Chapter - 2

07/04/23

VECTOR SPACES

2.1 Vector spaces and Subspaces :-

Vector Spaces :- A non-empty set V is said to be a vector space if it satisfies the following properties

1- $x+y = y+x$
2- $x+(y+z) = (x+y)+z$
3- There is a unique vector '0' (zero vector) such that $x+0 = x$ for all x.
4- For each x there is a unique vector $-x$ such that $x+(-x)=0$
5- $Ix = x$
6- $(c_1 c_2)x = c_1(c_2 x)$
7- $c(x+y) = cx + cy$
8- $(c_1+c_2)x = c_1 x + c_2 x$

where $x, y, z \in V$ and $c, c_1, c_2 \in \mathbb{R}$

Alternate Definition :-
A non-empty set V is said to be a vector space if it satisfies the following properties:

1- Vector Addition i.e.
$x, y \in V \Rightarrow x+y \in V$

2- Scalar multiplication i.e.
$c \in \mathbb{R}$, $x \in V \Rightarrow cx \in V$

Examples of Vector spaces, $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \cdots \mathbb{R}^n$

Ex:- Show that $\mathbb{R}^2$ is a vector space

Proof:- $\mathbb{R}^2$ contains infinitely many elements and the elements are pairs like $(0,0)$, $(1,2)$, $(1,2)$...
So, $\mathbb{R}^2$ is nonempty.

1. Vector Addition :-

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$

$$x+y = (x_1+y_1, x_2+y_2) \in \mathbb{R}^2 \text{ as}$$

$$x_1+y_1 \in \mathbb{R} \ \& \ x_2+y_2 \in \mathbb{R}$$

2. Scalar Multiplication :-

Let $c \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$

$$cx = (cx_1, cx_2) \in \mathbb{R}^2 \text{ as } cx_1, cy_2 \in \mathbb{R}$$

So, $\mathbb{R}^2$ is a vector space.

Ex:- Verify whether the 1st quadrant of $\mathbb{R}^2$ is a vector space or not.

Proof:- Let V be the 1st quadrant

$$V = \{(x_1, x_2) \in \mathbb{R} : x_1 \geq 0, x_2 \geq 0\}$$

$$= \{(0,0), (1,1), (1,2), (2,3) \ldots\}$$

V is non-empty

1. Vector Addition :- Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in V$

$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0$$

$x+y = (x_1+y_1, x_2+y_2) \in V$ as $x_1+y_1 \geq 0$ and $x_2+y_2 \geq 0$

2. Scalar Multiplication :- Let $c = -2 \in \mathbb{R}$ and $x = (1,2) \in V$
$cx = (-2,-4) \notin V$

So, V doesnot satisfy scalar multiplication property.
Hence, V i.e. 1st quadrant is not a vector space.

Ex:- Verify whether the 1st and 3rd quadrant of $\mathbb{R}^2$ is a vector space or not.

Proof:- Let V be 1st and 3rd quadrant
$V = \{(1,1), (1,2), (-1,-1), (-1,-2) \cdots\}$
So, V is nonempty.

1. Vector Addition :- Let $x = (-1,-2)$ and $y = (2,1) \in V$
$x+y = (1,-1) \notin V$

So, V doesnot satisfy vector addition property
Hence, V is not a vector space.

Subspace :- A subspace of a vector space is a non-empty subset that satisfies the requirements for a vector space.

Ex:- Show that $y = x$ line is a subspace of the vector space $\mathbb{R}^2$.

Proof:- Let V be $y=x$ line.
$V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$

$= \{(0,0), (1,1) (2,2), (-1,-1) \cdots \}$

So, V is a non-empty subset of $\mathbb{R}^2$

1. Vector Addition:-
Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in V$
Then $x_1 = x_2$ and $y_1 = y_2$

$x+y = (x_1+y_1, x_2+y_2) \in V$ as $x_1+y_1 = x_2+y_2$

2. Scalar Multiplication:-
Let $c \in R$ and $x = (x_1, x_2) \in V$
Then $x_1 = x_2$
$\Rightarrow cx_1 = cx_2$

$cx = (cx_1, cx_2) \in V$

So, V i.e. $y=x$ line is subspace of R^2

Vector space : R^2
Subspaces :
1. $\mathbb{R}^2$
2. Anyline passing through origin
3. Origin

Vector Space : R^3
Subspaces :
1. $\mathbb{R}^3$
2. Any plane passing through origin
3. Any line passing through origin
4. Origin

classmate

Date ____
Page ____

Points to remember :-

- Every vector space is a subspace of itself.
- A subspace is a vector space in its own right
- Every vector space is a largest subspace of itself and origin is the smallest subspace.

The Column Space :- Let A be an $m \times n$ matrix. Then the column space contains all the linear combinations of the columns of A. It is denoted by $C(A)$. It is a subspace of R^m.

Ex :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$$

$$x_1 = 0, \quad x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 1, \quad x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x_1 = 0, \quad x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$x_1 = 1, \quad x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdots \right\}$$

$$= R^2$$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$Ax = b$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$

$\Rightarrow x_1 = 0, \quad x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 1, \quad x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow x_1 = 0, \quad x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\Rightarrow x_1 = 1, \quad x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \dots \right\}$

$\Rightarrow y = 2x$ line in R^2

Ex:- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$Ax = b$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = b$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of R^2

Points to remember :-

1. If A is a 2nd order non-singular matrix, then $C(A) = \mathbb{R}^2$.

2. If A is a 2nd order non-zero singular matrix then $C(A)$ is a line passing through origin in $\mathbb{R}^2$

3. If A is a 2nd order zero matrix, then $C(A)$ is the origin of $\mathbb{R}^2$.

The Nullspace :- Let A be a matrix of order $m \times n$. The nullspace of matrix A consists of all vector x such that $Ax = 0$. It is denoted by $N(A)$. It is a subspace of $\mathbb{R}^n$.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0, \; x_2 = 0$ (only case)

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ i.e. origin of $\mathbb{R}^2$

Date
Page

Ex:
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0, \ x_2 = 0$

$\Rightarrow x_1 = 2, \ x_2 = -1$

$\Rightarrow x_1 = 4, \ x_2 = -2$

$\Rightarrow x_1 = 8, \ x_2 = -4$

$\vdots$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \end{bmatrix} \cdots\cdots \right\}$$

$y = -\frac{x}{2}$ in $\mathbb{R}^2$

Ex:-
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Any value of x_1 & x_2 can satisfy

So, $N(A) = \mathbb{R}^2$

Points to Remember :-

1. If A is a 2nd order nonsingular matrix, then $N(A)$ is the origin of $\mathbb{R}^2$

2. If A is a 2nd order non-zero singular matrix, then $N(A)$ is a line passing through origin in $\mathbb{R}^2$.

3. If A is a 2nd order zero matrix then $N(A) = \mathbb{R}^2$.

4. For the column space of a matrix 'A' we are collecting the 'b' of the system $Ax = b$.

5. For the null space of a matrix 'A', we are collecting the 'x' of the system $Ax = 0$.

6. If A is a 3rd order non singular matrix, then $C(A) = \mathbb{R}^3$ and $N(A) =$ origin of $\mathbb{R}^3$.

7. If A is a 3rd order zero matrix, then $C(A) =$ Origin of $\mathbb{R}^3$ and $N(A) = \mathbb{R}^3$

8. Let A be a 3rd order non-zero singular matrix. Then :-

i) $C(A)$ is a line passing through origin and $N(A)$ is a plane passing through origin if rank of $A = 1$.

ii) $C(A)$ is a plane passing through origin and $N(A)$ is a line passing through origin if rank of $A = 2$.

08/04/23

Date
Page

Problem Set 2.1:

If $|A| \neq 0$, then $C(A)$ = the whole space and $N(A)$ = {zero vector}

If A is a null matrix then $C(A)$ = {zero vector} and $N(A)$ = the whole space.

Q.1

a) Let $V = \{(u,v)$: u and v are the ratios p/q of integers where $q \neq 0\}$

$x, y \in V \Rightarrow x+y \in V$ and $x-y \in V$
So, V is closed under vector addition and subtraction.

But $\alpha = \sqrt{2} \in R$ and $x = \left(\frac{1}{2}, \frac{3}{2}\right) \in V$
$\alpha . x = \sqrt{2}\left(\frac{1}{2}, \frac{3}{2}\right) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) \notin V$
So, V is not closed under scalar multiplication

b) Let $V = \{ u, v$: where $u = 0$ or $v = 0\}$
$\alpha \in R$ and $x \in V \Rightarrow \alpha x \in V$
So, V is closed under scalar multiplication.

But $x = (2, 0) \in V$ and $y = (0, 3) \in V$

$x+y = (2,0) + (0,3) = (2,3) \notin V$

So, V is not closed under vector addition.

classmate
Date
Page

Q.2 Which of the following subsets of R^3 are actually subspaces?

a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.

b) The plane of vectors b with $b_1 = 1$

Sol^n

a) Let $V = \{$ The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0\}$. $= \{(0,0,0), (0,1,0), (0,1,2) \ldots\}$
So, V is a non empty subset of R^3

i) Let $b = (b_1, b_2, b_3) \in V$ and $c = (c_1, c_2, c_3) \in V$
Then $b_1 = 0$ and $c_1 = 0$

$b + c = (b_1 + c_1, b_2 + c_2, b_3 + c_3)$

$\Rightarrow b + c \in V$
V is closed under vector addition.

ii) Let $\alpha \in R$ and $b = (b_1, b_2, b_3) \in V$
Then $b_1 = 0$
$\Rightarrow \alpha b_1 = 0$

now, $\alpha b = (\alpha b_1, \alpha b_2, \alpha b_3) = (0, 0, 0) \in V$

$\Rightarrow V$ is closed under scalar multiplication.
$\therefore V$ is a subspace of R^3

b) Let $V = \{$ The plane of vectors $b = (b_1, b_2, b_3)$ with first component $b_1 = 1\}$. $= \{(1,0,0), (1,1,0), (1,1,2) \ldots\}$
So, V is a non empty subset of R^3

i) Vector addition

Let $b=(b_1, b_2, b_3)$ and $c=(c_1, c_2, c_3) \in V$. Then $b_1 = 1$ and $c_1 = 1$

$\Rightarrow b_1 + c_1 = 2$

$b + c = (b_1+c_1, b_2+c_2, b_3+c_3) \in V$ as $b_1 + c_1 \neq 1$

So, V doesnot satisfy vector addition property.

Hence, V is not a subspace of $\mathbb{R}^3$

Q.5 i) $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$Ax = b$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = b$

$\Rightarrow u\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -1 \\ 0 \end{bmatrix} = b$

$Ax = 0$

$\Rightarrow u\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u = 1, \ v = 1$

$N(A)$ is the line through $(1,1)$ i.e. $y = x$ line.

$u = 0, \ v = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u = 1, \ v = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u = 0, \ v = 1 \Rightarrow b = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \ldots \right\}$ which is X-axis of $\mathbb{R}^2$

ii) $B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$Bx = b$

$\Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = b$

$\Rightarrow u\begin{bmatrix} 0 \\ 1 \end{bmatrix} + v\begin{bmatrix} 0 \\ 2 \end{bmatrix} + w\begin{bmatrix} 3 \\ 3 \end{bmatrix} = b$

$u = 0, \quad v = 0, \quad w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u = 1, \quad v = 0, \quad w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u = 0, \quad v = 1, \quad w = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$u = 0, \quad v = 0, \quad w = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$C(B) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \ldots \right\}$

$C(B) = \mathbb{R}^2$

$Bx = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u\begin{bmatrix} 0 \\ 1 \end{bmatrix} + v\begin{bmatrix} 0 \\ 2 \end{bmatrix} + w\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u = -2, \quad v = 1, \quad w = 0$

$N(B)$ is the line through $(-2, 1, 0)$

classmate
Date
Page

iii)

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since C is a null matrix, so $C(C) = \{\text{zero vector}\}$ $= \{(0,0)\} = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ and $N(C) =$ the whole space $\mathbb{R}^3$

Q.24

a)
$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_1 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

condition for solvability are

$$\boxed{b_2 - 2b_1 = 0 \quad \text{and } b_3 + b_1 = 0}$$

b_1 is any finite real value
In general, $b = (c, 2c, -c)$

b)
$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 + b_1 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

condition for solvability are

$b_3 + b_1 = 0$ and $b_2 - 2b_1$ is any finite real value

Q.8 $Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$

$x_1 + 2x_3 = 0$

$\Rightarrow x_1 = -2x_3$

Let $x_3 = 1$. Then $x_1 = -2$ and $x_2 = -x_1 - x_3 = 2 - 1 = 1$

$\boxed{(-2, 1, 1)}$

$N(A)$ = line of vectors $(-2x, x, x)$

The solutions form:-

(b) a line

(d) a subspace

(e) the nullspace of A.

Q.26 $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$Ax = b$

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$$\Rightarrow x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}2\\0\\0\end{bmatrix} = b$$

$$\Rightarrow x_1=0,\ x_2=0 \Rightarrow b=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1=1,\ x_2=0 \Rightarrow b=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$x_1=0,\ x_2=1 \Rightarrow b=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$x_1=1,\ x_2=1 \Rightarrow b=\begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$C(A) = \left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix} \cdots\cdots \right)$$

i.e. X-axis of $\mathbb{R}^3$

$$B = \begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}$$

$Bx = b$

$$\Rightarrow \begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = b$$

$$\Rightarrow x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\2\\0\end{bmatrix} = b$$

classmate
Date
Page

$x_1 = 0 \quad , \quad x_2 = 0 \quad \Rightarrow \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1, \quad x_2 = 0 \quad \Rightarrow \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0, \quad x_2 = 1 \quad \Rightarrow \quad b = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 1, \quad x_2 = 1 \quad \Rightarrow \quad b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$C(B) = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdots \right)$

i.e. $z = 0$ i.e $\Rightarrow$ x-y plane.

$C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$

$Cx = b$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = b$

$x_1 = 0 \quad , \quad x_2 = 0 \quad \Rightarrow \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1 \quad , \quad x_2 = 0 \quad \Rightarrow \quad b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 0 \quad , \quad x_2 = 1 \quad \Rightarrow \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 1, \quad x_2 = 1 \quad \Rightarrow \quad b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$x_1 = 2, \quad x_2 = 1 \quad \Rightarrow \quad b = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

$$C(c) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \cdots \right\}$$

i.e. line of vectors $(x, 2x, 0)$

Q.7 Given: Plane (P) : $x + 2y + z = 6$

Equation of the plane P_0 through origin parallel to P is :-

$$P_0 : x + 2y + z = 0$$

P_0 is a subspace of $\mathbb{R}^3$ but P is not a subspace of $\mathbb{R}^3$

Q: Show that all combinations of a given vectors $(1,1,0)$ and $(2,0,1)$ is a subspace of $\mathbb{R}^3$

Proof:- Given : $(1,1,0)$ and $(2,0,1)$ two vectors.

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}$$

All combinations of the two vectors $(1,1,0)$ and $(2,0,1)$ means the column space $C(A)$ of the matrix A. From the definition of column space, we know that if A is a matrix of order $m \times n$, then the column space $C(A)$ is a subspace of $\mathbb{R}^m$. Here A is a matrix of order 3×2. So, column space of the given matrix A is a subspace of $\mathbb{R}^3$

$\Rightarrow$ All combinations of these two given vectors is a subspace of $\mathbb{R}^3$.

2.2. Solving $Ax=0$ and $Ax=b$:-

Given $Ax = b$ ---- ①

let $x = x_p$ be a solution of equation ①

Then

$Ax_p = b$ ----- ②

again,

$Ax = 0$ ---- ③

Let $x = x_n$ be a solution of equation ③. Then

$A x_n = 0$ ---- ④

Adding equations ② and ④, we have

$A(x_p + x_n) = b$ --- ⑤

Comparing equations ① & ⑤, we have

$x = x_p + x_n$,

which is the complete solution of equation ①
where x_p is the particular solution and
x_n is the nullspace solution.

Ex:- Write the complete solution of the system.

$y + z = 2$
$2y + 2z = 4$

Solⁿ

$y + z = 2$
$2y + 2z = 4$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

Since 1st column of the coefficient matrix has a pivot element, so the 1st variable y is the pivot variable.

$y \rightarrow$ pivot variable
$z \rightarrow$ free variable

$y+z=2$
$\Rightarrow y = 2-z$

$x = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2-z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the complete solution of the given system.

Ex:- Write the nullspace solution of the system

$y+z=0$
$2y+2z=0$

Sol^n Given, $y+z=0$
$2y+2z=0$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

classmate
Date
Page

$y \to$ Pivot variable
$z \to$ Free variable

$y + z = 0$
$\Rightarrow y = -z$

$\therefore \; x = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the nullspace solution of the given system.

Ex:- Write the nullspace solution of the given system :-

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solⁿ $Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \leftarrow R_3 - 2R_2$$

Since 1st and 3rd columns of the coefficient have pivot elements, so 1st and 3rd variables

Date
Page

u and w are pivot Variably.

u, w → pivot Variably
v, y → Free Variables

$3w + 3y = 0$
$\Rightarrow w = -y$

$u + 3v + 3w + 2y = 0$

$\Rightarrow u + 3v - 3y + 2y = 0$

$\Rightarrow u + 3v - y = 0$

$\Rightarrow u = -3v + y$

$\therefore x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is

the nullspace solution.

Echelon Form and Reduced row Echelon form:-

Ex:- convert the following matrix into echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

classmate
Date
Page

Sol^n

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2,$$

which is upper triangular form and also the echelon form.

$$= \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow 1/2\, R_1 \\ R_2 \leftarrow -1/8\, R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow R_1 - \frac{1}{2} R_3 \\ R_2 \leftarrow R_2 - \frac{1}{4} R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 \leftarrow R_1 - 1/2\, R_2,$$

which is the reduced row echelon form.

Ex:- Convert the following matrix into echelon form and reduced row echelon form.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}$$

classmate

Date
Page

Sol^{n}:- $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 + R_1$
which is the upper triangular form but not echlon form.

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$
which is echelon form.

$= \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \leftarrow \frac{1}{3} R_2$

$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 \leftarrow R_1 - 3R_2$,

which is reduced row echlon form.

* How to get an echlon form from the upper triangular form of the matrix?

Ans:- To get the echlon form from the upper triangular form of a matrix, we have to verify the following points :-

i) The ~~points~~ pivots are the first non-zero entries in their rows.

ii) Below each pivot is a column of zeroes, obtained by elimination.

iii) Each pivot lies to the right of the pivot in the row above.

This produces the staircase pattern and zero rows come last.

* How to get reduced row echelon form from the echelon form?

Ans:- To get reduced row echelon form from the echelon form, we have to follow the following steps :-

Steps I - Make the pivot elements one by dividing the pivot element with every element of that row

Step II - Make the elements zero which are present above the pivot places using the pivot place element.

Note:-

1- Every echelon form is upper triangular form but every upper triangular form may or may not be echelon form.

2- If A is a non-singular matrix then, the reduced row echelon form of A is an identity matrix of order same as A.

classmate
Date
Page

Ex:- Construction the 4x4 matrix $A = [a_{ij}]$, where $a_{ij} = (-1)^{ij}$. Also, convert A into echelon form and reduced row echelon form.

Solⁿ $A = [a_{ij}]$, where $a_{ij} = (-1)^{ij}$ and A is a 4x4 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 + R_1 \end{matrix}$$

$$= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \leftarrow R_4 - R_2$$

which is echelon form

$$= \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_1 \leftarrow -R_1 \\ R_2 \leftarrow \frac{1}{2} R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_1 \leftarrow R_1 + R_2$, which reduced row echelon form.

Rank of Matrices :-

Rank of a matrix :- The number of pivot elements in the echelon form of a matrix is known as rank of the matrix.

Ex:-

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 + R_2$$

, which is the echelon form.

The no. of pivot elements in the echelon form is 3. So, rank of $A = 3$.

Ex:- $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$, is the echelon form.

Since the no. of pivot elements in the echelon form is 2; so rank of $A = 2$.

Points to remember:-

1. Rank of a zero matrix is always zero.
2. Rank of a non-singular matrix is equal with its order.
3. If A is a non-zero matrix of order $m \times n$, then rank of $A \leq \min(m, n)$
4. Rank of a non-zero matrix is atleast one.

Problem set 2.2:-

Q.1 $u + v + 2w = 2$

$2u + 3v - w = 5$

$3u + 4v + w = c$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c-6 \end{bmatrix} \qquad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ c-7 \end{bmatrix} \qquad R_3 \leftarrow R_3 - R_2$$

The system is solvable for $c-7=0$

$\Rightarrow c=7$

Q.4 $Ax=b$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad R_2 \leftarrow R_2 - 2R_1$$

u, w are pivot variables

v is the free variable

$w = 2$

$u + 2v + 2w = 1$

$\Rightarrow u + 2v = -3 \quad \Rightarrow u = -3 - 2v$

$$\therefore x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2v \\ v \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$ is the complete solution

Date
Page

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

The system has no solution.

Q.5

(i) $$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

, is the echlon form

Since there are two pivot elements, so rank of $A = 2$.

ii) $$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 4R_1 \\ R_3 \leftarrow R_3 - 7R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

, is the echelon form.

Since there are two pivots, so rank of A = 2

Q.7 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $(Ax = b)$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - 2b_1 - 3b_2 \end{bmatrix} \quad R_3 \leftarrow R_3 - 3R_2$$

$\boxed{b_3 - 2b_1 - 3b_2 = 0}$ is the required constraints on b that turn the third equation into 0=0

$b_3 = 3b_2 + 2b_1$

$b_1 = 0, \ b_2 = 0 \Rightarrow b_3 = 0$

$b_1 = 1, \ b_2 = 0 \Rightarrow b_3 = 2$

$b_1 = 0, \ b_2 = 1 \Rightarrow b_3 = 3$

$b_1 = 1, \ b_2 = 1 \Rightarrow b_3 = 5$

⋮

The attainable right hand sides i.e. the column space is :-

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \ --- \right\}$$

Date
Page

Since there are two pivots in the echelon form of the matrix A, so rank of A = 2.

Q.13

(a) $Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_3 + 2x_4 = 0$ $\qquad$ x_1, x_3 are pivot variables

$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ $\qquad$ x_2, x_4 are free variables

$\Rightarrow$ $x_3 = -2x_4$

$x_1 + 2x_2 - 2x_4 = 0$

$\Rightarrow x_1 = -2x_2 + 2x_4$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ is}$$

the nullspace solution of $Ux = 0$

$Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

classmate

Date

Page

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftarrow R_1 - 3R_2$$

$\Rightarrow Rx = 0$, where R is the reduced row echelon form.

$x_1, x_3 \rightarrow$ Pivot variables
$x_2, x_4 \rightarrow$ Free variables

$x_3 + 2x_4 = 0$
$x_1 + 2x_2 - 2x_4 = 0$

$\Rightarrow x_3 = -2x_4$
$x_1 = -2x_2 + 2x_4$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_4 \\ x_2 + 0\cdot x_4 \\ 0\cdot x_2 - 2x_4 \\ 0\cdot x_2 + x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \text{ is}$$

the nullspace solution of $Rx = 0$

Q.44

i) $$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & 2 & -1 \\ 9 & 6 & q \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & q-3 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + \frac{1}{2}R_1 \\ R_3 \leftarrow R_3 - \frac{3}{2}R_1 \end{matrix}$$

$$= \begin{bmatrix} 6 & 4 & 2 \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

is the echelon form

for $q=3$, Rank of $A=1$

for $q\neq 3$, Rank of $A=2$

Rank of A will never be 3

ii) $B=\begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}$

$=\begin{bmatrix} 3 & 1 & 3 \\ 0 & 2-q/3 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - \frac{q}{3} R_1$

, echelon form

$2-\frac{q}{3}=0$

$\Rightarrow q=6$

For $q=6$, rank of $A=1$

For $q\neq 6$, rank of $A=2$

Q.36 i) $x+3y+3z=1$

$2x+6y+9z=5$

$-x-3y+3z=5$

$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_2$$

x, z are pivot variables
y is the free variable

$$3z = 3 \Rightarrow z = 1$$

$$x + 3y + 3z = 1 \Rightarrow x = -3y - 3z + 1$$
$$= -2 - 3y$$

$$\therefore x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 - 3y \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$ is the complete solution.

ii)
$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1$$

$$= \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

x, z are pivot variables
y, t are free variables

$2z + 4t = 1 \Rightarrow z = \frac{-1}{2} - 2t$

$x + 3y + z + 2t = 1 \Rightarrow x = -3y - z - 2t + 1$

$= -3y + \frac{1}{2} + 2t - 2t + 1$

$= \frac{3}{2} - 3y$

$\therefore \quad x = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - 3y + 0.t \\ 0 + y + 0.t \\ -1/2 + 0.y - 2t \\ 0 + 0.y + 1.t \end{bmatrix}$

$= \begin{bmatrix} 3/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

is the complete solution.

2.3. Linear Independence, Basis and Dimensions.

Linear independence or dependence of Vectors:-

Definition :- A set of vectors $v_1, v_2, \cdots v_n$ are said to be linearly independent if there exist scalars $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$$

$\Rightarrow$ all scalars $\alpha_i = 0$ for $i = 1, 2, 3 \cdots n$

classmate
Date
Page

Definition :- A set of vectors $v_1, v_2, \dots v_n$ are said to be linearly dependent if there exists scalars $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$$

$\Rightarrow$ atleast one $\alpha_i \neq 0$ for $i = 1, 2, 3 \dots n$

Ex :- Let $v_1 = (1,2)$ and $v_2 = (3,4)$
Let α_1 and α_2 be two scalars.

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$
$$\Rightarrow \alpha_1 (1,2) + \alpha_2 (3,4) = 0$$
$$\Rightarrow (\alpha_1, 2\alpha_1) + (3\alpha_2, 4\alpha_2) = 0$$
$$\Rightarrow \alpha_1 + 3\alpha_2 = 0$$
$$2\alpha_1 + 4\alpha_2 = 0$$
$$\Rightarrow \alpha_1 = 0, \quad \alpha_2 = 0$$

So, the two given vectors are linearly independent

Ex:- Let $v_1 = (1,2)$ and $v_2 = (2,4)$

Let α_1 and α_2 be two scalars
$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\Rightarrow \alpha_1 (1,2) + \alpha_2 (2,4) = 0$$
$$\Rightarrow \alpha_1 + 2\alpha_2 = 0$$
$$2\alpha_1 + 4\alpha_2 = 0$$

$$\Rightarrow \alpha_1 = 2, \quad \alpha_2 = -1$$

So, the given vectors are linearly dependent.

Alternate methods:-

Determinant Method:-

Ex:- Let $V_1 = (1,2)$ and $V_2 = (3,4)$

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$|A| = 4-6 = -2 \neq 0$$

So, the two given vectors are linearly independ

Ex:- Let $V_1 = (1,2)$ and $V_2 = (2,4)$

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$|A| = 4-4 = 0$$

So, the given vectors are linearly dependent.

Note: 1- This determinant method is appliable when A is a square matrix.

2- This determinant method is appliable when 2 vectors in $\mathbb{R}^2$, 3 vectors in $\mathbb{R}^3$ and so on.

Rank Method:-

Ex:- Let $V_1 = (1,2)$ and $V_2 = (3,4)$

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form.

No. of pivots = 2 = no. of columns.

$\Rightarrow$ The given two vectors are linearly independent

Ex:- $v_1 = (1, 1)$, $v_2 = (2, 3)$, $v_3 = (1, 2)$

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1$, echelon form.

No. of pivots = 2 $\neq$ no. of columns.

$\Rightarrow$ The three given vectors are linearly dependent.

Notes :-

1. Three or more vectors in R^2 are always linearly dependent.

2. Four or more vectors are always linearly dependent.

3. The set of n vectors in R^m must be linearly dependent if $n > m$.

classmate

Date
Page

Ex: Decide the dependence or independence of the vectors $(1,0,0)$, $(0,1,0)$ and $(-2,0,0)$

Sol^n Let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$ and $v_3 = (-2,0,0)$

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix A is already in echelon form.

No. of pivots $= 2 \neq$ no. of columns.

$\Rightarrow$ The three given vectors are linearly depende

Alternate method :-

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$|A| = 1(0-0) - 2(0-0) = 0$

So, the three given vectors are linearly dependen

Ex :- Let $V_1 = (1,2,0)$ and $V_2 = (0,1,-1)$

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (The two vectors are the two rows of the matrix)

The matrix A is already in echelon form.

No. of pivots $= 2 =$ no. rows.

$\Rightarrow$ The two vectors v_1 and v_2 are linearly independent.

Ex :- Given : $v_1 = (1, 2, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (0, 8, 0)$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

$|A| = 1(0-8) + 1(0-16) = -8-16 = -24 \neq 0$

$\Rightarrow$ The three vectors are linearly independent.

Ex :- Given : $v_1 = (1, 1, -1)$, $v_2 = (2, 3, 4)$, $v_3 = (4, 1, -1)$ and $v_4 = (0, 1, -1)$

The given vectors are in $\mathbb{R}^3$. We know that four or more vectors in $\mathbb{R}^3$ are always linearly dependent.

So, the given four vectors are linearly dependent.

Points to remember :-

1- Two vectors are dependent if they lie on the same line.

2- Three vectors are dependent if they lie on the same plane.

3- If the nullspace of a matrix is the zero vector only, then the columns of A are linearly independent.

4. The rows and the columns in which pivot elements present in the echelon form V and in the reduced row echelon form R of a matrix are linearly independent. The correspon rows and columns of the given matrix are also linearly independent.

5. In $\mathbb{R}^2$, maximum two linearly independent vectors are present.

6. In $\mathbb{R}^3$, maximum three linearly independent vectors are present.

7. In R^n, maximum n linearly independent vectors are present.

Ex:- Choose three linearly independent columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

Solⁿ

$$A = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{matrix} R_3 \leftarrow R_3 - 2R_2 \\ \text{, echelon form.} \end{matrix}$$

since the pivots are present in the 1st, 3rd and 4th column, of the echelon form, so the 1st, 3rd and 4th column of the given matrix are linearly independent.

The 2nd, 3rd & 4th columns of the given matrix are also linearly independent.

The given matrix has three linearly independent columns.

Spanning a Subspace:-

Let us know what is the meaning of a set of vectors to span a space.

Vector space $\mathbb{R}^2$:-

Minimum two linearly independent vectors are required to span the vector space $\mathbb{R}^2$ i.e. the linear combination of two linearly independent vectors of $\mathbb{R}^2$ can form the whole plane $\mathbb{R}^2$.

Vector space $\mathbb{R}^3$:-

Minimum three linearly independent vectors are required for the spanning of the space $\mathbb{R}^3$ i.e. the linear combination of 3 linearly independent vectors of $\mathbb{R}^3$ can form the whole space $\mathbb{R}^3$.

More than three vectors of $\mathbb{R}^3$ can also span the whole space $\mathbb{R}^3$ provided among the given vectors three are linearly independent. But two linearly independent. But two linearly independent vectors of $\mathbb{R}^3$ can not span the whole plane $\mathbb{R}^3$

Note :-

1- The column space of a matrix is spanned by its rows.

2- The row space of a matrix is spanned by its rows.

Ex:- Describe the subspace of $\mathbb{R}^2$ spanned by

a) the vectors (1,2) and (2,4)

b) the vectors (1,2) and (3,4)

c) the vector (1,2)

d) the vectors (1,2), (3,4) and (1,1)

Solⁿ

a) Given: the vectors (1,2) and (2,4)

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad R_2 \leftarrow R_2 - 2R_1, \text{ echelon form.}$$

Rank of A = 1.

The required subspace of $\mathbb{R}^2$ spanned by the given two vectors is a line passing through origin.

b) Given : the vectors (1,2) & (3,4)

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form.

Rank of A = 2

The required subspace of $\mathbb{R}^2$ spanned by the two given vectors is the whole plane $\mathbb{R}^2$ itself.

c) Given: the vector (1,2).

Let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form

Rank of A = 1

The required subspace of $\mathbb{R}^2$ spanned by the given vector is a line passing through the origin.

d) Given: the vectors (1,2), (3,4) & (1,1)

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$.

$= \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -1 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, echelon form.

Rank of $A = 2$

The required subspace of $\mathbb{R}^2$ spanned by the three given vectors is the whole plane $\mathbb{R}^2$

<u>Basis for a Vector Space :-</u>

A basis for a vector space V is a subset with a sequence of vectors having two properties at once.

1. The vectors are linearly independent (not too many vectors).

2. They span the space V. (not too few vectors)

<u>Points to remember :-</u>

1- A basis of a vector space is the maximal independent set.

2. A basis of a vector space is also a minimal spanning set.

3. Spanning involves the column space and independence involves the nullspace.

4. No elements of a basis will be wasted.

Ex:- Check whether the following sets are basis of $\mathbb{R}^3$ or not?

a) $B_1 = \{ (1,2,2), (-1,2,1), (0,8,0) \}$

b) $B_2 = \{ (1,2,2), (-1,2,1), (0,8,6) \}$

c) $B_3 = \{ (1,2,2), (-1,2,1) \}$

d) $B_4 = \{ (1,0,0), (0,1,0), (0,0,1) \}$

e) $B_5 = \{ (1,1,-1), (2,3,4), (4,1,-1), (0,1,-1) \}$

Solⁿ

a) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$

$|A| = 1(0-8) + 1(-16) = -24 \neq 0$

The vectors are linearly independent
So, B_1 is a basis of $\mathbb{R}^3$.

b) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix}$

$|A| = 1(12-8) + 1(12-16) = 4-4=0$

The vectors are linearly dependent
So, B_2 is not a basis of $\mathbb{R}^3$

c) B_3 is not a basis of $\mathbb{R}^3$ as it doesnot contain the maximum number of linearly independent vectors of $\mathbb{R}^3$.

d) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = 1 \neq 0$

The vectors of B_4 are linearly independent.

So, B_4 is a basis of $\mathbb{R}^3$ and is known as standard basis of $\mathbb{R}^3$

e) The elements of B_5 are linearly dependent as four or more vectors in $\mathbb{R}^3$ are always linearly dependent.

So, B_5 is not a basis of $\mathbb{R}^3$

<u>Dimension of vector spaces:-</u>

Dimension of a vector space is the maximum number of linearly independent vectors of the vector space

OR.

The no. of elements present in the basis of a vector space is known as dimensions of the vector space.

$\dim . \mathbb{R} = 1$, $\dim . \mathbb{R}^2 = 2$, $\dim . \mathbb{R}^3 = 3$, ~~dim~~
$\dim . \mathbb{R}^n = n$.

Dimension of the vector space $\mathbb{R}^3 = 3$
Any plane passing through the origin are two dimensional subspaces of $\mathbb{R}^3$.

Any line passing through origin are 1-dimensional subspace of $\mathbb{R}^3$

The origin $\{(0,0,0)\}$ is a 0-dimensional subset of $\mathbb{R}^3$.

★ **Note** :- A vector space has multiple spaces.

Problem Set 2.3 :-

Q.3 a) Given : vectors $(1,3,2)$, $(2,1,3)$ and $(3,2,1)$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

$$|A| = 1(1-6) - 2(3-4) + 3(9-2)$$
$$= -5 + 2 + 21$$
$$= 18 \neq 0$$

So, the given vectors are linearly independent.

b) Given: vectors $(1,-3,2)$, $(2,1,-3)$ and $(-3,2,1)$

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$$|A| = 1(1+6) - 2(-3-4) - 3(9-2) = 7+14-21 = 0$$
So, the given vectors are linearly dependent.

Date

Page

Q.5 Let w_1, w_2 and w_3 are linearly independent vectors.

Let $v_1 = w_2 - w_3$, $v_2 = w_1 - w_3$ and $v_3 = w_1 - w_2$

$$v_1 - v_2 + v_3 = w_2 - w_3 - w_1 + w_3 + w_1 - w_2$$
$$= 0$$

$\Rightarrow$ V_1, V_2 and V_3 are linearly dependent

Q.8 Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^3$

a) These four vectors are dependent because four or more vectors in $\mathbb{R}^3$ are always dependent.

b) The two vectors V_1 & V_2 will be dependent if one is a multiple of other.

c) The vectors v_1 and $(0,0,0)$ are dependent because $0 \cdot v_1 + c(0,0,0) = 0$ has a non-zero solution for any $c \neq 0$.

Q.9:- Given: Plane: $x + 2y - 3z - t = 0$ in $\mathbb{R}^4$

i) $x = -2y + 3z + t$

$y = 0$, $z = 0$, $t = 1$ $\Rightarrow x = 1$
$y = 0$, $z = 1$, $t = 0$ $\Rightarrow$ $x = 3$

So, $(1,0,0,1)$ and $(3,0,1,0)$ are two independent vectors.

ii) $y=1,\ z=0,\ t=0 \Rightarrow x=-2$

So, $(1,0,0,1)$, $(3,0,1,0)$ and $(-2,1,0,0)$ are three independent vectors.

iii) Since there are 3 free variables, so there will be not four independent vectors on the plane.

iv) The plane is the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$$

Q.4 Given: $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$|A| = 1(1-0) = 1 \neq 0$

So the vectors V_1, V_2 & V_3 are linearly independent

Let c_1, c_2, c_3 and c_4 be four scalars.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad c_1 + c_2 + c_3 + 2c_4 = 0$$
$$c_2 + c_3 + 3c_4 = 0$$
$$c_3 + 4c_4 = 0$$

$c_3 + 4c_4 = 0$
$\Rightarrow c_3 = -4c_4$

Let $c_4 = 1$, then $c_3 = -4$

$c_2 = -c_3 - 3c_4 = -(-4) - 3 = 4 - 3 = 1$

$c_1 = -c_2 - c_3 - 2c_4 = -1 + 4 - 2 = 1$

So, the four vectors are linearly dependent as atleast one of the scalars $c_i \neq 0$, $i = 1, 2, 3, 4$

Q.10 Let w_1, w_2, w_3 be independent vectors.

Let $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$ & $v_3 = w_1 + w_2$

Let c_1, c_2 and c_3 be three scalars.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$\Rightarrow c_1(w_2 + w_3) + c_2(w_1 + w_3) + c_3(w_1 + w_2) = 0$

$\Rightarrow (c_2 + c_3) w_1 + (c_1 + c_3) w_2 + (c_1 + c_2) w_3 = 0$

$\Rightarrow c_2 + c_3 = 0$, $c_1 + c_3 = 0$, $c_1 + c_2 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow v_1, v_2$ and v_3 are independent.

Q.16 Vector space $= \mathbb{R}^3$

(a) Given: vectors $(1, 1, -1)$ and $(-1, -1, 1)$

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 + R_1$, echelon form.

Rank of $A = 1$

The required subspace of $\mathbb{R}^3$ spanned by the two given vectors is a line passing through the origin.

(b). Given : vectors $(0,1,1)$, $(1,1,0)$ & $(0,0,0)$

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_1$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 + R_2$, echelon form.

Rank of $A = 2$

The subspace of $\mathbb{R}^3$ spanned by the three given vectors is a plane passing through origin.

(c). Given : the columns of a 3 by 5 echelon matrix with 2 pivots

Rank of matrix $= 2$

Date
Page

So the subspace of $\mathbb{R}^3$ spanned by the columns of matrix is a plane passing through the origin

d) Given :- All vectors with positive components.
Here all vectors with positive components in 1st octant contains three linearly independent vector. So, the subspace of $\mathbb{R}^3$ spanned by all vectors with positive components is the whole space $\mathbb{R}^3$

Q.19 Plane (P): $x-2y+3z=0$ in $\mathbb{R}^3$

$$x = 2y - 3z$$

$y=0,\ z=1 \Rightarrow x=-3$
$y=1,\ z=0 \Rightarrow x=2$

So, $\{(-3,0,1), (2,1,0)\}$ is a basis of P
xy-plane is $z=0$.

Intersection of the plane P with xy-plane is
$x-2y=0$, $z=0$ (a line in $\mathbb{R}^3$)
$x=2y$

A basis for the line is $\{(2,1,0)\}$

A basis for all vectors perpendicular to the plane is $\{(1,-2,3)\}$

classmate

Date

Page

Sol3 Let $v_1 = (1,1,0,0)$ $v_2 = (1,0,1,0)$ $v_3 = (0,0,1,1)$
$v_4 = (0,1,0,1)$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$\Rightarrow (c_1, c_1, 0, 0) + (c_2, 0, c_2, 0) + (0, 0, c_3, c_3) + (0, c_4, 0, c_4) = (0,0,0,0)$$

$$c_1 + c_2 = 0$$
$$c_1 + c_4 = 0$$
$$c_2 + c_3 = 0$$
$$c_3 + c_4 = 0$$

$\Rightarrow c_1 = c_3 = -1$ and $c_2 = c_4 = 1$

$\Rightarrow$ v_1, v_2, v_3 & v_4 are not linearly independent

Again,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = (0,0,0,1)$$

$$\Rightarrow \left.\begin{array}{l} c_1 + c_2 = 0 \\ c_1 + c_4 = 0 \\ c_2 + c_3 = 0 \\ c_3 + c_4 = 0 \end{array}\right] \Rightarrow \text{Impossible}$$

So, $v_1; v_2; v_3; v_4$ are not independent.
Hence, they donot span in $\mathbb{R}^4$

Q.3)

i) $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$

For $c=0$, $d=2$

$$A=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$
, echelon form.

since there are two pivot elements, so rank of A $=2$.

ii) $B=\begin{bmatrix} c & d \\ d & c \end{bmatrix}$, $|B| = c^2-d^2=0$
$= c = \pm d$.

for $c \neq \pm d$ the matrix B is non singular

Rank of a non-singular matrix is equal to its order. So, rank of $A=2$ for $c \neq \pm d$

Q.40

a) Vectors $(1,2,0)$ and $(0,1,-1)$

Let $B=\{(1,2,0),(0,1,-1)\}$

Since, exactly three linearly independent vectors are required for a basis of $\mathbb{R}^3$, so B is not a basis of $\mathbb{R}^3$.

(b) vectors : (1,1,-1), (2,3,4) (4,1,-1), (0,1,-1).

Let $B = \{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

Four or more vectors in R^3 are always linearly dependent. Since B contains four vectors, so B is not a basis.

(c) vectors : (1,2,2), (-1,2,1), (0,8,0)

Let $B = \{(1,2,2), (-1,2,1), (0,8,0)\}$

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$$

$$|A| = 1(0-8) + 1(0-16) = -24 \neq 0$$

So, the three vectors of B are linearly independent. Hence B is a basis of $\mathbb{R}^3$.

(d) vectors : (1,2,2), (-1,2,1), (0,8,6).

Let $B = \{(1,2,2), (-1,2,1), (0,8,6)\}$

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 6 \end{bmatrix}$$

$$|A| = 1(12-8) + 1(12-16) = 4-4 = 0$$

So, the vectors of B are linearly dependent. Hence B is not a basis.

Date
Page

2.4 The Four Fundamental Subspaces

Row Space :- The row space of a matrix A of order $m \times n$ contains all the linear combinations of the rows of A or columns of A^T. It is denoted by $C(A^T)$. It is a subspace of $\mathbb{R}^n$.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A^T y = b$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = b$

$y_1 = 0,\ y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 = 1,\ y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1 = 0,\ y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$y_1 = 1,\ y_2 = 1 \Rightarrow b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\vdots$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$

$= \mathbb{R}^2$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$A^T y = b$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = b$

$\Rightarrow y_1 = 0 , y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 = 1, y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow y_1 = 0 , y_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\Rightarrow y_1 = 1 , y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\vdots$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \cdots \right\}$

i.e $y = 2x$ line

Ex:- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^T y = b \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$

Date
Page

$$\Rightarrow y_1\begin{bmatrix}0\\0\end{bmatrix} + y_2\begin{bmatrix}0\\0\end{bmatrix} = b$$

$$\Rightarrow b = \begin{bmatrix}0\\0\end{bmatrix}$$

$C(A^T) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ i.e. origin of $\mathbb{R}^2$

Left Nullspace :- The left nullspace of a matrix A of order $m\times n$ contains all the vectors y such that $A^Ty=0$. It is denoted by $N(A^T)$. It is a subspace of $\mathbb{R}^m$.

Ex :- $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

$A^Ty = 0$

$$\Rightarrow \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow y_1\begin{bmatrix}1\\2\end{bmatrix} + y_2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow y_1 = 0, \quad y_2 = 0$$

$N(A^T) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ i.e. origin of $\mathbb{R}^2$

Ex :- $A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

$A^Ty = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 0 \;,\; y_2 = 0$

$y_1 = 2 \;,\; y_2 = -1$

$y_1 = 4 \;,\; y_2 = -2$

$y_1 = 6 \;,\; y_3 = -3$

$\vdots$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdots\cdots \right\}$$

which is the line $y = -\frac{x}{2}$ is $\mathbb{R}^2$

Ex:- $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Any value of y_1 and y_2 can satisfy

$\Rightarrow N(A^T) = \mathbb{R}^2$

classmate
Date
Page

Points to remember:-

1. If A is a non singular matrix of order n, then $C(A^T) = \mathbb{R}^n$ and $N(A^T)$ = origin of $\mathbb{R}^n$

2- If A is a zero matrix of order n, then $C(A^T)$ = origin of $\mathbb{R}^n$ and $N(A^T) = \mathbb{R}^n$

3- If A is a non zero singular matrix of order 2, then its row space as well as left nullspace is a line passing through origin in $\mathbb{R}^2$.

4- If A is a non-zero singular matrix of order 3, then its row space as well as left nullspace is either a line passing through the origin or a plane passing through origin in $\mathbb{R}^3$

Fundamental theorem of Linear Algebra, Part-I

Let A be a matrix of order $m \times n$ with rank r

Then dim. $C(A) = r$
dim. $C(A^T) = r$
dim. $N(A) = n - r$
dim. $N(A^T) = m - r$

Ex :- Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2\times 2}$

$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - 3R_1$
echelon form.

Rank of $A = r = 1$

$\dim C(A) = r = 1$, $\dim N(A) = n - r = 2 - 1 = 1$

$\dim C(A^T) = r = 1$, $\dim N(A^T) = m - r = 2 - 1 = 1$

Column space :

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} = b$$

$$\Rightarrow x_1 = 0 , x_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$x_1 = 1, x_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x_1 = 0 , x_2 = 1 \Rightarrow b = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$x_1 = 1, x_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$\vdots$

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix} \cdots \right\}$$

which is the line $y = 3x$ in $\mathbb{R}^2$

Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

Date
Page

Row space :-

$A^T y = b$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$

$\Rightarrow y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = b$

$y_1 = 0, \ y_2 = 0 \Rightarrow b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 = 1, \ y_2 = 0 \Rightarrow b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y_1 = 0, \ y_2 = 1 \Rightarrow b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$y_1 = 1, \ y_2 = 1 \Rightarrow b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\vdots$

$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} \cdots \right\}$

which is the line $y = 2x$ in $\mathbb{R}^2$

Basis of $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Nullspace :-

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0, \quad x_2 = 0$
$x_1 = 2, \quad x_2 = -1$
$x_1 = 4, \quad x_2 = -2$
$x_1 = 6, \quad x_2 = -3$
$\vdots$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdots \right\}$$

which is the line $y = -\frac{x}{2}$ in $\mathbb{R}^2$

Basis of $N(A) = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

Left Nullspace :-

$A^T y = 0$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad y_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 0, \quad y_2 = 0$
$y_1 = 3, \quad y_2 = -1$
$y_1 = 6, \quad y_2 = -2$
$y_1 = 9, \quad y_2 = -3$

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \end{bmatrix} \cdots \right\}$$

which is the line $y = -\frac{x}{3}$ in $\mathbb{R}^2$

Basis of $N(A^T) = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$

classmate

Date

Page

Note :- 1. The nullspace is called the kernel of A and its dimension $n-r$ is the nullity

2. The no. of independent columns = the no. of independent rows.

Existence of one-sided Inverse:-

Let A be a matrix of order $m \times n$ with rank r

i) If $r = m$, then A is a full row rank matrix. Right inverse C of A will exist and

$$\boxed{C = A^T (AA^T)^{-1}}$$

$$AC = I$$

ii) If $r = n$, then A is a full column rank matrix. Left inverse B of A will exist and

$$\boxed{B = (A^T A)^{-1} A^T}$$

$$BA = I$$

Ex:- $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}_{2\times 3}$

The matrix A is already in echelon form with two pivots. So, rank of $A = r = 2$.

Here $m = 2$ and $n = 3$.

$m = r = 2$

$\Rightarrow$ A is a full row rank matrix.

$\Rightarrow$ Right inverse C of A will exist.

$$C = A^T (AA^T)^{-1}$$

$$AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$$

$$(AA^T)^{-1} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix}$$

$$C = A^T (AA^T)^{-1}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

The matrix A is already in echelon form with 2 pivots. So, rank of $A = r = 2$

Here $m = 3$ and $n = 2$

$r = n = 2 \Rightarrow$ A is a full column rank matrix

$\Rightarrow$ Left inverse B of A will exist.

$$B = (A^T A)^{-1} A^T$$

$$A^T A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix}$$

$$B = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Matrices of rank 1:-</u>

Every matrix of rank 1 has the simple form
$A = u v^T$ column times row.

Ex:- $$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}$$

Every row of A is the multiple of the 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ Rank of $A = 1$

$$\cancel{A = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = u v^T$$

where $u = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Problem Set 2.4 :-

Q.1. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The column space of A is :-

$$C(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdots \right\}$$

which is $z=0$ i.e. x-y plane.

The row space of A is :-

$$C(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdots \right\}$$

which is $x=0$ i.e y-z plane.

The nullspace of A is:-

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \cdots \right\}$$

which is x-axis of $\mathbb{R}^3$

The left nullspace of A is :-

$$N(A^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \dots \right\}$$

which is z - axis in R^3

Q.2 $$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{3\times 4}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$, echelon form

Rank of $A = r = 2$

Here $m = 3$ and $n = 4$

Dim $C(A) = r = 2$

Dim $C(A^T) = r = 2$

Dim $N(A) = n - r = 4 - 2 = \cancel{3}\, 2$

Dim $N(A^T) = m - r = 3 - 2 = 1$

Basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

Basis of $C(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$Ax = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_2 + x_4 = 0$$
$$x_2 + x_3 = 0$$

$$\Rightarrow \quad x_2 = -x_3$$
$$x_1 = -2x_2 - x_4 = 2x_3 - x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis of $N(A)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$A^T y = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \quad y_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = 1, \; y_2 = 0, \; y_3 = -1.$$

Basis of $N(A^T)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

classmate
Date
Page

Q.3 $A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}_{2\times4}$ $(m=2, n=4)$

$= \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1$, echelon form

Rank of $A = r = 1$

dim. $C(A) = r = 1$, dim $N(A) = n-r = 4-1 = 3$
dim. $C(A^T) = r = 1$, dim $N(A^T) = m-r = 2-1 = 1$

Basis of $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Basis of $C(A^T)$ is $\left\{ [0, 1, 4, 0] \right\}$

$Ax = 0$

$\Rightarrow \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Basis of $N(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$A^T y = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 4 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow y_1 \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow y_1 = 2, \quad y_2 = -1$

Basis of $N(A^T)$ is $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

Q.6

i) $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$

Every row of A is a multiple of the 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ Rank of $A = 1$

$A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix} = u v^T$ = column times row

where $u = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}$.

ii) $A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$

Second row of A is a multiple of the 1st row.

$\Rightarrow$ dim. $C(A^T) = 1$

$\Rightarrow$ rank of $A = 1$

$A = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = u v^T$ = column times row

where $u = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Q.9
i)
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

The matrix A is already in echlon form with two pivots. So, rank of $A = r = 2$

Here $m = 2$ and $n = 3$

$r = m = 2$

$\Rightarrow$ A is a full row rank matrix

$\Rightarrow$ Right inverse of C (A) will exist.

$$C = A^T (AA^T)^{-1}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A \cdot A^T)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$C = A^T (AA^T)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_2$, echelon form

Rank of $A = r = 2$

$r = n = 2$

$\Rightarrow$ A is a full column rank matrix

$\Rightarrow$ Left inverse B of A will exist

$$B = (A^T A)^{-1} A^T$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$$