CH-16 VECTOR CALCULUS

Ex:- 16.1

(Q24) Find the gradient vector field of f.

$$f(x,y,z) = x\cos(y/z)$$

Ans:- $f_x(x,y,z) = \cos(y/z)$

$f_y(x,y,z) = -\frac{x}{z}\sin(y/z)$

$f_z(x,y,z) = \frac{xy}{z}\sin(y/z)$

So, gradient vector is,

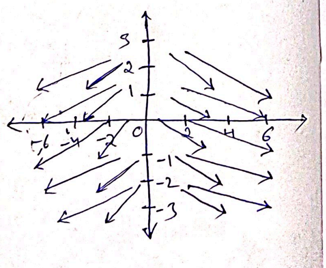
$$\left\langle \left[\cos(y/z)\right], \left[-\frac{x}{z}\sin(y/z)\right], \left[\frac{xy}{z}\sin(y/z)\right] \right\rangle$$

(Q25) Find the gradient vector field ∇f of f and sketch it.

$$f(x,y) = x^2 - y$$

Ans:- $f_x(x,y) = 2x$

$f_y(x,y) = -1$



Thus, gradient vector field is,

$$\nabla f(x,y) = 2x\,i - j$$

Ex:- 16.2

(4) Evaluate the line integral, where C is the given curve

$\int_C x \sin y \, ds$, C is the line segment from (0,3) to (4,6)

Ans:- Equn of a line passing through points (0,3) to (4,6)

$$\frac{y-3}{x-0} = \frac{6-3}{4-0}$$

$$\Rightarrow \frac{y-3}{x} = \frac{3}{4}$$

$$\Rightarrow x = \frac{4}{3}(y-3)$$

Let $y = t$

So, $x = \frac{4}{3}(t-3)$; $y = t$

Here t increases from 3 to 6

So, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \sqrt{\left(\frac{4}{3}\right)^2 + (1)^2}\, dt$$

$$\Rightarrow ds = \frac{5}{3} dt$$

So, $\int_C x \sin y \, ds = \int_3^6 \frac{4}{3}(t-3) \sin t \left(\frac{5}{3} dt\right)$

$= \frac{20}{9}\left[\int (t-3)\sin t\, dt\right]_3^6$

$= \frac{20}{9}\left[(t-3)\int \sin t\, dt - \int\left\{\frac{d(t-3)}{dt}\int \sin t\, dt\right\}dt\right]_3^6$

$= \frac{20}{9}\left[-(t-3)\cos t + \int \cos t\, dt\right]_3^6$

$= \frac{20}{9}\left[-(t-3)\cos t + \sin t\right]_3^6$

$= \frac{20}{9}\left[-(6-3)\cos 6 + \sin 6\right] - \frac{20}{9}\left[-(3-3)\cos 3 + \sin 3\right]$

$= \frac{20}{9}\left[\sin 6 - 3\cos 6 - \sin 3\right]$

(13) $\int_C xye^{yz}\,dy$; $C: x=t,\ y=t^2,\ z=t^3,\ 0 \le t \le 1$

Ans:- $dy = 2t\,dt$

So, $\int_0^1 (t)(t)^2 (e^{t^5})(2t)\,dt$

$= 2\int_0^1 t^4 e^{t^5}\,dt$

$= \frac{2}{5}\int_0^1 (e^u)\,du$ (As, $u = t^5$ $\Rightarrow du = 5t^4\,dt$)

$= \frac{2}{5}(e^1 - e^0)$

$= \frac{2}{5}(e-1)$

(21) Evaluate the line integral $\int_C F\cdot dr$, where C is given by the vector function $r(t)$.

$F(x,y,z) = \sin x\,i + \cos y\,j + xz\,k$

$r(t) = t^3 i - t^2 j + t k,\ 0 \le t \le 1$

Ans:- $\frac{dr}{dt} = 3t^2 i - 2t j + k$

$\Rightarrow dr = (3t^2 i - 2t j + k)\,dt$

$F(x,y,z) = \sin x\,i + \cos y\,j + xz\,k$

$\Rightarrow F(r(t)) = \sin(t^3)i + \cos(-t^2)j + t^4 k$

$\Rightarrow F(r(t)) = \sin(t^3)i + \cos(t^2)j + t^4 k$

So, $\int_C F\cdot dr = \left[\sin(t^3)i + \cos(t^2)j + t^4 k\right]\cdot\left[3t^2 i - 2t j + k\right]dt$

$= \int_0^1 3t^2\sin(t^3) - 2t\cos(t^2) + t^4\,dt$

$= \left[-\cos(t^3) - \sin(t^2) + \frac{t^5}{5}\right]_0^1$

$= \left(-\cos(1) - \sin(1) + \frac{1}{5}\right) - (-1 - 0 + 0)$

$= \frac{6}{5} - \cos(1) - \sin(1)$

Ex:- 16-3

(9) Determine whether or not F is a conservative vector field. If it is, find a function f such that $F = \nabla f$

$$F(x,y) = (\ln y + 2xy^3)i + (3x^2y^2 + x/y)j$$

Ans: Let $P = (\ln y + 2xy^3)$ and $Q = 3x^2y^2 + \frac{x}{y}$

$$\frac{dP}{dy} = \frac{1}{y} + 6xy^2$$

$$\frac{dQ}{dx} = \frac{1}{y} + 6xy^2$$

Here f is conservative

Here $\frac{dP}{dy} = \frac{dQ}{dx}$

So, $f_x(x,y) = \ln y + 2xy^3$

$f_y(x,y) = 3x^2y^2 + \frac{x}{y}$

Integrating $f_x(x,y)$ w.r.t x.

$$f(x,y) = x\ln y + x^2y^3 + g(y)$$

So, $f_y(x,y) = \frac{x}{y} + 3x^2y^2 + g'(y)$

Here $g(y) = C$; some constant

So, F is $f(x,y) = x\ln y + x^2y^3 + C$

(20) Show that the line integral is independent of path and evaluate the integral.

$$\int_C (1 - ye^{-x})dx + e^{-x}dy$$

C is any path from (0,1) to (1,2)

Ans:- Here $f_x = (1 - ye^{-x})$

$F_x = x - ye^{-x} + g(y)$

$f_y = e^{-x}$

$F_y = e^{-x} + g(x)$

let $g(x) = g(y) = K$ be a constant

$F_y = e^{-x} + K$

So, $F(1,2) - F(0,1)$

$\Rightarrow \frac{1}{e} + K - (-1 + K)$

$\Rightarrow \frac{1}{e} + 1 = \frac{1+e}{e}$

Ex: 16.4

(2) Evaluate the line integral by two methods: (a) directly and (b) using Green's theorem.

$$\oint_C xy\,dx + x^2\,dy$$

C is the rectangle with vertices (0,0), (3,0) (3,1) and (0,1)

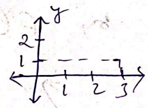
Ans:

Let $P = xy$

$\frac{dP}{dy} = x$

and $Q = x^2$

$\frac{dQ}{dx} = 2x$

Here $0 \le x \le 3$
$0 \le y \le 1$ (As, )

So, $\iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$ (Green's theorem)

$= \int_0^3 \int_0^1 (2x - x)\,dy\,dx$

$= \int_0^3 2xy - xy \Big|_0^1 dx$

$= \int_0^3 x\,dx = \frac{9}{2}$

(10) Use Green's theorem to evaluate:

$$\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy,$$

C is the boundary of the region between the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Ans: Let $P = 1-y^3$

So, $\frac{dP}{dy} = -3y^2\,dy$

and $Q = x^3 + e^{y^2}$

$\frac{dQ}{dx} = 3x^2\,dx$

So, $\int_0^{2\pi}\int_2^3 (3x^2+3y^2)\,dy\,dx$

$\cancel{= \int_0^{2\pi} 3x^2y + y^3 \Big|_2^3 dx}$

$= \int_0^{2\pi}\int_2^3 3r^3\,dr\,d\theta$ (As, $x = r\cos\theta$, $y = r\sin\theta$)

$= \int_0^{2\pi} \frac{3}{4} r^4 \Big|_2^3 d\theta$

$= \int_0^{2\pi} \frac{195}{4}\,d\theta$

$= \frac{195\pi}{2}$

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Ex-16.5

(8) $F(x,y,z) = \left\langle \frac{x}{y}, \frac{y}{z}, \frac{z}{x} \right\rangle$

Find (a) the curl and (b) the divergence of the vector field

Ans: (a) curl $F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ \frac{x}{y} & \frac{y}{z} & \frac{z}{x} \end{vmatrix}$

$= \left[\frac{d}{dy}\left(\frac{z}{x}\right) - \frac{d}{dz}(y/z)\right]\hat{i} - \left[\frac{d}{dx}\left(\frac{z}{x}\right) - \frac{d}{dz}\left(\frac{x}{y}\right)\right]\hat{j}$

$+ \left[\frac{d}{dx}\left(\frac{y}{z}\right) - \frac{d}{dy}\left(\frac{x}{y}\right)\right]\hat{k}$

$= \left(0 + \frac{y}{z^2}\right)\hat{i} - \left(-\frac{z}{x^2} - 0\right)\hat{j} + \left(0 + \frac{x}{y^2}\right)\hat{k}$

$= \left\langle \frac{y}{z^2}, \frac{z}{x^2}, \frac{x}{y^2} \right\rangle$

(b) Divergence $= \nabla \cdot F$

$= \frac{d}{dx}\left(\frac{x}{y}\right) + \frac{d}{dy}\left(\frac{y}{z}\right) + \frac{d}{dz}\left(\frac{z}{x}\right)$

$= \frac{1}{y} + \frac{1}{z} + \frac{1}{x}$

Ex-16.6

(25) Find a parametric representation for the surface the part of the cylinder $y^2+z^2=16$ that lies b/w the planes $x=0$ and $x=5$

Ans: Let $y = 4\cos u$ and $z = 4\sin u$ which fits for eqn $y^2+z^2=16$

As, $(4\cos u)^2 + (4\sin u)^2$

$= 16(\cos^2 u + \sin^2 u) = 16$

So, the parametric representation lies b/w the planes $x=0$ and $x=5$ is

$(v, 4\cos u, 4\sin u)$

where $v \in [0,5]$ and $u \in [0, 2\pi)$

(40) Find the area of the surface the part of the plane with vector equation.

$r(u,v) = \langle u+v, 2-3u, 1+u-v \rangle$ that is given by $0 \le u \le 2$, $-1 \le v \le 1$.

Ans:

$r(u,v) = \langle u+v, 2-3u, 1+u-v \rangle$

$\frac{dr}{du} = \langle 1, -3, 1 \rangle$ and $\frac{dr}{dv} = \langle 1, 0, -1 \rangle$

So, $\left|\frac{dr}{du} \times \frac{dr}{dv}\right| = \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix}$

$= \langle 3, 2, 3 \rangle$

So, $\left\|\frac{dr}{du} \times \frac{dr}{dv}\right\| = \sqrt{3^2+2^2+3^2} = \sqrt{22}$

So, $A(S) = \iint_T \left\|\frac{dr}{du} \times \frac{dr}{dv}\right\| du\,dv$

$= \int_{-1}^{1}\int_0^2 \sqrt{22}\,du\,dv$

$= \int_{-1}^{1} \left[\sqrt{22}\,u\right]_0^2 dv$

$= \int_{-1}^{1} 2\sqrt{22}\,dv$

$= \left[2\sqrt{22}\,v\right]_{-1}^{1} = 4\sqrt{22}$

Ex: 16.7

(8) Evaluate the surface integral

$\iint_S (x^2+y^2)\,dS$

S is the surface with vector equation.

$r(u,v) = \langle 2uv, u^2-v^2, u^2+v^2 \rangle$, $u^2+v^2 \le 1$

Ans:

$r_u = 2v\hat{i} + 2u\hat{j} + 2u\hat{k}$

$r_v = 2u\hat{i} - 2v\hat{j} + 2v\hat{k}$

$r_u \times r_v = 8uv\hat{i} + 4(u^2-v^2)\hat{j} - 4(v^2+u^2)\hat{k}$

$|r_u \times r_v| = \sqrt{32(u^2+v^2)}$

$\int_0^{2\pi}\int_0^1 \sqrt{32}(u^2+v^2)^{\frac{3}{2}}\,dv\,du$

$= \sqrt{32}\int_0^{2\pi}\int_0^1 (r^2)^{\frac{3}{2}}\, r\,dr\,d\theta$

$= \sqrt{32}\int_0^{2\pi} \frac{1}{5}r^5\Big|_0^1\, d\theta$

$= \sqrt{32}\left(\frac{2\pi}{8}\right) = \sqrt{2}\,\pi$

Ex: 16.8

(3) Use Stokes' Theorem to evaluate. $\iint_S \text{curl } F \cdot dS$

$F(x,y,z) = x^2z^2 i + y^2z^2 j + xyz k$

S is the part of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2 = 4$ oriented upward.

Ans:

$z = x^2+y^2$

So, $x^2+y^2 = 4 = z$

So, $r(t) = 2\cos t\, i + 2\sin t\, j + 4k$

$\frac{d r(t))}{dr} = (-2\sin t\, i + 2\cos t\, j + 0k)dt$

So, $F(x,y,z) = x^2z^2 \hat{i} + y^2z^2\hat{j} + xyz\hat{k}$

So, $F(r(t)) = (4\cos^2 t)(16) i + (4\sin^2 t)(16) j + (16 \sin t \cos t) k$

So, $\iint_S \text{curl} \cdot F \cdot dS = \int_C F \cdot dr$

$= \int_0^{2\pi} (64\cos^2 t\, i + 64 \sin^2 t\, \hat{j} + 16 \sin t \cos t\, k) \cdot (-2\sin t\, i + 2\cos t\, j + 0k)dt$

$= \cancel{\int_0^{2\pi} -128 \sin t \cos^2 t}$

$= \int_0^{2\pi} -128 \sin t \cos^2 t + 128 \cos t \sin^2 t\, dt$

$= \frac{128}{3} \cdot \int_0^{2\pi} 3\cos t \sin^2 t - 3 \sin t \cos^2 t\, dt$

$$= \frac{128}{3}\left[\cos^3 t + \sin^3 t\right]_0^{2\pi}$$

(As, $\frac{d}{dt}[\sin^3 t] = 3\sin^2 t \cos t$

and $\frac{d}{dt}[\cos^3 t] = -3\cos^2 t \sin t$)

$$= 0$$

Ex:- 16.9

(5) Use the Divergence Theorem to calculate the surface integral $\iint_S F\cdot dS$; that is, calculate the flux of F across S.

$$F(x,y,z) = xye^z i + xy^2z^3 j - ye^z k.$$

planes $x=3, y=2$, and $z=1$

Ans:- divergence $\vec{F} = \frac{d}{dx}(xye^z) + \frac{d}{dy}(xy^2z^3) + \frac{d}{dz}(-ye^z)$

$$= ye^z + 2xyz^3 - ye^z = 2xyz^3$$

So, $\int_0^3 \int_0^2 \int_0^1 2xyz^3 \, dz\, dy\, dx$ (where, $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$)

$$= \int_0^3 \int_0^2 \left(\frac{xy}{2}\right) dy\, dx$$

$$= \int_0^3 x \, dx = \frac{9}{2}$$

(7) $F(x,y,z) = 3xy^2 i + xe^z j + z^3 k.$

S is the surface of the solid bounded by the cylinder $y^2+z^2=1$ and the planes $x=-1$ and $x=2$

Ans → divergence $\vec{F} = \frac{d}{dx}(3xy^2) + \frac{d}{dy}(xe^z) + \frac{d}{dz}(z^3)$

$$= 3y^2 + 3z^2 = 3(y^2+z^2)$$

So, $3\int_{-1}^{2}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (y^2+z^2)\, dz\, dy\, dx$

(where $-1 \le x \le 2$
$-1 \le y \le 1$
$-\sqrt{1-y^2} \le z \le \sqrt{1-y^2}$)

$$= 3\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2\pi} (r^2\cos^2\theta + r^2\sin^2\theta)\, r\, d\theta\, dr\, dx$$

$$= 3\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2\pi} r^3\, d\theta\, dr\, dx$$

$$= 6\pi\int_{-1}^{2}\int_{0}^{1} r^3\, dr\, dx$$

$$= \frac{3\pi}{2}\int_{-1}^{2} dx$$

$$= \frac{9\pi}{2}$$