CHAPTER- 1

17/03/23

Matrices & Gaussian Elimination :-

→ To solve system of equations using normal elimination method and to know about the number of solutions.

1.1 (Introduction)

Eg- 1:- $x + 2y = 3$
$4x + 5y = 6$

$\Rightarrow y = \dfrac{3-x}{2}$ and $y = \dfrac{6-4x}{5}$

Equating both, we get :-

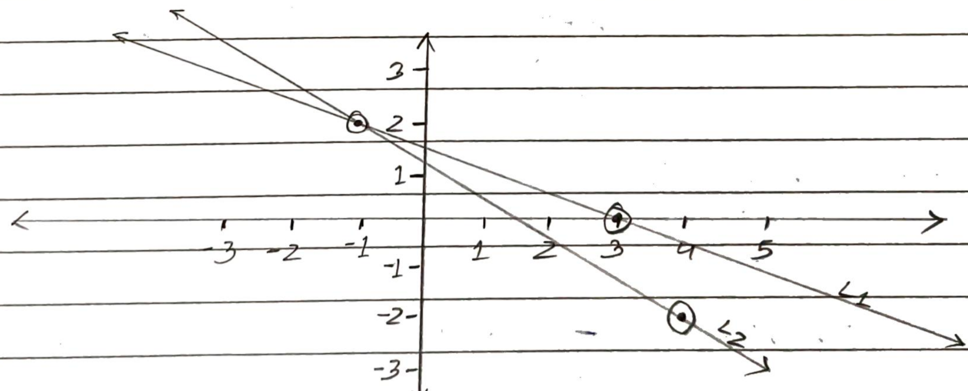
$\dfrac{3-x}{2} = \dfrac{6-4x}{5}$

$\Rightarrow 12 - 8x = 15 - 5x$
$\Rightarrow 3x = 3$
$\Rightarrow x = -1$

Putting $x = -1$
$\Downarrow$
$-1 + 2y = 3$
$2y = 4$
$y = 2$

$\therefore$ Both the lines intersect at $(-1, 2)$ and has an unique solution

$\boxed{\text{Unique solution}}$



From the above equation :-

$L_1 : x + 2y = 3 \Rightarrow (3,0), (-1,2)$
$L_2 : 4x + 5y = 6 \Rightarrow (-1,2), (4,-2)$

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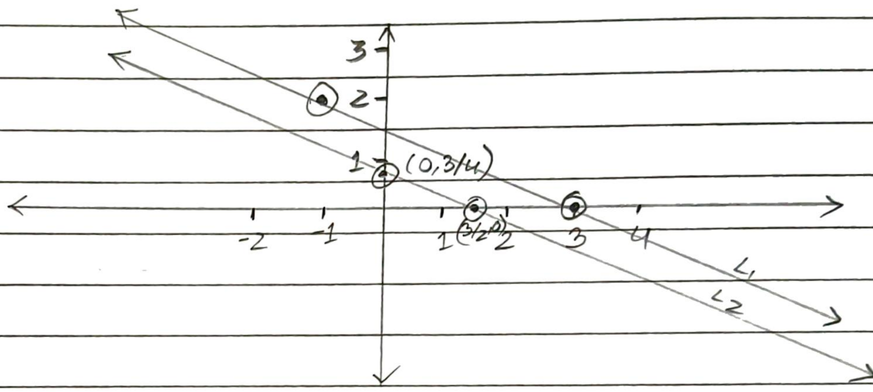
Eg:2 $x+2y=3$ ---- ①
$4x+8y=6$ ---- ②

Eqn ② $-4\times$ Eqn ① gives:-

$0=-6$ (which is not possible)

∴ The system has <u>no solution</u> ⇒ zero no. of solution

★ $\boxed{\text{NO solution}}$



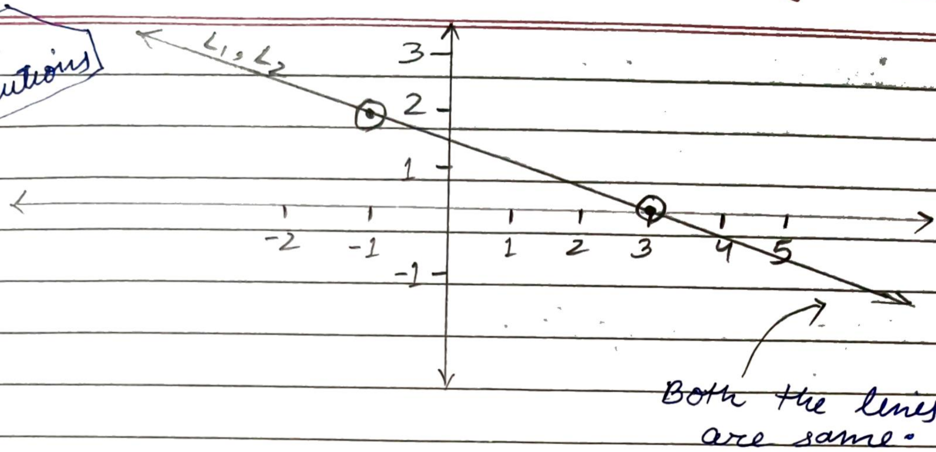
From the above equation,
L_1: $x+2y=3$ ⇒ $(3,0)$, $(-1,2)$
L_2: $4x+8y=6$ ⇒ $\left(\frac{3}{2},0\right)$, $\left(0,\frac{3}{4}\right)$

Eg:3:- $x+2y=3$ ---- ①
$4x+8y=12$ ---- ②

Equation ② $-$ $4\times$ Equation ① gives:-
$0=0$
(which is an identity)
[Both the lines are same / overlapping]

∴ The system has <u>infinite many solutions</u> or whole line of solution

* Infinite Many solutions



From the above equation,

$L_1 : x + 2y = 3 \Rightarrow (3,0) , (-1,2)$

$L_2 : 4x + 8y = 12 \Rightarrow (3,0) , (-1,2)$

[Infinite many solutions ~~to~~ other than these two]

In matrix form:-

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

$$[AX = B]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Coefficient Matrix — Solution Vector — Non-Homogenous Vector

LHS — RHS

Case - 1 : Non - Singular

If $\frac{a_{11}}{a_{21}} \neq \frac{a_{12}}{a_{22}}$ then $\Rightarrow$ Unique Soluti

Case - 2 : Singular

If $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}$

i) $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2}$ then $\Rightarrow$ Infinite Many Solutions

ii) $\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} \neq \frac{b_1}{b_2}$ then $\Rightarrow$ No solution or zero no. of solution

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1.2 :- The Geometry of Linear Equations :-

Eg :- $2x - y = 1$ ------ ①

$x + y = 5$ ------ ②

$\Rightarrow x = 5 - y$ $\Rightarrow$ Putting $x = 5 - y$ in ①

$\Downarrow$

$10 - 2y - y = 1$

$\Rightarrow y = 3$

$\therefore x = 2$

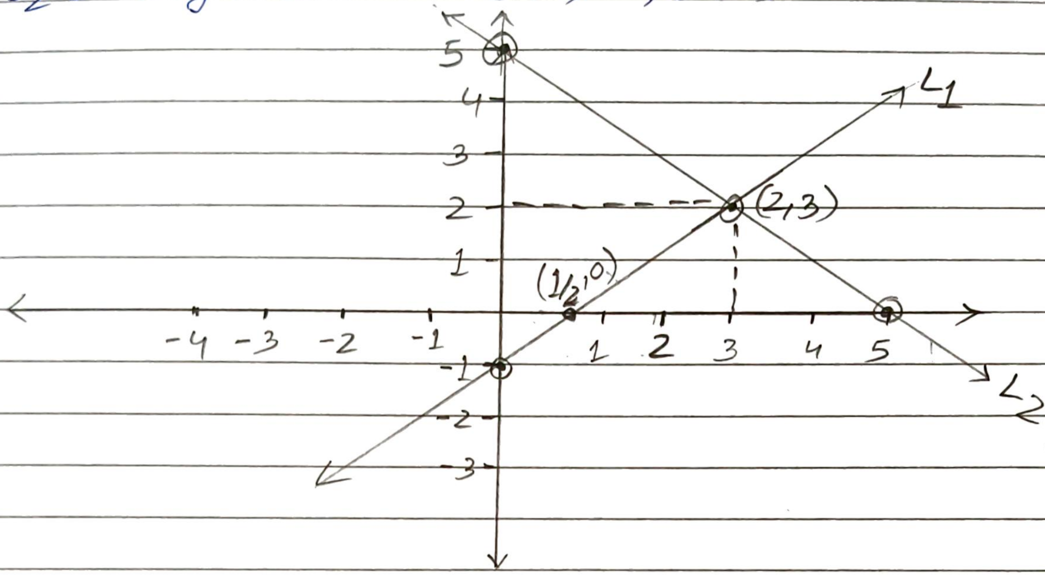
$\therefore$ The lines intersect at $x = 2$ & $y = 3$.

$(2, 3)$

Row Picture :-

$L_1 : 2x - y = 1 \longrightarrow (0,-1), (1/2, 0)$

$L_2 : x + y = 5 \longrightarrow (5,0), (0,5)$



Column Picture :-

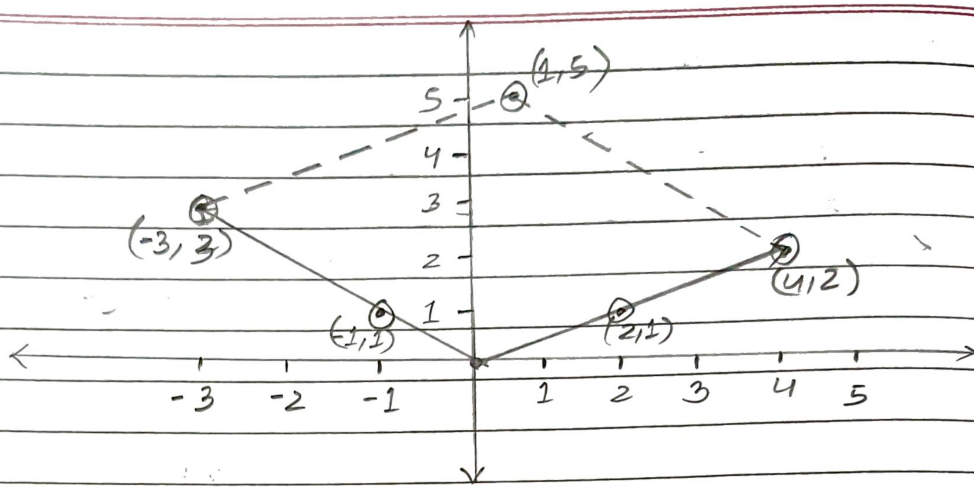
$$2x - y = 1$$
$$x + y = 5$$

$$\Rightarrow x\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + y\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\Rightarrow 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$ (which is true)



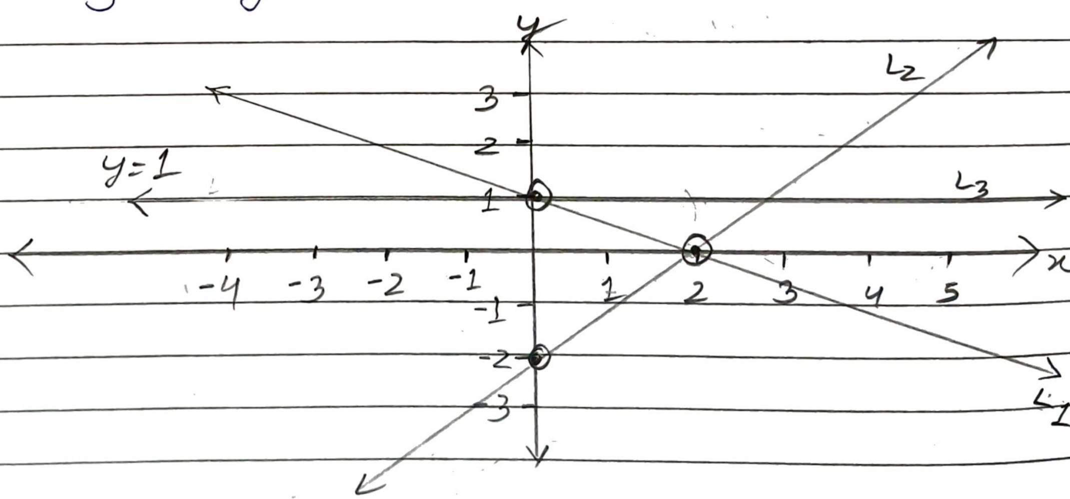
Problem Set 1.2 :-

Q.2 Given

$L_1 : x+2y=2 \Rightarrow (2,0), (0,1)$

$L_2 : x-y=2 \Rightarrow (2,0), (0,-2)$

$L_3 : y=1$



From the graph it is clear that the three lines are not passing through any point. So, the system is not solvable.

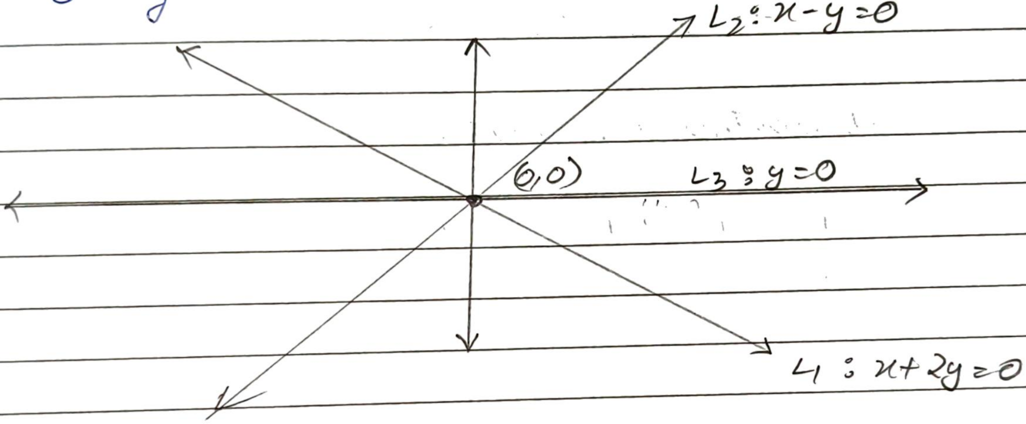
If all the right hand sides are zero, then the lines will pass through the origin and (0,0) will be the solution of the new system.

$\Rightarrow$ If all RHS are 0:-

$L_1: x+2y=0 \qquad (0,0)$

$L_2: x-y=0 \qquad \Rightarrow x=y=0 \quad (0,0)$

$L_3: y=0 \qquad (0,0)$

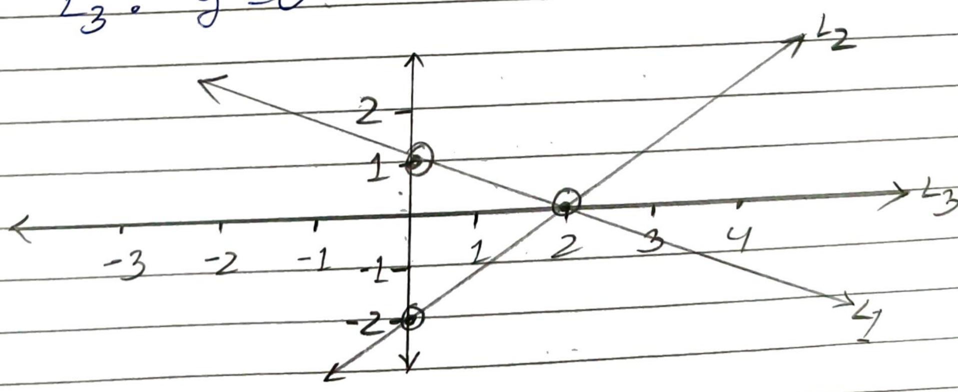


If the RHS is 2, 2, 0 :

$L_1: x+2y=2 \Rightarrow (2,0),\ (0,1)$

$L_2: x-y=2 \Rightarrow (2,0),\ (0,-2)$

$L_3: y=0$



Yes, the non-zero choice (2,2,0) of the RHS allows the three lines to intersect at common point (2,0).

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Q. 7. Given:

$$u + v + w = 2 \quad \text{------} \ (1)$$
$$u + 2v + 3w = 1 \quad \text{------} \ (2)$$
$$v + 2w = 0 \quad \text{------} \ (3)$$

(1) - (2) + (3) gives $0 = 1$ [which is not possible]

∴ → The system has no solution

→ The system is singular

Representing the equation in matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$R_1 + R_3 = R_2$ but $1 \neq 2 + 0$

∴ To make the system to have infinitely many solutions, we should replace 0 by -1 so that it will satisfy the condition $R_1 + R_3 = R_2$

∴ The new system is:

$$u + v + w = 2$$
$$u + 2v + 3w = 1$$
$$v + 2w = -1$$

$\Rightarrow v = -1 - 2w$

Let $w = 0$, then $v = -1$

$$u = 2 - v - w = 2 + 1 - 0$$
$$u = 3$$

$\therefore$ So, $(3,-1,0)$ is a solution of (u,v,w).

Q.8:- Points: $(0,y_1)$, $(1,y_2)$, $(2,y_3)$

The points lie on a straight line means the slope of the line joining $(0,y_1)$, $(1,y_2)$ and $(1,y_2)$, $(2,y_3)$ must be the same.

$$\Rightarrow \frac{y_2-y_1}{1-0} = \frac{y_3-y_2}{2-1}$$

$$\Rightarrow y_2-y_1 = y_3-y_2$$

$\Rightarrow y_1-2y_2+y_3=0$ [condition for the points to be in a straight line]

Q.11:- Column Picture:-

$$u\begin{bmatrix}1\\1\\0\end{bmatrix} + v\begin{bmatrix}1\\2\\1\end{bmatrix} + w\begin{bmatrix}1\\3\\2\end{bmatrix} = b$$

Given that b is the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$

Expressing the third column as a combination of first two.

$$x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\3\\2\end{bmatrix}$$

$-c_1 \qquad 2c_2 \qquad c_3$

$x + y = 1$

$x + 2y = 3$

$y = 2$

$\Rightarrow x = 1 - 2 = -1 \qquad \therefore x = -1, \; y = 2$

$\therefore$ The combination is :- $2C_2 - C_1 = C_3$

The equations are:-

$$u + v + w = 0$$
$$u + 2v + 3w = 0$$
$$v + 2w = 0$$
$$\Rightarrow v = -2w$$

$$u + (-2w) + w = 0$$
$$\Rightarrow u = w$$

$\therefore$ For every value of w the vector $(w, -2w, w)$ represents a solution of it.

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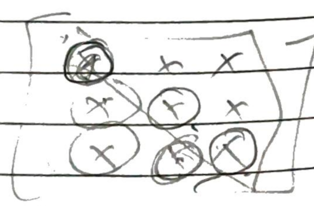
1.3 Gaussian Elimination :-

Ex-1

$2x - y = 1$

$x + y = 5$



The augmented matrix is :

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \left[\begin{array}{cc|c} \boxed{2} & -1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

$R_2 \leftarrow 2R_2 - R_1$

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Pivot

$$\approx \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 3 & 9 \end{array}\right]$$

$\Rightarrow$ $2x - y = 1 \longrightarrow \Rightarrow 2x - 3 = 1$

$3y = 9 \qquad \Rightarrow x = 2$

$\Rightarrow y = 3$

$\therefore$ So, the solution is $(2, 3)$

Ex-2

$2u + v + w = 5$

$4u - 6v = -2$

$-2u + 7v + 2w = 9$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

The augmented matrix is :-

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 + R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad R_3 \leftarrow R_3 + R_2$$

$$2u + v + w = 5$$
$$-8v - 2w = -12$$
$$w = 2$$

$$2u + v + w = 5$$
$$\Rightarrow 2u + 1 + 2 = 5$$
$$\Rightarrow u = 1$$

$$-8v - 2 \times 2 = -12$$
$$v = 1$$

$\therefore$ So, the solution is $(1, 1, 2)$

Ex-3

$$u + v + w = 1$$
$$2u + 2v + 5w = 2$$
$$4u + 6v + 8w = 6$$

The augmented matrix is:

$$\cancel{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 2 \\ 4 & 6 & 8 & 6 \end{array}\right]$$

Temporary Breakdown.

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{0} & 3 & 0 \\ 0 & 2 & 4 & 2 \end{array}\right] \qquad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$u + v + w = 1$$
$$2v + 4w = 2$$
$$3w = 0$$
$$\Rightarrow w = 0$$

$$2v + 4w = 2$$
$$2v = 2$$
$$v = 1$$

$$\cancel{u + 1 = 1}$$
$$\Rightarrow w$$

$u + 1 = 1$

$\Rightarrow u = 0$

$\therefore$ The solution is $(0, 1, 0)$

Ex-4

$u + v + w = 1$

$2u + 2v + 5w = 2$

$4u + 4v + 8w = 8$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 2 \\ 4 & 4 & 8 & 8 \end{array}\right]$$

Permanent Breakdown.

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

Using substitution we have:

$u + v + w = 1$

$3w = 0 \Rightarrow w = 0$

$4w = 4 \Rightarrow w = 1$

$\rightarrow$ Not possible

$\therefore$ So, the system has no solution.

Ex-5:- $u+v+w=1$

$2u+2v+5w=8$

$4u+4v+8w=12$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 5 & 8 \\ 4 & 4 & 8 & 12 \end{array}\right]$$

Permanent Breakdown.

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 4 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \end{array}$$

$u+v+w=1$

$3w=6 \Rightarrow w=2$

$4w=8 \Rightarrow w=2$

$u+v+w=1$

$\Rightarrow u+v=-1$

$\Rightarrow v=-1-u$

Let $u=1$, then $v=-2$

$\therefore$ So $(1,-2,2)$ is a solution and the system has infinite number of solutions.

The Breakdown of Elimination :-

If during Gaussian Elemination, a zero appears in pivot place, then there is a breakdown of the elimination. The breakdown may occur at the initial stage or at an intermediate stage during elimination
There are two types of breakdown:

1- Temporary Breakdown
2- Permanent Breakdown.

* Temporary Breakdown :-
In case of a temporary breakdown, the elimination algorithm needs repair. If we can repair i.e. to make the pivot place non-zero and will be able to get the full set of pivots then the breakdown is temporary.

* Permanent Breakdown :-
In case of a breakdown, if the elemination algorithm cannot be repairable, i.e. to the pivot place zero, we cannot make non-zero and will be not able to get full set of ~~piv~~ pivots then the breakdown is permanent.

Temporary Breakdown ⇒ The system is non-singuler and the solution is unique.

Permanent Breakdown ⇒ The system is singuler and the system has either no solution or infinite number of solutions.

Problem Set 1.3 :-

Q.4 :- $2x + 3y = 1$
$10x + 9y = 11$

The augmented matrix is :-

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 10 & 9 & 11 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -6 & 6 \end{array}\right] \qquad R_2 \leftarrow R_2 - 5R_1$$

$$2x + 3y = 1$$
$$-6y = 6$$
$$\Rightarrow y = -1$$

$$2x - 3 = 1 \quad \Rightarrow x = 2$$

The solution is $(2, -1)$

Q.8 $ax + 3y = -3$
$4x + 6y = 6$

$$\left[\begin{array}{cc|c} a & 3 & -3 \\ 4 & 6 & 6 \end{array}\right]$$

$$\approx \left[\begin{array}{cc|c} a & 3 & -3 \\ 0 & 6 - \frac{12}{a} & 6 + \frac{12}{a} \end{array}\right] \qquad R_2 \leftarrow R_2 - \frac{4}{a} R_1$$

For the elimination to breakdown permanently:-

$$6 - \frac{12}{a} = 0$$

$\Rightarrow$ $\underline{a = 2}$, elimination breaksdown permanently.

$\Rightarrow$ For $a = 0$, the elimination breaks down temporarily

$$\left[\begin{array}{cc|c} 0 & 3 & -3 \\ 4 & 6 & 6 \end{array}\right] \leftarrow \text{Temporary breakdown.}$$

Q. 14

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$3y + qz = t$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

$$\approx \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

The system is singular if

$$q + 4 = 0$$

$$\Rightarrow \boxed{q = -4}$$

The system has infinitely many solutions if :
$$t-5=0$$
$$\Rightarrow \boxed{t=5}$$

For the singular case,

$$3y-4z=5$$
$$\Rightarrow y=\frac{5+4z}{3}$$
$$\Rightarrow x=1+2z-\frac{4}{3}(5+4z)$$

Let $z=1$, Then $y=\frac{5+4}{3}=3$
$$x=1+2-\frac{4}{3}(5+4)$$
$$x=-9$$

$\therefore$ So, the solution is : $(-9, 3, 1)$

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Q-16

$$2x-y+z=0$$
$$2x-y+z=0$$
$$4x+y+z=2$$
$$\downarrow$$

The augmented matrix is :-

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 4 & 1 & 1 & 2 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

Here, the second pivot is missing hence the elimination breaks down

Interchanging row R_2 with R_3

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \leftarrow R_3 - 2R_1 \\ R_2 \leftrightarrow R_3 \end{array}$$

Here, the third pivot is missing. So, the breakdown is permanent.

Q.9 $Kx + 3y = 6$
$3x + Ky = -6$

The three numbers of K where elimination breaks down is :- $K = 0$

$K = 3$ $\qquad \frac{K}{3} = \frac{3}{K} \Rightarrow K^2 = 9$

$K = -3$ $\qquad \boxed{K = \pm 3}$

When $K = 0$

The augmented matrix is :-

$$\left[\begin{array}{cc|c} 0 & 3 & 6 \\ 3 & 0 & -6 \end{array}\right] \quad R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 3 & 0 & -6 \\ 0 & 3 & 6 \end{array}\right]$$

$\Rightarrow 3x = -6 \Rightarrow x = -2$

$3y = 6 \Rightarrow y = 2$

When $K = 3$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 3 & -6 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1 \Rightarrow \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 0 & -12 \end{array}\right]$$

Here $0 \neq -12$ (No. solution)

$K = -3$

$$\left[\begin{array}{cc|c} -3 & 3 & 6 \\ 3 & -3 & -6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 + R_1$$

$$3y - 3x = 6$$
$$y - x = 2$$
$$y = 2 + x$$

$(y-2, y)$ $y \in R$ [Infinite many solutions]
$(x, 2+x)$

Q.10 $x + by = 0$
$x - 2y - z = 0$
$y + z = 0$

$$\left(\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

Row exchange is possible if : $-2 - b = 0$

$\Rightarrow b = -2$

If to make the 2nd or 3rd pivot missing then :-

$$-2-b+1=0$$
$$\Rightarrow \boxed{b=-1}$$

Putting $b=-1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x-y=0$$
$$-y-z=0$$
$$\Rightarrow y=-z$$

$\therefore$ The ~~point~~ solution will be $(-z,-z,z)$

Q.12
$$2x+3y+z=8$$
$$4x+7y+5z=20$$
$$-2y+2z=0$$

The augmented matrix is :-

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \approx \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] \quad R_3 \leftarrow R_3 - 2R_2$$

$$\Downarrow$$

$2x+3y+z=8$	$z=1$	$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ unique solution
$y+3z=4$	$y=1$	
$8z=8$	$x=2$	

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1.4 Matrix Notation and Matrix Multiplication:-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \cdots\cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \cdots\cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \cdots\cdots a_{mn}x_n = b_m$$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$c_1x_1 + c_2x_2 + c_3x_3 = b_1$$

→ Diagonal Matrix :- $A_{n\times n}$ if $a_{ij} = 0 \quad \forall\, i \neq j$

Eg:- $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}_{3\times 3}$

→ Lower triangular Matrix:- if $a_{ij} = 0 \quad \forall\, i < j$

Eg:- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 15 & 18 & 9 \end{bmatrix}_{3\times 3}$

→ Upper triangular matrix:- if $a_{ij} = 0 \;\forall\, i > j$

Eg:- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 15 & 18 \\ 0 & 0 & 9 \end{bmatrix}$

→ Symmetric Matrix :- if $a_{ij} = a_{ji}$ $[A = A^T]$ $\forall$ i & j

Eg :- $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 15 & 18 \\ 3 & 18 & 9 \end{bmatrix}$

→ Skew - Symmetric Matrix :- if $a_{ij} = -a_{ji}$ $[A = -A^T]$ $\forall$ i & j

Eg :- $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 18 \\ 3 & -18 & 0 \end{bmatrix}$

Q- Compute the products :-

a → $\underset{3\times3}{\begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}} \times \underset{3\times1}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 4\times3 + 4\times0 + 5\times1 \\ 0\times3 + 1\times4 + 0\times5 \\ 4\times3 + 0\times4 + 1\times5 \end{bmatrix}} = \begin{bmatrix} 12+0+5 \\ 0+4+0 \\ 12+0+5 \end{bmatrix}$

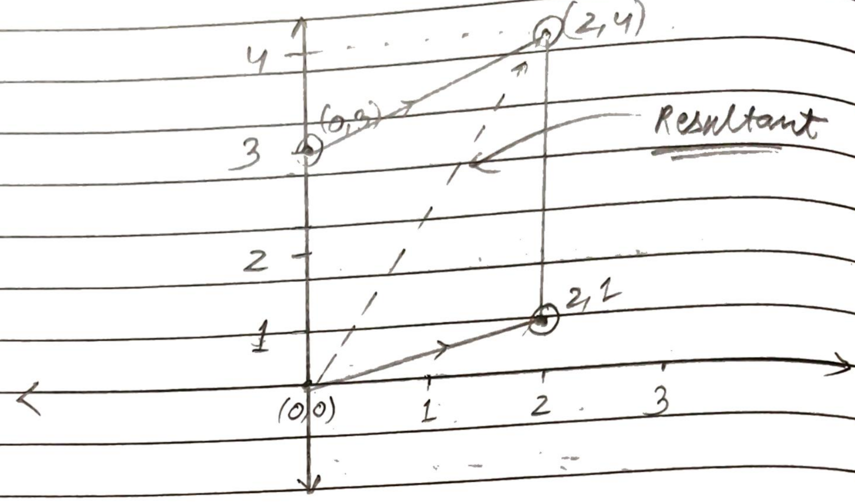
$= \underset{3\times1}{\begin{bmatrix} 17 \\ 4 \\ 17 \end{bmatrix}}$

b → $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$ → Same because it is multiplied with identity matrix.

Identity Matrix

c → $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Graphical representation of (c) :-



Q.21- $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Prove that: $A(\theta_1) \cdot A(\theta_2) = A(\theta_1 + \theta_2)$

$\Rightarrow A(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad A(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$\Rightarrow A(\theta_1) \cdot A(\theta_2) = \cancel{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \quad -\cos\theta_1 \sin\theta_2}$

$= \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1 \\ \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = A(\theta_1 + \theta_2)$

[Proved]

What is $A(\theta)$ times $A(-\theta)$?

$A(\theta) A(-\theta) = A(\theta - \theta) = A(0) = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Properties :-

$$A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$$

→ $AB \neq BA$ [Not commutative]
→ $(AB)C = A(BC)$ [Associative]
→ $A(B+C) = (A+B)C$ [Distributive]

Elementary Row operation :-

1) $R_i - KR_j \rightarrow R_i$

2) $R_i \longleftrightarrow R_j$

3) $i R_i \rightarrow R_i$

Elimination Matrices

Ex:- Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$-K = -2$

$K = 2$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1 \; (K_1 = 2)$
$R_3 \leftarrow R_3 + R_1 \; (K_2 = -1)$

$-K = 2$
$K = -2$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2 \; (K_3 = -1)$

$= U$

The Elemination Matrices are:-

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

In this example, 2,-1 and -1 are the multipliers of the 2nd row 1st column, 3rd row 1st column and 3rd row 2nd column places respectively.

We are writing the opposite value of ~~a~~ the ~~3rd order~~ multipliers in the respective places of a 3rd order identity matrise to get the elimination matrices.

* Note:- If a row operation is $R_i \leftarrow R_i - KR_j$ then 'K' is the multiplier for ith row and jth column place.

Q.27 $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$ $R_2 \leftarrow R_2 - 4R_1$ (4)
$R_3 \leftarrow R_3 + 2R_1$ (-2)

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad R_3 \leftarrow R_3 - 2R_2 \quad (2)$$

$= U$

The multipliers are:- 4, -2, 2

The elimination matrices are:-

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$\cancel{E_{21}\, E_{31}\, E_{32}}$

$E_{32}\, E_{31}\, E_{21}\, A = U$

$\Rightarrow \quad M A = U,$

Where $M = E_{32}\, E_{31}\, E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$

5/03/23

1.5 - Triangular Factorization and Row Exchanges

* Triangular Factorization:-

Given : $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 + R_1$ (2) & (-1)

$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 + R_2 (-1)$

$A = LU$

$= U$

The elementary matrices are:-

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$MA = U$

where $M = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$M^{-1} = L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$MA = U \Rightarrow A = M^{-1}U$

$\Rightarrow \boxed{A = LU}$ which is known as LU factorization ~~for~~ of the matrix A.

Eg:- Find the LU and LDU factorization of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol^n:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 3R_1$ (3)

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

$= U$

$EA = U$ $\quad A = E^{-1}U$ $\quad \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

LU-Factorization $\quad \boxed{A = LU}$

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} A = U$

$-2 \quad -2$

$LU = A$ $\quad A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

LDU - Factorization:-

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$L \times D \times \cancel{U} = A$

All diagonals are 1 in L & U
In D, write the diagonal of U and all other 0.
In L, write the k values accordingly.
In U, write the U but the diagonal should be 1

Eg:- Find the LU and LDU factorization of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

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Sol^n $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 2R_1$ (2)

$R_3 \leftarrow R_3 - 3R_1$ (3)

$\approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $R_3 \leftarrow R_3 - 2R_2$ (2)

$B = U$

$EA = U$

LU - Factorization :-

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\boxed{LU = A}$

LDU - Factorization :-

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{LDU = A}$

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Q- $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $R_2 \leftarrow R_2 - 3R_1$ (3)

$= U$

$EA = U$

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $\Rightarrow$ $A = E^{-1} U$

$= LU$

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$\boxed{LU = A}$

Q- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ $R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $R_3 \leftarrow R_3 - R_2$

$= U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{LU = A}$

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Q.7 $A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}$ $b = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$

$\approx \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$ $R_2 \leftarrow R_2 - 0R_1 (0)$
$R_3 \leftarrow R_3 - 3R_1 (3)$
$R_3 \leftarrow R_3 - 0R_2 (0)$

U

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ $c = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$A = LU = E_{31}^{-1} U$

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

$A = L \quad U$

Q.2 :- For any U to be non-singular, we need full set of pivot i.e. diagonal entries $\neq 0$.

Q.11 :-

(i)	(ii)	(iii)
$v - w = 2$	0	$v + w = 1$
$u - v = 2$	0	$u + v = 1$
$u - w = 2$	0	$u + w = 1$

i) $$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

$$\approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_1$$

$$\approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2$$

~~$0 \neq 2$ (No solution)~~

$0 \neq -2$ ~~(Infinite many solⁿ)~~

$0 \neq -2$ (No solution)

ii) $u = v = w$

$\Rightarrow$ solution matrix is : $\begin{bmatrix} w \\ w \\ w \end{bmatrix}$

iii) $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_1} = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \quad R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Non-singular
unique solution.

<u>Tridiagonal Matrix</u>:- All elements zero except on main diagonal & two adjacent diagonals

$$A = \begin{pmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+ \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

<u>Row Exchanges and Permutation Matrix</u>:-

During gaussian elimination, in case of breakdown problems, zero is appearing in the pivot place. To make that pivot place zero into non-zero, we are taking the help of row exchange.

For this row exchange, we will use permutation matrices.

Ex:- $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \times I = \begin{matrix} 2\,3 \\ 4\,5 \end{matrix} \quad \begin{matrix} 4\,5 \\ 2\,3 \end{matrix}$

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The 2nd order permutation matrices are :-

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = A$$

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

As we exchange the row in the ~~permutatio~~ identity matrix, the row in the matrix A gets exchanged if we multiply it.

Ex:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 5 & 8 \end{bmatrix} \quad R_2 \leftarrow R_2 - R_1$$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \quad R_3 \leftarrow R_3 - 2R_1$$

$\Downarrow R_2 \Leftrightarrow R_3$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Here $LU \neq A$

$\Rightarrow A$ has no LU Factorization

But ~~A~~ PA has LU Factorization, where the permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \quad (2) \\ R_3 \leftarrow R_3 - R_1 \quad (1) \end{matrix}$$

$= U$

LU Factorization :-

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boxed{LU = PA}$$

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1.6 Inverses and Transposes

Existence of Inverse :-

Inverse of a square matrix 'A' exists if it is non-singular. i.e $|A| \neq 0$. It is denoted by A^{-1}. If a square matrix has full set of pivots then it is also non-singular. So, inverse of a square matrix exists if it has full set of pivots.

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = 4 - 6 = -2 \neq 0$

Cofactor of $A = C = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$\boxed{C_{ij} = (-1)^{i+j}(\text{Det(Remaining)})}$

$(-1)^{1+1} (-1)^2 = 1$

$-1^{1+2} = -1$

Adjoint of $A = C^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$A^{-1} = \frac{Adj.A}{|A|} = \begin{bmatrix} 4/-2 & -2/-2 \\ -3/-2 & 1/-2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ or $\frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Ex. 2 - $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A| = 2(4-1) + 1(-2) = 6 - 2 = 4 \neq 0$

$\Rightarrow A^{-1}$ will exist.

Cofactor of $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$\text{Adj. } A = C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$A^{-1} = \dfrac{\text{Adj} A}{|A|} = \dfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

* Eg:- If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

Points to remember:-

1 - The inverse of a square matrix if exists is unique and is known as both sided inverse.

2 - $A A^{-1} = I = A^{-1} A$

3. $(AB)^{-1} = B^{-1}A^{-1}$, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

4. $Ax = b \Rightarrow x = A^{-1}b$

01/04/23 Gauss - Jordan Method : [The calculation of A^{-1}]

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$|A| = 4-6 = -2 \neq 0.$

$\Rightarrow A^{-1}$ will exist.

$[A \mid I]$

$= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$ $R_2 \leftarrow R_2 - 3R_1$

$= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right]$ $R_1 \leftarrow R_1 + R_2$

$= \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$ $R_2 \leftarrow -\frac{1}{2} R_2$

$= [I \mid A^{-1}]$

$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Ex-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$|A| = 1(6-4) - 1(3-2) + 1(2-2)$$
$$= 2 - 1 = 1 \neq 0$$

$\Rightarrow A^{-1}$ will exist

~~Minor of A = 3~~

$[A \mid I]$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$$= \left[I \mid A^{-1} \right]$$

where $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

03/04/23

Transpose :-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Symmetric matrix :- A square matrix is said to be symmetric if $A^T = A$

Skew - symmetric matrix :- A square matrix is said to be skew symmetric if $A^T = -A$

* If A is any real square matrix, then $A + A^T$ is always symmetric and $A - A^T$ is always skew symmetric.

Let $B = A + A^T$, then
$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$
$\Rightarrow B$ is symmetric

Let $C = A - A^T$. Then

$$C^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -C$$

$\Rightarrow C$ is skew symmetric

$$A = \underbrace{\frac{A+A^T}{2}}_{\text{Symmetric}} + \underbrace{\frac{A-A^T}{2}}_{\text{skew Symmetric}}$$

$\Rightarrow$ Any square matrix can be expressed as a sum of symmetric & skew symmetric matrix.

Eg:- Express the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as sum of symmetric and skew-symmetric matrix

Solⁿ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}}_{\text{Skew-Symmetric}}$$

Eg:- $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 12 & 18 \\ 5 & 18 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 3 & 5 \end{bmatrix}$ $R_2 \leftarrow R_2 - 3R_1$, $R_3 \leftarrow R_3 - 5R_1$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad R_3 \leftarrow R_3 - R_2 (1)$$

$$= U$$

LU - Factorization :-

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$LU = A$$

LDU - Factorization :-

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$LDU = A$$

Here, $U = L^T$ & $LDL^T = A$

In this example, since $A^T = A$, so $LDU = A$ is $LDL^T = A$

Note : If $A = A^T$ can be factored into $A = LDU$ without row exchanges, then U is the transpose of L and $A = LDL^T$.

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Points to remember :-

1. $(A+B)^T = A^T + B^T$
2. $(AB)^T = B^T A^T$, $(ABC)^T = C^T B^T A^T$
3. $(A^T)^T = A$
4. $(A^{-1})^T = (A^T)^{-1}$

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Problem set 1.6 :-

Q6 - a) If A is invertible and $AB = AC$, prove that $B = C$

Sol^n Proof :- Let A be invertible and $AB = AC$.

$$AB = AC$$
$$\Rightarrow A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C \quad \text{[By associative law]}$$
$$\Rightarrow IB = IC$$
$$\Rightarrow B = C$$

b) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ find an example with $AB = AC$ but $B \neq C$

Sol^n Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$AB = AC$ but $B \neq C$

Q.10 $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$[A_1 \mid I]$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_2 \leftarrow R_2 - R_1$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_2 \leftarrow R_2 - R_3$

$= [I \mid A_1^{-1}]$

$\therefore A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$A_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$[A_2 \mid I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + \frac{1}{2} R_1$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 + \frac{2}{3} R_2$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + \frac{3}{4} R_3$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 + \frac{2}{3} R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow 1/2 \; R_1 \\ R_2 \leftarrow 2/3 \; R_2 \\ R_3 \leftarrow 3/4 \; R_3 \end{array}$$

$$= \left[I \mid A_2^{-1} \right]$$

$$\therefore \quad A_2^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \left[A_3 \mid I \right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$$= [I \mid A_3^{-1}]$$

$$\therefore A_3^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Q.11 Let B be a square matrix

Let $A = B + B^T$ and $K = B - B^T$

$A^T = (B + B^T)^T = B^T + (B^T)^T = B^T + B = A$

$\Rightarrow A$ is symmetric

Again, $K^T = (B - B^T)^T = B^T - (B^T)^T = B^T - B$

$= -(B - B^T)$

$= -K$

$\Rightarrow K$ is skew-symmetric

$$A = B + B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

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$$K = B - B^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$B = \frac{B + B^T}{2} + \frac{B - B^T}{2}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{Skew-Symmetric}}$$

Q.15 $A = \begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}$

$|A| = c(0 - ef) = -cef$

A is singular if $|A| \neq 0$

$\Rightarrow -cef \neq 0$

$\Rightarrow cef \neq 0$

The required conditions for A to be invertible are :- [$a, b, c, d, e, f \in R$ such tha $c, e, f \neq 0$]

$B = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$,

$|B| = a(de - 0) - b(ce - 0) + 0 = e(ad - bc$

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B is invertible if $\Rightarrow |B| \neq 0$

$\Rightarrow \boxed{e(ad - bc) \neq 0}$,

$a, b, c, d, e \in R$

Q.17 a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore$ A+B is not invertible although A & B are invertible.

b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

A+B is invertible although A & B are not invertible.

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

All of A, B & A+B are invertible

Q.42

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

For $c = 0, 2, 7$, the matrix is not invertible as for these three values of c the determinant of the matrix is zero.

$c = 0 \Rightarrow$ zero column (or zero row)
$c = 2 \Rightarrow$ Identical rows
$c = 7 \Rightarrow$ Identical columns.