Matrices & Gaussian Elimination

Chapter - 1

1.2 The geometry of Linear Equations:-

Q- $x+2y=3$
$4x+5y=6$

$4(x+2y=3) \Rightarrow$ $4x+8y=12$
$4x+5y=6$

$x=-1$ $3y=6 \Rightarrow y=2$.

$\therefore$ Unique solution.

Q- $x+2y=3$
$4x+8y=6$ $4x+8y=12$
$4x+8y=6$

$\therefore$ No solution $0=6$

Q- $x+2y=3$
$4x+8y=12$

$0=0$

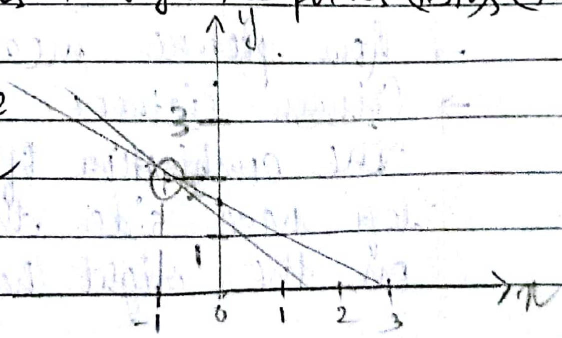
$\therefore$ Infinitely many solutions.

Geometrical Concept

$x+2y=3$
$4x+5y=6$

The line eqn 1 passes through the points $(3,0)$ & $(0,1.5)$
The line eqn 2 passes through the points $(1.5,0)$ & $(0,1.2)$

Since, the lines are intersecting, hence unique solution



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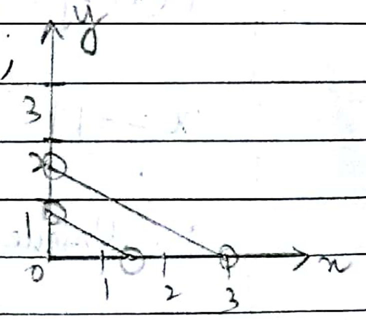
Q- $x+2y=3$
$4x+8y=6$

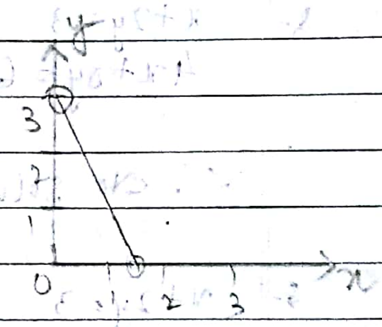
The line eq^n ① passes through points $(3,0)$ & $(0,1.5)$
The line eq^n ⓘⓘ passes through points $(0,0.75)$, $(1.5,0)$

Since the lines are parallel; hence no solution.



Q- $x+2y=3$
$4x+8y=12$
Line 1 :- $(3,0)$ & $(0,1.5)$
Line 2 :- $(0,1.5)$ & $(3,0)$

The two lines coincide each other. Hence, the given system has infinite no of solutions.



NOTE :- Consistent → solution exists → unique / infinitely many solutions

Inconsistent → no solution.

Methods to Solve System of Equations
→ Row Picture
→ Column Picture

→ Row picture means intersection of lines.
→ Column Picture :-
The combination of the column vectors on the left hand side that produces the vector on the right hand side.

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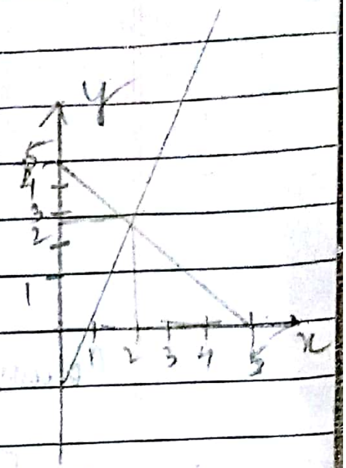
Q- Solve using Row & Column Picture.

$2x - y = 1$

$x + y = 5$



Points of eqn (1) are $(0,-1)$ & $(0.5, 0)$

Points of eqn (2) are $(5,0)$ & $(0,5)$

$x = 2,\ y = 3$

∴ Unique solution

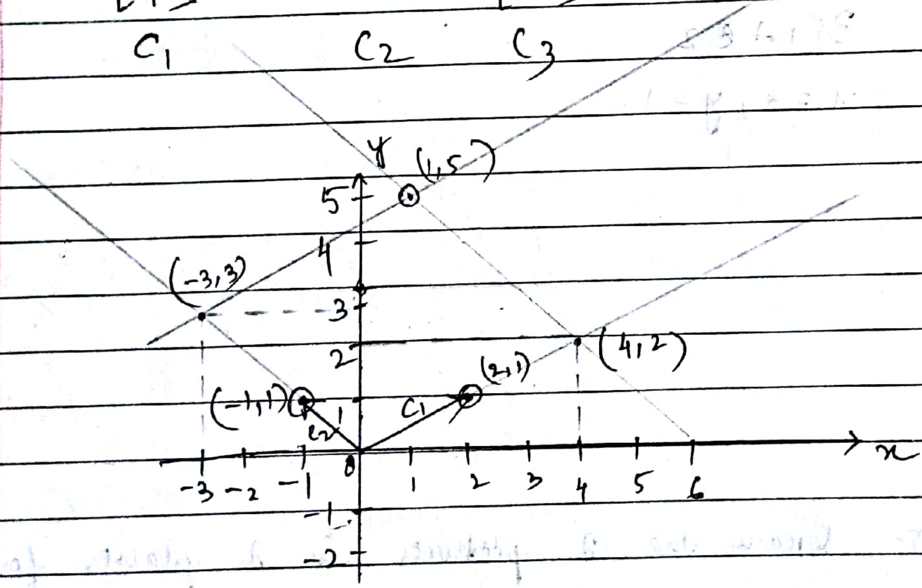
System is consistent.

(ii) Column Picture

The column form of the given system is:-

$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

$C_1 \qquad C_2 \qquad C_3$



$2C_1 + 3C_2$

$\Rightarrow x = 2$ & $y = 3$

1.2

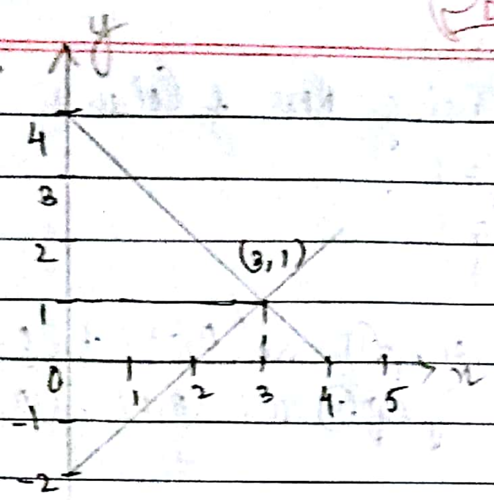
Q3 $x + y = 4$

$2x - 2y = 4$. Draw row & column picture.

Row:-

points on line 1 are $(4,0)$ & $(0,4)$

points on line 2 are $(2,0)$ & $(0,-2)$

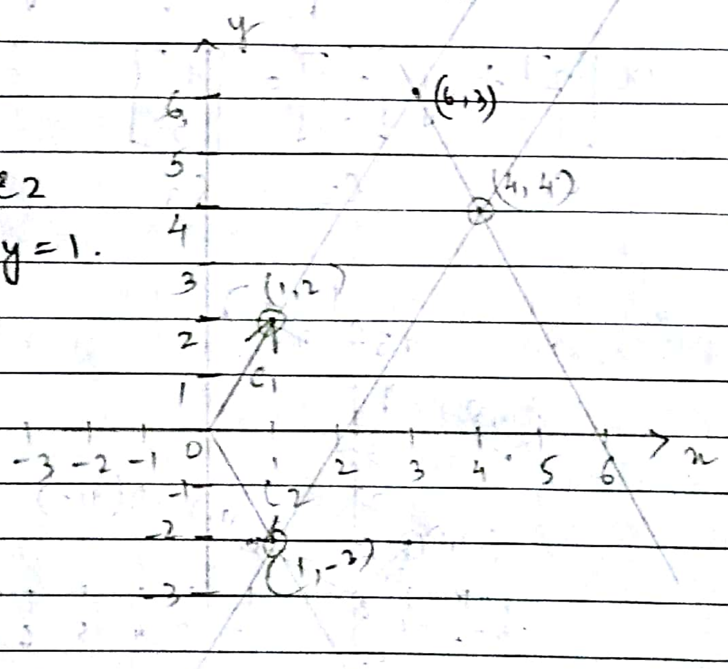


Column Picture:-

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}4\\4\end{bmatrix}$$

$3C_1 + C_2$

$x=3, y=1.$



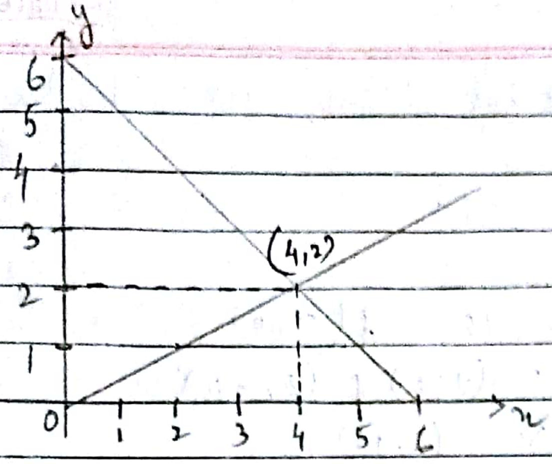
Q15- Draw the 2 pictures in 2 planes for:-

$x-2y=0$
$x+y=6$

Row Picture:-

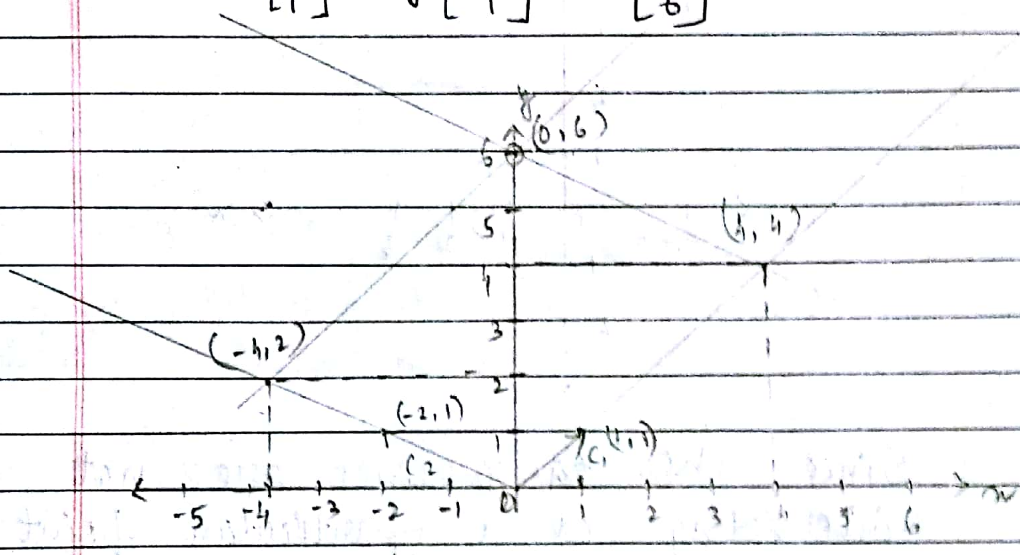
Line:- (0,0) & (2,1)

Line 2:- (6,0) & (0,6)



Column Picture :-

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$$



$4C_1 + 2C_2$

$x = 4$ & $y = 2$

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Singular < no solution / infinitely many solutions

Non Singular → unique solution.

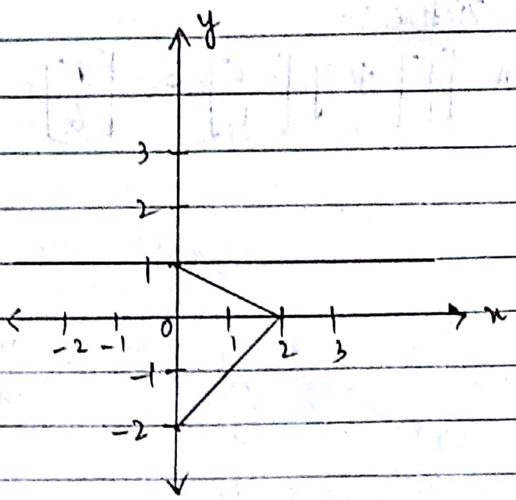
Q2 i) Sketch the three lines & decide if the equations are solvable.

ii) What happens if all right hand sides are (

iii) Is there any non-zero choice of right hand side that allows the 3 lines to intersect at the same point?

$x + 2y = 2$
$x - y = 2$
$y = 1$

i) Line 1:- $(0,1)$ & $(2,0)$
Line 2:- $(2,0)$ & $(0,-2)$
Line 3:- $(0,1)$



Since the ~~eq~~ 3 lines are not intersecting at a particular point, so the equations are not solvable.

ii) $x + 2y = 0$
$x - y = 0$
$y = 0$

If the right hand side of the system are all zero, then the solution is $x = 0$ & $y = 0$.

iii) If the right hand side of the first equation is 5, then all the three lines intersect at a point.

Q8- Under what condition on y_1, y_2, y_3 so that the points $(0, y_1)$, $(1, y_2)$, $(2, y_3)$ lie on a straight line?
a b c

The three points a, b, c lie on a straight line if slope of AB = slope of BC.

$$\Rightarrow \frac{y_2 - y_1}{1-0} = \frac{y_3 - y_2}{2-1}$$

$$\Rightarrow y_2 - y_1 = y_3 - y_2$$

$$\Rightarrow 2y_2 - y_1 - y_3 = 0.$$

$$\Rightarrow y_1 - 2y_2 + y_3 = 0.$$

Q7 i) Explain why the system is singular by finding a combination of the 3 equations that adds upto 0 = 1.

ii) What value should replace the last zero on the right side to allow the equations to have solutions?

$u + v + w = 2$ — 1

$u + 2v + 3w = 1$ — 2

$v + 2w = 0$ — 3

i) Here, by taking the combination of all equation i.e., 1 - 2 + 3, we get 0 = 1. Since, it is impossible, so it has no solutions. Hence, the system is singular.

ii) If the RHS of eqⁿ 3 is -1 i.e., $v + 2w = -1$, then by taking the combination 1-2+3, we have LHS = 0 & RHS = 0 i.e., infinitely many solutions. Hence, the system is singular.

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Q1 - i) Show that for the above system, the 3 columns on the left lie in the same plane by expressing the third column as a combination of the first two.

ii) What are the solutions, if the right hand side of the system is the 0 vector.

Ans

$u + v + w = 2$ — (i)

$u + 2v + 3w = 1$ — (ii)

$v + 2w = 0$ — (iii)

i)
$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$
$C_1 \quad C_2 \quad C_3 \quad C_4$

$$2C_2 - C_1 = C_3$$

$\therefore$ The three columns C_1, C_2, C_3 lie on a same plane.

ii) If the right hand side of the system is a zero vector then the column form is :-

$$u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + w \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2C_2 - C_1 - C_3 = C_4$$

By taking the above combination, the solution exists.

$u = 1, v = 2, w = -1$

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1.3 Gauss Elimination

Step 1:- Let us consider the system of equation in matrix form i.e., $Ax = b$.

Step 2:- Consider the matrix $[A \; b]$

Step 3:- Use elementary row operation to convert the coefficient matrix as upper triangular form by using pivot element.

Step 4:- Use Back Substitution method to obtain the required solution for the given system.

Upper Triangular Matrix

If all the elements below the diagonal are 0.

Ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Lower Triangular Matrix

If All the elements above the principal diagonal are 0.

Ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

Pivot Element

→ These elements cannot be 0.

→ Generally, the pivot position are the diagonal elements i.e., $a_{11}, a_{22}, a_{33}, \ldots$ etc.

Let the system of equation is ...

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

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The matrix form of the above system is $Ax = b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$$

Consider the matrix $[A \ b]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Here we have to eliminate a_{21}, a_{31}, a_{32}

Q Solve the system of eqn's by using Gauss Elimination Method.

$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$

Ans The given system of eqn's in matrix form is:-

$$Ax = b$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

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Consider the matrix $[A \cdot b]$ i.e.,

pivot element

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}_{3\times 4} \begin{array}{l} R_1 \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \rightarrow \cancel{R_3 - 8R_1} \rightarrow R_3 + R_2 \end{array}$$

pivot element

$$= \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

pivot element.

From the above matrix form, the corresponding eqn's are:-

$2u + v + w = 5$

$-8v - 2w = -12$

$w = 2$

$2u + 1 + 2 = 5$

$-8v - 4 = -12$

$2u + 3 = 5$

$\cancel{-8} v = \cancel{-8} \Rightarrow \boxed{v = 1}$

$\boxed{u = 1}$

1.3

Q32- Use elimination to solve :-

$u + v + w = 6$

$u + 2v + 2w = 11$

$2u + 3v - 4w = 3$

Ans

$Ax = b$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 11 \\ 3 \end{bmatrix}$$

Consider the matrix $[A \ b]$ i.e.,

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$$\left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 1 & 2 & 2 & 11 \\ 2 & 3 & -4 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & -6 & -9 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_2 - R_3 \end{array}$$

$$= \left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 7 & 14 \end{array}\right]$$

The corresponding eqn's are :-

$u + v + w = 6$

$v + w = 5$

$7w = 14$

$w = 2 \quad \Rightarrow \quad v = 3$

$u + 5 = 6$

$u = 1$

$\therefore u = 1$

$v = 3$

$w = 2$

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Special Cases in Gauss Elimination Method

$[A \quad b]$

i.e., $\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$

→ If $a_{33} = 0$ but $b_3 \neq 0$, then no solution.

→ If $a_{33} = b_3 = 0$, then infinitely many solutions.

→ If $a_{33} \neq 0$ but $b_3 = 0$, then unique solution.

→ If a_{33} & $b_3 \neq 0$, then unique solution.

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Q13 Explain whether the system is singular or non-singular.

$$2x - 3y = 3$$
$$4x - 5y + z = 7$$
$$2x - y - 3z = 5$$

Ans

$$Ax = b$$

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -5 & 2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$= \cancel{\begin{array}{cccc} 2 & -1 & 0 & 8 \\ 0 & -3 & 0 & 1 \end{array}}$$

$$= \left[\begin{array}{cccc} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

Since $a_{33} \neq 0$ and $b_3 = 0$, hence unique solution.

$$\boxed{z = 0}$$

$$y + z = 1 \Rightarrow y = 1$$
$$2x - 3y = 3$$
$$2x = 6 \Rightarrow x = 3.$$

$\therefore$ Hence, the system is non-singular.

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Q1 Choose a right hand side which gives no solution & another right hand side which gives infinitely many solutions. What are 2 of these solutions.

$$3x+2y=10$$
$$6x+4y=a$$

$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 10 \\ a \end{bmatrix}$

$\begin{bmatrix} 3 & 2 & 10 \\ 6 & 4 & a \end{bmatrix}$ R_1, $R_2 \to R_2 - 2R_1$

$= \begin{bmatrix} 3 & 2 & 10 \\ 0 & 0 & a-20 \end{bmatrix}$

$a-20=0$

ii) $\Rightarrow$ $\boxed{a=20}$ → then infinitely many solutions

i.e., when $a-20 \neq 0$,

i) When $a \neq 20$, then no solution.

iii) $\begin{bmatrix} 3 & 2 & 10 \\ 0 & 0 & 0 \end{bmatrix}$ Here, we get the solutions.

The eqn's are :-

$3x+2y=10$.

$x = \dfrac{10-2y}{3}$

When $y=2$, $x=2$.

When $y=\frac{1}{2}$, $x=3$.

Q3 i) Choose a coefficient b that makes the system singular.

ii) Choose a right hand side g that makes it solvable.

iii) Find 2 solutions in that singular case

$2x + by = 16$
$4x + 8y = g$

$A = \begin{bmatrix} 2 & b \\ 4 & 8 \end{bmatrix}$ $\quad b = \begin{bmatrix} 16 \\ g \end{bmatrix}$

$= \left[\begin{array}{ccc} 2 & b & 16 \\ 4 & 8 & g \end{array}\right]$ $\begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$

$= \left[\begin{array}{ccc} 2 & b & 16 \\ 0 & 8-2b & g-32 \end{array}\right]$

i) $8 - 2b = 0$
$\Rightarrow 2b = 8 \Rightarrow \boxed{b = 4}$ → The system is singular if $b = 4$

ii) $g - 32 = 0$
$\boxed{g = 32}$ → The system is solvable if $g = 32$

iii) Eq^n's are :
$2x + 4y = 16$
~~$4x + 8y = 32$~~
$\cancel{2}(x + 2y) = 16\,8$
$x = 8 - 2y$ When $y = 0, x = 8$
$y = 1, x = 6$

Q14- i) Which number q makes the system singular & which right hand side t gives infinitely many solutions ii) Find the solution that has $z = 1$

$x + 4y - 2z = 1$
$x + 7y - 6z = 6$
$3y + qz = t$

$-2q+16$

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$$A = \begin{bmatrix} 1 & 4 & -2 \\ 1 & 7 & -6 \\ 0 & 3 & q \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 6 \\ t \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \left[\begin{array}{cccc} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q-4 & t-5 \end{array}\right]$$

~~$16 - 2q = 0$~~
~~$\Rightarrow 2q = 16$~~
~~$\Rightarrow \boxed{q = 8}$~~

~~$t - 20 = 0$~~
~~$\Rightarrow \boxed{t = 20}$~~

$q - 4 = 0$
$\boxed{q = 4}$

$t - 5 = 0$
$\boxed{t = 5}$.

~~$\therefore \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$~~

~~The eqn's are:-~~
~~$x + 4y - 2z = 1$~~
~~$3y - 4z = 5$~~

$$\begin{matrix} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{matrix}$$

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Formula for the ERO :-

elimination row $\rightarrow$ $\boxed{\text{elimination row} - \dfrac{\text{eli. element} * \text{Pivot row}}{\text{pivot element}}}$

The Breakdown of Elimination

→ If a zero appears in the pivot position, then elimination breakdown temporarily or permanently.

→ If we can avoid it by interchanging the rows to obtain a pivot, then temporarily breakdown; otherwise permanently breakdown.

Breakdown
- Temporarily → Non-singular → Unique solution
- Permanently → singular
  - no solution
  - infinitely many solutions.

1.3 Q8 For which number, a does elimination breakdown?
i) temporarily
ii) permanently

$$ax + 3y = -3$$
$$4x + 6y = 6$$

$$A = \begin{bmatrix} a & 3 \\ 4 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

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$$A = \begin{bmatrix} a & 3 & -3 \\ 4 & 6 & 6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to aR_2 - 4R_1 \end{matrix}$$

$$A = \begin{bmatrix} a & 3 & -3 \\ 0 & 6a-12 & 6a+12 \end{bmatrix}$$

Case 1 :-

If $a=0$, then elimination breakdown temporarily

If $a=0$ $\begin{bmatrix} 0 & 3 & -3 \\ 4 & 6 & 6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$.

By interchanging R_1 & R_2, we have

$$\begin{bmatrix} 4 & 6 & 6 \\ 0 & 3 & -3 \end{bmatrix}$$

$4x + 6y = 6$

$3y = -3 \Rightarrow y = -1$.

$4x - 6 = 6 \Rightarrow x = 3$.

The system is non-singular.

Case 2 :-

If $a \neq 0$,

$$A = \begin{bmatrix} a & 3 & -3 \\ 0 & 6a-12 & 6a+12 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -3 \\ 0 & 0 & 12 \end{bmatrix}$$

$6a - 12 = 0$

$\Rightarrow 6a = 12$

$\Rightarrow \boxed{a=2}$.

The elimination breakdown permanently if $6a-12=0$ i.e., $a=2$.

Assgmt 1.3 ~~1~~, ~~3~~, 4, 7, ~~8~~, 9, 10, 12, ~~14~~, 16, ~~32~~

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Q Write the pivot elements for the matrix :-

i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix}$

Ans i) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

$\therefore$ 3 pivot elements.

ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to \\ R_3 \to R_3 - \frac{4}{3} R_2 \end{matrix}$

$\therefore$ ~~Only~~ 2 pivot elements i.e., 1 & 3.

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

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1.4 Matrix Notation and Matrix Multiplication

Let $A_{m\times n}$ and $B_{n\times r}$ are two matrices, then the product is of order $m \times r$ i.e., $AB_{m\times r}$

Multiplication by rows :-

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(1)+0(0)+2(-1) \\ 3(1)+1(0)+(-1)(-1) \\ 3(1)+0(0)+1(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Multiplication by column.

$$AB = 1 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Properties

→ Matrix multiplication is not commutative ($AB \neq BA$)

→ Matrix multiplication is associative i.e., $A(BC) = (AB)C$

→ Matrix operation is distributive i.e., $A(B+C) = AB + AC$

Q19. Find the power matrices $A, A^2, A^3, \dots, A^k$

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

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$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} & \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\ \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} & \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\therefore \quad A^K = A.$$

Q21 The matrix that rotates the x-y plane by an angle θ is $A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Verify that :-

i) $A(\theta_1) \cancel{\times} A(\theta_2) = A(\theta_1 + \theta_2)$.

ii) What is $A(\theta)$ times $A(-\theta)$.

Ans i) $A(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$ $\quad A(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$

$$A(\theta_1) + A(\theta_2) = \begin{bmatrix} \cos\theta_1 + \cos\theta_2 & -(\sin\theta_1 + \sin\theta_2) \\ \sin\theta_1 + \sin\theta_2 & \cos\theta_1 + \cos\theta_2 \end{bmatrix}$$

$$\cancel{A(\theta_1+\theta_2)}\ A(\theta_1) * A(\theta_2) = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\cos\theta_1\sin\theta_2 + \sin\theta_1 + \cos\theta_2) \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{bmatrix}$$

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$$A(\theta_1+\theta_2) = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$\therefore$ LHS = RHS (verified).

ii) $$A(\theta)\,A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q56 Find all matrices that satisfies

$$A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A.$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \text{ --- } (1)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \cancel{a+b} \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} \text{ --- } (2).$$

$$\cancel{a}+b = \cancel{a}+c$$
$$b = c.$$

$$a+\cancel{b} = \cancel{b}+d$$
$$a = d$$

The matrices which satisfy the given cond^n $AB = BA$ are:-

$\therefore a = d$ & $b = c$

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

* A matrix 'A' is said to be diagonal if $a_{ij} = 0$ for $i \neq j$

Ex:- $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Identity Matrix

$$a_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Ex:- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Upper Triangular Matrix

$$a_{ij} = 0 \text{ for } i > j$$

Ex:- $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

Lower Triangular Matrix

$$a_{ij} = 0 \text{ for } i < j$$

Ex:- $\begin{bmatrix} 6 & 0 \\ 7 & 9 \end{bmatrix}$

Elementary Matrix

An elementary matrix is a matrix which can be obtained by subtracting 'l' times row j from row i i.e., the matrix E_{ij}

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24/8/17.

1.3 Q- For which 3 numbers K, elimination breakdown which is fixed by a row exchange? In each case the no. of solutions 0, 1 &

$$Kx + 3y = 6$$
$$3x + Ky = -6.$$

$A = \begin{bmatrix} K & 3 \\ 3 & K \end{bmatrix}$ $\quad b = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$.

$= \begin{bmatrix} K & 3 & 6 \\ 3 & K & -6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to KR_2 - 3R_1 \end{matrix}$

$= \begin{bmatrix} K & 3 & 6 \\ 0 & K^2-9 & -(6K+18) \end{bmatrix}$ $\quad K = \pm 3$

Case 1:- Let $K = 0$

$\begin{bmatrix} 0 & 3 & 6 \\ 3 & 0 & -6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix} \xrightarrow{R_1 \sim R_2} \begin{bmatrix} 3 & 0 & -6 \\ 0 & 3 & 6 \end{bmatrix}$

The eqn's are:-

$3x = -6 \Rightarrow x = -2$

$3y = 6 \Rightarrow y = 2.$

Case 2:- Let $K \neq 0$.

$$K^2 - 9 = 0.$$
$$K^2 = 9$$
$$K = \pm 3 \quad \text{(No soln)}.$$

The above system is singular if $K^2 - 9 = 0$. i.e., $K = \pm 3$.

If $k=3$, $\begin{bmatrix} 3 & 3 & 6 \\ 0 & 0 & -12 \end{bmatrix}$ so no soln.

When $K = -3$,

$$\begin{bmatrix} -3 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$ i.e., infinitely many solutions.

Q Which elementary matrix put the coefficient matrix A as upper triangular matrix U.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to -3R_1 + R_2 \end{matrix}$$

$\downarrow E$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U$$

$$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

The elementary matrix is E_{21} i.e., $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Given:- $E \cdot A = U$.

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1) + 0(3) & 1(2) + 0(4) \\ -3(1) + 1(3) & -3(2) + 1(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U.$$

1.4 Q22 Which elementary matrices can put into A to make it upper triangular.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

E_{21}, E_{31}; E_{32} need to be eliminated

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$E = \begin{bmatrix} 1 & 1 & 0 \\ -4 & 6 & 1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ \cancel{x}-4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cancel{x}2 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \cancel{x}-2 & 1 \end{bmatrix}$$

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Verification . first calculate this

$EA = U$

$\Rightarrow E_{32}\, E_{31}\, (E_{21}\, A) = U$

$E_{21}\, A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(4)+0(-2) & 1(1)+0(6)+0(2) & 1(0)+0(1)+0(0) \\ -4(1)+1(4)+0(-2) & -4(1)+1(6)+0(2) & -4(0)+1(1)+0(0) \\ 0(1)+0(4)+1(-2) & 0(1)+0(6)+1(2) & 0(0)+0(1)+1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31} \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(0)+0(-2) & 1(1)+0(2)+0(2) & 1(0)+0(1)+0(0) \\ 0(1)+1(0)+0(-2) & 0(1)+1(2)+0(2) & 0(0)+1+0 \\ 2-2 & 2+2 & 2(0)+0(1)+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

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E_{32} . $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 1+0+0 & 0+0+0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$ Verified.

Q28 $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$. Find elementary matrix.

pivot element

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \rightarrow R_2 + \frac{1}{2} R_1 \quad \cancel{2R_2 + R_1} \\ R_3 \\ R_4 \end{matrix}$$

E_{21} , E_{32} , E_{43}

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \rightarrow R_3 + \frac{2}{3} R_2 \\ R_4 \end{matrix}$$

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$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \to R_4 + \frac{3}{4} R_3 \end{array}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}$$

Assignment
1.4 Q – 4, 5, 11, 19, 21, 27, 28, 56

26/9/17 **Triangular Factors and Row Exchanges.**

The factorisation of a matrix 'A' can be written as :- $A = LU$ with no exchanges of rows and where 'L' is a lower triangular matrix with diagonal entries as 1 and U is an upper triangular matrix.

Q Factorise the matrix A as LU where 'A' matrix is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\boxed{L = E^{-1}}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U$$

→ Inverse of $-3 = +3$

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}}_{U}$$

1.5 Q9. a) Apply elimination to produce the factors L and U for A.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - \frac{1}{3}R_1 \\ R_3 \to R_3 - \frac{1}{3}R_1 \end{matrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - \frac{2}{8}R_2 \end{matrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix} = U$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 2/8 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{bmatrix}$$

L U

Ans

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$ R_1, $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{bmatrix}$ R_1, R_2, $R_3 \to R_3 - R_2$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix} = U$

The factorization of matrix is given by:-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

L U

Ans

System of Equations by LU factorization

→ Let us consider the system $Ax = b$.
→ Factorise the matrix 'A' as LU i.e., $A = LU$
→ The given system becomes $LUx = b$.
→ Put $Ux = C$ which gives the system as $LC = b$.
→ Solve $LC = b$ to find c.
→ Solve $Ux = C$ to obtain the required solution x

1.5

Q) Use LU factorization to solve the following:

$$2u + 3v + 3w = 2$$
$$5v + 7w = 2$$
$$6u + 9v + 8w = 5$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \quad x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} = U.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

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$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

The given system can be written by:-

$A = LU$.

$Ax = b$

$LUx = b$.

Let $Ux = C \Rightarrow LC = b$.

Now, we need to solve.

$LC = b$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \\ 3c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$c_1 = 2$

$c_2 = 2$

$3c_1 + c_3 = 5 \Rightarrow 3(2) + c_3 = 5 \Rightarrow c_3 = -1$.

$$\therefore C = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Now consider $Ux = C$.

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2u + 3v + 3w \\ 5v + 7w \\ -w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

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$$2u + 3v + 3w = 2$$
$$5v + 7w = 2$$
$$-w = -1$$

$\Rightarrow$ $w = 1$

$5v + 7 = 2$

$5v = -5 \Rightarrow v = -1$

$2u + \cancel{3}(-1) + 3 = 2$

$2u - \cancel{3} + \cancel{3} = 2$

$u = 1.$

$\therefore$ $u = 1, \; v = -1, \; w = 1$ Ans.

C.W 28/8/17.

LDU (or LDV) factorisation

The factorisation of a matrix 'A' can be written as $A = LDU$ with no exchanges of rows and where 'L' is a lower triangular matrix with diagonal entries as 1.

'D' is a diagonal matrix of pivot elements.

'U' is an upper triangular matrix with diagonal entries as 1.

$A \to U$

L U

L D U

Ex.-

$$U = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 3/2 & 1/4 \\ 0 & 0 & -5 \end{bmatrix} \begin{array}{l} R_1 \to R_1/2. \\ R_2 \to \frac{2}{3} R_2 \\ R_3 \to \frac{-R_3}{5} \end{array}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 5/2 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

D U

1.5.

Q30 Apply elimination to produce the factors L, D fo

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = U$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{matrix} R_1 \to R_1/2 \\ R_2 \to R_2/3 \end{matrix}$$

L U

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

L D U

Q $A = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$ Find LDU factorization.

$$= \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 4R_1 \\ R_3 \end{matrix}$$

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$$= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}} \begin{array}{l} R_1 \\ R_2 \to -\frac{R_2}{4} \\ R_3 \to R_3/4 \end{array}$$

$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$ Ans

Q27- i) Compute L and U for :-

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

ii) Find 4 conditions on A, B, c & d to get $A = LU$ with 4 pivots.

Ans. $$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$b-a-b+a$

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~~2nd~~ For $a \neq 0$.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$= \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$= \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to \\ R_4 \to R_4 - R_3 \end{array}$$

$$= \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} = U.$$

The conditions are :-

1- $a \neq 0$

2- $b-a \neq 0$

3- $c-b \neq 0$

4- $d-c \neq 0$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{U}$$

Ans.

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Permutation Matrix

→ It is a matrix which has single '1' in each row & column.

→ The common permutation matrix is an identity matrix.

→ There are 'n'- factorial permutation matrices of a matrix of order $n \times n$.

→ The product of 2 permutation matrices is again a permutation matrix.

Ex.- 2X2

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

row 1 & 2 are exchanged. $P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3X3

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Interchange 2 & 3 in P_{12}. $P_{23}P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Interchange 2 & 3 in P_{13}. $P_{23}P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

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$$P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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* If 0 in the pivot position, then permutation matrix

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Q Which permutation matrix can make 'A' as upper triangular ?

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_1 \end{array}$$

$$= \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = U$$

Hence it is P_{12} to make A as upper triangular.

$$P_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P_{12} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = U$$

$\therefore P_{12} A = U$.

Q ~~As~~ Which matrix can put into A to make upper triangular.

$$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_1 \end{array}$$

$$A = \begin{bmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_2 \end{array}$$

$$A = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} = U.$$

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* If not getting U directly, apply general ERO method to make it U.

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

verification.

$$P_{23} P_{13} A = U.$$

$$P_{23} P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P \text{ (Let)}$$

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix}$$

$$= \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix} = U$$

$\therefore$ $P_{23} P_{13} A = U$ (verified)

1.5

Q- $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_1 \end{matrix}$$

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$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U.$$

$$\cancel{P_{12}} = \cancel{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$P_{23}A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} = U'$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \cancel{= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 8 & 5 & 8 \end{bmatrix} = U}.$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

1.5 Q10 Which permutation makes PA upper triangular.

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_1 . \\ R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 0 & 4 & 5 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_3 \\ R_3 \to R_2 . \end{matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = U$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$P_{23} P_{12} A$

$$P_{23} P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P \text{ (Let)}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = U$$

→ The factorisation of a matrix 'A' can be written as $PA = LU$, - - - - - - - and $PA = LDU$ with exchanges of rows & where P is a permutation matrix.

Q19- Find LDU factorization for the matrix.

$$A = \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \to R_1 \\ R_3 \end{array}$$

1.5 – 2, 7, 9, 11, 21, 27, 28, 30, 32, 40, 41

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_1 \end{array} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} = U' \text{ (Let)}.$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_1 \end{array} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Ans

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1.6 Inverses and Transposes:

→ Inverse of a Matrix:

• The inverse of a matrix A is denoted by A^{-1} Such that $AA^{-1} = I$ and $A^{-1}A = I$.

→ • The matrix A has an inverse B such that $AB = I$ & $BA = I$.

• The matrix A can't have 2 different inverses.

Properties of Matrix:

→ $(AB)^{-1} = B^{-1}A^{-1}$

→ $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

→ $(A^{-1})^{-1} = A$.

→ Singular Matrix: A square matrix is said to be singular if $|A| = 0$.

→ Non-Singular Matrix: A matrix 'A' is non-singular if $|A| \neq 0$.

NOTE:

a) Matrix A is invertible if A is a non-singular matrix.

b) Full set of pivots.

c) A is not invertible if A is a singular matrix, not full set of pivots.

~~Formula 1~~:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big/ ad-bc$$

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Q Find the inverse of the matrix A:

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} / 0+1$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

Q- $$A = \begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} / 8+15$$

$$= \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} / 23 = \begin{bmatrix} 4/23 & -3/23 \\ 5/23 & 2/23 \end{bmatrix}$$

Formula-2

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & 0 \cdots & 0 \\ 0 & 0 & d_3 \cdots & 0 \end{bmatrix}_{n \times n}$$

provided no diagonal entries are given.

$$A^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$$

Q- Find the inverse of the matrix:-

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1/-5 \end{bmatrix}$$

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→ Gauss Jordan Method :- (To find inverse of matrix)

Consider the matrix AI.

$$[A\ I] \xrightarrow{\text{Use ERO}} [I\ A^{-1}]$$

Ex-1.6.

Q10 - Use Gauss Jordan to find the inverse of matrix

i- $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Consider the matrix AI.

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - R_3 \\ R_3 \end{array}$$

I

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Hence, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Ans

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inverse of

ii) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Consider $[A\,I]$

$$= \left[\begin{array}{ccc} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

inverse of matrix:

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to (R_2 + R_1/2) \\ R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + 2/3\, R_2 . \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -1 & 1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 + \frac{3}{4} R_3 \\ R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{2}{3} R_2 \\ R_2 \\ R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & 1 & 1/2 \\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4 \\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1/2 \\ R_2 \to \frac{2}{3} R_2 \\ R_3 \to \frac{3}{4} R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix}$$ Ans

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Transpose of a Matrix

The transpose of a matrix A is denoted by A^T which is obtained by interchanging the rows into columns.

Properties:-

i- $(AB)^T = B^T A^T$
ii- $(A+B)^T = A^T + B^T$
iii- $(A^{-1})^T = (A^T)^{-1}$
iv- $(A^T)^T = A$.

Q Prove $(A^{-1})^T = (A^T)^{-1}$

We know that $AA^{-1} = I$.
Taking transpose on both sides.

$\Rightarrow (AA^{-1})^T = I^T$

$\Rightarrow (A^{-1})^T A^T = I^T$ by property ①.

$\Rightarrow (A^{-1})^T A^T = I \rightarrow I$

$\Rightarrow (A^{-1})^T A^T (A^T)^{-1} = I (A^T)^{-1}$ $\qquad A \cdot A^{-1} = I$.

$\Rightarrow (A^{-1})^T \cancel{I} = \cancel{I}(A^T)^{-1}$

$\Rightarrow (A^{-1})^T = (A^T)^{-1}$ proved.

Symmetric Matrix:

A square matrix 'A' is said to be symmetric if $A^T = A$.

Skew-Symmetric Matrix:

A square matrix 'A' is said to be skew symmetric if $A^T = -A$.

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NOTE!

→ A matrix is symmetric if all the corresponding elements below or above the diagonal are same.

→ A matrix is skew-symmetric if :-

i- all the diagonal entries are 0.

ii- all the corresponding elements below and above the diagonal are same but opposite in sign.

Q10 If A & B are symmetric matrices, then prove that $A^2 - B^2$ is symmetric.

Given :- $A^T = A$ & $B^T = B$.

To prove :- $A^2 - B^2$ is symmetric
$\Rightarrow (A^2 - B^2)^T = A^2 - B^2$.

LHS $(A^2 - B^2)^T$

$= [(A+B)(A-B)]^T$

$= (A-B)^T (A+B)^T$ by property (1)

$= (A^T - B^T)(A^T + B^T)$

$= (A-B)(A+B)$ [Given :- $A^T = A$ & $B^T = B$]

$= A^2 - B^2 =$ RHS proved

Q11 i) If B is a square matrix, then show that $A = B + B^T$ is always symmetric and $K = B - B^T$ is always skew symmetric.

ii) Find the matrices A & K when $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

iii) Write 'B' as the sum of symmetric & skew symmetric matrix.

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Ans i) Given:- B is a square matrix.

To prove: $A^T = A$

LHS A^T

$= (B+B^T)^T$

$= B^T + (B^T)^T$

$= B^T + B$

$= B+B^T$

$= A$ = RHS.

$\therefore A = B+B^T$ is symmetric

To prove:- ~~$B^T = -B$~~ $K^T = -K$

LHS ~~B^T~~ K^T

$= (B-B^T)^T$

$= B^T - (B^T)^T$

$= B^T - B$

$= -K$

$\therefore K = B-B^T$ is skew-symmetric.

ii) $B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

$A = B+B^T = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$

$K = B-B^T = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

iii) $B = \begin{bmatrix} 2 & 6 \\ 2 & 2 \end{bmatrix}$ i.e., $A+K$

or, $B = \frac{B+B^T}{2} + \frac{B-B^T}{2} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

The square matrix 'B' can be expressed as a sum of symmetric and skew symmetric ~~g~~ as follow:-

$$\boxed{B = \frac{B+B^T}{2} + \frac{B-B^T}{2}.}$$

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Q. Express 'A' as a sum of symmetric & Skew-symmetric.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 3 & 5 \end{bmatrix}$$

$$K = \frac{A+A^T}{2} = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 3 & 5 \end{bmatrix} = \frac{\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 10 \end{bmatrix}}{2}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$B = \frac{A-A^T}{2} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore A = K+B = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

proved.

Q42 For which 3 numbers, 'C', the given matrix is not invertible.

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

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Assignment 1.6 2, 4, 5, 6, 10, 11, 12, 15, 17, 37, 41, 42, 52, 54, 58

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Ans 'A' is not invertible if $|A| = 0$.

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

$|A| = 2(c^2 - 7c) - c(c^2 - 8c) + c(-c) = 0$

$\Rightarrow 2c^2 - 14c - c^3 + 8c^2 - c^2 = 0$

$\Rightarrow -c^3 + 9c^2 - 14c = 0$

$\Rightarrow c^3 - 9c^2 + 14c = 0$

$\Rightarrow c(c^2 - 9c + 14) = 0$

$\Rightarrow c = 0$ or $c^2 - 9c + 14 = 0$

$c^2 - 7c - 2c + 14 = 0$

$c(c-7) - 2(c-7) = 0$

$(c-7)(c-2) = 0$

$c = 7$ & $c = 2$

$\therefore c = 0, 2, 7$ then A is not invertible.

Vector Spaces & Sub-Spaces (2.1 - 2.6) → except 2.5

8/9/17 Chapter - 2

2.1 Vector Space:

A real set of vectors is a vector space if it is closed under vector addition and scalar multiplication.

Let 'V' be a vector space if :-

(i) Let $x \in V$, $y \in V$, then $x+y \in V$

(ii) Let $x \in V$, ~~&~~ then there exist a scalar $\alpha \in R$, then $\alpha x \in V$

Subspace:

A non-empty subset of a vector space is a subspace if it satisfy all the requirements of a vector space.

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* The spaces are denoted by $R^1, R^2, \ldots, R^n$ where $R^n \rightarrow n$ dimensional space which consist of all column vectors with n-components.

2.1

Q2 i- Whether the following subset of R_3 is a subspace?

The plane of vectors (b_1, b_2, b_3) with 1st component is 0.

Ans Let $V = \{(b_1, b_2, b_3) / b_1 = 0\}$

vector addition.

Let $x = \{(x_1, x_2, x_3) / x_1 = 0\} \in V$

$y = \{(y_1, y_2, y_3) / y_1 = 0\} \in V$.

$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

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Now $x_1 + y_1$

$= 0 + 0$

$= 0$.

So, $x + y \in V$.

Hence it is closed under vector addition.

Scalar Multiplication

Let $x = \{(x_1, x_2, x_3) / x_1 = 0\} \in V$.

Let $\alpha \in R$

$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3)$

Now αx_1

$= \alpha \cdot 0$

$= 0$

So, $\alpha x \in V$.

Hence, it is closed under scalar multiplication.

$\therefore$ The given set of vectors satisfies all the requirements of vector space & hence it is a subspace.

Q Whether the following subset of R_2 with 2nd component as 1 is a subspace?

Ans Let $V = \{(b_1, b_2) / b_2 = 1\}$

Vector addition

Let $x = \{(x_1, x_2) / x_2 = 1\} \in V$

Let $y = \{(y_1, y_2) / y_2 = 1\} \in V$.

$x + y = (x_1 + y_1, x_2 + y_2)$

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Now $x_2 + y_2$
$= 1+1$
$= 2$
$\therefore x+y \notin V$
Hence it is not closed under vector addition.

Scalar multiplication

Let $x = \{(x_1, x_2)/x_2 = 1\} \in V$
Let $\alpha \in R$.

$\alpha x = (\alpha x_1, \alpha x_2)$
Now αx_2
$= \alpha . 1$
$= \alpha$
$\therefore \alpha x \notin V$
Hence it is not closed under scalar multiplication.

$\therefore$ The given set of vectors does not satisfies the conditions & hence, it is not a ~~subsp~~ vector space & hence it is not a subspace.

Q. Whether the following subset of R_3 is a subspace?

The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

Ans Let $V = \{(b_1, b_2, b_3) / b_3 - b_2 + 3b_1 = 0\}$

Vector addition.

Let $a = \{(a_1, a_2, a_3) / a_3 - a_2 + 3a_1 = 0\} \in V$.
Let $c = \{(c_1, c_2, c_3) / c_3 - c_2 + 3c_1 = 0\} \in V$.

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$a+c = (a_1+c_1, a_2+c_2, a_3+c_3)$

Now $a_3 - a_2 + 3a_1$

$= a_3+c_3 - a_2 - c_2 + 3a_1 + 3c_1$

$= (a_3 - a_2 + 3a_1) + (c_3 - c_2 + 3c_1)$

$= 0 + 0$

$= 0$ $\therefore a+c \in V$.

$\therefore$ It is closed under vector addition.

Scalar multiplication

Let $x = \{(x_1, x_2, x_3) / x_3 - x_2 + 3x_1 = 0\} \in V$

Let $\alpha \in R$.

$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3)$

Now, $\alpha x_3 - \alpha x_2 + 3\alpha x_1$

$= \alpha (x_3 - x_2 + 3x_1)$

$= \alpha . 0$

$= 0$. $\therefore \alpha x \in V$

$\therefore$ It is closed under scalar multiplication.

$\therefore$ Since, it satisfies the conditions of vector space, therefore, it is a subspace.

Subspace of R^2

→ The whole space R^2

→ The line through origin

→ The zero vector i.e., $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

→

Imp Q largest subspace of R^2 — smallest subspace of R^2
Ans whole space of R^2. — Ans origin

Subspaces of R^3

→ The whole space R^3
→ The plane through origin
→ The line through origin
→ The zero vector i.e., $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$

* All vectors which lie in the 3rd quadrant can never be in subspace.

11/9/17: **Column Space:**

The column space of matrix 'A' is denoted by C(A) which consist of all linear combination of the columns of A.

NOTE: * The system $Ax=b$ is solvable iff the vector 'b' can be expressed as a combination of the columns of A, then b is in column space of A i.e., $b \in C(A)$.

* $C(A)$ is a subspace of R^m.

$$A = \begin{array}{cc} c_1 & c_2 \end{array}$$
$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$C(A) = c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$$

2.1 Q26- Describe the column space for the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

R^m (R^3)

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$$C(A) = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$C(A) = \begin{bmatrix}c_1\\0\\0\end{bmatrix} + \begin{bmatrix}2c_2\\0\\0\end{bmatrix}$$

$$C(A) = \begin{bmatrix}c_1+2c_2\\0\\0\end{bmatrix} = \begin{bmatrix}d\\0\\0\end{bmatrix} \in R^3$$

This represent a line in x-axis.

$$d\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$\therefore$ C(A) is the line or all the vectors $(x,0,0)$.

Q26 ii) Describe the column space of matrix

$$A = \begin{bmatrix}1 & 0\\0 & 2\\0 & 0\end{bmatrix}$$

$$C(A) = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\2\\0\end{bmatrix}$$

$$C(A) = \begin{bmatrix}c_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\2c_2\\0\end{bmatrix}$$

$$C(A) = \begin{bmatrix}c_1\\2c_2\\0\end{bmatrix} = \begin{matrix}x\\y\end{matrix}\begin{bmatrix}1\\2\\0\end{bmatrix} \in R^3$$

$\begin{bmatrix}x\\y\\0\end{bmatrix} \in R^3$ $\therefore$ ~~This represents a ~~line~~ plane or all the vectors $(x,2y,0)$~~.

$\therefore$ C(A) is the x-y plane or all the vectors of $(x,y,0)$

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Q24- For which right hand side, the following system is solvable.

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\quad A \qquad\qquad x \qquad\qquad b$

The system $Ax = b$ is solvable if b is in $C(A)$.

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}$$

$$C(A) = C_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix} + C_3 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

$$C(A) = \begin{bmatrix} C_1 \\ 2C_1 \\ -C_1 \end{bmatrix} + \begin{bmatrix} 4C_2 \\ 8C_2 \\ -4C_2 \end{bmatrix} + \begin{bmatrix} 2C_3 \\ 4C_3 \\ -2C_3 \end{bmatrix}$$

$$C(A) = \begin{bmatrix} C_1 + 4C_2 + 2C_3 \\ 2C_1 + 8C_2 + 4C_3 \\ -C_1 - 4C_2 - 2C_3 \end{bmatrix} \rightarrow \text{It represents the whole space } R^3.$$

$$C(A) = b \quad \begin{bmatrix} C_1 + 4C_2 + 2C_3 \\ 2(C_1 + 4C_2 + 2C_3) \\ -(C_1 + 4C_2 + 2C_3) \end{bmatrix} = \begin{bmatrix} C \\ 2C \\ -C \end{bmatrix}$$

Hence the system is solvable if,

$b_1 = C$

$b_2 = 2C = 2b_1$

$b_3 = -C = -b_1$

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Q For which right hand side, system is solvable. Q5

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}$$

$$C(A) = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 9 \\ -4 \end{bmatrix}$$

$$C(A) = \begin{bmatrix} c_1 + 4c_2 \\ 2c_1 + 9c_2 \\ -c_1 - 4c_2 \end{bmatrix} = \begin{bmatrix} c_1+4c_2 \\ 2c_1+8c_2+c_2 \\ -[c_1+4c_2] \end{bmatrix} = \begin{bmatrix} c \\ 2c+c_2 \\ -c \end{bmatrix}$$

System is solvable if

$C(A) = b$.

$c = b_1$

$b_2 = 2c + c_2 = 2b_1 + c_2$

$b_3 = -c = -b_1$

Null Space of a Matrix

The null space of a matrix 'A' is denoted by N(A) which contains all vector 'x' such that

upper triangular form $\leftarrow AX = 0$.

NOTE:- The N(A) is a subspace of R^n.

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2.1 ~~1, 2, 5, 6, 8, 21, 25, 28~~

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Q5. Describe the column space & null space of the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$C(A) = \begin{bmatrix} c_1 - c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \in R^2$$

C(A) represents a line or all vectors $(x, 0)$.

The N(A) contains all vectors X such that $AX = 0$.

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 = 0.$$

$$\Rightarrow x_1 = x_2$$

Hence, $X = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in R^2$

N(A) is a line to (1, 1) vector.

Q $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$. Find N(A) & C(A)

$C(A) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in R^2$. It represents zero vector. It represents origin.

N(A) $AX = 0$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 = 0$

$0 = 0$ $\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3$

$0 = 0$

N(A) describes the whole space R^3.

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2.2 Complete Solution of the System $Ax = b$

Echelon form (upper triangular form) (U)

→ The pivots are the first non-zero entries in their row.

→ Below each pivot is a column of zeros obtained by elimination.

→ Each pivot lies to the right of the pivot in the row above which produces the staircase pattern and zero rows come last.

Row-reduced Echelon form (R)

→ All pivots are 1 in the echelon form.

→ Use the pivot row to produce 0 above the pivot.

Q Convert into echelon form & row reduced echelon form.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$R = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - R_2 \\ R_2 \\ R_3 \end{array} \quad R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to \frac{R_2}{3} \\ R_3 \end{array}$$

$$R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underline{Ans}.$$

Q Find 'R' of the matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - \frac{4}{3} R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2/3 \\ R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \\ R_3 \end{matrix}$$

$$\therefore R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{Ans}.$$

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Pivot Variables & Free Variables

Variables correspond to columns with pivot are called pivot variables and columns without pivot are called free variable.

Rank of a Matrix

→ The rank of a matrix is 'R' when the elimination reduces A to U or R with R r-pivot rows & r-pivot columns.

* The no. of pivot elements in a matrix is called rank of a matrix.

2.2 Q15- i) Find 'R' & rank of the following matrix:-

The 3X4 matrix of all 1's.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U = R.$$

Rank :- 1. (since no. of pivots is 1).

ii) Find R and rank :

The 4X4 matrix with $a_{ij} = (-1)^{ij}$

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

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$$= \begin{array}{cl} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{array} \end{array}$$

$$= \begin{array}{cl} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} & \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \to R_4 - R_2 \end{array} \end{array}$$

$$= \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$= \begin{array}{cl} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} R_1 \to \frac{R_1}{-1} \\ R_2 \to R_2/2 \\ R_3 \\ R_4 \end{array} \end{array}$$

$$= \begin{array}{cl} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} R_1 \to R_1 + R_2 \\ R_2 \\ R_3 \\ R_4 \end{array} \end{array}$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank :- 2

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Steps to find Complete Solution :-

→ Write the system of eqn's in matrix form i.e., $Ax = b$.

→ Consider the matrix $[A \ b]$

→ Use ERO to convert coefficient matrix 'A' as U or R.

→ Identify the pivot variables and free variables.

→ Use back substitution method to find the solution.

→ Then the complete solution to the given system can be written in the following form:

$$\boxed{X = X_p + X_n}$$

where, X_p is a particular solution which can be evaluated by putting free variable as 0.

X_n is the null space solution to the system

only form
2 pivot
1 free

NOTE: The special solution to the null space equation $Ax = 0$ can be obtained by putting each free variable as 1 and other variable as 0.

always express i of free

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2.2 Q4- Solve the following eqn to find complete solution.

$$u + 2v + 2w = 1$$
$$2u + 4v + 5w = 4$$

Ans The given eqn in matrix form is :-

$$Ax = b$$

where, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

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Consider the matrix $[A\ b]$

i.e., $= \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \rightarrow R_2 - 2R_1 \end{matrix}$

$= \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = U.$

only from A.

2 pivot variables
1 free variable

$R_3 = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \rightarrow R_1 - 2R_2 \\ R_2 \end{matrix}$

$R = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Here u & w are the pivot variables and v is a free variable.

The corresponding eq'ns are from U are :-

$u + 2v + 2w = 1$ —(1)

$w = 2$

Putting $w = 2$ in (1)

$u + 2v = -3$

$u = -3 - 2v$

always express in terms of free variable

Hence, the solution to the given system is :

$X = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3-2v \\ v \\ 2 \end{bmatrix}$

$X = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2v \\ v \\ 0 \end{bmatrix}$

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$$X = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + v \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$ Ans.

$\downarrow$ X_p $\quad$ $\downarrow$ X_n

2.2

Q36- Find the complete solution of the following system.

$x + 3y + z + 2w = 1$
$2x + 6y + 4z + 8w = 3$
$2z + 4w = 1$.

Ans

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Consider $[A \ b]$

$$= \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

Here, x & z are pivot variables & y & w are free variables.

The equations from U are :-

$$x + 3y + z + 2w = 1 \quad \text{— (i)}$$
$$2z + 4w = 1 \quad \text{— (ii)}$$

$$z = \frac{1-4w}{2}$$

$x = 1 - 2w - 3y - \left(\frac{1-4w}{2}\right)$

$x = \frac{2 - 4w - 6y - 1 + 4w}{2}$

$x = \frac{-6y + 1}{2}$

$x = \frac{1}{2} - 3y$

The solution to the given system is :-

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1/2 - 3y \\ y \\ (1-4w)/2 \\ w \end{bmatrix}$$

$$X = \begin{bmatrix} -3y + 1/2 \\ y \\ 1/2 - 2w \\ w \end{bmatrix} = \begin{bmatrix} -3y + 1/2 \\ y \\ 1/2 - 2w \\ w \end{bmatrix}$$

$$X = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3y \\ y \\ -2w \\ w \end{bmatrix}$$

$$X = \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{X_n}$$

Ans

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Special Solution:-

If there are 2 free variables, then put one as 0 and other as 1.

When $y=1$ & $w=0$

$$= \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

When $y=0$ & $w=1$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

2.3 Linear Independence, Basis & Dimension

Linearly Independent Vectors (LI):

A set of vectors $v_1, v_2, \ldots, v_n$ are said to be LI if there exist scalars $c_1, c_2, \ldots, c_n$ such that their linear combination i.e.,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

where, all $c_i = 0$, $i = 1, 2, 3, \ldots, n$

Linearly Dependent Vectors (LD):

A set of vectors $v_1, v_2, \ldots, v_n$ are said to be LD if there exist scalars $c_1, c_2, \ldots, c_n$ such that their linear combination i.e.,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

where not all $c_i = 0$, $i = 1, 2, \ldots, n$

Ex:- v_1, v_2, v_3

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

Suppose $\left.\begin{matrix} c_1 = 1 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}\right\}$ LD. If $\left.\begin{matrix} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{matrix}\right\}$ LI.

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$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

$$Ax = b$$

$$A = [v_1, v_2, \ldots, v_n] \qquad x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad b = [0]$$

$Ax = 0$

↳ Null space equation.

*NOTE: The columns of A are independent exactly when null space of A i.e., $N(A) = \{\text{zero vector}\}$

Test for LI, LD :-

→ Let $A = [v_1, v_2, \ldots, v_n]_{n \times n}$. Find $|A|$.

If $|A| = 0 \Rightarrow LD$

If $|A| \neq 0 \Rightarrow LI$

→ If the rank of a matrix is equal to nos. of vectors, then the vectors are LI.

If rank is less than no. of vectors $\Rightarrow LD$.

2.3 Q. Decide the dependence or independence of the vector.

(1, 3, 2), (2, 1, 3), (3, 2, 1)

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$|A| = 1(-5) - 2(-1) + 3(7)$$
$$= -5 + 2 + 21 = 18$$
$$|A| = 18$$
$$\therefore |A| \neq 0.$$

Since $|A| \neq 0$, So LI.

∴ Hence, v_1, v_2, v_3 are LI.

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Rank - 1 ⇒ line
2 ⇒ plane

Spanning a Subspace

A Set of vectors are said to span a space if their linear combination produce the whole Space.

2.3

Q16- i) Describe the subspace of R^3 span by the vectors $(1,1,-1)$ & $(-1,-1,1)$.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$U = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \to \text{echelon form}$$

Rank = 1.

The subspace of R^3 by the given 2 vectors is,

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 - c_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$$

Hence, it represents a line in R^3 space.

ii) Describe subspace of R^3 span by the vectors $(0,1,1)$, $(1,1,0)$ & $(0,0,0)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_1 \\ R_3 \end{array} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \end{array}$$

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$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Since its rank = 2, so it represents a plane.

$\therefore$ Hence, It represents a plane (x,y) in R^3 space.

or Subspace of R^3 is given by:-

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 + C_2 \\ C_2 \\ 0 \end{bmatrix}$$

$\therefore$ It represents the xy-plane in R^3 space.

Basis for a Vector Space

A basis for the vector space 'V' is a sequence of vectors having the following 2 properties:

→ The vectors are linearly independent (LI).

→ The vectors span the space V (not too few vectors).

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40- i) Whether the following vectors are basis for R^3:
$(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)$

$$= \begin{bmatrix} 1 & 2 & 4 & 0 \\ 1 & 3 & 1 & 1 \\ -1 & 4 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 6 & 3 & -1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - 6R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 21 & -7 \end{bmatrix} = U \quad \text{Rank} = \cancel{4}\ 3$$

Rank $= 3$

$n = 4$ (no. of variables)

$\therefore$ Since Rank $< n \Rightarrow$ LD

So the given set of vectors are not basis for R^3

iii) Are basis for R^3?
$(1,2,2), (-1,2,1), (0,8,0)$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 8 \\ 2 & 1 & 0 \end{bmatrix}$$

$|A| = 1(-8) - 1(-16)$

$= -8 + 16 = 8$

$|A| = 8$.

Since $|A| \neq 0$, so LI

The given set of vectors are LI
Also they span the space R^3. Hence the vectors are the basis of R^3.

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4 fundamental subspaces

Fundamental theorem of Linear Algebra :-

1- Column Space: The columnspace of matrix A is denoted by $C(A)$ and the dimension is the rank 'r'.

2- Null Space:- The null space of A is denoted by $N(A)$ which contains all vectors x such that $AX=0$

$\dim(N(A)) = n - r$

3- Row Space:- The row space of A is the column space of $C(A^T)$ i.e.,

$\dim(C(A^T)) = r$

4- Left Null Space:-

The left null space of A is the null space of A^T. It is denoted by $N(A^T)$ which contains all vector x such that $A^Tx = 0$.

dimension of $N(A^T) = m - r$

for R^3

2.3 Dimension for a Vector Space

The no. of basis vectors of a vector space 'V' is known as the dimension of V.

NOTE:- The columns that contains pivots are a basis for the column space.

→ Dimension means no. of basis vectors.

2.3 Q15- Find dimension of the column space of matrix A:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$$

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2-3 Q - 1, 3, 4, 5, 8, 9, 10, 13, 15, 16, 19, 23, 31, 32, 40

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$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

No. of pivot variables = 2

$\therefore$ No. of basis vector = 2

$\therefore$ Dimension, V = 2.

The independent vectors are (1, 1, 3) & (1, 3, 1).

$\therefore$ Basis for the column space of A i.e.,

$C(A) = \{(1,1,3), (1,3,1)\}$

$\therefore$ Dim of $C(A) = 2$ Ans.

2.4

Q2 i) Find the basis and dimension for 4 fundamental Subspaces for the matrix A.

Ans. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2\times 2}$ $m = 2$, $n = 2$.

a) C(A)

$$= \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 3R_1 \end{array} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = U$$

$\therefore$ No. of pivots = 1.

set of independent vectors.

Basis for $C(A) = \{(1,3)\}$
dim of $C(A) = 1$
Rank, $r = 1$.

b) $N(A)$
It contains all the vector X such that $AX = 0$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \cancel{\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\Rightarrow \cancel{\begin{bmatrix} x_1 + 2(x_2) \\ 3(x_1 + 2x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}}$$

Consider the matrix $[A \;\; 0]$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \Rightarrow R_2 - 3R_1 \end{matrix}$$

$$= \begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_1 is the pivot variable & x_2 is the free variable.

The corresponding eqn is :-

$$x_1 + 2x_2 = 0$$
$$2x_2 = -x_1$$
$$x_1 = -2x_2.$$

$$\therefore N(A) : X = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

putting $x_2 = 1$

$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow$ basis for the null space.

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Basis for $N(A) = \{(-2,1)\}$
dim of $N(A) = 1$

$n - r$
$= 2 - 1 = 1$.

c) $C(A^T)$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$

$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = U$

No. of pivots $= 1$

$\therefore$ Basis for $C(A^T) = \{(1,2)\}$
dim of $C(A^T) = 1$
$r = 1$.

d) $N(A^T)$.

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

Consider $A^T X = 0$.

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

In $[A \; 0]$

$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$

$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The corresponding eqn's are:
$x_1 + 3x_2 = 0$.
$x_1 = -3x_2$

$N(A)$: $X = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

putting $x_2 = 1$

$X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Basis for $N(A) = \{(-3,1)\}$

dim of $N(A) = 1$.

ii) Find basis & dimension for the 4 fundamental subspaces for A

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

a) $C(A)$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

No. of pivots = 2

Basis for $C(A) = \{(1,0,1),(2,1,2)\}$

dim of $C(A) = 2$

$r = 2$.

b) $N(A)$

Consider $AX = 0$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{3\times 4} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1}$$

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Free variables x_3 & x_4

Consider matrix $[A : 0]$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

The corresponding eqn's are :-

$x_1 + 2x_2 + x_4 = 0$

$x_2 + x_3 = 0$

$\Rightarrow x_2 = -x_3$

$x_1 - 2x_3 + x_4 = 0$

$x_1 = 2x_3 - x_4$

$$N(A) : X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Basis for $N(A) = \{(2,-1,1,0), (-1,0,0,1)\}$

dim for $N(A) = 2$

rank, $r = 2$

c) $C(A^T)$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{4\times 3} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \\ R_4 \to R_4 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \end{array} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

No. of pivot elements = 2.

$\therefore$ Basis for $C(A) = \{(1,2,0,1), (0,1,1,0)\}$

dim for $C(A^T) = 2$

d) $N(A^T)$

Consider $[A^T \ 0]$ matrix. $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \\ R_4 \to R_4 - R_1 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \end{array} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variable:- x_3

pivot variables:- x_1, x_2

The corresponding eq^n's are :-

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 = -x_3$

$$\therefore N(A^T): \ X = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $N(A^T) = \{(-1,0,1)\}$

$\therefore$ dim for $N(A^T) = 1$

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Here we can directly apply the formula since it has not asked to find basis.

Q Find dimension for 4 fundamental subspaces for:-

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{m \times n} \quad m = 3,\ n = 4.$$

dim of $C(A) = r$ dim of $C(A^T) = r$

dim of $N(A) = n - r$ dim of $N(A^T) = m - r$

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \Rightarrow r = 2$$

According to the fundamental theorem of linear algebra

$\therefore$ dim of $C(A) = 2$

dim of $N(A) = n - r = 4 - 2 = 2$

dim of $C(A^T) = 2$

dim of $C(A^T) = m - r = 3 - 2 = 1$

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Existence of Inverse :-

Let 'A' be a matrix of order $m \times n$

i) Right Inverse

If rank $(A) = m$ = full row rank; then the given matrix A has a right inverse B such that $AB = I$ and where B equals to $A^T (AA^T)^{-1}$

$$\boxed{B = A^T (AA^T)^{-1}}$$

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ii) Left Inverse :- If rank (A) = n = full column rank, then the given matrix A has a left inverse B Such that BA = I where B equals to $(A^TA)^{-1} A^T$

$$\boxed{B = (A^TA)^{-1} A^T}$$

2.4 Q) Find the inverse of the matrix A.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}_{\substack{2\times3 \\ m\times n}} = V.$$

Rank (A) = 2 = m. = full row rank.

⇒ Right Inverse exists.

So, the given matrix A has a right inverse B such that AB = I and where AB = $A^T(AA^T)^{-1}$

$$A^T = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}_{3\times2}$$

$$AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 25 \end{bmatrix}$$

$$(AA^T)^{-1} = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix}_{2\times2}$$

$$A^T(AA^T)^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}_{3\times2} \begin{bmatrix} 1/16 & 0 \\ 0 & 1/25 \end{bmatrix}_{2\times2} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix} \text{ Ans.}$$

For verification :-
AB = I

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Verification :-

$AB = I$.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}_{2\times3} \begin{bmatrix} 1/4 & 0 \\ 0 & 1/5 \\ 0 & 0 \end{bmatrix}_{3\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

ii) Find inverse of matrix A.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times2} \quad \begin{matrix} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{matrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = U$$

$Rank(A) = 2 = n$ = full column rank

$\Rightarrow$ Left inverse exits

So, given matrix A has left inverse B such that $AB = I$ & $B = (A^T A)^{-1} A^T$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times3}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

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2.4 — Q. 2, 3, 6, 9, 11, 13, 17, 18, 24, 28, 29, 31, 32

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$$(A^TA)^{-1}A^T = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times3}$$

$$= \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$ → This is the inverse of A.

Verification:-

$BA = I$.

$$\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}_{2\times3} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.6 Linear Transformation

Transformation of a Plane by 4 matrices :

1- Stretching Matrix:-

The stretching matrix with a stretching factor of c is $A = c \cdot I$.

i.e., $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

2- Rotation Matrix (Q):-

The rotation matrix with an angle of 'θ' is Q_θ

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

3- Reflection Matrix (H):-

The reflection matrix with an angle of 'θ' is H_θ.

$$H_\theta = \begin{bmatrix} 2\cos^2\theta-1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta-1 \end{bmatrix}$$

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iv) Projection Matrix (P) :-

The projection matrix with an angle of 'θ' is

$$P_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Q Transform the vector $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by using :-

(i) stretching matrix with stretching factor of 2.

(ii) rotating with an angle of 90°

(iii) reflection through the line $y = x$.

(iv) projecting onto x - axis.

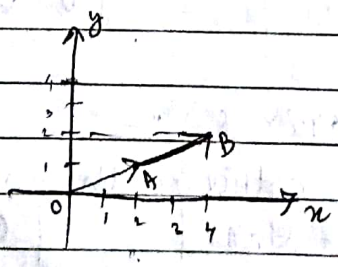
Sol^n :- (i) Stretching matrix, $c = 2$.

The stretching matrix with $c=2$ is $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$\therefore X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{2\times1} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\therefore T \begin{bmatrix} 2 \\ 1 \end{bmatrix} \xrightarrow[\text{by factor of 2}]{\text{Stretching}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



$OA \rightarrow OB$

(ii) $$Q_\theta = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A.$$

$$AX = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

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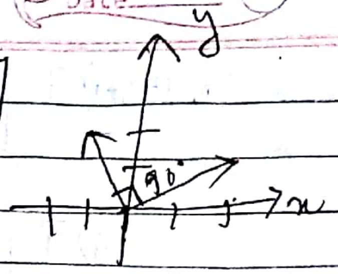
$\therefore\ T\begin{bmatrix}2\\1\end{bmatrix} \xrightarrow{\text{rotating with } \theta=90^\circ} \begin{bmatrix}-1\\2\end{bmatrix}$



iii) ~~H_θ = 2co~~ $y = x$

$m = 1$

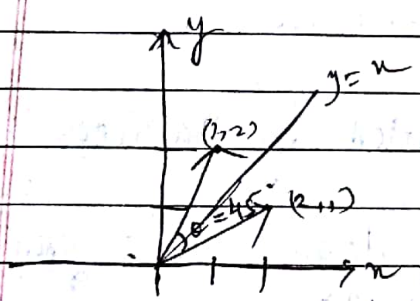
$\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) \Rightarrow \theta = 45^\circ$

$$H_\theta = \begin{bmatrix}2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1\end{bmatrix}$$

$$H_\theta = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = A$$

$$AX = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$\therefore\ T\begin{bmatrix}2\\1\end{bmatrix} \xrightarrow{\text{reflecting with } \theta=45^\circ} \begin{bmatrix}1\\2\end{bmatrix}$



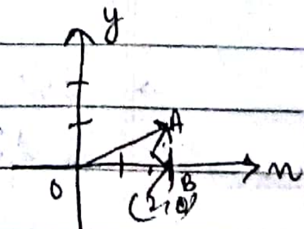
iv) projecting onto x-axis.

$\theta = 0^\circ$

$$P_\theta = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} = A$$

$$AX = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

$\therefore\ T\begin{bmatrix}2\\1\end{bmatrix} \xrightarrow{\text{projecting with } \theta=0^\circ} \begin{bmatrix}2\\0\end{bmatrix}$



Projection vector — OB

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Same ques

NOTE:- i) $T\begin{bmatrix}x\\y\end{bmatrix} \xrightarrow[\text{factor of } c]{\text{Stretching with}} \begin{bmatrix}cx\\cy\end{bmatrix}$

ii) $T\begin{bmatrix}x\\y\end{bmatrix} \xrightarrow{\text{rotating with } \theta=90^\circ} \begin{bmatrix}-y\\x\end{bmatrix}$

iii) $T\begin{bmatrix}x\\y\end{bmatrix} \xrightarrow[\theta=45^\circ]{\text{reflection with}} \begin{bmatrix}y\\x\end{bmatrix}$

iv) $T\begin{bmatrix}x\\y\end{bmatrix} \xrightarrow[\text{with } \theta=0^\circ]{\text{projecting}} \begin{bmatrix}x\\0\end{bmatrix}$

14/10/15

* A transformation is said to be linear if it satisfies the following 2 properties :
(i) $T(u+v) = T(u) + T(v)$
(ii) $T(\alpha u) = \alpha T(u)$, $\alpha \to$ scalar

Transformation represented by Matrices :

→ Let a polynomial of degree n is given by $P_n(t) = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$

→ The standard basis for a polynomial of degree n, $P_n(t) \to \{1, t, t^2, t^3, \ldots, t^n\}$

→ The dimension for the polynomial $P_n(t) \to n+1$

→ The operation of differentiation is linear i.e., $T(P_n) \to P_{n-1}$

→ The operation of integration from 0 to t is linear i.e., $T(P_n) \to P_{n+1}$

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→ Multiplication by a fixed polynomial like $(2+3t)$ is also linear.

2.6 Q20 Whether the following transformation is linear?
$T(v) = (v_2, v_1)$ when $v = (v_1, v_2)$

given, i.e., $T(v_1, v_2) = (v_2, v_1)$

(i) For linearity of the transformation, let us consider the vectors $u = (u_1, u_2)$, $v = (v_1, v_2)$.

$\Rightarrow T(u) = (u_2, u_1)$

$\Rightarrow T(v) = (v_2, v_1)$

Claim:- To prove $T(u+v) = T(u) + T(v)$

LHS

$$T(u+v)$$
$$= T(u_1+v_1, u_2+v_2)$$
$$= (u_2+v_2, u_1+v_1)$$
$$= (u_2, u_1) + (v_2, v_1)$$
$$= T(u) + T(v).$$

Claim:-

(ii) T.P.- $T(\alpha u) = \alpha T(u)$

LHS $T(\alpha u)$
$$= T(\alpha u_1, \alpha u_2)$$
$$= (\alpha u_2, \alpha u_1)$$
$$= \alpha (u_2, u_1)$$
$$= \alpha T(u)$$

Hence, the given transformation is linear.

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$P_n(t)$

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Q19 Whether the transformation $T(V) = (0,1)$, where $V = (V_1, V_2)$ is linear?

$$T(V_2, V_1) = (0, 1)$$

Let the considers the vectors $u = (u_1, u_2)$ & $v = (V_1, V_2)$

$$T(u) = (0,1)$$
$$T(V) = (0,1)$$

(i) Claim: $T(u+v) = T(u) + T(v)$

LHS $T(u+v)$

$$= T(u_1 + \cancel{V_1}, u_2 + V_2)$$
$$= (0,1) \neq RHS$$

$$T(u+v) \neq T(u) + T(v)$$

Hence the given transformation is not linear.

Q17 Whether the following transformation is linear? $T(v) = V_1 + V_2 + V_3$, where $V = (V_1, V_2, V_3)$

$$T(V_1, V_2, V_3) = V_1 + V_2 + V_3$$

Let the consider the vectors $u = (u_1, u_2, u_3)$ & $v = (V_1, V_2, V_3)$

$$T(u) = u_1 + u_2 + u_3$$
$$T(V) = V_1 + V_2 + V_3$$

(i) Claim :- $T(u+v) = T(u) + T(V)$

LHS $T(u+v)$

$$= T(u_1 + V_1, u_2 + V_2, u_3 + V_3)$$

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$= (u_1 + v_1 + v_2 + u_2 + u_3 + v_3)$

$= (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3)$

$= T(u) + T(v) = RHS$

(ii) Claim :- $T(\alpha u) = \alpha T(u)$

LHS

$T(\alpha u)$

$= T(\alpha u_1, \alpha u_2, \alpha u_3) = \alpha u_1 + \alpha u_2 + \alpha u_3$

$= \alpha (u_1 + u_2 + u_3)$

$= \alpha T(u) = RHS$.

Hence, the given transformation is linear.

Q26 The cyclic transformation T is defined by :-
$T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. what is :-
i) $T(T(T(v)))$
ii) $T^{100}(v)$

Ans Given : $T(v) = (v_2, v_3, v_1)$

i) $T(T(v)) = T(v_2, v_3, v_1)$

$\Rightarrow T(T(v)) = (v_3, v_1, v_2)$

$\Rightarrow T(T(T(v))) = T(v_3, v_1, v_2)$

$= (v_1, v_2, v_3)$ Ans

ii) Given :- $T^{100}(v)$

$= T(T^{99}(v))$

$= T(v_1, v_2, v_3)$

$= (v_2, v_3, v_1)$ Ans

This is called cyclic transformation.

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Q i) Construct a differentiation matrix for a polynomial of degree 3 and ii) also find transformation of $T(2+3t-5t^3)$.

Ans i) The standard basis for a polynomial of degree 3 i.e., $P_3(t) \to \{1, t, t^2, t^3\}$ → no. of columns

$T(P_3) \xrightarrow{\text{differentiation}} P_2(t)$

The standard basis for $P_2(t) \to \{1, t, t^2\}$ → no. of rows

$$T_{diff} = \begin{array}{c|cccc} & 1 & t & t^2 & t^3 \\ \hline 1 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 2 & 0 \\ t^2 & 0 & 0 & 0 & 3 \end{array}_{3\times 4}$$

ii) By using the above differentiation matrix, the coefficients of standard basis of $T(P_3)$

$$T(2+3t-5t^3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4} \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix}_{4\times 1} = \begin{array}{c} \begin{bmatrix} 3 \\ 0 \\ -15 \end{bmatrix}_{3\times 1} \end{array} \begin{array}{c} 1 \\ t \\ t^2 \end{array}$$

$= 3 - 15t^2$

Q i) Construct a differentiation matrix for a polynomial of degree 3 and the operator for the transformation is $\frac{d^2}{dt^2}$.

ii) Find transformation of i.e., $T(2+3t-5t^3)$

Ans i) Standard basis for the polynomial of degree 3 i.e,

$P_3(t) \to \{1, t, t^2, t^3\}$ → $m = 4$

$T(P_3) \xrightarrow{d^2/dt^2} P_1(t)$

The standard basis for $P_1(t) \to \{1, t\}$ → $n = 2$.

$$1 \to 0 \to 0,\quad t \to 1 \to 0,\quad t^2 \to 2t \to 2,\quad t^3 \to 3t^2 \to 6t$$

$$T_{diff} = \begin{array}{c} \\ 1 \\ t \end{array}\begin{array}{c} \begin{array}{cccc} 1 & t & t^2 & t^3 \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}_{2\times 4} \end{array}$$

ii) ~~By using the above differentiation,~~

$$\cancel{T(2+3t-5t^2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}_{2\times 4} \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{2\times 1} \begin{array}{c} 1 \\ t \end{array}}$$

$$\cancel{= 3}$$

ii) By using above differentiation,

$$T(2+3t-5t^2) = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -30 \end{bmatrix} \begin{array}{c} 1 \\ t \end{array}$$

$$= -30t \quad \text{Ans}$$

Q Find an integration matrix for a polynomial of degree 3. Then find $T(2+3t-5t^2)$.

Ans Standard basis for polynomial of degree 3 i.e., $P_3(t) \to \{1, t, t^2, t^3\}$ $\to m=4$

$$T(P_3) \xrightarrow{\int_0^t dt} P_4(t)$$

Standard basis for $P_4(t) \to \{1, t, t^2, t^3, t^4\}$ $\to n=5$

$$1 \to t,\quad t \to \tfrac{t^2}{2},\quad t^2 \to \tfrac{t^3}{3},\quad t^3 \to \tfrac{t^4}{4},\quad \cancel{t^4 \to \tfrac{t^5}{5}}$$

$$~~Standa~~\; T_{int} = \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}_{5\times 4}$$

2.6 Q → 2, 5, 11, 17, 19, 20, 25, 26, 28, 29

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$$T(2+3t-5t^3) = \underset{5\times 4}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix}} = \begin{array}{c} \begin{bmatrix} 0 \\ 2 \\ 3/2 \\ 0 \\ -5/4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array}$$

$$= 2t + \frac{3}{2}t^2 - \frac{5t^4}{4}$$ Ans.

2.6 Q11 Find a transformation matrix for a polynomial of degree 3 by multiplying a polynomial $(2+3t)$. Then find $T(2+3t-5t^3)$.

Ans. $P_3(t) \to \{1, t, t^2, t^3\} \Rightarrow m=4$

$$T(P_3) \xrightarrow[(2+3t)]{\text{multiply by}} P_4(t)$$

Now, standard basis for $P_4(t) \to \{1, t, t^2, t^3, t^4\}$ → n=5

$1 \to 2+3t$, $t \to 2t+3t^2$, $t^2 \to 2t^2+3t^3$, $t^3 \to 2t^3+3t^4$

1) $T_{\text{mult by } 2+3t} =$

$$\begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array} \underset{5\times 4}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

$$T(2+3t-5t^3) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 9 \\ -10 \\ -15 \end{bmatrix} \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array}$$

$$= 4 + 12t + 9t^2 - 10t^3 - 15t^4$$

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Orthogonality

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3.1 Orthogonal Vectors & the Subspaces.

Length of a Vector

Let $x = (x_1, x_2, \ldots, x_n)$ a vector whose length is given by norm x ($\|x\|$) & defined by $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Length Squared :-

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

i.e., $\boxed{\|x\|^2 = x^T x}$

$x = (x_1, x_2)$

$\|x\| = \sqrt{x_1^2 + x_2^2}$

$\|x\|^2 = x_1^2 + x_2^2$

$= x^T x.$

~~3.2~~ ~~3.3~~ Inner Product :-

The inner product of 2 vectors x & y is denoted by $x^T y$ and defined as :

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

NOTE:-

→ if $x^T y = 0$, then the angle in between the 2 vectors is $90°$ ($\theta = 90°$).

→ if $x^T y < 0$, then angle in between 2 vectors is $> 90°$ ($\theta > 90°$)

→ if $x^T y > 0$, then angle in between 2 vectors is $< 90°$ ($\theta < 90°$)

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Q7 Find length and inner product of the vectors $x = (1, 4, 0, 2)$ & $y = (2, -2, 1, 3)$.

i) $\|x\| = \sqrt{1^2 + (4)^2 + 0^2 + 2^2}$
$= \sqrt{21}$

$\|y\| = \sqrt{4 + 4 + 1 + 9}$
$= \sqrt{18}$

ii) Inner product:-

$$x^T y = \begin{bmatrix} 1 & 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix} = 2 + (-8) + 0 + 6$$
$$= 0$$

Orthogonal Vectors.

Two vectors x & y are orthogonal if their inner product is 0 i.e., $x^T y = 0$.

Mutually Orthogonal Vectors

If every vector is $\perp$ to every other vector.

Orthonormal Vectors

A mutually orthogonal unit vectors is known as orthonormal vectors.

3) Q) Which pairs are orthogonal among the vectors v_1, v_2, v_3, v_4..

$v_1 = (1, 2, -2, 1)$
$v_2 = (4, 0, 4, 0)$
$v_3 = (1, -1, -1, -1)$
$v_4 = (1, 1, 1, 1)$

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$v_1^T v_2 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} = 4 - 8$
$= -4 \neq 0$.

v_1 & v_2 are not orthogonal.

$v_1^T v_3 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \cancel{1} - \cancel{2} + \cancel{2} - \cancel{1}$
$= 0$

v_1 & v_3 are orthogonal

$v_1^T v_4 = \begin{bmatrix} 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 + \cancel{2} - \cancel{2} + 1 = 2 \neq 0$.

v_1 & v_4 are not orthogonal.

$v_2^T v_3 = \begin{bmatrix} 4 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = 4 - 4$
$= 0$

v_2 & v_3 are orthogonal.

$v_2^T v_4 = 8$.
$v_3^T v_4 = -2$.

$\therefore$ Hence, (v_1, v_3) & (v_2, v_3) are orthogonal vectors.

Ex: Whether the vectors $v_1 = (\cos\theta, \sin\theta)$ & $v_2 = (-\sin\theta, \cos\theta)$ are orthonormal?

$v_1^T v_2 = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta$
$= 0$

v_1, v_2 are orthogonal.

$\|v_1\| = 1$

$\|v_2\| = 1$

These are unit vectors

$\therefore$ Hence (v_1, v_2) are orthonormal.

Q12. Show that $(x-y) \perp (x+y)$ iff $\|x\| = \|y\|$

TS:- $(x-y) \perp (x+y) \Leftrightarrow \|x\| = \|y\|$

Proof:-

Part 1:- Given:- $(x-y) \perp (x+y)$

$\Leftrightarrow (x-y)^T \cdot (x+y) = 0$

$\Leftrightarrow (x^T - y^T)(x+y) = 0$

$\Leftrightarrow x^T x - x y^T + x^T y - y^T y = 0$

$\Leftrightarrow x^T x - y^T y - x y^T + x^T y = 0$

$\Leftrightarrow \|x\|^2 - \|y\|^2 = 0$

$\Leftrightarrow \|x\|^2 = \|y\|^2$

Sq. root both sides

$\Leftrightarrow \|x\| = \|y\|$

$\left[\because x^T y = y^T x\right.$ since inner product is commutative

~~Part 2:- Given $\|x\| = \|y\|$~~

~~$\Rightarrow$ Sq. both sides.~~

Orthogonal Subspace

Every vector in one subspace must be orthogonal to every vector in the other subspace.

Fundamental Theorem of Orthogonality:-

The row space is orthogonal to the null space in R^n and the column space is orthogonal to the left null space in R^m.

* $C(A^T) \perp N(A)$

* $C(A) \perp N(A^T)$

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Q2. Find a vector x orthogonal to the row space of A & a vector y orthogonal to the column space of A & a vector z orthogonal to the null space of A.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$$

i) We know, $N(A) \perp C(A^T)$

$N(A)$ contains all vectors x such that $Ax = 0$

consider the matrix $[A \; b]$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 0 \\ 3 & 6 & 4 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \end{array}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

No. of pivot variables $= 2$ $(x_1$ & $x_3)$

No. of free variables $= 1$ (x_2).

$x_1 + 2x_2 + x_3 = 0$

$x_3 = 0$

$x_1 + 2x_2 = 0$

$x_1 = -2x_2$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

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Hence, the vector orthogonal to $C(A^T)$ is $x = (-2, 1, 0)$.

iv) We know $N(A^T) \perp C(A)$.

Consider $[A^T \; b]$ matrix

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 1 & 3 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = U.$$

x_1 & x_2 are pivot variables
x_3 is free variable

$$x_1 + 2x_2 + 3x_3 = 0$$
$$x_2 + x_3 = 0$$

$$x_2 = -x_3 \quad — (1).$$

$$x_1 + -2x_3 + 3x_3 = 0$$
$$x_1 - 2x_3 + 3x_3 = 0$$
$$x_1 + x_3 = 0$$
$$x_1 = -x_3$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Hence, the vector orthogonal to $C(A)$ is $x = (-1, -1, 1)$

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iii) We know $C(A^T) \perp N(A)$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = U$$

No. of pivots = 1

Hence, the vector orthogonal to $C(A^T)$ is $x = (1, 2, 1)$.

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Orthogonal Complement

Given V of R^n, - - - then the space of all vectors orthogonal to V is called the orthogonal complement of V. It is denoted by $V^\perp$.

Q9 i) Find a basis for the orthogonal complement of the $C^T(A)$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$.

ii) Split $x = (3, 3, 3)$ into a row space component x_r & a null space component x_n.

Ans i) We know, $(C(A^T))^\perp = N(A)$.

The $N(A)$ contains all vector x such that $Ax = 0$

consider $[A\ 0]$

$$= \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \to R_2 - R_1 \end{matrix}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} = U. \end{array}$$

free variables :- x_3

$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$.

$x_2 + x_3 = 0$.

$x_2 = -2x_3$

3.1 1, 2, 7, 9, 10, 11, 12, 18, 33

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

Hence, the basis for $(C(A^T))^\perp = \{(-2, -2, 1)\}$

ii) $x = (3, 3, 3)$

$\Rightarrow x = x_r + x_n$.

$$\Rightarrow \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = x_r + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_r = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

3.2 Cosines & Projections onto a line

Inner Product & Cosine

The cosine of the angle between any non-zero vectors a & b is :-

$$\cos\theta = \frac{a^T b}{\|a\| \cdot \|b\|}$$

3.2 Q13

Schwarz Inequality

It states that :-

$$|a^T b| \leq \|a\| \cdot \|b\|$$

Proof :-

$$|\cos\theta| \leq 1$$

$$\Rightarrow \left| \frac{a^T b}{\|a\| \cdot \|b\|} \right| \leq 1$$

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$$\Rightarrow \frac{|a^T b|}{\|a\| \cdot \|b\|} \leq 1$$

$$\Rightarrow |a^T b| \leq \|a\| \cdot \|b\| \quad \text{proved.}$$

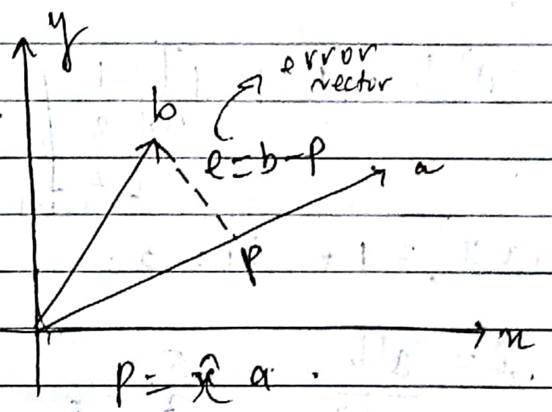
Triangle Inequality:-

It states that:-

$$\|a+b\| \leq \|a\| + \|b\|$$

Projection onto a line

The projection of the vector 'b' onto a line through a is 'p' where, $\boxed{p = \hat{x} a}$ and an estimate $\hat{x}$ is given by $\hat{x} = \frac{a^T b}{\|a\|^2}$



$p = \hat{x} a$.

3) Q17 Find projection of b onto a line through 'a' where $b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The projection of b onto a is $p = \hat{x} a$, where $\hat{x} = \frac{a^T b}{\|a\|^2}$

$a^T b = \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}_{2\times 1} = [\cos\theta]$.

$\|a\|^2 = 1^2 + 0^2 = 1$.

$\hat{x} = \frac{\cos\theta}{1} = \cos\theta$.

Hence, $p = \hat{x}\, a = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$ Ans

Q19 i) Find projection of b onto a line through 'a'

$b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

ii) Check that $e \perp a$.

Ans i) $p = \hat{x}\, a$.

$a^T b = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{3\times 1} = [1+2+2] = 5$.

Q5

$\|a\|^2 = 1+1+1 = 3$.

$\hat{x} = \frac{a^T b}{\|a\|^2} = \frac{5}{3}$

$\therefore p = \hat{x}\, a = \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$

ii) $e \perp a$.

$e = b - p$

$e = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$e \perp a$ means $e^T a = 0$.

$$e^T a = \begin{bmatrix} -2/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{-2}{3} + \frac{2}{3} = 0.$$

Since $e^T a = 0$, hence $e \perp a$.

Projection Matrix

To project b onto a, multiply by the projection matrix 'P' i.e., $\boxed{p = Pb}$

$p = \hat{x} a$

$p = a\hat{x}$

$p = \dfrac{a\, a^T b}{\|a\|^2}$

$p = Pb \Rightarrow \boxed{P = \dfrac{aa^T}{\|a\|^2}}$

Q5 Find the projection matrix onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$P = \frac{aa^T}{\|a\|^2}$$

$\|a\|^2 = 1 + 9 = 10$.

$$aa^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2\times1} \cdot \begin{bmatrix} 1 & 3 \end{bmatrix}_{1\times2} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$P = \frac{aa^T}{\|a\|^2} = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$ Ans.

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→ P is a symmetric matrix

→ $P^2 = P$

Trace

Sum of diagonal element.

v.v.imp.

Q8 Prove that the trace of a projection matrix is always equal to 1.

$$P = \frac{aa^T}{\|a\|^2}$$

Let $a = (a_1, a_2, \dots\dots, a_n)$

$$aa^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}_{1\times n}$$

$$= \begin{bmatrix} a_1^2 & a_1a_2 & \dots & a_1a_n \\ a_2a_1 & a_2^2 & \dots & a_2a_n \\ - & - & & - \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{bmatrix}_{n\times n}.$$

$$\|a\|^2 = a_1^2 + a_2^2 + \dots + a_n^2 .$$

$$\therefore P = \frac{1}{a_1^2 + a_2^2 + \dots + a_n^2} \begin{bmatrix} a_1^2 & a_1a_2 & \dots & a_1a_n \\ a_2a_1 & a_2^2 & \dots & a_2a_n \\ - & - & - & - \\ a_na_1 & a_na_2 & \dots & a_n^2 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{a_1^2}{\|a\|^2} & \frac{a_1a_2}{\|a\|^2} & \dots & \frac{a_1a_n}{\|a\|^2} \\ \frac{a_2a_1}{\|a\|^2} & \frac{a_2^2}{\|a\|^2} & \dots & \frac{a_2a_n}{\|a\|^2} \\ \frac{a_na_1}{\|a\|^2} & \frac{a_na_2}{\|a\|^2} & \dots & \frac{a_n^2}{\|a\|^2} \end{bmatrix}$$

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Trace of P : $\frac{a_1^2}{\|a\|^2} + \frac{a_2^2}{\|a\|^2} + \cdots + \frac{a_n^2}{\|a\|^2}$.

$= \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{\|a\|^2}$

$= \frac{\cancel{\|a\|^2}}{\cancel{\|a\|^2}} = 1$

$\therefore$ P = 1 proved.

3.2 - Q - ~~1~~, ~~3~~, ~~5~~, ~~8~~, ~~9~~, 11, ~~17~~, ~~19~~.

3.3 Projections and Least Squares

→ The least square solution to a problem $ax = b$ in one unknown is an estimate $\hat{x}$ of $\hat{x} = \frac{a^T b}{\|a\|^2}$

→ The least square solution to a problem $Ax = b$ in several variables is an estimate $\hat{x}$ of $\hat{x} = (A^TA)^{-1}A^Tb$

→ The projection of b onto the column space of a is the nearest point $A\hat{x}$ i.e.,
$\boxed{p = A\hat{x}}$

Projection Matrix :-

To project the vector b onto the column space of a, multiply by the projection matrix P.
i.e., $p = Pb$ where $P = A(A^TA)^{-1}A^T$

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3.3 Q1 i) Find the projection of 'b' onto the column space of A.

ii) Solve $Ax = b$ by using least square method.

iii) Verify that the error is $\perp$ to the column of A.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Ans. $b - C(A)$.

$p = a\hat{x}$

$$\hat{x} = \frac{a^T b}{\|a\|^2}$$

i) Projection of b onto column space of A is

$p = A\hat{x}$, where

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$\hat{x} = \left[\underset{2\times3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \cdot \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \right]^{-1} \underset{2\times3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$\hat{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x} = \underset{2\times2}{\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$\therefore p = A\hat{x}$

$$= \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

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ii) By using least square method, the solution of system $Ax=b$ is $\hat{x}$.

$$\hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

iii) To verify:- $e \perp C(A)$.

$$e = b - p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$e = (2/3, 2/3, -2/3)$

$a_1 = (1,0,1)$ & $a_2 = (0,1,1)$.

TP:- $e^T a_1 = 0$

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \cdot \cancel{[1\ 0\ 1]} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$\therefore e \perp a_1$.

TP:- $e^T a_2 = 0$

$$\begin{bmatrix} 2/3 & 2/3 & -2/3 \end{bmatrix}_{1\times 3} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{3\times 1} = 0$$

$\therefore e \perp a_2$

$\therefore e \perp C(A)$.

Least Squares fitting of data

→ Let the measurements $b_1, b_2, \ldots, b_n$ are given at the distinct points $t_1, t_2, \ldots, t_n$.

→ Let the straight line equation fit to the data is $b = C + Dt$.

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→ For the given measurements, we have the following system of equations:

$$b_1 = C + Dt_1$$
$$b_2 = C + Dt_2$$
$$\cdots$$
$$b_n = C + Dt_n$$

→ The above system in matrix form can be written as $Ax = b$ where

$$A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_n \end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

→ Use least square method to find the solution of the system $Ax = b$.

$$\hat{x} = (A^TA)^{-1} A^T b$$

Q9- Find the best straight line fit to the measurements $b = 4$ at $t = -2$.

$b = 3$ at $t = -1$

$b = 1$ at $t = 0$

$b = 0$ at $t = 2$.

Solⁿ:- Let the straight line fit to the data is $b = C + Dt$.

For the given measurements, we have the following equations:

$$4 = C - 2D \quad \text{— (i)}$$
$$3 = C - D \quad \text{— (ii)}$$
$$1 = C \quad \text{— (iii)}$$
$$0 = C + 2D \quad \text{— (iv)}$$

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3.3 — ~~1~~, 2, 4, 6, ~~9~~, 12

4 + 1 + 4

The above system in matrix form can be written as $Ax = b$ where,

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

By using least square method, the solution is $\hat{x}$.

$$\hat{x} = (A^TA)^{-1} A^Tb.$$

$$\hat{x} = \left(\underset{2\times4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}} \underset{4\times2}{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}} \right)^{-1} \left(\underset{2\times4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}} \right)$$

$$\hat{x} = \begin{bmatrix} 4 & -1 \\ -1 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$$\hat{x} = \underset{2\times2}{\begin{bmatrix} 9/35 & 1/35 \\ 1/35 & 4/35 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 8 \\ -11 \end{bmatrix}} = \begin{bmatrix} 61/11 \\ -36/35 \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

Hence the straight line eqn is $b = C + Dt$.

$$\boxed{b = \frac{61}{35} - \frac{36}{35}t} \quad \text{Ans}$$

3.4 Orthogonal Bases & Gram Schmidt

Orthonormal Basis

The vector $q_1, q_2, \dots, q_n$ are orthonormal if $q_i^T q_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

Orthogonal Matrix (Q):

A matrix with orthonormal columns is called an orthogonal matrix.

Properties:-

→ If Q is a square matrix with orthonormal columns, then $Q^T = Q^{-1}$.

→ If Q is a rectangular matrix with orthonormal columns, then Q^T = left inverse

→ An identity matrix is an orthogonal matrix.

→ Any permutation matrix 'P' is an orthogonal matrix.

→ The standard basis $e_1, e_2, \ldots, e_n$ are orthonormal vectors where

$e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, \ldots, 0)$

$e_n = (0, 0, \ldots, 1)$.

26/10/17. **The Gram Schmidt Process:-**

Let q_1, q_2, q_3 are the orthonormal vectors corresponding to the vectors a, b, c respectively.

1st vector:-

$$q_1 = \frac{a}{\|a\|}$$

2nd vector:-

$$q_2 = \frac{B}{\|B\|}$$

where $B = b - (q_1^T b) q_1$

3rd vector:-

$$q_3 = \frac{C}{\|C\|}$$

where $C = c - (q_1^T c) q_1 - (q_2^T c) q_2$

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The Factorization $A = QR$:-

The factorization of matrix 'A' can be written as $A = QR$ where Q is an orthogonal matrix & R is an upper triangular matrix.

$$Q = [q_1 \; q_2 \; q_3]$$

$$R = \begin{bmatrix} q_1^Ta & q_1^Tb & q_1^Tc \\ 0 & q_2^Tb & q_2^Tc \\ 0 & 0 & q_3^Tc \end{bmatrix}$$

Q Whether the following matrix is orthogonal or not.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

An Let $q_1 = (\cos\theta, \sin\theta)$

$q_2 = (-\sin\theta, \cos\theta)$

$$q_1^T q_2 = [\cos\theta \;\; \sin\theta] \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0$$

$q_2^T q_1 = 0$.

Length of vector :- $\sqrt{\cos^2\theta + \sin^2\theta} = 1$.

$$q_i^T q_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$i, j = 1, 2$.

Hence the given matrix Q is an orthogonal matrix.

Q Find the inverse of the matrix, $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$Q^{-1} = Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The given matrix is an orthogonal matrix.
Hence, $Q^{-1} = Q^T$.

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Find inverse of matrix A.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Since it is a permutation matrix, so the matrix A is an orthogonal matrix.

$$\therefore Q^{-1} = Q^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

3.4 Q9 Apply the gram schmidt process to the vectors.

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{a}{\|a\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{B}{\|B\|} \qquad B = b - (q_1^T b) q_1$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \left\{ \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\therefore q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - [1]_{1\times 1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\|B\| = 1$$

$$q_3 = \frac{C}{\|C\|}$$

$$C = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left\{ \left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} - \left\{ \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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$$C = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$\|C\| = 1$

$$\therefore\ q_3 = \frac{C}{\|C\|} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Q20 Apply gram schmidt to vector, $a = \begin{bmatrix}1\\2\\2\end{bmatrix}$ & $b = \begin{bmatrix}1\\3\\1\end{bmatrix}$

$$q_1 = \frac{a}{\|a\|} \qquad \|a\| = \sqrt{1+4+4} = 3$$

$$q_1 = \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix} \qquad q_2 = \frac{B}{\|B\|}$$

$$B = b - (q_1^T b) q_1$$

$$B = \begin{bmatrix}1\\3\\1\end{bmatrix} - \left\{ \left(\begin{bmatrix}1/3 & 2/3 & 2/3\end{bmatrix} \begin{bmatrix}1\\3\\1\end{bmatrix} \right) \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix} \right\}$$

$$B = \begin{bmatrix}1\\3\\1\end{bmatrix} - \begin{bmatrix}3\\9\\3\end{bmatrix} \cdot \begin{bmatrix}1/3\times3\\2/3\times3\\2/3\times3\end{bmatrix} \qquad \cancel{\Rightarrow B = \begin{bmatrix}1\\3\\1\end{bmatrix} - \begin{bmatrix}1\\2\\2\end{bmatrix}}$$

$$\cancel{\Rightarrow B = \begin{bmatrix}0\\-3\\-1\end{bmatrix}}$$

$$\cancel{\|B\| = \sqrt{9+1} = \sqrt{10}}$$

$$\cancel{\therefore B =}$$

$$B = \begin{bmatrix}1\\3\\1\end{bmatrix} - \begin{bmatrix}1\\2\\2\end{bmatrix} \Rightarrow B = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\|B\| = \sqrt{2}.$$

$$\therefore\ q_2 = \begin{bmatrix}0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} \quad \text{Ans.}$$