CHAPTER - 3

ORTHOGONALITY

3.1 Orthogonal Vectors and Subspaces :-

Length of a Vector :-

* Let $x = (x_1, x_2)$
  Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2}$

  Length of square of $x = \|x\|^2 = x_1^2 + x_2^2$

* Let $x = (x_1, x_2, x_3)$
  Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$
  Length of square of $x = \|x\|^2 = x_1^2 + x_2^2 + x_3^2$

* Let $x = (x_1, x_2, x_3 \ldots x_n)$
  Length of $x = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2 \ldots x_n^2}$
  Length of square of $x = \|x\|^2 = x_1^2 + x_2^2 + x_3^2 \ldots x_n^2$

Inner product :-

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$
Inner product of x with y is denoted by $x^T y$ and is denoted by :-

$$x^T y = [x_1 \ x_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= x_1 y_1 + x_2 y_2$$

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$

Then $x^T y = [x_1, x_2, x_3]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$

Let $x = (x_1, x_2 \ldots x_n)$ and $y = (y_1, y_2 \ldots y_n)$

Then $x^T y = [x_1, x_2, x_3 \ldots x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$$= x_1 y_1 + x_2 y_2 + x_3 y_3 \ldots x_n y_n$$

$x^T x = [x_1, x_2, x_3 \ldots x_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$= x_1^2 + x_2^2 + x_3^2 \ldots x_n^2$$
$$= \|x\|^2$$

$\Rightarrow$ Inner product of a vector with itself is equal to the length square of the vector.

$x^T y > 0 \Rightarrow$ angle between x and y is less than $90°$
$x^T y < 0 \Rightarrow$ angle between x and y is greater than $90°$
$x^T y = 0 \Rightarrow$ angle between x and y is $90°$
$\Rightarrow x$ is orthogonal to y.
i.e. $x \perp y$.

Notes : 1. Zero is the only vector with length
2. Zero is the only vector orthogonal to itself.

Definition :- Two vectors x and y are said to be orthogonal to each other iff $x^T y = 0$.

Ex :- Let $x = (2, 2, -1)$ and $y = (-1, 2, 2)$

$$\|x\| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{4+4+1} = 3$$

$$\|y\| = \sqrt{(-1)^2+2^2+2^2} = \sqrt{1+4+4} = 3$$

Inner product of x with y is :-

$$x^T y = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -2+4-2 = 0$$

$\Rightarrow x \perp y$

$\Rightarrow x$ is orthogonal to y.

<u>Orthogonal vectors in $\mathbb{R}^2$</u> :-

Let $v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$

$$\|v_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\|v_2\| = \sqrt{(-\sin\theta)^2 + \cos^2\theta} = 1$$

$$v_1^T v_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

$\Rightarrow v_1 \perp v_2$

$\Rightarrow v_1$ is orthogonal to v_2

Here assigning different values to θ, we can generate so many orthogonal vectors in $\mathbb{R}^2$

<u>Orthogonal subspaces</u> :-

Two subspaces v and w of the same space $\mathbb{R}^n$ are orthogonal if every vector of v is orthogonal to every vector of w.

Ex:- 1. x-axis and y-axis are subspaces of $\mathbb{R}^2$ and every vector of x-axis is orthogonal to every vector of y-axis. So, x-axis $\perp$ y-axis in $\mathbb{R}^2$

2. $y = x$ line $\perp$ $y = -x$ line in $\mathbb{R}^2$

3. All the three axis in $\mathbb{R}^3$ are orthogonal to each other.

<u>Notes</u> :-

1. The subspace $\{0\}$ is orthogonal to all subspaces.

2. A line is orthogonal to another line or it can be orthogonal to a plane but a plane cannot be orthogonal to a plane.

<u>The fundamental theorem of orthogonality</u> :-

The row space is orthogonal to the nullspace in $\mathbb{R}^n$ and the column space is orthogonal to the left nullspace in $\mathbb{R}^m$ for a matrix of order $m \times n$.

In symbol, $C(A) \perp N(A^T)$ in $\mathbb{R}^m$
and $C(A^T) \perp N(A)$ in $\mathbb{R}^n$
for the matrix A of order $m \times n$

Ex: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$C(A)$ is xy-plane
$C(A^T)$ is yz-plane
$N(A)$ is x-axis of $\mathbb{R}^3$
$N(A^T)$ is z-axis of $\mathbb{R}^3$

xy plane $\perp$ z axis in $\mathbb{R}^3$

$\Rightarrow$ $C(A) \perp N(A^T)$ in $\mathbb{R}^3$

yz plane $\perp$ x-axis in $\mathbb{R}^3$

$\Rightarrow$ $C(A^T) \perp N(A)$ in $\mathbb{R}^3$

Orthogonal Complement:-

Two subspaces V and W of the space $\mathbb{R}^n$ are said to be orthogonal complement of each other if $V \perp W$ and $\dim V + \dim W = n$

orthogonal complement of V is denoted by $V^\perp$

Ex:- 1- x-axis is the orthogonal complement of y-axis in $\mathbb{R}^2$.

2- $y = x$ line is orthogonal complement of $y = -x$ line in $\mathbb{R}^2$

3- x-axis is the orthogonal complement of yz-plane in $\mathbb{R}^3$

Fundamental Theorem of Linear Algebra. Part-II

The nullspace is the orthogonal complement of row space in $\mathbb{R}^n$ and the left nullspace is the orthogonal complement of column space in $\mathbb{R}^m$.

In symbol,

$$N(A) = (C(A^T))^\perp \text{ in } \mathbb{R}^n$$
$$N(A^T) = (C(A))^\perp \text{ in } \mathbb{R}^m$$

Problem Set 3.1 :-

Q.1 $v_1 = (1, 2, -2, 1)$, $v_2 = (4, 0, 4, 0)$, $v_3 = (1, -1, -1, -1)$
$v_4 = (1, 1, 1, 1)$

v_1 v_3, v_2 v_3 are orthogonal pairs since $v_1^T v_3 = 0$ and $v_2^T v_3 = 0$

Q.2 $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$

i) $x = (-2, 1, 0)$ is a vector orthogonal to the row space of A.

ii) $y = (-1, -1, 1)$ is a vector orthogonal to the column space of A.

iii) $z = (1, 2, 1)$ is a vector orthogonal to the nullspace of A.

$N(A) \perp C(A^T) \qquad z \in C(A^T)$

so, $z \perp x$

Q.7 $x = (1, 4, 0, 2)$, $y = (2, -2, 1, 3)$

$\|x\| = \sqrt{1^2 + 4^2 + 0^2 + 2^2} = \sqrt{21}$

$\|y\| = \sqrt{2^2 + (-2)^2 + 1^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

$x^T y = 2 - 8 + 0 + 6 = 0$

$\Rightarrow x \perp y$.

Q.9 $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$

row space $= C(A^T)$

$(C(A^T))^\perp = N(A)$

Basis for the orthogonal complement of the row space of A is same as basis for nullspace N(A).

$Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow u = -2, \ v = -2, \ w = 1$

Basis of $N(A) = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

i.e. Basis of $(C(A^T))^\perp = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$

$x = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = x_r + x_n$

$= x_r + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow x_r = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$

Q.10 Plane (P) $= x + 2y - z = 0$

A vector perpendicular to P is $(1, 2, -1)$

$A = [1, 2, -1]$

$N(A) = P$

$x + 2y - z = 0$

$\Rightarrow z = x + 2y$

$x = 1, \ y = 1 \Rightarrow z = 3$

$x = 0, \ y = 1 \Rightarrow z = 2$

$B = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

$C(B^T) = P$

Q.11 Given : vectors $(1, 4, 4, 1)$, $(2, 9, 8, 2)$

Let $A^T = \begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix}$

Row space $C(A^T) \perp$ nullspace $N(A)$

$A^T x = 0$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 4 & 1 \\ 2 & 9 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 8 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ldots \right\}$$

The vectors of the nullspace $N(A)$ are perpendicular to the two given vectors.

Q.12 Show that $x-y$ is orthogonal to $(x+y)$ if and only if $\|x\| = \|y\|$.

Proof :- Let $(x-y)$ is orthogonal to $(x+y)$

$\Leftrightarrow (x-y) \perp (x+y)$

$\Leftrightarrow (x-y)^T (x+y) = 0$

$\Leftrightarrow (x^T - y^T)(x+y) = 0$

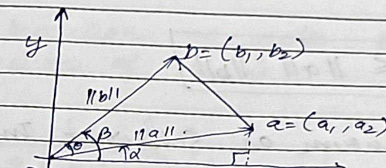
$\Leftrightarrow (x^T x + \cancel{x^T y} - y^T x - y^T y) = 0$

$\Leftrightarrow \|x\|^2 - \|y\|^2 = 0 \quad (\because x^T y = y^T x)$

$\Leftrightarrow \|x\| = \|y\|$

3.2 Cosines and Projections onto Lines :-

Inner product and cosines :-



Let the vectors a and b make angles α & β with x-axis respectively. The angle between a and b is θ. From the figure, it is clear that $\theta = \beta - \alpha$

$$\Rightarrow \cos\theta = \cos(\beta - \alpha)$$
$$= \cos\beta \cdot \cos\alpha + \sin\beta \sin\alpha$$
$$= \frac{a_1 b_1 + a_2 b_2}{\|a\| \, \|b\|}$$
$$= \frac{a^T b}{\|a\| \, \|b\|}$$

$$\Rightarrow \boxed{\cos\theta = \frac{a^T b}{\|a\| \, \|b\|}}$$

we know that :-

$$|\cos\theta| \le 1$$

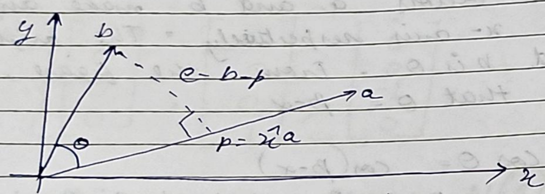
$$\Rightarrow \left| \frac{a^T b}{\|a\| \, \|b\|} \right| \le 1$$

$$\Rightarrow \frac{|a^T b|}{\|a\| \, \|b\|} \le 1$$

$$\Rightarrow \boxed{|a^T b| \le \|a\| \, \|b\|}$$

This is known as Schwarz Inequality.

Projection onto a line :-



From the figure, it is clear that:-

$$e \perp a$$
$$\Rightarrow (b-p) \perp a$$
$$\Rightarrow (b - \hat{x} a) \perp a$$
$$\Rightarrow a^T(b - \hat{x} a) = 0$$
$$\Rightarrow a^T b - \hat{x} a^T a = 0$$

$$\Rightarrow \hat{x} = \frac{a^T b}{a^T a}$$

The projection of the vector b onto the line in the direction of a is :-

$$p = \hat{x} a$$

$$\Rightarrow \boxed{p = \frac{a^T b}{a^T a} a}$$

The projection of vector b onto the line in the direction of a is :-

$$p = \hat{x} a$$
$$\Rightarrow \boxed{p = \frac{a^T b}{a^T a} a}$$

Note :- Equality holds in Schwarz inequality

$|a^T b| \le \|a\| \, \|b\|$ if and only if b is a multiple of a.

The angle $\theta = 0^\circ$ or 180° and $\cos\theta = 1$ or -1. In this case b is identical with its projection p ; and the distance between b and the line is zero.

Ex :- Find the projection of $b = (1, 2, 3)$ onto the line through $a = (1, 1, 1)$ and verify Schwarz inequality.

Sol^n Given : $a = (1, 1, 1)$, $b = (1, 2, 3)$

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1+2+3}{3} = \frac{6}{3} = 2$$

The projection is $p = \hat{x} a = (2, 2, 2)$

$\|a\| = \sqrt{3}$, $\|b\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$

Schwarz Inequality is :-

$|a^T b| \leq \|a\| \|b\|$

$\Rightarrow |6| \leq \sqrt{3} \cdot \sqrt{14}$

$\Rightarrow 6 \leq \sqrt{42}$

$\Rightarrow \sqrt{36} \leq \sqrt{42}$, which is true.

Projection matrix of Rank 1 :-

$p = \hat{x} a$

$= \dfrac{a^T b}{a^T a} a = a \dfrac{a^T b}{a^T a}$

$= \dfrac{a a^T}{a^T a} b = Pb$

where $P = \dfrac{a a^T}{a^T a}$ is the projection matrix.

Notes:-

1. P is a symmetric matrix
2. $P^2 = P$

Ex:- Find the projection matrix that projects any vector onto the line through $a = (1, 1, 1)$

Sol^n Given : $a = (1, 1, 1)$

$a^T a = \|a\|^2 = 3$

$$a a^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The required projection matrix is :-

$$P = \frac{a a^T}{a^T a} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Ex:- Find the projection matrix that projects any vector onto the line through $a = (\cos\theta, \sin\theta)$.

Sol^n Given : $a = (\cos\theta, \sin\theta)$

$a^T a = \|a\|^2 = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$$a a^T = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \sin\theta \cos\theta & \sin^2\theta \end{bmatrix}$$

The required projection matrix is

$$P = \frac{a a^T}{a^T a} = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \sin\theta \cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ sc & s^2 \end{bmatrix}$$

Problem Set 3.2 :-

Q.1

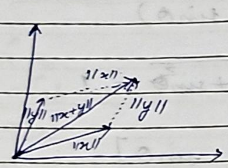
a) Let $a = (\sqrt{y}, \sqrt{x})$ and $b = (\sqrt{x}, \sqrt{y})$ where x and y are positive numbers.

Schwarz inequality is :-

$$|a^Tb| \le \|a\| \, \|b\|$$

$$\Rightarrow \sqrt{xy} + \sqrt{xy} \le \sqrt{y+x}\,\sqrt{x+y}$$
$$\Rightarrow 2\sqrt{xy} \le x+y$$
$$\Rightarrow \sqrt{xy} \le \frac{x+y}{2}$$
$$\Rightarrow GM \le AM.$$

b)



Triangle inequality is :-

$$\|x+y\| \le \|x\| + \|y\|$$

$$\Rightarrow \|x+y\|^2 \le (\|x\| + \|y\|)^2$$
$$\Rightarrow (x+y)^T(x+y) \le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$
$$\Rightarrow x^Tx + x^Ty + y^Tx + y^Ty \le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$
$$\Rightarrow \|x\|^2 + 2x^Ty + \|y\|^2 \le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$
$$\Rightarrow 2x^Ty \le 2\|x\|\,\|y\|$$
$$\Rightarrow x^Ty \le \|x\|\,\|y\|$$

$\Rightarrow |x^Ty| \le \|x\|\,\|y\|$ which is Schwarz inequality

Q.3 Let $a = (a_1, a_2, a_3 \dots a_n)$ and $b = (1, 1, 1 \dots 1) \in \mathbb{R}^n$

The Schwarz inequality is :-

$$|a^Tb| \le \|a\|\,\|b\|$$

$$\Rightarrow (a_1 + a_2 + \dots + a_n) \le (a_1^2 + a_2^2 \dots + a_n^2)^{1/2} (\underbrace{1+1+1 \dots +1}_{n \text{ times}})^{1/2}$$

$$\Rightarrow (a_1 + a_2 + \dots + a_n)^2 \le n(a_1^2 + a_2^2 + \dots + a_n^2)$$

Equality will hold when $a_1 = a_2 = \dots = a_n$.

Q.5 Projection matrix P_1 onto the line through

$a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $P_1 = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

The matrix P_2 that projects onto the line perpendicular to 'a' is

$$P_2 = I - P_1$$

$$P_2 = I - P_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

$$= \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

So, $P_1 + P_2 = I$ and $P_1 P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q.8 Let $a = (a_1, a_2, a_3 \ldots a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{a^Ta} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [a_1, a_2 \; a_3 \ldots a_n]$$

$$= \frac{1}{a^Ta} \begin{bmatrix} a_1^2 & a_1 a_2 & \ldots & a_1 a_n \\ a_{21} & a_2^2 & \ldots & a_2 a_n \\ \vdots & & & \\ a_n a_1 & a_n a_2 & \ldots & a_n^2 \end{bmatrix}$$

$$\text{Trace} = \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{a^Ta} = \frac{a^Ta}{a^Ta} = 1$$

Q.9 Line $= x + 2y = 0$

It passes through the point $(-2, 1)$
The matrix that projects onto the line through $(-2, 1)$ is $P = \frac{aa^T}{a^Ta}$

$$= \frac{1}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} [-2 \;\; 1]$$

$$= \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

Q.11 $a = (1, 1, 1)$

Let $p = \hat{x} a$ be closest to the point $b = (2, 4, 4)$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{a^Tb}{\|a\|^2}$$

$$= \frac{1}{3} [1 \; 1 \; 1] \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \frac{10}{3}$$

$\therefore p = \hat{x} a = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3} \right)$

The point closest to a on the line through b is :-

$$p = \frac{b^Ta}{b^Tb} b$$

$$= \frac{1}{36} [2 \; 4 \; 4] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} b$$

$$= \frac{10}{36} b = \frac{10}{36} [2, 4, 4] = \frac{5}{18} [2, 4, 4]$$

$$= \left[\frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right]$$

Q.17

a) $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = a^Tb \quad (\because a^Ta = 1)$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos\theta$$

Projection of b onto a is :-

$$p = \hat{x}a = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix} = (\cos\theta, 0)$$

$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{1}{2} a^Tb = \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \times 0 = 0$$

The projection of b onto a is :-

$$p = \hat{x} a = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0, 0)$$

Q.19

(a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{1}{3} \begin{bmatrix} 1, & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{5}{3}$$

Projection of the vector b onto the line through 'a' is $p = \hat{x}a$

$$= \frac{5}{3}(1, 1, 1) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$$

$$e = b - p = \left(1 - \frac{5}{3}, 2 - \frac{5}{3}, 2 - \frac{5}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$e^Ta = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$$

$\Rightarrow e \perp a$

(b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{1}{11} \begin{bmatrix} -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \frac{-11}{11} = -1$$

Projection of the vector b onto the line through a is:-

$$p = \hat{x}a = -1(-1, 3, -1) = (1, 3, 1)$$

$$\cancel{= \frac{5}{3}(1, 1, 1) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)}$$

$$e = b - p = (0, 0, 0)$$

$\cancel{e = b - p = \left(1 - \frac{5}{3}, 2 - \frac{5}{3},\right.}$ $e^Ta = 0 \Rightarrow e \perp a$

Q.21 $a_1 = (-1, 2, 2)$, $a_2 = (2, 2, -1)$

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{1}{9}\begin{bmatrix}-1\\2\\2\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}1 & -2 & -2\\-2 & 4 & 4\\-2 & 4 & 4\end{bmatrix}$$

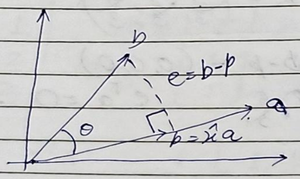
$$P_2 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{1}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix}\begin{bmatrix}2 & 2 & -1\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}4 & 4 & -2\\4 & 4 & -2\\-2 & -2 & 1\end{bmatrix}$$

$$P_1 P_2 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$ zero matrix since $a_1 \perp a_2$

3.3 Projections and Least Squares :-

Least square problems with single variable :



The least square solution to a problem $ax = b$ in one unknown is :-

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

Ex:- $2x = b_1$
$3x = b_2$ $\Rightarrow ax = b$,
$4x = b_3$ where $a = (2, 3, 4)$ and
$b = (b_1, b_2, b_3)$

This is solvable when b_1, b_2, b_3 are in the ratio $2:3:4$. The solution x will exist only if b is on the same line as column $a = (2, 3, 4)$

For $b = (4, 6, 8)$, $x = 2$, which is an exact solution.

For $b = (6, 9, 12)$, $x = 3$, which is an exact soln also.

But for $b = (4, 5, 8)$, there is no exact solution of the system $ax = b$ as the components of b are not obeying the ratio $2:3:4$.

In this case, b is not on the line passing through a. So, we have to find the least square solution. The least square solution is :-

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{2\times4 + 3\times5 + 4\times8}{2^2 + 3^2 + 4^2} = \frac{55}{29}$$

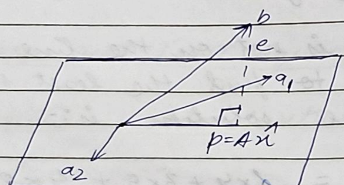
Least Square Problems with several variables:-

Given : $Ax=b$

The problem is to choose $\hat{x}$ so as to minimize the error and again this minimization will be done in the least square sense. The error is $E = \|Ax-b\|$ and this is exactly the distance from b to the point Ax in the column space. Searching for the least-squares solution $\hat{x}$, which minimizes E, is the same as locating the point $p = A\hat{x}$ that is closer to b than any other point in the column space. The error vector $e = b-p = b-A\hat{x}$ must be perpendicular to the column space.

To find: 1. To find the least square solⁿ $\hat{x}$

2. The projection $p = A\hat{x}$ onto $C(A)$



Given: $Ax=b$ (inconsistent i.e. $b \notin C(A)$)

All vectors perpendicular to column space lie in the left nullspace, So, the error vector $e = b-p = b - A\hat{x}$ must lie in the nullspace of A^T.

$$A^T(b - A\hat{x}) = 0$$

$$\Rightarrow A^Tb - A^TA\hat{x} = 0$$

$$\Rightarrow A^TA\hat{x} = A^Tb$$

$$\Rightarrow \boxed{\hat{x} = (A^TA)^{-1}A^Tb}$$ is the least square solution.

This least-square solution is also known as the best estimate.

The projection of b onto the column space is the nearest point $A\hat{x}$.

$$p = A\hat{x}$$

$$\Rightarrow \boxed{p = A(A^TA)^{-1}A^Tb}$$

Ex:- Solve $Ax=b$ by least squares and find the projection of b onto the column space of A, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Solⁿ $Ax = b$,

where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

The given system is inconsistent as $b \notin C(A)$.

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 13 & -5 \\ -5 & 2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}$$

The least square solution is

$$\hat{x} = (A^TA)^{-1} A^Tb$$

$$= \begin{bmatrix} 13 & -5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 23 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The projection of b onto the column space of A is $p = A\hat{x}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

$p \in C(A)$

Projection Matrices :-

$p = A\hat{x}$

$\Rightarrow p = A(A^TA)^{-1}A^Tb$

$\Rightarrow p = Pb$,

where $P = A(A^TA)A^T$ is the projection matrix that projects any vector b onto the column space of A.

Q. 12 $a_1 = (1, 0, 2)$, $a_2 = (1, 1, -1)$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Projection matrix $P = A(A^TA)^{-1}A^T$

$$A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad (A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & -1/3 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 5/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & -1/3 \\ 1/6 & -1/3 & 5/6 \end{bmatrix}$$

Least - Squares fitting of Data :-

Suppose we do a series of experiments and expect the output b to be a linear function of the input t. We look for a straight line $b = C + Dt$.

For example:-

At different times we measure the distance to a satellite on its way to Mars. In this case t is the time and b is the distance. unless the motor was left on or gravity is strong, the satellite should move with nearly constant velocity v: $b = b_0 + vt$.

Eg:- Find the best straight line fit (least squares) to the measurements.

$b = 1$ at $t = -1$, $b = 1$ at $t = 1$, $b = 3$ at $t = 2$

Sol^n Let $b = C + Dt$.

$b = 1, \ t = -1 \Rightarrow 1 = C - D$

$b = 1, \ t = 1 \Rightarrow 1 = C + D$

$b = 3, \ t = 2 \Rightarrow 3 = C + 2D$

$Ax = b$

$\Rightarrow C - D = 1$

$C + D = 1$

$C + 2D = 3$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\boxed{\hat{x} = (A^TA)^{-1} A^Tb}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

$$(A^Tb) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1} \cdot A^Tb$$

$$= \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3/7 & -1/7 \\ -1/7 & 3/14 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9/7 \\ 4/7 \end{bmatrix} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$$

The best solution is $\hat{C} = \frac{9}{7}$ and $\hat{D} = \frac{4}{7}$ and the best line is $\frac{9}{7} + \frac{4}{7}t$

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Problem Set 3.3 :-

Q.1 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\hat{x} = (A^TA)^{-1}A^Tb$

$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$(A^TA)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$, $A^Tb = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\hat{x} = (A^TA)^{-1}A^Tb$

$= \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

$\therefore p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

$e = b - p = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$ is perpendicular to both columns.

Q.2 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x = \begin{bmatrix} u \\ v \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$Ax = b \Rightarrow u = 1$
$v = 3$
$u + v = 4$

$E^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$

$\frac{\partial E^2}{\partial u} = 2(u-1) + 2(u+v-4) = 0$

$\frac{\partial E^2}{\partial v} = 2(v-3) + 2(u+v-4) = 0$

$\Rightarrow 2u + v = 5$
$u + 2v = 7$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ - - - - ①

same

$A^Tb = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A^TA\hat{x} = A^Tb = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ - - - ②

$(A^TA)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$

$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$e = b - p = 0$
$\Rightarrow b = p$

Q.4 $Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$$

$\Rightarrow$ $C - D = 4$ $\qquad$ $b=4$ at $t=-1$

$C = 5$ $\qquad \Rightarrow$ $b=5$ at $t=0$

$C+D = 9$ $\qquad$ $b=9$ at $t=1$

$$\hat{x} = (A^TA)^{-1} A^Tb$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 18 \\ 5 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1} A^Tb = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 18 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5/2 \end{bmatrix}$$

which is the best estimate.

Best line is $6 + \frac{5}{2} t$

$$p = A\hat{x}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 6 \\ 17/2 \end{bmatrix}$$

Q.6. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1} A^Tb$$

$$= \frac{1}{22} \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -11 \\ 27 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 9 \\ 37 \end{bmatrix}$$

$$p = A\hat{x}$$

$$= \frac{1}{22} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 37 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 46 \\ -28 \\ 130 \end{bmatrix} = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

$$b = p + q$$

$$\Rightarrow q = b - p = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

$$q \perp C(A)$$

$$\Rightarrow q \in N(A^T)$$