1. Prove the following statement using method of induction.

For an $\forall n \geq 1$ and $n, k \in R^+$, $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n+k-1}{k} \right\rfloor$

Sol^n:- For any integer $n \geq 1$

Given:- Let P_n be the statement $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n+k-1}{k} \right\rfloor$

IB: For $n=1$

$$LHS = \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{1}{k} \right\rceil = 1 \quad (\text{as } \lceil 0.x \rceil = 1)$$

$$RHS = \left\lfloor \frac{n+k-1}{k} \right\rfloor = \left\lfloor \frac{1+k-1}{k} \right\rfloor = \lfloor 1 \rfloor = 1$$

$\therefore$ LHS = RHS. Hence P_1 is true

IH: For $n = m$, $k = l$, let P_k be true

$$\left\lceil \frac{m}{l} \right\rceil = \left\lfloor \frac{m+l-1}{l} \right\rfloor$$

$$\cancel{\Rightarrow \left\lceil 1 \right\rceil = \left\lfloor \frac{2k-1}{k} \right\rfloor}$$

IS: For $n = m+1$, $k = l+1$, we need to prove P_{k+1} is true

$$\left\lceil \frac{m+1}{l+1} \right\rceil = \left\lfloor \frac{m+1+l-1}{l+1} \right\rfloor$$

$$\Rightarrow \left\lceil \frac{m+1}{l+1} \right\rceil = \left\lfloor \frac{m+l+1}{l+1} \right\rfloor$$

$\therefore n = m, k = l+1$

2. Prove the following statement using method of Induction.

For every non-negative integer n, $\sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2}$

Given,

For any $n \geq 1$, let P_n be the statement $\sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2}$

IB For $n=1$

LHS:- $\sum_{i=0}^{1} 3^i = 3^0 + 3^1 = 1+3 = 4$

RHS:- $\frac{3^{n+1}-1}{2} = \frac{3^2-1}{2} = 4$

$\therefore$ LHS = RHS

Hence, P_1 is true

IH:- For $n=K$, let P_K be true

$$\sum_{i=0}^{K} 3^i = \frac{3^{K+1}-1}{2}$$

IS:- For, $n=K+1$ we need to prove P_{K+1} is true

$$\sum_{i=0}^{K+1} 3^i = \frac{3^{(K+1)+1}-1}{2}$$

$$\Rightarrow \sum_{i=0}^{K} 3^i + 3^{K+1} \Rightarrow \frac{3^{K+1}-1}{2} + 3^{K+1}$$

$$\Rightarrow \frac{(3^{K+1}-1) + 2 \cdot 3^{K+1}}{2} \Rightarrow \frac{3 \cdot 3^{K+1}-1}{2}$$

$$\Rightarrow \frac{3^1 \cdot 3^{K+1}-1}{2} \Rightarrow \frac{3^{(K+1)+1}-1}{2}$$

Hence, P_{K+1} is true.

Hence, for all values of $n \geq 1$, P_n will be true

3. Prove the following statement using method of Induction

Every non-negative integer can be written as the sum of distinct powers of 2. For example: $23 = 2^4 + 2^2 + 2^1 + 2^0$

Given:- For any $n \geq 1$, let P_n be the statement that every non-negative integer can be written as the sum of distinct powers of 2.

IB:- For $n=1$

$1 = 2^0$ which is sum of distinct powers of 2.

IH:- Suppose that every natural number $j \leq K$ can be written as the sum of distinct powers of 2

IS:- To prove $K+1$ can be written as the sum of distinct powers of 2.

$(K+1)$ is either odd or even.

Case 1:- If $K+1$ is even then $(K+1)/2$ is a natural number less than or equal to K. By our (IH) this means there exists distinct powers $p1, p2, \ldots, pn$ such that $0 \leq p1 \leq \ldots \leq pn$ and $(K+1)/2 = 2^{p1} + 2^{p2} + \ldots 2^{pn}$

multiplying both sides by 2.

$$2\left(\frac{K+1}{2}\right) = 2\left(2^{p1} + 2^{p2} + \ldots + 2^{pn}\right)$$

$$= 2^{p1+1} + 2^{p2+1} + \ldots + 2^{pn+1}$$ which is sum of distinct powers of 2

Case 2:- If $K+1$ is odd then K is even and thus when K is expressed as the sum of distinct powers of 2, each power is atleast 1, so that they are all divisible by 2. That is

$K = 2^{p1} + 2^{p2} + \ldots + 2^{pn}$ where $0 < p1 < \ldots < p^n$. Thus

$K+1 = 2^{p1} + 2^{p2} + \ldots + 2^{pn} + 1 = 2^{p1} + 2^{p2} + \ldots + 2^{pn} + 2^0$ is the sum of distinct powers of 2.

Since the statement is true for $K+1$ in both cases, it is true by the principal of mathematical induction for all natural numbers, n

4. A binary tree is full if every node has either two children (internal node) or no children (leaf node). Prove that in any full binary tree, the no. of leaves is exactly one more than the no. of internal nodes.

Given:- To prove, if n is total no. of internal nodes then there are $n+1$ no. of leaves.

IB:- If $n=1$, no. of leaves $= 1+1 = 2$
(smallest full binary tree).

IH:- If $n=K$, then $K+1$ leaves. (Let the statement be true

~~Where~~

IS:- To prove for $n=K+1$, there are $(K+1)+1$ leaves.

adding 1 internal node to full binary there will bring 2 leave

$\therefore$ The new internal node was a leaf before, the no. of leaves after adding a new node increases by 1 (~~R7~~)

Hence proved.

5. Prove by mathematical induction that a decimal number is divisible by 3 if and only if the sum of its digits is divisible by 3.

Ans:- Given, to prove that a decimal number is divisible by 3 if and only if the sum of the digits is divisible by 3.

IB:- The smallest no. divisible by 3 is 0. The sum of its digit is 0, So, it is true

~~K = abc~~

IH:- Let K be a number which is divisible by 3. Then sum of its digits is also divisible by 3.

Let sum of the digits of K be SM.

IS:- To prove the statement is true for ~~all~~ other numbers other than K. ($n = abc$)

For any numbers

$$n = (10^2 \times a) + (10^1 \times b) + (10^0 \times c)$$

$$= (99+1)a + (9+1)b + c$$

$$= 99a + 9b + a + b + c$$

$$\frac{n}{3} = 33a + 3(b) + \frac{(a+b+c)}{3}$$

$\therefore$ n is divisible by 3 when sum of digits is divisible by 3.

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(b) $T(n) = n^{1/3} T(n^{2/3}) + \Theta(n)$

$n^{\log_b a} = n^{\log_1 n^{1/3}}$

$f(n)$ $\Theta(n)$

$$\boxed{T(n) = n^{\log_1 n^{1/3}}}$$

(h) $T(n) = \frac{n}{n-5} T(n-1) + 1.$

$T(n) = n^{\log_{(1-\frac{1}{n})} \frac{n}{n-5}}$

6 GCD(a,b)

if b == 0 then

return a

else

return GCD(b, a%b)

endif

GCD(a,b) in time $O(\log(a+b))$, a%b takes 1 unit of time

7. The Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 ... are recursively defined as follows:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

Prove the following statements using mathematical induction.

(i) For all non-negative integers n, F_n is even if and only if n is divisible by 3

Given:- To prove for all non-negative integers n, F_n is even if and only if n is divisible by 3.

IB:- Smallest integer divisible by 3 is 3.

So $F_3 = F_2 + F_1$
$= 1 + 1 = 2$.

So LHS = RHS.
Hence, basic step is true

IH:- Let the statement be true for K, where K = 3m (divisible by 3)

$F_K = F_{K-1} + F_{K-2}$ is even
$= 2m$

IS:- To prove the statement is true for K+3.

$\Rightarrow F_{K+3} = F_{K+3-1} + F_{K+3-2}$

$\Rightarrow F_{K+2} + F_{K+1}$

$\Rightarrow F_{K+1} + F_K + \cancel{F_K} + F_{K+1}$

$= 2F_{K+1} + F_K = 2F_{K+1} + 2m = 2(F_{K+1} + m)$

So, the statement is true as F_K is also even.

Therefore, f_n is even if and only if ~~f_n is even~~. n is divisible by 3.

(b) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

IB:- For $n=1$

$\Rightarrow \sum_{i=0}^{1} F_i = F_0 + F_1$

$= 0 + 1$

$= 1$

RHS $= F_{1+2} - 1$

$= F_3 - 1$

$= 2 - 1 = 1$

IH:- Let the statement be true for $n = K$.

$\Rightarrow \sum_{i=0}^{K} F_i = F_{K+2} - 1$

IS:- To prove for $n = K+1$ is true

$\sum_{i=0}^{K+1} F_i = F_{(K+1)+2} - 1$

$\Rightarrow \sum_{i=0}^{K} F + F_{K+1} = F_{K+2} - 1 + F_{K+1}$

$= F_{K+2} + F_{K+1} - 1$

$= F_{K+3} - 1$

$= F_{(K+1)+2} - 1$

Hence, proved that $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ is true for $n \geq 1$

(c) $F_n^2 - F_{n+1} \cdot F_{n-1} = (-1)^{n+1}$

IB: For $n=1$

$= F_1^2 - F_{(1+1)} \cdot F_{(1-1)} = (-1)^{1+1}$

$\Rightarrow 1 - 0 = (-1)^2$

$\Rightarrow 1 = 1$

LHS = RHS

Hence, inductive ~~step~~ base is true for $n=1$

IH:- Let the statement be true for $n=K$. Hence,

$$F_K^2 - F_{(K+1)} \cdot F_{(K-1)} = (-1)^{K+1}$$

$$\Rightarrow F_K^2 = (-1)^{K+1} + F_{(K+1)} \cdot F_{(K-1)}$$

IS:- To prove the statement is true for $n=K+1$.

$$F_{K+1}^2 - F_{((K+1)+1)} \cdot F_{((K+1)-1)} = (-1)^{K+1+1}$$

LHS:

$$F_{(K+1)}^2 - F_{(K+2)} \cdot F_K = (-1)^{K+2}$$

$$F_{(K+1)}^2 = (-1)^{K+2} + F_{(K+2)} \cdot F_K$$

So,

$$F_K \cdot F_{(K+2)} + (-1)^{K+2} = F_K \cdot (F_{K+1} + F_K) + (-1)^{K+2}$$

$$= F_K \cdot F_{K+1} + F_K^2 + (-1)^{K+2}$$

$$= F_K (F_K + F_{K-1}) + F_K^2 + (-1)^{K+2}$$

$$= F_K^2 + F_K \cdot F_{K-1} + (F_K^2 - (-1)^{K+1})$$

$$= F_K^2 + F_K \cdot F_{K-1} + F_{K+1} \cdot F_{(K-1)}$$

$$= F_K^2 + F_K \cdot F_{K-1} + (F_K + F_{K-1}) \cdot F_{K-1}$$

$$= F_K^2 + F_K \cdot F_{K-1} + F_K \cdot F_{K-1} + F_{K-1}^2$$

$$= F_K^2 + 2F_K \cdot F_{K-1} + F_{K-1}^2 = (F_K + F_{K-1})^2 = F_{(K+1)}^2$$

Hence proved.

(d) If n is an integer multiple of m, then F_n is an integer multiple of F_m.

By defination of Fibonacci numbers:

$$F(n) = F(n-1) + F(n-2)$$

Given, to prove $F(n)$ is divisible by $F(m)$ if n is divisible by m.

or

$F(kn)$ is divisible by $F(n)$, where k is an integer.

IB:- For $k=1$,

Since $F(n)$ is divisible $F(n)$, so the statement is true.

IH:- Let the statement is true.

Hence, $F(kn)$ is divisible by $F(n)$.

IS: To prove $F((k+1)n)$ is divisible by $F(n)$.

$$\begin{aligned}
&F(kn+n) \\
&= F(kn+n-1) + F(kn+n-2) \\
&= 2F(kn+n-2) + F(kn+n-3) \\
&= 3F(kn+n-3) + 2F(kn+n-4) \\
&\vdots \\
&= F(t+1)F(kn+n-t) + F(t)F(kn+n-[t+1])
\end{aligned}$$

$$\therefore F(kn+n) = F(n)F(kn+1) + F(n-1)F(kn)$$

Hence, $F(kn+n)$ is divisible by $F(n)$.

Hence, proved.

8. Prove by mathematical induction that the sum of the cubes of three successive natural numbers is divisible by 9.

Given, to prove the statement that the sum of the cubes of three successive natural numbers is divisible by 9.

which is $P_n = n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

ultiple of

IB: For n=1

$P_1 = 1^3 + (1+1)^3 + (1+2)^3$

$= 1+8+27 = 36$ which is divisible by 9

IH:- Let P(n) be true for n=K

$\therefore P_K = K^3 + (K+1)^3 + (K+2)^3 = 9m$, $K \in N$

m is +ve integer

IS:- To prove, P_{K+1} is true

$P_{K+1} = (K+1)^3 + (K+2)^3 + (K+3)^3$

Now, $P_{K+1} - P_K = (K+1)^3 + (K+2)^3 + (K+3)^3 - \{K^3 + (K+1)^3 + (K+2)^3\}$

$= (K+3)^3 - K^3$

$= K^3 + 27 + 9K^2 + 27K - K^3$

$= 9(K^2 + 3K + 3)$

$P_{K+1} = P_K + 9(K^2 + 3K + 3)$

$= 9m + 9(K^2 + 3K + 3)$

$= 9(m + K^2 + 3K + 3)$ which is divisible by 9.

Hence, proved that for all $n \geq 1$, the sum of the cubes of three successive natural number is divisible by 9.

Q10. To prove that $|A \times B| = |A| \times |B|$ for all finite sets A and B.

Given, to prove that if A and B are finite sets, A with n elements and B with m elements, then $A \times B$ has nm elements

IB:- Let n=1 and B has m elements

$|A \times B| = |1 \times m|$
$= |m|$
$= m$

$|A| \times |B|$
$= 1 \times m$
$= m$

LHS = RHS.

Hence, true for n=1.

RHS:- Let $n = K$ and m elements in B.

$\therefore |A \times B| = |A| \times |B|$

$= |K \times m| = |K| \times |m|$

$\Rightarrow \underline{Km}$

Now, we know $mn = nm$, so proving for $n+1$ or $m+1$ both will be same. So change in m or n will not affect

LS:- Let $n = K+1$ and m elements in B.

$\therefore |A \times B| = |A| \times |B|$

$\cancel{= |(K+1)m| = |(K+1) \times m|}$

$\cancel{\Rightarrow |Km + m| = Km + m}$

$\leq$

$\Rightarrow |A| \times |B| = |(K+1)| \times |(m)|$

$= (K+1) \times m$

$= Km + m$

$= |Km + m| = |(K+1)m|$

$= |A \times B|$

Hence, proved.

<u>COUNTER Examples</u>:- Prove or disprove using counter example

11. For every $n \in N$, $n^2 - n + 11$ is prime.

The given statement is false because if $n = 11$ then $(11)^2 - 11 + 11 = 121$ which is not a prime number

12. For every $x, y \in N$, $2^{2x} + 3^{2y+1}$ is prime

No, the following statement is not true because if

$x = 2$ and $y = 2$

then $2^4 + 3^5 = 16 + 243 = 259$ which is divisible by 7 and hence not prime

13. For every $m \in I$, there exists $n \in N$, such that $\left|\frac{1}{m} - \frac{1}{n}\right| > \frac{1}{2}$

Statement is not true

$m = 1$, $n = 1$

$\left|\frac{1}{1} - \frac{1}{1}\right| = 0 < \frac{1}{2}$ ~~then~~

14. If n is an integer and n^2 is divisible by 4, then n is divisible by 4.

The given statement is false because if $n = 6$.

$6^2 = 36$ is divisible by 4 but 6 is not divisible by 4.

15. $(a+b)^2 = a^2 + b^2$ is not an algebraic identity $\forall a, b \in R$

If $a = 1$, $b = 2$ then

$(a+b)^2 = (1+2)^2 = 3^2 = 9$ and $a^2 + b^2 = 1^2 + 2^2 = 5$

So, when $a = 1$, $b = 2$ then $(a+b)^2 \neq a^2 + b^2$, hence the statement is not an identity

16. For $\forall a \in N$, $\frac{1}{x+2} = \frac{1}{x} + \frac{1}{2}$ is not an identity

If $x = 1$, then $\frac{1}{1+1} = \frac{1}{2}$ and $\frac{1}{1} + \frac{1}{2} = 3/2$

So, when $x = 1$, then $\left(\frac{1}{x+2} \neq \frac{1}{x} + \frac{1}{2}\right)$, hence the statement is not an identity

17. If $pq = x$, then $p = x/q$ $\forall p, q \in R$

If $pq = x$, then $p = x/q$ is not true for all real numbers p, q and x

Let $p=1, q=0$ and $x=0$

Then $(1)(0)=0$ but $1 \neq \frac{0}{0}$, since division is undefined

18. Show using counter example that, $\forall a,b \in R$, $a+b$ can be less than $\min(a,b)$

$\Rightarrow$ If $a=-1, b=-2$ then $a+b=-1-2=-3$ and $\min(-1,-2)$

$=-2$.

So, when $a=-1$ and $b=-2$ then $a+b$ is less than $\min(a,b)$

$\forall a,b \in R$. So, the statement is false

19. For all real number x, if x^2 is rational then x is rational.

This statement is false.

Using counterexample, let $x=\sqrt{2}$. Since, $x^2=(\sqrt{2})^2=2$ is a rational number and $x=\sqrt{2}$ is an irrational number.

Hence, false

20. For all real number x, y, $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$

If $x=2.51$ and $y=1.31$.

$\lceil x+y \rceil = \lceil 2.51+1.31 \rceil = \lceil 3.82 \rceil = 4$ and

$\lceil x \rceil + \lceil y \rceil = \lceil 2.51 \rceil + \lceil 1.31 \rceil = 3+2=5$.

So, when $x=2.51$ and $y=1.31$ then $\lceil x+y \rceil \neq \lceil x \rceil + \lceil y \rceil$.

Hence statement is false

21. Input: n, a positive integer

Output: b, an array of bits, the bin repr. of n, starting with the least significant bits

```
t := n;
k := 0;
while (t>0) do
    b[k] := t mod 2;
    t := t div 2;
    k := k+1;
```

Initialization : prior to the first iteration, the array $A[n-1 : 1]$ contains the value 1 so, loop works 1 time

Maintenance :- The while loop works $n > 0$, so, n time

Termination :- The loop terminates when $n = 0$, converting all no's to bits hence terminating

27. State TRUE or FALSE justifying your answer with proper reason.

(a) ~~f(n)~~ $2n^2 + 1 = O(n^2)$

$\Rightarrow 2n^2 + 1 \cdot n^0$ so by defination of Big-Oh we know that the statement true. (TRUE)

(b) $n^2(1+\sqrt{n}) = O(n^2)$

$\Rightarrow n^2 + n^2\sqrt{n} = O(n^2)$ (TRUE)

(By the defination of Big-Oh.)

(c) $n^2(1+\sqrt{n}) = O(n^2 \log n)$

$\rightarrow$ FALSE.

By defination of Big-Oh notation $(n^2 + n^2\sqrt{n})$

$= n^2 + n^{3/2}$

So, $O(n^2)$.

(d) $3n^2 + \sqrt{n} = O(n + n\sqrt{n} + \sqrt{n})$

$\Rightarrow$ False

By defination of Big-Oh notation $(3n^2 + \sqrt{n})$

$= O(n^2)$

(e) $\sqrt{n} \log n = O(n)$

True

$\rightarrow$ let $n = 4$

$2 \log 4 = 4.77$.

$O(n) = O(4)$ $\boxed{\text{So, true}}$

(f) $\log n \in O(n)$

<u>True</u>, time complexity of $\log n = O(\log n)$. But $O(n)$ is greater than $O(\log n)$. So, $O(\log n)$ can belong to $O(n)$.

By defination of Big-Oh notation.

(g) $n \in O(n \lg n)$

True, time complexity of $n = O(n)$. But $O(n\log n)$ is greater than $O(n)$. So, $O(n\log n)$ can belong to $O(n\log n)$.

(By defination of Big-Oh notation).

(h) $n\lg n \in O(n^2)$

True, time complexity of $n\log n = O(n\log n)$. But $O(n^2)$ is greater than $O(n\log n)$. So, $O(n\log n)$ can belong to $O(n^2)$.

(By defination of Big-oh notation)

(i) $2^n \in \Omega(6^{\ln 2})$

True, but defination of Big-~~Theta~~ Omega

(j) $\lg^3 n \in o(n^{0.5})$

<u>False</u>

<u>26</u>:- For each of the following pairs of functions, determine whether $f(n) = O(g(n))$ or $g(n) = O(f(n))$

(i) $f(n) = n(n-1)/2$ and $g(n) = 6n$

(b) $f(n) = n + 2\sqrt{n}$ and $g(n) = n^2$

(c) $f(n) = n + \log n$ and $g(n) = n\sqrt{n}$

(d) $f(n) = n\log n$ and $g(n) = n\sqrt{n}/2$

(e) $f(n) = 2(\log n)^2$ and $g(n) = \log n + 1$

greater

(a) $f(n) = \frac{n^2 - n}{2}$, $g(n) = n$

$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(n^2-n)}{2n}$

$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \left(\frac{n-1}{2}\right)$

$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

$\therefore$ As $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, so $f(n)$ has more growth rate than $g(n)$, hence we can write $g(n) = O(f(n))$.

(b) $f(n) = n + 2\sqrt{n}$ and $g(n) = n^2$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n+2\sqrt{n}}{n^2}$

$= \lim_{n\to\infty} \left(\frac{1}{n} + \frac{2}{n\sqrt{n}}\right)$.

$= \frac{1}{\infty} + \frac{1}{\infty} = 0$.

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, so $g(n)$ has more growth rate than $f(n)$, hence $f(n) = O(g(n))$

(c) $f(n) = n + \log n$ and $g(n) = n\sqrt{n}$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n+\log n}{n\sqrt{n}}$

$= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{\log n}{n\sqrt{n}}\right)$

$= 0$.

So, $g(n)$ has more growth rate than $f(n)$, hence $f(n) = O(g(n))$.

(d) $f(n) = n\log n$ and $g(n) = n\sqrt{n}/2$

$= \lim_{n\to\infty} \frac{f(n)}{g(n)} = \cancel{\frac{n\log n}{n}} \lim_{n\to\infty} \frac{n\log n \times 2}{n\sqrt{n}}$

$= \lim_{n\to\infty} \frac{2\log n}{\sqrt{n}} = 0$

So, $g(n)$ has more growth rate than $f(n)$, hence $f(n) = O(g(n))$.

(e) $f(n) = 2(\log n)^2$ and $g(n) = \log n + 1$

$= \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2(\log n)^2}{\log n + 1}$

$= \lim_{n\to\infty} \frac{2\log n}{1 + 1/\log n} = \infty$

So, $f(n)$ has more growth rate than $g(n)$, hence $g(n) = O(f(n))$.

28 For each of the following pair of functions $f(n)$ and $g(n)$, determine whether $f(n) = O(g(n))$ or $f(n) = \Theta(g(n))$ or $f(n) = \Omega(g(n))$

(a) $f(n) = \sqrt{n}$, $g(n) = \log(n+3)$

(b) $f(n) = n\sqrt{n}$, $g(n) = n^2 - n$

(c) $f(n) = 2^n - n^2$, $g(n) = n^4 + n^2$

(d) $f(n) = n^2 + 3n + 4$ & $g(n) = 6n^2$

(e) $f(n) = n + n\sqrt{n}$, $g(n) = 4n\log(n^2+1)$

(a) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\log(n+3)}$

$= \infty$

$\Rightarrow f(n) = O(g(n))$

also when

$\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$ then $f(n) = \Omega(g(n))$

(b) $f(n) = n\sqrt{n}$, $g(n) = n^2 - n$

$\cancel{\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{n\sqrt{n}}{n^2-n}}$ let $n = 4$

$\cancel{= \lim_{n\to\infty} \frac{\sqrt{n}}{n-1}}$ $4 \times 2 = 8$

$16 - 4 = 12$

$\therefore f(n) = O(g(n))$

(c) $f(n) = 2^n - n^2$, $g(n) = n^4 + n^2$

let $n = 5$.

$2^5 - 5^2 = 7$, $5^4 + 5^2 = 650$

$\therefore f(n) = O(g(n))$

(d) $f(n) = n^2 + 3n + 4$

$g(n) = 6n^2$

let $n = 2$

$2^2 + 6 + 4 = 14$

$6 \times 2^2 = 24$

$f(n) = \Omega(g(n))$

(e) $f(n) = n + n\sqrt{n}$.

$g(n) = 4n\log(n^2+1)$

$\lim_{n\to\infty} \frac{n + n\sqrt{n}}{4n\log(n^2+1)} = \frac{1+\sqrt{n}}{4\log(n^2+1)}$

$= \infty$

$f(n) = \Theta(g(n))$

30. Given $f(n) = (n+a)^b$ for any real constants a and b, where $b>0$

Prove that $f(n) \in \Theta(n^b)$.

$(n+a)^b = n^b + c_1 n^{b-1} a + \dots = \Theta(n^b)$

$0 \le c_1 n^b \le (n+a)^b \le c_2 n^b \mid \forall\ n \ge n_0$

$n+a \ge n - |a|$

$n+a \ge n/2 \quad |n| \le n/2$

let $f(n) = (n+a)^b$ and $g(n) = n^b$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(n+a)^b}{n^b} = \lim_{n\to\infty} \frac{n^b(1+a/n)^b}{n^b}$$

$$= \lim_{n\to\infty} \left(1+\frac{a}{n}\right)^b = 1^b \quad \text{ie. const}$$

$\therefore f(n) = O(g(n))$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n^b}{(n+a)^b} = \lim_{n\to\infty} \frac{n^b}{n^b(1+a/n)^b} = \frac{1}{1^b} \quad \text{(ie const)}$$

$\Rightarrow f(n) = \Omega(g(n))$ — ②

$\therefore$ Both ① and ② gives that -

$f(n) \in \Theta(g(n)) \Rightarrow (n+a)^b \in \Theta(n^b)$

29:- (a) $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$

As per defination of Big-Omega, let us assume variables n_1 and n_2

Suppose for all $n \ge n_1$ ~~$f_1(n)$~~ $0 \le c_1 \cdot g_1(n) \le f_1(n)$ and for all $n \ge n_2$,

$0 \le c_2 \cdot g_1(n) \le f_2(n)$

~~adding eq^n~~ for $n \ge 0$ $(n_0 = \min\{n_1, n_2\},\ c_0 = \min\{c_1, c_2\}$

$0 \le c_1 \cdot g_1(n) + c_2 \cdot g_2(n) \le f_1(n) + f_2(n)$

$\le c_0 (g_1(n) + g_2(n))$

$\Rightarrow f_1(n) + f_2(n) = \Omega(g_1(n) + g_2(n))$.

(b) If $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$ then $f_1(n) + f_2(n) = \Omega(\min(g_1(n) + g_2(n)))$

As per the defination of Big-~~Theta~~ Omega, assume variable c_0, n_1, n_2

For all $n \geq n_1$, $0 \leq c_1 \cdot g_1(n) \leq f_1(n)$ and $n \geq n_2$ $0 \leq c_2 \cdot g_2(n) \leq f_2(n)$

$n_0 = \min\{n_1, n_2\}$

$c_0 = c_1 + c_2$

$0 \leq c_1 \cdot g_1(n) + c_2 \cdot g_2(n) \leq f_1(n) + f_2(n)$

$\Rightarrow (c_1 + c_2) g_1(n) + (c_1 + c_2) g_2(n) \leq f_1(n) + f_2(n)$

$\Rightarrow f_1(n) + f_2(n) = \Omega(\min(g_1(n), g_2(n)))$.

(c) If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$.

As per defination of Big-Oh, assume variables n_1, n_2, c_0.

For all $n \geq n_1$, $f_1(n) \leq c_1 \cdot g_1(n)$ and $n \geq n_2$, $f_2(n) \leq c_2 \cdot g_2(n)$.

Let $n_0 = \max\{n_1, n_2\}$

$c_3 = c_1 \cdot c_2$.

$f_1(n) \cdot f_2(n) \leq c_1 \cdot g_1(n) \cdot c_2 \cdot g_2(n)$

$f_1(n) \cdot f_2(n) \leq c_3 \cdot (g_1(n) \cdot g_2(n))$

$f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$

(d) If $f(n) = O(g(n))$, then $f(n)^k = O(g(n)^k)$

As per defination of Big-Oh notation

$f(n) \leq c_1 \cdot g(n)$

Let K be some constant

$f^k(n) \leq c^k \cdot g^k(n)^k$

$\Rightarrow f^k(n) = O(g(n)^k)$

(e) If $f_1(n) = \Theta(g_1(n))$, $f_2(n) = \Theta(g_2(n)) \ldots, f_k(n) = \Theta(g_k(n))$ where $K \in I^+$ then prove that $\sum_{i=1}^{k} f_i(n) = \Theta(\max_{1 \le j \le k} (g_j(n)))$

As per defination of Big-Theta let us assume variables $c_1, c_2, c_3 \ldots c_k$

$$0 \le c_1(g_1(n)) \le f_1(n) \le c_1'(g_1(n))$$

$$\vdots$$

$$0 \le c_k(g_k(n)) \le f_k(n) \le c_k' \, g_k(n))$$

$$c_0 = \max\{c_1; c_2, c_3 \ldots c_k\}$$

$$0 \le c_1(g_1(n) + g_2(n) + \ldots + g_k(n)] \le f_1(n) + \ldots + f_k(n) \le c_0'(g_1(n) + g_2(n) + \ldots g_k(n))$$

$$\sum_{i=1}^{k} f_i(n) = \Theta\left[\max_{1 \le j \le k} (g_j(n))\right]$$

31. Prove that, $a^n \in o(b^n)$ for $b > a > n$

let $f(n) = a^n$, $g(n) = b^n$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{a^n}{b^n} = \text{invalid.}$$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{b^n}{a^n} = \text{invalid}$$

$\Rightarrow f(n) = o(g(n))$

$\therefore \boxed{a^n \in o(b^n).}$

32:- Prove that $\lg n = O(\sqrt{n})$, however $\sqrt{n} \neq O(\lg n)$

By defination of Big-Oh,

$$\lg(n) \leq c \cdot \sqrt{n}$$

let $n = 4$

$0 \leq 0.602 \leq 2$

$O(2)$.

but

$2 \neq O(0.602)$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{\log n}{\sqrt{n}} = \text{constant value}$$

Hence, $\lg n = O(\sqrt{n})$

33.

10	11	. . . .		70
0	1			60

(i) In binary search,

if midterm == target, then $O(1)$.

(ii) if (midterm < target) or (midterm > target)

then, total $O(7)$, 7 comparisions will be needed as again the array will be divided to midterms.

35:- function (int n)

{ if (n==1) ——— 1

return 1; ——— 1

else ——— 1

function (n/3); function (n/3); function (n/3) — $3(2^n)$

for (i=1; i<=n; i++) — n

x = x+1; — n-1

}

$$T(n) = \begin{cases} 2, & \text{if } (n=1) \\ 3(2^n) + n + n-1 + 1, & \text{if } n > 1 \end{cases}$$

$\Rightarrow T(n) = 2 + 2n + 3(2^n)$

$S(n) = O(n^2)$

36. void function (int n) {

temp = 1; ——— 1

Repeat ——— n+1

for i=1 to n ——— n

temp = temp+1 — n-1

n = n/2; — n-1

Until n<=1

}

$$T(n) = \begin{cases} 1, & \text{temp} = 1 \\ 4n+1 & \text{temp} > n \end{cases}$$

$\Rightarrow T(n) = O(4n)$

$S(n) = 12 + n$

$= O(n)$.

(37) int function (int n){

if (n<=2)

return 1;

else

return (function (floor (sqrt(n)))) + 1

}

$$T(n) = \begin{cases} 1, & n \le 2 \\ 1 + T(n-1) + k & \text{if } n > 2 \end{cases}$$

$T(n) = T(n-1) + k$

$S(n) = O(n+1)$

38 if (n=1)

return 1; -1

else

for (i=1; i<=8; i++) — 9

function (n/2); — 2^n

for (i=1; i<=n^3; i++) — n^3

count = count+1; — n^3-1

}

$$T(n) = \begin{cases} 1, & n=1 \\ 9+2^n & n>1 \\ +n^3 & \\ +n^3-1 & \end{cases}$$

$T(n) = O(n^3)$

$S(n) = 16+4n$

$O(S(n))$

39 Linear Search(A, n, el)

1. for i=1 to n do
2. if A(i) = el then
3. return i
4. return NIL

$T(n) = T(2^n) + O(n)$

$S(n) = O(1)$

40(a) $T(n) = \begin{cases} 1 & n = 2 \\ 1 & n = 4 \\ T(\frac{n}{4}) + 2T(\frac{n}{4}) + \theta(n^2) & n > 4 \end{cases}$

$\Rightarrow T(\frac{n}{2}) = T(\frac{n}{4}) + T(\frac{n}{8}) + \theta(\frac{n^2}{4})$

$T(\frac{n}{4}) = T(\frac{n}{8}) + T(\frac{n}{16}) + \theta(\frac{n^2}{16})$

$T(n) = T(\frac{n}{4}) + 4T(\frac{n}{8}) + 4T(\frac{n}{16}) + \theta(\frac{n^2}{4}) + 2\theta(\frac{n^2}{16})$

$T(n) = 5T(\frac{n}{2^k}) + 6T(\frac{n}{2^{k+1}}) = \theta(\frac{n^2}{2^{k-1}}) + 3\theta(\frac{n^2}{2^{k+1}})$

$T(2) = 1 \qquad T(4) = 1$

$\frac{n}{2^k} = 2, \frac{n}{2^{k+1}} = 2, \frac{n}{2^k} = 4, \frac{n}{2^{k+1}} = 4$

$n = 2^{k+1}, n = 2^{k+2}, n = 2^{k+2}, n = 2^{k+3}$

$\log n = k+1, \log n = k+2, \log n = k+3$

~~(b) $T(n) = n^{1/3}T(n^{2/3}) + \theta(n)$~~

$T(n) = 5T(4) + 6T(2) + \theta\left(\frac{2^{2k+4}}{2^{k-2}}\right) + 3\theta\left(\frac{2^{2k+4}}{2^{k+1}}\right)$

$\Rightarrow 11 + \theta(2^{k+6}) + 2\theta(2^{k+3})$

$\Rightarrow 11 + 4\theta(2^{2k+9}) \Rightarrow \theta(2^{2k+9})$

c)

$$T(n) = n^{1/2}\, T(n^{1/2}) + \log n$$

$$T(2^k) = n^{k/2}\, T(n^{k/2}) + \log(2^k)$$

Let $T(2^k) = S(k)$

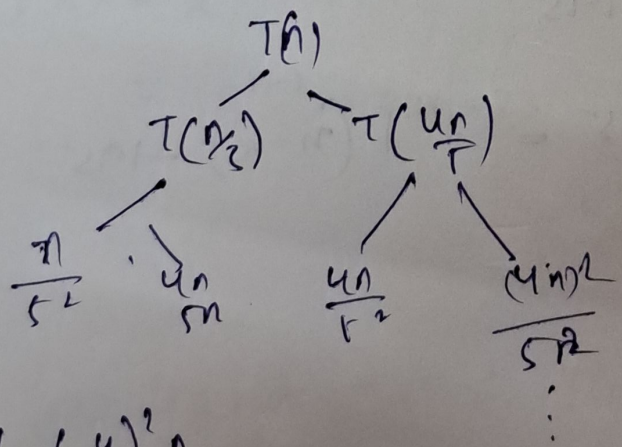
$$S(k) = n^{k/2}\, S(k/2) + k$$

By master's method.

$$h(n) = \frac{k}{n^{\log_k n^{k/2}}}$$

$$T(n) = \frac{n^{\log n^{k/2}} (\log_2 n)^{\cos n^{k/2}}}{n^{\log n^{k/2}}} = (\log n)^{\log n^{k/2}}$$

d) $T(n) = T(n/5) + T\left(\frac{4n}{5}\right) + \Theta(n)$



$$= \left(\frac{4}{5}\right)^2 n$$

$$= \left(\frac{4}{5}\right)^i n = \left(\frac{4}{5}\right)^i = 1$$

$$\log\left(\frac{4}{5}\right)^i = \log\left(\frac{1}{n}\right) = i\log\left(\frac{4}{5}\right) = \log\left(\frac{1}{n}\right)$$

$$= i\log\left(\frac{4}{5}\right) = \log\left(\frac{1}{n}\right)$$

$$\Rightarrow i = \log_{5/4} n$$

$$\therefore T(n) = \Theta\left(n^{\log_{5/4} 2}\right)$$

$$n(n) = \frac{Cn^2}{n^{\log n^3}} = C^n$$

$$\therefore \quad u(n) = O(C^n)$$

$$T(n) = n^{\log n^3} \, O(C^n) = Cn^2$$

f).

$$T(n) = 3T(n-1) - 15$$

$$T(n-1) = 3T(n-2) - 15$$

$$T(n-2) = 3T(n-3) - 15$$

$$\therefore \quad T(n) = 3\left[3T(n-2) - 15\right] - 15$$

$$T(n) = 9\left[3T(n-3) - 15\right] - 45 - 15$$

$$27T(n-3) - 195$$

$$T(n) = 3^3 T(n-3) - 195 = 3^k T(n-k) - 195$$

$$T(1) = 8.$$

$$n - k = 8$$

$$k = n - 8$$

$$T(n) = 3^{n-8} T(n - n + 8) - 195$$

$$= 3^{n-8} - 195$$

$$= O(3^{n-8})$$

g).

$$T(n-1) = 2T(n-2) + 3(n-1) + 1$$

$$T(n-2) = 2T(n-3) + 3(n-2) + 1$$

$$\therefore T(n) = 2\left[2T(n-2) + 3(n-1) + 1\right] + 3n + 1$$

$$T(n) = 4\left[2T(n-3) + 3n - 6 + 1\right] + 6n - 6 + 2 + 3n + 1$$

$$T(n) = \cancel{8\left[T\right.}\ 8T(n-3) + 12n - 24 + 4 + 6n - 6 + 2 + 3n + 1$$

$$T(n) = 2^3 T(n-3) + 21n - 23$$

$$= 2^k T(n-k) + 21n - 23$$

$$T(1) = 5$$

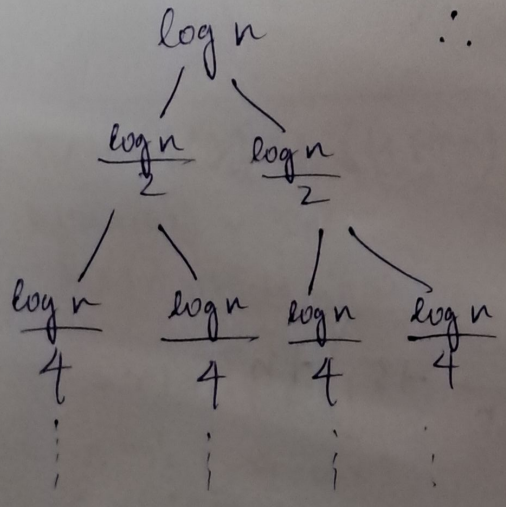
$$n - k = 1$$

$$k = n - 1$$

$$T(n) = 2^{n-1} T(5) + 21(n) - 23$$

$$= 2^{n-1} + 21n - 23 = O(2^{n-1})$$

i)



$$\therefore T(n) = O\left(\sum_{i=0}^{\log(n)} \log^{i+1}(n)\right)$$

$$= O(\log n)$$

(l)

By Master Method

$$h(n) = \frac{1}{\sqrt{n}} \times \frac{1}{n^{\log_{4/3} 1}} = \frac{1}{\sqrt{n}} = n^{-1/2}$$

$$\therefore u(n) = O(1)$$

$$T(n) = n^{\log_{4/3} 1} \, O(1) = O(1)$$

(m)

$$T(n) = 3\left[3T\left(\frac{n}{9}+5\right) + \frac{n}{6}\right] + n/2$$

$$T(n) = 9T\left(\frac{n}{9}+5\right) + n$$

$$\therefore T(n) = 3^k T\left(\frac{n}{3^k}+5\right) + n$$

$$T(1) = 1$$

$$n/3^k + 5 = 1$$

$$n = -4(3^k)$$

$$\log n + \log 4 = k \log 3$$

$$\log n = k$$

$$\therefore T(n) = 3^{\log 3} T\left(\frac{n}{3^{\log n}} + 5\right) + n$$

$T(n) = 3^{\log n} + n = O(3^{\log n})$

n)

$$T(n/2) = 2T(n/4) + \frac{n/2}{\log\log n/2}$$

$$\therefore T(n) = 2\left[2T\left(\frac{n}{4}\right) + \frac{n/2}{\log\log n/2}\right] + \frac{n}{\log\log n}$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2\left(\frac{n}{\log\log n}\right)$$

$T(1) = 1$

$\frac{n}{2^k} = 1$

$n = 2^k$

$\log n = k$

$$T(n) = 2^{\log n} + 2\frac{n}{\log\log n}$$

$$T(n) = O\left(2^{\log n}\right)$$

(o) $T(n) = \begin{cases} 1 & \text{for } n=0 \\ T(n-2) + 2\log n & n>0 \end{cases}$

$T(n-2) = T(n-4) + 2\log(n-2)$

$\therefore T(n) = T(n-4) + 2[\log(n-2) + \log(n)]$

$T(n) = T(n-2^n) + 2[\log(n-2) + \log n]$

$T(0) = 1$

$\therefore n-2^k = 0$, $n = 2^k$

$\log n = k$

$\therefore T(n) = T(n-2^{\log n}) + 2(\log(n-1 \times n))$

$T(n) = O(\log n - 1)$

(P) $T(n) = 2T(n/2) + n^2(1+\sin n)$

by master's method

$$h(n) = \frac{n^2(1+\sin n)}{n^{\log_2 2}} = (n+n\sin n)$$

$\therefore u(n) = n(1+\sin n)$

$T(n) = n^{\log_2 2}(n(1+\sin n))$

$= n^2(1+\sin n)$

22. The while loop works in the sub array A[0, n-1] comparing all the

Ini :- The loop invariant holds, since there is only one ~~loop~~ element in array which is the more of array.

Main :- Assume that Invariant holds for any kth iteration.

∴ A[1], ---, A[k-1] - The if condition, checks if the next element is greater than A[k-1] or not if yes ∴ A, -- A(k-1) ... A(k)

Ter :- The loop terminate when all elements have been check

Q3 Reversing the array A[0 : n-1]

In :- The loop invariant holds, since there is only one element in the array so it can be reverse

Mai :- The statement under while loop moves the cursor to next element. So, Invariant holds until j reaches n-1

Ter :- When all elements are reversed.

Q4 Checking required element A[0 : len(n)-1]

In : The loop invariant holds, 1 element in array

Main :- There are 3 condition to check weather the elem in the begin, mid or the end of arr.

The loop works until the whole

Ter The loop terminate when the whole array has started getting one the required elem