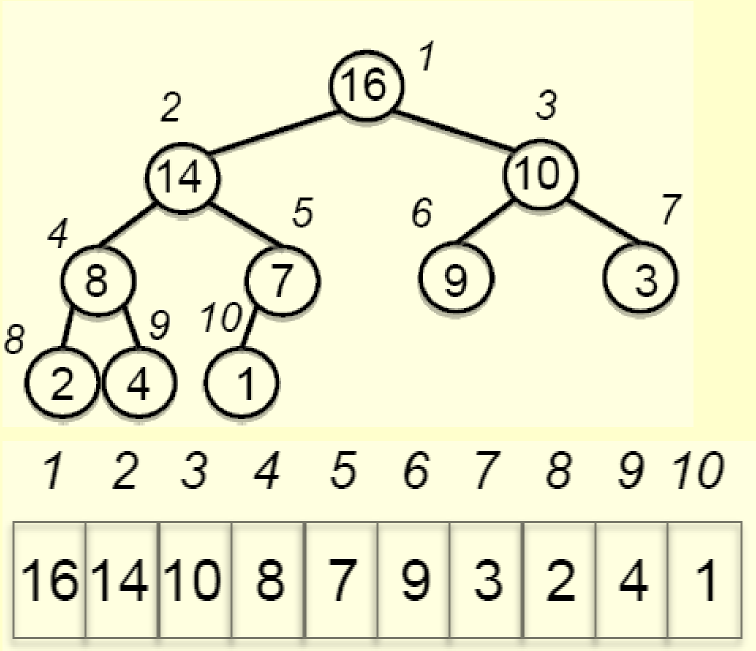
Heapsort

Heapsort

- Combines the better attributes of merge sort and insertion sort.
  - » Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - » Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - » Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - » Priority Queues

Data Structure Binary Heap

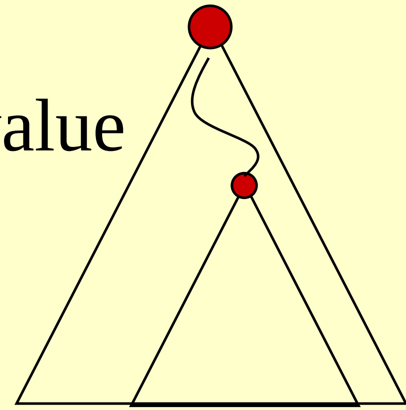
- Array viewed as a nearly complete binary tree.
  - » Physically – linear array.
  - » Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - » Root – $A[1]$
  - » Left[i] – $A[2i]$
  - » Right[i] – $A[2i+1]$
  - » Parent[i] – $A[\lfloor i/2 \rfloor]$



- length[A] – number of elements in array A.
- heap-size[A] – number of elements in heap stored in A.
  - » heap-size[A] $\leq$ length[A]

Heap Property (Max and Min)

- Max-Heap
  - » For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



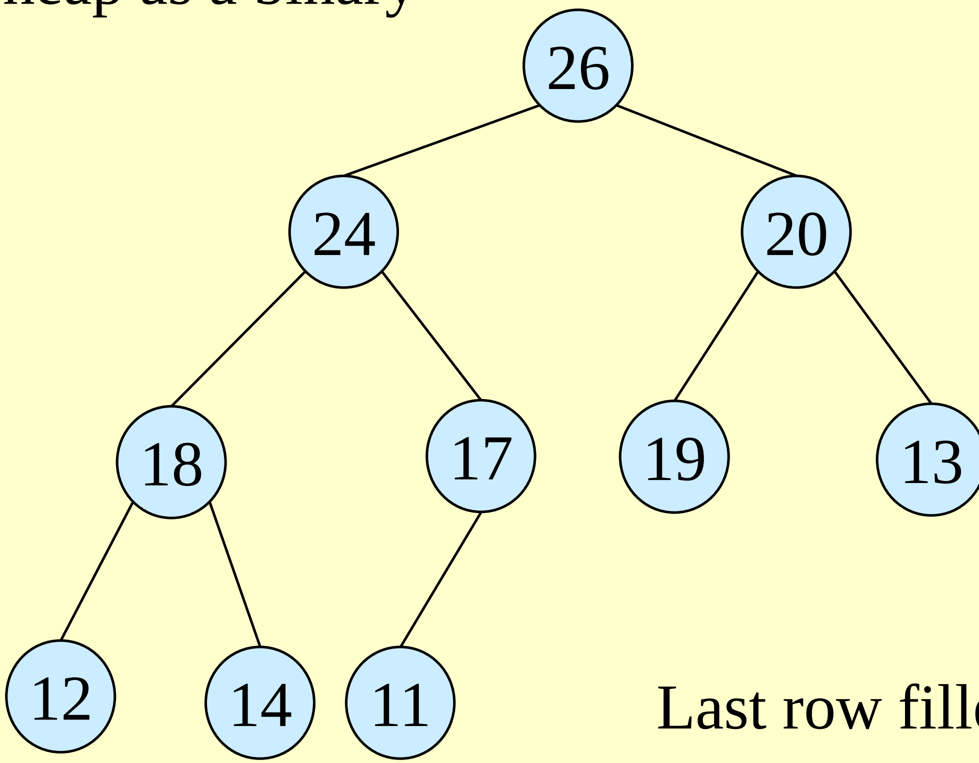
- Min-Heap
  - » For every node excluding the root, value is at least that of its parent: $A[parent[i]] \leq A[i]$
- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree root

Heaps – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

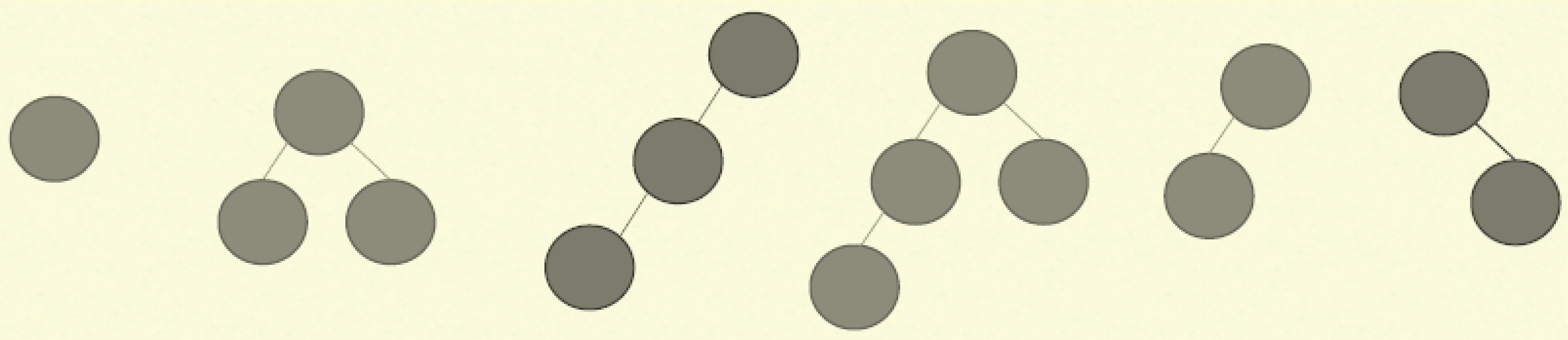
Max-heap as an array.

Max-heap as a binary tree.

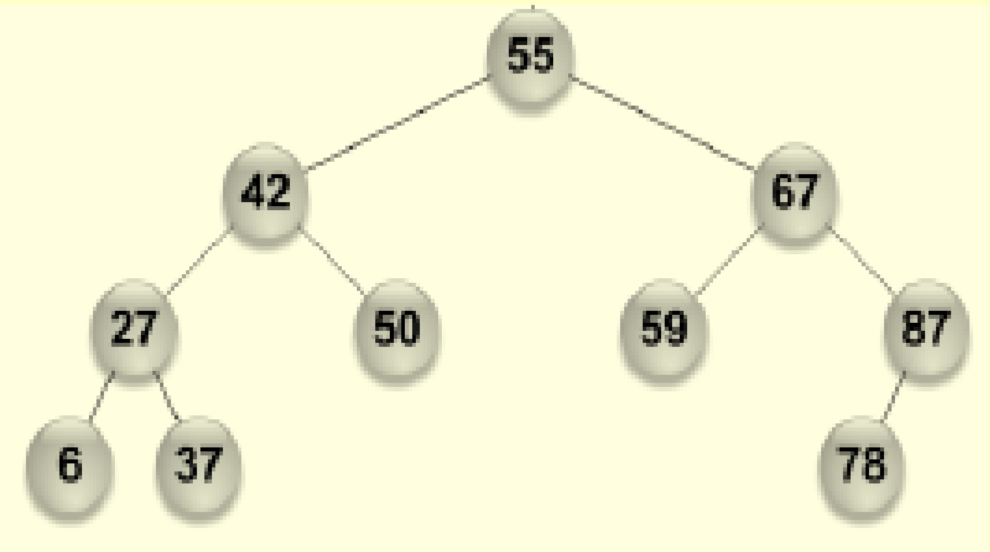


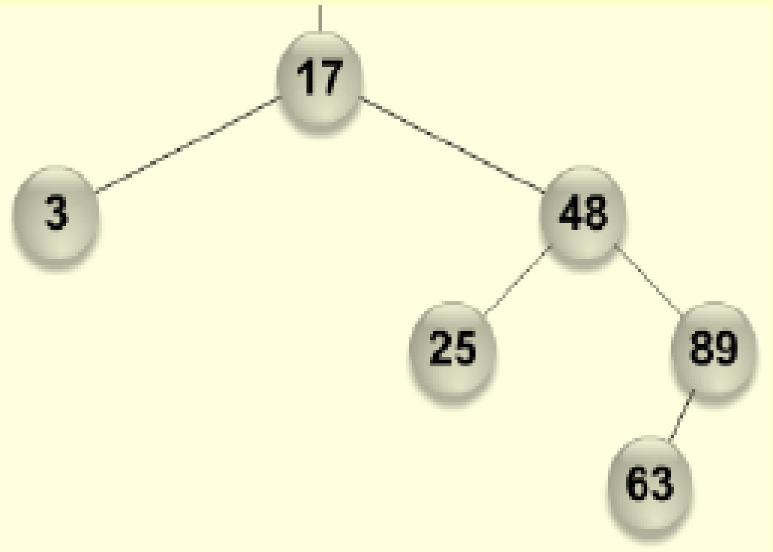
Last row filled from left to right.

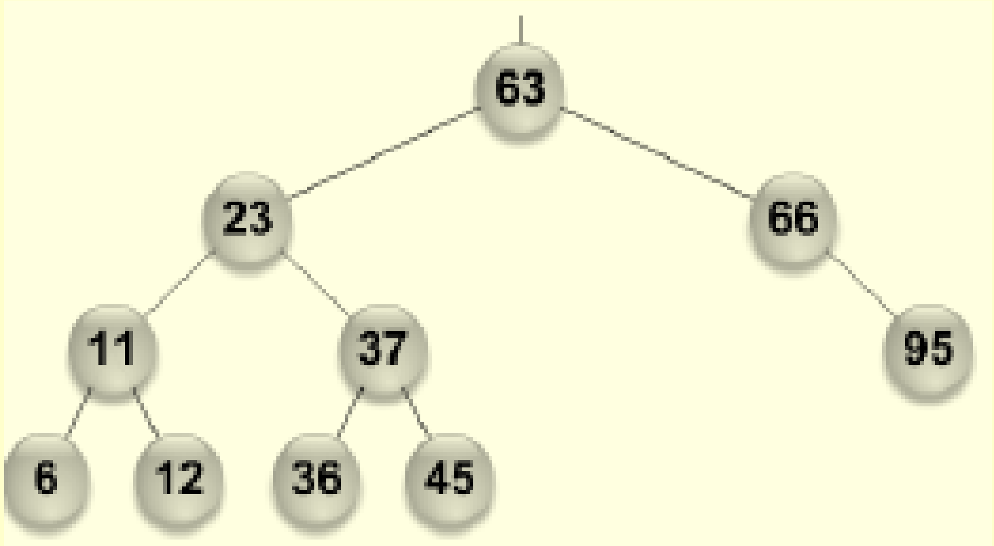
Heap or Not?



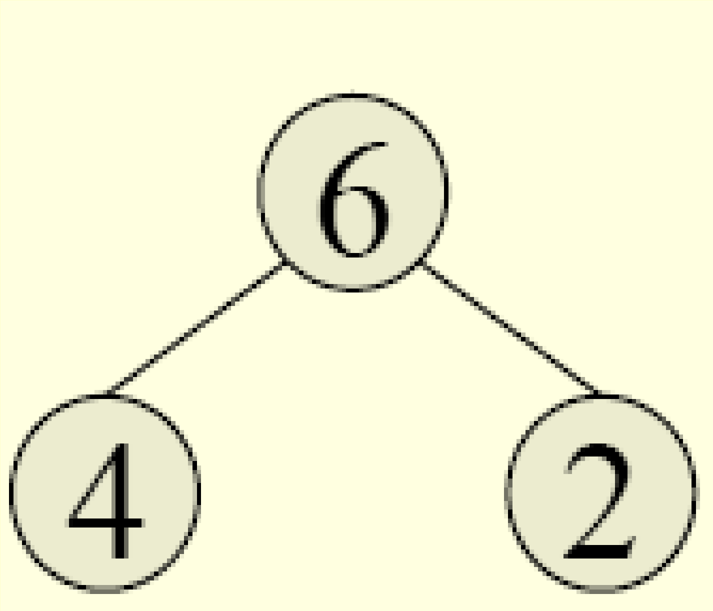
Heap or Not?

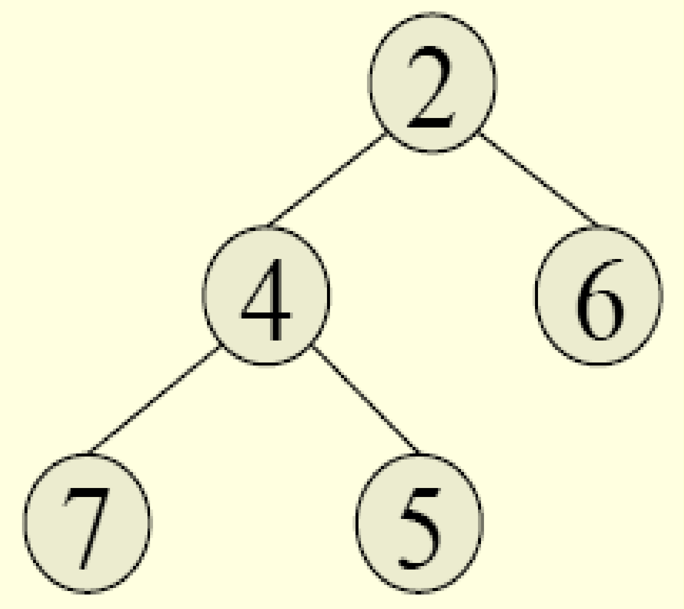


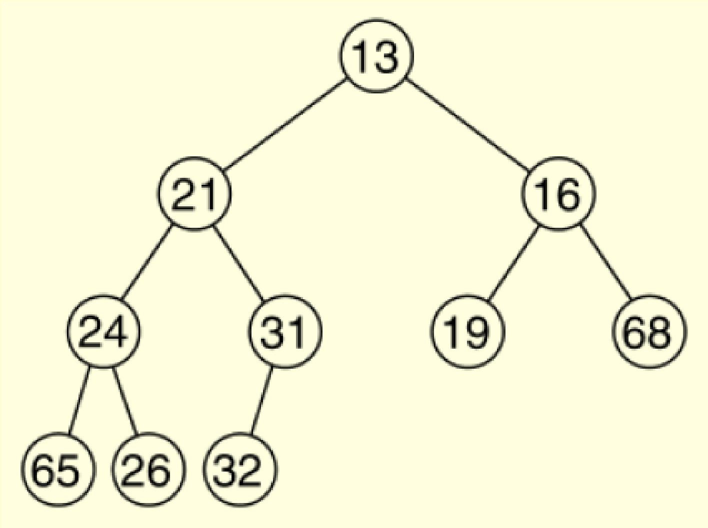


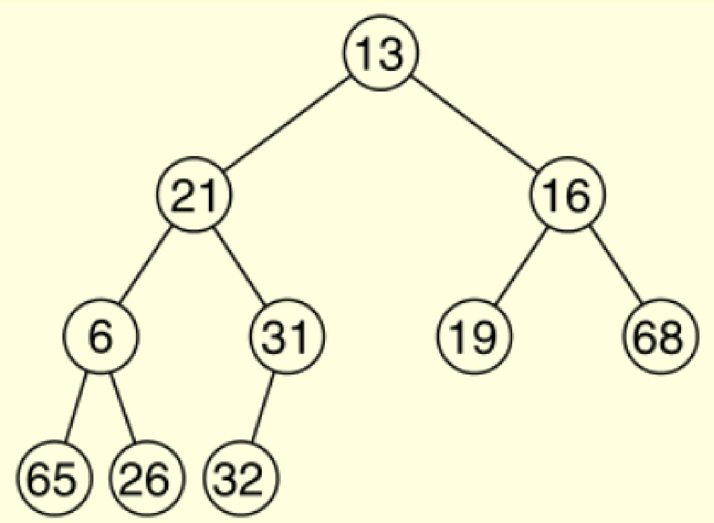


Heap or Not?



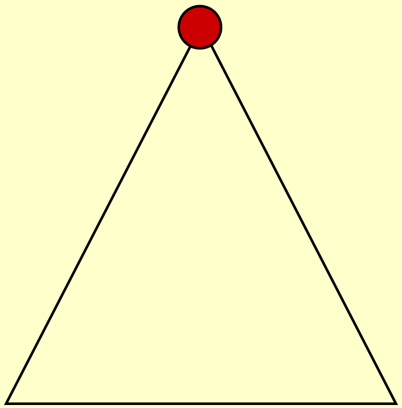






Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- Height of a heap: $\lfloor \lg n \rfloor$
  - » Basic operations on a heap run in $O(\lg n)$ time

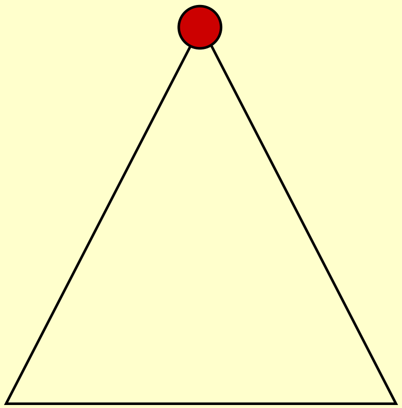


Heaps in Sorting

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  - » Convert the given array of size n to a max-heap (*BuildMaxHeap*)
  - » Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n - 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n - 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

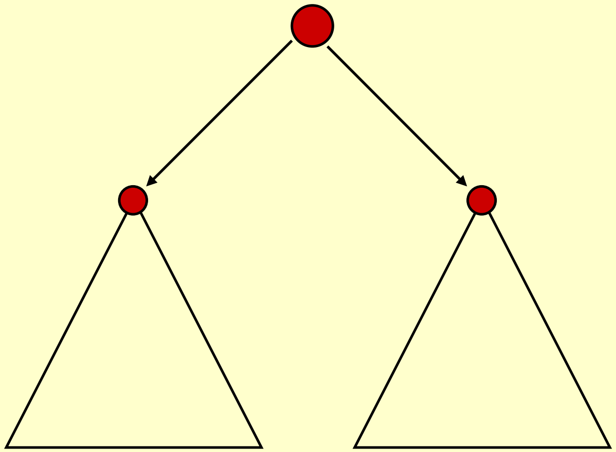
Heap Characteristics

- *Height* $= \lfloor \lg n \rfloor$
- No. of *leaves* $= \lceil n/2 \rceil$
- No. of nodes of height h $\leq \lceil n/2^{h+1} \rceil$



Maintaining the heap property

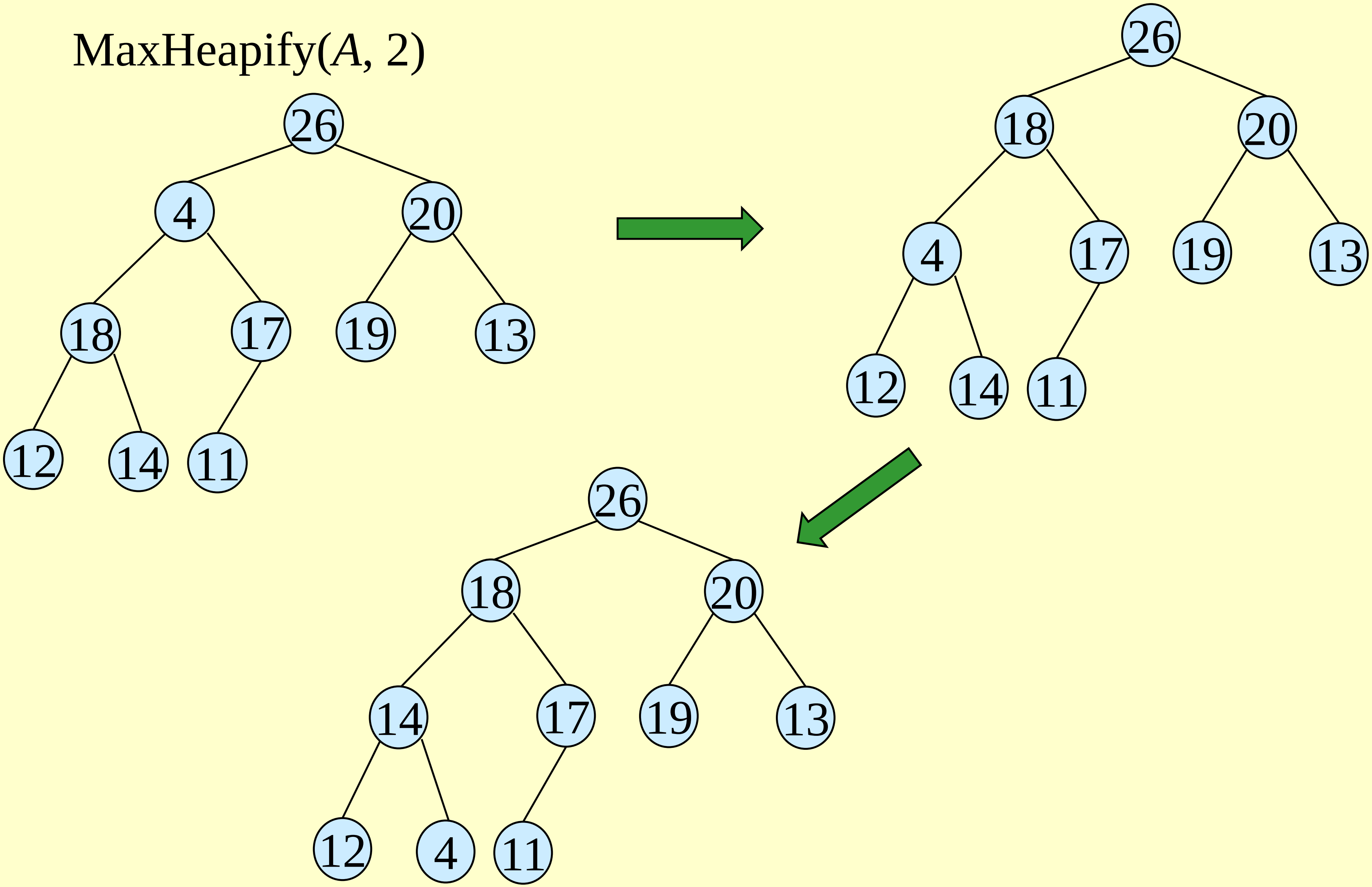
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - » May lead to the subtree at the child not being a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(A, 2)



Procedure MaxHeapify

MaxHeapify(*A*, *i*)

```
l ← left(i)
r ← right(i)
if l ≤ heap-size[A] and A[l] > A[i]
    then largest ← l
    else largest ← i
if r ≤ heap-size[A] and A[r] > A[largest]
    then largest ← r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
            MaxHeapify(A, largest)
```

Assumption:
Left(i) and Right(i) are max-heaps.

Running Time for MaxHeapify

<u>*MaxHeapify*(*A*, *i*)</u>

1. $l \leftarrow$ left(i)
2. $r \leftarrow$ right(i)
3. **if** $l \leq$ *heap-size*[A] and $A[l] > A[i]$
4. **then** *largest* $\leftarrow l$
5. **else** *largest* $\leftarrow i$
6. **if** $r \leq$ *heap-size*[A] **and** $A[r] > A[largest]$
7. **then** *largest* $\leftarrow r$
8. **if** *largest* $\neq i$
9. **then** exchange $A[i] \leftrightarrow A[largest]$
10. *MaxHeapify*(A, *largest*)

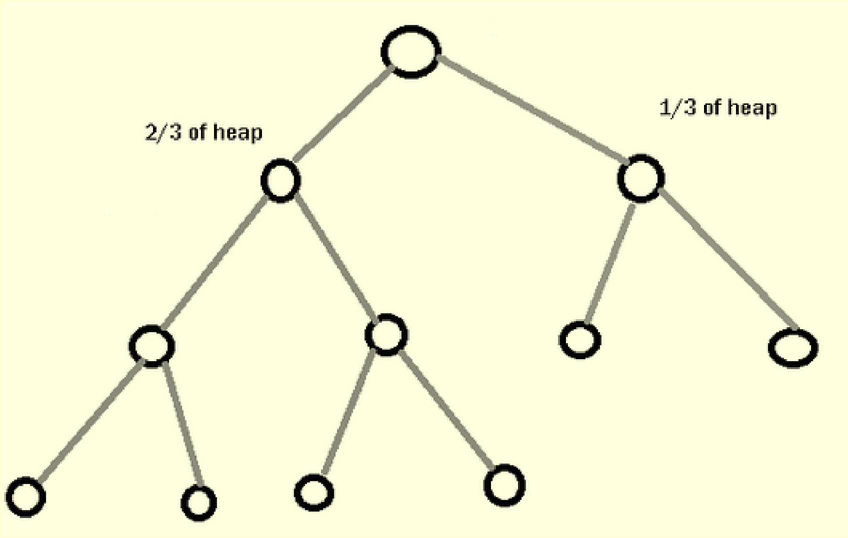
Time to fix node *i* and its children = **Θ(1)**

PLUS

Time to fix the subtree rooted at one of *i*'s children = *T*(size of subree at *largest*)

Running Time for MaxHeapify(A, n)

- $T(n) = T(largest) + \Theta(1)$
- $largest \le 2n/3$ (worst case occurs when the last row of tree is exactly half full)
- $T(n) \le T(2n/3) + \Theta(1) \Rightarrow$
  $T(n) = O(\lg n)$



- Alternately, MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied

Building a heap

- Use *MaxHeapify* to convert an array A into a max-heap.
- Call MaxHeapify on each element in a bottom-up manner.

BuildMaxHeap(A)

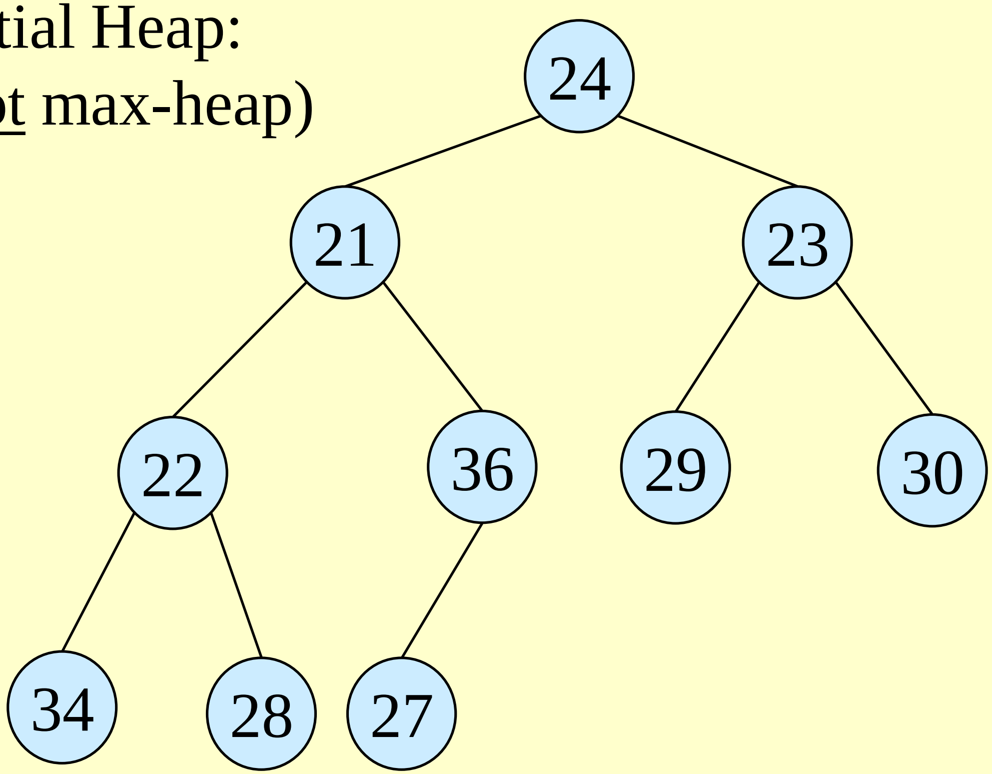
1. $\textit{heap-size}[A] \leftarrow \textit{length}[A]$
2. **for** $i \leftarrow \lfloor \textit{length}[A]/2 \rfloor$ **downto** 1
3. **do** *MaxHeapify*(A, i)

BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27

Initial Heap:
(not max-heap)



BuildMaxHeap – Example

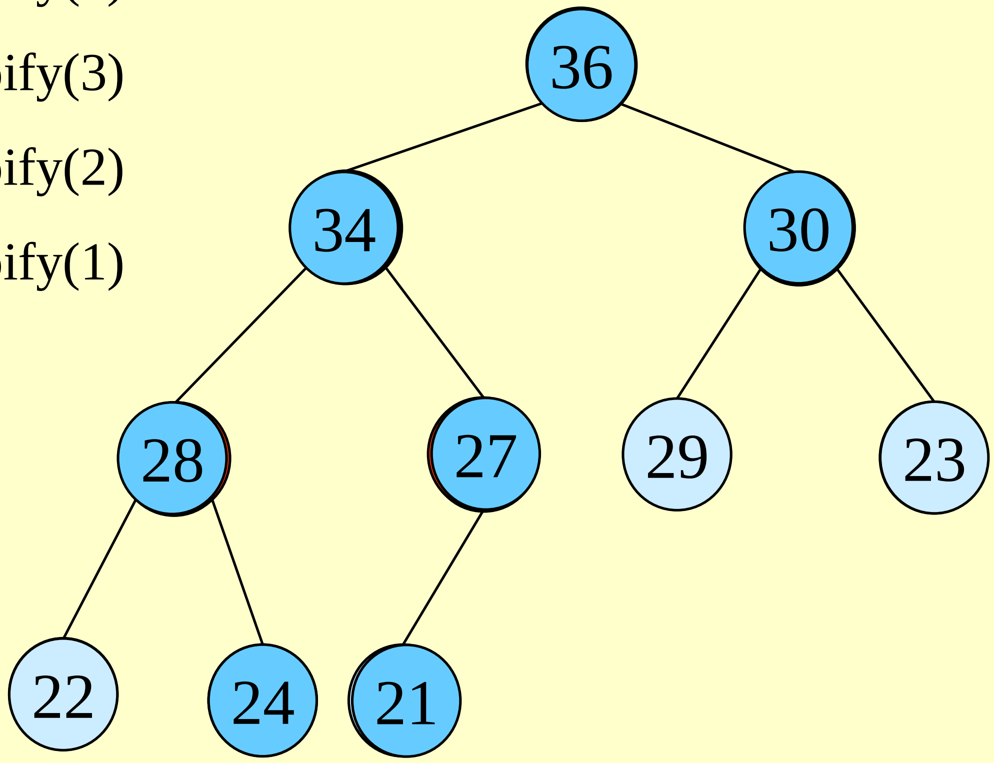
MaxHeapify($\lfloor 10/2 \rfloor = 5$)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)



Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node i+1, i+2, …, n is the root of a max-heap.
- Initialization:
  - » Before first iteration $i = \lfloor n/2 \rfloor$
  - » Nodes $\lfloor n/2 \rfloor$+1, $\lfloor n/2 \rfloor$+2, …, n are leaves and hence roots of max-heaps.
- Maintenance:
  - » By LI, subtrees at children of node i are max heaps.
  - » Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - » Decrementing i reestablishes the loop invariant for the next iteration.
- Termination: For i = 0, All nodes i+1, i+2+…n are max heaps.

Running Time of *BuildMaxHeap*

- Loose upper bound:
  - » Cost of a *MaxHeapify* call × No. of calls to *MaxHeapify*
  - » $O(\lg n) \times O(n) = O(n\lg n)$
- Tighter bound:
  - » Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.
  - » Height of most nodes smaller than $\lg n$.
  - » Height of nodes h ranges from 0 to $\lfloor \lg n \rfloor$.
  - » No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for *T*(*BuildMaxHeap*)

T(*BuildMaxHeap*)

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$

$$\le \sum_{h=0}^{\infty} \frac{h}{2^h} \quad , x = 1/2 \text{ in (A.8)}$$

$$= \frac{1/2}{(1-1/2)^2}$$

$$= 2$$

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}$$

for $|x| < 1$.

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

Can build a heap from an unordered array in linear time

Heapsort

- Sort by maintaining the as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A.
  - » Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - » Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - » Decrement heap-size$[A]$.
- Restore the max-heap property on $A[1..n-1]$.
  - » Call *MaxHeapify*$(A, 1)$.
- Repeat until heap-size$[A]$ is reduced to 2.

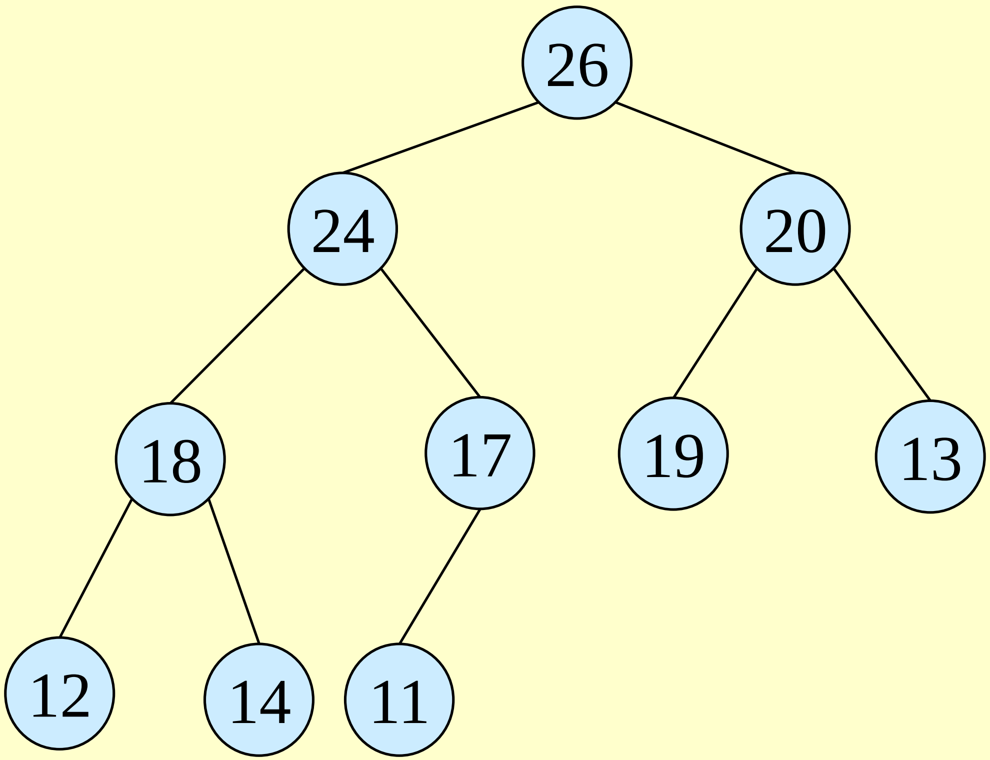
Heapsort(A)

HeapSort(A)

1. Build-Max-Heap(A)
2. **for** $i \leftarrow \mathit{length}[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $\mathit{heap\text{-}size}[A] \leftarrow \mathit{heap\text{-}size}[A] - 1$
5. $\mathit{MaxHeapify}(A, 1)$

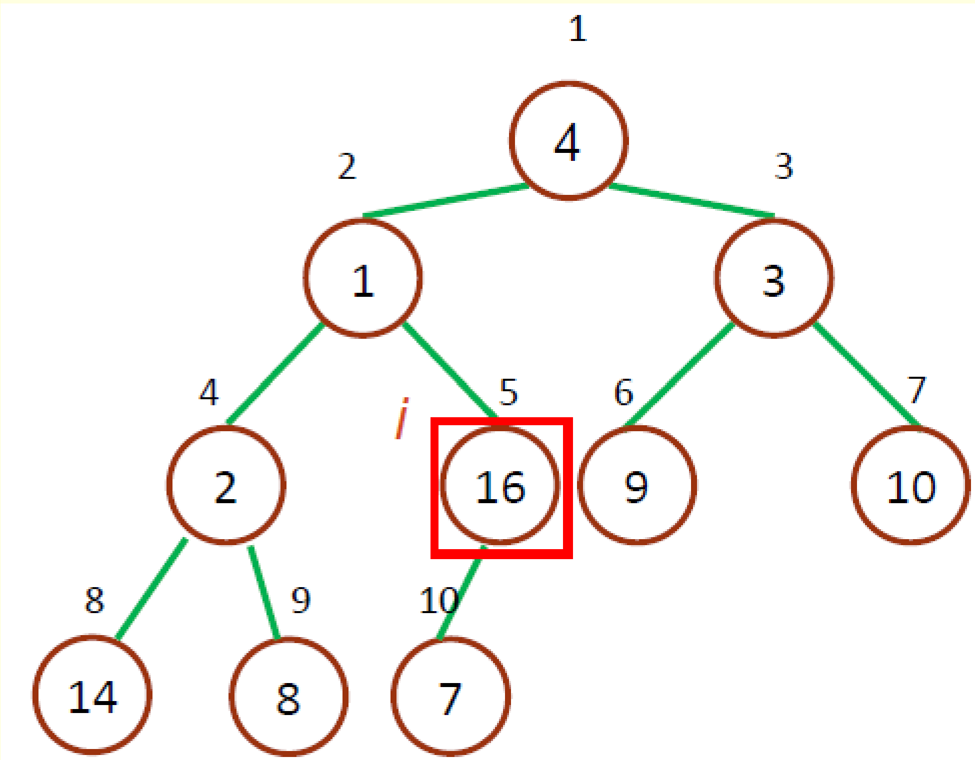
Heapsort – Example

1	2	3	4	5	6	7	8	9	10
26	24	20	18	17	19	13	12	14	11



Heapsort – Example

	1	2	3	4	5	6	7	8	9	10
A	4	1	3	2	16	9	10	14	8	7



BuildMaxHeap(*A*)

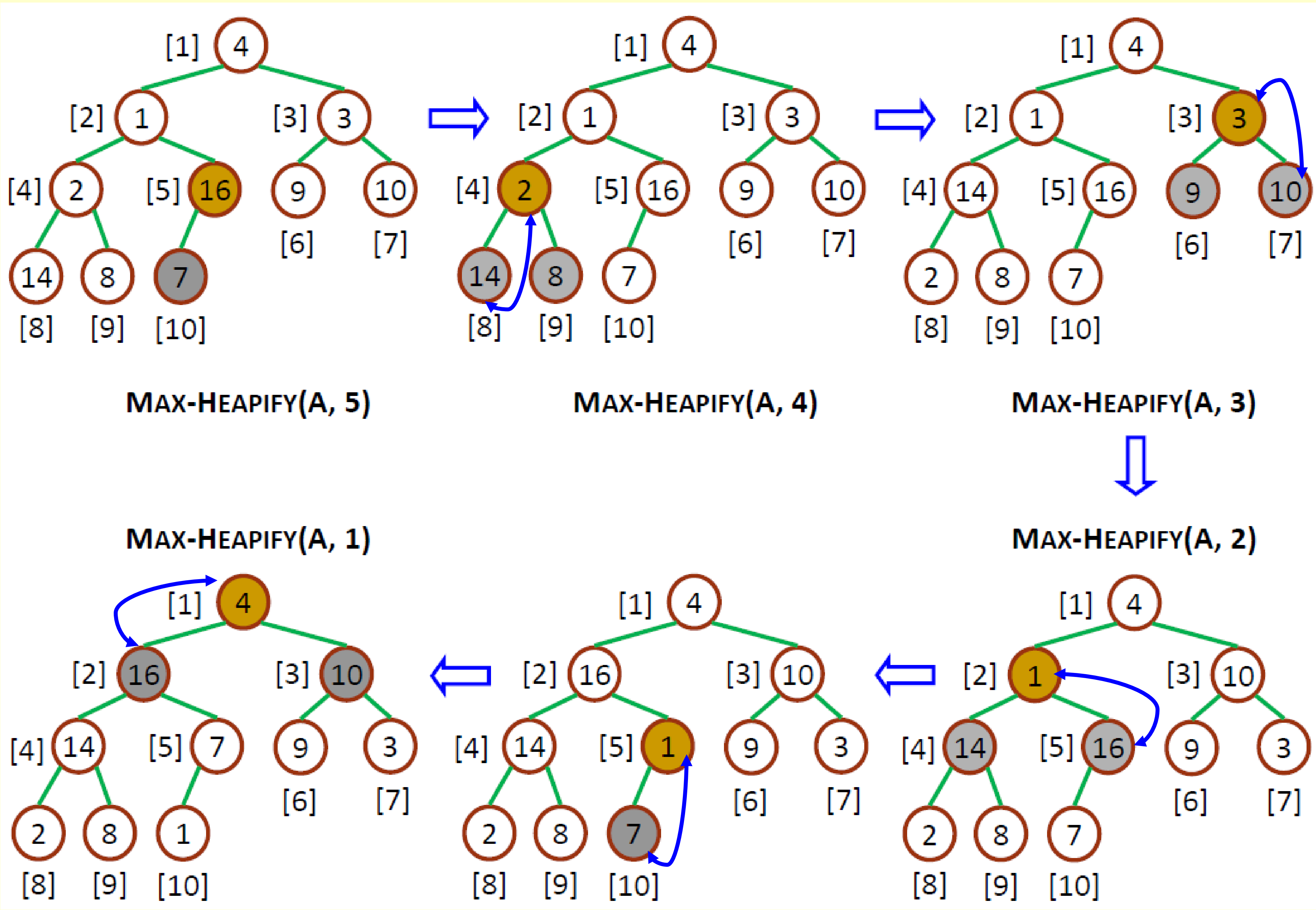
1. *heap-size*[*A*] $\leftarrow$ *length*[*A*]
2. **for** *i* $\leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3. **do** *MaxHeapify*(*A*, *i*)

Applying *BuildMaxHeap*(*A*)

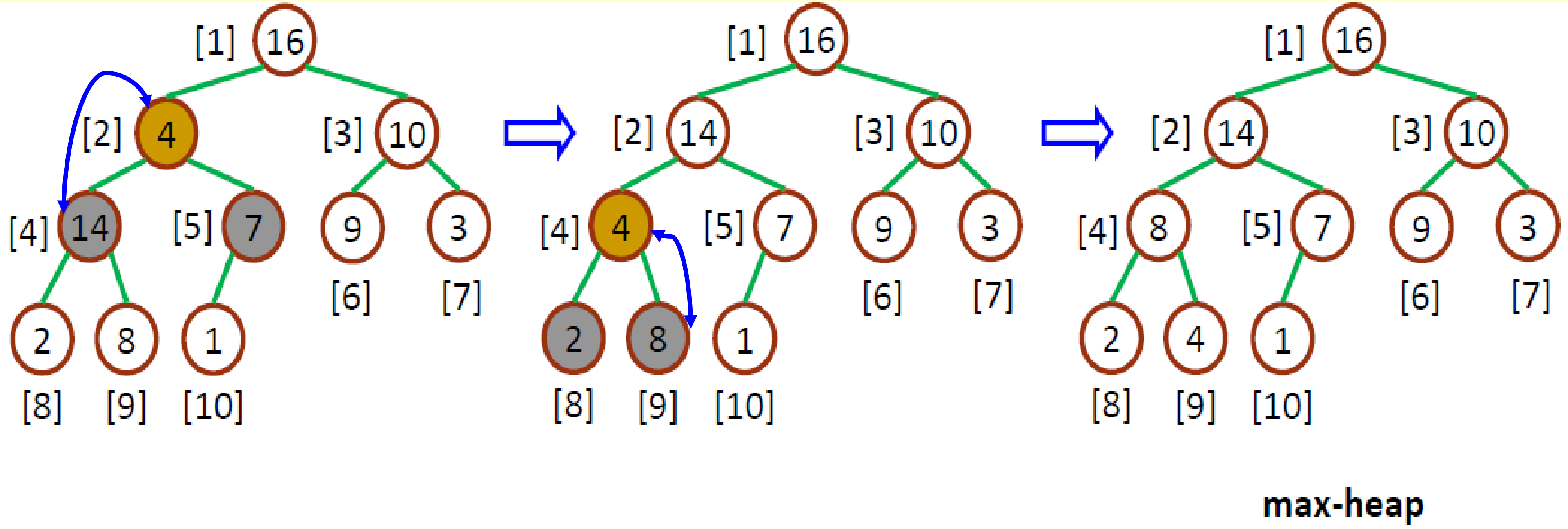
heap-size[*A*] =10

i = 10/2 = 5 to 1

BuildMaxHeap(*A*)



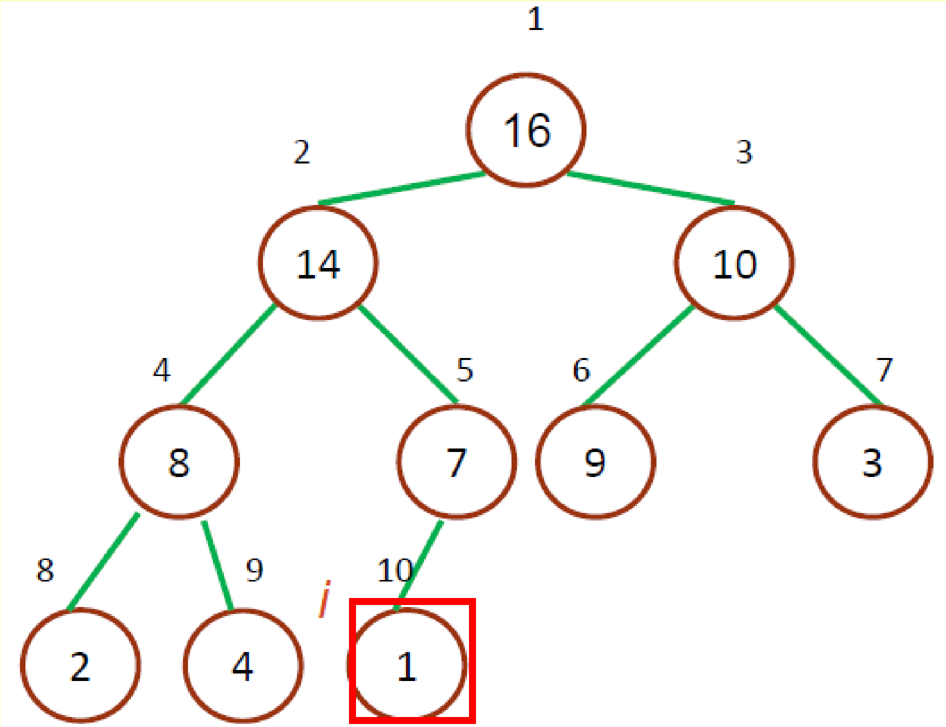
BuildMaxHeap(A)



Now we have a max heap.

16	14	10	8	7	9	3	2	4	1
1	2	3	4	5	6	7	8	9	10

HeapSort(A)



Now we have a max heap.

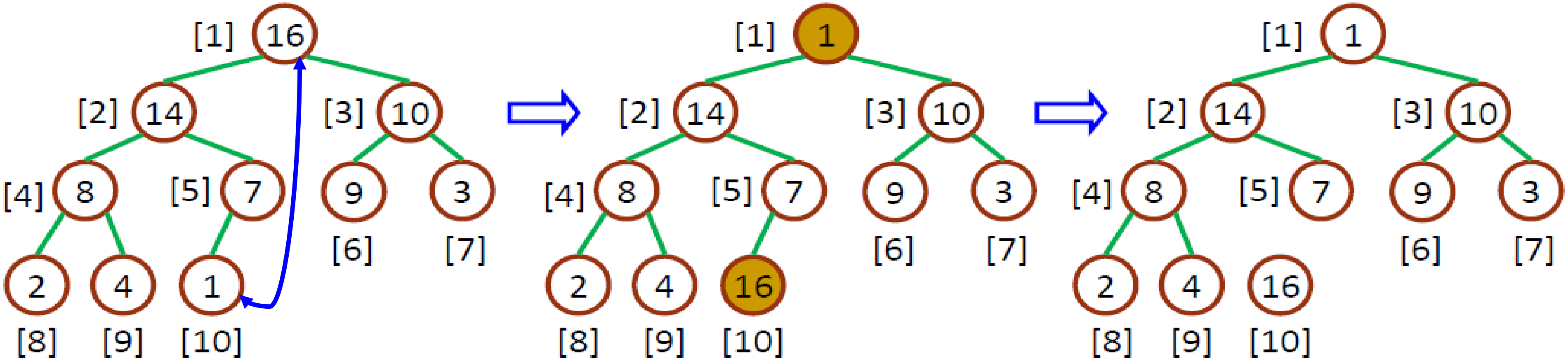
16	14	10	8	7	9	3	2	4	1
1	2	3	4	5	6	7	8	9	10

i = 10 to 2

HeapSort(A)

1. Build-Max-Heap(A)
2. **for** $i \leftarrow$ *length*[A] **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. *heap-size*[A] $\leftarrow$ *heap-size*[A] – 1
5. *MaxHeapify*(A, 1)

HeapSort(A)



Initial heap

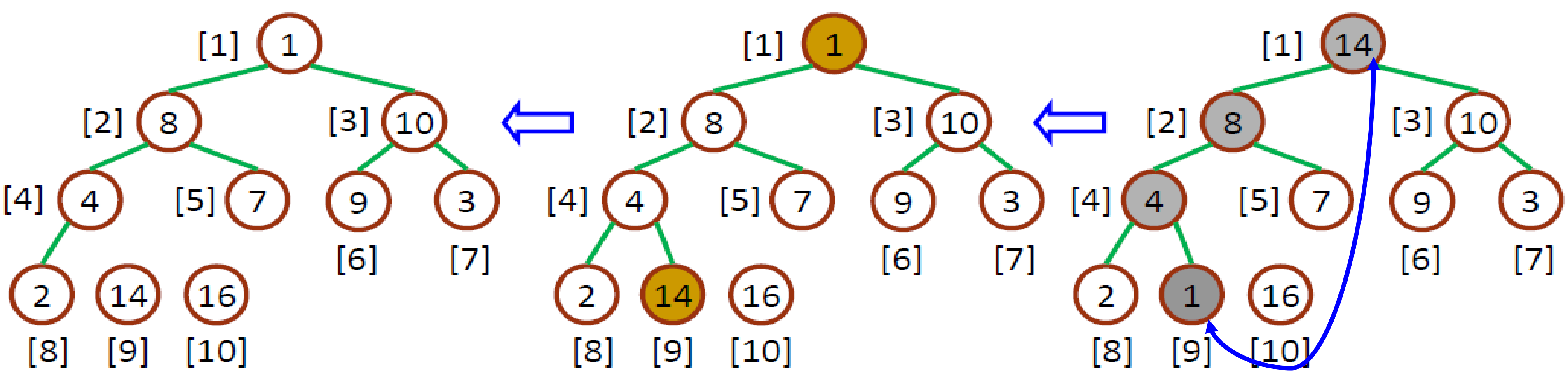
Exchange
Heap size = 10
Sorted=[16]

Discard
Heap size = 9
Sorted=[16]

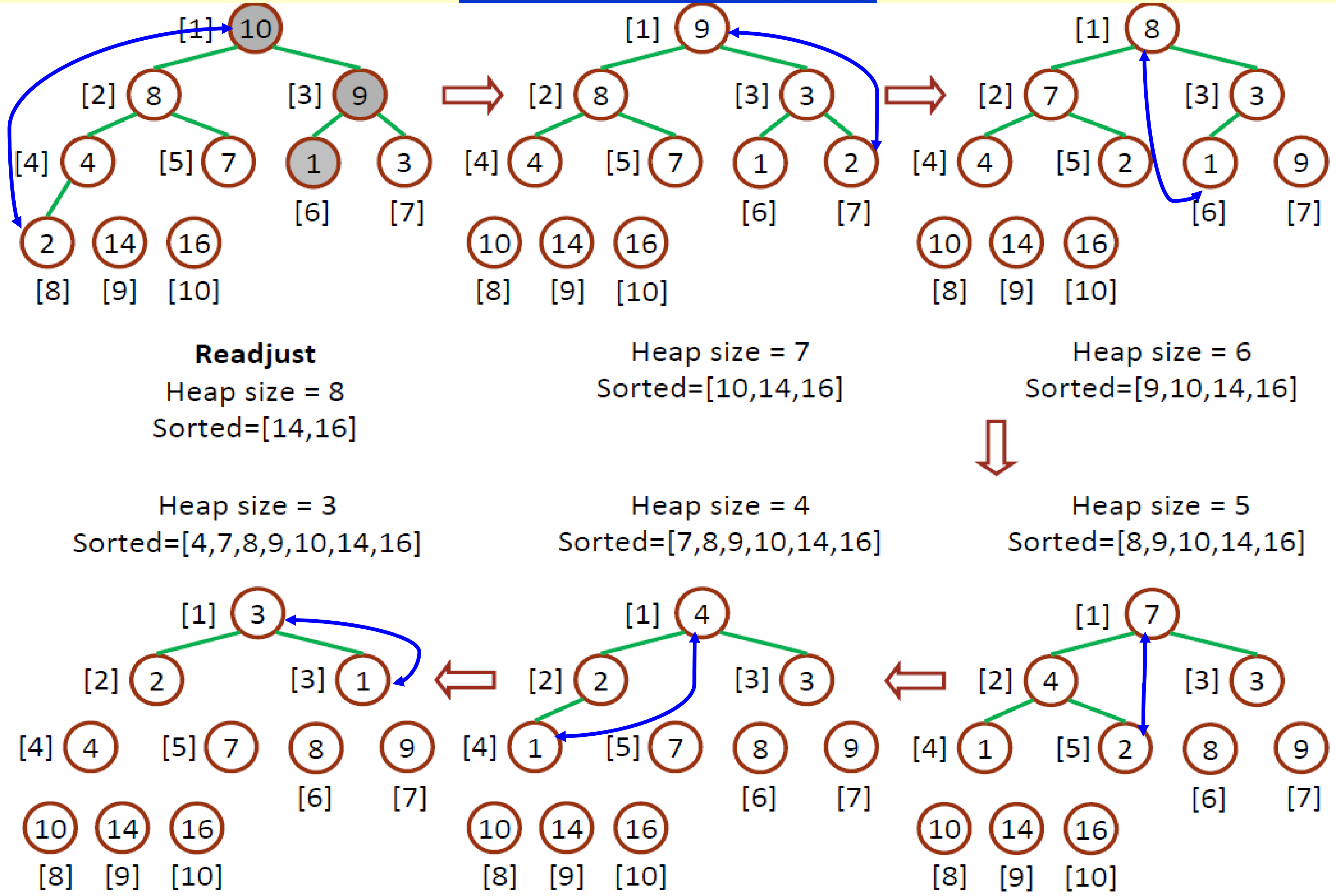
Readjust
Heap size = 9
Sorted=[16]

Exchange
Heap size = 9
Sorted=[14,16]

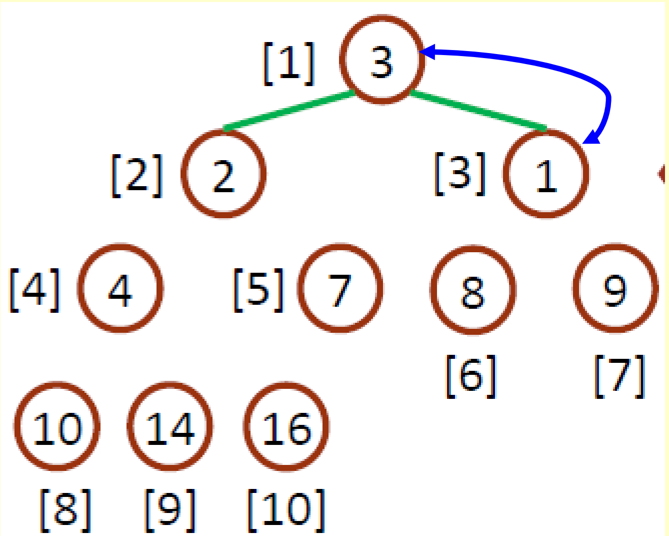
Discard
Heap size = 8
Sorted=[14,16]

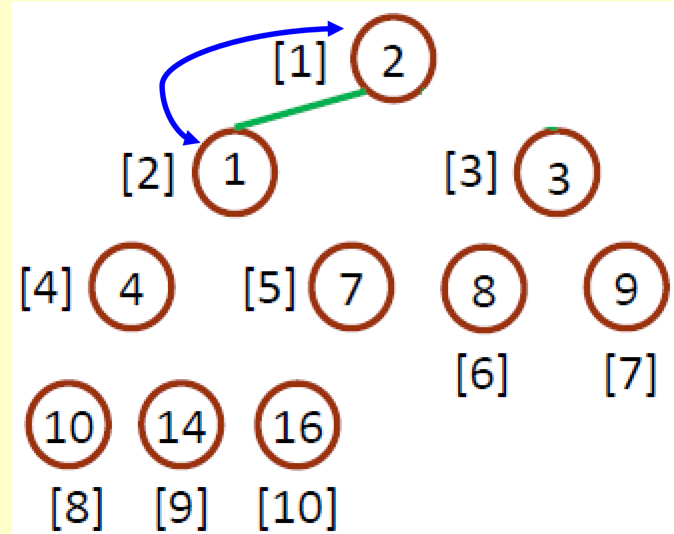


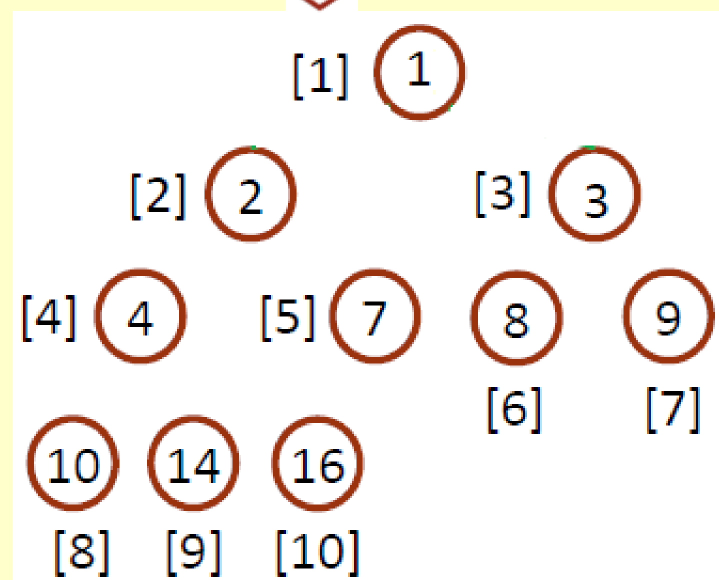
HeapSort(A)


Readjust
Heap size = 8
Sorted=[14,16]
Heap size = 7
Sorted=[10,14,16]
Heap size = 6
Sorted=[9,10,14,16]
Heap size = 5
Sorted=[8,9,10,14,16]
Heap size = 4
Sorted=[7,8,9,10,14,16]
Heap size = 3
Sorted=[4,7,8,9,10,14,16]

HeapSort(A)







Now we have a sorted array.

1	2	3	4	7	8	9	10	14	16
1	2	3	4	5	6	7	8	9	10

Algorithm Analysis

<u>*HeapSort*(*A*)</u>

1. Build-Max-Heap(A)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

- In-place
- Not Stable
- Build-Max-Heap takes $O(n)$ and each of the n-1 calls to Max-Heapify takes time $O(\lg n)$.
- Therefore, $T(n) = O(n \lg n)$

Heap Procedures for Sorting

- MaxHeapify $O(\lg n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \lg n)$

Priority Queue

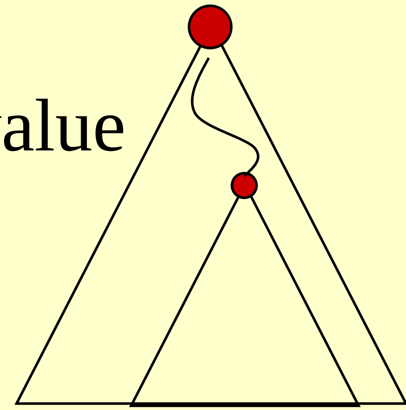
- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - » Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - » In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - » Insert(S, x) - inserts the element x into the set S
    - $S \leftarrow S \cup \{x\}$.
  - » Maximum(S) - returns the element of S with the largest key.
  - » Extract-Max(S) - removes and returns the element of S with the largest key.
  - » Increase-Key(S, x, k) – increases the value of element x's key to the new value k.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - » For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



- Min-Heap
  - » For every node excluding the root, value is at least that of its parent: $A[parent[i]] \leq A[i]$
- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree root

Heap-Extract-Max(A)

Implements the Extract-Max operation.

```
Heap-Extract-Max(A)
1.  if heap-size[A] < 1
2.      then error "heap underflow"
3.  max ← A[1]
4.  A[1] ← A[heap-size[A]]
5.  heap-size[A] ← heap-size[A] - 1
6.  MaxHeapify(A, 1)
7.  return max
```

Running time : Dominated by the running time of MaxHeapify
$= O(\lg n)$

Heap-Insert(A, *key*)

<u>*Heap-Insert*(A, *key*)</u>

1. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
2. $i \leftarrow heap\text{-}size[A]$
4. **while** $i > 1$ **and** $A[\text{Parent}(i)] < key$
5. **do** $A[i] \leftarrow A[\text{Parent}(i)]$
6. $i \leftarrow \text{Parent}(i)$
7. $A[i] \leftarrow key$

Running time is $O(\lg n)$

The path traced from the new leaf to the root has length $O(\lg n)$

Heap-Increase-Key(A, i, key)

Heap-Increase-Key(A, i, key)

If $key < A[i]$
    then error “new key is smaller than the current key”
$A[i] \leftarrow key$
while $i > 1$ **and** $A[\text{Parent}[i]] < A[i]$
        do exchange $A[i] \leftrightarrow A[\text{Parent}[i]]$
            $i \leftarrow \text{Parent}[i]$

Heap-Insert(A, key)

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
Heap-Increase-Key(A, $heap\text{-}size[A]$, key)

Example: Heap-Insert(A, 15)

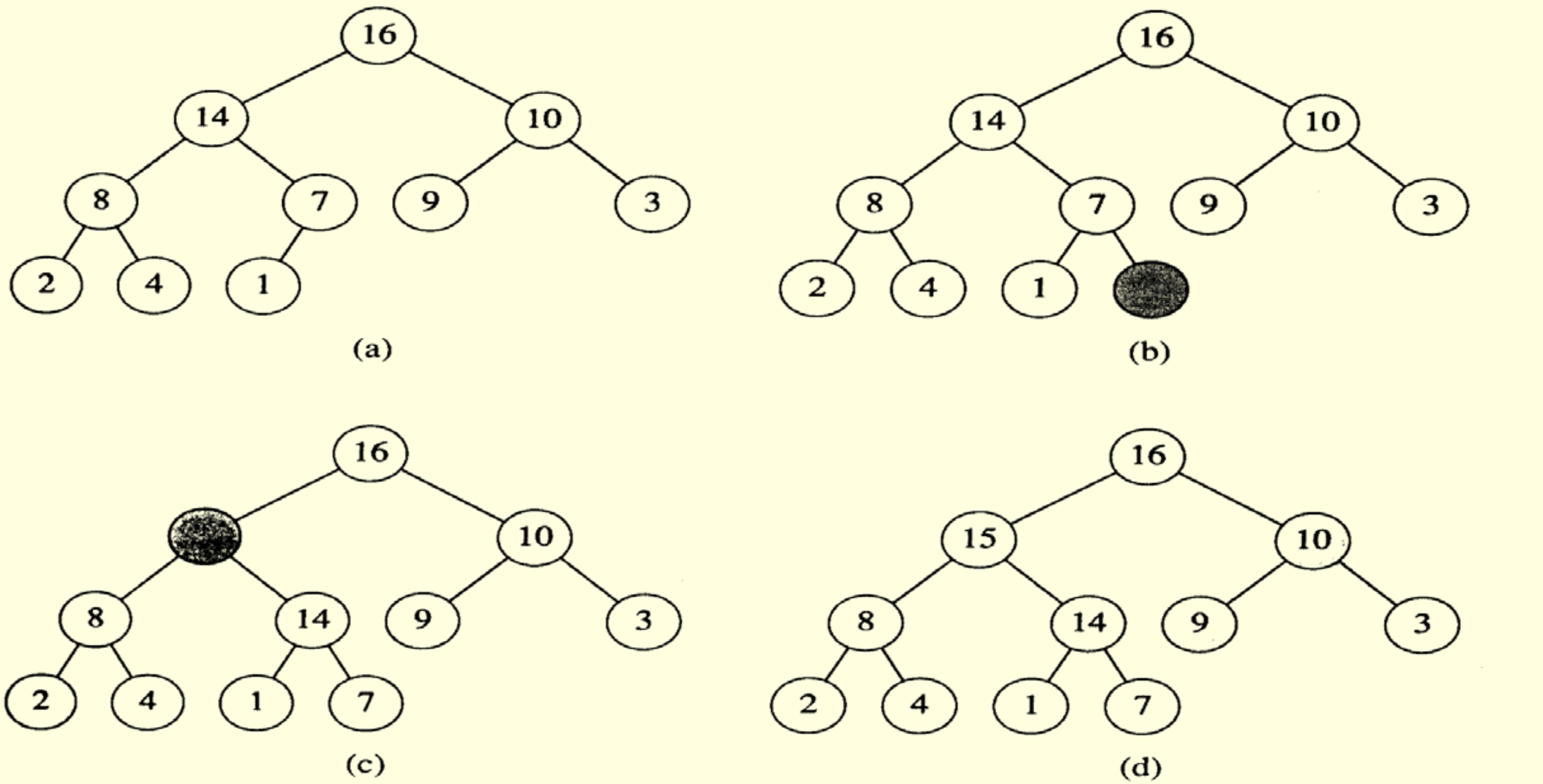


Figure 7.5 The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.

Example:Heap-Increase-Key(*A*, *9*, *15*)