Hypergraphs with no tight cycles

Shoham Letzter*

Abstract

We show that every r-uniform hypergraph on n vertices which does not contain a tight cycle has at most $O(n^{r-1}(\log n)^5)$ edges. This is an improvement on the previously best-known bound, of $n^{r-1}e^{O(\sqrt{\log n})}$, due to Sudakov and Tomon, and our proof builds up on their work. A recent construction of B. Janzer implies that our bound is tight up to an $O((\log n)^4 \log\log n)$ factor.

1 Introduction

It is well known, and easy to see, that the maximum number of edges in a graph on n vertices with no cycles is $n-1$. It is natural to consider an analogous problem for hypergraphs: what is the maximum possible number of edges in an r-uniform hypergraph (henceforth *r-graph*) on n vertices which does not contain a cycle? Unlike the graph case, there are multiple natural notions of cycles in hypergraphs, the most notable of which are Berge cycles, loose cycles and tight cycles.

A *Berge cycle* of length ℓ is a sequence $(v_1, e_1, \ldots, v_\ell, e_\ell)$ such that $v_1, \ldots, v_\ell$ are distinct vertices, $e_1, \ldots, e_\ell$ are distinct edges, and $v_i \in e_{i-1} \cap e_i$ (subtraction of indices is taken modulo ℓ). We claim that the maximum possible number of edges in an n-vertex r-graph with no Berge cycles is $\left\lfloor \frac{n-1}{r-1} \right\rfloor$. For the upper bound, it suffices to show that the edges of an r-graph with no Berge cycles can be ordered as $e_1, \ldots, e_m$ so that $|e_i \cap (e_1 \cup \ldots \cup e_{i-1})| \le 1$ for every $i \in [m]$, which is not hard to prove. To see the lower bound, form an r-graph on at most n vertices by taking $\left\lfloor \frac{n-1}{r-1} \right\rfloor$ pairwise disjoint sets of size $r-1$, and joining each of them to the same new vertex.

A *loose cycle* of length ℓ is a sequence $(e_1, \ldots, e_\ell)$ of distinct edges such that two consecutive edges (as well as the first and last) have exactly one vertex in common, and non-consecutive edges are disjoint. Frankl and Füredi [3] showed that any n-vertex r-graph with no loose triangles (i.e. loose cycles of length 3) has at most $\binom{n-1}{r-1}$ edges, whenever n is sufficiently large. Note that there exists an n-vertex r-graph with no loose cycles with this number of edges: take its edges to be all r-sets that contain a certain vertex u. It thus follows that the answer to the above question for loose cycles is $\binom{n-1}{r-1}$.

*Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK. Email: s.letzter@ucl.ac.uk. Research supported by the Royal Society.

An r-uniform *tight cycle* of length ℓ is a sequence $(v_1, \ldots, v_\ell)$ of distinct vertices, satisfying that $(v_i, \ldots, v_{i+r-1})$ is an edge for every $i \in [\ell]$ (with addition of indices taken modulo ℓ). Denote the family of all tight cycles by $\mathcal{C}$, and let $\mathrm{ex}_r(n, \mathcal{C})$ be the maximum possible number of edges in an n-vertex r-graph with no tight cycles. The question from the first paragraph, for tight cycles, can be restated as follows: what is $\mathrm{ex}_r(n, \mathcal{C})$?

It might be tempting to guess that $\mathrm{ex}_r(n, \mathcal{C}) = \binom{n-1}{r-1}$, similarly to the loose cycles case. Indeed, this was conjectured by Sós and, independently, Verstraëte (see [10, 13]). This conjecture was disproved by Huang and Ma [7], who showed that for every r there exists $c = c(r) \in (1, 2)$ such that $\mathrm{ex}_r(n, \mathcal{C}) \geq c \cdot \binom{n-1}{r-1}$. Very recently, B. Janzer improved this lower bound on $\mathrm{ex}_r(n, \mathcal{C})$ substantially, showing that $\mathrm{ex}_r(n, \mathcal{C}) = \Omega(n^{r-1} \cdot \frac{\log n}{\log \log n})$.

Until recently, the best upper bound on $\mathrm{ex}_r(n, \mathcal{C})$ for general r was $\mathrm{ex}_r(n, \mathcal{C}) = O(n^{r-2^{-(r-1)}})$, which follows from a result of Erdős [2] about the extremal number of a complete r-partite r-graph with vertex classes of size 2. For $r = 3$, an unpublished result of Verstraëte regarding the extremal number of a tight cycle of length 24 implies that $\mathrm{ex}_3(n, \mathcal{C}) = O(n^{5/2})$. A recent result of Tomon and Sudakov [12] shows that $\mathrm{ex}_r(n, \mathcal{C}) \leq n^{r-1} e^{O(\sqrt{\log n})}$, greatly improving on previous bounds, and thus establishing that $\mathrm{ex}_r(n, \mathcal{C}) = n^{r-1+o(1)}$.

We prove the following result about the extremal number of tight cycles in r-graphs, which lowers the $e^{O(\sqrt{\log n})}$ error term in Sudakov and Tomon's bound to a polylogarithmic term.

Theorem 1. *Suppose that $\mathcal{H}$ is an r-graph on n vertices which does not contain a tight cycle. Then $\mathcal{H}$ has $O(n^{r-1}(\log n)^5)$ edges.*

In other words, we show that $\mathrm{ex}_r(n, \mathcal{C}) = O(n^{r-1}(\log n)^5)$. In light of Janzer's result [8], this is tight up to an $O((\log n)^4 \log \log n)$ factor.

We give an overview of our proof in Section 2, mention relevant tools and definitions from [12] in Section 3, and prove our main result in Section 4. We conclude the paper in Section 5 with some closing remarks. Throughout the paper, logarithms are understood to be in base 2, and floor and ceiling signs are often dropped.

2 Overview of the proof

Our proof builds up on ideas Sudakov and Tomon's work [12]. They introduce the notions of r-line-graphs, which are graphs that correspond naturally to r-partite r-graph, and expansion in such graphs. They show that, given a dense enough r-partite r-graph $\mathcal{H}$, the r-line-graph that corresponds to $\mathcal{H}$ contains a dense expander G. Next, they define σ-paths and σ-cycles, which correspond to tight paths and cycles in the original hypergraph $\mathcal{H}$. It thus suffices to show that every r-line-graph which is a dense expander contains a σ-cycle. Sudakov and Tomon are not able to prove this. Instead, they show that every expander contains either a σ-cycle or a very dense subgraph, and proceed via a density increment argument.

Our main contribution is to show that every r-line-graph which is a dense expander indeed contains a σ-cycle (see Theorem 6). A key step in our proof is to show that in such an expander G, for every vertex $x \in V(G)$, almost every other vertex $y \in V(G)$ can be reached from x via a short σ-path $P(x, y)$ in a 'robust' way, meaning that no vertex in the underlying r-graph $\mathcal{H}$ meets too many of the paths $P(x, y)$ (see Lemma 5). If the robustness requirement is dropped, we obtain a lemma from [12]. To prove the robust version, we use the non-robust version from [12] as a black box, along with another lemma from the same paper, which asserts that the removal of a small number of vertices from the underlying r-graph $\mathcal{H}$ does not ruin the expansion.

To find a σ-cycle, let $P(x, y)$ be paths as above, defined for almost every $x, y \in V(G)$. Note that while we are guaranteed that, for every $x \in V(G)$, no vertex v of $\mathcal{H}$ meets too many paths $P(x, y)$, we do not have any control over the number of times v meets a path $P(x, y)$, for a given y. Nevertheless, since the paths $P(x, y)$ are short, for every $y \in V(G)$ there are few vertices in $\mathcal{H}$ that meet many path $P(x, y)$; denote the set of such vertices in $\mathcal{H}$ by $F(y)$. Using tools mentioned above, for every y and almost every x there is a short σ-path $Q(y, x)$ from y to x that avoids $F(y)$. To complete the proof, we note that the robustness implies that for almost every $x, y \in V(G)$ the path $Q(y, x)$ is defined, and there are linearly many $z \in V(G)$ for which $P(x, z)P(z, y)$ is a σ-path from x to y. Using robustness and the choice of $Q(y, x)$, the concatenation $P(x, z)P(z, y)Q(y, x)$ is a σ-cycle for linearly many $z \in V(G)$.

3 Expansion in r-line-graphs

We say that G is an *r-line-graph* if the vertex set of G is a set of r-tuples in $A_1 \times \ldots \times A_r$, where $A_1, \ldots, A_r$ are pairwise disjoint, and x and y are joined by an edge if and only if x and y differ in exactly one coordinate. Observe that an r-partite r-graph naturally corresponds to an r-line-graph.

Let G be an r-line-graph with $V(G) \subseteq A_1 \times \ldots \times A_r$. We will refer to the vertices of $A_1 \cup \ldots \cup A_r$ as *coordinates*. For a set of vertices X in G, let $\mathrm{co}(X)$ be the set of coordinates that appear in tuples in X. For a vertex x we write $\mathrm{co}(x)$ as a shorthand for $\mathrm{co}(\{x\})$.

For a vertex x and $i \in [r]$, define $N^{(i)}(x)$ to be the set of vertices y in G that differ from x in the i-th coordinate only. An *i-block* in G is a set of form $\{x\} \cup N^{(i)}(x)$, for $x \in V(G)$ and $i \in [r]$. Let $p(G)$ be the number of blocks in G, and define the *density* of G, denoted $\mathrm{dens}(G)$, as

$$\mathrm{dens}(G) = \frac{\sum_B |B|}{p(G)} = \frac{r|G|}{p(G)}, \tag{1}$$

where the sum is over all blocks B in G. In words, the density is the average size of a block.

The *i-degree* of a vertex x, denoted $d_G^{(i)}(x)$, is defined to be $|N^{(i)}(x)| + 1$. The *minimum degree* of G, denoted $\delta(G)$, is defined to be the minimum of $d^{(i)}(x)$, over $x \in V(G)$ and $i \in [r]$ (this is not quite the same as the usual notion of a minimum degree of a graph).

For a graph H, say that H is a *λ-expander* if every set of vertices X with $|X| \leq \frac{1}{2}|H|$ satisfies

$|N(X)| \geq \lambda|X|$, where $N(X)$ is the set of vertices in $V(H) \setminus X$ that are neighbours of at least one vertex in X. For an r-line-graph G, say that G is a (λ, d)-*expander* if G is a λ-expander and $\delta(G) \geq d$.

The following lemma from [12] allows us to find expanders in r-line-graphs that are sufficiently dense. It is reminiscent of a similar result of Shapira and Sudakov [11] about the existences of expanders in graphs.

Lemma 2 (Lemma 3.3 in [12])**.** *Let G be an r-line-graph on n vertices with density at least d, and suppose that $0 < \lambda \leq \frac{1}{2\log n}$. Then G contains a subgrah of density at least $d(1 - \lambda \log n)$ which is a $(\lambda, \frac{d}{2r})$-expander.*

The following lemma, also from [12], shows that the notion of expansion is robust, in the sense that the removal of a small number of coordinates does not affect the expansion too much.

Lemma 3 (Lemma 3.5 in [12])**.** *Let r, u, d be positive integers, let $\lambda \in (0, 1)$ and suppose that $u \leq \frac{\lambda d}{4r}$. Let G be an r-line-graph on n vertices with $V(G) \subseteq A_1 \times \ldots \times A_r$ which is a (λ, d)-expander. Suppose that H is a subgraph of G obtained by removing at most u coordinates in $A_1 \cup \ldots \cup A_r$ from G (along with edges of G that meet these coordinates). Then H is an r-line graph on at least $(1 - \frac{u}{\delta})n$ vertices which is a $(\frac{\lambda}{2}, \frac{d}{2})$-expander.*

Next, we need the notions of σ-neighbours, σ-paths and σ-cycles. Let G be an r-line-graph with $V(G) \subseteq A_1 \times \ldots \times A_r$. Given a permutation $\sigma \in S_r$ and vertices $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_r)$ in G, we say that y is a σ-*neighbour* of x if $\mathrm{co}(x)$ and $\mathrm{co}(y)$ are disjoint, and the r-tuples $z_0, \ldots, z_r$, defined as follows, are vertices in G.

$$(z_i)_j = \begin{cases} x_j & \sigma^{-1}(j) > i \\ y_j & \sigma^{-1}(j) \leq i. \end{cases}$$

Note that $z_0 = x$ and $z_r = y$. If σ is the identity permutation, we have $z_i = (y_1, \ldots, y_{i-1}, x_i, \ldots, x_r)$. Observe that $z_i \in N^{(\sigma(i))}(z_{i-1})$ for $i \in [r]$. Also note that if y is a σ-neighbour of x then the sequence $(x_{\sigma(1)}, \ldots, x_{\sigma(r)}, y_{\sigma(1)}, \ldots, y_{\sigma(r)})$ is a tight path in the r-graph that corresponds to G.[1]

A σ-path in G is a sequence $(x_1, \ldots, x_k)$ of vertices in G whose coordinate sets are pairwise disjoint, and such that x_{i+1} is a σ-neighbour of x_i for $i \in [k-1]$. Similarly, a σ-*cycle* is a sequence $(x_1, \ldots, x_k)$ of vertices in G whose coordinate sets are pairwise disjoint, such that x_{i+1} is a σ-neighbour of x_i, for $i \in [k]$ (with indices taken modulo k). Writing $x_i = (x_{i,1}, \ldots, x_{i,r})$, if $x_1, \ldots, x_r$ is a σ-path (σ-cycle), then $(x_{1,\sigma(1)}, \ldots, x_{1,\sigma(r)}, \ldots, x_{k,\sigma(1)}, \ldots, x_{k,\sigma(r)})$ is a tight path (cycle) in the r-graph corresponding to G. It would thus be useful to show that r-line-graphs that are dense expanders have σ-cycles; we do so in Theorem 6 below.

The *order* of a σ-path or σ-cycle $(x_1, \ldots, x_k)$ is k. If there is a σ-path $(x_1, \ldots, x_k)$ in G, we say that x_k *can be reached from* x_1 by a σ-path of order k. The following lemma from [12] shows that, given

[1]For the purpose of this paper it suffices to fix σ to be any particular permutation in S_r. We state the definitions and results for general σ to mirror [12].

a vertex x in an r-line-graph G which is a dense expander, almost every vertex in G can be reached from x by a relatively short σ-path.

Lemma 4 (Lemma 4.4 in [12]). *Let $\sigma \in S_r$, let $\varepsilon, \lambda \in (0,1)$ and let n and d be positive integers such that $500r^4 \log n < \varepsilon^2 \lambda^2 d$. Suppose that G is an r-line-graph on n vertices which is a (λ, d)-expander, and let $x \in V(G)$. Then at least $(1-\varepsilon)n$ vertices in G can be reached from x by a σ-path of length at most $\frac{5r \log n}{\varepsilon \lambda}$.*

4 Existence of σ-cycles in expanders

Recall that $\mathrm{co}(X)$, where X is a set of vertices in an r-line-graph, is the set of coordinates in tuples in X. The following key lemma is the first new ingredient in our proof. It shows that for every vertex x in an r-line-graph G which is a dense expander, almost every vertex in G can be reached from x by a short σ-path, such that no coordinate (other than the coordinates in x) is met by too many such σ-paths.

Lemma 5. *Let $\sigma \in S_r$, let $\varepsilon, \lambda \in (0,1)$ and let n, d, ℓ, t be positive integers such that $\ell = \frac{10r \log n}{\varepsilon \lambda}$, $t \le \frac{\lambda d}{4r\ell}$, $4000r^4 \log n < \varepsilon^2 \lambda^2 d$ and $\frac{\lambda}{4r} \le \varepsilon$. Suppose that G is an r-line-graph on n vertices, with $V(G) \subseteq A_1 \times \ldots \times A_r$, which is a (λ, d)-expander, and let $x \in V(G)$. Then there is a set $Y \subseteq V(G)$ of size at least $(1-2\varepsilon)n$ such that every $y \in Y$ can be reached from x by a σ-path $P(y)$ of order at most ℓ, and every $w \in (A_1 \cup \ldots \cup A_r) \setminus \mathrm{co}(x)$ is in $\mathrm{co}(P(y))$ for at most $\frac{n}{t}$ values of y.*

Proof. Write $A = A_1 \cup \ldots \cup A_r$ and $u = t\ell$. So $u \le \frac{\lambda d}{4r}$ and $\frac{u}{d} \le \varepsilon$.

Let Y_0 be a subset of $V(G)$ of maximum size such that there exists a collection of σ-paths $(P(y))_{y \in Y_0}$, such that $P(y)$ is a σ-path from x to y of order at most ℓ for $y \in Y_0$, and every $w \in A \setminus \mathrm{co}(x)$ is in $\mathrm{co}(P(y))$ for at most $\frac{n}{t}$ values of y; fix such a collection $(P(y))_{y \in Y_0}$. Our task is to show that $|Y_0| \ge (1-2\varepsilon)n$, so suppose otherwise.

Let F be the set of coordinates $w \in A \setminus \mathrm{co}(x)$ such that $w \in \mathrm{co}(P(y))$ for exactly $\frac{n}{t}$ values of $y \in Y_0$. By choice of F and the upper bound on the order of $P(y)$, we have

$$\frac{|F|n}{t} \le \sum_{y \in Y_0} |\mathrm{co}(P(y))| \le \ell n. \tag{2}$$

It follows that $|F| \le t\ell = u$.

Let H be the graph obtained from G by removing the vertices that meet the set F. By Lemma 3, H is an r-line-graph on at least $(1-\frac{u}{d})n \ge (1-\varepsilon)n$ vertices which is a $(\frac{\lambda}{2}, \frac{d}{2})$-expander. Note that x is in H because F is disjoint of $\mathrm{co}(x)$. Thus, by Lemma 4, there is a subset $Y_1 \subseteq V(H)$, with $|Y_1| \ge (1-\varepsilon)|H| \ge (1-\varepsilon)^2 n \ge (1-2\varepsilon)n$, such that the vertices in Y_1 can be reached from x by a σ-path in H of order at most ℓ (here we use the inequality $500r^4 \log |H| \le 500r^4 \log n < \varepsilon^2 \left(\frac{\lambda}{2}\right)^2 \left(\frac{d}{2}\right)$). By assumption on the size of Y_0, there is a vertex $y \in Y_1 \setminus Y_0$. Let $P(y)$ be a σ-path in H from x to

y whose order is at most ℓ; so $P(y)$ is a path in G that avoids F. It follows that every $w \in A \setminus \mathrm{co}(x)$ is in $\mathrm{co}(P(y))$ for at most $\frac{n}{t}$ values of y in $Y_0 \cup \{y\}$. This is a contradiction to the maximality of Y_0. Thus $|Y_0| \geq (1-2\varepsilon)n$, as required. $\square$

We now prove the main ingredient in our proof, namely that r-line-graphs which are dense expanders contain (short) σ-cycles.

Theorem 6. *Let $\sigma \in S_r$, let $\varepsilon, \lambda \in (0,1)$, and let n and d be positive integers such that $d \geq \frac{4000r^4(\log n)^2}{\varepsilon^3\lambda^3}$, $\frac{\lambda}{4r} \leq \varepsilon < \frac{1}{12}$ and n is sufficienlty large. Let G be an r-line-graph on n vertices which is a (λ, d)-expander. Then G contains a σ-cycle of order at most $\frac{30r\log n}{\varepsilon\lambda}$.*

Proof. Let $A_1, \ldots, A_r$ be disjoint sets such that $V(G) \subseteq A_1 \times \ldots \times A_r$ and write $A = A_1 \cup \ldots \cup A_r$. Let $u = \frac{\lambda d}{4r}$, write $\ell = \frac{10r\log n}{\varepsilon\lambda}$ and let $t = \frac{u}{\ell}$. We claim that the following inequalities hold: $\frac{u}{d} \leq \varepsilon$ and $\frac{r\ell}{t} \leq \varepsilon$. The former is easy to check by the definition of u and the lower bound on ε. The latter is more tedious but follows directly from the choices of u, ℓ, t and the lower bound on t.

For each vertex x in G, let $Y(x) \subseteq V(G)$ be a set of size at least $(1-2\varepsilon)n$ and let $P(x,y)$ be a σ-path of length at most ℓ in G from x to y, for $y \in Y(x)$, such that

$$\text{every } w \in A \setminus \mathrm{co}(x) \text{ is in } \mathrm{co}(P(x,y)) \text{ for at most } \frac{n}{t} \text{ vertices } y \text{ in } Y(x), \text{ for } x \in V(G). \tag{3}$$

Such set $Y(x)$ and paths $P(x,y)$ exist by Lemma 5. For each vertex y in G, let $F(y)$ be the set of elements $w \in A \setminus \mathrm{co}(y)$ that appear in more than $\frac{n}{t}$ sets $\mathrm{co}(P(x,y))$ with $x \in V(G)$. Using a calculation as in (2), it is easy to see that $|F(y)| \leq u$ for every $y \in V(G)$. Let $G(y)$ be the graph obtained from G by removing all vertices that meet $F(y)$. It follows from Lemma 3 that $G(y)$ is an r-line-graph on at least $(1-\frac{u}{d})n \geq (1-\varepsilon)n$ vertices, and it is also a $(\frac{\lambda}{2}, \frac{d}{2})$-expander. By Lemma 4, there is a subset $X(y)$ of $V(G(y))$ with $|X(y)| \geq (1-\varepsilon)^2 n \geq (1-2\varepsilon)n$, and σ-paths $Q(y,x)$ in $G(y)$ from y to x whose order is at most ℓ, for $x \in X(y)$.

Consider a vertex x in G. Let $D(x)$ be a directed graph on vertices $V(G)$ where yz is an edge if paths $P(x,y)$ and $P(y,z)$ are defined and $\mathrm{co}(P(x,y)) \cap \mathrm{co}(P(y,z)) = \mathrm{co}(y)$; equivalently, yz is an edge if the concatenation of $P(x,y)$ and $P(y,z)$ forms a σ-path in G from x to z. Given y for which $P(x,y)$ is defined, the number of vertices z for which $P(y,z)$ is defined but yz is not an edge in $D(x)$ is at most $\frac{r\ell n}{t} \leq \varepsilon n$, by (3). Since $P(y,z)$ is defined for at least $(1-2\varepsilon)n$ vertices z, this implies that every vertex in $X(y)$ has out-degree at least $(1-3\varepsilon)n$. It follows that the number of edges in $D(x)$ is at least $(1-2\varepsilon)n \cdot (1-3\varepsilon)n \geq (1-5\varepsilon)n^2$, and thus there are at least $(1-10\varepsilon)n$ vertices in G with in-degree at least $\frac{n}{2}$ in $D(x)$.

The previous paragraph implies that the number of pairs (x,y) with $x, y \in V(G)$, such that y has in-degree at least $\frac{n}{2}$ in $D(x)$, is at least $(1-10\varepsilon)n^2$. Recall that the number of pairs (x,y) with $x, y \in V(G)$, such that $Q(y,x)$ is defined, is at least $(1-2\varepsilon)n^2$. It follows that there are at least $(1-12\varepsilon)n^2$ pairs (x,y) such that y has in-degree at least $\frac{n}{2}$ in $D(x)$ and $Q(y,x)$ is defined. We claim that every such pair yields a σ-path in G that passes through x and y.

To see this, fix a pair (x,y) as in the previous paragraph. Write $S = \mathrm{co}(Q(y,x)) \setminus (\mathrm{co}(x) \cup \mathrm{co}(y))$. Then $|S| \le r\ell$, and S is disjoint of $F(y)$, by choice of $Q(y,x)$. Let Z be the in-neighbourhood of y in $D(x)$; so $|Z| \ge \frac{n}{2}$. We claim that there is a vertex z in Z such that $P(x,z)$ and $P(z,y)$ both avoid S. To see this, first note that, by (3), there are at most $\frac{r\ell n}{t} \le \varepsilon n$ vertices z in Z such that $P(x,z)$ intersects S. Similarly, as S is disjoint of $F(y)$ and by choice of $F(y)$, there are at most $\frac{r\ell n}{t} \le \varepsilon n$ vertices z in Z such that $P(z,y)$ meets S. It follows that there are at least $|Z| - 2\varepsilon n \ge \frac{n}{4}$ vertices $z \in Z$ such that $\mathrm{co}(P(x,y))$ and $\mathrm{co}(P(y,z))$ are disjoint of S. Fix such z. The concatenation $P(x,z)P(z,y)Q(y,x)$ is a σ-cycle in G (of order at most 3ℓ). □

Finally, we prove our main result, Theorem 1. It follows easily from the results above.

Proof of Theorem 1. Let $\mathcal{H}$ be an r-graph on N vertices which does not contain a tight cycle. By considering a random partition of $V(\mathcal{H})$ into r parts, we can find an r-partite subgraph $\mathcal{H}'$ of $\mathcal{H}$ with at least $\frac{r!}{r^r} \cdot e(\mathcal{H})$ edges.

Write $e(\mathcal{H}') = dN^{r-1}$, $n = e(\mathcal{H}')$, $\lambda = \frac{1}{2\log n}$ and $\varepsilon = \frac{1}{20}$. Consider the r-line-graph G that corresponds to $\mathcal{H}'$. Then $\mathrm{dens}(G) = \frac{rn}{p(G)} \ge \frac{rdN^{r-1}}{rN^{r-1}} = d$ (see (1)). By Lemma 2, there is a subgraph G' of G which is an r-line-graph and a $(\lambda, \frac{d}{2r})$-expander. By Theorem 6, we find that

$$\frac{d}{2r} < \frac{4000r^4(\log n)^2}{\varepsilon^3\lambda^3} \le 2^8 \cdot 10^6 \cdot r^4(\log n)^5.$$

Indeed, otherwise Theorem 6 yields a σ-cycle in G' (of length at most $r \cdot \frac{30r\log n}{\varepsilon\lambda} \le 1200r^3(\log n)^2$), which corresponds to a tight cycle in $\mathcal{H}'$, contradicting the assumption on $\mathcal{H}$. It follows that $d \le 10^9 r^5(\log n)^5 \le 10^9 r^{10}(\log N)^5$ (using $n \le N^r$), implying that

$$e(\mathcal{H}) \le \frac{r^r}{r!} \cdot e(\mathcal{H}') \le \frac{10^9 r^{r+10}}{r!} \cdot N^{r-1}(\log N)^5 = O(N^{r-1}(\log N)^5),$$

as required. □

5 Conclusion

We proved that the maximum possible number of edges in an n-vertex r-graph with no tight cycles is at most $O(n^{r-1}(\log n)^5)$, thus pinning down this extremal number up to a polylogarithmic factor. Specifically, we showed that every r-line-graph G which is a (λ, d)-expander, with d sufficiently large, contains a σ-cycle. In fact, our proof implies that there is a σ-cycle between almost every two vertices in G. However, it is not clear if the same should hold for every two vertices in G whose coordinate sets are disjoint. Even the following, slightly weaker question, remains open: in an r-line-graph which is a dense expander, can every two vertices which do not share coordinates be joined by a σ-path?

It is natural to consider a similar question to the one discussed in this paper, where instead of forbidding all tight cycles, we forbid a tight cycle of given length ℓ. This was addressed for ℓ which is linear in n by Allen, Böttcher, Cooley and Mycroft [1], and an unpublished result of Verstraëte considered the case $\ell = 24$ and $r = 3$. When ℓ is not divisible by r, there exist n-vertex r-graphs with $\Omega(n^r)$ edges and no tight cycles of length ℓ; indeed, any dense r-partite r-graph would do. Conlon (see [10]) asked the following question for fixed ℓ which is divisible by r.

Question 7 (Conlon)**.** *Given $r \geq 3$, is there $c = c(r)$ such that whenever $\ell > r$ and ℓ is divisible by r, every n-vertex r-graph with no tight cycle of length ℓ has at most $O(n^{r-1+c/\ell})$ edges?*

We note that a lot more is known about the number of edges in an r-graph with no Berge or loose cycle of given lengths; see, e.g., [3, 4, 5, 6, 9].

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