SUBLINEAR EXPANDERS

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UCL

BCC
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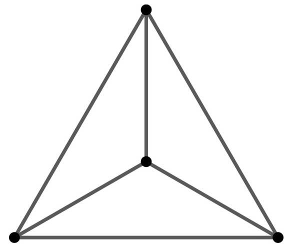
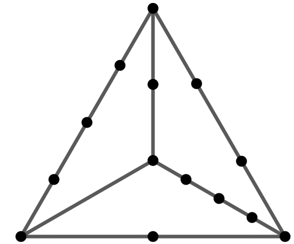
Joint with António Girão, Tao Jiang, Abhishek Methuku, and Liana Yepremyan

Subdivisions:



K_4



K_4-subdivision

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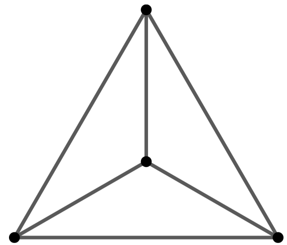

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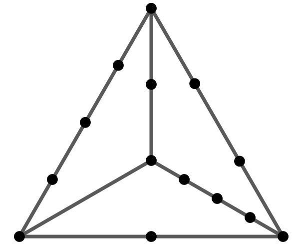

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Question (Mader '67, Erdős-Hajnal '69): What is

$$\mathrm{sub}(k) := \min\{d : \text{if } d(G) \geq d \text{ then } G \text{ contains a } K_k\text{-subdivision}\}?$$

average degree of G

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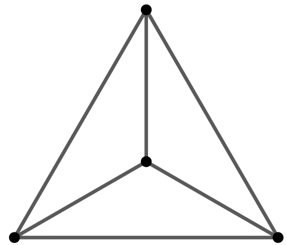

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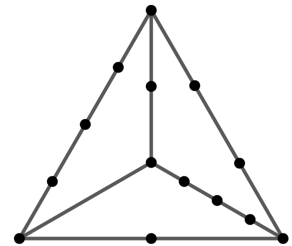

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$d(G)$: average degree of G

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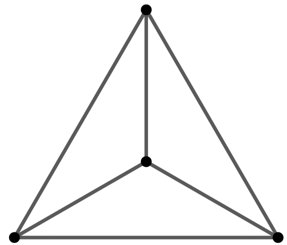

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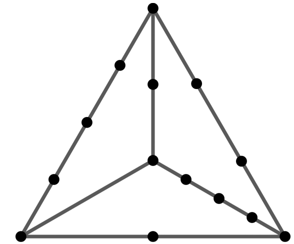

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Komlós–Szemerédi '96 / Bollobás–Thomason '96: $\mathrm{sub}(k) = O(k^2)$.

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* Have desirable properties, making them useful in combinatorics and related fields.
* Can be found in random graphs, with high probability.

G is an **(ε, t)-expander** if $|N(U)| \geq \rho_{\varepsilon,t}(|U|) \cdot |U|$ for $U \subseteq V(G)$ with $\frac{t}{2} \leq |U| \leq \frac{|G|}{2}$, where $\rho_{\varepsilon,t}(x) = \dfrac{\varepsilon}{\left(\log\left(\frac{15x}{t}\right)\right)^2}$.

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    (so every graph on $< \frac{t}{2}$ vertices is an (ε,t)-expander).

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- <u>Moshkowitz–Shapira '18</u>: Cannot guarantee an expander with expansion rate $\Omega\left(\frac{1}{(\log n)^{1-c}}\right)$.

- <u>Shapira–Sudakov '15</u>: Can always find an expander with expansion rate $\Omega\left(\frac{1}{\log n\,(\log\log n)^2}\right)$.

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  This works because between every two vertices there is a path of length $O((\log n)^3)$ avoiding any smallish set of vertices.

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<u>**Best bounds:**</u> $(1+o(1))\frac{9k^2}{64} \le \mathrm{sub}(k) \le (1+o(1))\frac{10k^2}{23}$.

bipartite random graphs

Kühn–Osthus '06

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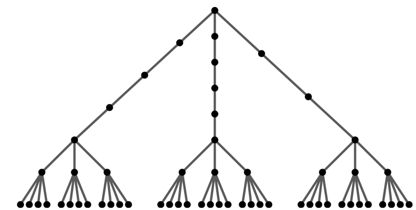

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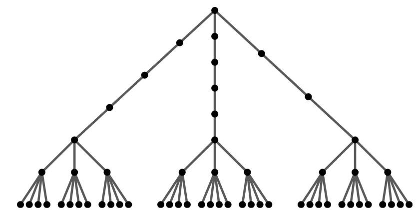

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    - Montgomery also proves an analogous result for minors, improving on Shapira-Sudakov '15.

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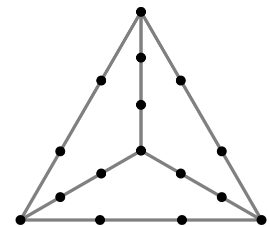

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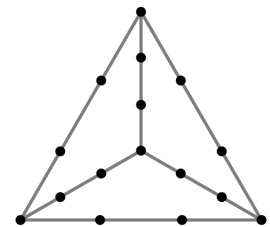

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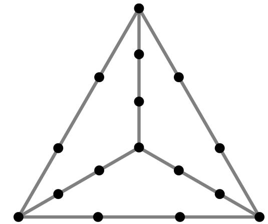

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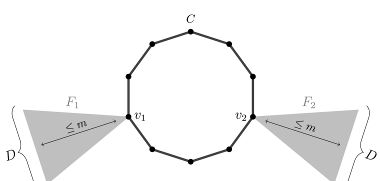
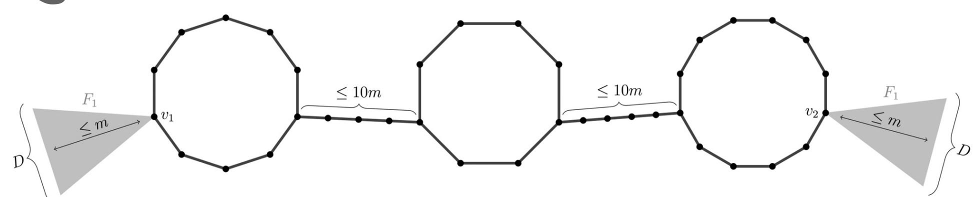
Uses ideas from Liu–Montgomery '23 resolving Erdős and Hajnal's odd cycle problem.



simple adjuster



adjuster

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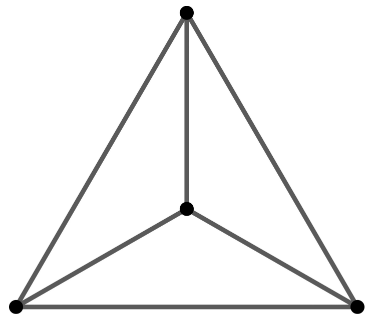

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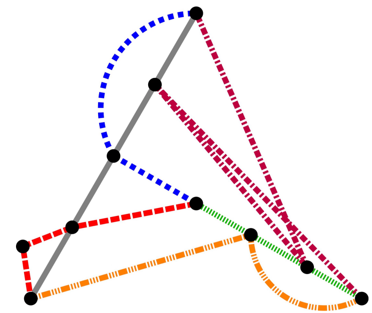

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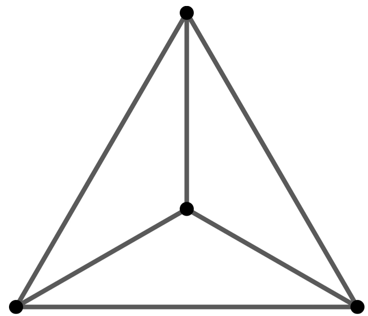

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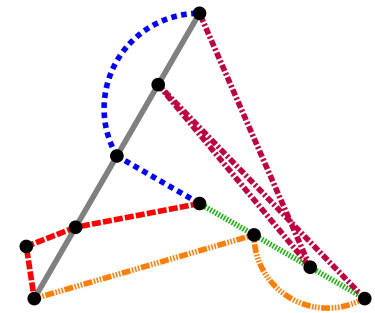

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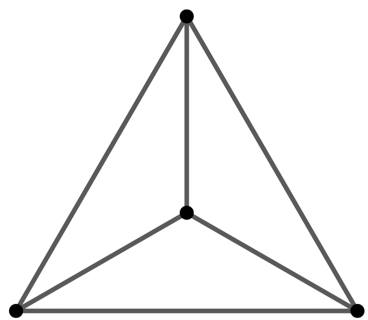

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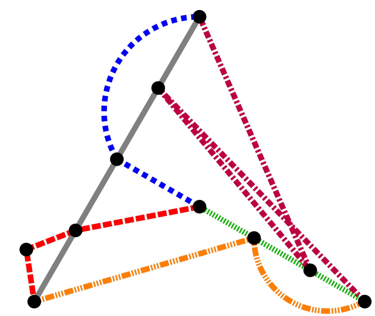

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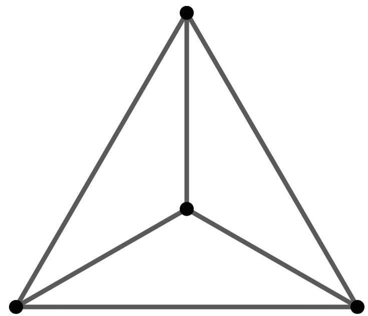

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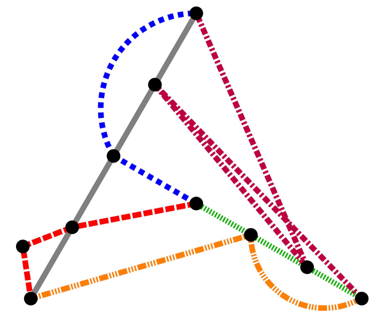

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Dvořák–Yepremyan '18: $\delta(G) \geq 11k+7 \Rightarrow K_k$-immersion.

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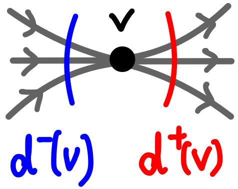
* Liu-Wang-Yang '22: Yes, if G is $K_{s,t}$-free.

Question. If $d(G) \geq \mathrm{imm}(k)+\varepsilon$, is there a K_k-immersion on $O_\varepsilon(\log|G|)$ vertices?

Question. Is there $c>0$ s.t. if $d(G) \geq ck$ then G has a 'balanced' K_k-immersion?

A directed graph G is **Eulerian** if $d^+(v)=d^-(v)$ for every vertex v.



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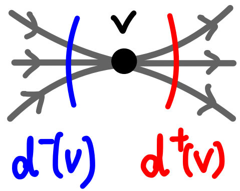

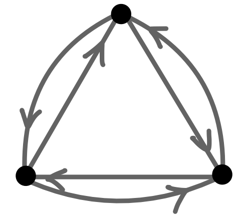

$\vec{K}_3$

Girão-L. '23: If G is an Eulerian directed graph with min deg $\geq ck$ then it has a $\vec{K}_k$-immersion.

Complete digraph on k vertices

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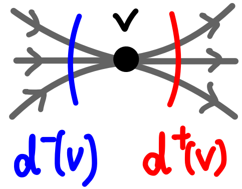

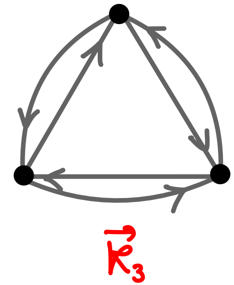

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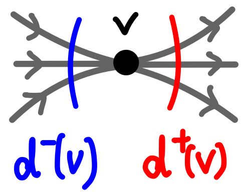

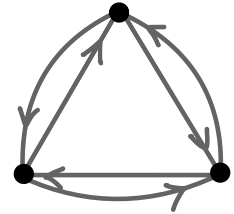

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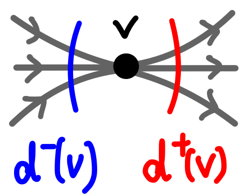

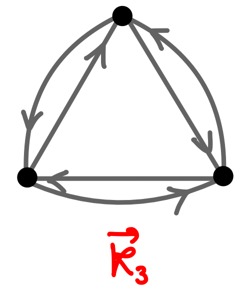

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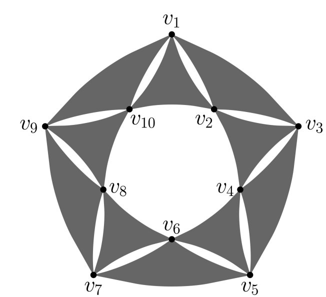
An r-uniform **tight cycle** of length ℓ has vertices $\{v_1, \ldots, v_\ell\}$ and edges $\{v_i \ldots v_{i+r-1 \pmod{\ell}} : i \in [\ell]\}$.



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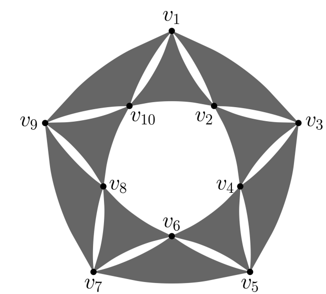

a tight 3-uniform cycle of length 10

Question (Sós/Verstraëte): What is the maximum number of edges in an n-vertex r-uniform hypergraph with no tight cycles?

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* Strengthened above property to:

  For every edge e in an expander H, can reach almost every edge via a short tight path so that no vertex (outside e) is overused.

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Question: (Keevash–Mubayi–Sudakov–Verstraëte '06)
What is the maximum number of edges in a properly-edge-coloured n-vertex graph with no rainbow cycles?

Question: Let $k \geq 3$. What is the maximum number of edges in a properly-edge-coloured n-vertex graph with no rainbow K_k-subdivision?

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O. Janzer '22: $O(n (\log n)^4)$. [Counting homomorphisms.]

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* Found almost regular expanders
* Used the fast mixing time of random walks in expanders.

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Wang '21: $O\left(n(\log n)^{2+o(1)}\right)$.

* Optimised Tomon's approach for this particular problem.

Back to rainbow cycles

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More results

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* Haslegrave-Kim-Liu '22: subdivisions of sparse graphs.
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