Packing subgraphs in regular graphs

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Abstract

An *H-packing* in a graph G is a collection of pairwise vertex-disjoint copies of H in G. We prove that for every $c > 0$ and every bipartite graph H, any $\lfloor cn \rfloor$-regular graph G admits an H-packing that covers all but a constant number of vertices. This resolves a problem posed by Kühn and Osthus in 2005. Moreover, our result is essentially tight: the conclusion fails if G is not both regular and sufficiently dense, it is in general not possible to guarantee covering all vertices of G by an H-packing, and if H is non-bipartite then G need not contain any copies of H.

We also prove that for all $c > 0$, integers $t \geq 2$, and sufficiently large n, all the vertices of every $\lfloor cn \rfloor$-regular graph can be covered by vertex-disjoint subdivisions of K_t. This resolves another problem of Kühn and Osthus from 2005, which goes back to a conjecture of Verstraëte from 2002.

Our proofs combine novel methods for balancing expanders and super-regular subgraphs with a number of powerful techniques including properties of robust expanders, regularity lemma, and blow-up lemma.

1 Introduction

Given two graphs H and G, an *H-packing* (or an *H-tiling*) in G is a collection of pairwise vertex-disjoint copies of H in G. An H-packing in G is called *perfect* if it covers all the vertices of G. Note that when H consists of a single edge, an H-packing is a graph matching. Tutte's theorem characterizes those graphs which have a perfect H-packing if H is an edge, but no such characterisation is known for other connected graphs H. In fact, it is known [19] that the decision problem of whether a graph G has a perfect H-packing is NP-complete if and only if H has a component which contains at least three vertices. So it is natural to seek simple sufficient conditions that guarantee the existence of a perfect H-packing in a given graph.

A fundamental result in this area is the Hajnal–Szemerédi theorem [13] from 1970 which states that every graph whose order n is divisible by t and whose minimum degree is at least $(1 - \frac{1}{t})n$ contains a perfect K_t-packing. For arbitrary graphs, Komlós, Sárközy and Szemerédi [22] showed that for every graph H, there exists a constant $C = C(H)$ such that every graph G whose order n is divisible by $|V(H)|$ and whose minimum degree is at least $(1 - \frac{1}{\chi(H)})n + C$ contains a perfect H-packing. It turns out that there are graphs H for which the bound on the minimum degree can be improved significantly i.e., one can often replace $\chi(H)$ by a smaller parameter called the *critical chromatic number* $\chi_{cr}(H)$ of H, which is defined as $\chi_{cr}(H) := (\chi(H) - 1)\frac{|H|}{|H| - \sigma(H)}$. Here $\sigma(H)$ denotes the minimum size of the smallest colour class in an optimal colouring of H. It is easy to see that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$. In 2000, Komlós [20] showed that the critical chromatic number is the parameter that governs the existence of almost perfect packings in graphs of large minimum degree. Finally, in 2009, Kühn and Osthus [32] determined, up to an additive constant, the minimum degree of a graph G that ensures the existence of a perfect H-packing in G, for every graph H. More precisely, they proved that for every graph H, either its critical chromatic number or its chromatic number is the parameter which governs the existence of perfect H-packings in graphs of large minimum degree (where the exact classification depends on a parameter called the *highest common factor* of H).

1.1 H-packings in dense regular graphs

In view of the above results, rather surprisingly, Kühn and Osthus [31] showed that if we restrict our attention to packings in regular graphs, then *any* linear bound on the degree guarantees an almost perfect H-packing.

Theorem 1.1 (Kühn and Osthus [31])**.** *For every bipartite graph H and every $0 < c, \alpha \leq 1$, there exists $n_0 = n_0(H, c, \alpha)$ such that every d-regular graph G of order n, where $d \geq cn$ and $n \geq n_0$, has an H-packing that covers all but at most αn vertices of G.*

It is easy to see that the restriction to (sufficiently) regular graphs G in Theorem 1.1 is necessary. Indeed, if G is, say, a complete bipartite graph $K_{n/4,3n/4}$ and H is an edge, then it is clearly impossible to find an H-packing covering almost all the vertices of G. The restriction to bipartite graphs H in Theorem 1.1 is also necessary because if $c \leq 1/2$, then G could be a bipartite graph (in which case it cannot contain any subgraph H which is non-bipartite).

This raises the natural question of whether the bound αn on the number of uncovered vertices in Theorem 1.1 can be lowered significantly to obtain an H-packing which is close to being perfect. Indeed, Kühn and Osthus [31] proposed the following general problem in 2005.

Problem 1.2 (Kühn and Osthus [31])**.** *Is it true that, for every bipartite graph H and every $0 < c \leq 1$, there is a constant $C = C(H, c)$ such that every d-regular graph G of order n, where $d \geq cn$, has an H-packing that covers all but at most C vertices of G?*

Kühn and Osthus [31] resolved Problem 1.2 in the special case when the parts of H have unequal sizes. More precisely, they showed the following.

Theorem 1.3 (Kühn and Osthus [31])**.** *For every bipartite graph H whose vertex classes have different size and for every $0 < c \leq 1$, there is a constant $C = C(H, c)$ such that every d-regular graph G of order n, where $d \geq cn$, has an H-packing which covers all but at most C vertices of G.*

Despite extensive research on graph packings, this problem (Problem 1.2) remained open for nearly twenty years. Our main result is a complete resolution of Problem 1.2 as follows.

Theorem 1.4. *For every bipartite graph H and every $0 < c \leq 1$, there is a constant $C = C(H, c)$ such that every d-regular graph G of order n, where $d \geq cn$, has an H-packing that covers all but at most C vertices of G.*

As noted in [31], it is necessary to allow for uncovered vertices in Theorem 1.4, and the upper bound C on the number of uncovered vertices must depend on c and H (even if n is divisible by $|V(H)|$), in sharp contrast to the result of Komlós, Sárközy and Szemerédi [22] mentioned earlier. Indeed, for instance, if G is the vertex-disjoint union of cliques of order $k|V(H)| - 1 = d + 1$ (for some positive integer k), then G is d-regular, and it is easy to see that we must have at least $(|V(H)| - 1) \cdot \frac{n}{d+1}$ uncovered vertices in any H-packing of G. Moreover, this example also shows that d must be linear in n in order for the number of uncovered vertices to be bounded by a constant C, so the requirement that the vertex degrees of G are linear in n in Theorem 1.4 is also necessary.

It is easy to reduce Theorem 1.4 to the special case when H is a complete bipartite graph $K_{t,t}$. This follows, for example, from the fact that any bipartite graph H can be perfectly packed into the complete bipartite graph $K_{|H|,|H|}$. To prove Theorem 1.4 when $H = K_{t,t}$, we start by partitioning the vertex set of our dense regular graph G into a small number of expanders using a structural decomposition result from the work of Gruslys and Letzter [11], which goes back to the work of Kühn, Lo, Osthus and Staden [27, 28] on Hamilton cycles in dense regular graphs. These expanders are, in turn, closely related to a powerful notion called *robust expanders* that was introduced by Kühn, Osthus and Treglown [34] and has since been used to prove several longstanding conjectures (see, e.g., [29, 30, 33]).

One of the key contributions of our paper is a novel method for 'balancing' these expanders. This method enables us to obtain expanders that are either balanced bipartite or far from bipartite by removing a carefully constructed, small $K_{t,t}$-packing. Our main idea is to track a certain subgraph, which initially consists of the edges between expanders. We then sequentially move vertices between expanders and update the subgraph accordingly, ensuring that by the end, it has bounded maximum degree while still containing enough edges to construct a $K_{t,t}$-packing that balances

the expanders (up to an additive constant, which is sufficient for our purposes). Note that while balancing expanders also plays a key role in [27, 28]—in resolving a problem of Bollobás and Häggkvist on Hamilton cycles in regular graphs—and in [11], which proves a conjecture of Magnant and Martin [39] for dense graphs, our approach differs significantly from the methods used in these works, so it is of independent interest. See Section 2.1 for further details on our balancing procedure.

After balancing the expanders, the original problem reduces to finding an almost perfect $K_{t,t}$-packing either in a balanced bipartite expander or in an expander that is far from bipartite. To address the latter case, our first goal is to apply the regularity lemma to obtain vertex-disjoint super-regular pairs covering all but a small set of exceptional vertices. For this, rather than directly seeking a perfect matching in the reduced graph, we exploit the Hamiltonicity of expanders to find a perfect *fractional* matching (inspired by [16, 17, 25]). This fractional matching is used to construct a collection of edges and odd cycles covering all vertices of the reduced graph. We then refine the clusters by splitting them each into two equal parts and then remove a small number of vertices from the new clusters to obtain vertex-disjoint super-regular pairs covering all but a small exceptional set of vertices, as required.

Second, we cover the exceptional vertices with a $K_{t,t}$-packing, while also ensuring that none of the super-regular pairs are overused. Third, we construct another small $K_{t,t}$-packing, whose purpose is to ensure that the number of remaining vertices in each part of every super-regular pair is divisible by t. Ideally, at this point, one could attempt to complete the proof by applying the blow-up lemma to find a perfect $K_{t,t}$-packing covering the uncovered portion of each super-regular pair. However, this is not immediately feasible because the number of remaining vertices in the two parts of a super-regular pair may differ. To address this, we introduce another novel balancing strategy: we construct a suitable matching within a small subgraph induced by the uncovered vertices, once again exploiting the Hamiltonicity of expanders. This matching serves as a 'template' for constructing a carefully designed balancing $K_{t,t}$-packing, whose removal allows us to find a perfect $K_{t,t}$-packing covering all remaining vertices using the blow-up lemma, as intended. We give a detailed sketch of this balancing strategy in Section 2.2, and a more detailed outline of our methods in Section 2.

1.2 Packing subdivisions in dense regular graphs

Given a graph F, a *subdivision* of F, denoted by TF, is a graph obtained by replacing the edges of F with pairwise internally vertex-disjoint paths between the corresponding ends, whose interiors avoid the vertices of F. In this case, we refer to the vertices of F as the *branch vertices* of TF. For instance, a subdivision TK_t of the complete graph of order t consists of t (branch) vertices $\{v_1, \ldots, v_t\}$ and $\binom{t}{2}$ pairwise internally vertex-disjoint paths $P_{i,j}$, $1 \le i < j \le t$, such that $P_{i,j}$ joins v_i and v_j and avoids all other vertices in $\{v_1, \ldots, v_t\}$.

The notion of subdivisions has played an important role in topological graph theory since the seminal result of Kuratowski [35] from 1930 showing that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$. One of the most classical results in this area is due to Mader [38], who showed that there is some $d = d(t)$ such that every graph with average degree at least d contains a subdivision of the complete graph K_t. Mader [38], and independently Erdős and Hajnal [9] conjectured that $d(t) = O(t^2)$. In the 1990s, Komlós and Szemerédi [23, 24], and independently, Bollobás and Thomason [4] confirmed this conjecture. Since then, various extensions and strengthenings of this result have been studied, see, e.g., [10, 37, 40].

Given graphs F and G, a TF-*packing* in G is a collection of pairwise vertex-disjoint copies of subdivisions of F in G (which are not required to be isomorphic). In 2002, Verstraëte [44] made the following conjecture.

Conjecture 1.5 (Verstraëte [44]). *For every graph F and every $\eta > 0$, there exists $d_0 = d_0(F, \eta)$ such that, for all integers $d \ge d_0$, every d-regular graph G of order n contains a TF-packing which covers all but at most ηn vertices of G.*

Quite a lot of research has established Conjecture 1.5 in several natural special cases. A result of Kelmans, Mubayi and Sudakov [18] proved the conjecture when F is a tree, and in 2003, Alon [1] proved it when F is a cycle. In 2005, Kühn and Osthus [31] observed that when G is dense (i.e., $d = \Omega(n)$) the conjecture follows from their results on packings in dense regular graphs. Making significant progress towards the general case, Letzter, Methuku and

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Sudakov [36] applied their work on nearly-Hamilton cycles in sublinear expanders to show that the conjecture holds for all graphs F when G has degree at least polylogarithmic in n. Finally, very recently, Montgomery, Petrova, Ranganathan and Tan [42] resolved the conjecture in full.

For subdivisions of complete graphs K_t, Kühn and Osthus [31] went even further and showed that one can actually guarantee perfect packings in dense regular graphs for $t = 4$ and $t = 5$. More precisely, they showed that for every $0 < c \leq 1$, there is a positive number $n_0 = n_0(c)$ such that every d-regular graph G of order n, with $d \geq cn$ and $n \geq n_0$, has a perfect TK_t-packing for $t = 4$ and $t = 5$. By a result of Gruslys and Letzter [11], this also holds for $t = 2$ and $t = 3$.

In 2005, Kühn and Osthus asked whether the same holds for $t \geq 6$.

Problem 1.6 (Kühn and Osthus [31])**.** Given $t \geq 6$ and $0 < c \leq 1$, does every d-regular graph G of order n, where $d \geq cn$ and n is sufficiently large, have a perfect TK_t-packing?

Kühn and Osthus's result [31] actually guarantees a perfect TK_t-packing, with $t \in \{4, 5\}$, even when G is only assumed to be almost regular. In contrast, for $t = 3$ or $t \geq 6$, the restriction to regular graphs G in Problem 1.6 is necessary. Indeed, it is easy to see that the complete bipartite graph $K_{m,m+1}$ does not have a perfect TK_t-packing if $t = 3$ (note that in this case a TK_3-packing corresponds to a cycle partition, i.e., a collection of pairwise vertex-disjoint cycles covering all vertices). Interestingly, Kühn and Osthus showed that $K_{m,m+1}$ also does not have a perfect TK_t-packing if $t \geq 6$ (see Proposition 5.1 in [31]).

Our second result is the complete resolution of Problem 1.6. In fact, we prove a stronger result, showing that there is a perfect TF-packing, for every graph F.

Theorem 1.7. *For every graph F and $0 < c \leq 1$, there is a positive number $n_0 = n_0(c)$ such that every d-regular graph G of order n, where $d \geq cn$ and $n \geq n_0$, has a perfect TF-packing.*

It is unclear whether the requirement that the vertex degrees in G are linear in n in Theorem 1.7 is necessary. In this direction, Kühn and Osthus [31] gave an example showing that for all $t \geq 3$ the vertex degrees in G must be at least $\sqrt{n/2}$ to guarantee a perfect TK_t-packing in G, even if G is regular. Interestingly, the case $t = 2$ is different. It is a simple exercise to show that every d-regular graph admits a perfect TK_2-packing, i.e., its vertex set can be partitioned into vertex-disjoint paths of positive length. (In fact, if d is even, then by Petersen's 2-factor theorem [43] it also admits a perfect TK_3-packing.) It is therefore natural to ask how few paths are required to achieve such a partition. In this direction, Magnant and Martin [39] conjectured that the vertices of every d-regular graph on n vertices can be covered by at most $n/(d + 1)$ pairwise vertex-disjoint paths. This conjecture of Magnant and Martin was confirmed for $d \leq 5$ by Magnant and Martin themselves and for dense graphs G by Gruslys and Letzter [11]. Montgomery, Müyesser, Pokrovskiy and Sudakov [41] showed that almost all vertices of every n-vertex d-regular graph can be covered by at most $n/(d + 1)$ vertex-disjoint paths. Very recently, Christoph, Draganić, Girão, Hurley, Michel and Müyesser [6] proved that every d-regular graph can be covered by at most $2n/(d + 1)$ vertex-disjoint paths, thereby proving the Magnant–Martin conjecture up to a constant factor and confirming a conjecture of Feige and Fuchs (see also [5]).

To prove Theorem 1.7, we begin by partitioning the vertex set of the dense regular graph G into a small number of expanders, following the approach used in the proof of Theorem 1.4. In this setting, however, we employ a lemma from [11] that provides a small linear forest H that balances the expanders, ensuring that each expander contains either zero or two leaves of H. We then modify this linear forest so that at most one component of the new linear forest is associated with each expander, with both leaves of the component contained within that expander. Next, we construct a pair of small, vertex-disjoint F-subdivisions in each expander, while also ensuring that their union remains balanced. Finally, we *absorb* the components of H, along with any remaining uncovered vertices in each expander, into these F-subdivisions. This absorption step again relies on the Hamiltonicity of expanders and a robust connectivity property, which guarantees that any two vertices can be joined by a short path that avoids a small set of forbidden vertices. See Section 7 for a more detailed sketch of these ideas.

1.3 Organisation of the paper

In the next subsection, we present the notation used throughout the paper. In Section 2, we outline a detailed proof sketch of our main result, Theorem 1.4, on almost perfect packings in dense regular graphs. In Section 3 we collect some tools and lemmas that are used throughout the paper. Key to our proof of Theorem 1.4 are two lemmas – Lemma 4.1 (for balancing expanders) and Lemma 5.1 (for packing $K_{t,t}$'s in expanders). We prove Lemma 4.1 in Section 4 and Lemma 5.1 in Section 5, and complete the proof of Theorem 1.4 in Section 6. In Section 7, we prove our second result, Theorem 1.7, which addresses perfect packings of subdivisions in dense regular graphs. A detailed sketch of its proof is given at the beginning of Section 7.

1.4 Notation

Let $[m]$ denote the set $\{1, \ldots, m\}$. We write $c = a \pm b$ if $a - b \leq c \leq a + b$. We use the '$\ll$' notation to state many of our results. We write $a \ll b$ to mean that there exists a non-decreasing function $f : (0, 1] \mapsto (0, 1]$ such that the subsequent result holds for all $a, b \in (0, 1]$ with $a \leq f(b)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way and are to be read from right to left.

For a graph G, let $e(G)$ denote the number of edges in G and, for two sets $X, Y \subseteq V(G)$, we let $G[X, Y]$ denote the subgraph of G with the vertex set $X \cup Y$, whose edges are the edges of G with one end in X and the other end in Y, and we let $e_G(X, Y)$ denote the number of edges in $G[X, Y]$. If $X = Y$, for convenience, we write $G[X]$ instead of $G[X, X]$, and $e_G(X)$ instead of $e_G(X, X)$. The maximum degree of a graph G is denoted by $\Delta(G)$. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbours of v in G. For a set $S \subseteq V(G)$ and $v \in V(G)$, let $d_G(v, S)$ denote the number of edges in G between v and S. If the host graph is clear from the context, sometimes we drop the subscript and write $e(X, Y), e(X), d(v, S), N(v)$ instead of $e_G(X, Y), e_G(X), d_G(v, S), N_G(v)$ respectively. For a set $U \subseteq V(G)$, let $\overline{U}$ denote the set $V(G) - U$, and let $G - U$ denote the subgraph of G induced by $V(G) - U$.

For a graph G, a *cut* of G is a partition $\{X, Y\}$ of $V(G)$, where X and Y are both non-empty. We say that a cut $\{X, Y\}$ is α-*sparse* if $e_G(X, Y) \leq \alpha|X||Y|$. If G is an n-vertex graph, we say that G is α-*almost-bipartite* if there exists a partition $\{X, Y\}$ of $V(G)$ such that G has at most αn^2 edges that are not in $G[X, Y]$. Otherwise we say that G is α-*far-from-bipartite*.

For a graph G and vertices x, y in G, an (x, y)-*path* in G is a path with ends x and y. If P is an (x, y)-path and Q is a (y, z)-path, we denote by $xPyQz$ the concatenation of the paths P and Q.

Using a slight abuse of notation, for a collection $\mathcal{K}$ of subgraphs in a graph G, we denote the set of vertices covered by these subgraphs as $V(\mathcal{K})$. Throughout this paper, we often omit floor and ceiling signs when dealing with large numbers, whenever they are not crucial.

2 Sketch of the proof of Theorem 1.4

In this section, we present a sketch of the proof of our main result, Theorem 1.4. At a high level, our approach begins by partitioning the vertex set of the regular graph G into subsets with certain expansion properties, which we refer to as expanders. We then move some vertices between these expanders and remove a carefully constructed collection of $K_{t,t}$'s to 'balance' these expanders, using our first key lemma (Lemma 4.1). Next, we find almost perfect $K_{t,t}$-packings in each of the resulting expanders with the help of our second key lemma (Lemma 5.1). However, several significant challenges arise in implementing this approach, which we now address in detail.

Let H be a bipartite graph, and let $0 < c \leq 1$. Let G be a d-regular graph of order n, where $d \geq cn$. Let $t = |V(H)|$. Recall that Theorem 1.4 aims to find an H-packing that covers all but at most a constant number of vertices. It is easy to see that in order to find such a packing, it suffices to find a $K_{t,t}$-packing in G covering all but at most C vertices of G for some constant $C = C(t, c)$ because $K_{t,t}$ has a perfect H-packing.

Our proof begins by partitioning the vertex set of the regular graph G into a small number of subsets $Z_1, \ldots, Z_r$ such that each induced subgraph $G[Z_i]$, for $i \in [r]$, has strong expansion properties, and the number of edges between different sets Z_i is small. (See Lemma 3.2 for the precise statement.)

2.1 Balancing the expanders

Given the partition $\{Z_1, \ldots, Z_r\}$ of the vertices of G, a natural strategy for obtaining an almost perfect $K_{t,t}$-packing in G is to find an almost perfect $K_{t,t}$-packing in each of the expanders $G[Z_i]$. However, it is possible that some of these expanders are rather imbalanced bipartite graphs, rendering this plan impossible. To overcome this obstacle, in our first key lemma (Lemma 4.1), we modify the sets Z_i, and remove a small collection of vertex-disjoint $K_{t,t}$-copies and a very small number of additional vertices, so that the remainder of each modified set Z_i spans an expander which is either close to a balanced bipartite graph, or is far from being bipartite.

If the graph consisting of edges between expanders has small maximum degree, then it is relatively easy to realise this plan. Indeed, one can iteratively find vertex-disjoint $K_{t,t}$-copies in G between pairs of expanders, so that the number of $K_{t,t}$'s between a given pair of expanders is proportional to the number of edges between them, using an unbalanced variant of the Kővári–Sós–Turán theorem. Removing these $K_{t,t}$-copies results in expanders that are either close enough to being balanced bipartite, or are far from being bipartite. Of course, it is possible that the maximum degree of edges between expanders is large, even after moving some vertices between expanders. Nevertheless, we emulate the small minimum degree situation as follows. We initially set H to be the subgraph of edges with ends in distinct sets Z_i, $i \in [r]$, and then follow a carefully designed procedure (see Section 4.1) that moves vertices of high degree between expanders, and modifies H so as to satisfy certain invariants. After performing this procedure, H 'essentially' has small maximum degree[1]. Moreover, the invariants guarantee that only few vertices are moved throughout the procedure (which is essential for showing that the modified expanders are, indeed, expanders) and that H still has enough edges to construct a suitable $K_{t,t}$-packing, as described (vaguely) at the beginning of this paragraph.

Balancing expanders (in a different context) was also a key step in the proofs of Kühn, Lo, Osthus and Staden in [27, 28] and of Gruslys and Letzter in [11]. However, the method we develop here to construct a balanced $K_{t,t}$-packing differs significantly from these previous approaches, so it is of independent interest.

2.2 Packing $K_{t,t}$'s in expanders

Balancing the expanders (using Lemma 4.1) reduces the original problem to the problem of finding an almost perfect $K_{t,t}$-packing (covering all but a constant number of vertices) in an expander G that is either bipartite and balanced or far from bipartite. We address this reduced problem in our second key lemma, Lemma 5.1, where we construct a perfect $K_{t,t}$-packing under the assumption that the number of vertices is divisible by $2t$, using the following approach. First, we apply the regularity lemma (Lemma 3.5) to G to obtain vertex-disjoint clusters V_i, $i \in [m]$, covering all but a small number of exceptional vertices in G, and a subgraph G' with vertex set $V_1 \cup \ldots \cup V_m$. We will then find vertex-disjoint super-regular subgraphs $G'[U_{2i-1}, U_{2i}]$, $i \in [m]$, covering almost all vertices in $V_1 \cup \ldots \cup V_m$, such that the sets $U_1, \ldots, U_{2m}$ have the same size; for this, we reply on the Hamiltonicity of expanders to find a perfect *fractional* matching in the reduced graph (we explain this in more detail in the next paragraph). Next, we find three vertex-disjoint collections $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ of vertex-disjoint $K_{t,t}$-copies in G such that $\mathcal{K}_1$ covers all vertices outside of $U_1 \cup \ldots \cup U_{2m}$ while using very few vertices from each set U_i, $i \in [2m]$; $\mathcal{K}_2$ is very small and is chosen so that the number of vertices in U_i, $i \in [2m]$, not covered by $\mathcal{K}_1$ or $\mathcal{K}_2$ is divisible by t; and $\mathcal{K}_3$ is chosen so that the number of vertices in U_i that are uncovered by $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ is the same for all $i \in [2m]$. We then apply the blow-up lemma to find a perfect $K_{t,t}$-packing in each super-regular graph $G'[U'_{2i-1}, U'_{2i}]$, $i \in [m]$, where U'_i denotes the set of uncovered vertices in U_i. These latter $K_{t,t}$-packings, together with $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$, form the desired perfect $K_{t,t}$-packing in G.

For the construction of the sets $U_1, \ldots, U_{2m}$, we consider the 'reduced graph' Γ whose vertices are $V_1, \ldots, V_m$, where V_i is adjacent to V_j if and only if $G'[V_i, V_j]$ is ε-regular with large enough density. We will first find a 'perfect 2-matching' in Γ, that is, a collection of vertex-disjoint edges and cycles covering the vertices of Γ. This perfect 2-matching then corresponds to a perfect matching in the 'lift' of Γ, so if we partition each cluster V_i into two sets of equal size, then we obtain a perfect matching in the reduced graph whose vertices are these new clusters. Finally, by moving a small number of vertices from the new clusters into the exceptional set, we obtain the desired vertex-disjoint super-regular pairs $G'[U_{2i-1}, U_{2i}]$ for $i \in [m]$, covering all but a small fraction of the vertices in $V_1 \cup \cdots \cup V_m$.

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[1]More precisely, the subgraph of H consisting of edges between any two distinct expanders either has small maximum degree, or it has very few edges. In particular, the number of large degree vertices in H is at most a constant.

To find a perfect 2-matching in the reduced graph Γ, it suffices to find a perfect fractional matching (see, e.g., Fact 5.4 and its proof). Since $G'[V_1 \cup \cdots \cup V_m]$ is an almost regular expander, this in turn follows from the fact that almost regular expanders are Hamiltonian (provided they are far from bipartite, or are balanced and bipartite). The Hamiltonicity of almost regular expanders has many applications; see, for example, [11, 27, 28], and it follows from the work of Kühn, Osthus, and Treglown [34] on robust expanders. The idea of seeking perfect 2-matchings in reduced graphs was used previously in, e.g., [16, 17, 25].

One of the main novelties in our proof of the key lemma, Lemma 5.1, is in the construction of the collection $\mathcal{K}_3$ of vertex-disjoint $K_{t,t}$-copies in G for 'balancing' the number of uncovered vertices in the sets U_i. Our strategy here is to construct a suitable matching in G and use it as a template to construct the collection $\mathcal{K}_3$. More precisely, we consider a random subgraph H (with vertex set disjoint from $V(\mathcal{K}_1 \cup \mathcal{K}_2)$) such that $|V(H) \cap U_i| = \frac{|U_i|}{2t}$ for every $i \in [2m]$ and show that H 'inherits' the expansion properties of G. This implies that H is *robustly matchable*, i.e. that there is a perfect matching in H even after the removal of any small set of vertices (where the set we remove must balance the two parts of H if it is bipartite). In particular, if S is a set of vertices in H such that $|S \cap U_i| = \frac{|V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap U_i|}{t}$ for every $i \in [2m]$, then $H - S$ has a perfect matching $\mathcal{M}$. We now use $\mathcal{M}$ as a template to construct the collection $\mathcal{K}_3$ of $K_{t,t}$'s, by picking, for every distinct $i, j \in [2m]$, as many $K_{t,t}$'s between U_i and U_j as there are edges of $\mathcal{M}$ between U_i and U_j. This ensures that, for every $i \in [2m]$,

$$|V(\mathcal{K}_3) \cap U_i| = t \cdot |(V(H) - S) \cap U_i| = t \cdot \left(\frac{|U_i|}{2t} - \frac{|V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap U_i|}{t} \right) = \frac{|U_i|}{2} - |V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap U_i|,$$

since there are $|(V(H) - S) \cap U_i|$ edges in $\mathcal{M}$ that are incident to the vertices of U_i. Hence, taking $U_i' := U_i - V(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$, we have

$$|U_i'| = |U_i| - |V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap U_i| - |V(\mathcal{K}_3) \cap U_i| = \frac{|U_i|}{2},$$

showing that the sets U_i' with $i \in [2m]$ all have the same size. Now, we can indeed find a perfect $K_{t,t}$-packing in each super-regular graph $G'[U_{2i-1}', U_{2i}']$ using the blow-up lemma. As discussed earlier, these $K_{t,t}$-packings together with $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$ yield the desired perfect $K_{t,t}$-packing in G.

3 Useful lemmas and tools

3.1 Concentration inequality

We need the following concentration inequality for the hypergeometric distribution. Let $m, n, N \in \mathbb{N}$ such that $\max\{m, n\} < N$. Recall that a random variable X has *hypergeometric distribution* with parameters N, n, m if $X := |S \cap [m]|$, where S is a random subset of $[N]$ of size n.

Lemma 3.1 (Theorem 2.10 in [14])**.** *Let $X \sim \mathrm{Hypergeo}(N, m, n)$ be a hypergeometric random variable with mean $\mathbb{E}[X] = nm/N$. Then for all $0 \le \varepsilon \le 3/2$, we have*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \ge \varepsilon \mathbb{E}[X]\big) \le 2e^{-\varepsilon^2 \mathbb{E}[X]/3}.$$

3.2 Expanders in dense regular graphs

We will use the following structural result (Lemma 3.2) due to Gruslys and Letzter [11], which goes back to the work of Kühn, Lo, Osthus, and Staden [27, 28]. This result serves as the first step in the proofs of both Theorem 1.4 and Theorem 1.7. It shows that the vertices of a dense regular graph G can be partitioned into a small number of sets such that the subgraphs induced by these sets exhibit strong expansion properties and have a large minimum degree. Throughout the remainder of the paper, we refer to these subgraphs as *expanders*. The notion of expansion used here is characterized by the absence of sparse cuts. Specifically, a graph G has no ζ-sparse cuts if, for every partition $\{X, Y\}$ of $V(G)$, the number of edges between X and Y is at least $\zeta|X||Y|$. This notion of expansion was first introduced (using different terminology and with ζ as a function of n) in a paper by Conlon, Fox, and Sudakov [7].

We will distinguish between expanders that are close to bipartite and those that are far from bipartite. Recall that a graph G on n vertices is said to be γ-*almost-bipartite* if one can make G bipartite by removing up to γn^2 edges, and otherwise, G is called γ-*far-from-bipartite*.

Lemma 3.2 (Lemma 2.1 from [11])**.** *Let $c \in (0,1)$ and $n_0 \in \mathbb{N}$ satisfy $1/n_0 \ll c$. Let G be a d-regular graph on n vertices, where $n \geq n_0$ and $d \geq cn$. Then, there exist positive numbers $r \leq \lceil 1/c \rceil$ and $\eta, \beta, \gamma, \zeta, \delta$, where $1/n_0 \ll \eta \ll \beta \ll \gamma \ll \zeta \ll \delta \ll c$, and a partition $\{Z_1, \ldots, Z_r\}$ of $V(G)$ that satisfies the following properties.*

(E1) *G has at most ηn^2 edges with ends in distinct Z_i's.*

(E2) *$G[Z_i]$ has minimum degree at least δn, for $i \in [r]$.*

(E3) *$G[Z_i]$ has no ζ-sparse cuts, for $i \in [r]$.*

(E4) *$G[Z_i]$ is either β-almost-bipartite or γ-far-from-bipartite, for $i \in [r]$.*

The following lemma is used multiple times throughout the paper. Its proof is very similar to that of Lemma 2.2 in [11], and relies on results concerning Hamiltonicity of robust expanders, due to Kühn, Osthus and Treglown [34]. The lemma shows that expanders far from being bipartite are Hamiltonian and remain Hamiltonian even after the removal of any small set of vertices. However, expanders that are bipartite may not be Hamiltonian – for example, an imbalanced bipartite graph cannot be Hamiltonian. In this case, the lemma shows that such expanders still become Hamiltonian after the removal of any small set of vertices that balances its two parts. We prove Lemma 3.3 in Appendix B.

Lemma 3.3. *Let $1/n \ll \eta \ll \xi \ll \gamma, \zeta \ll c$, and let $d \geq cn$. Suppose G is an n-vertex graph with maximum degree at most d and average degree at least $d - \eta n$, such that G does not have ζ-sparse cuts. Additionally, assume that G is bipartite or γ-far-from-bipartite. Then the following holds.*

(HC1) *If G is bipartite with the bipartition $\{X, Y\}$, then, for every subset $W \subseteq V(G)$ of size at most ξn satisfying $|X - W| = |Y - W|$ and any vertices $x \in X - W$, $y \in Y - W$, there is a Hamilton path in $G - W$ with ends x and y. In particular, $G - W$ has a Hamilton cycle.*

(HC2) *If G is γ-far-from-bipartite, then, for every subset $W \subseteq V(G)$ of size at most ξn and any two distinct vertices $x, y \in V(G) - W$, there is a Hamilton path in $G - W$ with ends x and y. In particular, $G - W$ has a Hamilton cycle.*

We use the following lemma from [11] in the proof of Theorem 1.7. When presented with a partition of a dense regular graph into expanders (by applying Lemma 3.2), Lemma 3.4 produces a small linear forest (i.e., a small collection of vertex-disjoint paths) whose removal balances the expanders that are close to being bipartite. A similar linear forest was also constructed in [27, 28].

Lemma 3.4 (Lemma 2.3 from [11])**.** *Let $\eta, \beta, \xi, \gamma, \zeta, \delta, c \in (0,1)$ and let $n \in \mathbb{N}$ satisfy $1/n \ll \eta \ll \beta \ll \xi \ll \gamma \ll \zeta \ll \delta \ll c$. Let G be a d-regular graph on n vertices, where $d \geq cn$. Suppose that $\{Z_1, \ldots, Z_r\}$ is a partition of $V(G)$ satisfying properties (E1)– (E4) in Lemma 3.2, where $r \leq \lceil 1/c \rceil$. For $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite, let $\{X_i, Y_i\}$ be a partition of Z_i maximising $e_G(X_i, Y_i)$. Then there is a linear forest H in G with the following properties.*

(P1) *$|V(H)| \leq \xi n$.*

(P2) *H has no isolated vertices.*

(P3) *For each $i \in [r]$, Z_i contains either zero or two leaves of H. Moreover, for each $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite and Z_i contains two leaves of H, one of the leaves is in X_i and the other leaf is in Y_i.*

(P4) *For each $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite, $|X_i - V(H)| = |Y_i - V(H)|$.*

3.3 The regularity lemma and the blow-up lemma

Given a graph G, and sets $X, Y \subseteq V(G)$, let $d(X,Y) := \frac{e(X,Y)}{|X||Y|}$. The density of a bipartite graph G with the bipartition $\{A, B\}$ is denoted by $d(A, B)$.

Given $\varepsilon > 0$, we say that a bipartite graph G with the bipartition $\{A, B\}$ is ε-*regular* if, for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d(A, B) - d(X, Y)| < \varepsilon$. The following is the degree form of Szemerédi's Regularity Lemma which can be easily derived from the classical version (see, e.g., [3] and [8]). It is a slight variation of Lemma 2.4 in [31].

Lemma 3.5 (regularity lemma). *For every $\varepsilon > 0$, there exists $M = M(\varepsilon)$ such that the following holds. Let $0 \leq \mu \leq 1$, and let G be an n-vertex graph. Then there is a partition $\{V_0, \ldots, V_m\}$ of $V(G)$ and a spanning subgraph G' of $G - V_0$, such that the following properties hold.*

(R1) $m \leq M$.

(R2) *If G is bipartite, then every V_i with $i \in [m]$ is contained in one of the two parts of G.*

(R3) $|V_0| \leq \varepsilon n$ and $|V_1| = \ldots = |V_m| \leq \varepsilon n$.

(R4) $d_{G'}(v) \geq d_G(v) - (\mu + \varepsilon)n$ for every $v \in V(G')$.

(R5) $e_{G'}(V_i) = 0$ for every $i \in [m]$.

(R6) *For all $1 \leq i, j \leq m$, the graph $G'[V_i, V_j]$ is ε-regular with density either 0 or more than μ.*

The sets $V_1, \ldots, V_m$ in Lemma 3.5 are called *clusters*, and the set V_0 is called the *exceptional set*. Given clusters and G' as in Lemma 3.5, the *reduced graph* Γ is the graph whose vertices are $V_1, \ldots, V_m$ (note that V_0 is omitted here) and in which V_i is adjacent to V_j whenever $G'[V_i, V_j]$ is ε-regular and has density more than μ. Thus, $V_i V_j$ is an edge of Γ if and only if G' has an edge between V_i and V_j.

Given $\varepsilon > 0$ and $d \in [0, 1]$, a bipartite graph G with the bipartition $\{A, B\}$ is called (ε, d)-*super-regular*, if all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ satisfy $d(X, Y) > d$, and moreover, every vertex in A has degree more than $d|B|$, and every vertex in B has degree more than $d|A|$.

We will need the following special case of the blow-up lemma of Komlós, Sárközy and Szemerédi [21].

Lemma 3.6 (Special case of the blow-up lemma [21]). *For every $d > 0$ and Δ, there exists a positive number $\varepsilon_0 = \varepsilon_0(d, \Delta)$ such that for all $\varepsilon \leq \varepsilon_0$ the following holds. If H is a balanced complete bipartite graph with n vertices in each part satisfying $\Delta(H) \leq \Delta$, then every (ε, d)-super-regular bipartite graph with n vertices in each part contains H as a subgraph.*

We also need the following standard fact concerning super-regular graphs.

Proposition 3.7 (Proposition 2.3 in [31]). *Every ε-regular bipartite graph G with the bipartition $\{A, B\}$ and density $d > 2\varepsilon$, can be made into a $(\varepsilon/(1 - \varepsilon), d - 2\varepsilon)$-super-regular graph by removing $\varepsilon|A|$ vertices from A and $\varepsilon|B|$ vertices from B.*

3.4 The Kővári–Sós–Turán theorem

The following is an unbalanced variant of the well-known Kővári–Sós–Turán theorem [26].

Lemma 3.8. *Let $t \geq 2$, let $\delta > 0$, let G be a bipartite graph on at most n vertices with the bipartition $\{X, Y\}$ such that $|X| \geq 2t \left(\frac{e}{\delta}\right)^t$ and every vertex in X has at least δn neighbours in Y. Then G contains a $K_{t,t}$.*

Proof. Suppose for a contradiction that G is $K_{t,t}$-free. A copy of $K_{1,t}$ in G is called a *t-star*, the vertex of degree t in a t-star is called its *center*, and the t vertices of degree 1 in a t-star are called its *leaves*. Since G is $K_{t,t}$-free, the number of t-stars in G whose centre is in X and whose t leaves are in Y is at most

$$(t-1)\binom{|Y|}{t} \leq t\left(\frac{e|Y|}{t}\right)^t \leq \frac{(en)^t}{t^{t-1}}.$$

On the other hand, since every vertex in X has at least δn neighbours in Y, the number of such t-stars in G is at least

$$|X|\binom{\delta n}{t} \geq |X|\left(\frac{\delta n}{t}\right)^t.$$

Thus, we obtain $|X| \leq t\left(\frac{e}{\delta}\right)^t$. This is a contradiction, proving the lemma. $\qquad\square$

4   Balancing the expanders

As discussed in the proof sketch, our proof of Theorem 1.4 starts by partitioning the vertex set of our dense regular graph G into a small number of expanders (using Lemma 3.2). A natural strategy for finding a $K_{t,t}$-packing in G is to find a $K_{t,t}$-packing in each of these expanders. However, if any of these expanders happens to be a rather imbalanced bipartite graph, then it is impossible to find a $K_{t,t}$-packing in it covering all but a constant number of vertices. To overcome this difficulty, we need the following key lemma which shows that we can partition the vertex set of any dense regular graph G into a set L that contains a perfect $K_{t,t}$-packing, a small set of constant size, and a small number of expanders, each of which is either far from bipartite or bipartite and *balanced*. Using Lemma 5.1 we can then find a $K_{t,t}$-packing in each of these expanders covering all but a constant number of its vertices, giving us the desired $K_{t,t}$-packing in G.

Lemma 4.1. *Let $0 < c \leq 1$, let $t \geq 2$ be an integer, and let n_0 be sufficiently large. Suppose that G is a d-regular graph on n vertices, where $d \geq cn$ and $n \geq n_0$. Then there exist positive numbers η, γ, ζ, r, with $r \leq \lceil 1/c \rceil$ and $1/n_0 \ll \eta \ll \zeta \ll \gamma \ll c, 1/t$ such that the following holds. There exist pairwise disjoint sets $Z_1, \ldots, Z_r, L \subseteq V(G)$ and a subgraph $G' \subseteq G$ with $V(G') = V(G)$ satisfying the following properties.*

(B1) $|V(G) - (Z_1 \cup \ldots \cup Z_r \cup L)| \leq 64r^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{8\zeta}\right)^t$.

(B2) *For every $i \in [r]$, $G'[Z_i]$ has average degree at least $d - \eta|Z_i|$.*

(B3) *For every $i \in [r]$, $G'[Z_i]$ has no ζ-sparse cuts.*

(B4) *For every $i \in [r]$, $G'[Z_i]$ is either bipartite and balanced, or one needs to remove at least γn^2 edges from $G'[Z_i]$ to make it bipartite.*

(B5) *$G[L]$ has a perfect $K_{t,t}$-packing.*

Proof of Lemma 4.1. Let c, t, n_0, n, d be as in the statement of the lemma, and let G be a d-regular graph on n vertices, where $d \geq cn$. By applying Lemma 3.2 to G (with $\min\{c, 1/t\}$ playing the role of c) we obtain positive numbers $r, \eta, \beta, \gamma, \zeta, \delta$, and a partition $\{Z_1, \ldots, Z_r\}$ of $V(G)$ satisfying $r \leq \lceil 1/c \rceil$ and (E1)–(E4), such that

$$1/n_0 \ll \eta \ll \beta \ll \gamma \ll \zeta \ll \delta \ll c, 1/t.$$

Let σ, ρ be positive numbers satisfying

$$\beta \ll \sigma \ll \rho \ll \gamma.$$

For every $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite, fix $\{X_i, Y_i\}$ to be a partition of Z_i that maximises the number of edges between X_i and Y_i. For every $i \in [r]$ such that $G[Z_i]$ is γ-far-from-bipartite, let $\{X_i, Y_i\}$ be a partition of Z_i satisfying

(G1) $|e_G(X_i) - e_G(Y_i)| \le \beta n^2$,

(G2) $\delta(G[X_i, Y_i]) \ge \delta n/3$.

Notice that this can easily be achieved by taking $\{X_i, Y_i\}$ to be a random partition of Z_i. Morever, these conditions are satisfied also when $G[Z_i]$ is β-almost-bipartite. Indeed, if $G[Z_i]$ is β-almost-bipartite, then $e_G(X_i) + e_G(Y_i) \le \beta n^2$, implying that (G1) holds. Moreover, since $\{X_i, Y_i\}$ is a partition maximising the number of edges in $G[X_i, Y_i]$ and since the minimum degree in $G[Z_i]$ is at least δn (by (E2)), the graph $G[X_i, Y_i]$ has minimum degree at least $\delta n/2$, showing (G2).

We claim that the following holds.

$$\big(e_G(X_i, \overline{Z_i}) - e_G(Y_i, \overline{Z_i})\big) + 2\left(e_G(X_i) - e_G(Y_i)\right) - d\left(|X_i| - |Y_i|\right) = 0. \tag{1}$$

Indeed, to see that (1) holds, we have the following equations counting the number of edges incident to X_i and Y_i, respectively.

$$d|X_i| = e_G(X_i, Y_i) + e_G(X_i, \overline{Z_i}) + 2e_G(X_i),$$
$$d|Y_i| = e_G(X_i, Y_i) + e_G(Y_i, \overline{Z_i}) + 2e_G(Y_i).$$

Subtracting one from the other yields (1).

Let H be the graph obtained from G by removing all edges between X_i and Y_i, and also arbitrarily removing $\min\{e_G(X_i), e_G(Y_i)\}$ edges from each of $G[X_i]$ and $G[Y_i]$ for all $i \in [r]$. Then H satisfies the following properties.

(H1) $\big(e_H(X_i, \overline{Z_i}) - e_H(Y_i, \overline{Z_i})\big) + 2\left(e_H(X_i) - e_H(Y_i)\right) - d\left(|X_i| - |Y_i|\right) = 0$,

(H2) $e_H(X_i, Y_i) = 0$ for every $i \in [r]$,

(H3) H has maximum degree at most $d - \delta n/3$,

(H4) $e(H) \le \eta n^2 + r\beta n^2 \le 2r\beta n^2$.

Note that for (H3) we used (G2), and for (H4) we used (E1) and (G1).

In the next subsection, we use a procedure for moving vertices between the expanders $G[Z_i]$, $i \in [r]$, in order to ensure that the degrees of vertices in H between the resulting subgraphs of G are appropriately bounded (whenever there are sufficiently many such edges), while also ensuring that the subgraphs have large minimum degree. More precisely, we will show that, when the procedure ends, (H1)–(H4) still hold, and (H5) and (H6) below also hold.

4.1   Procedure for moving vertices between the expanders for better degree control

Start with $H, (X_i)_{i\in[r]}, (Y_i)_{i\in[r]}, (Z_i)_{i\in[r]}$ satisfying (H1)–(H4), and modify them using the following procedure, by running it as long as possible. For convenience, in the rest of the proof we denote the values of these quantities before the start of the procedure by $H^{\text{start}}, (X_i^{\text{start}})_{i\in[r]}, (Y_i^{\text{start}})_{i\in[r]}, (Z_i^{\text{start}})_{i\in[r]}$, respectively.

In other words, whenever there are $i, j \in [r]$, $W_i \in \{X_i, Y_i\}$, $W_j \in \{X_j, Y_j\}$ and $v \in W_i$ such that $d_H(v, W_j) \ge \rho n$ and $e_H(W_i, W_j) \ge d$ (see (2)), we move v from W_i to U_j (where $\{U_j, W_j\} = \{X_j, Y_j\}$), we remove $d - d_H(v)$ edges between $W_i - \{v\}$ and $W_j - \{v\}$ from H, and we also remove all edges between v and W_j from H. Note that it is possible to remove $d - d_H(v)$ edges between $W_i - \{v\}$ and $W_j - \{v\}$ by the assumption that $e_H(W_i, W_j) \ge d \ge d - d_H(v) + d_H(v, W_j)$.

Claim 4.2. *Throughout the procedure Move-high-degree-vertices, (H1) to (H4) hold.*

Proof. Since we never add any edges to H during the procedure and (H3) and (H4) hold at the start, it is obvious that they hold throughout the procedure. It is also easy to check that (H2) is true, because it is true at the start,

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1: **procedure** MOVE-HIGH-DEGREE-VERTICES
2:     Start with $H, (X_i)_{i \in [r]}, (Y_i)_{i \in [r]}, (Z_i)_{i \in [r]}$ satisfying (H1)–(H4).
3:     **while** there exist $i, j \in [r], W_i \in \{X_i, Y_i\}, W_j \in \{X_j, Y_j\}$ and $v \in W_i$ such that

$$d_H(v, W_j) \geq \rho n \quad \text{and} \quad e_H(W_i, W_j) \geq d. \tag{2}$$

   do
4:         $W_i \leftarrow W_i - \{v\}$,
5:         $U_j \leftarrow U_j \cup \{v\}$, where $\{U_j, W_j\} = \{X_j, Y_j\}$,
6:         $Z_i \leftarrow X_i \cup Y_i$ and $Z_j \leftarrow X_j \cup Y_j$,
7:         $H \leftarrow H - (E_{i,j} \cup E_H(v, W_j))$, where $E_{i,j} \subseteq H[W_i - \{v\}, W_j - \{v\}]$ is a set of exactly $d - d_H(v)$ edges, and $E_H(v, W_j)$ is the set of all edges in H between v and W_j.
8:     **end while**
9: **end procedure**

and after moving a vertex v to U_j, we remove all edges between v and W_j. Hence, it suffices to show that if (H1) holds for a given $H, (X_i)_{i \in [r]}, (Y_i)_{i \in [r]}$, then (H1) holds after performing one step of the procedure as well. Let H', $(X_i')_{i \in [r]}, (Y_i')_{i \in [r]}$ denote the corresponding quantities after performing one step of the procedure. We need to show that for all $i \in [r]$,

$$(e_{H'}(X_i', \overline{Z_i'}) - e_{H'}(Y_i', \overline{Z_i'})) + 2(e_{H'}(X_i') - e_{H'}(Y_i')) - d(|X_i'| - |Y_i'|) = 0. \tag{3}$$

Without loss of generality, we first prove it when $(W_i, W_j) = (X_i, Y_j)$ with $i \neq j$ and then we prove it when $(W_i, W_j) = (X_i, X_i)$. Indeed, by symmetry, this covers all cases since $e_H(X_i, Y_i) = 0$, and X_i and Y_i play the same role for every $i \in [r]$.

First, consider the case when $(W_i, W_j) = (X_i, Y_j)$ with $i \neq j$. In this case, note that the following twelve equations hold.

$$e_{H'}(X_i', \overline{Z_i'}) - e_H(X_i, \overline{Z_i}) = -(d - d_H(v)) - d_H(v, \overline{Z_i}) + d_H(v, X_i)$$
$$e_{H'}(Y_i', \overline{Z_i'}) - e_H(Y_i, \overline{Z_i}) = 0$$
$$e_{H'}(X_i') - e_H(X_i) = -d_H(v, X_i)$$
$$e_{H'}(Y_i') - e_H(Y_i) = 0$$
$$|X_i'| - |X_i| = -1$$
$$|Y_i'| - |Y_i| = 0.$$

$$e_{H'}(X_j', \overline{Z_j'}) - e_H(X_j, \overline{Z_j}) = d_H(v, \overline{Z_j}) - d_H(v, X_j)$$
$$e_{H'}(Y_j', \overline{Z_j'}) - e_H(Y_j, \overline{Z_j}) = -(d - d_H(v)) - d_H(v, Y_j)$$
$$e_{H'}(X_j') - e_H(X_j) = d_H(v, X_j)$$
$$e_{H'}(Y_j') - e_H(Y_j) = 0$$
$$|X_j'| - |X_j| = 1$$
$$|Y_j'| - |Y_j| = 0.$$

Multiplying the first six equations by $1, -1, 2, -2, -d, d$, respectively, and adding them all up, we find that

$$(e_{H'}(X_i', \overline{Z_i'}) - e_{H'}(Y_i', \overline{Z_i'})) + 2(e_{H'}(X_i') - e_{H'}(Y_i')) - d(|X_i'| - |Y_i'|)$$
$$= (e_H(X_i, \overline{Z_i}) - e_H(Y_i, \overline{Z_i})) + 2(e_H(X_i) - e_H(Y_i)) - d(|X_i| - |Y_i|) = 0.$$

Similarly, multiplying the last six equations by $1, -1, 2, -2, -d, d$, respectively, and adding them up, gives

$$\left(e_{H'}(X_j', \overline{Z_j'}) - e_{H'}(Y_j', \overline{Z_j'})\right) + 2\left(e_{H'}(X_j') - e_{H'}(Y_j')\right) - d\left(|X_j'| - |Y_j'|\right)$$
$$= \left(e_H(X_j, \overline{Z_j}) - e_H(Y_j, \overline{Z_j})\right) + 2\left(e_H(X_j) - e_H(Y_j)\right) - d\left(|X_j| - |Y_j|\right) = 0.$$

It is easy to see that for $\ell \in [r] - \{i, j\}$, we have $e_{H'}(X_\ell', \overline{Z_\ell'}) = e_H(X_\ell, \overline{Z_\ell})$, $e_{H'}(Y_\ell', \overline{Z_\ell'}) = e_H(Y_\ell, \overline{Z_\ell})$, $e_{H'}(X_\ell') = e_H(X_\ell)$, $e_{H'}(Y_\ell') = e_H(Y_\ell)$, $|X_\ell| = |X_\ell'|$, and $|Y_\ell| = |Y_\ell'|$, proving (3) in this case.

Now consider the case when $(W_i, W_j) = (X_i, X_i)$. In this case, note that the following six equations hold.

$$e_{H'}(X_i', \overline{Z_i'}) - e_H(X_i, \overline{Z_i}) = -d_H(v, \overline{Z_i})$$
$$e_{H'}(Y_i', \overline{Z_i'}) - e_H(Y_i, \overline{Z_i}) = d_H(v, \overline{Z_i})$$
$$e_{H'}(X_i') - e_H(X_i) = -d_H(v, X_i) - (d - d_H(v))$$
$$e_{H'}(Y_i') - e_H(Y_i) = 0$$
$$|X_i'| - |X_i| = -1$$
$$|Y_i'| - |Y_i| = 1.$$

As before, multiplying these equations by $1, -1, 2, -2, -d, d$, respectively, and adding up the resulting equations gives

$$\left(e_{H'}(X_i', \overline{Z_i'}) - e_{H'}(Y_i', \overline{Z_i'})\right) + 2\left(e_{H'}(X_i') - e_{H'}(Y_i')\right) - d\left(|X_i'| - |Y_i'|\right)$$
$$= \left(e_H(X_i, \overline{Z_i}) - e_H(Y_i, \overline{Z_i})\right) + 2\left(e_H(X_i) - e_H(Y_i)\right) - d\left(|X_i| - |Y_i|\right) = 0.$$

Again, it is easy to see that for $\ell \in [r] - \{i\}$, we trivially have $e_{H'}(X_\ell', \overline{Z_\ell'}) = e_H(X_\ell, \overline{Z_\ell})$, $e_{H'}(Y_\ell', \overline{Z_\ell'}) = e_H(Y_\ell, \overline{Z_\ell})$, $e_{H'}(X_\ell') = e_H(X_\ell)$, $e_{H'}(Y_\ell') = e_H(Y_\ell)$, $|X_\ell| = |X_\ell'|$, and $|Y_\ell| = |Y_\ell'|$, proving (3) in this case and completing the proof of the claim. □

Let us bound the number of steps taken by the procedure MOVE-HIGH-DEGREE-VERTICES by the time it ends, that is, by the time (2) is no longer satisfied by any choice of parameters. Notice that after each step of the procedure, the number of edges in H decreases by at least $d - d_H(v) \geq \delta n/3$ (where the inequality follows because H satisfies (H3) by Claim 4.2). Since H satisfies (H4) at the beginning of the procedure, this implies that the total number of steps taken before the procedure ends is at most $2r\beta n^2/(\delta n/3) = (6r\beta/\delta)n \leq \rho n/2$ (using $r \leq \lceil 2/c \rceil$ and $\beta \ll \rho, \delta, c$). Since at most one vertex is moved at each step of the procedure we obtain the following.

(M) At most $(6r\beta/\delta)n \leq \rho n/2$ vertices are moved during the procedure MOVE-HIGH-DEGREE-VERTICES.

In the rest of the proof, let $(X_i)_{i \in [r]}$, $(Y_i)_{i \in [r]}$, $(Z_i)_{i \in [r]}$ and H denote the parts and the graph obtained at the end of the procedure MOVE-HIGH-DEGREE-VERTICES. Then we claim that the following properties hold.

(H5) $\delta(G[X_i, Y_i]) \geq \rho n/2$ for $i \in [r]$.

(H6) For every $i, j \in [r]$, $W_i \in \{X_i, Y_i\}$, $W_j \in \{X_j, Y_j\}$ either $e_H(W_i, W_j) < d$ or $\Delta(H[W_i, W_j]) \leq \rho n$.

Indeed, note that for every vertex v in $X_i \cup Y_i$, there is a step during the procedure MOVE-HIGH-DEGREE-VERTICES where its degree is at least ρn in $G[X_i^*, Y_i^*]$, where X_i^* and Y_i^* denote the parts corresponding to X_i and Y_i, respectively, in that step. (Here we used that if v is moved into X_i^* or Y_i^* at the previous step, then it has degree at least ρn in $G[X_i^*, Y_i^*]$, and if v has never been moved during the procedure, then we use that (G2) holds at the beginning of the procedure, and that $\rho \ll \delta$.) Moreover, by (M), at most $\rho n/2$ vertices are moved during the procedure. Hence, the degree of v changes by at most $\rho n/2$ when the procedure ends, proving that (H5) holds (note that (H5) pertains to the degree in G, not in H). Moreover, it is easy to see that (H6) holds, because otherwise the procedure would not have stopped.

In the next subsection, we carefully construct a small collection of vertex-disjoint copies of $K_{t,t}$ whose removal makes the subgraphs $G[X_i, Y_i]$, for $i \in [r]$, more balanced.

4.2 Constructing a collection $\mathcal{K}$ of $K_{t,t}$'s in G for balancing the subgraphs $G[X_i, Y_i]$

In the rest of this proof, let

$$T := \frac{4t}{c}\left(\frac{4e}{\rho}\right)^t.$$

By repeatedly applying an unbalanced variant of the Kővári–Sós–Turán theorem (Theorem 3.8), we prove the following claim. Recall that the vertex set of a given copy K of $K_{t,t}$ in G is denoted by $V(K)$, and the set of vertices contained in a given collection $\mathcal{K}$ of copies of $K_{t,t}$ in G is denoted by $V(\mathcal{K})$.

Claim 4.3. *There exists a collection $\mathcal{K} := \cup_{1 \le i \le j \le r}\big(\mathcal{K}(X_i, X_j) \cup \mathcal{K}(X_i, Y_j) \cup \mathcal{K}(Y_i, X_j) \cup \mathcal{K}(Y_i, Y_j)\big)$ of pairwise vertex-disjoint copies of $K_{t,t}$ in G, such that for every $i, j \in [r]$ with $i \le j$ and every $W_i \in \{X_i, Y_i\}$, $W_j \in \{X_j, Y_j\}$, the following two properties hold.*

- *If $e_H(W_i, W_j) \ge Td$ then $\big|\mathcal{K}(W_i, W_j)\big| = \left\lfloor \frac{e_H(W_i, W_j)}{d} \right\rfloor$ and, otherwise, $\mathcal{K}(W_i, W_j) = \emptyset$. In particular, $|\mathcal{K}| \le \sum_{i \le j} \frac{e_H(W_i, W_j)}{d} \le \frac{e(H)}{d}$.*

- *Every copy $K \in \mathcal{K}(W_i, W_j)$ satisfies $V(K) \subseteq Z_i \cup Z_j$, and for every $\ell \in \{i, j\}$, we have*

$$|W_\ell \cap V(K)| - |U_\ell \cap V(K)| = \begin{cases} 1 & \text{if } i \ne j \\ 2 & \text{if } i = j, \end{cases} \qquad (4)$$

where $U_\ell = Z_\ell - W_\ell$.

Proof. Let σ be an ordering of the pairs (W_i, W_j), with $1 \le i \le j \le r$ and $e_H(W_i, W_j) \ge Td$, in increasing order of $e_H(W_i, W_j)$. We will show by induction that for every such pair (W_i, W_j) there is a collection $\mathcal{K}$ of pairwise vertex-disjoint copies of $K_{t,t}$ in G satisfying the following: for every $(W_{i'}, W_{j'})$ such that $(W_{i'}, W_{j'}) \le_\sigma (W_i, W_j)$, $\mathcal{K}$ consists of exactly $\left\lfloor \frac{e_H(W_{i'}, W_{j'})}{d} \right\rfloor$ copies K of $K_{t,t}$ satisfying $V(K) \subseteq Z_{i'} \cup Z_{j'}$ and (4).

To this end, fix (W_i, W_j) such that $1 \le i \le j \le r$ and $e_H(W_i, W_j) \ge Td$, and suppose that there is a collection $\mathcal{K}$ of pairwise vertex-disjoint copies of $K_{t,t}$ in G satisfying the following: for every $(W_{i'}, W_{j'})$ such that $(W_{i'}, W_{j'}) <_\sigma (W_i, W_j)$, $\mathcal{K}$ consists of exactly $\left\lfloor \frac{e_H(W_{i'}, W_{j'})}{d} \right\rfloor$ copies K of $K_{t,t}$ satisfying $V(K) \subseteq Z_{i'} \cup Z_{j'}$ and (4) (and there are no other $K_{t,t}$-copies). Let $f := e_H(W_i, W_j)$, and let $\mathcal{K}'$ be a maximal collection of at most $\lfloor f/d \rfloor$ copies K of $K_{t,t}$ that are pairwise vertex-disjoint and vertex-disjoint from the copies of $K_{t,t}$ in $\mathcal{K}$ and satisfy $V(K) \subseteq Z_i \cup Z_j$ and (4).

If $|\mathcal{K}'| = \lfloor f/d \rfloor$, then we are done with the proof of the induction step, so suppose $|\mathcal{K}'| < \lfloor f/d \rfloor$. Let V be the set of all vertices contained in a copy of $K_{t,t}$ in $\mathcal{K} \cup \mathcal{K}'$. Then $|V| \le (2r)^2 \cdot (f/d) \cdot 2t = 8r^2 ft/d$ because each pair $(W_{i'}, W_{j'})$ that precedes (W_i, W_j) in σ satisfies $e_H(W_{i'}, W_{j'}) \le f$ by the choice of σ, and there are at most $(2r)^2$ such pairs $(W_{i'}, W_{j'})$. Thus, the number of edges in $H[W_i, W_j]$ incident to a vertex from V is at most $\rho n \cdot |V| \le \rho n \cdot 8r^2 ft/d \le f/2$ (using that $\Delta(H[W_i, W_j]) \le \rho n$ by (H6), that $r \le \lceil 1/c \rceil$ and that $\rho \ll c, 1/t$). This implies that there is a vertex v in W_i with at least $\frac{f}{2n} \ge \frac{Td}{2n} \ge 2t\left(\frac{4e}{\rho}\right)^t$ neighbours in $W_j - V$; denote the set of these neighbours by A. Observe that $G[W_j - V, U_j - V]$ has minimum degree at least $\rho n/4$ because (H5) holds (by Claim 4.2) and $|V| \le 2t \cdot e(H)/d \le 2t \cdot 2r\beta n^2/d \le \rho n/4$ because (H4) holds, $d \ge cn$ and $\beta \ll \rho, c, 1/t$. Thus, by the Kővári–Sós–Turán theorem (Lemma 3.8), there is a copy of $K_{t,t-1}$ in $G[A, U_j - V]$, with t vertices in A and $t - 1$ vertices in $U_j - V$. Together with v, this copy of $K_{t,t-1}$ forms a copy K of $K_{t,t}$ in G satisfying $V(K) \subseteq Z_i \cup Z_j$ and (4), contradicting the maximality of the collection $\mathcal{K}'$, and proving the claim. $\qquad \square$

Let $\mathcal{K}$ be the collection of copies of $K_{t,t}$ guaranteed by Claim 4.3. We claim that, for each $i \in [r]$, we have

$$|X_i \cap V(\mathcal{K})| - |Y_i \cap V(\mathcal{K})| = |X_i| - |Y_i| \pm 8rT. \qquad (5)$$

Indeed, let us fix $i \in [r]$, and note that Claim 4.3 ensures that for every $j \in [r]$, $W_i \in \{X_i, Y_i\}$, $W_j \in \{X_j, Y_j\}$ such that $e_H(W_i, W_j) \ge Td$ and $W_j \ne Z_i - W_i$, there are exactly $\left\lfloor \frac{e_H(W_i, W_j)}{d} \right\rfloor$ copies $K \in \mathcal{K}$ satisfying $V(K) \subseteq Z_i \cup Z_j$

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and (4). Therefore, it is easy to see that the following holds (where $\mathbb{1}\{E\}$ denotes the indicator function of E i.e., $\mathbb{1}\{E\} = 1$ if E holds, and $\mathbb{1}\{E\} = 0$, otherwise).

$$
\begin{aligned}
&|X_i \cap V(\mathcal{K})| - |Y_i \cap V(\mathcal{K})| \\
&= \sum_{W_j \in \{X_j, Y_j\}, \, j \in [r] - \{i\}} \left(\left\lfloor \frac{e_H(X_i, W_j)}{d} \right\rfloor \mathbb{1}\{e_H(X_i, W_j) \geq Td\} - \left\lfloor \frac{e_H(Y_i, W_j)}{d} \right\rfloor \mathbb{1}\{e_H(Y_i, W_j) \geq Td\} \right) \\
&\quad + \left(2 \left\lfloor \frac{e_H(X_i)}{d} \right\rfloor \mathbb{1}\{e_H(X_i) \geq Td\} - 2 \left\lfloor \frac{e_H(Y_i)}{d} \right\rfloor \mathbb{1}\{e_H(Y_i) \geq Td\} \right) \\
&= \sum_{W_j \in \{X_j, Y_j\}, \, j \in [r] - \{i\}} \left(\frac{e_H(X_i, W_j)}{d} - \frac{e_H(Y_i, W_j)}{d} \right) + \left(\frac{2e_H(X_i)}{d} - \frac{2e_H(Y_i)}{d} \right) \pm 8rT \\
&= \frac{1}{d} \cdot \left(e_H(X_i, \overline{Z_i}) - e_H(Y_i, \overline{Z_i}) + 2e_H(X_i) - 2e_H(Y_i) \right) \pm 8rT = |X_i| - |Y_i| \pm 8rT,
\end{aligned}
$$

where for the last equality we used that (H1) holds (by Claim 4.2). Hence (5) holds.

Note that by (5), for every $i \in [r]$, we have

$$
|X_i - V(\mathcal{K})| = |Y_i - V(\mathcal{K})| \pm 8rT.
$$

For every $i \in [r]$, let $X_i' \subseteq X_i - V(\mathcal{K})$ and $Y_i' \subseteq Y_i - V(\mathcal{K})$ be subsets of equal size, obtained by removing at most $8rT$ vertices from $X_i - V(\mathcal{K})$ and from $Y_i - V(\mathcal{K})$. Write $Z_i' = X_i' \cup Y_i'$. Then, since $\{Z_1, \ldots, Z_r\}$ partition $V(G)$, we have

$$
|V(G) - (Z_1' \cup \ldots \cup Z_r')| \leq |V(\mathcal{K})| + 8rT \cdot 2r = |V(\mathcal{K})| + 64r^2 \cdot \frac{t}{c} \cdot \left(\frac{4e}{\rho} \right)^t. \tag{6}
$$

Let G' be the spanning subgraph of G obtained by removing the edges of G within X_i' and Y_i' for all $i \in [r]$ such that $G[Z_i^{\text{start}}]$ is β-almost-bipartite. This ensures that the subgraph $G'[Z_i']$ is bipartite and balanced for all $i \in [r]$ for which $G[Z_i^{\text{start}}]$ is β-almost-bipartite. (Recall that here the sets Z_i^{start} for $i \in [r]$, are defined before running the procedure MOVE-HIGH-DEGREE-VERTICES.)

Let us make a simple observation concerning the number of vertices added to or removed from the set Z_i^{start} in order to form the set Z_i' for every $i \in [r]$. Note that, for every $i \in [r]$, by (M) and the definition of Z_i', we have $|Z_i^{\text{start}} - Z_i'| \leq (6r\beta/\delta)n + |V(\mathcal{K})| + 16rT$ and $|Z_i' - Z_i^{\text{start}}| \leq (6r\beta/\delta)n$. Moreover, since $r \leq \lceil 1/c \rceil$ and $\beta \ll \sigma, c, \delta$, we have $(6r\beta/\delta)n \leq \sigma n/4$, and by Claim 4.3, (H4) and the fact that $\beta \ll \sigma, c, 1/t$, we have $|V(\mathcal{K})| \leq 2t \cdot \frac{e(H)}{d} \leq 2t \cdot \frac{2r\beta n^2}{d} \leq \sigma n/4$. Therefore,

$$
|Z_i^{\text{start}} - Z_i'|, |Z_i' - Z_i^{\text{start}}| \leq \sigma n. \tag{7}
$$

4.3 Showing that expansion is preserved after balancing

In this subsection, we will prove the following claim, which shows that the subgraphs $G'[Z_i']$ for $i \in [r]$ (which are obtained after balancing) still have good expansion properties.

Claim 4.4. *For every $i \in [r]$, the graph $G'[Z_i']$ has no $\rho/8$-sparse cuts.*

Proof. Fix $i \in [r]$, and, for convenience, write $Z' := Z_i^{\text{start}}$ and $Z := Z_i'$. Consider a partition $\{A, B\}$ of Z, where A and B are non-empty and $|A| \geq |B|$. We will show that $e_{G'}(A, B) \geq (\rho/8)|A||B|$, thereby proving the claim.

Suppose first that $|B| \leq \rho n/8$. Recall that $G[X_i, Y_i]$ has minimum degree at least $\rho n/2$ (by (H5)). Since $|X_i - X_i'|, |Y_i - Y_i'| \leq \sigma n \leq \rho n/4$, by (7) and $\sigma \ll \rho$, the graph $G[X_i', Y_i']$ has minimum degree at least $\rho n/4$. As this is a spanning subgraph of $G'[Z_i'] = G'[Z]$, the graph $G'[Z]$ also has minimum degree at least $\rho n/4$. Hence

$$
e_{G'}(A, B) \geq |B| \cdot \left(\rho n/4 - |B| \right) \geq |B| \cdot \rho n/8 \geq (\rho/8) \cdot |A||B|,
$$

Suppose, instead, that $|B| > \rho n/8$. Write $U := Z - Z'$, $W := Z' - Z$, $A' := (A - U) \cup W$ and $B' := B - U$. Then $\{A', B'\}$ is a partition of Z'. Since $G[Z']$ has no ζ-sparse cuts, we have

$$e_G(A', B') \geq \zeta |A'||B'| \geq \zeta \cdot |A - U| \cdot |B - U| \geq \zeta \cdot (|A| - \sigma n) \cdot (|B| - \sigma n) \geq (\zeta/4) \cdot |A||B|. \tag{8}$$

For the last inequality, we used $|A| \geq |B| \geq \rho n/8 \geq 2\sigma n$ (using $\sigma \ll \rho$). We now upper bound the number of edges in $G[Z']$ that are not in $G'[Z']$. Write $X' := X_i^{\text{start}}$, $Y' := Y_i^{\text{start}}$, $X := X_i$ and $Y := Y_i$. Recall that if $G'[Z]$ and $G[Z]$ differ, then $e_G(X') + e_G(Y') \leq \beta n^2$ and $G'[Z]$ is obtained by removing some of the edges in $G[X]$ and $G[Y]$ from $G[Z]$. Assuming $G[Z]$ and $G'[Z]$ differ, we thus have

$$\begin{aligned}
e_G(Z) - e_{G'}(Z) = e_G(X) + e_G(Y) &\leq e_G(X') + |X - X'| \cdot n + e_G(Y') + |Y - Y'| \cdot n \\
&\leq \beta n^2 + 2\sigma n^2 \leq 3\sigma n^2,
\end{aligned} \tag{9}$$

using (7) and $\beta \ll \sigma$. Hence

$$\begin{aligned}
e_{G'}(A, B) = e_{G'}(A \cup W, B) - e_{G'}(W, B) &\geq e_{G'}(A', B') - \sigma n^2 \\
&\geq e_G(A', B') - 4\sigma n^2 \\
&\geq (\zeta/4) \cdot |A||B| - 4\sigma n^2 \geq (\zeta/8) \cdot |A||B| \geq (\rho/8) \cdot |A||B|.
\end{aligned}$$

Here we used $|W| \leq \sigma n$ for the first inequality, (9) for the second one, (8) for the third, $(\zeta/4) \cdot |A||B| \geq (\zeta \rho^2/256) \cdot n^2 \geq 8\sigma n^2$ for the fourth, and that $\rho \ll \zeta$ for the last inequality. $\qquad \square$

4.4 Putting everything together and completing the proof of Lemma 4.1

To complete the proof of Lemma 4.1, we will show that the following properties hold for all $i \in [r]$.

(a) $G'[Z_i']$ has no $\rho/8$-sparse cuts.

(b) Either $G'[Z_i']$ is bipartite and balanced, or one needs to remove at least $\gamma n^2/4$ edges from $G'[Z_i']$ to make it bipartite.

(c) The average degree of $G'[Z_i']$ is at least $d - \frac{144\sigma}{\rho^2}|Z_i'|$.

Indeed, Claim 4.4 shows (a).

For proving (b), notice that if $G[Z_i^{\text{start}}]$ is β-almost-bipartite, then $G'[Z_i']$ is indeed bipartite and balanced, by the choice of X_i' and Y_i'. So consider $i \in [r]$ such that $G[Z_i^{\text{start}}]$ is γ-far-from-bipartite. Recall that $|Z_i^{\text{start}} - Z_i'| \leq \sigma n \leq \gamma n/4$ by (7) and the fact that $\sigma \ll \gamma$. Suppose for a contradiction that one can remove fewer than $\gamma n^2/4$ edges from $G'[Z_i'] = G[Z_i']$ to make it bipartite. But this means one can remove fewer than $\gamma n^2/4 + |Z_i^{\text{start}} - Z_i'| \cdot n \leq \gamma n^2/2$ edges from $G[Z_i^{\text{start}}]$ to make it bipartite. (Indeed, we can remove all edges incident to $Z_i^{\text{start}} - Z_i'$, and fewer than $\gamma n^2/4$ edges from $G[Z_i^{\text{start}} \cap Z_i']$ to make it bipartite.) This contradicts $G[Z_i^{\text{start}}]$ being γ-far-from-bipartite, and proves (b).

Finally, for proving (c), first recall that $e_G(Z_i') - e_{G'}(Z_i') \leq 3\sigma n^2$, by (9). Second, by (7), and since the sets Z_i^{start} satisfy (E1), we have

$$e_G(Z_i', \overline{Z_i'}) \leq e_G(Z_i^{\text{start}}, \overline{Z_i^{\text{start}}}) + |Z_i^{\text{start}} - Z_i'|n + |Z_i' - Z_i^{\text{start}}|n \leq \eta n^2 + 2\sigma n^2 \leq 3\sigma n^2.$$

Combining the above two inequalities we have

$$d|Z_i'| - 2e_{G'}(Z_i') \leq e_G(Z_i', \overline{Z_i'}) + 2(e_G(Z_i') - e_{G'}(Z_i')) \leq 9\sigma n^2 \leq \frac{144\sigma}{\rho^2}|Z_i'|^2, \tag{10}$$

where in the last inequality we used $|Z_i'| \geq \rho n/4$, which follows from $G[Z_i']$ having minimum degree at least $\rho n/4$, as pointed out at the beginning of the proof of Claim 4.4. Hence, by (10), the average degree of $G'[Z_i']$ is at least $d - \frac{144\sigma}{\rho^2}|Z_i'|$, proving (c).

Using (a), (b) and (c), it is now easy to check that (B2), (B3) and (B4) of the lemma hold with Z_i' playing the role of Z_i for $i \in [r]$, $V(\mathcal{K})$ playing the role of L and with $\frac{144\sigma}{\rho^2}$, $\rho/8$ and $\gamma/4$ playing the roles of η, ζ and γ, respectively (in particular, notice that these new parameters satisfy $\eta \ll \zeta \ll \gamma \ll c$, as $\sigma \ll \rho \ll \gamma \ll c$). Since, by definition, $\mathcal{K}$ is a collection of pairwise vertex-disjoint copies of $K_{t,t}$ in G, (B5) of the lemma immediately follows. Finally, by (6) and using that $\rho/8$ plays the role of ζ, we have

$$\left| V(G) - \left(Z_1' \cup \ldots \cup Z_r' \cup V(\mathcal{K}) \right) \right| \leq 64r^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{2\zeta} \right)^t ,$$

proving (B1), and completing the proof of Lemma 4.1. $\qquad\square$

5   Packing $K_{t,t}$'s in expanders

Our second key lemma for proving Theorem 1.4 is as follows. This lemma shows that expanders admit a perfect $K_{t,t}$-packing if their number of vertices is divisible by $2t$. It will be used to find a $K_{t,t}$-packing that covers all but at most $2t - 1$ vertices in each of the expanders provided by Lemma 4.1.

Lemma 5.1. *Let $t \geq 2$, n_0 be integers, let $0 < c, \gamma, \zeta, \eta < 1$ and suppose that $1/n_0 \ll \eta \ll \zeta \ll \gamma \ll c, 1/t$. Let $n \geq n_0$ be an integer divisible by $2t$, let $d \geq cn$ and let G be an n-vertex graph with the following properties.*

(Q1) *G has average degree at least $d - \eta n$ and maximum degree at most d.*

(Q2) *G has no ζ-sparse cuts.*

(Q3) *G is either bipartite and balanced, or γ-far-from-bipartite.*

Then G has a perfect $K_{t,t}$-packing.

Note that the following corollary, where we drop the assumption of divisibility from Lemma 5.1 and allow for up to $2t - 1$ uncovered vertices, follows directly from Lemma 5.1. Indeed, given G as in the next corollary, remove up to $2t - 1$ vertices to obtain a graph G' whose number of vertices n' is divisible by $2t$. It is easy to see that the properties (Q1)–(Q3) above hold for G', with slightly worse parameters, say, with $n_0 - 2t, c/2, \gamma/2, 2\eta, \zeta/2, n', d$ in place of $n_0, c, \gamma, \eta, \zeta, n, d$ respectively.

Corollary 5.2. *Let $t \geq 2$, n_0 be integers, let $0 < c, \gamma, \zeta, \eta < 1$ and suppose that $1/n_0 \ll \eta \ll \zeta \ll \gamma \ll c, 1/t$. Let $n \geq n_0$, let $d \geq cn$ and let G be an n-vertex graph with the following properties.*

(1) *G has average degree at least $d - \eta n$ and maximum degree at most d.*

(2) *G has no ζ-sparse cuts.*

(3) *G is either bipartite and balanced, or γ-far-from-bipartite.*

Then, G has a $K_{t,t}$-packing covering all but at most $2t - 1$ vertices.

Let us now proceed with the proof of Lemma 5.1.

Proof of Lemma 5.1. Let ε, μ, ξ be positive numbers satisfying

$$1/n_0 \ll \varepsilon \ll \mu \ll \eta \ll \xi \ll \zeta.$$

Apply the regularity lemma (Lemma 3.5) to G with parameters ε and μ, to obtain positive numbers $M = M(\varepsilon)$, m with $m \leq M$, a subgraph G' and a partition $\{V_0, \ldots, V_m\}$ of $V(G)$ into clusters $V_1, \ldots, V_m$ and an exceptional set V_0 satisfying properties (R1)–(R6). Let Γ be the corresponding reduced graph, namely, it has vertex set $\{V_1, \ldots, V_m\}$

and $V_i V_j$ is an edge in Γ if and only if $G'[V_i, V_j]$ is ε-regular with density more than μ. Equivalently, $V_i V_j$ is an edge in Γ if and only if there is an edge of G' between V_i and V_j (by (R6)).

Notice that if G is bipartite, then both G' and Γ are bipartite. Indeed, if G is bipartite then G' is bipartite since it is a subgraph of G, and Γ is bipartite because every V_i with $i \in [m]$ is contained in one of the parts of G by (R2).

In the rest of the proof, we will assume that m is even, and that if G is bipartite and balanced, then both G' and Γ are bipartite and balanced. Indeed, since all clusters V_i with $i \in [m]$ are equal in size — each of size at most εn by (R3) — and each V_i lies entirely within one of the two parts of the bipartition of G by (R2), this adjustment can be made, if necessary, by moving the vertices of a single cluster V_i (for some $i \in [m]$) from G' into V_0. Moreover, after (potentially) moving these vertices, it still follows from (R3) that

$$|V_0| \leq 2\varepsilon n, \tag{11}$$

and by (R4), for every $v \in V(G')$ we have

$$d_{G'}(v) \geq d_G(v) - (\mu + 2\varepsilon)n. \tag{12}$$

We claim that, after this potential adjustment, G' satisfies the following properties.

(G'1) G' has average degree at least $d - 3\eta n$.

(G'2) G' has no $\zeta/4$-sparse cuts. (In particular, G' has minimum degree at least $\zeta/4 \cdot (|V(G')| - 1) \geq \zeta n/5$.)

(G'3) If G is bipartite and balanced, then G' is bipartite and balanced. Otherwise, one needs to remove at least $\gamma n^2/2$ edges from G' to make it bipartite. (In particular, G' is $\gamma/2$-far-from-bipartite.)

Indeed, (G'1) follows directly from (12) and the fact that the average degree of G is at least $d - \eta n$ and $\varepsilon, \mu \ll \eta$. To prove (G'2), we use (12) together with the fact that G has no ζ-sparse cuts, as follows. For every partition $\{X, Y\}$ of $V(G')$ with $|X| \leq |Y|$, we have

$$
\begin{aligned}
e_{G'}(X, Y) &\geq e_G(X, V(G) - X) - |X|(\mu + 2\varepsilon)n \\
&\geq \zeta|X| \cdot (n - |X|) - |X|(\mu + 2\varepsilon)n \\
&\geq \left(\frac{\zeta}{2} - (\mu + 2\varepsilon)\right) \cdot |X| \cdot n \geq \frac{\zeta}{4} \cdot |X| \cdot |Y|,
\end{aligned}
$$

using $|X| \leq n/2$ for the third inequality, and $\varepsilon, \mu \ll \zeta$ for the last inequality. This proves (G'2). Finally, to prove (G'3), observe that if G is bipartite and balanced, then, as noted before, we can assume that G' is also bipartite and balanced. Otherwise, by our assumption in Lemma 5.1, G is γ-far-from-bipartite, in which case we claim that at least $\gamma n^2/2$ edges need to be removed from G' to make it bipartite. Indeed, by (11) and (12), the number of edges in G that are not present in G' is at most $|V_0| \cdot n + n \cdot (\mu + 2\varepsilon)n \leq (\mu + 4\varepsilon)n^2 \leq \gamma n^2/2$. Since G is γ-far-from-bipartite, at least γn^2 edges must be removed to make it bipartite. Therefore, even after accounting for the fact that at most $\gamma n^2/2$ of these edges are missing in G', one still needs to remove at least $\gamma n^2/2$ edges from G', proving (G'3).

5.1 Finding vertex-disjoint super-regular subgraphs that cover all vertices outside V_0

Recall that the clusters V_i, $i \in [m]$, have the same size by (R3). For each $i \in [m]$, if $|V_i|$ is odd, move one vertex from V_i to V_0. Now for each $i \in [m]$, partition V_i into two sets V_i', V_i'' of equal size, let Γ_{ref} be the graph with the vertex set $\{V_1', V_1'', \ldots, V_m', V_m''\}$ whose edges are all the pairs AB with $A \in \{V_i', V_i''\}$, $B \in \{V_j', V_j''\}$ where $V_i V_j$ is an edge in Γ. (In particular, Γ_{ref} is the 2-lift of Γ.) Then note that every edge in Γ_{ref} corresponds to a 3ε-regular subgraph of G' with a density more than $\mu - \varepsilon \geq \mu/2$.

We claim that Γ_{ref} contains a perfect matching. To see this, we first prove the following claim. A 2-*matching* is a collection of pairwise vertex-disjoint edges and odd cycles. A *perfect* 2-*matching* in a graph is a 2-matching covering all vertices in the graph.

Claim 5.3. Γ *has a perfect 2-matching.*

Before we prove the claim, recall that a *fractional matching* of a graph G is an edge weighting $w : E(G) \to [0,1]$, such that $\sum_{u \in N_G(v)} w(uv) \leq 1$, for every vertex v in G. A *perfect fractional matching* is a fractional matching where we have equality $\sum_{u \in N_G(v)} w(uv) = 1$ for every vertex v in G. We need the following useful fact.

Fact 5.4. *Let G be a graph which has a perfect fractional matching. Then it has a perfect 2-matching.*

This is a standard fact, but for completeness we prove it in Appendix A.

Proof of Claim 5.3. By (G'1)–(G'3) and Lemma 3.3, the graph G' has a Hamilton cycle, which we denote by $C = (v_1 \ldots v_{n'})$, where $n' = |V(G')|$. Let $w : E(\Gamma) \to \mathbb{R}^{\geq 0}$ be the edge-weighting where $w(V_i V_j)$ is the number of edges in C with one end in V_i and the other in V_j, divided by $2|V_1|$. We claim that w is a perfect fractional matching in Γ; that is, $\sum_{j \in [m]} w(V_i V_j) = 1$ for every $i \in [m]$. Indeed, for every $i \in [m]$, since G' has no edges with both ends in V_i, the number of edges in C with an end in V_i is exactly $2|V_i| = 2|V_1|$, so $\sum_{j \in [m]} w(V_i V_j) = 1$ for every $i \in [m]$, as claimed. By Fact 5.4, this implies that Γ has a perfect 2-matching, as required. $\square$

Let $\mathcal{M}$ be a perfect 2-matching in Γ. Replace each isolated edge $V_i V_j$ in $\mathcal{M}$ with two vertex-disjoint edges $V_i' V_j''$ and $V_i'' V_j'$ in Γ_{ref} and each odd cycle $(V_{i_1}, \ldots, V_{i_{2k+1}})$ (for some $k \geq 1$) in $\mathcal{M}$ with the edges $V_{i_j}' V_{i_{j+1}}'' \in E(\Gamma_{\mathrm{ref}})$ for $j \in [2k+1]$ (where addition in the indices is taken modulo $2k+1$), resulting in a perfect matching $\mathcal{M}_{\mathrm{ref}}$ in Γ_{ref}.

Let $U_1 U_2, \ldots, U_{2m-1} U_{2m}$ denote the edges of $\mathcal{M}_{\mathrm{ref}}$. Since every edge in Γ_{ref} corresponds to a 3ε-regular subgraph of G' with density more than $\mu/2$, it follows that for every $i \in [m]$, $G'[U_{2i-1}, U_{2i}]$ is 3ε-regular with density more than $\mu/2$. Furthermore, by (R6), for all $i, j \in [2m]$, the graph $G'[U_i, U_j]$ is 3ε-regular with density either 0 or more than $\mu/2$.

By Proposition 3.7, we can move $3\varepsilon|U_i|$ vertices from U_i (for every $i \in [2m]$) into the set V_0, so that the resulting subgraphs $G'[U_{2i-1}, U_{2i}]$ are $(4\varepsilon, \mu/4)$-super-regular for every $i \in [m]$. By moving less than $2t$ additional vertices from each U_i to V_0, we also assume that $|U_i|$ is divisible by $2t$ for every $i \in [2m]$. We claim that after moving these vertices, the following properties hold.

(G'4) We have $\bigcup_{i=1}^{2m} U_i = V(G) - V_0 = V(G')$, $|U_1| = \ldots = |U_{2m}|$, and $|V_0|, |U_1|, \ldots, |U_{2m}|$ are divisible by $2t$. Furthermore, if G is bipartite and balanced with the bipartition $\{X, Y\}$, then we can assume that $\bigcup_{i=1}^{m} U_{2i-1} = X - V_0$ and $\bigcup_{i=1}^{m} U_{2i} = Y - V_0$ (and hence G' is still bipartite and balanced).

(G'5) For every $i \in [m]$, $G'[U_{2i-1}, U_{2i}]$ is $(5\varepsilon, \mu/5)$-super-regular.

(G'6) For all $i, j \in [2m]$, the graph $G'[U_i, U_j]$ is 5ε-regular with density either 0 or more than $\mu/5$.

Items (G'5) and (G'6) follow easily from the discussion above. To see why (G'4) holds, recall that, by (R3), the clusters V_i, $i \in [m]$ have the same size, and by definition, the sets U_i, $i \in [2m]$, had the same size before an equal number of vertices from each of these sets are moved to V_0. Furthermore, if G is bipartite and balanced with the bipartition $\{X, Y\}$, then every cluster V_i, $i \in [m]$ is contained in one of the parts of G by (R2), and moreover, as noted earlier, G' and the reduced graph Γ are both bipartite and balanced. Hence, the 2-matching $\mathcal{M}$ in Γ is, in fact, a perfect matching in this case. Hence, $\mathcal{M}_{\mathrm{ref}}$ is also a perfect matching. Thus, without loss of generality, even after moving the vertices, we may assume $\bigcup_{i=1}^{m} U_{2i-1} = X - V_0$ and $\bigcup_{i=1}^{m} U_{2i} = Y - V_0$, as desired, proving (G'4).

Since a total of at most $m + 3\varepsilon n + 4tm \leq 4\varepsilon n$ vertices have been moved into V_0 from G', by (11), we have that

$$|V_0| \leq 6\varepsilon n, \tag{13}$$

and, by (G'1), we have that

(G'7) G' has average degree at least $d - 3\eta n - 4\varepsilon n \geq d - 6\eta n$.

Furthermore, we claim that the following two properties hold.

(G'8) G' has no $\zeta/12$-sparse cuts. Moreover, G' has minimum degree at least $\zeta n/6$.

(G'9) If G is bipartite and balanced, then G' is bipartite and balanced. Otherwise, one needs to remove at least $\gamma n^2/4$ edges from G' to make it bipartite.

To see why (G'8) holds, first note that after moving at most $4\varepsilon n$ vertices from G' to V_0, the minimum degree of G' is at least $\zeta n/5 - 4\varepsilon n \geq \zeta n/6$ by (G'2). Consider a partition $\{X, Y\}$ of G' with $|X| \leq |Y|$ and note that if $|X| \leq \zeta n/12$, then, by the minimum degree condition, $e_{G'}(X, Y) \geq \zeta n/12 \cdot |X| \geq \zeta/12 \cdot |X| \cdot |Y|$, as desired. So we may assume that $|X|, |Y| \geq \zeta n/12$. But then $e_{G'}(X, Y) \geq \zeta/4 \cdot |X| \cdot |Y| - 4\varepsilon n^2 \geq \zeta/12 \cdot |X| \cdot |Y|$ since G' had no $\zeta/4$-sparse cuts (by (G'2)) and at most $4\varepsilon n^2$ edges are removed from G' by moving at most $4\varepsilon n$ vertices from G' to V_0. This shows that G' has no $\zeta/12$-sparse cuts, completing the proof of (G'8). Finally, to prove (G'9), note that if G is bipartite and balanced, then by (G'4) and the above process of moving vertices, G' remains bipartite and balanced after moving the vertices. Otherwise, we claim that one needs to remove at least $\gamma n^2/4$ edges from G' to make it bipartite. Indeed, this follows from (G'3) and the fact that at most $4\varepsilon n^2 \leq \gamma n^2/4$ edges are removed from G' after moving at most $4\varepsilon n$ vertices to V_0, proving (G'9).

5.2 Splitting the super-regular subgraphs

As shown in the previous subsection, the vertices outside V_0 can be covered by super-regular subgraphs $G'[U_{2i-1}, U_{2i}]$ for $i \in [m]$. In this subsection, we will show that if we randomly partition U_i into five sets $\left\{U_i^{(1)}, \ldots, U_i^{(5)}\right\}$ of appropriate sizes for each $i \in [2m]$, then the subgraph H of G' induced by the set $\bigcup_{i \in [2m]} U_i^{(3)}$ still has good expansion properties (see (U6)) and, moreover, H is either bipartite and balanced or is far from being bipartite (see (U7)).

Claim 5.5. *For every $i \in [2m]$, there is a partition $\left\{U_i^{(1)}, \ldots, U_i^{(5)}\right\}$ of U_i satisfying the following properties.*

(U1) $\left|U_i^{(1)}\right| = \left|U_i^{(2)}\right| = \xi|U_i|$ *for $i \in [2m]$.*

(U2) $\left|U_i^{(3)}\right| = \frac{|U_i|}{2t}$ *for $i \in [2m]$.*

(U3) $\left|U_i^{(4)}\right| = \frac{2|U_i|}{3}$ *for $i \in [2m]$.*

(U4) $\left|U_i^{(5)}\right| = (\frac{1}{3} - \frac{1}{2t} - 2\xi)|U_i|$ *for $i \in [2m]$.*

(U5) *For every $v \in V(G)$, $i \in [2m]$ and $j \in [5]$, if $d_G(v, U_i) \geq \varepsilon|U_i|$, then $\frac{d_G\left(v, U_i^{(j)}\right)}{d_G(v, U_i)} = (1 \pm \varepsilon)\frac{|U_i^{(j)}|}{|U_i|}$.*

Similarly, for every $v \in V(G')$, $i \in [2m]$ and $j \in [5]$, if $d_{G'}(v, U_i) \geq \varepsilon|U_i|$ then $\frac{d_{G'}\left(v, U_i^{(j)}\right)}{d_{G'}(v, U_i)} = (1 \pm \varepsilon)\frac{|U_i^{(j)}|}{|U_i|}$.

(U6) *The graph $H := G'\left[\bigcup_{i \in [2m]} U_i^{(3)}\right]$ has no $\frac{\zeta}{20}$-sparse cuts.*

(U7) *If G is bipartite and balanced, then $H = G'\left[\bigcup_{i \in [2m]} U_i^{(3)}\right]$ is also bipartite and balanced. Otherwise, one needs to remove at least $\frac{\gamma}{256t^2}n^2$ edges from H to make it bipartite.*

Proof of claim. For every $i \in [2m]$, let $\left\{U_i^{(1)}, \ldots, U_i^{(5)}\right\}$ be a random partition of U_i into sets of sizes $\xi|U_i|$, $\xi|U_i|$, $(1/2t)|U_i|$, $(2/3)|U_i|$, $(1/3 - 1/2t - 2\xi)|U_i|$, respectively. Then (U1)–(U4) hold by definition. Moreover, by Lemma 3.1 and a union bound, (U5) holds with probability at least $1 - o(1)$; fix an outcome such that it holds.

For proving (U6), let $\{X, Y\}$ be a partition of $V(H)$ with $|X| \leq |Y|$. Notice that by (U5) and (G'8), H has minimum degree at least $(\zeta/7) \cdot |V(H)|$. Thus, if $|X| \leq (\zeta/14) \cdot |V(H)|$, then every vertex in X has at least $(\zeta/14) \cdot |V(H)|$ neighbours in Y, showing that $e_H(X, Y) \geq |X| \cdot (\zeta/14) \cdot |V(H)| \geq (\zeta/14) \cdot |X| \cdot |Y|$, as required.

So suppose that $|X| \geq (\zeta/14) \cdot |V(H)|$. Let $\{X', Y'\}$ be a partition of $V(G')$ such that, for every $i \in [2m]$, we have

$$\frac{|X \cap U_i^{(3)}|}{|U_i^{(3)}|} = \frac{|X' \cap U_i|}{|U_i|}. \tag{14}$$

By (U2) and (14), it follows that $|X \cap U_i| = |X \cap U_i^{(3)}| = \frac{|X' \cap U_i|}{2t}$ for every $i \in [2m]$. Thus, $|X| = \frac{|X'|}{2t}$. Similarly, $|Y| = \frac{|Y'|}{2t}$.

We claim that for any $i, j \in [2m]$, we have

$$e_H(X \cap U_i, Y \cap U_j) \geq \frac{1}{4t^2} \cdot e_{G'}(X' \cap U_i, Y' \cap U_j) - 10\varepsilon |U_i||U_j| \tag{15}$$

Indeed, if $|X' \cap U_i| \leq 10t\varepsilon|U_i|$ or $|Y' \cap U_j| \leq 10t\varepsilon|U_j|$, this holds trivially, as then the right-hand side of (15) is at most $(1/4t^2) \cdot 10t\varepsilon|U_i||U_j| - 10\varepsilon|U_i||U_j| < 0$. If, instead, $|X' \cap U_i| \geq 10t\varepsilon|U_i|$ and $|Y' \cap U_j| \geq 10t\varepsilon|U_j|$, then $|X \cap U_i| = |X \cap U_i^{(3)}| \geq 5\varepsilon|U_i|$ and $|Y \cap U_j| = |Y \cap U_j^{(3)}| \geq 5\varepsilon|U_j|$ by (14) and (U2), so by 5ε-regularity of $G'[U_i, U_j]$ (see (G'6)), we have

$$\frac{e_H(X \cap U_i, Y \cap U_j)}{|X \cap U_i| \cdot |Y \cap U_j|} \geq \frac{e_{G'}(X' \cap U_i, Y' \cap U_j)}{|X' \cap U_i| \cdot |Y' \cap U_j|} - 10\varepsilon.$$

This implies that

$$e_H(X \cap U_i, Y \cap U_j) \geq e_{G'}(X' \cap U_i, Y' \cap U_j) \cdot \frac{|X \cap U_i|}{|X' \cap U_i|} \cdot \frac{|Y \cap U_j|}{|Y' \cap U_j|} - 10\varepsilon \cdot |X \cap U_i| \cdot |Y \cap U_j|$$

$$\overset{(14)}{\geq} \frac{1}{4t^2} \cdot e_{G'}(X' \cap U_i, Y' \cap U_j) - 10\varepsilon|U_i||U_j|,$$

as required for (15).

Hence, we obtain the following lower bound on the number of edges between X and Y in H.

$$e_H(X, Y) = \sum_{i,j \in [2m]} e_H(X \cap U_i, Y \cap U_j)$$

$$\overset{(15)}{\geq} \frac{1}{4t^2} \cdot \sum_{i,j \in [2m]} e_{G'}(X' \cap U_i, Y' \cap U_j) - 10\varepsilon \cdot \sum_{i,j \in [2m]} (|U_i| \cdot |U_j|)$$

$$= \frac{1}{4t^2} \cdot e_{G'}(X', Y') - 10\varepsilon \cdot |V(G')|^2$$

$$\overset{(G'8)}{\geq} \frac{1}{4t^2} \cdot \frac{\zeta}{12} \cdot |X'| \cdot |Y'| - 10\varepsilon n^2$$

$$= \frac{\zeta}{12} \cdot |X| \cdot |Y| - 10\varepsilon n^2$$

$$\geq \frac{\zeta}{20} \cdot |X| \cdot |Y|,$$

using that $|X|, |Y| \geq (\zeta/14)|V(H)| \geq (\zeta/56t)n$ and that $\varepsilon \ll 1/t, \zeta$ for the last inequality. This proves (U6).

Finally, it remains to prove (U7). Note that by (G'4) and (U2), if G is bipartite and balanced, then H is also bipartite and balanced, as desired. Now suppose that G is γ-far-from-bipartite. Then, by (G'9), one needs to remove at least $\gamma n^2/4$ edges from G' to make it bipartite. Let $\{X, Y\}$ be an arbitrary partition of $V(H)$. To prove (U7), we need to show that $e_H(X) + e_H(Y) \geq \frac{\gamma n^2}{256t^2}$. Define the sets X', Y' as in (14). Then, analogously to (15), for every $i, j \in [2m]$, we have

$$e_H(X \cap U_i, X \cap U_j) \geq \frac{1}{4t^2} \cdot e_{G'}(X' \cap U_i, X' \cap U_j) - 10\varepsilon|U_i||U_j|. \tag{16}$$

It follows that

$$
\begin{aligned}
e_H(X) &= \sum_{1 \leq i < j \leq 2m} e_H(X \cap U_i, X \cap U_j) \\
&\overset{(16)}{\geq} \frac{1}{4t^2} \cdot \sum_{1 \leq i < j \leq 2m} e_{G'}(X' \cap U_i, X' \cap U_j) - 10\varepsilon \cdot \sum_{1 \leq i < j \leq 2m} (|U_i| \cdot |U_j|) \\
&\geq \frac{1}{4t^2} \cdot e_{G'}(X') - 10\varepsilon n^2.
\end{aligned}
$$

Analogously, $e_H(Y) \geq \frac{1}{4t^2} \cdot e_{G'}(Y') - 10\varepsilon n^2$. Notice that $e_{G'}(X) + e_{G'}(Y) \geq \gamma n^2/4$ by (G'9) and the assumption that G is not bipartite. Hence, altogether, we have

$$
e_H(X) + e_H(Y) \geq \frac{1}{4t^2}\big(e_{G'}(X') + e_{G'}(Y')\big) - 20\varepsilon n^2 \geq \frac{\gamma}{16t^2} \cdot n^2 - 20\varepsilon \cdot n^2 \geq \frac{\gamma}{256t^2} \cdot n^2,
$$

using $\varepsilon \ll \gamma, 1/t$ for the last inequality. This proves (U7). $\qquad\square$

5.3 Covering V_0 using a well-distributed collection of copies of $K_{t,t}$

In the rest of the proof, let $U^{(j)} := \bigcup_{i \in [2m]} U_i^{(j)}$ for $j \in [5]$. In this subsection, we find a collection of vertex-disjoint copies of $K_{t,t}$ in $G\big[V_0 \cup U^{(1)} \cup U^{(2)}\big]$ that covers V_0 such that the number of uncovered vertices in each U_i is divisible by $2t$. In the rest of the proof, using a slight abuse of notation, we denote the set of vertices contained in a given collection $\mathcal{K}$ of copies of $K_{t,t}$ in G by $V(\mathcal{K})$.

First, we cover the vertices in V_0 using vertex-disjoint $K_{t,t}$'s in $G\big[V_0 \cup U^{(1)}\big]$ (ignoring the divisibility requirement).

Claim 5.6. *There is a collection $\mathcal{K}_1$ of pairwise vertex-disjoint $K_{t,t}$'s in $G[V_0 \cup U^{(1)}]$ such that $V_0 \subseteq V(\mathcal{K}_1)$ and each copy of $K_{t,t}$ in $\mathcal{K}_1$ contains exactly one vertex of V_0.*

Proof. Let $\mathcal{K}_1$ be a maximal collection of pairwise vertex-disjoint copies of $K_{t,t}$ in $G[V_0 \cup U^{(1)}]$, where each copy of $K_{t,t}$ contains exactly one vertex from V_0. We will show that $V_0 \subseteq V(\mathcal{K}_1)$, proving the claim. Suppose not, and let $v \in V_0 - V(\mathcal{K}_1)$. Write $U := U^{(1)}$ and $U' := U - V(\mathcal{K}_1)$.

We claim that every vertex u in G satisfies $d_G(u, U') \geq (\xi\zeta/8)n$. Indeed, given a vertex u in G, let I be the set of indices $i \in [2m]$ such that $d_G(u, U_i) \geq \varepsilon|U_i|$. Then

$$
\begin{aligned}
d_G(u, U) = \sum_{i \in [2m]} d_G\big(u, U_i^{(1)}\big) &\overset{(U5)}{\geq} \sum_{i \in I}(1 - \varepsilon) \cdot \frac{\big|U_i^{(1)}\big|}{|U_i|} \cdot d_G(u, U_i) \\
&\geq (1 - \varepsilon) \cdot \xi \cdot \left(d_G(u) - d_G(u, V_0) - \sum_{i \in [2m] - I} d_G(u, U_i)\right) \\
&\geq (1 - \varepsilon) \cdot \xi \cdot \left(\frac{\zeta n}{2} - |V_0| - \sum_{i \in [2m] - I} \varepsilon|U_i|\right) \\
&\overset{(13)}{\geq} (1 - \varepsilon) \cdot \xi \cdot \left(\frac{\zeta}{2} - 7\varepsilon\right) \cdot n \geq \frac{\xi\zeta}{4} \cdot n.
\end{aligned}
$$

For the third inequality we used that G has minimum degree at least $\zeta(n - 1) \geq \frac{\zeta}{2}n$, which follows from the fact that there are no ζ-sparse cuts in G. Since $|V(\mathcal{K}_1)| \leq 2t|V_0| \leq 12\varepsilon tn$, every vertex u in G satisfies $d_G(u, U') \geq d_G(u, U) - |V(\mathcal{K}_1)| \geq (\frac{\xi\zeta}{4} - 12\varepsilon t)n \geq (\xi\zeta/8)n$, as desired.

Fix a subset $W \subseteq N(v, U')$ of size $(\xi\zeta/16)n$ and write $Z := U' - W$. Since every vertex u in G satisfies $d_G(u, U') \geq (\xi\zeta/8)n$, every vertex in W has at least $(\xi\zeta/16)n$ neighbours in Z. By Lemma 3.8, the graph $G[W, Z]$ has a $K_{t,t}$, say, with vertex set $A \cup B$ where $A \subseteq W$ and $B \subseteq Z$. Let B' be a subset of B of size $t - 1$. Then $G[A, B' \cup \{v\}]$ is a copy

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of $K_{t,t}$ in $G[V_0 \cup U^{(1)}]$, which is vertex-disjoint from $V(\mathcal{K}_1)$ and contains exactly one vertex from V_0, contradicting the maximality of $\mathcal{K}_1$. □

Next, we find a small number of vertex-disjoint $K_{t,t}$'s in $G[U^{(2)}]$ that will allow us to achieve the desired divisibility condition. Recall that the *support* of a function $f : A \to B$, denoted supp(f), is the set of elements a in A with $f(a) \neq 0$.

Claim 5.7. *Let $f : [2m] \to \{0, \ldots, t-1\}$ be a function such that $\sum_{i \in [2m]} f(i)$ is divisible by t, and if G is bipartite, then $\sum_{i \in [m]} f(2i)$ is also divisible by t. Then there is a collection $\mathcal{K}_2$ of pairwise vertex-disjoint $K_{t,t}$'s in $G[U^{(2)}]$ such that $|U_i \cap V(\mathcal{K}_2)| \le 3t \cdot |\operatorname{supp}(f)|$ and $|U_i \cap V(\mathcal{K}_2)| \equiv f(i) \pmod{t}$ for $i \in [2m]$.*

Proof of claim. We prove the claim by induction on the size of supp(f). If $|\operatorname{supp}(f)| = 0$, then we can take $\mathcal{K}_2 = \emptyset$. So, suppose that supp$(f) \neq \emptyset$, and let $a \in [2m]$ satisfy $f(a) \neq 0$. By the assumption that $\sum_{i \in [2m]} f(i)$ is divisible by t, there is $b \in [2m] - \{a\}$ such that $f(b) \neq 0$, and, if G is bipartite, we may take b to have the same parity as a.

Define $f' : [2m] \to \{0, \ldots, t-1\}$ as follows.

$$f'(i) = \begin{cases} f(i) & i \neq a, b, \\ 0 & i = a, \\ f(a) + f(b) & i = b. \end{cases}$$

Notice that $\sum_{i \in [2m]} f'(i)$ is divisible by t, $\sum_{i \in [m]} f'(2i)$ is also divisible by t if G is bipartite and $|\operatorname{supp}(f')| < |\operatorname{supp}(f)|$. Thus, by the induction hypothesis, there is a collection $\mathcal{K}'$ of pairwise vertex-disjoint $K_{t,t}$'s in $G[U^{(2)}]$ such that $|U_i \cap V(\mathcal{K}')| \le 3t \cdot |\operatorname{supp}(f')|$ and $|U_i \cap V(\mathcal{K}')| \equiv f'(i) \pmod{t}$ for $i \in [2m]$.

We claim that there is a walk W of even length from U_a to U_b in Γ_{ref} that does not visit a vertex more than three times. Indeed, notice that Γ_{ref} is connected since G' has no $\zeta/12$-sparse cuts (see (G'8)). If G is bipartite, any path in Γ_{ref} from U_a to U_b suffices: such a path exists by connectivity, and because a and b have the same parity, its length is necessarily even. If G is not bipartite, then (G'9) (and the fact that $\mu \ll \gamma$) implies that Γ_{ref} is also not bipartite. Let C be an odd cycle in Γ_{ref}, and let $v \in V(C)$. By connectivity of Γ_{ref}, there are paths P_a and P_b from a to v and from b to v. If the lengths of P_a and P_b have the same parity, then $aP_a v P_b b$ is a walk of even length visiting each vertex at most twice. Otherwise, $aP_a v C v P_b b$ is a walk of even length that visits each vertex at most three times, as desired.

Let W be a walk in Γ_{ref} as in the previous paragraph, and write $W = U_{i_0}, \ldots, U_{i_{2\ell}}$, where $U_{i_0} = U_a$ and $U_{i_{2\ell}} = U_b$. Since W visits each vertex in Γ_{ref} at most three times and $|U_i \cap V(\mathcal{K}')| \le 3t \cdot |\operatorname{supp}(f')| \le 6tm$ for $i \in [2m]$, we can apply Lemma 3.8 to iteratively choose pairwise vertex-disjoint $K_{t,t}$'s in $U^{(2)} - V(\mathcal{K}')$, denoted $K_1, \ldots, K_\ell$, such that for every $j \in [\ell]$, K_j consists of t vertices in $U^{(2)}_{i_{2j-1}}$, $f(a)$ vertices in $U^{(2)}_{i_{2j-2}}$ and $t - f(a)$ vertices in $U^{(2)}_{i_{2j}}$. Write $\mathcal{K}_2 := \mathcal{K}' \cup \{K_1, \ldots, K_\ell\}$. It is easy to check that $\mathcal{K}_2$ satisfies the requirements of the induction hypothesis since $|U_i \cap V(\mathcal{K}_2)| \le 3t \cdot |\operatorname{supp}(f')| + 3t \le 3t|\operatorname{supp}(f)|$ and $|U_i \cap V(\mathcal{K}_2)| \equiv f(i) \pmod{t}$ for $i \in [2m]$. This completes the induction step and proves the claim. □

Let $\mathcal{K}_1$ be a collection of copies of $K_{t,t}$ as guaranteed by Claim 5.6. For $i \in [2m]$, let $f(i) := -|U_i \cap V(\mathcal{K}_1)| \pmod{t} = |U_i^{(1)} \cap V(\mathcal{K}_1)| \pmod{t}$. Then

$$-\sum_{i \in [2m]} f(i) \equiv \sum_{i \in [2m]} |U_i \cap V(\mathcal{K}_1)| \equiv |V(\mathcal{K}_1)| - |V_0| \equiv 0 \pmod{2t}, \tag{17}$$

where the second equality uses that $V(G) = V_0 \cup U_1 \cup \cdots \cup U_{2m}$ and $V_0 \subseteq V(\mathcal{K}_1)$, and the third follows from the fact that both $|V(\mathcal{K}_1)|$ and $|V_0|$ are divisible by $2t$, with the divisibility of $|V_0|$ given by (G'4).

If G is bipartite (and balanced), then, denoting by $\{X, Y\}$ the bipartition of G with $X - V_0 = \bigcup_{i \in [m]} U_{2i-1}$ and $Y - V_0 = \bigcup_{i \in [m]} U_{2i}$,

$$-\sum_{i \in [m]} f(2i) \equiv \sum_{i \in [m]} |U_{2i} \cap V(\mathcal{K}_1)| \equiv |(Y - V_0) \cap V(\mathcal{K}_1)| \equiv \frac{|V(\mathcal{K}_1)| - |V_0|}{2} \equiv 0 \pmod{t}. \tag{18}$$

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The first equality follows by the definition of f, and the second equality follows from the fact that $Y - V_0 = \bigcup_{i \in [m]} U_{2i}$. For the third equality, first recall that $|U_1| = \ldots = |U_{2m}|$ by (G'4), so $|X - V_0| = |Y - V_0|$. This implies that $|X \cap V_0| = |Y \cap V_0|$ since G is bipartite and balanced. Combining this with the fact that $V_0 \subseteq V(\mathcal{K}_1)$, and that every copy of $K_{t,t}$ in $\mathcal{K}_1$ contains exactly one vertex of V_0, we get $|(X - V_0) \cap V(\mathcal{K}_1)| = |(Y - V_0) \cap V(\mathcal{K}_1)|$ from which the third equality follows easily. The final equality follows from $|V_0|$ and $|V(\mathcal{K}_1)|$ being divisible by $2t$. Using (17), (18) and Claim 5.7, we can find a collection $\mathcal{K}_2$ of pairwise vertex-disjoint copies of $K_{t,t}$ in $G[U^{(2)}]$ satisfying $|U_i^{(2)} \cap V(\mathcal{K}_2)| \equiv f(i) \equiv -|U_i^{(1)} \cap V(\mathcal{K}_1)| \pmod{t}$ for $i \in [2m]$, which implies that $|U_i \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)|$ is divisible by t, as desired.

5.4   Covering the remaining vertices of G with vertex-disjoint copies of $K_{t,t}$

In this subsection, our aim is to cover the remaining vertices of G with vertex-disjoint copies of $K_{t,t}$. To this end, we will find a collection $\mathcal{K}_3$ of copies of $K_{t,t}$ such that $V(\mathcal{K}_3) \subseteq V(G') - V(\mathcal{K}_1 \cup \mathcal{K}_2)$ and such that the sets $U_i' := U_i - V(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$, $i \in [2m]$, have the same size. This allows us to apply the blow-up lemma to find perfect $K_{t,t}$-packings in $G'[U_{2i-1}', U_{2i}']$ for every $i \in [m]$, thereby covering all of the remaining vertices, as desired.

Let us now carry out the above strategy. For $i \in [2m]$, let $U_i'^{(3)}$ be an arbitrary subset of $U_i^{(3)}$ such that

$$\left| U_i'^{(3)} \right| = \left| U_i^{(3)} \right| - \frac{\left| U_i \cap V(\mathcal{K}_1 \cup \mathcal{K}_2) \right|}{t}. \tag{19}$$

(Notice that the right-hand-side of (19) is an integer due to the choice of $\mathcal{K}_2$.) Recall that $U^{(3)} = \bigcup_{i \in [2m]} U_i^{(3)}$. Write $U'^{(3)} := \bigcup_{i \in [2m]} U_i'^{(3)}$. In the rest of this section, let $H := G'\left[U^{(3)}\right]$, $H' := G'\left[U'^{(3)}\right]$ and let $d' := d/2t$.

Claim 5.8. *There is a perfect matching in H'.*

Proof of claim. We claim that the following properties hold.

(a) H has maximum degree at most $d'(1 + \eta)$.

(b) H has average degree at least $d'(1 + \eta) - 32\eta t \cdot |V(H)|$.

(c) H has no $\zeta/20$-sparse cuts.

(d) H is either bipartite and balanced, or one needs to remove at least $\frac{\gamma}{256t^2} n^2$ edges from H to make it bipartite.

(e) $|V(H) - V(H')| \leq 8t\xi \cdot |V(H)|$.

(f) If G is bipartite and balanced, then H' is also bipartite and balanced.

Indeed, by (U2) and (U5), any vertex v in H has degree at most $\sum_{i \in [2m]} \left((1 + \varepsilon) \frac{d_{G'}(v, U_i)}{2t} \right) + \varepsilon n \leq d'(1 + \varepsilon) + \varepsilon n \leq$

$d'(1 + \eta)$, proving (a). For proving (b), we have the following sequence of inequalities.

$$2e(H) = \sum_{v \in V(H)} \sum_{i \in [2m]} d_{G'}(v, U_i^{(3)}) \overset{(U5)}{\geq} (1 - \varepsilon) \sum_{v \in V(H)} \sum_{i \in [2m]} d_{G'}(v, U_i) \cdot \frac{|U_i^{(3)}|}{|U_i|}$$

$$\overset{(U2)}{=} \frac{1 - \varepsilon}{2t} \sum_{v \in V(H)} d_{G'}(v)$$

$$= \frac{1 - \varepsilon}{2t} \left(\sum_{v \in V(G')} d_{G'}(v) - \sum_{v \in V(G') - V(H)} d_{G'}(v) \right)$$

$$\overset{(G'7)}{\geq} \frac{1 - \varepsilon}{2t} \cdot \left((d - 6\eta n) \cdot |V(G')| - d \cdot (|V(G')| - |V(H)|) \right)$$

$$= \frac{1 - \varepsilon}{2t} \cdot \left(d \cdot |V(H)| - 6\eta n \cdot |V(G')| \right)$$

$$\geq d' \cdot |V(H)| - \varepsilon d' \cdot |V(H)| - 6\eta n \cdot |V(H)|$$

$$= d'(1 + \eta) \cdot |V(H)| - (\varepsilon d' + 6\eta n + \eta d') \cdot |V(H)|$$

$$\geq d'(1 + \eta) \cdot |V(H)| - 8\eta n \cdot |V(H)|$$

$$\geq d'(1 + \eta) \cdot |V(H)| - 32\eta t \cdot |V(H)|^2.$$

It follows that H has average degree at least $d'(1 + \eta) - 32\eta t \cdot |V(H)|$, as required for (b). Notice that (c) is the same as (U6) and (d) is the same as (U7). For proving (e), note that $|V(H) - V(H')| \leq |U^{(1)}| + |U^{(2)}| \leq 2\xi n \leq 8t\xi \cdot |V(H)|$. Finally, for proving (f), suppose that G is bipartite and balanced, and denote its bipartition by $\{X, Y\}$. Recall that, by (G'4), $|V_0 \cap X| = |V_0 \cap Y|$ (since $|X| = |Y|$). By the choice of $\mathcal{K}_1$ and $\mathcal{K}_2$ (see Claim 5.6 and Claim 5.7), we have $|V(G') \cap (V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap X)| = |V(G') \cap (V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap Y)|$, and thus

$$|V(H') \cap X| = \sum_{i \in [m]} \left(|U_{2i-1}^{(3)}| - \frac{|U_{2i-1} \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)|}{t} \right)$$

$$\overset{(U2)}{=} \left(\sum_{i \in [m]} \frac{|U_{2i-1}|}{2t} \right) - \frac{|V(G') \cap (V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap X)|}{t}$$

$$\overset{(G'4)}{=} \left(\sum_{i \in [m]} \frac{|U_{2i}|}{2t} \right) - \frac{|V(G') \cap (V(\mathcal{K}_1 \cup \mathcal{K}_2) \cap Y)|}{t}$$

$$= \sum_{i \in [m]} \left(|U_{2i}^{(3)}| - \frac{|U_{2i} \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)|}{t} \right) = |V(H') \cap Y|,$$

proving that H' is balanced, as required for (f).

Thus, by applying Lemma 3.3 with H, $d'(1 + \eta)$, $c/2t$, $32\eta t$, $8t\xi$, $\gamma/(256t^2)$, $\zeta/20$, $V(H) - V(H')$, $V(H)$ playing the roles of G, d, c, η, ξ, γ, ζ, W, n respectively, we find that there is a Hamilton cycle in H', which contains a perfect matching in H' (since $|V(H')|$ is even), thus proving the claim. $\qquad\square$

Let $\mathcal{M}'$ be a perfect matching in H' (guaranteed by Claim 5.8). For $i, j \in [2m]$, denote by $f(i, j)$ the number of edges in $\mathcal{M}'[U_i^{(3)}, U_j^{(3)}]$.

Claim 5.9. *There is a collection $\mathcal{K}_3$ of pairwise vertex-disjoint copies of $K_{t,t}$ in $G'\left[U^{(4)}\right]$ such that for every $i, j \in [2m]$, $\mathcal{K}_3$ consists of $f(i, j)$ copies of $K_{t,t}$ in $G'[U_i, U_j]$ (and there are no other $K_{t,t}$'s).*

Proof. Let $\mathcal{K}_3$ be a maximal collection of pairwise vertex-disjoint $K_{t,t}$'s in $G'\left[U^{(4)}\right]$, such that for every $i, j \in [2m]$, $\mathcal{K}_3$ consists of at most $f(i, j)$ copies of $K_{t,t}$ in $G'[U_i^{(4)}, U_j^{(4)}]$ (and no other copies of $K_{t,t}$). Suppose towards a

contradiction that there exist $i, j \in [2m]$ such that the number of copies of $K_{t,t}$ in $\mathcal{K}_3$ that are in $G'[U_i, U_j]$ is strictly smaller than $f(i,j)$ (implying that $f(i,j) > 0$). Write $U''_\ell := U^{(4)}_\ell - V(\mathcal{K}_3)$ for every $\ell \in [2m]$.

We claim that $|U''_\ell| \geq |U_\ell|/6$ for every $\ell \in [2m]$. Indeed, note that for every $\ell \in [2m]$ we have $|U^{(4)}_\ell \cap V(\mathcal{K}_3)| \leq t \cdot \sum_{j \in [2m]} f(\ell, j) = t \cdot |U'^{(3)}_\ell| \leq t \cdot |U^{(3)}_\ell| = |U_\ell|/2$, where in the last equality we used (U2). Thus, by (U3), for every $\ell \in [2m]$, $|U''_\ell| = |U^{(4)}_\ell| - |U^{(4)}_\ell \cap V(\mathcal{K}_3)| \geq (2/3)|U_\ell| - (1/2)|U_\ell| = |U_\ell|/6$, as claimed.

Now recall that, by (G'6), for all $i, j \in [2m]$, the graph $G'[U_i, U_j]$ is 5ε-regular, and it has density more than $\mu/5$ if it has at least one edge. Hence, we have $e_{G'}(U''_i, U''_j) > (\mu/5 - 5\varepsilon) \cdot |U''_i| \cdot |U''_j|$, so by the Kővári–Sós–Turán theorem (Lemma 3.8), there is a $K_{t,t}$ in $G'[U''_i, U''_j]$, contradicting the maximality of $\mathcal{K}_3$ and proving the claim. $\qquad\square$

Let $\mathcal{K}_3$ be a collection of copies of $K_{t,t}$ as guaranteed by Claim 5.9. For $i \in [2m]$, write $U'_i := U_i - V(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)$. For $i \in [2m]$, by the choice of $U'^{(3)}_i$ as in (19) and the fact that $|U_i \cap V(\mathcal{K}_3)| = t \cdot \sum_{j \in [2m]} f(i,j) = t \cdot |U'^{(3)}_i|$, we have

$$|U'_i| = |U_i - V(\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3)| = |U_i| - |U_i \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)| - |U_i \cap V(\mathcal{K}_3)|$$
$$= |U_i| - |U_i \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)| - t \cdot \left(|U^{(3)}_i| - \frac{|U_i \cap V(\mathcal{K}_1 \cup \mathcal{K}_2)|}{t} \right) = |U_i| - t|U^{(3)}_i| = \frac{|U_i|}{2}.$$

Recall that the sets U_i for $i \in [2m]$ have the same size, which is divisible by $2t$ by (G'4). Hence, the sets U'_i for $i \in [2m]$ have the same size, which is divisible by t. Moreover, by (U4), (U5), (G'5) and the fact that $U^{(5)}_i$ is a subset of U'_i for each $i \in [2m]$, we have that $G'[U'_{2i-1}, U'_{2i}]$ is $(10\varepsilon, \mu/50)$-super-regular for $i \in [m]$. Therefore, it follows from the blow-up lemma (Lemma 3.6) that $G'[U'_{2i-1}, U'_{2i}]$ has a perfect $K_{t,t}$-packing (since $\varepsilon \ll \mu, 1/t$). These perfect $K_{t,t}$-packings of $G'[U'_{2i-1}, U'_{2i}]$ for $i \in [m]$, along with $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$, form a perfect $K_{t,t}$-packing in G. This completes the proof of Lemma 5.1. $\qquad\square$

6 H-packings in dense regular graphs

In this section, we combine Lemma 4.1 and Corollary 5.2 to prove Theorem 1.4, which asserts that for every bipartite graph H and every constant $0 < c \leq 1$, there is a constant $C > 0$ such that every d-regular graph G of order n, with $d \geq cn$, has an H-packing that covers all but at most C vertices of G. Denote the sizes of the two parts of H by a and b, and write $t := a + b$. Then, it is easy to see that there are two vertex-disjoint copies of H covering all the vertices of the complete bipartite graph $K_{t,t}$. Hence, Theorem 1.4 follows immediately from the following.

Theorem 6.1. *Let $0 < c \leq 1$, and let $t \geq 2$ be an integer. Then there exists a constant $C = C(t, c)$ such that every d-regular graph G of order n, where $d \geq cn$, has a $K_{t,t}$-packing that covers all but at most C vertices of G.*

Proof of Theorem 6.1. Let $n_0 \in \mathbb{N}$ be chosen sufficiently large depending on c and t, that is, $1/n_0 \ll c$. If $n < n_0$, then the conclusion holds trivially for a suitably large constant $C = C(t, c)$. So, assume $n \geq n_0$. Then by applying Lemma 4.1 to G, we obtain positive numbers η, γ, ζ, r with $r \leq \lceil 1/c \rceil$, a subgraph $G' \subseteq G$ with $V(G') = V(G)$ and pairwise disjoint sets $Z_1, \ldots, Z_r, L \subseteq V(G)$ satisfying properties (B1)–(B5) such that

$$1/n_0 \ll \eta \ll \zeta \ll \gamma \ll c, 1/t.$$

For all $i \in [r]$, by (B2), $G'[Z_i]$ has average degree at least $d - \eta|Z_i|$, by (B3), $G'[Z_i]$ has no ζ-sparse cuts and, by (B4), either $G'[Z_i]$ is bipartite and balanced, or one needs to remove at least $\gamma n^2 \geq \gamma|Z_i|^2$ edges to make it bipartite, and $d \geq cn \geq c|Z_i|$. Thus, for every $i \in [r]$, we can apply Corollary 5.2 with $G'[Z_i]$ playing the role of G, to obtain a $K_{t,t}$-packing of $G'[Z_i]$ covering all but at most $2t - 1$ vertices in Z_i. Moreover, $G[L]$ has a perfect $K_{t,t}$-packing by (B5), and $|V(G) - (Z_1 \cup \ldots \cup Z_r \cup L)| \leq 64r^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{8\zeta} \right)^t$ by (B1). This yields a $K_{t,t}$-packing of G covering all but at most the following number of vertices.

$$r \cdot (2t - 1) + 64 \cdot r^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{8\zeta} \right)^t \leq 66 \cdot r^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{8\zeta} \right)^t \leq 66 \left(\left\lceil \frac{1}{c} \right\rceil \right)^2 \cdot \frac{t}{c} \cdot \left(\frac{e}{8\zeta} \right)^t.$$

This completes the proof of Theorem 6.1. □

7 Packing subdivisions in dense regular graphs

In this section, we prove Theorem 1.7. We start by giving a detailed sketch of its proof here. Let F be a graph, and let $0 \leq c \leq 1$. Let G be a d-regular graph of order n, where $d \geq cn$. We start, as before, by partitioning the vertex set of G into sets $Z_1, \ldots, Z_r$ such that the subgraphs $G[Z_i]$, $i \in [r]$ induced by these sets are expanders with few edges between them (using Lemma 3.2). Let $\{X_i, Y_i\}$ be a bipartition of G maximising the number of edges in $G[X_i, Y_i]$. Let G' be the graph obtained from G by removing all edges within X_i and Y_i for all $i \in [r]$ such that $G[Z_i]$ is close to being bipartite. This ensures that for all $i \in [r]$, $G'[Z_i]$ is either bipartite with the bipartition $\{X_i, Y_i\}$ (with roughly the same average degree and minimum degree as $G[Z_i]$) or it is far from being bipartite.

Next, we find short paths $P_1, \ldots, P_r$ (where each P_i has both of its ends in Z_i) whose removal 'balances' the expanders that are close to being bipartite. More precisely, for every $i \in [r]$, if $G[Z_i]$ is close to being bipartite, then, writing $Q := P_1 \cup \ldots \cup P_r$, we have $|X_i - V(Q)| = |Y_i - V(Q)|$, and P_i has one end in X_i and the other end in Y_i. The key tool for building these paths is Lemma 3.4 due to Gruslys and Letzter [11], which produces a small linear forest H that balances the expanders that are close to being bipartite, while also ensuring that each expander contains zero or two leaves of H. To obtain the desired paths $P_1, \ldots, P_r$, we iteratively merge pairs of components of H whose leaves lie in the same expander and, if necessary, we define some paths P_i as arbitrary edges within $G'[Z_i]$.

Our next step is to find two small, vertex-disjoint F-subdivisions within each expander $G'[Z_i]$, denoted F_i and F_i', which are disjoint from Q and whose union is balanced; that is, if $G[Z_i]$ is close to being bipartite, then $|X_i \cap V(F_i \cup F_i')| = |Y_i \cap V(F_i \cup F_i')|$. We construct F_i and F_i' greedily: we begin by selecting $|V(F)|$ vertices from X_i to serve as the branch vertices of F_i, and $|V(F)|$ vertices from Y_i to serve as the branch vertices of F_i'. We then iteratively connect pairs of branch vertices (corresponding to the edges of F) one pair at a time, to complete each subdivision.

Finally, for each $i \in [r]$, we absorb the path P_i and the remaining uncovered vertices in Z_i into the subdivision F_i. To do this, we replace an arbitrary edge $x_i y_i$ in F_i with a path that starts at x_i, connects to one end of P_i, traverses P_i, and then continues through all the remaining uncovered vertices in Z_i before returning to y_i.

In the last three steps, we rely on the fact that the expanders $G'[Z_i]$ are *robustly connected* via short paths; that is, any two vertices can be joined by a short path that avoids any given small set of forbidden vertices (see Lemma 7.1). In the final step, we also use the *robust Hamiltonicity* property of the expanders $G'[Z_i]$, meaning that for any small set of forbidden vertices W (which is balancing if $G'[Z_i]$ is bipartite), and any two vertices $x, y \in Z_i - W$ (lying in different parts of $G'[Z_i]$ if it is bipartite), there exists a Hamilton path in $G'[Z_i] - W$ with ends x and y.

The proof of Theorem 1.7 is given in Section 7.3, after establishing two preparatory lemmas in the following two subsections.

7.1 Expanders are robustly connected via short paths

In the following lemma, we show that our expanders are 'robustly connected' via short paths, a property we will use several times in the proof.

Lemma 7.1. *Let n be a positive integer, and let $\beta, \xi, \zeta, \delta \in (0, 1)$ satisfy $1/n \ll \xi, \beta \ll \zeta \ll \delta$. Let H be a graph on at most n vertices with minimum degree at least δn and no ζ-sparse cuts. If H is β-almost-bipartite, let H' be a largest bipartite subgraph of H; otherwise, set $H' := H$. Then, H' has no $\zeta/2$-sparse cuts. Moreover, if $W \subseteq V(H)$ is a subset of size at most ξn, then $H'' := H' - W$ satisfies the following properties.*

(i) *H'' has no $\zeta/2$-sparse cuts.*

(ii) *For every pair $p, p' \in V(H'')$, there exists a (p, p')-path in H'' of length at most $15/\delta$.*

Proof. Notice that H' has minimum degree at least $\delta n/2$, and H'' has minimum degree at least $\delta n/4$. First, we show that H' has no $\zeta/2$-sparse cuts. Consider a partition $\{X, Y\}$ of $V(H')$ with $|X| \leq |Y|$. If $|X| \leq \delta n/4$, then, since H' has minimum degree at least $\delta n/2$, we have $e_{H'}(X,Y) \geq \frac{\delta n}{4} \cdot |X| \geq \frac{\delta}{4} \cdot |X| \cdot |Y| \geq \frac{\zeta}{2} \cdot |X| \cdot |Y|$. So we may assume that $|X|, |Y| \geq \delta n/4$. Then, since H has no ζ-sparse-cuts, we have $e_{H'}(X,Y) \geq e_H(X,Y) - \beta n^2 \geq \zeta \cdot |X| \cdot |Y| - \beta n^2 \geq \zeta/2 \cdot |X| \cdot |Y|$, showing that H' has no $\zeta/2$-sparse cuts, as desired. To prove (i), we use a very similar argument. Consider a partition $\{X, Y\}$ of $V(H'')$ with $|X| \leq |Y|$. If $|X| \leq \delta n/8$, then, since H'' has minimum degree at least $\delta n/4$, we have $e_{H''}(X,Y) \geq \frac{\delta n}{8} \cdot |X| \geq \frac{\delta}{8} \cdot |X| \cdot |Y| \geq \frac{\zeta}{2} \cdot |X| \cdot |Y|$. So we may assume that $|X|, |Y| \geq \delta n/8$. But then, since H has no ζ-sparse-cuts,

$$
\begin{aligned}
e_{H''}(X,Y) = e_{H'}(X,Y) = e_{H'}(X, Y \cup W) - e_{H'}(X, W) \\
\geq (e_H(X, Y \cup W) - \beta n^2) - \xi n^2 \\
\geq \zeta \cdot |X| \cdot |Y| - \beta n^2 - \xi n^2 \\
\geq \zeta/2 \cdot |X| \cdot |Y|.
\end{aligned}
$$

since $\xi, \beta \ll \delta, \zeta$. This shows that H'' has no $\zeta/2$-sparse cuts, proving (i).

Now we prove (ii). Let $p, p' \in V(H'')$. By (i), H'' is connected, so it contains a (p, p')-path. Let P be a shortest (p, p')-path in H''. We claim that P has length at most $15/\delta$. Indeed, suppose otherwise, and let P be the path $u_0 \ldots u_t$, where $u_0 = p$ and $u_t = p'$ such that $t > 15/\delta$. Set $U := \{u_0, u_3, \ldots, u_{3q}\}$, where $q := \lceil 4/\delta \rceil$ (this is well defined because $t > \frac{15}{\delta} \geq 3 \cdot (\frac{4}{\delta} + 1) \geq 3 \cdot \lceil \frac{4}{\delta} \rceil = 3q$). Since H'' has minimum degree at least $\delta n/4$, the number of edges of H'' incident to vertices in U is at least $(q + 1) \cdot \delta n/4 > n$, implying that there is a vertex v in H'' with at least two neighbours in U, say u_{3i} and u_{3j}, where $i < j$. Denote by W the walk $u_0 u_1 \ldots u_{3i} v u_{3j} \ldots u_t$. Note that W is a (p, p')-walk in H''. Let P' be a (p, p')-path contained in W. Since W is shorter than P, it follows that P' is also shorter than P, a contradiction to the minimality of P. This proves (ii), completing the proof of Lemma 7.1. $\square$

7.2 Balancing the expanders

As further preparation for the proof of Theorem 1.4, we state the following corollary of Lemma 3.4, which will allow us to balance the expanders using a suitable collection of paths. This is somewhat similar to Lemma 4.1, where we also balance the expanders, but using a collection of vertex-disjoint $K_{t,t}$'s instead of paths.

Lemma 7.2. *Let* $\eta, \beta, \xi, \gamma, \zeta, \delta, c \in (0, 1)$ *and* $n \in \mathbb{N}$ *satisfy* $1/n \ll \eta \ll \beta \ll \xi \ll \gamma \ll \zeta \ll \delta \ll c$. *Let* G *be a* d-*regular graph on* n *vertices, where* $d \geq cn$. *Suppose that* $\{Z_1, \ldots, Z_r\}$ *is a partition of* $V(G)$ *satisfying properties* (E1)–(E4) *in Lemma 3.2, where* $r \leq \lceil 1/c \rceil$. *For* $i \in [r]$ *such that* $G[Z_i]$ *is* β-*almost-bipartite, let* $\{X_i, Y_i\}$ *be a partition of* Z_i *maximising* $e_G(X_i, Y_i)$. *Then there are vertex-disjoint paths* $P_1, \ldots, P_r$ *in* G *satisfying the following properties, where* $Q := \bigcup_{i \in [r]} P_i$.

(P'1) *For every* $i \in [r]$, *we have* $2 \leq |V(P_i)| \leq \xi n$.

(P'2) *For every* $i \in [r]$, *both leaves of* P_i *are in* Z_i. *Moreover, for each* $i \in [r]$ *such that* $G[Z_i]$ *is* β-*almost-bipartite, one of the leaves of* P_i *is in* X_i *and the other leaf is in* Y_i.

(P'3) *For each* $i \in [r]$ *such that* $G[Z_i]$ *is* β-*almost-bipartite,* $|X_i - V(Q)| = |Y_i - V(Q)|$.

Proof. Let H be the linear forest guaranteed by Lemma 3.4, satisfying properties (P1)–(P4). Let G' be the subgraph of G obtained by removing all edges with both endpoints in X_i or both endpoints in Y_i, for each $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite.

We will iteratively add some edges from G' to H using the following procedure to construct subgraphs $H_0 := H, H_1, \ldots, H_r$ satisfying $|V(H_i)| \leq \xi n + 15i/\delta$ for every $i \in [r]$.

For $i \in [r]$, suppose that H_{i-1} is already defined such that $|V(H_{i-1})| \leq \xi n + 15(i-1)/\delta$; we construct H_i as follows. If H_{i-1} has two distinct components, say P and P', with leaves p, p' in Z_i, say $p \in V(P) \cap Z_i$, $p' \in V(P') \cap Z_i$, let L_i be a shortest (p, p')-path in $G'[Z_i] - (V(H_{i-1}) - \{p, p'\})$. Applying (ii) of Lemma 7.1 with $G[Z_i]$, $G'[Z_i]$,

$V(H_{i-1}) - \{p, p'\}$, 2ξ playing the roles of H, H', W, ξ, respectively, we obtain that the path L_i has length at most $15/\delta$. (Note that Lemma 7.1 indeed applies because $|V(H_{i-1})| \leq \xi n + 15(i-1)/\delta \leq 2\xi n$.) Define $H_i := H_{i-1} \cup L_i$. Note that $|V(H_i)| \leq |V(H_{i-1})| + 15/\delta \leq \xi n + 15i/\delta$, as required.

If $G[Z_i]$ is β-almost-bipartite, then, by (P3), one of the two vertices p, p' is in X_i and the other one is in Y_i. Therefore, we have the following.

$$\text{If } G[Z_i] \text{ is } \beta\text{-almost-bipartite, then } |V(L_i) \cap X_i| = |V(L_i) \cap Y_i|. \tag{20}$$

Notice that for every $i \in [r]$, the number of leaves of H_j in Z_i is the same as the number of leaves of H_{j-1} in Z_i, unless $j = i$, in which case it either remains the same, or it decreases from 2 to 0. Moreover, if H_i has two leaves in Z_i, then they must belong to the same component of H_i, which is then also a component of H_r; denote this component by P_i. For every $i \in [r]$ such that P_i is not defined (which means that H_r has no components with leaves in Z_i), define P_i to be an edge in $G'[Z_i] - V(H_r)$. This is indeed possible because

$$|V(H_r)| \leq \xi n + 15r/\delta \leq 2\xi n, \tag{21}$$

which implies that $G'[Z_i] - V(H_r)$ contains at least one edge.

Notice that $P_1, \ldots, P_r$ are pairwise vertex-disjoint. We claim that they satisfy (P'1)–(P'3) (with 2ξ playing the role of ξ for (P'1)). Indeed, by (21) we have $|V(P_i)| \leq \max\{2, |V(H_r)|\} \leq 2\xi n$, and since P_i contains two leaves, we have $|V(P_i)| \geq 2$, proving (P'1). Note that (P'2) holds when P_i is a component of H_r by the discussion in the previous paragraph together with (P3); otherwise, it holds by the choice of P_i as an edge in $G'[Z_i] - V(H_r)$. For proving (P'3), recall that $Q = P_1 \cup \ldots \cup P_r$ and consider $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite. Also recall that if L_i is defined, then its leaves are p, p'. Then we have,

$$V(Q) \cap Z_i = \begin{cases} V(H) \cap Z_i & \text{if } P_i \subseteq H_r \\ (V(H) \cap Z_i) \uplus V(P_i) & \text{if } P_i \nsubseteq H_r \text{ and } L_i \text{ is not defined} \\ (V(H) \cap Z_i) \uplus V(P_i) \uplus (V(L_i) - \{p, p'\}) & \text{if } P_i \nsubseteq H_r \text{ and } L_i \text{ is defined.} \end{cases}$$

(Here $A \uplus B$ denotes the union of the disjoint sets A and B.) Hence, (P'3) follows from (P4), (20), and the choice of P_i as an edge in $G'[Z_i] - V(P_r)$ when $P_i \nsubseteq H_r$, completing the proof of Lemma 7.2. $\square$

7.3 Proof of Theorem 1.7

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. Let F be a graph with at least one edge, let $n_0 \in \mathbb{N}$ be such that $1/n_0 \ll c$, and let G be a d-regular graph of order $n \geq n_0$ and let $d \geq cn$. Our aim is to show that G has a perfect TF-packing.

By applying Lemma 3.2 to G, we obtain positive numbers $\eta, \beta, \gamma, \zeta, \delta$ where

$$1/n_0 \ll \eta \ll \beta \ll \gamma \ll \zeta \ll \delta \ll c$$

and a partition $\{Z_1, \ldots, Z_r\}$ of $V(G)$ satisfying properties (E1)–(E4) such that $r \leq \lceil 1/c \rceil$.

Let $\xi \in (0, 1)$ satisfy

$$\beta \ll \xi \ll \gamma.$$

For each $i \in [r]$, let X_i, Y_i be a partition of Z_i that maximizes $e_G(X_i, Y_i)$. Let G' be the subgraph of G obtained by removing all edges with both endpoints in X_i or both in Y_i, for each $i \in [r]$ such that $G[Z_i]$ is β-almost-bipartite. Let $P_1, \ldots, P_r$ be vertex-disjoint paths satisfying (P'1)–(P'3), guaranteed by Lemma 7.2, and let $Q := P_1 \cup \ldots \cup P_r$.

7.3.1 Constructing a balanced pair of F-subdivisions in each expander

For every $i \in [r]$, we will construct two vertex-disjoint subdivisions of F in $G'[Z_i] - V(Q)$ as follows.

Claim 7.3. *For every $i \in [r]$, there exist two vertex-disjoint F-subdivisions F_i, F_i' in $G'[Z_i] - V(Q)$ such that $|V(F_i)|, |V(F_i')| \leq e(F) \cdot 15/\delta$. Moreover, if $G[Z_i]$ is β-almost-bipartite, then*

$$|X_i \cap (V(F_i \cup F_i'))| = |Y_i \cap (V(F_i \cup F_i'))|. \tag{22}$$

Proof. Fix $i \in [r]$. Let $S, S' \subseteq Z_i - V(Q)$ be disjoint sets of size $|V(F)|$, where $S \subseteq X_i$ and $S' \subseteq Y_i$ if $G[Z_i]$ is β-almost-bipartite, and let $\phi : S \to V(F)$ and $\phi' : S' \to V(F)$ be two injections chosen arbitrarily. Let T be the set of unordered pairs $\{s, s'\}$ with $s, s' \in S$ such that $\phi(s)\phi(s')$ is an edge in F, and similarly, let T' be the set of unordered pairs $\{s, s'\}$ with $s, s' \in S'$ such that $\phi'(s)\phi'(s')$ is an edge in F.

Let $\mathcal{P}$ be a maximal collection of pairwise internally vertex-disjoint paths in $G'[Z_i] - V(Q)$, with internal vertices in $Z_i - (V(Q) \cup S \cup S')$, whose length is at most $15/\delta$ and such that each of the paths in $\mathcal{P}$ is an (s, s')-path for a distinct pair $\{s, s'\} \in T \cup T'$. We claim that $|\mathcal{P}| = 2e(F)$. Indeed, suppose otherwise. Then there is a pair $(s, s') \in T \cup T'$ for which there is no (s, s')-path in $\mathcal{P}$. Fix such a pair (s, s'), and let $W := V(\mathcal{P}) \cup V(Q) \cup S \cup S'$. Then

$$|W| \leq 2e(F) \cdot 15/\delta + \xi n + 2|V(F)| \leq 2\xi n.$$

Thus, by applying (ii) of Lemma 7.1 (with 2ξ playing the role of ξ), we obtain an (s, s')-path P in $G'[Z_i] - W$ of length at most $15/\delta$. But then $\mathcal{P} \cup \{P\}$ contradicts the maximality of $\mathcal{P}$. This shows that $|\mathcal{P}| = 2e(F)$, as claimed.

Pick a collection $\mathcal{P}$ as guaranteed by the previous paragraph. Take F_i to be the subgraph consisting of the paths in $\mathcal{P}$ with ends in S, and take F_i' to be the subgraph consisting of the paths in $\mathcal{P}$ with ends in S'. We claim that F_i and F_i' satisfy the requirements of Claim 7.3. Indeed, first note that F_i is an F-subdivision in $G'[Z_i] - V(Q)$ whose branch vertices are in S, and similarly F_i' is an F-subdivision in $G'[Z_i] - V(Q)$ whose branch vertices are in S'. Second, notice that F_i and F_i' are vertex-disjoint, since the paths in $\mathcal{P}$ are internally vertex-disjoint whose internal vertices are in $Z_i - (V(Q) \cup S \cup S')$. Third, we have $|V(F_i)|, |V(F_i')| \leq e(F) \cdot 15/\delta$, since every path in $\mathcal{P}$ has length at most $15/\delta$ by our choice of $\mathcal{P}$. Finally, if $G[Z_i]$ is β-almost-bipartite, then for every path $P \in \mathcal{P}$ with ends in S, we have

$$|V(P^\circ) \cap X_i| = |V(P^\circ) \cap Y_i| - 1,$$

where P° denotes the interior of the path P. Hence

$$|V(F_i) \cap X_i| - |V(F_i) \cap Y_i| = \sum_{P \text{ a path in } \mathcal{P} \text{ with ends in } S} \left(|V(P^\circ) \cap X_i| - |V(P^\circ) \cap Y_i| \right) + |S|$$
$$= -e(F) + |V(F)|,$$

and analogously,

$$|V(F_i') \cap Y_i| - |V(F_i') \cap X_i| = -e(F) + |V(F)|.$$

Altogether, since F_i and F_i' are vertex-disjoint,

$$|V(F_i \cup F_i') \cap X_i| - |V(F_i \cup F_i') \cap Y_i|$$
$$= |V(F_i) \cap X_i| - |V(F_i) \cap Y_i| + |V(F_i') \cap X_i| - |V(F_i') \cap Y_i| = 0.$$

This proves (22), completing the proof of Claim 7.3. $\qquad \square$

7.3.2 Absorbing the paths $P_1, \ldots, P_r$ and all the uncovered vertices into the subdivisions F_i

For every $i \in [r]$, let F_i, F_i' be the two vertex-disjoint subdivisions of F in $G'[Z_i] - V(Q)$, as guaranteed by Claim 7.3. Denote the leaves of P_i by u_i and v_i, where $u_i \in X_i$ and $v_i \in Y_i$ if $G[Z_i]$ is β-almost-bipartite. Let $x_i y_i$ be an arbitrary edge of F_i, where $x_i \in X_i$ and $y_i \in Y_i$ if $G[Z_i]$ is β-almost-bipartite. Let Q_i be a shortest (y_i, u_i)-path in $G'[Z_i]$ whose interior avoids $V(Q) \cup V(F_i) \cup V(F_i')$. By (P'1),

$$|V(Q)| + |V(F_i)| + |V(F_i')| \leq r\xi n + 2|V(F)| \leq \lceil 1/c \rceil \xi n + 2|V(F)| \leq \sqrt{\xi} n. \tag{23}$$

Hence, it follows that Q_i exists and is of length at most $15/\delta$ by applying (ii) of Lemma 7.1 (with $\sqrt{\xi}$ playing the role of ξ). Let $H \coloneqq G[Z_i]$, and let $H' \coloneqq G'[Z_i]$. Then, by Lemma 7.1, H' has no $\zeta/2$-sparse cuts. Now let

$$W_i \coloneqq Z_i \cap \Big(\big(V(Q) \cup V(F_i) \cup V(F_i') \cup V(Q_i) \big) - \{x_i, v_i\} \Big).$$

By (23) and the fact that $|V(Q_i)| \leq \frac{15}{\delta} + 1$, we have $|W_i| \leq \frac{15}{\delta} + 1 + \sqrt{\xi}n \leq 2\sqrt{\xi}n$.

Note that H' has maximum degree at most d since it is a subgraph of G. By (E1), and the fact that $|V(H')| = |Z_i| \geq \delta n$ (by (E2)), we have

$$2e(H') \geq d \cdot |V(H')| - \eta n^2 - \beta n^2 \geq d \cdot |V(H')| - \frac{2\beta}{\delta^2} \cdot |V(H')|^2,$$

showing that H' has average degree at least $d - (2\beta/\delta^2)|V(H')|$. Moreover, H' is either bipartite or γ-far-from-bipartite by definition. Therefore, using $|W_i| \leq 2\sqrt{\xi}n$, we can apply Lemma 3.3 with H', W_i, $2\beta/\delta^2$, 2ξ, $\zeta/2$ playing the roles of G, W, η, ξ, ζ, respectively, to find a Hamilton path Q_i' in $H' - W_i = G'[Z_i] - W_i$ with ends x_i and v_i. Note that Lemma 3.3 indeed applies because when $G[Z_i]$ is β-almost-bipartite (so H' is bipartite), (P'3) and (22) together imply that $|X_i - W_i| = |Y_i - W_i|$.

Now replace the edge $x_i y_i$ in F_i with the path $x_i Q_i' v_i P_i u_i Q_i y_i$ to obtain a subgraph F_i'' which is a subdivision of F. Note that $V(F_i'') \cup V(F_i') = (Z_i - V(Q)) \cup V(P_i)$. Hence, $\{F_i', F_i'' \mid i \in [r]\}$ is a collection of F-subdivisions that are pairwise vertex-disjoint covering all vertices of G, completing the proof of Theorem 1.7. $\qquad\square$

8 Concluding remarks

Recall that our main result (Theorem 1.4) states that for every bipartite graph H, every dense regular graph G contains an H-packing covering all but $O(1)$ vertices. As noted in the introduction, this was already known for unbalanced bipartite graphs H by Kühn and Osthus [31] (see Theorem 1.3). In fact, they proved a slightly stronger statement in this case: the host graph G need only be *almost regular* rather than regular. More precisely, for every bipartite H and every $c > 0$, there exist $\varepsilon > 0$ and C such that any n-vertex graph G in which every vertex has degree between $(c - \varepsilon)n$ and $(c + \varepsilon)n$ contains an H-packing covering all but at most C vertices. This fails, however, for balanced bipartite H (for example, if G is a slightly unbalanced bipartite graph) so our result cannot, in general, be extended to the almost regular setting.

Keevash proved an analogous result for hypergraphs (see Theorem 5.14 in [15]).[2] Specifically, he showed that for every 3-partite 3-graph H whose parts are not all of equal size, and for every $0 < c_1, c_2 < 1$, there exist $\varepsilon > 0$ and $C > 0$ such that if G is a 3-graph on n vertices in which every vertex has degree between $(1 - \varepsilon)c_1 n^2$ and $(1 + \varepsilon)c_1 n^2$, and every pair of vertices has degree more than $c_2 n$, then G contains an H-packing covering all but at most C vertices. Here, the *degree of a pair* $\{u, v\}$ of vertices in G is the number of vertices $w \in V(G) - \{u, v\}$ such that $\{u, v, w\} \in E(G)$.

It is plausible that, analogous to Theorem 1.4, the conclusion holds for *all* tripartite 3-graphs H when G is assumed to be regular.

Question 8.1. *Is it true that for every tripartite 3-graph H and every $0 < c < 1$, there exist $\varepsilon > 0$ and $C > 0$ such that the following holds? If G is a 3-graph on n vertices in which all vertices have the same degree at least cn^2, and every pair of vertices has degree at least cn, then G contains an H-packing covering all but at most C vertices.*

The problem of partitioning the vertex set of edge-coloured complete graphs into a small number of monochromatic subgraphs has a very rich history; see [12] for a recent survey. An early example of a problem of this kind is Lehel's conjecture. An interesting problem in this direction, suggested by Matija Bucić, is to consider a variant of Theorem 1.4 where the edges of G are colored and we seek packings using monochromatic H-copies (where different copies of H may receive different colors).

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[2]Keevash's theorem is stated and proved for 3-uniform hypergraphs, but the analogous statement for higher uniformities can be obtained by the same methods.

Question 8.2. *Is it true that for every bipartite graph H, integer $r \geq 2$, and $0 < c < 1$, there exists $C > 0$ such that the following holds? If G is an n-vertex d-regular graph with $d \geq cn$, whose edges are colored with r colors, then G contains a collection of vertex-disjoint monochromatic copies of H covering all but at most C vertices.*

The problem of finding a large monochromatic H-packing in a graph with a given minimum degree was studied by Balogh, Freschi, and Treglown [2].

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A   Perfect fractional matching implies perfect 2-matching

Proof of Fact 5.4. Suppose that G is a graph with a perfect fractional matching. Let $w : E(G) \mapsto [0,1]$ be a perfect fractional matching in G that minimises the number of edges whose weight is neither 0 nor 1, and denote by G' the subgraph of G whose edges have non-zero weight in w. We claim that G' is a 2-matching; that is, every connected component of G' is either a single edge or an odd cycle. Notice that G' is spanning, because w is a perfect fractional matching. Thus, if G' is a 2-matching, this would show that G' is a perfect 2-matching in G as required.

Therefore, it remains to show that G' is a 2-matching. Suppose for a contradiction that it is not, and let F be a component of G' which is neither an edge nor an odd cycle.

First, notice that F has minimum degree at least 2. Indeed, otherwise, let v be a vertex with degree 1 in F, let u be its unique neighbour, and let x be a neighbour of u which is not v. Then $w(vu), w(ux) > 0$, implying that $w(vu) < 1$, contradicting the assumption that w is a perfect fractional matching.

Next, we claim that F does not have even cycles. Indeed, suppose that $C = (v_1 \ldots v_{2s})$ is an even cycle in F. For $x > 0$, let $w_x : E(G) \mapsto [0,1]$ be the weighting of the edges of G defined as follows (with the addition of indices taken modulo $2s$).

$$
w_x = \begin{cases} w(e) & e \notin E(C) \\ w(e) + x & e = v_{2i-1}v_{2i} \text{ for } i \in [s] \\ w(e) - x & e = v_{2i}v_{2i+1} \text{ for } i \in [s]. \end{cases}
$$

Note that the edges $v_i v_{i+1}$, with $i \in [2s]$, have weight strictly between 0 and 1, and so there exists $x > 0$ such that all weights in w_x are in $[0,1]$. Let x' be the maximum x with this property. Then at least one of the edges in C has weight either 0 or 1 in $w_{x'}$, showing that $w_{x'}$ is a perfect fractional matching in G with fewer edges whose weight is neither 0 nor 1 compared to w, a contradiction to the choice of w. This shows that F does not have even cycles, as desired. Moreover, this shows that F cannot be a cycle since we assumed that F is not an odd cycle.

Since F has minimum degree at least 2 and is not a cycle, it contains two distinct cycles C_1 and C_2. Because F has no even cycles, these cycles are odd and share at most one vertex (otherwise their union contains an even cycle). Let P be a shortest path with one end in C_1 and the other end in C_2. (Note that P is a singleton if the cycles share a vertex.) Write $C_1 = (v_1 \ldots v_{2s+1})$, $C_2 = (u_1 \ldots u_{2t+1})$, and suppose that P is a path from v_1 to u_1. Now, for $x > 0$, let $w_x : E(G) \mapsto [0,1]$ be the weighting of the edges of G obtained from w by making the following modifications.

- Decrease the weight of edges $v_{2i-1}v_{2i}$, with $i \in [s+1]$, by x (where addition in the indices is taken modulo $2s+1$).

- Increase the weight of edges $v_{2i}v_{2i+1}$, with $i \in [s]$, by x.

- Increase the weight of the first, third, fifth, etc. edges of P by $2x$ (starting from v_1).

- Decrease the weight of the second, fourth, sixth, etc. edges of P by $2x$ (starting from v_1).

- If P has odd length, decrease the weight of edges $u_{2i-1}u_{2i}$ with $i \in [t+1]$ by x (where addition of indices is taken modulo $2t+1$). Otherwise, increase the weight of these edges by x.

- If P has odd length, increase the weight of edges $u_{2i}u_{2i+1}$, with $i \in [t]$, by x. Otherwise, decrease the weight of these edges by x.

As before, there exists $x > 0$ such that all weights in w_x are in $[0,1]$, and let x' be the maximum x with this property. Then $w_{x'}$ is a perfect fractional matching of G with fewer edges whose weight is neither 0 nor 1 compared to w, a contradiction.

In conclusion, the assumption that F is a component of G' that is neither an odd cycle nor an edge leads to a contradiction, so G' is a 2-matching (and thus G' is a perfect 2-matching in G), as required. $\square$

B Hamiltonicity of clusters

In this section, we present the proof of Lemma 3.3. Our proof closely follows that of Lemma 4 in [11], but we include it here for completeness and to address differences in the lemma's statement.

The proof is based on known results concerning the Hamiltonicity of *robust out-expanders*, a concept introduced by Kühn, Osthus and Treglown [34]. Before stating the relevant result, we first introduce some necessary definitions.

Given a digraph G on n vertices, a set of vertices S and a parameter $\nu \in (0,1)$, the *robust ν-out-neighbourhood* of S in G, denoted $\mathrm{RN}_{\nu,G}^+(S)$, is the set of vertices in G that have at least νn in-neighbours in S; we omit the subscript G when it is clear from the context. Given $0 < \nu \le \tau < 1$, we say that G is a *robust (ν, τ)-out-expander* if $|\mathrm{RN}_\nu^+(S)| \ge |S| + \nu n$ for every set of vertices S with $\tau n \le |S| \le (1-\tau)n$. We shall also use the following undirected version of a robust out-neighbourhood. In a graph G on n vertices, the *robust ν-neighbourhood* of a set of vertices S, denoted $\mathrm{RN}_{\nu,G}(S)$, is the set of vertices in G with at least νn neighbours in S. As before, we may omit the subscript G when it is clear from the context. We say that G is a *robust (ν, τ)-expander* if $|\mathrm{RN}_{\nu,G}(S)| \ge |S| + \nu n$ for every set of vertices S with $\tau n \le |S| \le (1-\tau)n$.

We will need the following theorem from [34]. The minimum semi-degree of a digraph G, denoted $\delta^0(G)$, is defined as $\delta^0(G) = \min\{\delta^+(G), \delta^-(G)\}$, where $\delta^+(G), \delta^-(G)$ are the minimum out-degree and in-degree of G, respectively.

Theorem B.1 (Kühn, Osthus and Treglown [34]). *Let $n \in \mathbb{N}$ and let ν, τ, γ be reals such that $1/n \ll \nu \le \tau \ll \gamma < 1$. Let G be a digraph on n vertices with $\delta^0(G) \ge \gamma n$ which is a robust (ν, τ)-out-expander. Then G contains a Hamilton cycle.*

We shall use the following corollary of Theorem B.1.

Corollary B.2. *Let ν, τ, γ be reals and let n be an integer such that $1/n \ll \nu \le \tau \ll \gamma < 1$. Let G be a digraph on n vertices with $\delta^0(G) \ge \gamma n$ which is a robust (ν, τ)-out-expander. Then for every choice of distinct vertices x and y, there is a Hamilton path in G with ends x and y.*

Proof. Given vertices x and y, form G' by adding the edge xy to G, removing the edge yx (if it exists), and removing all the edges directed toward y or away from x. Next, form G'' by contracting the edge xy. It is easy to check that G'' is a robust $(\nu/2, 2\tau)$-out-expander. Thus, by Theorem B.1, it contains a Hamilton cycle. This cycle corresponds to a Hamilton cycle in G' which contains the edge xy, which in turn corresponds to a Hamilton path in G with ends x and y. $\qquad\square$

Proof of Lemma 3.3. Let G be a graph with n vertices, maximum degree at most d, and average degree at least $d - \eta n$. Suppose G has no ζ-sparse cuts and is either γ-far-from-bipartite or bipartite with a bipartition $\{X, Y\}$. Let $W \subseteq V(G)$ be a set of size at most ξn, which satisfies $|X - W| = |Y - W|$ if G is bipartite. Define $H = G - W$, $V = V(H)$, and, if G is bipartite, then also define $X' = X - W$ and $Y' = Y - W$. Then H has maximum degree at most d, average degree at least $d - 2\xi n$, and no $\zeta/2$-sparse cuts.

The following claim will allow us to use Corollary B.2 above.

Claim B.3. *Let $S \subseteq V$ be a set satisfying $\xi^{1/7}n \le |S| \le (1 - \xi^{1/7})n$ if G is γ-far-from-bipartite, or $\xi^{1/7}n \le |S| \le (1/2 - \xi^{1/7})n$ if G is bipartite. Then $|\mathrm{RN}_{\xi,H}(S)| \ge |S| + \xi n$.*

Proof. Define $S_1 = S - \mathrm{RN}_\xi(S)$, $S_2 = S \cap \mathrm{RN}_\xi(S)$, $T_1 = \mathrm{RN}_\xi(S) - S$, $T_2 = V - (S \cup T_1)$. Assume, towards a contradiction, that $|\mathrm{RN}_\xi(S)| < |S| + \xi n$, which implies that $|T_1| < |S_1| + \xi n$.

Given sets $A, B \subseteq V$, denote by $e(A, B)$ the number of ordered pairs ab such that ab is an edge of H and $a \in A, b \in B$. Then

$$e(S_1, V - T_1) \le e(S_1, S) + e(S_1, T_2) \le (|S_1| + |T_2|) \cdot \xi n \le \xi n^2, \tag{24}$$

using that the vertices in $S_1 \cup T_2$ are not in $\mathrm{RN}_\xi(S)$. By the maximum and average degree assumption on H, we have

$$e(S_1, T_1) = \sum_{v \in S_1} d_H(v) - e(S_1, V - T_1) \ge d|S_1| - 2\xi n^2 - \xi n^2 = d|S_1| - 3\xi n^2.$$

As $|T_1| < |S_1| + \xi n$, we obtain the following bound on $e(T_1, V - S_1)$.

$$e(T_1, V - S_1) \le d|T_1| - e(S_1, T_1) \le d(|T_1| - |S_1|) + 3\xi n^2 \le 4\xi n^2. \tag{25}$$

Consider the quantity $e(S_1 \cup T_1, V - (S_1 \cup T_1))$. By (24) and (25), this quantity is at most $5\xi n^2$, and since H has no $\zeta/2$-sparse cuts, it is at least $(\zeta/2) \cdot |S_1 \cup T_1| \cdot |V - (S_1 \cup T_1)|$. As $\xi \ll \zeta$, we find that either $|S_1 \cup T_1| \le \xi^{1/3}n$ or $|V - (S_1 \cup T_1)| \le \xi^{1/3}n$.

First, suppose that $|S_1 \cup T_1| \le \xi^{1/3}n$. Then

$$e(S_2, V - S_2) \le e(S_1 \cup T_1, V) + e(S_2, T_2) \le \xi^{1/3}n^2 + |T_2|\xi n \le 2\xi^{1/3}n^2,$$

using that $|S_1 \cup T_1| \le \xi^{1/3}n$ and $T_2 \cap \mathrm{RN}_\xi(S_2) = \emptyset$. However, since H does not have $\zeta/2$-sparse cuts and given the assumption of the size of S, we have

$$e(S_2, V - S_2) \ge (\zeta/2) \cdot |S_2| \cdot |V - S_2| \ge \frac{\zeta}{2} \cdot (\xi^{1/7}n - |S_1|) \cdot \frac{\xi^{1/7}n}{2} \ge \frac{\zeta}{2} \cdot \frac{\xi^{2/7}n^2}{4} > 2\xi^{1/3}n^2,$$

a contradiction.

Next, suppose that $|V - (S_1 \cup T_1)| \le \xi^{1/3}n$. If G is bipartite, then using that $|T_1| < |S_1| + \xi n$ and $|S| \le (1/2 - \xi^{1/7})n$, we have

$$|V| = |V - (S_1 \cup T_1)| + |S_1 \cup T_1| < \xi^{1/3}n + |S_1| + (|S_1| + \xi n)$$
$$\le \xi^{1/3}n + n - 2\xi^{1/7}n + \xi n < n - \xi n \le n - |W|,$$

a contradiction. Hence, we may assume that G is γ-far-from-bipartite. Note that G can be made bipartite by removing edges incident to $W \cup S_2 \cup T_2$ or within S_1 or T_1. However, there are at most $(\xi + \xi^{1/3})n^2$ edges of the former type (since $|W| \le \xi n$ and $|V - (S_1 \cup T_1)| = |S_2 \cup T_2| \le \xi^{1/3}n$) and at most $5\xi n^2$ edges of the latter type (by (24) and (25)), so there are fewer than γn^2 edges in total (using $\xi \ll \gamma$). This contradicts the assumption that G is γ-far-from-bipartite, thus completing the proof of the claim. $\qquad\square$

It follows from Claim B.3 that H is a robust $(\xi, 2\xi^{1/7})$-expander. Let $x, y \in V$, where $x \in X$ and $y \in Y$ if G is bipartite. Our goal is to show that H contains a Hamilton path with ends x and y. First, consider the case where G is γ-far-from-bipartite. Form a digraph D by replacing each edge uv of H with the two directed edges uv and vu. It follows from Claim B.3 that D is a robust $(\xi, 2\xi^{1/7})$-out-expander. Then, Corollary B.2 implies the existence of a Hamilton path in D with ends x and y, which corresponds to a Hamilton path in H with the same ends.

Now, suppose that G is bipartite. Note that in this case H is a balanced bipartite graph (by our choice of the set W). We first show that H has a perfect matching by verifying Hall's condition i.e., we will show that for every $S \subseteq X'$, we have $|N_H(S)| \ge |S|$. Notice that H has minimum degree at least $\zeta(n-1) - \xi n \ge \zeta n/2$, by the assumption that G has no ζ-sparse cuts and by $|W| \le \xi n$. Thus, if $|S| < \zeta n/2$ (and $S \ne \emptyset$) then $|N_H(S)| \ge \zeta n/2 \ge |S|$. Similarly, if $|S| > |V|/2 - \zeta n/2$ then $|N_H(S)| = |V|/2 \ge |S|$. Hence, we may assume that $\zeta n/2 \le |S| \le |V|/2 - \zeta n/2$, which implies that $\xi^{1/7}n \le |S| \le (1/2 - \xi^{1/7})n$, but then Claim B.3 implies that $|N_H(S)| \ge |S|$, showing that H has a perfect matching.

Let $\{a_1b_1, \ldots, a_tb_t\}$ be a perfect matching in H, where $t = |X'|$ and $a_i \in X', b_i \in Y'$ for $i \in [t]$. We assume for convenience that a_ib_i is not the edge xy (if the edge xy exists) for $i \in [t]$ – this is possible because removal of the edge xy from H does not affect the arguments above. Without loss of generality, $a_1 = x$ and $b_t = y$. Form an auxiliary directed graph D with vertex set $\{v_1, \ldots, v_t\}$ where v_iv_j is a directed edge whenever b_ia_j is an edge of H. It follows from Claim B.3 that D is a robust $(\xi, 3\xi^{1/7})$-out-expander. Then, by Corollary B.2, there is a Hamilton path in D starting at v_1 and ending at v_t. Without loss of generality, suppose that this path is $v_1, \ldots, v_t$. This path corresponds to the Hamilton path $x = a_1, b_1, \ldots, a_t, b_t = y$ in H, as desired. $\qquad\square$