9/9/23

Languages : Finite State Machines

→ **Alphabet** :

A non-empty finite set of symbols is called alphabet denoted by Σ.

Eg: $\Sigma = \{0, 1\}$ $\Sigma = \{a, b, c, d, e\}$

NOTE: 1 In any alphabet Σ, we do not list elements that can be formed from other elements of Σ by juxta position.

Juxta position : if $a, b \in \Sigma$, then the string ab is the juxtaposition of a, b.

Def 2

→ If Σ is an alphabet and $n \in \mathbb{Z}^+$, we define the powers of Σ recursively as follows.

i) $\Sigma^1 = \Sigma$

ii) $\Sigma^{n+1} = \{xy \mid x \in \Sigma, y \in \Sigma^n\}$ where xy denotes juxtaposition of x, y.

$\Sigma = \Sigma^1 = \{0, 1\}$

$\Sigma^2 = \{00, 11, 01, 10\}$

$\Sigma^3 = \{xy \mid x \in \Sigma, y \in \Sigma^2\}$

$\Sigma = \{a, b\}$ b

→ $\Sigma^3 = \{xy \mid x \in \Sigma, y \in \Sigma^2\}$

In general (for all) $\forall n \in \mathbb{Z}^+$, we can easily find that $|\Sigma^n| = |\Sigma|^n$

Def 3 :-

→ For an alphabet Σ, we define $\Sigma^0 = \{\lambda\}$, where λ denotes the empty strings.

i.e., the string consisting of no symbols consisting ~~of~~ in Σ.

Def 4

→ If Σ is an alphabet, we define a) $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n$

$= \bigcup_{n=\mathbb{Z}^+} \Sigma^n$

b) $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$

→ Def 5 :

→ If $w_1, w_2 \in \Sigma^+$, we can say

$w_1 = x_1, x_2 \dots x_m$

$w_2 = y_1, y_2 \dots y_n$ for $m, n \in \mathbb{Z}^+$

$x_1, x_2 \dots x_m, y_1, y_2 \dots y_n \in \Sigma$

The two strings w_1 & w_2 will be equal i.e., $w_1 = w_2$

if $x_i = y_i \ \forall \ 1 \leq i \leq m$

and $m = n$

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$\lambda \notin \Sigma$ $\quad |\{\lambda\}| = 1$ $\quad \{\lambda\} \not\subseteq \Sigma$

~~$\phi \in \Sigma$~~ $\quad |\phi| = 0$ $\quad \phi \subseteq \Sigma$

NOTE 1 : $\lambda \notin \Sigma$

→ Def 6 :

Let $w = x_1, x_2 \dots x_m \in \Sigma^+$, $x_i \in \Sigma$, $1 \leq i \leq m$

the length of w will be denoted by $\| w \|$

$\| w \| = m$, $\| \lambda \| = 0$

→ Def 7 :-

Let $x, y \in \Sigma^+$ with $x = x_1, x_2 \dots x_m$ and

$y = y_1, y_2 \dots y_n$ so that for each x_i, $1 \leq i \leq m$

and each y_j, $1 \leq j \leq n$ is in Σ, the ~~conta~~

the concatination of x and y (xy) is the string

$x_1 x_2 x_3 \dots x_m y_1 y_2 \dots y_n$

NOTE: $x\lambda = x$, $\lambda x = x$, $\lambda\lambda = \lambda$

→ Defⁿ 8

For each $x \in \Sigma^*$, we define the powers of x by $x^0 = \lambda$, $x^1 = x$, $x^2 = xx$, $x^3 = xx^2$

$$\boxed{x^{n+1} = \underbrace{x x^n}_{\text{concatenation}}}$$

Eg: $\Sigma = \{0, 1\}$

$\Sigma^0 = \lambda$, $\Sigma^1 = \{0, 1\}$, $\Sigma^2 = \{00, 01, 10, 11\}$

$\Sigma^3 = \{000, 111, 001, 010, 011, 100, 101, 110\}$

$\Sigma^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 111, \dots\}$

Eg: $x = 01$ $\quad x^2 = 0101$

$x^0 = \lambda$ $\quad x^3 = 010101$

$\|x\| = 2$ $\quad \|x^2\| = 4 = 2.2 = 2\|x\|$

$\|x^3\| = 6 = 2.3 = 3\|x\|$

$\|x^4\| = 8 = 2.4 = 4\|x\|$

if $\|x\| = n$

$\|x^m\| = mn$ or $\|x^n\| = n\|x\|$

Ex: 6.1

Q3. If $x \in \Sigma^*$ and $\|x^3\| = 36$, what is $\|x\|$?

$\|x^3\| = 3\|x\| \Rightarrow \|x\| = \frac{36}{3} = 12$

→ Def 9

If x and $y \in \Sigma^*$ and $w = xy$, then the String x is called a prefix of w and if $y \neq \lambda$ then x is said to be a proper prefix.

Similarly, string y is called suffix of w, it is a proper suffix if $x \neq \lambda$

→ Def^n 10

For a given Σ, any subset of Σ^* is called a language over Σ. ϕ will be called an empty language

Eg: $\Sigma = \{0, 1\}$ $\Sigma^* = \{0, 1, 11, 00, \dots\}$

$A = \{000, 10\}$ $A \subseteq \Sigma^*$

For a ~~lan~~ alphabet Σ, if $A, B \subseteq \Sigma^*$ the contatination of A & B, denoted by AB is $\{ab \mid a \in A, b \in B\}$

Eg: $\Sigma = \{x, y, z\}$ $A = \{x, xy, z\}$ $B = \{\lambda, y\}$

$|AB| \neq |BA|$ $AB = \{x, xy, z, xyy, zy\}$

$BA = \{x, xy, z, yx, yxy, yz\}$

Q1. $\Sigma = \{a, b, c, d, e\}$ $|\Sigma^0| = 1$

$|\Sigma^2| = 5^2 = 25$ $|\Sigma^3| = 5^3 = 125$ $|\Sigma^4| = 5^4 = 625$

$|\Sigma^5| = 5^5 = 3125$

$$\begin{array}{r} 3125 \\ 155 \\ \hline 3280 \\ +\ 625 \\ \hline 3905 \\ +\ 1 \\ \hline 3906 \end{array}$$

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Finite State Machines

A finite state machine is a 5 tuple. $M = (S, I, O, v, w)$

$S \to$ set of internal states for M.

$I \to$ represents i/p alphabet for M.

$O \to$ " o/p " " M.

$v \to$ $v : S \times I \to S$ is the next state func^n

$w \to$ $S \times I \to O$ is the output func^n

Eg 1. $M = (S, I, O, v, w)$

$S = \{s_0, s_1, s_2\}$

$I = O = \{0, 1\}$

	v		w	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_2	s_1	0	0
s_2	s_0	s_1	0	1

$v(s_1, 1) = s_1$ $\quad w(s_2, 1) = 1$

$v(s_2, 0) = s_0$ $\quad w(s_1, 1) = 0$

M 1010 $\to$ 0010

~~1001001 → 0000001~~

state	s_0	s_1	s_2	s_1
Input	1	0	1	0
output	0	0	1	0

101001000 $\to$

1001001

state	s_0	s_1	s_2	s_0	s_1	s_2	s_0
input	1	0	0	1	0	0	1
output	0	0	0	0	0	0	0

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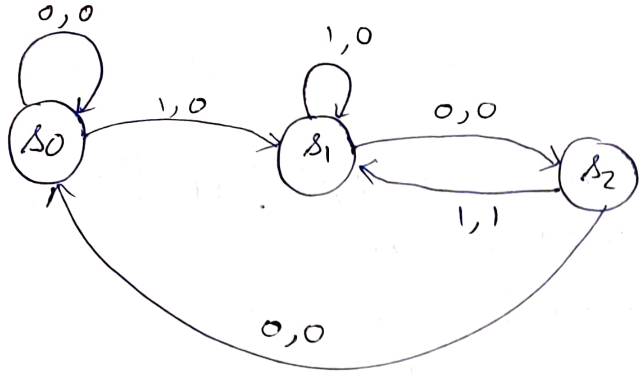
00110101

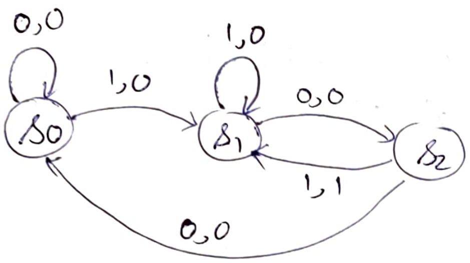
state	s_0	s_0	s_0	s_1	s_1	s_2	s_1	s_2
i/p	0	0	1	1	0	1	0	1
o/p	0	0	0	0	0	1	0	1

H/W

Eg 6,19

Exe-3





	v: 0	v: 1	w: 0	w: 1
s_0	s_0	s_1	0	0
s_1	s_2	s_1	0	0
s_2	s_0	s_1	0	1

Chap - 2

Def^n 1

For sets A, B, any subset of $A \times B$ is called a relation from A to B. For sets A, B any subset of $A \times A$ is called a relation on A.

Def^n 2

Let's Σ be an alphabet with language $A \subseteq \Sigma^*$ for $x, y \in A$, we define $x R y$ if x is a prefix of y.

Def^n 3

A Relation R on a set A is called reflexive if $\forall x \in A$, $(x, x) \in R$

Q. Is $R = \{(x,y) \mid x, y \in A, x \leq y\}$ reflexive?

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3),$
$(3,4), (4,4)\}$

Defn. 4

Relation R on set A is called ~~syy~~ symmetric if $(x, y) \in R \Rightarrow (y, x) \in R$

Eg: $A = \{1, 2, 3\}$

$R = \{(1,2), (2,1), (3,1), (1,3)\}$

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Eg 3: Given a finite set A. $|A| = n$. we have $|A \times A| = n^2$. so there are 2^{n^2} relations on A. How many of this are reflexive?

Soln. If $A = \{a_1, a_2 \dots a_n\}$, a relation R on A is reflexive iff $\{(a_i, a_i) \mid 1 \leq i \leq n\} \subseteq R$

Considering the other $n^2 - n$ ordered pairs in $A \times A$ (those of the form (a_i, a_j) where $i \neq j$, $1 \leq i, j \leq n$).

As we construct a Relation R on a, we either include or exclude each of these ordered pairs. so, by the rule of product, there are $2^{n^2 - n}$ reflexive relation on A.

Defn 5

A Relation R on set A is called transitive if

$\forall x, y, z \in A \quad \forall (x, y), (y, z) \in R$
$(x, z) \in R$

Eg. $A = \{1, 2, 3\}$.

a) $R_1 = \{(1,2), (2,1) (1,3), (3,1)\}$

R ✗ S ✓ T ✗

b) $R_2 = \{(1,1) (2,2) (3,3) (2,3)\}$

R ✓ S ✗ T ✓

c) $R_3 = \{(1,1), (2,2), (3,3)\}$
R ✓ S ✓ T ✓

d) $R_4 = \{(1,1), (2,3), (3,3)\}$
R × S × T ✓

e) $R_5 = \{(1,1), (2,3), (3,4), (2,4)\}$

EX- 7.1

1. $A = \{1, 2, 3, 4\}$

a) R ✓ S ✓ T ×
$\{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (1,2), (2,1)\}$

b) R ✓ T ✓ S ×
$\{(1,1), (2,2) \ (3,3) \ (4,4) \ (1,2)\}$

c) R × S ✓ T ✓

~~$\{(1,2) \ (2,1) \ (1,3) \ (3,1)\}$~~

~~$(1,1)$~~

$\{(1,2) \ (2,1) \ (1,1), (2,2)\}$

23/9/23.

5. a) $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ where aRb if a/b (a divides b)

$\{(2,4), (2,2), (1,1) \ldots\}$

Reflexive ✓
symmetric × $(2,4)$ exists but not $(4,2)$

R is antisymmetric if $\forall (a,b) \in A \rightarrow a = b$ ×
a/b & b/a

iv) transitive a/b & $b/c \Rightarrow b = k_1 a \Rightarrow c = k_2 b$
$c = k_1 k_2 a$
$\Rightarrow a/c$

b) R is relation on z wher a R b if a/b

i) Refleseive (0,0) ∉

ii) symmetric X

iii) Anti symmetric X

iv) transitive a/b , b/c => a/c

→ A Relation R on A is called partial ordering relation if R is reflesuive, antisymmetric , transitive.

→ A Relation R on A is equivalence relation if R is refleseive, symmetric , transitive.

5

c) A R B $A \cap C = B \cap C$

i) Refleseive ✓ ∀ A R A $A \cap C = A \cap C$ => A R A

ii) symmetric ✓ A R B → $A \cap C = B \cap C$ => $B \cap C = A \cap C$

iii) Antisymmetric X A R B & B R A => B R A

$A = \{1, 2, 3, 4\}$ $C = \{1, 2\}$ $B = \{1, 2, 5\}$

X

iv) A R B & B R D

$A \cap C = B \cap C$ & $B \cap C = D \cap C$

$A \cap C = D \cap C$ => A R D

d) Refleseive l_1 R l_1 X

Symmetric $l_1 \perp l_2$, $l_2 \perp l_1$ ✓

Antisymmetric X

transitive X

e) R is relation on z where x R y if x+y odd.

Refleseive X

Symmetric ✓

Anti Symmetric X

transitive X → 1+2 = 3, 2+3 = 5 but 1+3 = 4

f) $x - y$ is even.

Reflexive ✓

symmetric ✓

Antisymmetric ✗

transitive ✓

H/w for 7.2

Eg: 7.8, 7.9, 7.11, 7.12

5) g, h, 6)

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7.2 Computer Recognition

Zero one matrices and Directed Graphs

★ **Def.ⁿ 1**

If A, B, C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ the the Composite relation $R_1 \circ R_2$ is a relation from A to C defined by

$$R_1 \circ R_2 = \{ (x, z) \mid x \in A, z \in C \text{ and } \exists\, y \in B \text{ with } (x, y) \in R_1, (y, z) \in R_2 \}$$

Eg. $A = \{1, 2, 3, 4\}$ $\quad R_1 = \{ (1, x) (2, x), (3, y) (3, z) \}$

$B = \{w, x, y, z\}$ $\quad R_2 = \{ (w, 5) (x, 6) \}$

$C = \{5, 6, 7\}$

$R_1 \circ R_2 = \{ (1, 6) (2, 6) \}$

$R_3 = \{ (w, 5) (w, 6) \} \Rightarrow R_1 \circ R_3 = \phi$

★ **Def.ⁿ 2**

Given a set A and a Relation R on A, we define the powers of R recursively by

i) $R^1 = R$

ii) $R^2 = R \circ R$

iii) $R^{n+1} = R \circ R^n$, where $n \in \mathbb{Z}^+$.

Eg. $A = \{1, 2, 3, 4\}$. Find R^2, R^3, R^4

$R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\}$

$R^2 = R \circ R = \{(1,4)\ (1,2)\ (3,4)\}$

$R^3 = R \circ R^2 = \cancel{\phi}\{(1,4)\}$

$R^4 = R \circ R^3 = \phi$ $\qquad R^n = \phi, \forall\ n \geq 4$

★ **Def.^{n} 3**

An $m \times n$ zero one matrix $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers, arranged in m rows and n columns, where each e_{ij} for $1 \leq i \leq m$, $1 \leq j \leq n$, denotes the entry in the ith row and jth column of E and each such entry is 0 or 1.

Eg. $A = \{1, 2, 3, 4\}$ $\quad B = \{w, x, y, z\}$ $\quad C = \{5, 6, 7\}$

$R_1 = \{(1, x)\ (2, x), (3, y), (3, z)\}$

$R_2 = \{(w, 5), (x, 6)\}$ $\qquad R_3 = \{(w, 5)\ (w, 6)\}$

$$M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R_1 \circ R_2 = \{(1,6)\ (2,6)\}$

$$M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M(R_3) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_1 \circ R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow M(R_1)\, M(R_2) = M(R_1 \circ R_2)$$

★ $\boxed{M(R_1)\ M(R_2) = M(R_1 \circ R_2)}$

EX – 7.2

1) $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (1,3), (2,4), (4,4)\}$ $\quad P = \{(1,1), (1,2), (1,3), (2,3), (2,4)\}$

$R \circ P = \{(1,4), (1,3)\}$ $\quad P \circ R = \{(1,2)\ (1,3)\ (1,4)\ (2,4)\}$

$R^2 = R \circ R = \{(1,4)\ \cancel{(\ )}\ (2,4), (4,4)\}$

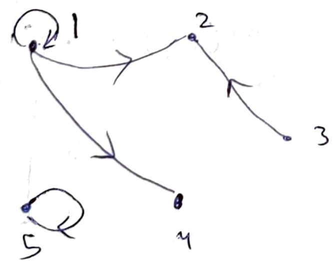
$R^3 = R \circ R^2 = \{(1,4), (2,4), (4,4)\}$

$P^2 = \{(1,1)\ (1,2)\ (1,3), (1,4)\}$ $\quad P^3 = P \circ P^2 = \{(1,2)\ (1,3)\ (1,4)\}$

30/9/23 **Def.^{n} 4**

Let V be a non-empty finite set, a directed graph G on V is made up of the elements of V, called the vertices or nodes of G, ~~and~~ and a subset E, of V × V that contains the edges of G. The set V is called vertex set of G and set E is called the edge set of G. and will denote the graph $G = (V, E)$

Eg. $E = \{(1,1)\ (1,2)\ (1,4)\ (3,2)\}$ $\quad V = \{1, 2, 3, 4, 5\}$



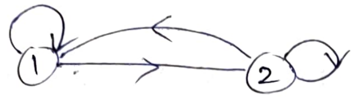
Def.^{n} 5

A directed Graph G on V is called strongly connected if $\forall x, y \in V$, where $x \neq y$, there is a path of directed edges from x to y, i.e., either the directed edge (x,y) is in G or for $n \in \mathbb{Z}^+$ and distinct vertices ~~called~~ is $v_1, v_2 \ldots v_n \in V$

$(x_1, v_1), (v_1, v_2) \ldots\ldots (v_n, y)$ are in G.

Eg. $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (1,2), (2,1) (2,2), (3,3) (3,4) (4,4)\}$



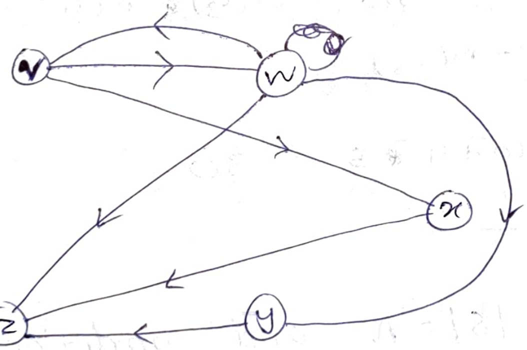
EX-7.2

18) $A = \{v, w, x, y, z\}$

a) $M(R) =$

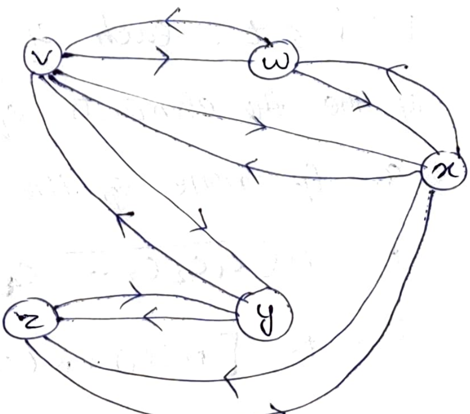
$$\begin{array}{c|ccccc} & v & w & x & y & z \\ \hline v & 0 & 1 & 1 & 0 & 0 \\ w & 1 & 0 & 1 & 1 & 1 \\ x & 0 & 0 & 0 & 0 & 1 \\ y & 0 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 0 & 0 & 0 \end{array}$$

$R = \{(v,w), (v,x), (w,v)$
$(w,x), (w,z), (x,z)$
$(y,z)\}$



(b) $M(R) =$

$$\begin{array}{c|ccccc} & v & w & x & y & z \\ \hline v & 0 & 1 & 1 & 1 & 0 \\ w & 1 & 0 & 1 & 0 & 0 \\ x & 1 & 1 & 0 & 0 & 1 \\ y & 1 & 0 & 0 & 0 & 1 \\ z & 0 & 0 & 1 & 1 & 0 \end{array}$$



5/10/23 Ch-8

H.W. (7.2)
Eg: 7.22, 7.25, 7.27, 7.28
Ex - 4, 5, 19

The principles ~~on~~ of Inclusion and Exclusion.

Ex.

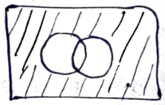
$N = 100$ $\qquad N(C_1) : 35$ $\qquad N(C_1 C_2) = 9$

C_1 : IDM $\qquad N(C_2) : 30$ $\qquad N(C_1 \bar{C}_2) = 35 - 9$

C_2 : PIP $\qquad\qquad\qquad = 26$

$$N(\bar{C}_1 C_2) = N(C_2) - N(C_1 C_2)$$
$$= 30 - 9 = 21$$

$$N(\bar{C}_1 \bar{C}_2) = \cancel{100 - N(C_1) - N(C_2)} \quad 100 - 65 + 9$$
$$= \cancel{100 - 35 - 30} \quad = 44$$



$$= N - N(C_1) - N(C_2) + N(C_1 C_2)$$

$N(C_3) = 30$ $\qquad N(C_2 C_3) = 10$ $\qquad N(C_3 C_1) = 11$

$N(C_1 C_2 C_3) = 5$

$$N(\overline{C_1 C_2 C_3}) = N + N(C_1 C_2) + N(C_2 C_3) + N(C_1 C_3) - N(C_1 C_2 C_3) -$$
$$= N(C_1) - N(C_2) - N(C_3)$$

$$= 100 - 95 + 9 + 10 + 11 - 5 = 30$$

Consider a set S with $|S| = N$ and condition C_i, $1 \leq i \leq t$, each of which may be satisfied by ~~some~~ sum of the the elements of S. The no. of elements of S that satisfy none of the conditions C_i is denoted by $\bar{N}$

$$\bar{N} = N(\overline{C_1 C_2 C_3 \cdots C_t})$$

$$\bar{N} = N - [N(C_1) + N(C_2) \cdots + N(C_t)] + [N(C_1 C_2) + N(C_1 C_3) + \cdots + N(C_{t-1} C_t)]$$

$$- N[N(C_1 C_2 C_3) + N(C_1 C_2 C_4) + \cdots + N(C_1 C_2 C_t) + N(C_1 C_3 C_t) + \cdots N(C_1 C_3 C_t) + N(C_{t-2} C_{t-1} C_t)]$$

$+ (-1)^t N(c_1 c_2 c_3 \dots c_t)$

H.W. 8.1 Eg: 8.4, 8.5
EX : 2, 6

Q. Determine the no. of +ve integers n where $1 \le n \le 100$ and n is not divisible by 2, 3, or 5.

Sol^n. $N(c_1) = 50$ $N(c_2) = 33$ $N(c_3) = 20$

~~$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = N$~~

$N(c_1 c_2) = \left\lfloor \frac{100}{6} \right\rfloor = 16$

$N(c_2 c_3) = \left\lfloor \frac{100}{15} \right\rfloor = 6$

$N(c_3 c_1) = \left\lfloor \frac{100}{10} \right\rfloor = 10$

$N(\bar{c}_1 \bar{c}_2 \bar{c}_3) = 100 - 50 - 33 - 20 + 16 + 6 + 10 - 3$

$= 26$

EX-8.1

Q.3. $N = 100$ $N(c_1) = 35$, $N(c_2) = 30$, $N(c_3) = 30$,

$N(c_1 c_2) = 9$, $N(c_1 c_3) = 11$, $N(c_2 c_3) = 10$, $N(c_1 c_2 c_3) = 5$,

$N(c_4) = 41$, $N(c_3 c_4) = 10$, $N(c_1 c_2 c_4) = 6$, $N(c_2 c_3 c_4) = 6$,

$N(c_1 c_2 c_3 c_4) = 4$ $N(c_1 c_4) = \overset{13}{\cancel{30}}$ $N(c_2, c_4) = 14$ $N(c_1, c_3 c_4) = 6$

$N(c_3 c_4) = 10$

☆ $\boxed{N(\bar{c}_2 \bar{c}_3 \bar{c}_4) = N(c_1 \bar{c}_2 \bar{c}_3 \bar{c}_4) + N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)}$

(i) $N(c_3 \bar{c}_1 \bar{c}_2 \bar{c}_4) = N(\bar{c}_1 \bar{c}_2 \bar{c}_4) - N(\bar{c}_1 \bar{c}_2 \bar{c}_3 \bar{c}_4)$

7/10/23.

Generating Functions

① Swaym
② Jyoti
③ Akash

S	J	A
4	4	4
5	4	3
4	3	5
3	4	5
4	5	3
4	2	6
4	6	2
5	2	5

S	J	A
5	5	2
5	3	4
6	2	4
6	3	3
6	4	2
7	3	2
7	2	3
8	2	2

Ex - 9.1

a) $c_1 + c_2 + c_3 + c_4 = 20$, $0 \leq c_i \leq 7 \ \forall \ 1 \leq i \leq 4$

b) $c_1 + c_2 + c_3 + c_4 = 20$, $0 \leq c_i \ \forall \ 1 \leq i \leq 4$ with c_2 & c_3 even.

a). $0 \leq c_1 \leq 7$ $(1 + x + \dots + x^7)$

" c_2 " "

" c_3 " "

" c_4 " "

"

$\Rightarrow$ $(1 + x + x^2 + \dots x^7)^4$

Q. If there is an unlimited no. (or atleast 24 of each colour) of red, green, white & black jellybeans. In how many ways can swaym select 24 of these candies so that he has an even no. of white beans and atlest 6 black ones.

soln. red, green $\rightarrow$ $(1 + x^1 + x^2 + \dots + x^{24})$

white $\rightarrow$ $(1 + x^2 + x^4 + \dots x^{24})$

black $\rightarrow$ $(x^6 + x^7 + \dots + x^{24})$

28/10/23

① $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 \cdots\cdots + \binom{n}{n}x^n$

② $\dfrac{(1-x^{n+1})}{1-x} = 1 + x + x^2 + \cdots + x^n$

③ $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$

④ $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$

⑤ $\dfrac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots$

Ex

a) $\binom{8}{0}\binom{8}{1}\binom{8}{2}\cdots\binom{8}{8} \rightarrow (1+x)^8$

b) $\binom{8}{1},\ 2\binom{8}{2},\ 3\binom{8}{3},\ \cdots\ 8\binom{8}{8} \rightarrow 8(1+x)^{\cancel{8}\,7}$

c) $1, -1, 1, -1, 1, -1 \cdots\cdots \quad \dfrac{1}{1+x}$

d) $0, 0, 0, 6, -6, 6, -6, 6 \ldots \rightarrow \frac{6x^3}{1+x}$

e) $1, 0, 1, 0, 1, 0 \ldots \rightarrow \frac{1}{1-y^2}$

f) $0, 0, 1, a, a^2, a^3 \ldots, a \neq 0 \rightarrow \frac{x^2}{1-ax}$

a) $(2x-3)^3$

$(a-b)^3 = \cancel{(a-b)}$ *
$a^3 - 3a^2b + 3ab^2 - b^3$

$= 8x^3 - 3(2x)^2 3 + 3(2x)(3)^2 - (3)^3$

$= 8x^3 - 36x^2 + 54x - 27$

$-27, 54, -36, 8, \ldots$

2/11/23

b) $f(x) = \frac{x^4}{1-x} = 0, 0, 0, 0, 1, 1, 1 \ldots$

c) $f(x) = \frac{x^3}{1-x^2} =$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots$

$\frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + x^9 \ldots$

$0, 0, 0, 1, 0, 1, 0, 1 \ldots$

d) $f(x) = \frac{1}{1+3x} = \frac{1}{1-x} = 1 + x + x^2 + x^3 \ldots$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$

$\frac{1}{1+3x} = 1 - 3x + 3^2x^2 - 3^3x^3 + \ldots$

$= 1, -3, 3^2, -3^3, 3^4 \ldots$

e) $f(x) = \frac{1}{3-x} = \frac{1}{3\left(1-\frac{x}{3}\right)} = 1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \ldots$

$= 1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} \ldots$

f) $f(x) = \frac{1}{1-x} + 3x^7 - 11 = (1 + x + x^2 + x^3 \ldots) + 3x^7 - 11$

$= -10 + x + x^2 + x^3 + \ldots x^6 + 4x^7 + x^8 + \ldots$

$= -10, 1, 1, 1, 1, 1, 1, 4, 1 \ldots$

4. Determine the constant in $(3x^2-(2/x))^{15}$

Sol^n: $(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + \dots + {}^nC_n\, a^0 b^n$

$(3x^2 - \frac{2}{x})^{15} = {}^{15}C_0 (3x^2)^{15} (-\frac{2}{x})^0 + {}^{15}C_1 (3x^2)^{14} (-\frac{2}{x})^1 + \dots$

$\underbrace{{}^{15}C_{10} (3x^2)^5 (-\frac{2}{x})^{10}}_{\text{const. term}} + {}^{15}C_{15} (3x^2)^0 (-\frac{2}{x})^{15}$

$\Rightarrow {}^{15}C_{10}\ 3^5\ 2^{10}$

Recurrence Relation

$a, ar, ar^2, ar^3 \dots$

common ratio r

$a_0, a_1, a_2, a_3 \dots$

$1, 2, 2^2, 2^3, 2^4 \dots \Rightarrow \frac{a_1}{a_0} = 2$

$5, 5^2, 5^3, 5^4 \dots$

$\frac{a_2}{a_1} = 2$

$\boxed{a_n = 2a_{n-1}, \ n \geq 0}$

$2, 10, 50, 250, \dots$

$a_n = 5a_{n-1}, \ n \geq 0$

9/11/23

$(1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r \binom{n+r-1}{r} x^r = \binom{n-1}{0} - \binom{n}{1} x + \binom{n+1}{2} x^2$

Find the coefficient of x^{15} in $(x^2 + x^3 + \dots)^4$

$= \{x^2(1 + x + x^2 + \dots)\}^4$

$= x^8 \{1 + x + x^2 + \dots\}^4$

$= x^8 \frac{1}{(1-x)^4} = x^8 (1-x)^{-4}$

$(1+(-x))^{-4} = \sum_0^{\infty} (-1)^r \binom{3+r}{r} (-x)^r$

$= (-1)^7 \binom{10}{7} (-x)^7 = {}^{10}C_7$

Q. Find coefficient of x^{50} in $(x^7+x^8+x^9+\cdots\cdots)^6$

Sol^n.

$\{x^7(\cancel{x}1+x+x^2\cdots\cdots)\}^6$

$x^{42}\ (1+x+x^2\cdots\cdots)^6$

$x^{42}\ (1-x)^{-6}$

$(1+(-x))^6 = (-1)^8 \binom{5+8}{8} (-1)^8 = {}^{13}C_8$

Q. Find coefficient of x^{15} in $x^3(1-2x)^{10}$

Sol^n. $(1-2x)^{10} = (1-y)^{10} = {}^{10}C_0 - {}^{10}C_1 y + {}^{10}C_2 y^2 - \cdots\cdots + {}^{10}C_{10} y^{10}$

$= \underbrace{{}^{10}C_0 - {}^{10}C_1\, 2x + {}^{10}C_2 (2x)^2 - \cdots\cdots + {}^{10}C_{10}(2x)^{10}}$

Here Max^n power of $x=10$

when multiplied with x^3

The max power can touch x^{13} so, there is no coefficient of x^{15}. so $\underline{0}$ (Ans)

Q. Find the coefficient of x^{15} in $\dfrac{(x^3-5x)}{(1-x)^3}$

Sol^n.

$\cancel{\dfrac{(x^3-5x)(1-x)^{-3}}{1-x^3-3x+3x^2}}$ $\qquad (x^3-5x)(1-x)^{-3}$

$\cancel{(x^3-5x)\{(-1)\}}$ $\qquad (x^3-5x)\sum_{r=0}^{\infty}(-1)^r\binom{3+r-1}{r}(-x)^n$

$\cancel{(x^3-5x)(-1)^{12}\ 3+12-1}$

$\binom{3+12-1}{12} + (-5)\binom{14+3-1}{14}$

${}^{14}C_{12} - 5\ {}^{16}C_{14}$

Q. Find coefficient of x^{15} in $(1+x)^4/(1-x)^4$

Sol^n. $\left(\binom{4}{0}+\binom{4}{1}x+\cdots\cdots\binom{4}{4}x^4\right)(1-x)^{-4}$

$\sum_0^{\infty}(-1)^r\binom{4+r-1}{r}(-x)^r \qquad [r=11]$

$(-1)^{11}\binom{14}{11}(-x)^{11}$

${}^{14}C_{11}$

${}^4C_0 \times {}^{18}C_{15} + {}^4C_1\ {}^{17}C_{14} +$

${}^4C_2\ {}^{16}C_{13} + {}^4C_3\ {}^{15}C_{12} + {}^4C_4\ {}^{14}C_{11}$

Recurrence Relation

$\boxed{a_{n+1} = d a_n, \; n \geq 0}$, where d is a constant and $a_0 = A$

is given by $\boxed{a_n = A d^n}$, $n \geq 0$

a) 2, 10, 50, 250 .... $a_0 = 0$; $a_{n+1} = 5a_n$, $n \geq 0$

b) 6, −18, 54, −162 .... $a_{n+1} = (-3)a_n$, $a_0 = 6$, $n \geq 0$

$a_n = (-3)a_{n-1}$, $a_0 = 6$, $n \geq 1$ / $n \geq 0$

c) 7, 14/5, 28/25, $\frac{56}{125}$, ... $a_{n+1} = \frac{2}{5} a_n$, $a_0 = 7$, $n \geq 0$

Q. Find the unique sol.ⁿ for the recurrence relation

a) $a_{n+1} - 1.5 a_n = 0$, $n \geq 0$ $\quad a_n = a_0 (1.5)^n$, $n \geq 0$

b) $4a_n - 5a_{n+1} = 0$, $n \geq 1$ $\quad a_n = a_0 (5/4)^n$, $n \geq 0$

c) $3a_{n+1} - 4a_n = 0$, $n \geq 0$, $a_1 = 5$ $\quad a_n = a_0 (4/3)^n$, $n \geq 0$

$3a_1 = 4a_0 \Rightarrow a_0 = \frac{3}{4} \times 5 = \frac{15}{4}$

$a_n = \left(\frac{15}{4}\right)\left(\frac{4}{3}\right)^n$, $n \geq 0$

d) $2a_n - 3a_{n-1} = 0$, $n \geq 1$, $a_4 = 81$

$a_n = a_0 \left(\frac{3}{2}\right)^n$, $n \geq 0$

$a_4 = a_0 \left(\frac{3}{2}\right)^4$

$81 = a_0 \frac{(3)^4}{16} \Rightarrow a_0 = 16$

3. If a_n, $n \geq 0$ is the unique sol.ⁿ of the recurrence relation $a_{n+1} - d a_n = 0$ and $a_3 = 153/49$, $a_5 = \frac{1377}{2401}$ what is d?

sol.ⁿ $a_n = a_0 d^n$; $n \geq 0$

$a_3 = a_0 d^3 \Rightarrow \frac{153}{49} = a_0 d^3$

$a_5 = a_0 d^5 \Rightarrow \frac{1377}{2401} = a_0 d^5$

$\Rightarrow \frac{1}{9/49} = \frac{1}{d^2}$

$d^2 = \frac{9}{49}$

$d = 3/7$

$a_n + a_{n-1} - 6a_{n-2} = 0, \quad n \geq 2$

$a_0 = -1$

$a_1 = 8$

$a_{n+2} + a_{n+1} + 6a_n = 0, \quad n \geq 0$

$a_n = Ar^n$

$Ar^n + Ar^{n-1} - 6Ar^{n-2} = 0 \quad \Rightarrow r^n + r^{n-1} - 6r^{n-2} = 0$

$\Rightarrow r^2 + r - 6 = 0 \qquad r^n\left(+\frac{1}{r} - \frac{6}{r^2}\right) = 0$

$\frac{r^n}{r^2}(r^2 + r - 6) = 0$

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$a_{n+1} + a_n = 5a_{n-1}$

$\boxed{a_n = dr^n}$

$dr^{n+1} + dr^n = 5dr^{n-1}$

$r^{n+1} + r^n = 5r^{n-1}$

$r^n\{r + 1 - \frac{5}{r}\}$

$\frac{r^n}{r}\{r^2 + r - 5\}$

a) real < equal, unequal

b) complex

$\alpha + i\beta$

$\alpha - i\beta$

unequal r_1, r_2

$a_n = C_1 r_1^n + C_2 r_2^n$

Equal : r_1, r_1

$a_n = C_1 r_1^n + C_2 n r_1^n$

Eg $x + iy$

$r = \sqrt{x^2 + y^2}$

Q. $a_n + a_{n-1} - 6a_{n-2} = 0, \quad n \geq 2, \quad a_0 = -1, \quad a_1 = 8, \quad a_n = cr^n$

$a_1 = a_0 d^1$

$8 = -1(d)$

$d = -8$

$a_n = dr^n$

$dr^n + dr^{n-1} - 6dr^{n-2}$

$r^n + r^{n-1} - 6r^{n-2} = 0$

$r^{n-2}(r^2 + r - 6) = 0$

$r^2 + r - 6 = 0$

$r^2 + 3r - 2r - 6 = 0$

$r(r+3) - 2(r+3) = 0$

$r = -3, 2$

$a_n = cr_1^n + C_2 r_2^n$

$a_n = C_1(-3)^n + C_2(2)^n$

$a_0 = c_1 \cdot 1 + c_2$

$\boxed{a_n = (-2)(-3)^n + (1)(2)^n}$

~~&~~

$a_1 = -3c_1 + 2c_2$

$\downarrow$

sol^n

$c_1 + c_2 = -1$

$-3c_1 + 2c_2 = 8$

$2c_1 + 2c_2 = -2$

$-3c_1 + 2c_2 = 8$

$(+) \quad (-) \quad (-)$

$5c_1 = -10$

$c_2 = -1 + 2$

$= 1$

$\boxed{c_1 = -2}$

Q. $F_{n+2} = F_{n+1} + F_n \quad n \geq 0, \; F_0 = 0, \; F_1 = 1$

$dr^{n+2} = dr^{n+1} + dr^n$

$dr^n + dr^{n+1} - dr^{n+2} = 0$

$dr^n (r^2 - r - 1) = 0$

$r^2 - r - 1 = 0$

$r = -0.6, \; 1.6$

$F_0 = c_1 r^0 + c_2 r_2^0$

$0 = c_1 + c_2$

$c_1 = -c_2$

$1 = c_1(0.6) + c_2(1.6)$

$-0.6c_1 + 1.6c_2 = 1$

$-0.6c_1 - 1.6c_1 = 1$

$-2.2c_1 = 1$

$\boxed{c_1 = -\frac{1}{2.2}}$

$\boxed{c_2 = \frac{1}{2.2}}$

Ex - 10.2

1. a) $a_n = 5a_{n-1} + 6a_{n-2}, \; n \geq 2 \quad a_0 = 1, \; a_1 = 3$

$a_n - 5a_{n-1} - 6a_{n-2} = 0$

$a_{n+2} - 5a_{n+1} - 6a_n = 0, \; n \geq 0$

$dr^{n+2} - 5dr^{n+1} - 6dr^n = 0$

$r^n (r^2 - 5r - 6) = 0$

$r^2 - r - 6 = 0$

$r^2 - 3r + 2r - 6 = 0$

$r(r-3) + 2(r-3) = 0$

$r = -2, 3$

Ch - 14

25/11/23

Ring and Modular Arithmetic

Let R be a non-empty set on which we have 2 closed binary operations denoted by $\cdot$ and $+$ then $(R, +, \cdot)$ is a ring if $\forall a, b, c \in R$, the following conditions are satisfied.

a) $a+b = b+a$ (Commutative law of +)

b) $a+(b+c) = (a+b)+c$ (Associative law)

c) There exists $z \in R$ such that $a+z = z+a = a$ for every $a \in R$
(existence of identity under +)

d) For each $a \in R$, there is an element $b \in R$
$a+b = b+a = z$ (existence of inverse under +)

e) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associative law of $\cdot$)

f) $a(b+c) = a \cdot b + a \cdot c$
$(a+b) \cdot c = a \cdot c + b \cdot c$ [Distributive law of $\cdot$ over +]

Eg. $\mathbb{Z}$ - integers

a) $a+b = b+a$ ✓

b) $a+(b+c) = (a+b)+c$ ✓

c) $a+z = z+a = a$ ✓

d) $a+b = b+a = z = 0$ ✓

e) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ ✓

f) ✓

$\mathbb{Z}$ is a ring under $\cdot$ and $+$ $(\mathbb{Z}, +, \cdot)$

NOTE:
Eg. Under ordinary addition and multiplication $\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{R}$ are rings. In all these rings 0 is identity element. The additive inverse of each element x is $-x$.