Q. If A is a hermitian matrix, then show that iA is a skew-hermitian matrix :-

Proof:- Let A be a Hermitian matrix.
Then $A^H = A$

$(iA)^H = \bar{i} A^H = -iA$

$\Rightarrow iA$ is a skew-Hermitian matrix.

Q. Express the following matrix as a sum of hermitian and skew-hermitian matrix.

$$A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$$

Solⁿ:- Given: $A = \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix}$

$$A^H = \bar{A}^T = \begin{bmatrix} 2-3i & 5 \\ 7i & 1+i \end{bmatrix}$$

$$A = \frac{A+A^H}{2} + \frac{A-A^H}{2}$$

$$\Rightarrow \begin{bmatrix} 2+3i & -7i \\ 5 & 1-i \end{bmatrix} = \begin{bmatrix} 2 & \frac{5}{2} - \frac{7}{5}i \\ \frac{5}{2} + \frac{7}{5}i & 1 \end{bmatrix} \Rightarrow \text{Hermitian}$$

$$+ \begin{bmatrix} 3i & \frac{-5}{2} - \frac{7}{2}i \\ \frac{5}{2} + \frac{7}{2}i & -i \end{bmatrix} \Rightarrow \text{Skew-Hermitian.}$$

CH-6

POSITIVE DEFINITE MATRICES

6.2 Test for Positive Definiteness :-

Positive Definite :-

Each of the following test is a necessary & sufficient condition for the real symmetric matrix A to be positive definite :-

Test-1 :- $x^T A x > 0$ for all nonzero real vector x

Test 2 :- All the eigenvalues of A are positive

Test III :- All the principal submatrices have +ve determinants

Test IV :- All the pivots (without row exchange) are positives

Eg:- Check the positive definiteness of the following matrix. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solⁿ Given $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow$ It is real symmetric

The principal submatrices are:-

$A_1 = [2]$, $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$|A_1| = 2$, $|A_2| = 4-1 = 3$, $|A_3| = |A|$
$= 2(4-1) + 1(-2-0)$
$= 6-2 = 4$

Since the determinants of all the principal submatrices are positive, so the given matrix A is positive definite.

Positive semidefinite :-
Each of the following test is a necessary and sufficient condition for a real symmetric matrix A to be positive semidefinite.

Test 1 :- $x^T A x \geq 0$ for all vectors x
Test 2 :- All the eigenvalues of A are either +ve or 0
Test 3 :- The determinant of all the principal submatrices are either positive or zero.
Test 4 :- No pivots are negative.

Eg :- ~~Ho~~ Test for positive semidefiniteness of the following matrix :

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solⁿ 1st method : (Test-III)

Given : $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Rightarrow$ It is a real symmetric.

The principal submatrices are:-
$A_1 = [2]$ $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $A_3 = A$

$|A_1| = 2$, $|A_2| = 4 - 1 = 3$, $|A_3| = |A| = 2(4-1) + 1(-2-1) - 1(1+2)$
$= 6 - 3 - 3 = 0$

Since the determinants of the principal submatrices are either positive or zero, so A is a positive semidefinite matrix.

2nd method : (Test-IV)

Given $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Rightarrow$ It is real symmetric

The principal submatrices are :- $A_1 = 2$, $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
$A_3 = A$

$|A_1| = 2$, $|A_2| = 3$, $|A_3| = |A| = 0$

The pivot elements of A are :-

$$d_1 = \frac{|A_1|}{|A_0|} = \frac{2}{1} = 2 \quad d_2 = \frac{|A_2|}{|A_1|} = \frac{3}{2}$$

$$d_3 = \frac{|A_3|}{|A_2|} = \frac{0}{3} = 0$$

Since no pivots are negative, i.e. the pivots are either positive or zero, so the matrix A is positive semidefinite.

Ex :- Check the positive definiteness of the following matrix :

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ It is real symmetric.

Solⁿ :- The principal submatrices are :-
$A_1 = [2]$, $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $A_3 = A$

$|A_1| = 2, \quad |A_2| = 3, \quad |A_3| = 4$

The pivot elements of A are :-

$d_1 = \frac{|A_1|}{|A_0|} = \frac{2}{1} = 2 \qquad d_2 = \frac{|A_2|}{|A_1|} = \frac{3}{2},$

$d_3 = \frac{|A_3|}{|A_2|} = \frac{4}{3}$

Since all the pivots are positive, so the matrix A is positive definite.

Q.34 i) Given : $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The principal submatrices are :-

$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad A_3 = A$

$|A_1| = 1 \quad |A_2| = 0 \quad |A_3| = |A| = 1(0-1) - 1(0-1) + 1(1-1)$

$= -1 + 1 + 0$

$= 0$

Since the determinants of the principal submatrices are 0 or +ve, so the given matrix A is +ve semidefinite.

ii) Given : $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

The principal submatrices are :-

$A_1 = [2], \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad A_3 = \cancel{A}\ B$

$|A_1| = 2 \quad |A_2| = 2 - 1 = 1 \quad |A_3| = |B| = 2(2-1) - 1(2-2) + 2(1-2)$

$= 2 - 0 - 2$

$= 0$

Since no pivots are negative, so it is positive semidefinite.

Problem Set 6.2 :-

Q.3: Let A and B be positive definite

Then $x^T A x > 0$ and $x^T B x > 0$ for any $x \neq 0$

Now, $x^T A x + x^T B x > 0$

$\Rightarrow x^T (A+B) x > 0$

$\Rightarrow A+B$ is positive definite

Q.11 Let $A = R^T R$

$|x^T A y|^2 = |x^T R^T R y|^2$

$= |(Rx)^T R y|^2 \leq \|Rx\|^2 \|Ry\|^2$

$= (Rx)^T Rx \cdot (Ry)^T Ry$

$= (x^T R^T R x)(y^T R^T R y)$

$= (x^T A x)(y^T A y)$

$\Rightarrow |x^T A y|^2 \leq (x^T A x)(y^T A y)$

Q.25 (a) $\cancel{f} = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3)$

The symmetric coefficient matrix is :-

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

The principal submatrices are :-

$A_1 = [2] \quad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad A_3 = A$

$|A_1| = 2$, $|A_2| = 3$, $|A_3| = 4$

The pivots are :-

$$d_1 = \frac{|A_1|}{|A_0|} = \frac{2}{1} = 2$$

$$d_2 = \frac{|A_2|}{|A_1|} = \frac{3}{2}$$

$$d_3 = \frac{|A_3|}{|A_2|} = \frac{4}{3}$$

Since all the pivots are positive, so the matrix A is positive definite.

(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$

The symmetric coefficient matrix is :-

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The principal submatrices are :-

$A_1 = [2]$, $A_2 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $A_3 = A$

$|A_1| = 2$ $|A_2| = 3$ $|A_3| = 0$

Since A is singular, so the matrix A is not positive definite

6.3 :- Singular Value Decomposition (SVD) :

Singular Value Decomposition :-

Any $m \times n$ matrix A can be factored into

$A = U \Sigma V^T$ = (orthogonal)(diagonal)(orthogonal)

where U is an $m \times m$ matrix whose columns are normalized form of the eigenvectors of AA^T, V is an $n \times n$ matrix whose columns are the normalized form of the eigenvectors of A^TA and Σ is a diagonal matrix whose 1st diagonal elements are the square root of the positive eigenvalues of both AA^T and A^TA

If $A = U \Sigma V^T$ is the singular value decomposition of A, then its pseudo-inverse is $A^+ = V \Sigma^+ U^T$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \rightarrow \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 1/\sigma_3 \end{bmatrix}$$

where none of σ_1, σ_2 and σ_3 is zero.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \rightarrow \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 \\ 0 & 1/\sigma_2 \\ 0 & 0 \end{bmatrix}$$

Q.14:- Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$|AA^T - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda(\lambda - 2) = 0$

$\Rightarrow \lambda = 2, 0$

For $\lambda = 2$: $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 1, \; x_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 0$:- $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0, \; x_2 = 1 \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 - 1 = 0$

$\lambda - 1 = \pm 1$

$\lambda = 1 \pm 1$

$= 2, 0$

For $\lambda = 2$:

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 1, \; x_2 \, 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

For $\lambda = 0$:-

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 1, \; x_2 = -1 \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

$V \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$

The pseudoinverse of A is:-

$A^+ = V \Sigma^+ U^T$

$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$

Q. Find the singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol^n Given : $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The eigenvalues of AA^T are 1 are 1
The eigenvectors of AA^T are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively.

$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The eigenvalues of A^TA are 1 and 1

The eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively

$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q. Find the singular value decomposition of the matrix :-
$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Sol^n Given : $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The eigenvalues of AA^T are 1 and 1
The eigenvectors of AA^T are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^TA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^TA - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$

$|A^TA - \lambda I| = 0$
$\Rightarrow (\lambda-1)(\lambda-1)(-\lambda) = 0$
$\Rightarrow \lambda = 1, 1, 0$ are eigenvalues of A^TA

For $\lambda = 1$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x_1\begin{bmatrix}0\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 1,\ x_2 = 0,\ x_3 = 0$ or $x_1 = 0, x_2 = 1, x_3 = 0$.

So, $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ are the eigenvectors for $\lambda = 1$

For $\lambda = 0$:-

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 0,\ x_2 = 0,\ x_3 = 1$

So, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is the eigenvector for $\lambda = 0$

$$V = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\Sigma = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}.$$

$$= \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$$

The pseudoinverse of A is $A^+ = V\Sigma^+U^T$

$$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}.$$

$$= \begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}$$

Q.3 Find the SVD and the pseudoinverse of the matrix $A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}$

Solⁿ $A^T = \begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}$ $AA^T = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$

$$\Rightarrow |AA^T - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)^2 - 1 = 0$$

$$\Rightarrow \lambda - 2 = \pm 1$$

$$\Rightarrow \lambda = 2 \pm 1$$

$= 3, 1$ are the eigenvalues of AA^T

For $\lambda = 3$:-

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = 1$$

So, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector of AA^T for $\lambda = 3$

For $\lambda = 1$:-

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 = 1, \quad x_2 = -1$$

So, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the eigenvector of AA^T for $\lambda = 1$.

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(1-\lambda) - 1] - 1(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[[(\lambda-2)(\lambda-1) - 1] - 1] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 3\lambda) = 0$$

$$\Rightarrow (\lambda-3)(\lambda-1)\lambda = 0$$

$\Rightarrow \lambda = 3, 1, 0$ are the eigenvalues of A^TA.

for $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\-1\\1\end{bmatrix}+x_3\begin{bmatrix}0\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 1, \quad x_2 = 2, \quad x_3 = 1$

So, $\begin{bmatrix}1\\2\\1\end{bmatrix}$ is the eigenvector of A^TA for $\lambda = 3$

For $\lambda = 1$

$$\begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}0\\1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\1\\1\end{bmatrix}+x_3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 1, \quad x_2 = 0, \quad x_3 = -1$

So, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ is the eigenvector for A^TA for $\lambda = 1$

For $\lambda = 0$:-

$$\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_1\begin{bmatrix}1\\1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\\1\end{bmatrix}+x_3\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1 = 1, \quad x_2 = -1, \quad x_3 = 1$

So, $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is the eigenvector of A^TA for $\lambda = 0$

$$V = \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{6}\\ 2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \times V^T$$

$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & \sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\end{bmatrix} = A$$

The pseudoinverse of A is:-

$$A^+ = V\Sigma^+U^T$$

$$= \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\\ 2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}1/\sqrt{3} & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

$$= \begin{bmatrix}1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\\ 2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}1/\sqrt{6} & 1/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}2/3 & -1/3\\ 1/3 & 1/3\\ -1/3 & 2/3\end{bmatrix}$$

Q- Find the SVD and the pseudoinverse of the matrix $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Sol^n Given: $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$|AA^T - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda-2)^2 - 1 = 0$

$\Rightarrow \lambda - 2 = \pm 1$

$\Rightarrow \lambda = 2 \pm 1$

$= 3, 1$ are the eigenvalues of AA^T

For $\lambda = 3$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = -1, \ x_2 = 1$

So, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the eigenvector of AA^T for $\lambda = 3$

for $\lambda = 1$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 1, \ x_2 = 1$

So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector of AA^T for $\lambda = 1$

$U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$|A^TA - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]-1(1-\lambda)=0$$
$$\Rightarrow (1-\lambda)\left[(\lambda-2)(\lambda-1)-1\right]-1(1-\lambda)=0$$
$$\Rightarrow (1-\lambda)(\lambda^2-3\lambda)=0$$
$$\Rightarrow (\lambda-3)(\lambda-1)\lambda=0$$
$\Rightarrow \lambda = 3,1,0$ are the eigenvalues of A^TA

For $\lambda=3$:-
$$\begin{bmatrix}-2 & -1 & 0\\ -1 & -1 & -1\\ 0 & -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\Rightarrow x_1\begin{bmatrix}-2\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}-1\\-1\\-1\end{bmatrix}+x_3\begin{bmatrix}0\\-1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$\Rightarrow x_1 = 1,\ x_2=-2,\ x_3=1$

$\Rightarrow$ So, $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is the eigenvector of A^TA for $\lambda=3$

For $\lambda=1$:-
$$\begin{bmatrix}0 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\Rightarrow x_1\begin{bmatrix}0\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}-1\\1\\-1\end{bmatrix}+x_3\begin{bmatrix}0\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$\Rightarrow x_1=-1,\ x_2=0,\ x_3=1$

$\Rightarrow$ So, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ is the eigenvector of A^TA for $\lambda=1$

For $\lambda=0$:-
$$\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$\Rightarrow x_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}-1\\2\\-1\end{bmatrix}+x_3\begin{bmatrix}0\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$\Rightarrow x_1=1,\ x_2=1,\ x_3=1$

So, $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is the eigenvector of A^TA for $\lambda=0$

$$V=\begin{bmatrix}1/\sqrt6 & -1/\sqrt2 & 1/\sqrt3\\ -2/\sqrt6 & 0 & 1/\sqrt3\\ 1/\sqrt6 & 1/\sqrt2 & 1/\sqrt3\end{bmatrix}$$

$$\Sigma=\begin{bmatrix}\sqrt3 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$U\Sigma V^T=\begin{bmatrix}-1/\sqrt2 & 1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}\begin{bmatrix}\sqrt3 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt6 & -2/\sqrt6 & 1/\sqrt6\\ -1/\sqrt2 & 0 & 1/\sqrt2\\ 1/\sqrt3 & 1/\sqrt3 & 1/\sqrt3\end{bmatrix}$$
$$=\begin{bmatrix}-1/\sqrt2 & 1/\sqrt2\\ 1/\sqrt2 & 1/\sqrt2\end{bmatrix}\begin{bmatrix}1/\sqrt2 & -\sqrt2 & 1/\sqrt2\\ -1/\sqrt2 & 0 & 1/\sqrt2\end{bmatrix}$$
$$=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}=A$$

The pseudoinverse of A is :- $A^+ = V\Sigma^+U^T$

$$= \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{18} & -1/\sqrt{2} \\ -2/\sqrt{18} & 0 \\ 1/\sqrt{18} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix}$$