On a problem of Brown, Erdős and Sós

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Abstract

Let $f^{(r)}(n; s, k)$ be the maximum number of edges in an n-vertex r-uniform hypergraph not containing a subhypergraph with k edges on at most s vertices. Recently, Delcourt and Postle, building on work of Glock, Joos, Kim, Kühn, Lichev and Pikhurko, proved that the limit $\lim_{n\to\infty} n^{-2} f^{(r)}(n; k+2, k)$ exists for all $k \geq 2$, solving an old problem of Brown, Erdős and Sós (1973). Meanwhile, Shangguan and Tamo asked the more general question of determining if the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ exists for all $r > t \geq 2$ and $k \geq 2$.

Here we make progress on their question. For every even k, we determine the value of the limit when r is sufficiently large with respect to k and t. Moreover, we show that the limit exists for $k = 5$ and $k = 7$.

1 Introduction

Amedeo: A better spacing for the notation
k^--configuration could be
k^-–configuration or
k^--configuration?

Shoham: I like the first best

An (s, k)-configuration in an r-uniform hypergraph (henceforth *r-graph*) is a collection of k edges spanning at most s vertices. Brown, Erdős and Sós [1] started the investigation of the function $f^{(r)}(n; s, k)$, defined as the maximum number of edges in an n-vertex r-graph not containing an (s, k) configuration. In particular, they showed that $f^{(r)}(n; s, k) = \Omega(n^{(rk-s)/(k-1)})$ for all $s > r \geq 2$ and $k \geq 2$. Suppose now that the exponent $t := (rk - s)/(k - 1)$ is an integer, then $s = k(r-t)+t$. Observe that this is the number of vertices spanned by a k-edge r-graph where the edges can be ordered so that all but the first edge share exactly t vertices with the previous edges. In particular, any set of vertices of size t which is contained in k distinct edges creates a $(k(r-t)+t, k)$-configuration. Therefore

$$f^{(r)}(n; k(r-t) + t, k) = O(n^t)\,.$$

A major open problem is the following conjecture, which was proposed by Shangguan and Tamo [7] and generalises an old conjecture of Brown, Erdős and Sós [1] (corresponding to $r = 3$ and $t = 2$).

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Conjecture 1.1. *For any r, k and t, the limit*

$$\pi(r,t,k) := \lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$$

exists.

We can assume $k \geq 2$ and $t \in [r-1]$, as otherwise Conjecture 1.1 is trivial. Moreover, when $t = 1$, it is easy to establish that the limit exists and that $\pi(r,1,k) = \frac{k-1}{(k-1)(r-1)+1}$, as already observed in [4]. Indeed, the extremal constructions are vertex-disjoint unions of loose trees with $k-1$ edges, while the upper bound follows from the fact any collection of k edges, which can be ordered in such a way that all but the first edge share at least one vertex with the previous ones, is a $(k(r-1)+1, k)$-configuration. Therefore we can in fact assume $k \geq 2$ and $t \in [2, r-1]$, and we will do so in the rest of the paper. Recently, significant progress has been made towards Conjecture 1.1 and we now summarise the main developments.

We start by discussing the results concerning the original conjecture of Brown, Erdős and Sós, that is Conjecture 1.1 for $r = 3$ and $t = 2$, i.e. the existence of $\pi(3,2,k)$. They studied the case $k = 2$ and showed [1] that the limit is $\pi(3,2,2) = 1/6$. More than 40 years later, Glock [3] proved the conjecture for $k = 3$ and determined that $\pi(3,2,3) = 1/5$. Very recently, Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] proved the conjecture for $k = 4$ and determined that $\pi(3,2,4) = 7/36$. In the concluding remarks, they also claim that their methods can be adapted to show that $\pi(3,2,k) = 1/5$ for $k \in \{5,7\}$. Finally, Delcourt and Postle [2] proved the Brown–Erdős–Sós conjecture in full, i.e. they showed that $\pi(3,2,k)$ exists for all $k \geq 2$, although their method does not provide an expliict value for the limit.

Shanguann and Tamothis [7] adapted the methods in [3] to any uniformity and showed that $\pi(r,2,3) = 1/(r^2 - r - 1)$, and Shanguann [6] adapted [2] to any uniformity and showed that $\pi(r,2,k)$ exists (with the explicit value not known).

Concerning the general conjecture, the case $k = 2$ follows from the work of Rödl [5] on the existence of asymptotic Steiner systems and we have $\pi(r,t,2) = \frac{1}{t!}\binom{r}{t}^{-1}$. Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] settled the cases $k = 3$ and $k = 4$ by showing that $\pi(r,t,3) = \frac{2}{t!}\left(2\binom{r}{t} - 1\right)^{-1}$ for every $r \geq 2$ and $\pi(r,t,4) = \frac{1}{t!}\binom{r}{t}^{-1}$ for every $r \geq 4$ (note that the case $r = 3$ (and $t = 2$) is covered by one of the results mentioned above and does not follow the same pattern).

Our first result provides the exact value of the limit when k is even and r is sufficiently large in terms of k and t.

Amedeo: Added explicit bound on r.

Theorem 1.2. *Let k be an even positive integer and $t \geq 2$ an integer. Then, for $r \geq t + 4(k^3 \cdot t!)^{1/t}$, we have that $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)) = \frac{1}{t!}\binom{r}{t}^{-1}$.*

We remark that, as mentioned above, this was already known for $k = 2$ [5] and $k = 4$ [4] (and any r). Moreover, it is interesting to observe that the behaviour for odd k is potentially different. For example, from [4], it holds that $\pi(r,t,3) = \frac{2}{t!}\left(2\binom{r}{t} - 1\right)^{-1}$. Therefore, we now focus on the case of k being odd.

Firstly, we completely settle Conjecture 1.1 for $k = 5$.

Theorem 1.3. *For every $t \in [2, r-1]$, the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 5(r-t)+t, 5)$ exists.*

Next, we settle Conjecture 1.1 for $k = 7$. (Our proof does not work when $r = 3$ and $t = 2$, but this case has been resolved for all k in [2].)

Shoham: our proof now works for all cases except $(r, t) = (3, 2)$

Theorem 1.4. *Let r, t be integers satsifying $r > t \geq 2$ and $(r, t) \neq (3, 2)$. Then the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t)+t, 7)$ exists.*

More about the case of odd k is discussed in Section 6.

Shoham: state result

Organisation. The rest of the paper is organised as follows. Section 2 introduces the relevant notation and collects some preliminary results, including our key proposition (Proposition 2.5) needed for the proofs of Theorems 1.3 and 1.4, which are proved in Section 4 and Section 5, respectively. Section 3 provides the proof of Theorem 1.2 and finally Section 6 contains some concluding remarks.

2 Preliminaries and notation

Since the values of r and t will be always clear from the context, we introduce the following terminology. A *k-configuration* denotes a $(k(r-t)+t, k)$-configuration, while a *k^--configuration* denotes a $(k(r-t)+t-1, k)$-configuration. Moreover, we say that a hypergraph is *k-free* (resp. *k^--free*) if it does not contain any k-configuration (resp. k^--configuration).

2.1 Lower bounds

In order to build k-free r-graphs with many edges, the strategy of Glock, Joos, Kim, Kühn, Lichev and Pikhurko [4] consisted of packing many copies of a carefully chosen k-free r-graph of constant size, while making sure not to create any k-configurations using edges from different copies. Before stating their main technical result, we introduce some definitions.

Recall that the *t-shadow* of a hypergraph $\mathcal{F}$, denoted $\partial_t\mathcal{F}$, is the t-graph on $V(\mathcal{F})$ whose edges are all t-subsets of edges in $\mathcal{F}$.

Definition 2.1. Given an r-graph $\mathcal{F}$ and a t-graph J, we say that J is a *supporting t-graph* of $\mathcal{F}$ if $V(J) = V(\mathcal{F})$ and J contains the t-shadow of $\mathcal{F}$. For $\mathcal{F}$ and J as above, we define the *non-edge girth* of $(\mathcal{F}, J)$ to be the smallest $g \geq 1$ for which there exists a g-configuration in $\mathcal{F}$ whose vertex set contains a non-edge of J. Equivalently, it is the largest $g \geq 1$ such that for every ℓ-configuration S in $\mathcal{F}$ with $\ell < g$, all t-subsets of $V(S)$ are edges of J. If no such g exists, we set the non-edge girth of $(\mathcal{F}, J)$ to be infinity.

Here is the main technical result in [4].

Theorem 2.2 (Theorem 3.1 in [4])**.** *Fix $k \geq 2$, $r \geq 3$ and $t \in [2, r-1]$. Let $\mathcal{F}$ be an r-graph which is k-free and ℓ^--free for all $\ell \in [2, k-1]$. Let J be a supporting t-graph of $\mathcal{F}$ such that the non-edge girth of $(\mathcal{F}, J)$ is greater than $k/2$. Then,*

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{|\mathcal{F}|}{t!\,|J|}\,.$$

Amedeo: We do not need g, just f. Should we state it with f and remove the comment or leave it as it is?

Shoham: I removed the definition of g, I think it might not be needed

In particular, by choosing $\mathcal{F}$ to be a single r-uniform edge and $J = \binom{V(\mathcal{F})}{t}$, the hypotheses of Theorem 2.2 hold and we get the following corollary.

Corollary 2.3 (Corollary 3.2 in [4])**.** *Fix $k \geq 2$, $r \geq 3$ and $t \in [2, r-1]$. Then we have*

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t!\binom{r}{t}}.$$

2.2 A useful lemma

In order to apply the density argument of Proposition 2.5, for any given k-free n-vertex r-graph $\mathcal{F}$, we need to find a subhypergraph which is ℓ^--free for each $\ell \in [2, k-1]$ and satisfies some additional properties. It turns out that, for some values of ℓ, there is a simple argument which shows that $\mathcal{F}$ can be made ℓ^--free by removing only $O(n^{t-1})$ edges. This is established, together with additional properties, by the following lemma.

Amedeo: Maybe the last sentence is not needed.

Shoham: commented out

Lemma 2.4. *Let r, k and t be fixed positive integers. Let $\mathcal{F}$ be a k-free n-vertex r-graph. Then there exists a subhypergraph $\mathcal{F}'$ of $\mathcal{F}$ such that the following holds.*

(P1) *$\mathcal{F}'$ is ℓ^--free for every $\ell \in [2, k]$ with $\ell|(k-1)$ or $\ell|k$;*

(P2) *for every positive integers a and b with $a + b = k$, every a^--configuration and every b-configuration of $\mathcal{F}'$ are edge-disjoint;*

(P3) *$|\mathcal{F}'| \geq |\mathcal{F}| - O(n^{t-1})$.*

Proof. We show that we can get a subhypergraph of $\mathcal{F}$ which satisfies (P1) and (P2) by removing $O(n^{t-1})$ edges, which in turn will imply (P3) as well.

Observe that, since $\mathcal{F}$ is k-free, $\mathcal{F}$ is also k^--free. Next, we show that $\mathcal{F}$ can made $(k-1)^-$-free by removing $O(n^{t-1})$ edges. Let $\mathcal{S}$ be a maximal collection of pairwise edge-disjoint $(k-1)^-$-configurations of $\mathcal{F}$. If $|\mathcal{S}| > \binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ which is contained in the $(t-1)$-shadow of two $(k-1)^-$-configurations S_1 and S_2 in $\mathcal{S}$. Let $e \in S_2$ be an edge such that $T \subseteq e$. Then $S_1 \cup \{e\}$ is a k-configuration in $\mathcal{F}$, being a collection of k edges spanning at most $(k-1)(r-t)+t-1+(r-|T|) = k(r-t)+t$ vertices, a contradiction. Therefore $|\mathcal{S}| \leq \binom{n}{t-1}$ and, by removing from $\mathcal{F}$ all the edges of each $S \in \mathcal{S}$, we obtain a subhypergraph $\mathcal{F}_0$ of $\mathcal{F}$ which is $(k-1)^-$-free and satisfies $|\mathcal{F}_0| \geq |\mathcal{F}| - O(n^{t-1})$.

Let $2 \leq \ell < k-1$ with $\ell|(k-1)$ (resp. $\ell|k$). Let $j > 1$ be the positive integer such that $\ell \cdot j = k-1$ (resp. $\ell \cdot j = k$). Let $\mathcal{S}_\ell$ be a maximal collection of pairwise edge-disjoint ℓ^--configurations in $\mathcal{F}_0$. If $|\mathcal{S}_\ell| > (j-1)\cdot\binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ which is contained in the vertex set of j distinct ℓ^--configurations $S_1, \ldots, S_j$ in $\mathcal{S}_\ell$. Then $S_1 \cup \cdots \cup S_j$ is a $(k-1)^-$-configuration of $\mathcal{F}_0$, being a collection of $\ell \cdot j = k-1$ edges spanning at most $j[\ell(r-t)+t-1]-(j-1)|T| = j\ell(r-t)+t-1 = (k-1)(r-t)+t-1$ vertices (resp. a k^--configuration). This is a contradiction to $\mathcal{F}_0$ being $(k-1)^-$-free (resp. k^--free). Therefore $|\mathcal{S}_\ell| = O(n^{t-1})$ for all relevant ℓ.

Let a and b be positive integers with $a+b=k$. Let H_a be the collection of edges involved in a^--configurations of $\mathcal{F}_0$ and let $\mathcal{S}_{a,b}$ be a maximal collection of pairwise edge-disjoint b-configurations of $\mathcal{F}_0$ containing an edge of H_a. If $|\mathcal{S}_{a,b}| \geq a \cdot \binom{n}{t-1}$, then there exists a set $T \subseteq V(\mathcal{F})$ of size $t-1$ and $a+1$ distinct b-configurations $S_1, \dots, S_{a+1}$ in $\mathcal{S}_{a,b}$ such that there exists $e_i \in S_i \cap H_a$ with $T \subseteq e_i$ for every $i \in [a+1]$. By definition of H_a, there exists $f_2, \dots, f_a \in \mathcal{F}_0$ such that $S' := \{e_1, f_2, \dots, f_a\}$ is an a^--configuration and, without loss of generality, we assume that S_{a+1} and S' are edge-disjoint. Then $S_{a+1} \cup S'$ is a k-configuration of $\mathcal{F}_2$, being a collection of $b+a=k$ edges spanning at most $[b(r-t)+t]+[a(r-t)+t-1]-|T| = k(r-t)+t$, a contradiction to $\mathcal{F}_0$ being k-free. Therefore $|\mathcal{S}_{a,b}| = O(n^{t-1})$.

Let $\mathcal{F}'$ be the subhypergraph of $\mathcal{F}_0$ which is obtained by removing all the edges of each $S \in \mathcal{S}_\ell$ for every $2 \leq \ell < k-1$ with $\ell | (k-1)$ or $\ell | k$, and all the edges of each $S \in \mathcal{S}_{a,b}$ for every positive integers a and b with $a+b=k$. Then $\mathcal{F}'$ satisfies (P1), (P2) and (P3). $\square$

2.3 Density argument

The approach of Delcourt and Postle [2], while proving Conjecture 1.1 for $r=3$, $t=2$ and any $k \geq 2$, relies on the following reduction: they show that in any sufficiently dense k-free 3-graph, it is possible to find a subgraph with almost the same density which is additionally ℓ^--free for each $\ell \in [2, k-1]$, i.e. having ℓ^--configurations is 'inefficient' for the extremal k-free graph. Here we provide another density-type argument. Before stating the result, we introduce some notation. Given an r-graph $\mathcal{F}$, define $J(\mathcal{F})$ to be the t-graph with $V(\mathcal{F})$ as vertex set and where a t-subset $T \subseteq V(\mathcal{F})$ is an edge of $J(\mathcal{F})$ if and only if there exists an ℓ-configuration for some $\ell \in [\lfloor k/2 \rfloor]$ whose vertex set contains T. Observe that, since every edge is a 1-configuration, $J(\mathcal{F})$ contains the t-shadow of $\mathcal{F}$. Therefore $J(\mathcal{F})$ is a supporting t-graph of $\mathcal{F}$ (recall Definition 2.1). Moreover, its non-edge girth is greater than $\lfloor k/2 \rfloor$.

Amedeo: New statement to account that in applications we first remove $O(n^{t-1})$ edges.

Amedeo: Should we make clear that this is different from Problem 3.3 in [4]?

Shoham: Sure

Proposition 2.5. *Suppose that for every $\varepsilon > 0$ and large enough n, for every k-free n-vertex r-graph $\mathcal{F}$ with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$ there exist subhypergraphs $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$ such that $|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1})$, $\mathcal{F}_2$ is ℓ^--free for every $\ell \in [2, k-1]$, and*

$$\frac{|\mathcal{F}_2|}{|J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|}. \tag{1}$$

Then the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ exists.

Proof. Define α to satisfy $\frac{\alpha}{t!} = \limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)$ and observe that, since Corollary 2.3 gives that $\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t!}\binom{r}{t}^{-1}$, it holds that $\alpha \geq \binom{r}{t}^{-1}$. If we have equality, we are done. Therefore, we can assume the inequality is strict and thus for small enough $\varepsilon > 0$ we have $\alpha > \binom{r}{t}^{-1} + \varepsilon$. Given the definition of α, for every $n \in \mathbb{N}$, there exist $m \geq n$ and an m-vertex r-graph $\mathcal{F}$ with $|\mathcal{F}| \geq (\alpha - \varepsilon)\binom{m}{t}$. Owing to the assumptions of the proposition, there exist subhypergraphs $\mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}$ such that $|\mathcal{F}_1| \geq |\mathcal{F}| - O(m^{t-1})$, $\mathcal{F}_2$ is ℓ^--free for every $\ell \in [2, k-1]$, and $\frac{|\mathcal{F}_2|}{|J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|}$. As observed above, $J(\mathcal{F}_2)$ is a supporting t-graph of $\mathcal{F}_2$ and its non-edge girth is greater than $k/2$. Therefore, by Theorem 2.2, we have

$$\begin{aligned}\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) &\geq \frac{|\mathcal{F}_2|}{t! \cdot |J(\mathcal{F}_2)|} \geq \frac{|\mathcal{F}_1|}{t! \cdot |J(\mathcal{F}_1)|} \\ &\geq \frac{(\alpha-\varepsilon)\binom{m}{t} - O(m^{t-1})}{t! \cdot \binom{m}{t}} = \frac{\alpha - \varepsilon}{t!} - O(m^{-1}),\end{aligned}$$

using that $|J(\mathcal{F}_1)| \leq \binom{m}{t}$ for the last inequality. Since ε can be made arbitrarily small and m arbitrarily large, the conclusion easily follows from

$$\liminf_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{\alpha}{t!} = \limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k).$$ □

Amedeo: Added a short paragraph concerning the cases $k = 2, 3$.

Observe that Proposition 2.5 offers short proofs that Conjecture 1.1 holds for $k = 2$ and $k = 3$. Indeed, the case $k = 2$ is immediate as any 2-free graph is also 2^--free. For $k = 3$, given a 3-free r-graph $\mathcal{F}$, Lemma 2.4 gives a subhypergraph $\mathcal{F}' \subseteq \mathcal{F}$ which is 2^--free and satisfies $|\mathcal{F}'| \geq |\mathcal{F}| - O(n^{t-1})$. We can then take $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}'$.

Remark 2.6. *We remark that if $\mathcal{F}_1$ is an r-graph with $|\mathcal{F}_1| \geq \binom{r}{t}^{-1}|J_1|$ and $\mathcal{F}_2 \subseteq \mathcal{F}_1$, then the condition*

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_2)| \geq \binom{r}{t}(|\mathcal{F}_1| - |\mathcal{F}_2|) \tag{2}$$

implies Condition (1). Indeed, writing $x_1 = |J(\mathcal{F}_1)|$, $y_1 = |\mathcal{F}_1|$, $x_2 = |J(\mathcal{F}_2)|$, $y_2 = |\mathcal{F}_2|$ and $\alpha = \binom{r}{t}$, we have $x_1 \leq \alpha y_1$, $x_2 \leq x_1$ and $y_2 \leq y_1$ by assumption. Moreover, by (2), $x_2 \leq x_1 - \alpha(y_1 - y_2) \leq \alpha y_2$. Therefore, using $x_2 \leq \alpha y_2$ and (2), which is equivalent to $\alpha y_1 - x_1 \leq \alpha y_2 - x_2$,

$$\begin{aligned}\alpha(x_2 y_1 - x_1 y_2) &= x_2(\alpha y_1 - x_1) + x_1 x_2 - \alpha x_1 y_2 \\ &\leq x_2(\alpha y_2 - x_2) + x_1(x_2 - \alpha y_2) = (x_1 - x_2)(x_2 - \alpha y_2) \leq 0.\end{aligned}$$

This implies $x_1 y_2 \geq x_2 y_1$, which in turn is equivalent to (1).

3 Proof of Theorem 1.2 (Conjecture 1.1 for k even)

We first state and prove some claims.

Claim 3.1. *Let $\mathcal{F}$ be a k-free r-graph and $e \in \mathcal{F}$. Then the number of 2-configurations of $\mathcal{F}$ containing e is at most $k-2$.*

Proof. If there were $k-1$ distinct 2-configurations $\{e, e_i\}$ for $i \in [k-1]$, then $S = \{e, e_1, \ldots, e_{k-1}\}$ would be a k-configuration of $\mathcal{F}$, being a collection of k edges spanning at most $r + (k-1)(r-t) = k(r-t)+t$ vertices, a contradiction. □

Claim 3.2. *Let k be an even integer, $\mathcal{F}$ be a k-free r-graph and $T \subseteq V(\mathcal{F})$ with $|T| = t$. Then the number of 2-configurations whose vertex set contains T is at most $k(k-2)$.*

Proof. Let $\mathcal{S}$ be a maximal collection of pairwise edge-disjoint 2-configurations of $\mathcal{F}$ whose vertex set contains T. We have $|\mathcal{S}| < k/2$, as otherwise there would exist distinct 2-configurations $S_1, \dots, S_{k/2}$ in $\mathcal{S}$ and $S_1 \cup \dots \cup S_{k/2}$ would give a k-configuration of $\mathcal{F}$, being a collection of k edges spanning at most $(2r-t)k/2 - (k/2-1)|T| = k(r-t)+t$ vertices. By maximality of $\mathcal{S}$, any 2-configuration of $\mathcal{F}$ whose vertex set contains T must contain an edge which belongs to some $S \in \mathcal{S}$. There are $2|\mathcal{S}| < k$ such edges and, for each of them, by Claim 3.1, the number of 2-configurations of $\mathcal{F}$ containing it is at most $k-2$. Therefore the number of 2-configurations of $\mathcal{F}$ whose vertex set contains T is at most $k(k-2)$. $\square$

Given a hypergraph $\mathcal{F}$, we say that a t-set T of $V(\mathcal{F})$ is *covered* exactly i times, if T is contained in exactly i edges of $\mathcal{F}$. We denote by $J_i(\mathcal{F})$ the set of t-sets of $V(\mathcal{F})$ covered exactly i times, and by $J_{\geq i}(\mathcal{F})$ the set of t-sets of $V(\mathcal{F})$ covered at least i times.

Claim 3.3. *Let k be an even integer and $\mathcal{F}$ be a k-free and 2^--free r-graph. Then we have*

$$k(k-2)|J_0| \geq \left\{ \binom{2r-t}{t} - \left[2\binom{r}{t} - 1 \right] - \left[(k-3)\binom{2t}{t} \right] \right\} \cdot |J_{\geq 2}| .$$

Proof. We use a double counting argument on the set

$$\mathcal{Q} := \left\{ (S,T) : \begin{array}{c} S \text{ is a 2-configuration of } \mathcal{F}, \\ T \in J_0 \text{ and } T \subseteq V(S) \end{array} \right\} .$$

Fix $T \subseteq V(\mathcal{F})$ with $|T| = t$. By Claim 3.2, the number of 2-configurations of $\mathcal{F}$ whose vertex set contains T is at most $k(k-2)$. We conclude that

$$|\mathcal{Q}| \leq k(k-2)|J_0| . \tag{3}$$

Now fix a 2-configuration $S = \{f_1, f_2\}$ and observe that, since $\mathcal{F}$ is 2^--free, S spans precisely $2r-t$ vertices and f_1 and f_2 share precisely t vertices. We now estimate the number of t-sets $T \subseteq V(S)$ with $T \in J_0$. Since $T \subseteq V(S)$, either T is fully contained in f_1 or in f_2, or intersects both $f_1 \setminus f_2$ and $f_2 \setminus f_1$.

If T is fully contained in f_1, then it does not belong to J_0, as it is covered at least once (by the edge f_1). Clearly, the same argument applies to any T which is fully contained in f_2. Moreover, the number of such t-sets is $2\binom{r}{t} - 1$.

Now we consider those T intersecting both $f_1 \setminus f_2$ and $f_2 \setminus f_1$. If $T \notin J_0$, then there exists $e \in \mathcal{F}$ with $T \subseteq e$, and clearly $e \neq f_1, f_2$. Since $\mathcal{F}$ is 2^--free, then $|e \cap f_i| \leq t$ for each $i = 1, 2$, and therefore the number of t-subsets of $V(S)$ contained in e is at most $\binom{2t}{t}$. It follows that among the t-sets of $V(S)$ intersecting both $f_1 \setminus f_2$ and $f_2 \setminus f_1$, all but at most $\binom{2t}{t} \cdot (k-3)$ belong to J_0. Otherwise, using what we observed above, there would be pairwise distinct t-sets $T_1, \dots, T_{k-2} \subseteq V(S)$ and pairwise distinct edges $e_1, \dots, e_{k-2}$ with $T_i \subseteq e_i$ for each $i \in [k-2]$. However $\{e_1, \dots, e_{k-2}, f_1, f_2\}$ would be a k-configuration, being a collection of k edges spanning at most $(r-t)(k-2) + 2r - t = k(r-t)+t$ vertices.

Therefore, for a given 2-configuration S, the number of $T \subseteq V(S)$ with $T \in J_0$ is at least

$$\binom{2r-t}{t} - \left[2\binom{r}{t} - 1\right] - \left[(k-3)\binom{2t}{t}\right], \tag{4}$$

where the first term stands for the number of t-sets of $V(S)$, while the rest accounts for the arguments above.

Finally, observe that every $T \in J_{\geq 2}$ gives rise to a 2-configuration $\{f_1, f_2\}$ with $T = f_1 \cap f_2$ (we have $T \subseteq f_1 \cap f_2$ by definition, with equality because $\mathcal{F}$ is 2^--free). Thus different t-sets in $J_{\geq 2}$ give rise to distinct 2-configurations in $\mathcal{F}$), showing that the number of 2-configurations of $\mathcal{F}$ is at least $|J_{\geq 2}|$. Using (4), we conclude that

$$|\mathcal{Q}| \geq \left\{\binom{2r-t}{t} - \left[2\binom{r}{t} - 1\right] - \left[(k-3)\binom{2t}{t}\right]\right\} \cdot |J_{\geq 2}|. \tag{5}$$

The claim follows from (3) and (5). $\square$

The final claim we shall use contains a calculation.

Claim 3.4. *Suppose that r, t, k are integers satisfying $t, k \geq 2$ and $r \geq t + 4(k^3 \cdot t!)^{1/t}$. Then*

$$\binom{2r-t}{t} - \left[2\binom{r}{t} - 1\right] - \left[(k-3)\binom{2t}{t}\right] \geq k(k-2)^2.$$

Proof. Notice that

$$\begin{aligned}
\binom{2r-t}{t} &= \frac{1}{t!} \cdot \prod_{i=0}^{t-1}(2r-t-i) = \frac{1}{t!} \cdot (2r-2t+1)(2r-2t+2) \cdot \prod_{i=0}^{t-3}(r-i+r-t) \\
&\geq \frac{1}{t!} \cdot 2 \cdot (r-t+1)(r-t+2) \cdot \left[\prod_{i=0}^{t-3}(r-i) + \prod_{i=0}^{t-3}(r-t)\right] \\
&\geq 2\binom{r}{t} + \frac{2}{t!}(r-t)^t,
\end{aligned}$$

where in the second line we used that $2 \leq t \leq r-1$, which in turn gives

$$\binom{2r-t}{t} - 2\binom{r}{t} \geq \frac{2}{t!} \cdot (r-t)^t \geq 2 \cdot k^3 \cdot 4^t \geq k(k-2)^2 + (k-3)\binom{2t}{t},$$

where the second inequality follows from $r \geq t + 4(k^3 \cdot t!)^{1/t}$ and the last one from $\binom{2t}{t} \leq 2^{2t}$. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let k be an even integer, $t \geq 2$ an integer and let r be an integer satisfying $r \geq t + 4(k^3 \cdot t!)^{1/t}$. From Corollary 2.3 we get

$$\liminf_{n \to \infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \geq \frac{1}{t! \binom{r}{t}}. \tag{6}$$

Let $\mathcal{F}$ be a k-free r-graph. By Lemma 2.4, there exists a subhypergraph $\mathcal{F}'$ of $\mathcal{F}$ which is 2^--free and satisfies $|\mathcal{F}'| \geq |\mathcal{F}| - O(n^{t-1})$. Set $J_i := J_i(\mathcal{F}')$ and $J_{\geq i} := J_{\geq i}(\mathcal{F}')$ and observe that an application of Claim 3.3 to $\mathcal{F}'$ and Claim 3.4 give

$$k(k-2)|J_0| \geq \left\{ \binom{2r-t}{t} - \left[2\binom{r}{t} - 1 \right] - \left[(k-3)\binom{2t}{t} \right] \right\} \cdot |J_{\geq 2}| \geq k(k-2)^2 |J_{\geq 2}| \,.$$

Therefore, $|J_0| \geq (k-2) \cdot |J_{\geq 2}|$. Now consider the following set

$$\mathcal{P} := \{(e, T) : e \in \mathcal{F}', T \subseteq e \text{ with } |T| = t\} \,.$$

Then $|\mathcal{P}| = |\mathcal{F}'| \cdot \binom{r}{t}$ and

$$\begin{aligned} |\mathcal{P}| = \sum_{i \geq 1} i \cdot |J_i| &\leq |J_1| + (k-1) \cdot |J_{\geq 2}| \\ &= |J_{\geq 0}| + (k-2) \cdot |J_{\geq 2}| - |J_0| \leq \binom{n}{t} \,, \end{aligned}$$

where in the first inequality we used that a t-set covered at least k times gives a k-configuration and thus, since $\mathcal{F}'$ is k-free, we have $J_{\geq k} = \emptyset$, while in the last inequality we used $|J_{\geq 0}| = \binom{n}{t}$ and $|J_0| \geq (k-2) \cdot |J_{\geq 2}|$. We conclude that

$$|\mathcal{F}| \leq |\mathcal{F}'| + O(n^{t-1}) = |\mathcal{P}| \cdot \binom{r}{t}^{-1} + O(n^t) \leq \binom{n}{t} \cdot \binom{r}{t}^{-1} + O(n^{t-1}) \,,$$

which allows to establish that

$$\limsup_{n \to \infty} n^{-t} f^{(r)}(n; k(r-t)+t, k) \leq \frac{1}{t! \binom{r}{t}} \,. \tag{7}$$

The proposition follows from (6) and (7). □

4 Proof of Theorem 1.3 (Conjecture 1.1 for $k = 5$)

In this section we prove Theorem 1.3, asserting that the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 5(r-t)+5)$ exists, for every integers $2 \leq t < r$.

Proof of Theorem 1.3. Let $\varepsilon > 0$ and $\mathcal{F}$ be a 5-free n-vertex r-graph with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$. By applying Lemma 2.4, we get a subhypergraph $\mathcal{F}_1 \subseteq \mathcal{F}$ which is 2^--free, 4^--free and 5-free, satisfies

$$|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1}) \geq \binom{r}{t}^{-1} \binom{n}{t} \geq \binom{r}{t}^{-1} |J(\mathcal{F}_1)|, \tag{8}$$

and where any 2-configuration and any 3^--configuration are edge-disjoint.

Let $\{S_1, \ldots, S_m\}$ be a maximal collection of pairwise edge-disjoint 3^--configurations. Define $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus S_i$ for $i \in [m]$.

Claim 4.1. *For every $i \in [m]$ we have $|J(S_i)| = 3\binom{r}{t}$ and $J(S_i) \subseteq J(\mathcal{F}_i) \setminus J(\mathcal{F}_{i+1})$.*

Proof. Observe that, by definition of $J(S_i)$, we have that $T \in J(S_i)$ if and only there exists an edge $e \in S_i$ with $T \subseteq e$ or there exists a 2-configuration S in S_i whose vertex set contains T. Since in $\mathcal{F}_1$ any 2-configuration and any 3^--configuration are edge-disjoint, we can rule out the second option. Moreover, a set of size t cannot be in more than one edge of S_i as, otherwise, S_i would contain a 2-configuration, which cannot happen for the same reason. Since S_i has three edges, we conclude that $|J(S_i)| = 3\binom{r}{t}$.

Let $T \in J(S_i)$. That $T \in J(\mathcal{F}_i)$ is obvious. To show that $T \notin J(\mathcal{F}_{i+1})$, note that $T \in J(\mathcal{F}_{i+1})$ if and only if there exists an edge $e \in \mathcal{F}_{i+1}$ with $T \subseteq e$ or there exists a 2-configuration S of $\mathcal{F}_{i+1}$ whose vertex set contains T. The first option cannot happen as, otherwise, $S_i \cup \{e\}$ would be a 4^--configuration of $\mathcal{F}_i$, being a collection of four edges spanning at most $(3r-2t-1)+r-t = 4(r-t)+t-1$ vertices, a contradiction. Similarly, we can rule out the second option as, otherwise, using that S_i and S are edge-disjoint, $S \cup S_i$ would be a 5^--configuration of $\mathcal{F}_i$, being a collection of five edges spanning at most $(3r-2t-1)+(2r-t)-t = 5(r-t)+t-1$ vertices, a contradiction. □

Using Claim 4.1, we can establish the following for $i \in [m]$.

$$|J(\mathcal{F}_i)| - |J(\mathcal{F}_{i+1})| \geq |J(S_i)| = 3\binom{r}{t} = \binom{r}{t}\left(|\mathcal{F}_i| - |\mathcal{F}_{i+1}|\right) .$$

Summing up these inequalities for $i \in [m]$, we get that

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_{m+1})| \geq \binom{r}{t}\left(|\mathcal{F}_1| - |\mathcal{F}_{m+1}|\right) .$$

By Remark 2.6 and (8), it follows that

$$\frac{|\mathcal{F}_{m+1}|}{|J(\mathcal{F}_{m+1})|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|} .$$

The conclusion now follows from Proposition 2.5 (applied with $\mathcal{F}_{m+1}$ in place of $\mathcal{F}_2$). □

5 Proof of Theorem 1.4 ($k = 7$)

Before proving the theorem, we prove a simple inequality.

Claim 5.1. *Let r, t be integers such that $2 \leq t < r$. Then*

$$\binom{3r-2t}{t} - 4 \geq 3\binom{r}{t} .$$

Proof. The claimed inequality can be checked directly for $t = 2$, so suppose that $t \geq 3$.

$$\begin{aligned}\binom{3r-2t}{t} &= \frac{1}{t!} \cdot \prod_{i=0}^{t-1}(3r-2t-i) \\ &= \frac{1}{t!} \cdot (3r-3t+1)(3r-3t+2)(3r-3t+3) \cdot \prod_{i=0}^{t-4}(r-i+2r-2t) \\ &\geq \frac{1}{t!} \cdot 4(r-t+1)(r-t+2)(r-t+3) \cdot \prod_{i=0}^{t-4}(r-i) \\ &= 4\binom{r}{t} \geq 3\binom{r}{t} + 4\,.\end{aligned}$$

Here in the first inequality we used the inequality $(3x+1)(3x+2)(3x+3) \geq 4(x+1)(x+2)(x+3)$ for $x \geq 1$, which can be checked directly. In the last inequality we used that $\binom{r}{t} \geq \binom{t+1}{t} \geq t+1 \geq 4$. $\square$

Claim 5.2. *Let r, t be integers such that $3 \leq t < r$ or $t = 2$ and $r \geq 4$. Then*

$$\binom{2r-t}{t} \geq 2\binom{r}{t} + 2.$$

Shoham: add proof

We now prove Theorem 1.4, which asserts that the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t)+t, 7)$ exists for $r > t \geq 2$.

Proof of Theorem 1.4. Let $\varepsilon > 0$ and $\mathcal{F}$ be any 7-free n-vertex r-graph with $|\mathcal{F}| \geq \left(\binom{r}{t}^{-1} + \varepsilon\right)\binom{n}{t}$. Apply Lemma 2.4 to get a subhypergraph $\mathcal{F}_1 \subseteq \mathcal{F}$ which is 2^--free, 3^--free, 6^--free and 7-free, satisfies

$$|\mathcal{F}_1| \geq |\mathcal{F}| - O(n^{t-1}) \geq \binom{r}{t}^{-1}\binom{n}{t} \geq \binom{r}{t}^{-1}|J(\mathcal{F}_1)|, \tag{9}$$

and where any 2-configuration and any 5^--configuration are edge-disjoint, and any 3-configuration and any 4^--configuration are edge-disjoint. Now we prove some structural claims on $\mathcal{F}_1$.

Claim 5.3. *Let $\mathcal{G} \subseteq \mathcal{F}_1$ and suppose that S is a 3-configuration in $\mathcal{G}$ contained in a 4-configuration in $\mathcal{G}$. Take $\mathcal{G}' = \mathcal{G} \setminus S$. Then*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right)\,.$$

Proof. Write $S = \{e_1, e_2, e_3\}$ and let $e_4 \in \mathcal{G}$ be such that $\{e_1, e_2, e_3, e_4\}$ is a 4-configuration. Let $T' = V(S) \cap e_4$ and observe that $|T'| = t$. Indeed, $|T'| \geq t$ follows from the fact that S is a 3-configuration but not 3^--configuration, implying that $|V(S)| = 3r - 2t$, and $S \cup \{e_4\}$ is a 4-configuration, while $|T'| \leq t$ follows from the fact that otherwise $S \cup \{e_4\}$ would be a 4^--configuration, a contradiction as any 3-configuration and any 4^--configuration of $\mathcal{F}_1$ being edge-disjoint.

We need to lower bound $|J(\mathcal{G})| - |J(\mathcal{G}')|$. Let $T \subseteq V(S)$ with $|T| = t$. Since S is a 3-configuration, we have $T \in J(\mathcal{G})$. For T to be in $J(\mathcal{G}')$ there must be an ℓ-configuration in $\mathcal{G}'$ with $\ell \in [3]$ whose vertex set contains T. We now prove the following assertions, to help us bound the number of times each of these options can happen.

(i) excluding T', no t-subset of $V(S)$ is contained in a 3-configuration of $\mathcal{G}'$;

(ii) at most one t-subset of $V(S)$ is contained in a 2-configuration of $\mathcal{G}'$;

(iii) excluding T', at most two t-subsets of $V(S)$ are contained in an edge of $\mathcal{G}'$.

We can show (i) as follows. Any 3-configuration S' in $\mathcal{G}'$ satisfies $|V(S) \cap V(S')| \leq t$ as otherwise $S \cup S'$ would be a 6^--configuration. Therefore if a t-subset $T \neq T'$ of $V(S)$ is contained in a 3-configuration S' in $\mathcal{G}'$, then $e_4 \notin S'$ and $S \cup S' \cup \{e_4\}$ would be a 7-configuration of $\mathcal{G}$, a contradiction.

For (ii) we argue as follows. First observe that any 2-configuration S' in $\mathcal{G}'$ satisfies $|V(S) \cap V(S')| \leq t$ as otherwise $S \cup S'$ would be a 5^--configuration of $\mathcal{G}$, which is a contradiction as any 5^--configuration and any 2-configuration of $\mathcal{G}$ are edge-disjoint. Now suppose there are two distinct t-subsets T_1 and T_2 of $V(S)$ and two 2-configurations S_1 and S_2 in $\mathcal{G}'$ with $T_i \subseteq V(S_i)$ for $i \in [2]$. Observe that $S_1 \neq S_2$ as any 2-configuration in $\mathcal{G}'$ intersects $V(S)$ in no more than t vertices, as argued above. If S_1 and S_2 were edge-disjoint, then $S_1 \cup S_2 \cup S$ would be a 7-configuration of $\mathcal{G}$, a contradiction. If S_1 and S_2 were not edge-disjoint, then $S_1 \cup S_2$ would be a 3-configuration intersecting $V(S)$ in more than t vertices, but then $S_1 \cup S_2 \cup S$ would be a 6^--configuration of $\mathcal{G}$, a contradiction.

Finally, we prove (iii). Any edge not in S intersects $V(S)$ in at most t vertices, as 3-configurations and 4^--configurations of $\mathcal{G}$ are edge-disjoint. Suppose there were three distinct t-subsets T_1, T_2 and T_3 of $V(S)$, all distinct from T', and three (distinct) edges f_1, f_2 and f_3 not in S with $T_i \subseteq f_i$ for $i \in [3]$. Then $e_4 \notin \{f_1, f_2, f_3\}$ and $S \cup \{e_4, f_1, f_2, f_3\}$ would be a 7-configuration of $\mathcal{G}$, a contradiction.

Recall that $|V(S)| = 3r - 2t$, and thus there are $\binom{3r-2t}{t}$ subsets of $V(S)$ of size t. Taking (i), (ii) and (iii) into account, and using Claim 5.1, we get

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{3r-2t}{t} - 1 - 3 \geq 3\binom{r}{t} = \binom{r}{t}(|\mathcal{G}| - |\mathcal{G}'|),$$

as claimed. □

By repeatedly applying Claim 5.3, we get a subhypergraph $\mathcal{F}_2 \subseteq \mathcal{F}_1$ satisfying

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_2)| \geq \binom{r}{t}(|\mathcal{F}_1| - |\mathcal{F}_2|), \tag{10}$$

which has no 3-configuration contained in a 4-configuration.

Claim 5.4. *Let $\mathcal{G} \subseteq \mathcal{F}_2$. Suppose that S is a 4^--configuration in $\mathcal{G}$. Then there exists a non-empty subset $S' \subseteq S$, such that the following holds for $\mathcal{G}' = \mathcal{G} \setminus S'$.*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}(|\mathcal{G}| - |\mathcal{G}'|). \tag{11}$$

Proof. We start by observing that, for $e, e' \in \mathcal{G}$, if $e \in S$ and $\{e, e'\}$ is a 2-configuration, then $e' \in S$. Indeed, otherwise, the 5^--configuration $S \cup \{e'\}$ and the 2-configuration $\{e, e'\}$ would not be edge-disjoint, a contradiction. Therefore, either S contains a 2-configuration or the edges of S

are not involved in any 2-configuration of $\mathcal{G}$. Since S contains no 3-configurations (by every 4^--configuration being edge-disjoint of all 3-configurations in $\mathcal{G}$), we have the following three cases: S contains no 2-configurations; S contains a single 2-configuration; and S can be partitioned into two 2-configurations. We consider each case separately.

Case 1. S contains no 2-configurations. Let e be an edge of S and $\mathcal{G}' := \mathcal{G} \setminus \{e\}$. We claim that $J(\mathcal{G}) \setminus J(\mathcal{G}')$ contains all t-subsets of $V(S')$, which would prove (11) in this case. Since any such t-subset belongs to $J(\mathcal{G})$, this follows once we show that

(i) no t-subset of e is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-subset of e is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) no t-subset of e is contained in an edge of $\mathcal{G}'$.

Fact (i) holds since, by assumption, any 3-configuration is edge-disjoint of S and thus, if it shares t vertices with e, its union with S would give a 7^--configuration, a contradiction. For (ii), recall that if there was a t-subset of e contained in a 2-configuration S' then $e \notin S'$ and e would belong to both the 3-configuration $S' \cup \{e\}$ and the 4^--configuration S, a contradiction. Finally, (iii) holds as otherwise e would belong to a 2-configuration of $\mathcal{G}$, a contradiction to the assumption that S contains no 2-configurations and its edges are thus not involved in 2-configurations in $\mathcal{G}$.

Case 2. S contains a single 2-configuration S'. Set $\mathcal{G}' := \mathcal{G} \setminus S'$. Let $T \subseteq V(S')$ with $|T| = t$. Since S' is a 2-configuration, we have $T \in J(\mathcal{G})$. For T to be in $J(\mathcal{G}')$, there must be an ℓ-configuration in $\mathcal{G}'$ with $\ell \in [3]$ whose vertex set contains T. We prove the following assertions, to help us bound the number of times this can happen.

(i) no t-subset of $V(S')$ is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-subsets of $V(S')$ are contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) at most one t-subset of $V(S')$ is contained in an edge of $\mathcal{G}'$.

Indeed, (i) can be proved as in the previous case. For (ii), if S'' is a 2-configuration of $\mathcal{G}'$ which intersects $V(S')$ in (at least) t vertices then, by the argument above, S'' and S are edge-disjoint, but then $S \cup S''$ is a 6^--configuration in $\mathcal{G}$, a contradiction. Finally, (iii) holds since any $e \in S \setminus S'$ intersect $V(S')$ in at most $t-1$ vertices (as otherwise $S' \cup \{e\}$ would be a 3-configuration, a contradiction to the fact that any 3-configuration and any 4^--configuration in $\mathcal{G}$ are edge-disjoint) and at most one edge outside of S intersects $V(S)$ in t vertices (as otherwise the union of any two such edges and S would give a 6^--configuration in $\mathcal{G}$, a contradiction).

By (i), (ii), (iii) and Claim 5.2, we have

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{2r-t}{t} - 1 \geq 2\binom{r}{t} \geq \binom{r}{t} \cdot (|\mathcal{G}| - |\mathcal{G}'|) \,.$$

Case 3. *S* **can be partitioned into two 2-configurations** S_1**,** S_2**.** Set $\mathcal{G}' = \mathcal{G} \setminus S$. Let $\mathcal{J}$ be the collection of t-sets which are subsets of either $V(S_1)$ or $V(S_2)$. Note that if T is in the t-shadow of S_1 then T is not a subset of $V(S_2)$ (otherwise, S would contain a 3-configuration). Thus,

$$|\mathcal{J}| \geq |\partial_t S_1| + \binom{|V(S_2)|}{t} = 2\binom{r}{t} - 1 + \binom{2r-t}{t} \geq 4\binom{r}{t} + 1, \tag{12}$$

using Claim 5.2. Then $\mathcal{J} \subseteq J(\mathcal{G})$, as its elements are t-subsets of vertex sets of 2-configurations. As usual, we claim that

(i) no t-set in $\mathcal{J}$ is contained in the vertex set of a 3-configuration of $\mathcal{G}'$;

(ii) no t-set in $\mathcal{J}$ is contained in the vertex set of a 2-configuration of $\mathcal{G}'$;

(iii) at most one t-set in $\mathcal{J}$ is contained in an edge of $\mathcal{G}'$.

Assertion (i) can be proved as in the first case. For (ii), if S' is a 2-configuration in $\mathcal{G}'$ whose vertex set contains a t-set in $\mathcal{J}$, then $S \cup S''$ is a 6^--configuration in $\mathcal{G}$, a contradiction. Finally, for (iii), if T and T' are distinct elements of $\mathcal{J}$ that are contained in edges e and e' in $\mathcal{G}'$, then e and e' are distinct (otherwise e would form a 3^--configuration together with either S_1 or S_2), but then $S \cup \{e, e'\}$ is a 6^--configuration, a contradiction.

It follows from (i), (ii), (iii) and (12) that

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq |\mathcal{J}| - 1 \geq 4\binom{r}{t} = \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right). \qquad \square$$

By repeatedly applying Claim 5.4, we get a subhypergraph $\mathcal{F}_3 \subseteq \mathcal{F}_2$ which is 4^--free and satisfies

$$|J(\mathcal{F}_2)| - |J(\mathcal{F}_3)| \geq \binom{r}{t}\left(|\mathcal{F}_2| - |\mathcal{F}_3|\right). \tag{13}$$

Claim 5.5. *Let $\mathcal{G} \subseteq \mathcal{F}_3$. If there exists a 5^--configuration S of $\mathcal{G}$ then, with $\mathcal{G}' := \mathcal{G} \setminus S$, it holds that*

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right).$$

Proof. Let $\mathcal{J}$ be the t-shadow of S. Then $\mathcal{J} \subseteq J(\mathcal{G})$ and $|\mathcal{J}| = 5\binom{r}{t}$, as a set of size t cannot be in more than one edge of S (as otherwise the 5^--configuration S would contain a 2-configuration, a contradiction).

Next we show that if $T \in \mathcal{J}$, then $T \notin J(\mathcal{G}')$; for that it is enough to prove that T is not contained in any ℓ-configuration of $\mathcal{G}'$ with $\ell \in [3]$. For $\ell = 1$, this follows from the fact that any 2-configuration and any 5^--configuration of $\mathcal{G}$ are edge disjoint. Similarly, $\ell = 2$ holds since any 2-configuration is edge-disjoint of S an thus, if it shares t vertices with S, its union with S would give a 7^--configuration, a contradiction. Finally, for $\ell = 3$, we use that, from Claim 5.3, a 3-configuration cannot be contained in any 4-configuration of $\mathcal{G}$. Therefore we get

$$|J(\mathcal{G})| - |J(\mathcal{G}')| \geq |\mathcal{J}| = 5\binom{r}{t} = \binom{r}{t}\left(|\mathcal{G}| - |\mathcal{G}'|\right). \qquad \square$$

By repeatedly applying Claim 5.4, we get a subhypergraph $\mathcal{F}_4 \subseteq \mathcal{F}_3$ which is 4^--free and satisfies

$$|J(\mathcal{F}_3)| - |J(\mathcal{F}_4)| \geq \binom{r}{t} (|\mathcal{F}_3| - |\mathcal{F}_4|) . \tag{14}$$

Then, by summing up (10), (13) and (14), we get

$$|J(\mathcal{F}_1)| - |J(\mathcal{F}_4)| \geq \binom{r}{t} (|\mathcal{F}_1| - |\mathcal{F}_4|) .$$

Thus, using Remark 2.6 and (9), we get

$$\frac{|\mathcal{F}_4|}{|J(\mathcal{F}_4)|} \geq \frac{|\mathcal{F}_1|}{|J(\mathcal{F}_1)|} .$$

Notice that $\mathcal{F}_4$ is ℓ^--free for $\ell \in \{2,3,4,5,6\}$ and 7-free. Thus, by Proposition 2.5, the limit $\lim_{n\to\infty} n^{-t} f^{(r)}(n; 7(r-t)+t, 7)$ exists. $\square$

6 Conclusion

Theorem 1.2 establishes that $\pi(r,t,k) = \frac{1}{t!}\binom{r}{t}^{-1}$ when k is even and $r \geq t + 6(k^3 \cdot t!)^{1/t}$. It would be interesting to prove if $\pi(r,t,k) = \frac{1}{t!}\binom{r}{t}^{-1}$ holds for every even k and $r \geq t+1$, i.e. without any further restriction on r. For odd k we can give a simple proof of the following upper bound when r is sufficiently large with respect to k and t.

Proposition 6.1. *Let k be an odd integer, $t \geq 2$ an integer and r sufficiently large with respect to k and t. Then*

$$\limsup_{n\to\infty} n^{-t} f^{(r)}(n; k(r-t)+t, k)) \leq \frac{2}{t!}\left[2\binom{r}{t} - 1\right]^{-1} .$$

Proof. Let $\mathcal{F}$ be any k-free r-graph. We construct the t-tight components of $\mathcal{F}$ as follows: starting from one component for every edge of $\mathcal{F}$, at every step, if there exists a pair of edges e and f of $\mathcal{F}$ in different components such that $|e \cap f| = t$, then we merge their components. We stop when there is not such pair.

For $i \in \{1,2\}$, let $\mathcal{F}_i$ be the set of the edges of $\mathcal{F}$ which belong to components of size i, and let $\mathcal{F}_3 := \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. Then we have that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ is a partition of $\mathcal{F}$. For $i \in \{1,2\}$, define $\mathcal{G}_i$ to be the t-shadow of $\mathcal{F}_i$, and define $\mathcal{G}_3$ to be the collection of t-subsets T of $V(\mathcal{F})$ which are contained in a unique 2-configuration of $\mathcal{F}_3$.

For each component C in $\mathcal{F}_3$, fix an arbitrary 2-configuration of C and denote it by S_C (it exists since C has size at least three). We claim that the following holds:

(i) at most $(k-2)\cdot\binom{2t}{t}$ elements of $\mathcal{G}_1$ are subsets of $V(S_C)$;

(ii) at most $k(k-2)$ elements of $\mathcal{G}_2$ are subsets of $V(S_C)$;

(iii) at most ? t-subsets of $V(S_C)$ are contained in the vertex set of two different 2-configurations S_C and $S_{C'}$.

First observe that, if e and f are edges in different components, then $|e \cap f| < t$ as otherwise e and f would be in the same t-component. We will make use of this property several times, without further mentioning it. Property (i) can be shown as follows. There are at most $k-2$ edges which intersect $V(S_C)$ in at least t vertices (as otherwise, together with S_C, they would give a k-configuration). Each of them intersects $V(S_C)$ in at most $2t$ vertices. Therefore the number of t-subsets of edges of $\mathcal{F}_1$ which are subsets of $V(S_C)$ is at most $(k-2) \cdot \binom{2t}{t}$. We now proceed to (ii). The number of pairwise edge-disjoint 2-configurations of $\mathcal{F}_2$ whose t-shadow intersects $V(S_C)$ in at least t vertices is at most $k/2$ (otherwise, we could get a k-configuration by taking S_C and choosing appropriately the right number of edges from these 2-configurations). From Claim 3.2, every edge belongs to at most $k-2$ distinct 2-configurations and (ii) follows. For (iii), any two components of $\mathcal{F}_3$ can intersect in at most tk vertices and thus they have at most $\binom{tk}{t}$ t-subsets in common.

Amedeo: I am not sure how to get a contradiction if there are many 2-configurations intersecting S_C in t vertices. This was fine in (ii), as we were looking at shadows. Here we are instead looking at vertex-subsets, so I do not see how to build a k-configuration starting from a bunch of 2-configurations whose vertex set intersects $V(S_C)$ in t vertices: k is odd, so for one 2-configuration I need to choose a single edge, but then there is no guarantee that such edge intersects $V(S_C)$ in t vertices?

Moreover we have

$$
\begin{aligned}
|\mathcal{G}_1| &= |\mathcal{F}_1| \cdot \binom{r}{t} \geq \frac{2\binom{r}{t}-1}{2} |\mathcal{F}_1|\,, \\
|\mathcal{G}_2| &= \frac{|\mathcal{F}_2|}{2} \cdot \left[2\binom{r}{t} - 1\right] \\
|\mathcal{G}_3 \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)| &\geq \frac{|\mathcal{F}_3|}{k} \cdot \left[\binom{2r-t}{t} - (k-2) \cdot \binom{2t}{t} - k(k-2) - ?\right] \geq \frac{2\binom{r}{t}-1}{2} |\mathcal{F}_3|\,,
\end{aligned}
$$

where in the last inequality we used that $\mathcal{F}_3$ has at least $|\mathcal{F}_3|/k$ distinct components and r is sufficiently large compared to k and t. Therefore

$$
\begin{aligned}
|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| &\leq 2\left[2\binom{r}{t} - 1\right]^{-1} \cdot (|\mathcal{G}_1| + |\mathcal{G}_2| + |\mathcal{G}_3 \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)|) \\
&\leq 2\left[2\binom{r}{t} - 1\right]^{-1} \cdot \binom{n}{t}\,,
\end{aligned}
$$

from which the result follows. $\square$

We wonder if the upper bound provided in Proposition 6.1 is optimal for every odd k, given that this is the case for $k = 3$, as proved by [4].

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