!}(-x)^2 + \dots + \dfrac{(-3)(-3-1)\dots\{-3-(r-1)\}}{r!}(-x)^r + \dots \right]$

$\therefore$ Coefficient of x^{15} in $g(x)$

$= \dfrac{(-3)(-3-1)\dots(-3-11)}{(12)!}(-1)^{12} - 5\,\dfrac{(-3)(-3-1)\dots(-3-13)}{(14)!}(-1)^{14}$

$= \dfrac{3\times 4\times \dots \times 14}{(12)!} - 5\left[\dfrac{3\times 4\times \dots\times 16}{(14)!} \right]$

$= \dfrac{14\times 13}{2} - 5\left(\dfrac{16\times 15}{2}\right)$

$= -509$ (Ans)

c) $h(x) = \dfrac{(1+x)^4}{(1-x)^4}$

$= (1+x)^4 \, (1-x)^{-4}$

$= \left(1 + 4x + 6x^2 + 4x^3 + x^4\right)$

$\left[1 + 4x + \dfrac{(-4)(-5)}{2!}(-x)^2 + \cdots + \dfrac{(-4)(-5)\cdots\{-4-(r-1)\}}{r!}(-x)^r\right]$

$\therefore$ Coeff. of x^{15}

$= \dfrac{(-4)(-5)\cdots(-18)}{(15)!}(-1)^{15} + 4\,\dfrac{(-4)(-5)\cdots(-17)}{(14)!}(-1)^{14}$

$+ 6\,\dfrac{(-4)(-5)\cdots(-16)}{(13)!}(-1)^{13} + 4\,\dfrac{(-4)(-5)\cdots(-15)}{(12)!}(-1)^{12}$

$+ \dfrac{(-4)(-5)\cdots(-14)}{(11)!}(-1)^{11}$ (Evaluate)

4) 2 dozen $= 24$

a) $x_1 + x_2 + x_3 + x_4 = 24$ —— (1)

$x_i \geq 3$ for $1 \leq i \leq 4$

$3 \leq x_i \Rightarrow 0 \leq x_i - 3$

$\Rightarrow 0 \leq y_i$ where $x_i - 3 = y_i$

$\Rightarrow x_i = y_i + 3$

$1 \leq i \leq 4$

$\therefore$ ① $\Rightarrow y_1+3+y_2+3+y_3+3+y_4+3 = 24$

$\Rightarrow y_1+y_2+y_3+y_4 = 12$

$y_i \geq 0$ for $1 \leq i \leq 4$.

$\therefore$ Required solution $= {}^{4+12-1}C_{12} = {}^{15}C_{12}$

b) $x_1+x_2+x_3+x_4 = 24$ ——— ②

$3 \leq x_i \leq 9$ for $1 \leq i \leq 4$.

$3 \leq x_i \leq 9 \Rightarrow 0 \leq x_i - 3 \leq 6$

$\Rightarrow 0 \leq y_i \leq 6$ where $y_i = x_i - 3$

$\Rightarrow x_i = y_i + 3$

$\therefore$ ② $\Rightarrow (y_1+3)+(y_2+3)+(y_3+3)+(y_4+3) = 24$

$0 \leq y_i \leq 6$

$\Rightarrow y_1+y_2+y_3+y_4 = 12$

$0 \leq y_i \leq 6$.

Let $c_1: y_1 > 6 \Rightarrow y_1 \geq 7$

$c_2: y_2 > 6 \Rightarrow y_2 \geq 7$

$c_3: y_3 > 6 \Rightarrow y_3 \geq 7$

$c_4: y_4 > 6 \Rightarrow y_4 \geq 7$

$\therefore$ Required Solution is

$$N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = N - [N(C_1) + N(C_2) + N(C_3) + N(C_4) - N(C_1C_2) - N(C_1C_3) - N(C_1C_4) - N(C_2C_3) - N(C_2C_4) - N(C_3C_4) + N(C_1C_2C_3) + N(C_1C_2C_4) + N(C_1C_3C_4) + N(C_2C_3C_4) - N(C_1C_2C_3C_4)]$$

To calculate N

$$y_1 + y_2 + y_3 + y_4 = 12$$

$$\therefore N = {}^{4+12-1}C_{12}$$

$$= {}^{15}C_{12}$$

To calculate $N(C_1)$

$$y_1 + y_2 + y_3 + y_4 = 12$$

$$y_1 \geq 7$$

$$\Rightarrow y_1 - 7 \geq 0$$

$$\Rightarrow z_1 \geq 0$$

$$z_1 + 7 + y_2 + y_3 + y_4 = 12$$

$$\Rightarrow z_1 + y_2 + y_3 + y_4 = 5$$

$$\therefore N(C_1) = {}^{4+5-1}C_5 = {}^{8}C_5$$

$$\therefore N = {}^{15}C_{12}$$

$$N(C_1) = N(C_2) = N(C_3) = N(C_4) = {}^{8}C_5$$

Rest of the Terms are all zero.

$$\therefore N(\bar{C_1}\bar{C_2}\bar{C_3}\bar{C_4}) = {}^{15}C_{12} - (4 \times {}^{8}C_5)$$ (Evaluate)

5) $(1-4x)^{-1/2}$

$$= 1 + \left(-\tfrac{1}{2}\right)(-4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-4x)^2$$

$$+ \cdots \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left\{-\frac{1}{2}-(r-1)\right\}}{r!}(-4x)^r + \cdots$$

$\therefore$ Coefficient of x^n

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left\{-\frac{1}{2}-(n-1)\right\}}{n!}(-4)^n$$

$$= (-1)^{2n}\frac{1}{n!}\left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\,\frac{7}{2}\cdots\frac{2n-3}{2}\cdot\frac{2n-1}{2}\right]4^n$$

$$= \frac{2^{2n}}{n!}\cdot\frac{1}{2^n}\left[1.3.5.7\ldots(2n-3)(2n-1)\right]$$

$$= \frac{2^n}{n!}\left[1\cdot3.5\ldots(2n-3)(2n-1)\right]$$

$$= \frac{2^n}{n!}\left[\frac{1.2.3.4.5\ldots(2n-4)(2n-3)(2n-2)(2n-1)(2n)}{2.4.6\ldots(2n-4)(2n-2)(2n)}\right]$$

$$= \frac{2^n}{n!}\,\frac{(2n)!}{2^n\,n!}$$

$$= \frac{(2n)!}{n!.n!} = {}^{2n}C_n = \binom{2n}{n}$$ [proved]

6) 8 % inverest compounded quarterly.

$P_0 = ?$

$P_n = \$ 7218.27$

Time $= 15$ years.

$= 60$ quarters.

$$P_{n+1} = P_n + \left(\frac{8}{100} \times \frac{1}{4}\right) P_n$$

$$P_{n+1} = 1.02 \, P_n$$

Solution to the above recurrence is $P_n = P_0 (1.02)^n$

$$\therefore \quad 7218.27 = P_0 (1.02)^{60}$$

$$\Rightarrow P_0 \approx 2200$$

$\therefore$ Paul's initial investment is $\$ 2200$. (Ans)

7) $a_{n+2} = a_{n+1} \, a_n \quad , \ n \geq 0 \ , \ a_0 = 1, \ a_1 = 2.$

$n = 0 \Rightarrow a_2 = a_1 \cdot a_0 = 2.1 = 2$

$n = 1 \Rightarrow a_3 = a_2 \cdot a_1 = 2 \cdot 2 = 2^2$

$n = 2 \Rightarrow a_4 = a_3 \cdot a_2 = 2^2 \cdot 2 = 2^3$

$n = 3 \Rightarrow a_5 = a_4 \cdot a_3 = 2^3 \cdot 2^2 = 2^5$

$n = 4 \Rightarrow a_6 = a_5 \cdot a_4 = 2^5 \cdot 2^3 = 2^8$.

and so on.

$\therefore\ a_n = 2^{F_n}$ where F_n is the n^{th} Fibonacci number, $n \geq 0$.

8) $2a_{n+2} = 5a_n - 3a_{n+1}$, $n \geq 0$, $a_0 = 1$, $a_1 = 0$

Let $a_n = cr^n$, $c, r \neq 0$.

$\Rightarrow 2cr^{n+2} = 5cr^n - 3cr^{n+1}$

$\Rightarrow 2r^2 = 5 - 3r$

$\Rightarrow 2r^2 + 3r - 5 = 0$

$\Rightarrow r = \dfrac{-3 \pm \sqrt{9+40}}{4} = \dfrac{-3 \pm 7}{4} = -\dfrac{5}{2}, 1$

$\therefore\ a_n = C_1\left(-\frac{5}{2}\right)^n + C_2(1)^n$

$a_0 = 1 \Rightarrow 1 = C_1 + C_2$

$a_1 = 0 \Rightarrow 0 = -\frac{5}{2}C_1 + C_2 \Rightarrow C_2 = \frac{5}{2}C_1$.

$\therefore\ C_1 + C_2 = 1 \Rightarrow C_1 + \frac{5}{2}C_1 = 1$

$\Rightarrow 7C_1 = 2$

$\Rightarrow C_1 = \frac{2}{7}$

$\Rightarrow C_2 = \frac{5}{7}$

$\therefore\ a_n = \frac{2}{7}\left(-\frac{5}{2}\right)^n + \frac{5}{7}$ (Ans)

9) $a_{n+3} + 2a_{n+2} - a_{n+1} - 2a_n = 0$

Let $a_n = cr^n$, $c, r \neq 0$

$\Rightarrow cr^{n+3} + 2cr^{n+2} - cr^{n+1} - 2cr^n = 0$

$\Rightarrow r^3 + 2r^2 - r - 2 = 0$

$\Rightarrow r^2(r+2) - 1(r+2) = 0$

$\Rightarrow (r+2)(r^2-1) = 0$

$\Rightarrow r = -2, -1, 1$

$\therefore a_n = c_1(-2)^n + c_2(-1)^n + c_3(1)^n$

$a_0 = 1 \Rightarrow 1 = c_1 + c_2 + c_3$

$a_1 = 0 \Rightarrow 0 = -2c_1 - c_2 + c_3$

$a_2 = -1 \Rightarrow -1 = 4c_1 + c_2 + c_3$

$\Rightarrow c_1 = -\frac{2}{3}, \ c_2 = \frac{3}{2}, \ c_3 = \frac{1}{6}$

$\therefore a_n = -\frac{2}{3}(-2)^n + \frac{3}{2}(-1)^n + \frac{1}{6}$ (Ans).

10) $a_{n+2} - 4a_{n+1} + 8a_n = 0$, $n \geq 0$, $a_0 = 0$, $a_1 = 2$.

Let $a_n = cr^n$, $c, r \neq 0$

$\Rightarrow cr^{n+2} - 4cr^{n+1} + 8cr^n = 0$

$\Rightarrow r^2 - 4r + 8 = 0$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm \sqrt{-16}}{2}$$

$$\Rightarrow r = \frac{4 \pm 4i}{2} \Rightarrow r = 2 \pm 2i$$

$$\therefore a_n = c_1 (2+2i)^n + c_2 (2-2i)^n$$

$$= 2^n \left[c_1 (1+i)^n + c_2 (1-i)^n \right]$$

$$= 2^n \cdot (\sqrt{2})^n \left[c_1 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)^n + c_2 \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)^n \right]$$

$$= (2\sqrt{2})^n \left[c_1 \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) + c_2 \left(\cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4} \right) \right]$$

$$\Rightarrow a_n = (2\sqrt{2})^n \left[(c_1 + c_2) \cos\left(\frac{n\pi}{4}\right) + i (c_1 - c_2) \sin\left(\frac{n\pi}{4}\right) \right]$$

$$\because a_0 = 0 \Rightarrow 0 = c_1 + c_2$$

$$a_1 = 2 \Rightarrow 2 = 2\sqrt{2} \left[(c_1 + c_2) \frac{1}{\sqrt{2}} + i (c_1 - c_2) \cdot \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow 1 = (c_1 + c_2) + i (c_1 - c_2)$$

$$\Rightarrow i (c_1 - c_2) = 1 \quad [\because c_1 + c_2 = 0]$$

$$\left.\begin{aligned} \Rightarrow \quad & c_1 - c_2 = -i \\ \text{and} \quad & c_1 + c_2 = 0 \end{aligned}\right\} \Rightarrow c_1 = -\frac{i}{2}, \; c_2 = \frac{i}{2}$$

$$\therefore a_n = (2\sqrt{2})^n \left[0 + i(-i) \sin\left(\frac{n\pi}{4}\right) \right]$$

$$= (2\sqrt{2})^n \sin\left(\frac{n\pi}{4}\right) \quad \text{(Ans)}.$$