7. Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, ... are recursively defined as:

$$F_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

Prove the following using induction

(i) For all non-ve integers n, F_n is even iff n is divisible by 3.

Base step:

$n = 3$

$F_3 = 2$ and $2 \equiv 0 \pmod 2$ $\longrightarrow$ divisible by 2 so even

$n = 3 \equiv 0 \pmod 3$ $\longrightarrow$ divisible by 3

Inductive hypothesis:

Let it is true for $n = k$, $k \geq 3$ so that F_k is even and k divisible by 3.

Inductive step:

$n = k + 1$

$F_{k+1} = F_k + F_{k-1}$

$= \text{even} + \text{odd}$

$= \text{odd}$

$(k-1)$ is not divisible by 3

F_{k-1} is odd

$n = k + 2$

$F_{k+2} = F_{k+1} + F_k = \text{odd} + \text{even}$

$= \text{odd}$

$n = k+3$

$F_{k+3} = F_{k+2} + F_{k+1}$

$= odd + odd$

$= even$

$k+3 \equiv 0 \pmod 3) \rightarrow$ divisible by 3

$F(n)$ is even iff n is divisible by 3

(ii) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$

Base step:

$n = 0$

LHS : $\sum_{i=0}^{0} F_i = F_0 = 0$

RHS : $F_{n+2} - 1 = F_2 - 1 = 0$

True

Inductive hypothesis

For $n = k$, $\sum_{i=0}^{k} F_i = F_{k+2} - 1$

Inductive step:

$n = k+1$

LHS : $\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} F_i + F_{k+1}$

$= F_{k+2} - 1 + F_{k+1}$

$= (F_{k+2} + F_{k+1}) - 1 = (F_{(k+3)-1} + F_{(k+3)-2}) - 1$

$= (F_{k+3}) - 1$

$= F_{(k+1)+2} - 1 = \text{RHS}$

(iii) $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

Base step:

$n = 0+1 = 1$ ($\because n \neq 0$ as F_{-1} is not possible)

LHS : $F_1^2 - F_2F_0 = 1$

RHS : $(-1)^2 = 1$

True

Inductive hypothesis:

$n = k$

$F_k^2 - F_{k+1}F_{k-1} = (-1)^{k+1}$

$\Rightarrow F_k^2 = (-1)^{k+1} + F_{k+1}F_{k-1}$

Inductive step: Let $n = k+1$

$F_{k+1}^2 = (-1)^{k+2} + F_{k+2}F_k$

$F_{k+2} \cdot F_k + (-1)^{k+2}$

$= F_k(F_{k+1} + F_k) + (-1)^{k+2}$

$= F_k \cdot F_{k+1} + F_k^2 + (-1)^{k+2}$

$= F_k(F_k + F_{k-1}) + F_k^2 + (-1)^{k+2}$

$= F_k^2 + F_kF_{k-1} + (F_k^2 - (-1)^{k-1})$

$= F_k^2 + F_kF_{k-1} + F_{k+1}F_{k-1}$

$= F_k^2 + F_kF_{k-1} + F_{k-1}(F_k + F_{k-1})$

$= F_k^2 + F_kF_{k-1} + F_{k-1}F_k + F_{k-1}^2$

$= F_k^2 + 2F_k F_{k-1} + F_{k-1}^2$

$= (F_k + F_{k-1})^2 = F_{k+1}^2$

(iv) If n is an integer multiple of m, F_n is integer multiple of F_m.

Base step:
$n = 6$, $m = 3$
$F_n = F_6 = 8$ $\quad F_m = F_3 = 2$

So, F_n is an integer multiple of F_m

Inductive hypothesis:
$m = k$ and $n = ka$, where $a \in Z^+$, $k > 0$
F_{ak} is an integer multiple of F_k

Inductive step:
Let $m = k+1$ and $n = (k+1)a = ka + a$

F_k is divisor of F_{ak}

So, F_{k+1} is divisor of $F_{(k+1)a} = F_{ka+a}$

So, F_{ka+a} is integer multiple of F_k.

Proved

8. Prove that sum of cubes of three successive numbers is divisible by 9