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Ch-03
Random Variables and Probability Distributions

3. Let W be random variable giving no. of heads minus no. of tails in three tosses of coin. List elements of sample space S for three tosses and to each sample point assign a value w of W

$S = \{HHH, HHT, HTH, THH, THT, TTH, HTT, TTT\}$

$w = \{3, 1, -1, -3\}$

w	3	1	-1	-3
$f(w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$f(w) = \begin{cases} 1/8 & , w = 3, -3 \\ 3/8 & , w = 1, -1 \\ 0 & \text{otherwise} \end{cases}$$

4. A coin is flipped until 3 heads in succession occur. List only those elements of sample space that require 6 or less tosses. Is this is a discrete sample space? Explain

$S = \{HHH, THHH, HTHHH, TTHHH, TTTHHH, HTTHHH, THTHHH, HHTHHH, \ldots\}$

The sample set is discrete as space contains finite number of possibilities with as many elements as they are whole numbers.

7. Total no. of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over period of one year is continuous random variable X with density function

$$f(x) = \begin{cases} x & , 0 < x < 1 \\ 2-x & , 1 \le x < 2 \\ 0 & , \text{elsewhere} \end{cases}$$

Find probability that over a period of one year, a family runs their vacuum cleaner

a) less than 120 hours

b) bet^n 50 and 100 hours

(a) $P(X < 1.2) = \int_{-\infty}^{1.2} f(x)\,dx$

$= \int_{-\infty}^{0} \cancel{f(x)}\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.2} f(x)\,dx$

$= \int_{0}^{1} x\,dx + \int_{1}^{1.2} (2-x)\,dx$

$= \left.\frac{x^2}{2}\right]_0^1 + \left.2x - \frac{x^2}{2}\right]_1^{1.2} = \frac{1}{2} + 2.4 - \frac{1.44}{2} - 2 + \frac{1}{2}$

$= 1.4 - 0.72 = 0.68$

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b) $P(\frac{1}{2} < X < 1) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} x\,dx$

$= \left.\frac{x^2}{2}\right]_{1/2}^{1} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8} = 0.375$

10 Find formula for probability distribution of random variable X representing outcome when single die is rolled once.

$S = \{1, 2, 3, 4, 5, 6\}$

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$f(x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$

11 A shipment of 7 T.V sets contains 2 defective sets. A hotel makes random purchase of 3 of sets. If x is no of defective sets purchased by hotel, find probability distribution of X. Express results graphically.

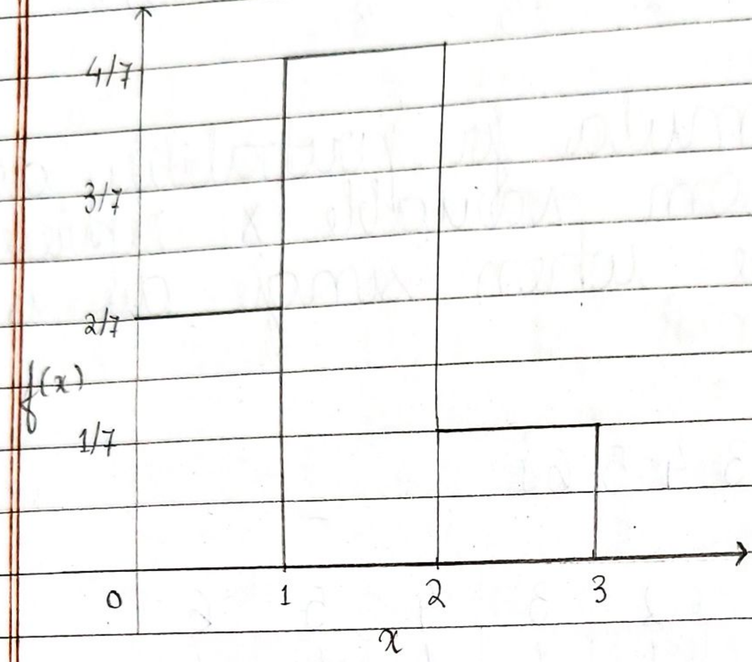
X : random variable
- no. of defective laptops

$X = \{0, 1, 2\}$

$F(0) = P(X = 0) = \frac{{}^5C_3 \times {}^2C_0}{{}^7C_3} = \frac{2}{7}$

$$F(1) = P(X=1) = \frac{{}^5C_2 \times {}^2C_1}{{}^7C_3} = \frac{4}{7}$$

$$F(2) = P(X=2) = \frac{{}^5C_1 \times {}^2C_2}{{}^7C_3} = \frac{1}{7}$$



x	0	1	2
$f(x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

12. An investment firm offers its customers municipal bonds that mature after varying no. of years. Given that the cumulative distribution function of T, no. of years to maturity for random selected bond, is

$$F(t) = \begin{cases} 0 & t < 1 \\ 1/4 & 1 \leq t < 3 \\ 1/2 & 3 \leq t < 5 \\ 3/4 & 5 \leq t < 7 \\ 1 & t \geq 7 \end{cases}$$

find

(a) $P(T = 5)$ (c) $P(1.4 < T < 6)$

(b) $P(T > 3)$ (d) $P(T \leq 5 \mid T \geq 2)$

(a) $P(T = 5) = F(5) - F(4)$

$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

(b) $P(T > 3) = 1 - P(T \leq 3)$

$= 1 - F(3)$

$= 1 - \frac{1}{2} = \frac{1}{2}$

(c) $P(1.4 < T < 6) = F(6) - F(1.4)$

$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

(d) $P(T \leq 5 \mid T \geq 2) = \dfrac{P((T \leq 5) \cap (T \geq 2))}{P(T \geq 2)}$

$= \dfrac{P(2 \leq T \leq 5)}{P(T \geq 2)} = \dfrac{F(5) - F(2)}{1 - P(T < 2)}$

$= \dfrac{3/4 - 1/4}{1 - 1/4} = \dfrac{2}{3}$

14. Waiting time, in hours, bet^n successive speeders spotted by radar unit is a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-8x} & x \geq 0 \end{cases}$$

Find probability of waiting less than 12 min bet^n successive speeders

a) using cumulative distribution

b) using probability density function

(a) $P(X < 0.2) = F(0.2)$
$= 1 - e^{-1.6} = 1 - 0.201$
$= 0.798$

(b) $P(X < 0.2) = \int_0^{0.2} f(x)\,dx$

$f(x) = F'(x) = 8e^{-8x}$

$P(X < 0.2) = 8\int_0^{0.2} e^{-8x}\,dx$

$= -e^{-8x}\Big]_0^{0.2}$
$= (-0.201) + (1)$
$= 0.7981$

21 Consider density function

$$f(x) = \begin{cases} k\sqrt{x} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

(a) Evaluate k.
(b) Find $F(x)$ and use it to evaluate $P(0.3 < X < 0.6)$

(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$= \int_{-\infty}^{0} \cancel{f(x)}^{0}\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} \cancel{f(x)}^{0}\,dx$

$= \int_0^1 f(x)\,dx = 1$

$= \int_0^1 k\sqrt{x}\,dx = 1$

$\Rightarrow k \cdot \frac{3x^{3/2}}{2} \Big]_0^1 = 1$

$k = \frac{3}{2}$

$$f(x) = \begin{cases} \frac{3}{2}\sqrt{x} , & 0 < x < 1 \\ 0 , & \text{elsewhere} \end{cases}$$

(b) $F(x) = \int_{-\infty}^{\infty} F(t)dt = \int_{-\infty}^{0} \cancel{F(t)}dt + \int_{0}^{x} F(t)\,dt$

$= \int_0^x \frac{3}{2}\sqrt{t}\,dt = t^{3/2}\Big]_0^x$

$= x^{3/2}$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^{3/2} & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

$P(0.3 < X < 0.6) = F(0.6) - F(0.3)$
$= (0.6)^{3/2} - (0.3)^{3/2}$
$= 0.3$

29. From production data in past, it has been determined that particle size distribution is characterized by

$$f(x) = \begin{cases} 3x^{-4} & x > 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) Verify that it's a density function
b) Evaluate $F(x)$

(c) What is the probability that a random particle from manufactured fuel exceeds 4 micrometers?

(a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} \cancel{f(x)}^{\,0}\,dx + \int_{1}^{\infty} f(x)\,dx$

$= \int_{1}^{\infty} 3x^{-4}\,dx = -\frac{3x^{-3}}{3}\Big]_{1}^{\infty}$

$= 1$

$f(x)$ is density function

(b) $F(x) = \int_{1}^{x} 3t^{-4}\,dt = 1 - x^{3}$

$F(x) = \begin{cases} 0 & x < 1 \\ 1 - x^{3} & x \geq 1 \end{cases}$

(c) $P(X > 4) = 1 - F(4)$
$= 4^{-3} = 0.015625$

30. Suppose the measurement error X of certain physical quantity is decided by density function
$f(x) = \begin{cases} k(3 - x^2) & -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

a) Determine k

b) Find probability that random error in measurement is less than 1/2

c) For this particular measurement, it is undesirable if magnitude of error (i.e $|x|$) exceeds 0.8, what's the probability that this occurs?

(a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} k(3-x^2)\,dx = 1$

$\Rightarrow k\left(3x - \frac{x^3}{3}\right)_{-1}^{1} = 1$

$\Rightarrow \frac{16}{3}k = 1 \Rightarrow k = \frac{3}{16}$

(b) $F(x) = \frac{3}{16}\int_{-1}^{x} (3-t^2)\,dt$

$= \frac{3}{16}\left(3t - \frac{t^3}{3}\right)_{-1}^{x}$

$= \frac{1}{2} + \frac{9x}{16} - \frac{x^3}{16}$

$P(X < \frac{1}{2}) = \frac{1}{2} + \frac{9}{16} \times \frac{1}{2} - \frac{1}{16} \times \frac{1}{8}$

$= \frac{1}{2} + \frac{9}{32} - \frac{1}{128} = \frac{64 + 36 - 1}{128}$

$= \frac{99}{128}$

(c) $P(|X| < 0.8) = P(X < -0.8)$ (from -1 to -0.8) $+ P(X > 0.8)$ (from 0.8 to 1)

$= F(-0.8) + 1 - F(0.8)$

$= 1 + \left(\frac{1}{2} - \frac{9}{16}(0.8) + \frac{1}{16}(0.8)^3\right) - \left(\frac{1}{2} + \frac{9}{16}(0.8) - \frac{(0.8)^3}{16}\right)$

$= 1 + 0.082 - 0.918 = 0.164$

35 X, the no. of cars that arrive at a specific intersection during 20 s time period, is characterized by following discrete probability function:

$$f(x) = \frac{e^{-6}\, 6^x}{x!}, \quad x = 0, 1, 2, \ldots$$

a) Find probability that more than 8 cars arrive at intersection

b) Find probability that only 2 cars arrive.

a) $P(X > 8) = 1 - P(X \leq 8)$

$$= 1 - \sum_{x=0}^{8} e^{-6} \frac{6^x}{x!}$$

$$= 1 - e^{-6}\left[\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!} \ldots + \frac{6^8}{8!}\right]$$

$$= 1 - e^{-6}\left[1 + 6 + 18 + 36 + 54 + 64.8 + 64.8 + 55.54 + 41.65\right]$$

$$= 1 - e^{-6}(341.79) = 1 - 0.84721$$

$$= 0.15278$$

(b) $P(X = 2) = e^{-6} \cdot \frac{6^2}{2!} = e^{-6} \cdot 18$

$$= 0.0446$$

38. Joint probability distribution of X and Y is given by

$$f(x, y) = \frac{x+y}{30}, \quad x = 0, 1, 2, 3; \quad y = 0, 1, 2,$$

Find

(a) $P(X \leq 2, Y = 1)$
(b) $P(X > 2, Y \leq 1)$
(c) $P(X > Y)$
(d) $P(X + Y = 4)$

(a) $P(X \leq 2, Y = 1) = f(0,1) + f(1,1) + f(2,1)$
$= \frac{6}{30} = \frac{1}{5}$

(b) $P(X > 2, Y \leq 1) = f(3,1) + f(3,0)$
$= \frac{4}{30} + \frac{3}{30} = \frac{7}{30}$

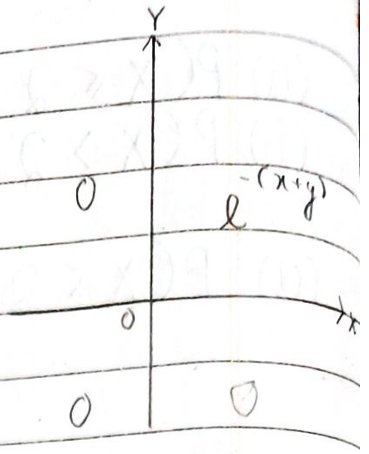
(c) $P(X > Y) = f(1,0) + f(2,0) + f(3,0) + f(2,1) + f(3,1) + f(3,2)$
$= \frac{1}{30} + \frac{2}{30} + \frac{3}{30} + \frac{3}{30} + \frac{4}{30} + \frac{5}{30}$
$= \frac{18}{30}$

(d) $P(X + Y = 4) = f(2,2) + f(3,1)$
$= \frac{4}{30} + \frac{4}{30} = \frac{8}{30}$

42. Let X and Y denote lengths of life of two components in electronic system. If joint density funⁿ is

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

find $P(0 < X < 1 \mid Y = 2)$



$P(a < X < b \mid Y = 2)$

$= \int_a^b f(x|y)\, dx$

$f(x|y) = \dfrac{f(x,y)}{h(y)}$

$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^{\infty} e^{-x} \cdot e^{-y}\, dx$

$= e^{-y} [e^{-x}]_0^{\infty} = e^{-y}$

$f(x|y) = e^{-x}$

$P(0 < X < 1 \mid Y = 2) = \int_0^1 e^{-x}\, dx = -e^{-x}\Big]_0^1$

$= 1 - e^{-1} = 0.63212$

44 Each rear tire on experimental airplane is filled to pressure of 40 pounds per sq. inch. Let X be actual air pressure for right tire & Y for left. Suppose X & Y are random variables

$f(x,y) = \begin{cases} k(x^2 + y^2), & 30 \leq x < 50,\ 30 \leq y < 50 \\ 0 & \text{elsewhere} \end{cases}$

a) Find k

b) Find $P(30 \leq X \leq 40$ and $40 \leq Y < 50)$

c) Find probability that both tires are underfilled.

a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

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$$\Rightarrow k\int_{30}^{50}\int_{30}^{50}(x^2+y^2)\,dx\,dy = 1$$

$$\Rightarrow k\int_{30}^{50}\left[\frac{x^3}{3}+xy^2\right]_{30}^{50} dy = 1$$

$$\Rightarrow k\int_{30}^{50}\frac{50^3}{3}+50y^2-\frac{30^3}{3}-30y^2\,dy = 1$$

$$\Rightarrow k\left(\frac{50^3-30^3}{3}\cdot y+\frac{20y^3}{3}\right)_{30}^{50} = 1$$

$$\Rightarrow k\left(\left[\frac{50^3-30^3}{3}\right]50+\frac{20\cdot 50^3}{3}-\left[\frac{50^3-30^3}{3}\right]30-\frac{20.30^3}{3}\right.$$

$$\Rightarrow 2k\left[\frac{50^3-30^3}{3}\right]\cdot 20 = 1$$

$$\Rightarrow \frac{3920000k}{3} = 1$$

$$\Rightarrow k = \frac{3}{3920000} = \frac{3}{392}\times 10^{-4}$$

(b) $P(30 \leq X \leq 40 \text{ and } 40 \leq Y < 50)$

$$= k\int_{30}^{40}\int_{40}^{50} x^2+y^2\,dy\,dx$$

$$= k\int_{30}^{40}\left[x^2y+\frac{y^3}{3}\right]_{40}^{50} dx$$

$$= k\int_{30}^{40} 50x^2+\frac{50^3}{3}-40x^2-\frac{40^3}{3}\,dx$$

$$= k\left[\left(\frac{50^3-40^3}{3}\right)x+\frac{10x^3}{3}\right]_{30}^{40} = 0.25$$

(c) $P(30 \le X \le 40,\ 30 \le Y \le 40)$

$$= k\int_{30}^{40}\int_{30}^{40} x^2 + y^2 \, dx\, dy$$

$$= k\int_{30}^{40} \left[\frac{x^3}{3} + xy^2\right]_{30}^{40} dy$$

$$= k\int_{30}^{40} \frac{40^3}{3} + 40y^2 - \frac{30^3}{3} - 30y^2 \, dy$$

$$= k\left[\frac{(40^3 - 30^3)}{3}y + \frac{10y^3}{3}\right]_{30}^{40}$$

$$= k\left[\frac{(40^3 - 30^3)40}{3} + 10\cdot\frac{40^3}{3} - \frac{(40^3 - 30^3)30}{3} - 10\cdot\frac{30^3}{3}\right]$$

$= 0.18877$

49. X is no. of times certain numerical control machine will malfunction: 1, 2, 3 times on any given day. Y is no. of times technician is called.

$f(x,y)$		$x=1$	$x=2$	$x=3$
y	1	0.05	0.05	0.10
	3	0.05	0.10	0.35
	5	0.00	0.20	0.10

a) Evaluate marginal distribution of X
b) Evaluate marginal distribution of Y
c) Find $P(Y=3 \mid X=2)$

(a) $g(x) = \sum_y f(x, y)$

X	1	2	3
g(x)	0.1	0.35	0.55

(b) $h(y) = \sum_x f(x, y)$

Y	1	3	5
h(y)	0.2	0.5	0.3

(c) $P(Y = 3 \mid X = 2) = \dfrac{f(x, y)}{g(2)} = \dfrac{f(2,3)}{g(2)}$

$$= \frac{0.10}{0.35} = \frac{10}{35} = \frac{2}{7}$$

50. Suppose X and Y have following joint probability distribution:

f(x,y)		x = 2	x = 4
y	1	0.1	0.15
	3	0.2	0.3
	5	0.1	0.15

a) Find marginal distribution of X
b) Find marginal distribution of Y

a) $g(x) = \sum_y f(x, y)$

$g(2) = f(2,1) + f(2,3) + f(2,5) = 0.4$
$g(4) = f(4,1) + f(4,3) + f(4,5) = 0.6$

X	2	4
g(x)	0.4	0.6

(b) $h(y) = \sum_{x} f(x,y)$

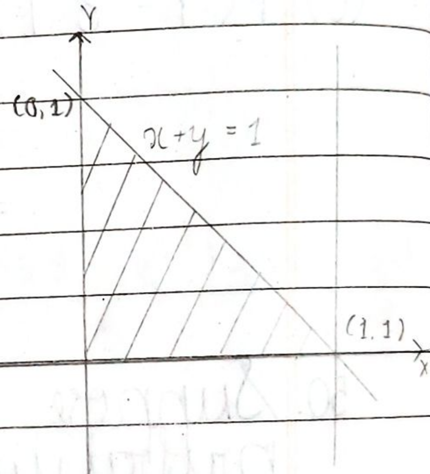
Y	1	3	5
h(y)	0.25	0.5	0.25

56. Joint density function of X & Y is

$$f(x,y) = \begin{cases} 6x, & 0 < x < 1,\ 0 < y < 1-x \\ 0, & \text{elsewhere} \end{cases}$$

a) Show X and Y aren't independent

b) Find $P(X > 0.3 \mid Y = 0.5)$



(a) $g(x) = \int_0^{1-x} f(x,y)\, dy$

$= \int_0^{1-x} 6x\, dy = 6x - 6x^2$

$h(y) = \int_0^1 6x\, dx$

$= 3$

$f(x,y) \neq g(x)h(y)$

So, X and Y aren't independent

(b) $P(X > 0.3 \mid Y = 0.5) = \int_{0.3}^{1} f(x|y)\, dx$

$f(x|y) = \dfrac{f(x,y)}{h(y)} = \dfrac{6x}{3} = 2x$

$P(X > 0.3 \mid Y = 0.5) = \int_{0.3}^{1} 2x\, dx = 0.91$

60. Joint probability density funⁿ of X, Y, Z is

$$f(x,y,z) = \begin{cases} \frac{4xyz^2}{9}, & 0 < x, y < 1, 0 < z < 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find
a) joint marginal density funⁿ of Y, Z
b) marginal density of Y
c) $P(1/4 < X < 1/2, Y > 1/3, 1 < Z < 2)$
d) $P(0 < X < 1/2 \mid Y = 1/4, Z = 2)$

(a) $$g(y,z) = \int_0^1 \frac{4}{9} xyz^2 \, dx = \frac{2}{9} yz^2$$

(b) $$h(y) = \frac{2}{9} \int_0^3 yz^2 \, dz = 2y$$

(c) $P(1/4 < X < 1/2, Y > 1/3, 1 < Z < 2)$

$$= \frac{4}{9} \int_1^2 \int_{1/3}^1 \int_{1/4}^{1/2} xyz^2 \, dx \, dy \, dz$$

$$= \frac{4}{9} \int_1^2 \int_{1/3}^1 \left. \frac{x^2}{2} yz^2 \right]_{1/4}^{1/2} dy \, dz$$

$$= \frac{2}{9} \times \frac{3}{16} \int_1^2 \left. \frac{y^2}{2} z^2 \right]_{1/3}^1 dz$$

$$= \frac{1}{3} \times \frac{1}{16} \times \frac{8}{9} \left. \frac{z^3}{3} \right]_1^2$$

$$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{9} \times \frac{1}{3} \times 7 = \frac{7}{162}$$

(d) $P(0 < X < \frac{1}{2} \mid Y = \frac{1}{4}, Z = 2)$

$$f(x \mid y,z) = \frac{f(x,y,z)}{g(y,z)} = \frac{\frac{4}{9} xyz^2}{\frac{2}{9} yz^2}$$

$$= 2x$$

$$P(0 < X < \frac{1}{2} \mid Y = \frac{1}{4}, Z = 2) = \int_0^{1/2} 2x \, dx$$

$$= \frac{1}{4}$$

62. An insurance company offers differ premium payment options. For randomly selected policyholder, let X be no. of months betⁿ successive payments.

$$f(x) = \begin{cases} 0 & x < 1 \\ 0.4 & 1 \leq x < 3 \\ 0.6 & 3 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1.0 & x \geq 7 \end{cases}$$

(a) What is the probability mass funⁿ of X?

(b) Compute $P(4 < X \leq 7)$

(a)

x	1	3	5	7
$f(x)$	0.4	0.2	0.2	0.2

(b) $P(4 < X \leq 7) = F(7) - F(4)$

$$= 1 - 0.6$$

$$= 0.4$$

66. Consider X and Y with joint density function

$$f(x,y) = \begin{cases} x+y, & 0 \le x, y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find marginal distributions of X, Y

b) Find $P(X > 0.5, Y > 0.5)$

(a) $$g(x) = \int_0^1 (x+y)\, dy = x + \frac{1}{2}$$

$$h(y) = \int_0^1 (x+y)\, dx = \frac{1}{2} + y$$

(b) $$P(X > 0.5, Y > 0.5) = \int_{0.5}^{1}\int_{0.5}^{1} (x+y)\, dx\, dy$$

$$= \int_{0.5}^{1} \frac{x^2}{2} + xy \Big]_{0.5}^{1} dy$$

$$= \int_{0.5}^{1} \frac{3}{8} + \frac{y}{2}\, dy = \frac{3}{8}y + \frac{y^2}{4}\Big]_{0.5}^{1}$$

$$= \frac{3}{8} + \frac{1}{4} - \frac{3}{16} - \frac{1}{16}$$

$$= \frac{3}{8}$$