Thresholds for constrained Ramsey and anti-Ramsey problems

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March 10, 2025

Let H_1 and H_2 be graphs. A graph G has the *constrained Ramsey property for* (H_1, H_2) if every edge-colouring of G contains either a monochromatic copy of H_1 or a rainbow copy of H_2. Our main result gives a 0-statement for the constrained Ramsey property in $G(n, p)$ whenever $H_1 = K_{1,k}$ for some $k \geq 3$ and H_2 is not a forest. Along with previous work of Kohayakawa, Konstadinidis and Mota, this resolves the constrained Ramsey property for all non-trivial cases with the exception of $H_1 = K_{1,2}$, which is equivalent to the anti-Ramsey property for H_2.

For a fixed graph H, we say that G has the *anti-Ramsey property for H* if any proper edge-colouring of G contains a rainbow copy of H. We show that the 0-statement for the anti-Ramsey problem in $G(n, p)$ can be reduced to a (necessary) colouring statement, and use this to find the threshold for the anti-Ramsey property for some particular families of graphs.

1 Introduction

For fixed graphs H_1, H_2, we say that a graph G has the *constrained Ramsey property for* (H_1, H_2), denoted $G \xrightarrow{\text{c-ram}} (H_1, H_2)$, if any edge-colouring of G contains either a monochromatic copy of H_1 or a rainbow copy of H_2, i.e. a copy of H_2 where each edge has a different colour. It is not hard to see that G can not have the constrained Ramsey property unless either H_1 is a star or H_2 is a forest. Indeed, label $V(G)$ (arbitrarily) as $v_1, \ldots, v_{v(G)}$ and define a colouring χ such that $\chi(v_i v_j) = \min\{i, j\}$ for every edge $v_i v_j$. Then the only monochromatic

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This research was supported by a PIMS postdoctoral fellowship at the University of Victoria.

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graphs in χ are stars (as every edge coloured i touches the vertex v_i), and every cycle in χ contains at least two edges of the same colour (if i is the smallest index of a vertex in a cycle C, then C has two edges coloured i). Hence we have $G \overset{\text{c−ram}}{\not\longrightarrow} (H_1, H_2)$ if H_1 is not a star and H_2 is not a forest.

In this paper, we are interested in determining when a graph typically has the constrained Ramsey property for a given pair (H_1, H_2). This is formalised by asking when the *random graph* $G(n, p)$ (which has n vertices and where each possible edge is included independently with probability p) is likely to have the constrained Ramsey property.

A *graph property* is a collection of graphs, and we say that a graph property $\mathcal{P}$ is *monotone (increasing)* if, whenever H and G are graphs satisfying $H \in \mathcal{P}$, $H \subseteq G$ and $V(G) = V(H)$, then $G \in \mathcal{P}$. Throughout this paper, we will say that a function $f : \mathbb{N} \to \mathbb{R}$ is a *coarse threshold function for* $\mathcal{P}$ *in* $G(n, p)$ if

$$\lim_{n\to\infty} \mathbb{P}[G(n, p) \in \mathcal{P}] = \begin{cases} 0 & \text{if } p = o(f(n)), \\ 1 & \text{if } p = \omega(f(n)), \end{cases}$$

and a *semi-sharp threshold function for* $\mathcal{P}$ *in* $G(n, p)$ if there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}[G(n, p) \in \mathcal{P}] = \begin{cases} 0 & \text{if } p \leq cf(n), \\ 1 & \text{if } p \geq Cf(n). \end{cases}$$

Bollobás and Thomason [5] showed that every non-trivial[1] monotone property $\mathcal{P}$ for $G(n, p)$ has a coarse threshold. We will use the term *0-statement* to refer to the statement for $p = o(f(n))$ or $p \leq cf(n)$ (depending on context), and *1-statement* to refer to the statement for $p = \omega(f(n))$ or $p \geq Cf(n)$.

Note that a 1-statement is equivalent to saying that with high probability (that is, with probability tending to 1 as n tends to infinity, which we will henceforth abbreviate to w.h.p.) a graph on n vertices with density much greater than $f(n)$ has the property $\mathcal{P}$, whereas the 0-statement says that w.h.p. a graph on n vertices with density much less than $f(n)$ does not have the property $\mathcal{P}$.

Question 1.1 (Constrained Ramsey)**.** *Let H_1, H_2 be graphs such that H_1 is a star or H_2 is a forest. What is a coarse/semi-sharp threshold function for the constrained Ramsey property for (H_1, H_2) in $G(n, p)$?*

Before discussing what is currently known for various H_1 and H_2, we need to introduce some definitions. For a graph H, we define $d(H) := \frac{e(H)}{v(H)}$ and the *density* of H to be

$$m(H) := \max\{d(J) : J \subseteq H \text{ and } v(J) \geq 1\}.$$

We say that H is *balanced* if $m(H) = d(H)$, and *strictly balanced* if for all proper subgraphs $J \subsetneq H$ with $v(J) \geq 1$, we have $d(J) < d(H)$.

[1] A property $\mathcal{P}$ is *non-trivial* if, for every large enough n, there exist n-vertex graphs H and G such that $H \in \mathcal{P}$ and $G \notin \mathcal{P}$.

Similarly, we define

$$d_2(H) \coloneqq \begin{cases} \frac{e(H)-1}{v(H)-2} & \text{if } e(H) \geq 1 \text{ and } v(H) \geq 3, \\ \frac{1}{2} & \text{if } H \cong K_2, \\ 0 & \text{otherwise,} \end{cases}$$

and the 2-*density of* H to be

$$m_2(H) \coloneqq \max\{d_2(J) : J \subseteq H\}.$$

We say that H is 2-*balanced* if $m_2(H) = d_2(H)$, and *strictly* 2-*balanced* if for all proper subgraphs $J \subsetneq H$, we have $d_2(J) < d_2(H)$.

When H_1 is connected and is not a star, the random Ramsey theorem of Rödl and Ruciński [16, 17, 18] states that the threshold for $G(n,p)$ having the Ramsey property for H_1 – that is, for all $r \geq 2$, for any r-colouring of the edges of $G(n,p)$ there is a monochromatic copy of H_1 – is $n^{-1/m_2(H_1)}$. Since when H_2 has at least three edges, any 2-colouring automatically avoids a rainbow copy of H_2, so in this case a 0-statement for the constrained Ramsey property for (H_1, H_2) holds at $n^{-1/m_2(H_1)}$. Collares, Kohayakawa, Moreira and Mota [7] proved a 1-statement for all forests H_2 with at least three edges, and further obtained the location of the threshold for all remaining cases where H_2 is a forest. Note however that the threshold is not always explicit: in some cases it is given in terms of the infimum of $m(F)$ taken over an infinite family of graphs F. See the full collection of known results about thresholds for the constrained Ramsey property for (H_1, H_2) in Table 1 in Section 4.

We therefore focus our attention exclusively on the case where H_1 is a star $K_{1,k}$ with $k \geq 2$ and H_2 is not a forest.

For any fixed $k \geq 2$, a natural candidate for the threshold function of the constrained Ramsey property for $(K_{1,k}, H_2)$ in $G(n,p)$ is $n^{-1/m_2(H_2)}$. Indeed, the expected number of edges of $G(n,p)$ is $\Theta(n^2 p)$ and the expected number of copies of H_2 in $G(n,p)$ is $\Theta(n^{v(H_2)} p^{e(H_2)})$. One can observe that if $p = n^{-1/m_2(H_2)}$ and H_2 is 2-balanced then these quantities are of the same order. Therefore if $p \leq cn^{-1/m_2(H_2)}$, for a small enough constant $c > 0$, then we have intuitively that every edge belongs to very few (possibly zero) copies of H_2. We can then hope to colour each copy of H_2 such that the colourings of different copies barely interact with each other. In particular, we can hopefully use for each copy H' of H_2 a colour $c_{H'}$ twice, such that for any two copies H' and H'' of H_2, we have $c_{H'} \neq c_{H''}$. This way we avoid a rainbow copy of H_2, and it suffices to show that this is possible without creating a monochromatic $K_{1,k}$.

The 1-statement for the constrained Ramsey property in the case where H_1 is a star is a result of Kohayakawa, Konstadinidis and Mota [10]. Note that in [10], the result is instead phrased in terms of the threshold for having the property that any locally r-bounded edge-colouring (for every vertex, no colour appears more than r times on the edges incident to it) produces a monochromatic copy of H. This is clearly equivalent to the constrained Ramsey property for $(K_{1,r+1}, H)$.

Theorem 1.2 ([10, Theorem 2.2])**.** *Let $k \geq 2$ and H be a graph. There exists a constant $C > 0$ such that if $p \geq Cn^{-1/m_2(H)}$ then*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (K_{1,k}, H)] = 1.$$

In the following two theorems we complete the picture for the constrained Ramsey problem in random graphs, with the exception of the case when $H_1 = K_{1,2}$. Note that the 1-statements of these results follow immediately from Theorem 1.2.

Theorem 1.3. *Let $k \geq 3$ and H be a graph that is not a forest. If $k \geq 4$, or $m_2(H) \neq 2$, or $m_2(H) = 2$ and H contains a strictly 2-balanced graph $J \neq K_3$ with $m_2(J) = 2$, then there exist constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (K_{1,k}, H)] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases}$$

The cases not covered by Theorem 1.3, where $k = 3$, $m_2(H) = 2$ and the unique strictly 2-balanced graph J with $m_2(J) = 2$ contained in H is K_3 (e.g. when H itself is K_3), require a weaker 0-statement.

Theorem 1.4. *Let H be a graph where $m_2(H) = 2$ and K_3 is the unique strictly 2-balanced graph with 2-density equal to 2 contained in H. There exists a constant $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (K_{1,3}, H)] = \begin{cases} 0 & \text{if } p = o\left(n^{-1/2}\right), \\ 1 & \text{if } p \geq Cn^{-1/2}. \end{cases}$$

Moreover, for any constant $c > 0$ there exists $\zeta = \zeta(c) > 0$ such that if $p = cn^{-1/2}$ then $\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (K_{1,3}, K_3)] \geq \zeta$.

The second part of the statement shows that for $H = K_3$ the threshold is coarse, and it cannot be improved to semi-sharp.

The proofs of Theorems 1.3 and 1.4 can be found in Section 3.

For a fixed graph H, we say that G has the *anti-Ramsey property for H*, written $G \xrightarrow{\text{a-ram}} H$, if any *proper* edge-colouring of G contains a rainbow copy of H. If $H_1 = K_{1,2}$, the star with two edges, then an edge-colouring of G avoiding a monochromatic copy of H_1 is precisely a proper colouring. That is, having the constrained Ramsey property for $(K_{1,2}, H_2)$ is equivalent to having the anti-Ramsey property for H_2.

The study of the anti-Ramsey problem was initiated by Rödl and Tuza [19] who focused on the case when H is a cycle. As with the constrained Ramsey problem, we consider the question of finding a threshold for $G(n,p)$ having the anti-Ramsey property for a given H.

Question 1.5 (Anti-Ramsey)**.** *Let H be a graph. Do there exist constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{a-ram}} H] = \begin{cases} 0 & \text{if } p \leq cn^{-1/m_2(H)}, \\ 1 & \text{if } p \geq Cn^{-1/m_2(H)}. \end{cases}$$

As before, we refer to the two parts of this question as the *0-statement for anti-Ramsey* and *1-statement for anti-Ramsey* respectively.

Note that, when $r = 2$, Theorem 1.2 is precisely the 1-statement of the anti-Ramsey problem, and so immediately yields a positive answer for the 1-statement in Question 1.5.

In [14], Nenadov, Person, Škorić and Steger introduced a general framework for proving 0-statements in various settings, and applied this to show that the 0-statement for anti-Ramsey holds for sufficiently long cycles and sufficiently large complete graphs. These results were extended in [3] for cycles and in [12] for complete graphs to obtain a 0-statement for anti-Ramsey for all cycles and complete graphs on at least four vertices.

The threshold here is $n^{-1/m_2(H)}$ for all cases except for when H is a copy of C_4 or K_4, where it is instead given by the threshold for the appearance of a graph which has the anti-Ramsey property for H. Notice that if there exists a finite graph G such that $m(G) < m_2(H)$ and $G \xrightarrow{\text{a-ram}} H$, then for any $p = \omega(n^{-1/m(G)})$ (in particular for $p = n^{-1/m_2(H)}$) G occurs in $G(n,p)$ w.h.p, and so $G(n,p) \xrightarrow{\text{a-ram}} H$ w.h.p. Therefore $n^{-1/m_2(H)}$ cannot be a threshold, and trivially a 1-statement holds at $n^{-1/m(G)} = o(n^{-1/m_2(H)})$. Such graphs G exist for C_4 and K_4.

Similarly, if H is a copy of K_3, then H is rainbow in any proper colouring of $G(n,p)$. It follows that the threshold for $G(n,p)$ having the anti-Ramsey property for K_3 corresponds with the threshold for the appearance of a copy of K_3 which is at $n^{-1/m(K_3)} = n^{-1}$ (see [4]). This idea was pursued further in [2] where it was demonstrated that $n^{-\frac{1}{m_2(H)}}$ is not a threshold for the anti-Ramsey property for graphs of the form $B_t \oplus F$: here B_t is the book graph formed by t triangles all sharing a common edge, F is any graph satisfying $1 < m_2(F) < 2$, and $\oplus$ is the graph operation of gluing two graphs together along an edge.

For the anti-Ramsey problem, we have some partial progress, reducing proving the 0-statement to a (necessary) colouring statement. By the reasoning in the previous paragraph, in order to prove that $n^{-1/m_2(H)}$ is a semi-sharp threshold, it is necessary to prove that for all graphs G with $m(G) \leq m_2(H)$, we have $G \stackrel{\text{a-ram}}{\not\longrightarrow} H$. For a coarse threshold one may replace $\leq$ by $<$ in the above statement. We prove that it is in fact sufficient to prove the above colouring statement, for most graphs H.

Theorem 1.6. *Let H be a strictly 2-balanced graph on at least five edges with $m_2(H) > 1$. To prove the 0-statement of Question 1.5 it suffices to prove for all graphs G with $m(G) \leq m_2(H)$ that $G \stackrel{\text{a-ram}}{\not\longrightarrow} H$.*

Note that when attempting to confirm the 0-statement for anti-Ramsey, it is sufficient to consider the case when H is strictly 2-balanced. Indeed, otherwise, take a minimal subgraph $H' \subseteq H$ such that $m_2(H) = d_2(H')$, so that H' is strictly 2-balanced. If a proper colouring of $G(n,p)$ contains no rainbow copy of H', it also contains no rainbow copy of H.

We prove Theorem 1.6 in Subsection 2.1. Note that the only strictly 2-balanced graphs excluded by the theorem are K_3, C_4 or a 'cherry' P_3 (the path with two edges). For all of these cases, the colouring statement within is not true anyway (see [3] for C_4, and notice that it fails trivially for K_3 and P_3 because every proper colouring of either of these graphs is rainbow), and the threshold is instead given by the appearance of a certain graph with the anti-Ramsey property.

We prove the colouring statement for some special cases of H, including when H is a d-regular graph on at least $4d$ vertices, in Subsection 2.2. However, Question 1.5 remains open in general.

Organisation

We prove Theorems 1.3 and 1.4 in Section 3 and we prove Theorem 1.6 in Section 2. In Section 4, we give tables which provide a list of all known results for the constrained Ramsey and anti-Ramsey thresholds, addressing what type of threshold occurs in each case, and we briefly discuss the remaining open cases for the anti-Ramsey threshold.

2 Anti-Ramsey results

In this section we prove Theorem 1.6 which reduces the 0-statement of Question 1.5 to a colouring statement (see Section 2.1). We then apply it to give a family of graphs H which satisfy the 0-statement of the question (see Section 2.2).

2.1 Reduction to the colouring statement

As preparation for the proof of Theorem 1.6, we list some well-known facts regarding strictly 2-balanced graphs.

Fact 2.1. Let H be a strictly 2-balanced graph with $m_2(H) > 1$. Then the following hold.

- H has minimum degree at least 2.
- H is 2-connected (see [15, Lemma 3.3]).

We call a graph H *spacious* if for every edge e in H there exists an edge f which is vertex disjoint from e. The following was observed by Rödl and Ruciński in [16]. We prove it here for completeness.

Lemma 2.2. *Let H be a strictly 2-balanced graph satisfying $m_2(H) > 1$ and $H \neq K_3$. Then H is spacious.*

Proof. Assume for a contradiction that H is not spacious. Then there is an edge xy such that every other edge contains x or y. As $m_2(H) > 1$, $H \neq K_2$. We claim that every vertex $z \in V(H) \setminus \{x, y\}$ is adjacent to both x and y and to no other vertices. Indeed, it has no neighbours other than x and y by assumption on xy, and it has degree at least 2 by Fact 2.1. Hence, H contains a triangle (namely xyz for any $z \in V(H) \setminus \{x, y\}$) and $e(H) = 1+2(v(H)-2)$, showing that

$$d_2(H) = \frac{1 + 2(v(H) - 2) - 1}{v(H) - 2} = 2 = m_2(K_3).$$

This contradicts the assumption that H is strictly 2-balanced and is not a triangle. □

Combining Fact 2.1 and Lemma 2.2 with some of the proof of [14, Theorem 7], we will be able to prove Theorem 1.6. In order for the proof to make sense, we require a number of definitions from the beginning of [14, Section 2]. Since we only deal with graphs – and not hypergraphs – in this paper, we will only state these definitions for graphs.

Definition 2.3 (H-equivalence)**.** Given graphs H and G, say that two edges $e_1, e_2 \in E(G)$ are H-*equivalent*, with notation $e_1 \equiv_H e_2$, if for every copy H' of H in G we have $e_1 \in E(H')$ if and only if $e_2 \in E(H')$.

Definition 2.4 (H-*closed* property)**.** For given graphs H and G, define the property of being H-*closed* as follows:

- an edge $e \in E(G)$ is *H-closed* if e belongs to at least two copies of H in G,
- a copy H' of H in G is *H-closed* if at least three edges from $E(H')$ are H-closed,
- a graph G is *H-closed* if every vertex and edge of G belongs to at least one copy of H and every copy of H in G is H-closed.

If the graph H is clear from the context, we simply write *closed*.

Definition 2.5 (*H-blocks*)**.** Given graphs H and G, we say that G is an *H-block* if G is H-closed and for every non-empty proper subset of edges $E' \subsetneq E(G)$ there exists a copy H' of H in G such that $E(H') \cap E' \neq \emptyset$ and $E(H') \setminus E' \neq \emptyset$ (in other words, there exists a copy of H which partially lies in E').

With these definitions we can now give the graph version of [14, Theorem 12] which we will need in order to prove Theorem 1.6.

Theorem 2.6 ([14, Theorem 12])**.** *Let H be a strictly 2-balanced graph. Then there exist constants $c, L > 0$ such that for $p \leq cn^{-1/m_2(H)}$, w.h.p. $G \sim G(n,p)$ satisfies that every H-block $B \subseteq G$ contains at most L vertices.*

We will also need the following corollary of Theorem 2.6.

Corollary 2.7 ([14, Corollary 13])**.** *Let H be a strictly 2-balanced graph. Then there exists a constant $c > 0$ such that for $p \leq cn^{-1/m_2(H)}$, w.h.p. $G \sim G(n,p)$ satisfies that for every H-block $B \subseteq G$ we have $m(B) \leq m_2(H)$. Moreover, if $p = o\left(n^{-1/m_2(H)}\right)$ then a strict inequality holds.*

Further, we need the following basic property of H-closed graphs. Here a *decomposition* of a graph G is a collection $H_1, \dots, H_k$ of subgraphs of G such that every edge of G is covered by exactly one of the subgraphs H_i.

Lemma 2.8 ([14, Lemma 14])**.** *Let H be a graph. Then if a graph G is H-closed, there exists a decomposition $B_1, \dots, B_k$ of G, such that every subgraph B_i is an H-block and every copy of H in G is entirely contained in some block B_i.*

Our contribution to the proof of Theorem 1.6 is the following claim. It essentially says that removing two edges from a strictly 2-balanced graph H on at least five edges does not yield a graph that is 'too far' from being spacious.

Proposition 2.9. *Let H be a strictly 2-balanced graph on at least five edges with $m_2(H) > 1$. Then if we remove any two edges from $E(H)$, the resulting graph contains two edges which are vertex disjoint.*

Proof. Recall from Fact 2.1 and Lemma 2.2 that H has minimum degree at least 2, H is 2-connected and H is spacious. Denote the removed edges by e and f. We have the following cases.

Case 1: *The edges e and f are vertex disjoint.* In this case, if an edge g is incident to both e and f in H then we are immediately done since H is spacious; there is some edge in H which is vertex disjoint from g, and is thus not e or f. Assume there is no such edge g. Then since $\delta(H) \geq 2$, there exists a collection E of four distinct edges, each containing exactly one vertex of $V(e) \cup V(f)$. Now, there are either two edges in E which are vertex disjoint, and we are done, or all the edges in E intersect in the same vertex v. Since H is 2-connected there exists some edge h which is not incident to v and not in $E \cup \{e, f\}$ (otherwise e and f lie in different components of $H \setminus \{v\}$), and h must be vertex disjoint from some edge from E.

Case 2: *The edges e and f form a path.* In this case, again, if an edge intersects both e and f then we are done by H being spacious. Assume not. Then, since $\delta(H) \geq 2$, there are edges e' and f' such that e' is incident to the vertex in $e \setminus f$ and f' is incident to the vertex in $f \setminus e$.

If e' and f' are vertex disjoint, we are done. Assume not. Then e' and f' meet at some vertex v forming a copy of C_4 with e and f. Since H has at least five edges, there exists at least one more edge h, which by assumption is not adjacent to $e \cap f$ and does not form a triangle with e and f. If h is vertex disjoint from either e' or f' we are done. Hence h contains v. Observe that, since H is 2-connected, there also exists an edge which is neither incident to v nor is one of e or f (otherwise $h \setminus \{v\}$ and e are in different components of $H \setminus \{v\}$). This edge is vertex disjoint from at least one of e' and f'. $\square$

We now adapt the proof of [14, Theorem 7] to prove Theorem 1.6.

Proof of Theorem 1.6. Let H be a strictly 2-balanced graph on at least five edges and c be a constant given by Corollary 2.7 when applied to H. Let $p \leq cn^{-1/m_2(H)}$ and $G \sim G(n, p)$. Then w.h.p. every H-block B in G satisfies $m(B) \leq m_2(H)$; let us assume that this holds for G. We use Algorithm Rainbow-Colour (see Figure 1) to find a proper colouring of G without a rainbow copy of H.

1: **procedure** Rainbow-colour($G = (V, E)$)
2: $\quad \hat{G} \leftarrow G$
3: $\quad \mathrm{col} \leftarrow 0$
4: $\quad$ **while** $\exists e_1, e_2 \in E(\hat{G}) : e_1 \equiv_H e_2$ in $\hat{G}$ and $e_1 \cap e_2 = \emptyset$ **do**
5: $\quad\quad$ colour e_1, e_2 with col
6: $\quad\quad \hat{G} \leftarrow \hat{G} \setminus \{e_1, e_2\}$ and $\mathrm{col} \leftarrow \mathrm{col} + 1$
7: $\quad$ **end while**
8: $\quad$ **while** $\exists e \in E(\hat{G})$: e does not belong to a copy of H **do**
9: $\quad\quad$ colour e with col
10: $\quad\quad \hat{G} \leftarrow \hat{G} \setminus \{e\}$ and $\mathrm{col} \leftarrow \mathrm{col} + 1$
11: $\quad$ **end while**
12: **end procedure**
13: Remove isolated vertices in $\hat{G}$.
14: $\{B_1, \ldots, B_k\} \leftarrow H$-blocks obtained by applying Lemma 2.8 to $\hat{G}$.
15: Colour (properly) each B_j without a rainbow copy of H using distinct sets of colours (cf. text why these last two lines are possible).

Figure 1: The implementation of algorithm Rainbow-colour.

To see the correctness of the algorithm, we first deduce that it suffices to argue that the graph $\hat{G}$ obtained in line 13 can be properly coloured without creating a rainbow copy of H. Indeed, to obtain $\hat{G}$ we only remove edges that are either in pairs of non-adjacent edges that are both contained in exactly the same H-copies (and can thus not be contained in a rainbow copy if we give them the same colour), or not contained in a copy of H (and can thus be coloured with a new colour without risk of creating a rainbow copy of H).

It thus remains to prove that lines 14 and 15 are indeed possible. For the former, we must show that the graph $\hat{G}$ is H-closed. Assume otherwise. Then there exists a copy H' of H which has at most two closed edges (as there are no vertices and edges which are not a part of a copy of H). But by applying Proposition 2.9 (and ensuring the closed edges are chosen to be removed), H' must also have two edges e_1, e_2 which are vertex disjoint and are not closed. So H' is the only copy of H that e_1 and e_2 belong to and hence $e_1 \equiv_H e_2$. This is a contradiction, as this pair of edges would have been removed in line 6 of Algorithm 1 (noting that the procedure in Lines 8 to 11 does not change the collection of H-copies in $\hat{G}$). Hence $\hat{G}$ is indeed H-closed and thus, by assumption, every H-block B in G satisfies $m(B) \leq m_2(H)$.

Now for line 15, by Lemma 2.8 and our initial assumptions, colouring one block B_i does not influence the colouring of any copy of H which does not lie in B_i and all blocks B_i are edge-disjoint. It follows that for line 15 to succeed (and to prove the 0-statement of Question 1.5) it suffices to prove that all graphs G with $m(G) \leq m_2(H)$ satisfy $G \overset{\text{a-ram}}{\not\longrightarrow} H$. □

2.2 Anti-Ramsey threshold in a special case

We now wish to use Theorem 1.6 to find the threshold for the anti-Ramsey property for a class of graphs including sufficiently sparse regular graphs.

Lemma 2.10. *Let H be a strictly 2-balanced graph where $1 < m_2(H) < \frac{\delta(H)(\delta(H)+1)}{2\delta(H)+1}$. Every graph B such that $m(B) \leq m_2(H)$ satisfies $B \overset{\text{a-ram}}{\not\longrightarrow} H$.*

Proof. Suppose for a contradiction that there exists some graph B with $m(B) \leq m_2(H)$ and $B \xrightarrow{\text{a-ram}} H$. Let B be such a graph with the minimum number of vertices. By minimality, every vertex $v \in B$ must be contained in a copy of H: if not, we could take a proper colouring of $B \setminus \{v\}$ that does not contain a rainbow copy of H and extend it to a proper colouring of B with no rainbow copy of H by colouring all the edges from v with distinct new colours. In particular, $\delta(B) \geq \delta(H)$.

Claim 2.11. *B contains two adjacent vertices of degree $\delta(H)$.*

Proof of claim. Suppose not, for a contradiction. Let X be the set of vertices of degree $\delta(H)$; by assumption, X is a (possibly empty) independent set. We have

$$2e(B) \geq |X|\delta(H) + (v(B) - |X|)(\delta(H) + 1) = v(B)(\delta(H) + 1) - |X|$$

and so $|X| \geq v(B)(\delta(H) + 1) - 2e(B)$. Moreover, as X is independent,

$$e(B) \geq |X|\delta(H) \geq \big(v(B)(\delta(H) + 1) - 2e(B)\big)\delta(H).$$

Rearranging, we see that

$$m_2(H) \geq m(B) \geq \frac{e(B)}{v(B)} \geq \frac{\delta(H)(\delta(H) + 1)}{2\delta(H) + 1},$$

which contradicts that $m_2(H) < \frac{\delta(H)(\delta(H)+1)}{2\delta(H)+1}$. $\square$

Therefore, we can take u, v to be two adjacent vertices of degree $\delta(H)$. As B is minimal, there is a proper colouring φ of $B \setminus \{u, v\}$ that does not contain a rainbow copy of H. As $B \xrightarrow{\text{a-ram}} H$, any extension of φ to a proper colouring φ' of B contains a rainbow copy of H.

The only copies of H that could be rainbow in such a φ' must contain $\{u, v\} \cup N(u) \cup N(v)$ (as $d(u) = d(v) = \delta(H)$). So if there exist $x \neq y$ such that $ux, vy \in E(B)$, then using colours distinct from φ, colouring ux and vy the same and giving every other edge a distinct colour will yield φ' with no rainbow copy of H. Therefore it must be the case that $N(u) \setminus v = N(v) \setminus u = \{w\}$, for some vertex w. However, by Fact 2.1 and the conditions on H, we have that H is 2-connected, and thus we must have that $H = K_3$. However, K_3 does not satisfy the condition that $m_2(H) < \frac{\delta(H)(\delta(H)+1)}{2\delta(H)+1}$, giving our desired contradiction. $\square$

Theorem 1.6 and Lemma 2.10 immediately imply the following.

Theorem 2.12. *Let H be a strictly 2-balanced graph where $1 < m_2(H) < \frac{\delta(H)(\delta(H)+1)}{2\delta(H)+1}$. There exists a constant c such that if $p \leq cn^{-1/m_2(H)}$ then*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{a-ram}} H] = 0.$$

As a corollary, we get that every relatively sparse, strictly 2-balanced regular graph satisfies the 0-statement in Question 1.1.

Corollary 2.13. *Let H be a d-regular strictly 2-balanced graph with $v(H) \geq 4d$. There exists a constant c such that if $p \leq cn^{-1/m_2(H)}$ then*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{a-ram}} H] = 0.$$

Proof. Since H is d-regular, $e(H) = dv(H)/2$. Then

$$m_2(H) = \frac{e(H)-1}{v(H)-2} = \frac{d}{2} + \frac{d-1}{v(H)-2}.$$

Rearranging, we find that $m_2(H) < \frac{d}{2} + \frac{d}{4d+2} = \frac{d(d+1)}{2d+1}$ if $v(H) > 4d - \frac{2}{d}$. Thus we can apply Theorem 2.12 to deduce the result. $\square$

As a concrete example of a family of graphs H satisfying the conditions of Corollary 2.13, consider for $k \in \mathbb{N}$ the family of k-blow-ups of cycles of length at least 8.

3 Constrained Ramsey

In this section we will prove the 0-statements of Theorems 1.3 and 1.4, regarding the constrained Ramsey problem for $(K_{1,k}, H)$, where $k \geq 3$. (Recall that the 1-statement of both theorems follows from Theorem 1.2.) The proof of the 0-statements splits into two parts, according to whether H is a triangle or not. [2]

[2] Technically, the two parts depend on whether there is a strictly 2-balanced graph $J \neq K_3$ with $m_2(J) = m_2(H) = 2$ contained in H.

3.1 Threshold when H is not a triangle

In this subsection we prove the following lemma, which implies the 0-statement of Theorem 1.3 in the case where H is not a triangle. Recall that[3] we may assume H is strictly 2-balanced.

Lemma 3.1. *Let H be a strictly 2-balanced graph such that $m_2(H) > 1$ and $H \neq K_3$. There exists a constant $c > 0$ such that if $p \leq cn^{-1/m_2(H)}$ then*

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \xrightarrow{\text{c-ram}} (K_{1,3}, H)] = 0.$$

Note that the same conclusion with $K_{1,3}$ replaced with $K_{1,k}$ for any fixed integer $k \geq 3$ follows trivially from Lemma 3.1. We will make use of the following result from [14]. Note that the original result was proven in more general terms regarding so-called 2-bounded colourings (these are colourings where each colour is used on at most two edges); here we will only need the weaker constrained Ramsey property for $(K_{1,3}, H)$.

Lemma 3.2 ([14, Lemma 26])**.** *Let H be a strictly 2-balanced graph on at least four vertices with $m_2(H) > 1$ such that $H \neq C_4$. Then for any graph B such that $m(B) \leq m_2(H)$ it holds that $B \not\xrightarrow{\text{c-ram}} (K_{1,3}, H)$.*

For $H = C_4$, since we are working with the $(K_{1,3}, H)$ constrained Ramsey property and not 2-bounded colourings, we can prove a similar lemma.[4] Its proof uses Vizing's theorem, that is, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Lemma 3.3. *Every graph B such that $m(B) \leq m_2(C_4)$ satisfies $B \not\xrightarrow{\text{c-ram}} (K_{1,3}, C_4)$.*

Proof. Assume for a contradiction that B is a minimal graph on n vertices such that $m(B) \leq m_2(C_4) = 3/2$ and $B \xrightarrow{\text{c-ram}} (K_{1,3}, C_4)$.

Suppose that there is a vertex v in B with degree at most 2. By the minimality assumption on B, colour $B \setminus \{v\}$ to witness that $B \setminus \{v\} \not\xrightarrow{\text{c-ram}} (K_{1,3}, C_4)$ and colour the (at most) two edges incident to v with the same (new) colour. This yields a colouring of B with no rainbow copy of C_4 and no monochromatic copy of $K_{1,3}$, contradicting our choice of B.

Thus B has minimum degree at least 3, so $e(B) \geq 3n/2$. Since $e(B) \leq m(B)n \leq 3n/2$, we find that $e(B) = 3n/2$, implying that $\Delta(B) = 3$.

Hence by Vizing's theorem, $\chi'(G) \in \{3, 4\}$. If $\chi'(G) = 3$, then we have a proper edge colouring on 3 colours, and thus trivially no rainbow copies of C_4 or monochromatic copies of $K_{1,3}$. If $\chi'(G) = 4$, set any edges of colour 4 to be colour 3. Then we have an edge colouring on 3 colours and thus no rainbow copies of C_4. Moreover, we cannot have created a monochromatic copy of $K_{1,3}$ as the original 4 edge-colouring of G was proper.

$\square$

We are now in a position to prove Lemma 3.1. Our proof will largely follow that of [14, Theorem 5].

[3] See just after the statement of Theorem 1.6 on page 5.

[4] Note that there does exist a graph B such that $m(B) = m_2(C_4)$ and any 2-bounded colouring of B contains a rainbow C_4 (see the proof of [14, Lemma 27]).

1: **procedure** Rainbow-colour-constrained($G = (V, E)$)
2: $\quad \hat{G} \leftarrow G$
3: $\quad$ col $\leftarrow 0$
4: $\quad$ **while** $\exists$ distinct $e_1, e_2 \in E(\hat{G}) : e_1 \equiv_H e_2$ in $\hat{G}$ **do**
5: $\qquad$ colour e_1, e_2 with col
6: $\qquad \hat{G} \leftarrow \hat{G} \setminus \{e_1, e_2\}$ and col $\leftarrow$ col $+1$
7: $\quad$ **end while**
8: $\quad$ **while** $\exists e \in E(\hat{G})$: e does not belong to a copy of H **do**
9: $\qquad$ colour e with col
10: $\qquad \hat{G} \leftarrow \hat{G} \setminus \{e\}$ and col $\leftarrow$ col $+1$
11: $\quad$ **end while**
12: **end procedure**
13: Remove isolated vertices in $\hat{G}$.
14: $\{B_1, \ldots, B_k\} \leftarrow H$-blocks obtained by applying Lemma 2.8 on $\hat{G}$.
15: Colour (properly) each B_j without a rainbow copy of H using distinct sets of colours (cf. text why the last two lines are possible).

Figure 2: The algorithm Rainbow-colour-constrained.

Proof of Lemma 3.1. Let c be a constant given by Corollary 2.7 when applied to H. Let $G \sim G(n,p)$ for $p \leq cn^{-1/m_2(H)}$. Then w.h.p. every H-block B in G satisfies $m(B) \leq m_2(H)$. Assume this is indeed the case. We use Algorithm Rainbow-colour-constrained (above) to find a colouring containing no monochromatic copy of $K_{1,3}$ and no rainbow copy of H. Notice that the only difference between Algorithms Rainbow-colour and Rainbow-colour-constrained is in the condition in line 4. We may proceed as in the proof of Theorem 1.6, noting that $\hat{G}$ is H-closed by construction. Indeed, every edge and vertex of $\hat{G}$ is contained in a copy of H by lines 8 and 13, and line 4 implies that every copy of H must contain at least three closed edges, namely the (at least three) edges that are in at least two copies of H (because otherwise, using $e(H) \geq 4$, there is a copy H' of H with at least two edges that do not appear in other copies of H, contradicting line 4). Now $\hat{G}$ being H-closed means line 14 succeeds, and so by applying Lemmas 3.2 and 3.3 we can colour each H-block B_i as desired. We do so by using disjoint sets of colours (that also avoid the colours already used by the algorithm) for different blocks, and hence w.h.p. Algorithm Rainbow-colour-constrained finds the desired colouring of G. $\square$

3.2 Reduction to colouring statement when H is a triangle

We cannot prove the 0-statements of Theorems 1.3 for $H = K_3$ and Theorem 1.4 using a similar strategy to the proof of Theorem 1.6. Indeed, one cannot apply Proposition 2.9 since it is only applicable to graphs with at least five edges. Furthermore, every edge of a K_3-block is closed, i.e. contained in at least two K_3-copies, a property too strong to aim for in a proof.

Instead of considering K_3-blocks, we will prove the 0-statements of Theorems 1.3 for $H = K_3$ and Theorem 1.4 by considering a broader class of structures we will call *triangle-connected* graphs.

We need the following definition. A *triangle sequence* $\bar{T}$ is a sequence $T_0, \ldots, T_\ell$ such that:

- T_0 is a triangle.

- For $i \in [\ell]$, there is a vertex v_i in $V(T) \setminus V(T_{i-1})$ and an edge e_i in T_{i-1} that together form a triangle in T, such that $V(T_i) = V(T_{i-1}) \cup \{v_i\}$ and $E(T_i)$ is the union of $E(T_{i-1})$ with the two edges between v_i and e_i, and possibly other edges of T between v_i and $V(T_{i-1})$.

Further, we say that i is a *regular step* if $\deg(v_i, T_{i-1}) = 2$. We say that a graph T is *triangle-connected* if T has no isolated vertices, every edge in T is in a triangle, and the 3-uniform hypergraph on $V(T)$ whose edges are the triangles in T is tightly-connected[5].

Observe also that if T is triangle-connected, then there is a triangle sequence $T_0, \dots, T_\ell$ with $T_\ell = T$ (start with an arbitrary triangle T_0 and always add all edges between v_i to T_{i-1} for some v_i which is in a triangle with an edge in T_{i-1}; this process will only stop when we have exhausted all vertices of T, by triangle-connectivity). In the next lemma we show that if p is sufficiently smaller than $n^{-1/2}$ then, w.h.p., every triangle-connected subgraph of $G(n,p)$ has low density. Using this, in order to prove the 0-statements of Theorem 1.3 for H being a triangle and Theorem 1.4, it suffices to show that every triangle-connected graph with low density can be coloured appropriately (see Lemmas 3.8 and 3.9).

Lemma 3.4.

- *If $p \le n^{-1/2}/20$ then, w.h.p., every triangle-connected subgraph T of $G(n,p)$ satisfies $e(T) \le 2v(T)$.*
- *If $p = o(n^{-1/2})$ then, w.h.p., every triangle-connected subgraph T of $G(n,p)$ satisfies $e(T) < 2v(T)$.*

Before we prove Lemma 3.4, we develop our understanding of triangle sequences.

For a triangle sequence $\bar{T} = (T_0, \dots, T_\ell)$, define

$$r(\bar{T}) = \sum_{i \in [\ell]} \big(\deg(v_i, T_{i-1}) - 2 \big), \tag{1}$$

and notice that $e(T_\ell) = 3 + 2\ell + r(\bar{T})$ and $v(T_\ell) = \ell + 3$.

We therefore have

★ If T is triangle-connected and $e(T) \ge 2v(T)$ then there is a triangle sequence $\bar{T} = (T_0, \dots, T_\ell)$ with $T_\ell = T$ and $r(\bar{T}) \ge 3$. If $e(T) > 2v(T)$ then there is a triangle sequence $\bar{T} = (T_0, \dots, T_\ell)$ with $T_\ell = T$ and $r(\bar{T}) \ge 4$.

Let ρ be a positive integer. Say that a triangle sequence $\bar{T} = (T_0, \dots, T_\ell)$ is *ρ-minimal* if $r(\bar{T}) \ge \rho$ and there is no triangle sequence $\bar{T}' = (T'_0, \dots, T'_{\ell'})$ with $r(\bar{T}') \ge \rho$ and $T'_{\ell'} \subsetneq T_\ell$. In the next claim we prove basic properties of ρ-minimal triangle sequences.

Proposition 3.5. *Let $\bar{T} = (T_0, \dots, T_\ell)$ be a ρ-minimal triangle sequence. Then*

(a) $r(\bar{T}) = \rho$.

[5] A hypergraph is tightly-connected if it can be obtained by starting with a hyperedge and adding hyperedges one by one, such that every added hyperedge intersects with one of the previous hyperedges in 2 vertices.

(b) There are at most ρ irregular steps.

(c) For all but at most $\rho(\rho+2)$ regular steps i, there is an edge f_i between v_i and e_i, such that f_i forms a triangle together with v_j, for some $j > i$.

Proof. For (a), notice that $r(\bar{T}) = r(T_0, \ldots, T_{\ell-1}) + \deg(v_\ell, T_{\ell-1}) - 2$. By minimality, we have $r(T_0, \ldots, T_{\ell-1}) < \rho$, showing that $\deg(v_\ell, T_{\ell-1}) \geq 3$. If $r(\bar{T}) > \rho$, then remove an edge between v_ℓ and $V(T_{\ell-1}) \setminus V(e_\ell)$ to obtain a graph T'_ℓ, and notice that $T_0, \ldots, T_{\ell-1}, T'_\ell$ is a triangle sequence with $r(T_0, \ldots, T_{\ell-1}, T'_\ell) \geq \rho$, contradicting minimality.

For (b), notice that every irregular step contributes at least 1 to $r(\bar{T})$, so by (a) there are at most ρ irregular steps.

Notice that the number of edges added during irregular steps is at most $\rho(\rho+2)$, using (b) to see that there are at most ρ irregular steps, and using (a) to see that at most $\rho+2$ edges are added during any single irregular step. Thus there are at most $\rho(\rho+2)$ regular steps i such that v_i is contained in an edge added during an irregular step.

Finally, consider a regular step i such that v_i is not contained in an edge added during an irregular step.

Claim 3.6. *Let i be a regular step such that v_i is not contained in an edge added during any irregular step. Then there exists $j > i$ such that $v_i \in e_j$.*

Proof. Suppose for contradiction that this fails for some regular step i. Define a new triangle sequence $\bar{T}' = (T'_0, \ldots, T'_{\ell-1})$, where

$$T'_j = \begin{cases} T_j & \text{if } j < i, \\ T_{j+1} \setminus \{v_i\} & \text{otherwise.} \end{cases}$$

Notice that $\bar{T}'$ is indeed a triangle sequence (using that no edge e_j with $j > i$ in the original sequence involved the vertex v_i). Moreover, $r(\bar{T}') = r(\bar{T})$, because i is a regular step in the original sequence and is not contained in an edge added during any irregular step. That is, it contributes 0 to the sum defining $r(\bar{T})$ (see (1)). The existence of the new sequence $\bar{T}'$ contradicts the minimality of $\bar{T}$, proving the claim. □

By Claim 3.6, let $j > i$ be minimal such that $v_i \in e_j$. Then, right before step j, the vertex v_i is incident only to the two edges joining it to e_i. This implies the existence of an edge f_i as in (c). □

In a ρ-minimal triangle sequence $T_0, \ldots, T_\ell$, for every regular step i let f_i be an edge as in Proposition 3.5 (c), if it exists. Define $J(i) := \{j > i : v_j \text{ and } f_i \text{ forms a triangle}\}$, with $J(i) := \emptyset$ if such f_i does not exist or if i is an irregular step. Then by Claim 3.5 (c) we have $J(i) \neq \emptyset$ for all but $\rho(\rho+2)$ regular steps i. In the following claim we show that for almost all regular i we have that $|J(i)| = 1$, and moreover for almost all i, the edge e_i is equal to f_j for some $j < i$ satisfying $|J(j)| = 1$. These properties will be used to count the number of non-isomorphic ρ-minimal triangle sequences.

Proposition 3.7. *Let $T_0, \ldots, T_\ell$ be a ρ-minimal triangle sequence, with $\rho \leq 4$. Then, the following holds.*

(a) For every $i \in [\ell]$, for all but at most 200 values of $i' < i$, it holds that $|J(i')| = 1$ and the single element j in $J(i')$ satisfies $j < i$.

(b) For all but at most 260 regular steps i, we have $e_i = f_j$ for some $j < i$ such that $|J(j)| = 1$.

Note that for clarity of exposition, we make no attempt to optimise the constants in the lemmas of this section.

Proof. Let x be the number of regular steps i' with $|J(i')| \geq 2$. Then the number of triangles in T_ℓ is at least

$$\sum_{i \in [\ell]} |J(i)| \geq \#\{\text{regular steps}\} - \rho(\rho+2) + x \geq \ell - \rho - \rho(\rho+2) + x \geq \ell - 28 + x.$$

For the first inequality we used that every pair (i, j) with $j \in J(i)$ gives rise to the triangle $f_i \cup \{v_j\}$, and these triangles are distinct for different pairs, and moreover we used that $J(i) \neq \emptyset$ for all but at most $\rho(\rho+2)$ regular steps i by Proposition 3.5 (c). For the second inequality we used that there are at most ρ irregular steps, by Proposition 3.5 (b). The number of triangles is also at most

$$1 + \ell + \rho \cdot \left(\binom{6}{2} - 1\right) \leq \ell + 57,$$

because regular steps (and step 0) give rise to exactly one new triangle each, and irregular steps, of which there are at most ρ, yield at most $\binom{6}{2}$ new triangles each. Altogether, we get that $x \leq 85$.

Fix $i \in [\ell]$. Let y be the number of regular steps $i' < i$ with $|J(i')| = 1$ for which the single element $j \in J(i')$ satisfies $j \geq i$. The number of triangles in T_ℓ which are not in T_{i-1} is at least

$$\sum_{i'} \left|J(i') \cap [i, \ell]\right| \geq y + \sum_{i' \geq i} |J(i')| \geq y + \ell - i - \rho - \rho(\rho+2) \geq \ell - i - 28 + y.$$

Here we used that every pair (i', j) with $j \geq i$ and $j \in J(i')$ gives rise to the triangle formed by $f_{i'} \cup \{v_j\}$ which is not in T_{i-1}, and we also used that $|J(i')| \geq 1$ for all but at most $\rho(\rho+2)$ regular steps i'. The number of triangles in T_ℓ which are not in T_{i-1} is also at most

$$\ell - i + 1 + \rho \cdot \left(\binom{6}{2} - 1\right) \leq \ell - i + 57,$$

where again we used that regular steps gives rise to a unique new triangle, and irregular steps yield at most $\binom{6}{2}$ new triangles. It follows that $y \leq 85$.

It is now easy to conclude (a). Indeed, every $i' < i$ satisfies one of the following: i' is irregular; $|J(i')| \geq 2$; $|J(i')| = 1$ and the unique element j in $J(i')$ satisfies $j \geq i$; i' is regular and $J(i') = \emptyset$; or $|J(i')| = 1$ and the single element j in $J(i')$ satisfies $j < i$. There are at most four irregular steps, at most 85 steps of the second type, at most 85 steps of the third type, at most $\rho(\rho+2) \leq 24$ steps of the fourth type, and so there are at least $i - (4 + 85 + 85 + 24) = i - 198 \geq i - 200$ steps of the fifth type, as claimed in (a).

For (b), notice that the number of regular i, for which there is $j < i$ with $e_i = f_j$ and $|J(j)| = 1$, is at least the number of triangles of the form $f_j \cup \{v_i\}$ with $|J(j)| = 1$ and $i \in J(j)$, minus the number of triangles of form $e \cup \{v_i\}$ with i irregular and $e \in E(T_{i-1})$. The first quantity is at least $(\ell-1)-200$, by (a) (with $i=\ell$). The second quantity is at most $\rho\cdot\binom{6}{2}\le 60$. It follows that the number of i as in (b) is at least $(\ell-200)-60\ge \ell-260$. $\square$

We are now in a position to prove Lemma 3.4.

Proof of Lemma 3.4. Let $m_{\ell,\rho}$ be the number of non-isomorphic ρ-minimal triangle sequences $T_0,\dots,T_\ell$. Then, for $\ell\ge 1$,

$$\begin{aligned} m_{\ell,\rho} &\le \binom{\ell}{\le 4}\cdot\binom{\ell}{\le 260}\cdot\binom{\ell+3}{\le 6}^{\rho}\cdot(2\ell+7)^{260}\cdot 200^{\ell} \\ &\le 2^{1+1+4+2\cdot 6\cdot 4+4\cdot 260}\cdot \ell^{4+260+24+260}\cdot 200^{\ell}\le 2^{1100}\cdot \ell^{600}\cdot 200^{\ell}. \end{aligned}$$

(Here $\binom{\ell}{\le i}$ stands for $\binom{\ell}{0}+\ldots+\binom{\ell}{i}$.) Indeed, in the first inequality, the term $\binom{\ell}{\le 4}$ bounds the number of ways to choose which steps are irregular, the term $\binom{\ell}{\le 260}$ bounds the number of ways to choose which regular steps i do not satisfy $e_i=f_j$ for some $j<i$ with $|J(j)|=1$ (using Proposition 3.7 (b)), the term $\binom{\ell+3}{\le 6}^{\rho}$ bounds the number of ways to choose the neighbours of v_i in T_{i-1} for irregular i, the term $(2\ell+7)^{260}$ bounds the number of ways to choose the edge e_i for regular i for which it is not the case that $e_i=f_j$ with $|J(j)|=1$ (using that $e(T_\ell)=3+2\ell+r(\bar{T})\le 2\ell+7$), and finally the term 200^{ℓ} bounds the number of ways to choose e_i for all other regular i (using Proposition 3.7 (a)). For the inequality we used $\binom{\ell}{\le i}\le \ell^i+1\le 2\ell^i$, $\ell+3\le 4\ell$, and $2\ell+7\le 16\ell$ for $\ell\ge 1$.

We now complete the proof of the lemma. First, suppose that $p=o(n^{-1/2})$. Then the expected number of 3-minimal triangle sequences $T_0,\dots,T_\ell$ with $T_\ell\subseteq G(n,p)$ is at most

$$\sum_{\ell\ge 1} m_{\ell,3}\cdot n^{\ell+3}\cdot p^{2\ell+6}\le \sum_{\ell\ge 1} 2^{1100}\cdot \ell^{600}\cdot\left(200np^2\right)^{\ell+3}=o(1).$$

Indeed, for the first inequality we used that $e(T_\ell)=3+2\ell+r(\bar{T})=2\ell+6$. Thus, w.h.p. $G(n,p)$ has no 3-minimal triangle sequences, implying that there are no triangle sequences $\bar{T}$ with $r(\bar{T})\ge 3$, which in turn shows that there are no subgraphs T which are triangle-connected and satisfy $e(T)\ge 2v(T)$ by ★.

Next, suppose that $p\le n^{-1/2}/20$. Then the expected number of 4-minimal triangle sequences $T_0,\dots,T_\ell$ with $T_\ell\subseteq G(n,p)$ is at most

$$\begin{aligned} \sum_{\ell\ge 1} m_{\ell,4}\cdot n^{\ell+3}\cdot p^{2\ell+7} &\le p\cdot\sum_{\ell\ge 1} 2^{1100}\cdot \ell^{600}\cdot\left(200np^2\right)^{\ell+3} \\ &\le n^{-1/2}\cdot\sum_{\ell\ge 1} 2^{1100}\cdot \ell^{600}\cdot\left(\frac{200}{400}\right)^{\ell+3}=o(1). \end{aligned}$$

As before, we conclude that, w.h.p., $G(n,p)$ has no 4-minimal triangle sequences, showing that there are no triangle sequences $\bar{T}$ with $r(\bar{T})\ge 4$, which in turn shows that there are no subgraphs T which are triangle-connected and satisfy $e(T)>2v(T)$ by ★. $\square$

3.3 Proof of colouring statements

Armed with Lemma 3.4, it suffices to prove the following two colouring statements to conclude the respective 0-statements of Theorem 1.3 (for $H = K_3$) and Theorem 1.4.

Lemma 3.8. *For any triangle-connected graph T with $e(T) \leq 2v(T)$ it holds that $T \overset{\text{c−ram}}{\not\longrightarrow} (K_{1,4}, K_3)$.*

Lemma 3.9. *For any triangle-connected graph T with $e(T) < 2v(T)$ it holds that $T \overset{\text{c−ram}}{\not\longrightarrow} (K_{1,3}, K_3)$.*

Proof of Lemma 3.8. Let B be a triangle-connected graph with $e(B) \leq 2v(B)$ and let $\bar{T} = (T_0, \dots, T_\ell)$ be a triangle sequence such that $T_\ell = B$. Write $\rho = r(\bar{T})$. Then $\rho \leq 3$, as $e(B) \leq 2v(B)$, $e(B) = 3 + 2\ell + \rho$ and $v(B) = \ell + 3$. Assume that $\bar{T}$ is chosen so that the first irregular step (if one exists) appears as early as possible, and similarly that each subsequent irregular step (if any exist) appears as early as possible given this. We now construct a sequence $\phi_0, \dots, \phi_\ell$ of partial colourings of $T_0, \dots, T_\ell$, such that: ϕ_i extends ϕ_{i-1} for $i \in [\ell]$ and uses the colour set $[0, i]$; every triangle in T_i has at least two edges coloured with the same colour in ϕ_i; and there is no monochromatic $K_{1,4}$. Note that this suffices to prove the lemma, as we may finish by colouring each edge in B that is left uncoloured by ϕ_ℓ with a unique new colour.

- For $i = 0$, the graph T_0 is a triangle. Colour two edges of T_0 by 0, and leave the remaining edge uncoloured.
- For $i \geq 1$, let U be the set of vertices u' in T_{i-1} such that there exists an edge $e \in E(T_{i-1})$ containing u' and forming a triangle with v_i (so $\delta(T_{i-1}[U]) \geq 1$).
  - If $|U| \leq 3$ and there are at most three edges from v_i to T_{i-1} then colour these edges with colour i. If $|U| \leq 3$ and $d(v_i, T_{i-1}) \geq 4$, colour all edges from v_i to U by colour i (leaving the other edges from v_i to T_{i-1} uncoloured).
  - If $|U| \geq 4$, define u to be the last vertex in U added to the sequence $\bar{T}$.

    We claim that $|U| = 4$. Indeed, otherwise $|U| = 5$ (as $\rho \leq 3$), and this is the first (and only) irregular step. Then u was added in a regular step, so it sends at most two edges to U. This and $\delta(T_{i-1}[U]) \geq 1$ imply that $U \setminus \{u\}$ spans an edge. But then we could have added v_i to the graph right before adding u (and joining v_i to $U \setminus \{u\}$), thus having the first irregular step appear earlier, a contradiction.

    It remains to consider the case $|U| = 4$. We claim that u was added in an irregular step. Indeed, notice that by the assumption on the sequence, we may assume that v_i was added right after u (the only other possibility is that there was another irregular step right between u and v_i, but then we can swap the order of these two irregular steps). Assuming u was added in a regular step, u sends at most two edges to U. This shows that $U \setminus \{u\}$ spans an edge, and thus v_i could have been added before u in an irregular step, a contradiction.

    So this step and the step j in which u was added are the only irregular steps, and u was added along with exactly three edges, which are coloured j, and no other edges are coloured j. Colour the edges between v_i and $U \setminus \{u\}$ by j. Then the colour class of j has maximum degree at most 3, and every triangle touching v_i has two edges coloured j. □

We now prove Lemma 3.9. The proof is similar to Lemma 3.8 but more involved.

Proof of Lemma 3.9. Let B be a triangle-connected graph with $e(B) < 2v(B)$. Here we use a variant of a triangle sequence, where we allow T_0 to be either a triangle, a K_4, or a K_5^- (a K_5 minus one edge).

Let $\bar{T} = (T_0, \ldots, T_\ell)$ be such a variant of a triangle sequence with $T_\ell = B$ with the following properties, where $\rho = r(\bar{T})$ is defined as in (1).

- If possible, T_0 is a K_5^-. In this case we have $\rho = 0$, i.e. there are no irregular steps.
- Otherwise, if possible, T_0 is a K_4. In this case we have $\rho \leq 1$, so there is at most one irregular step i, and then v_i has exactly three neighbours in T_{i-1}.
- Otherwise, B is K_4-free. If possible, avoid having irregular steps i with v_i having four neighbours in T_{i-1}. In this case $\rho \leq 2$, so there are at most two irregular steps.

We now define, for each $i \in [0, \ell]$, an edge-colouring ϕ_i of T_i, using colours from $[0, i] \times [0, 2]$ such that: the colourings ϕ_i are nested (i.e. ϕ_{i+1} extends ϕ_i for every $i \in [0, \ell - 1]$), and there are no rainbow triangles or monochromatic copies of $K_{1,3}$.

- Suppose that $i = 0$.
  - If T_0 is a triangle, colour two of its edges $(0, 0)$ and the third $(0, 1)$.
  - If T_0 is a K_4, colour a 4-cycle in it $(0, 0)$, colour one of the two other edges $(0, 1)$, and colour the final edge $(0, 2)$.
  - If T_0 is a K_5^-, colour a 5-cycle in it $(0, 0)$ and colour all other edges $(0, 1)$.
- Now suppose $i \geq 1$.
  - If i is a regular step, colour the two edges from v_i to T_{i-1} by $(i, 0)$ (and colour T_{i-1} according to ϕ_{i-1}).
  - If i is an irregular step with $\deg(v_i, T_{i-1}) = 3$, denote its three neighbours in T_{i-1} by u_1, u_2, u_3, and proceed according to the following cases.
    * If $\{u_1, u_2, u_3\}$ spans a single edge, then without loss of generality u_3 is isolated. Colour v_iu_1 and v_iu_2 by $(i, 0)$ and colour v_iu_3 by $(i, 1)$.
    * If $\{u_1, u_2, u_3\}$ forms a triangle, then without loss of generality u_1u_2 and u_2u_3 are coloured by the same colour c_1 (as there are no rainbow triangles).

      We claim that there are no other edges coloured c_1 in ϕ_{i-1}. Indeed, the set $\{v_i, u_1, u_2, u_3\}$ forms a K_4, so T_0 is not a triangle. Further, since i is an irregular step, T_0 is not a copy of K_5^-. Thus T_0 is a copy of K_4, and this is the first and only irregular step. This shows, by the procedures for $T_0 = K_4$ and $i = 0$, and for regular steps, that all colour classes, except for $(0, 0)$'s colour class, are paths of length at most 2. If $c_1 = (0, 0)$, then $u_1, u_2, u_3 \in V(T_0)$, but then $V(T_0) \cup \{v_i\}$ induces a K_5^-, a contradiction. Thus $c_1 \neq (0, 0)$, so c_1's colour class must be a path of length at most 2, and indeed there are no edges coloured c_1 other than u_1u_2 and u_2u_3.

      Colour v_iu_1 and v_iu_3 by c_1, and colour v_iu_2 by $(i, 0)$.

* Suppose that $u_1u_2u_3$ is a path coloured c_1, u_1u_3 is a non-edge, and this is the first irregular step.

  We claim that there are no other edges coloured c_1. Indeed, T_0 is either a triangle or a K_4 (since i is an irregular step), and if there is a colour with more than two edges then T_0 is a K_4, this colour is $(0,0)$, and vertices touching $(0,0)$ form a clique, so $c_1 \neq (0,0)$.

  Colour v_iu_1 by c_1, and colour v_iu_2 and v_iu_3 by $(i,0)$.

* Suppose that u_1u_2 and u_2u_3 are edges, coloured by distinct colours c_1 and c_2, u_1u_3 is a non-edge, and this is the first irregular step.

  Without loss of generality, the edges coloured c_1 form a path of length at most 2 (because so far we have at most one colour class which is not a path of length at most 2). Thus for some $j \in [2]$ the vertex u_j is not incident to other edges coloured c_1. If $j = 1$ then colour v_iu_1 by c_1 and colour v_iu_2 and v_iu_3 by $(i,0)$. If instead $j = 2$ then colour v_iu_1 by $(i,0)$ and colour v_iu_2 and v_iu_3 by c_1. (See Figure 3.)

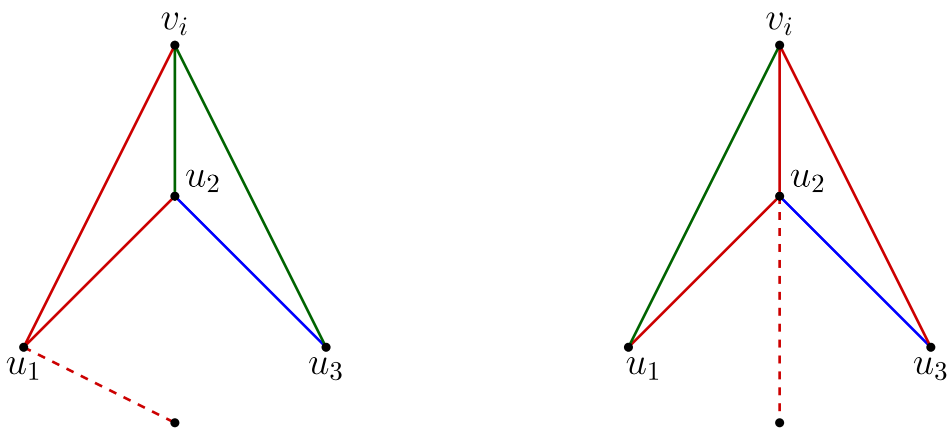

Figure 3: Colourings for $j = 1$ (left) and $j = 2$ (right) in the 4th subcase when $i \geq 1$ and i is an irregular step with $\deg(v_i, T_{i-1}) = 3$. Here $c_1 =$ red, $c_2 =$ blue and $(i,0) =$ green.

* Next, suppose that $u_1u_2u_3$ is a path coloured c_1, u_1u_3 is a non-edge, and this is the second irregular step.

  We claim that for some $j \in \{1,3\}$, the vertex u_j is not incident with other edges coloured c_1. Indeed, notice that in this case T_0 is a triangle and, from cases already considered, all colour classes are paths of length at most 4. This means that if u_1 and u_3 are both incident to other edges coloured c_1 then there are vertices u_0, u_4 such that $u_0 \dots u_4$ is a path of length 4 coloured c_1, but then u_1u_3 is an edge, a contradiction. Indeed, the only case where a path of length 4 is formed is in the previous case, and then the second and fourth vertices are adjacent[6].

[6] These are the vertices v_i and u_1 in the $j = 2$ subcase of the previous case (see the right image of Figure 3).

Without loss of generality, $j = 1$. Colour v_iu_1 by c_1, and colour v_iu_2 and v_iu_3 by $(i, 0)$.

* Finally, we may assume that u_1u_2 and u_2u_3 are edges coloured by distinct colours c_1 and c_2, u_1u_3 is a non-edge, and this is the second irregular step.

  Again, we may assume that the edges coloured c_1 form a path of length at most 2 (in this case T_0 is a triangle, so all colour classes, except for at most one[7], are paths of length at most 2).

  As before, if u_1 is not incident with other edges coloured c_1, then colour v_iu_1 by c_1 and v_iu_2 and v_iu_3 by $(i, 0)$. Otherwise, colour v_iu_1 by $(i, 0)$ and colour v_iu_2 and v_iu_3 by c_1.

- Now suppose i is an irregular step with $\deg(v_i, T_{i-1}) = 4$. In particular, this is the first and only irregular step and T_0 is a triangle. Let u_1, u_2, u_3, u_4 be the neighbours of v_i, chosen so that u_4 is the last vertex to be added to the sequence.

  Then $\{u_1, u_2, u_3\}$ is an independent set, because otherwise we could have added v_i to the sequence before u_4, thereby avoiding having irregular steps during which four edges are added. Note that u_4 has a neighbour in $\{u_1, u_2, u_3\}$, otherwise this is not a legal step of a triangle sequence. Assume without loss of generality u_3u_4 is an edge. Then u_4u_1 and u_4u_2 are non-edges. Indeed, suppose to the contrary that u_j is a neighbour of u_4 for $j \in \{1, 2\}$. This implies that u_ju_3 is an edge, because u_4 was added in a regular step, a contradiction.

  Colour v_iu_1 and v_iu_2 by $(i, 0)$ and v_iu_3 and v_iu_4 by $(i, 1)$. □

3.4 Tightness when $k = 3$ for Lemma 3.9

Lemma 3.9 shows that if G is a graph satisfying $m(G) < 2$ then $G \overset{\text{c−ram}}{\not\longrightarrow} (K_{1,3}, K_3)$. This bound is tight, as the following lemma shows, noting that $m(G) = 2 = m_2(K_3)$ for the graph G shown in Figure 4.

Lemma 3.10. *Let G be a copy of K_6 with three edges forming a triangle removed. Then every edge–colouring of G contains either a monochromatic $K_{1,3}$ or a rainbow triangle.*

Proof. Label the vertices of G with $1, \ldots, 6$ such that 13, 35 and 15 are non-edges, and all other pairs of vertices are edges, as shown in Figure 4.

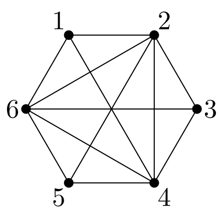

Figure 4: The graph G in the statement of Lemma 3.10

[7] This would be the colour c_1 from either the 3rd or 4th subcase.

Suppose for a contradiction that there exists a colouring of G containing no monochromatic $K_{1,3}$ and no rainbow triangle. Two of the edges of the triangle 246 must be assigned the same colour. By symmetry we can say without loss of generality that 46 and 26 are the same colour, say red.

As there is no monochromatic $K_{1,3}$, the edge 36 must receive a different colour, say blue. Triangles 346 and 236 are not rainbow so edges 23 and 34 must each be either red or blue. They cannot both be blue else there would be a blue $K_{1,3}$ centred on vertex 3, so at most one is blue. By symmetry we can say without loss of generality that 23 is red.

Consider now 25 and 56. These are both adjacent to two red edges at 2 and 6 respectively, so cannot be red. They must therefore be the same colour c, else triangle 256 is rainbow. Note that c may be blue.

Edge 45 cannot be colour c, else there would be a monochromatic $K_{1,3}$ centred on vertex 5. Therefore 45 must be red, else triangle 456 would be rainbow.

Edge 24 cannot be red, else there would be a red $K_{1,3}$ centred on vertex 2 (or 4). Therefore 24 must be colour c, else triangle 245 would be rainbow.

Edge 34 cannot be red, else there would be a red $K_{1,3}$ centred on vertex 4. Therefore 34 must be colour c, else triangle 234 would be rainbow.

Note that colour c must be equal to blue, else triangle 346 is rainbow.

To conclude, observe that none of edges 12, 14 or 16 may be red or blue without creating a monochromatic $K_{1,3}$. Triangle 124 is not rainbow, so edges 12 and 14 are the same colour. Moreover, triangle 146 is not rainbow, so edges 14 and 16 are the same colour. But then we have a monochromatic $K_{1,3}$ centred at 1, a contradiction. □

3.5 Proof of Theorems 1.3 and 1.4

We now combine the results from throughout this section to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. The 1-statement follows from Theorem 1.2.

First we consider the 0-statement when $k \geq 4$, $m_2(H) = 2$ and the only strictly 2-balanced graph with 2-density 2 contained in H is K_3. Let $p \leq n^{-1/2}/20$ (so $p \leq cn^{-1/m_2(H)}$ for $c = 1/20$). We wish to show that, w.h.p., $G = G(n,p)$ can be edge-coloured so that there are no rainbow triangles or monochromatic $K_{1,4}$'s (this would then imply the statement for all $k \geq 4$, noting that if there are no rainbow triangles then there are no rainbow copies of H). In fact, it suffices to prove this for every triangle-connected component[8] of G (namely a maximal triangle-connected subgraph of G), since we can use disjoint sets of colours for different components and colour each edge not contained in any triangle with its own unique colour. By Lemma 3.4 we may assume that every such component H satisfies $e(H) \leq 2v(H)$, and by Lemma 3.8, every such H can be coloured appropriately. This proves the 0-statement.

For the 0-statement when either $m_2(H) \neq 2$ or $m_2(H) = 2$ and H contains a strictly 2-balanced graph $J \neq K_3$ with $m_2(J) = 2$, note that it suffices to consider the case where H is strictly 2-balanced. If not, take a minimal subgraph $H' \subseteq H$ which is not a triangle such that $m_2(H) = d_2(H')$, so that H' is strictly 2-balanced. (Note that there exists such an H' which is

[8] Note that these 'components' are not necessarily vertex disjoint but are edge disjoint.

not a triangle by the conditions on H.) If a colouring of $G(n,p)$ contains no rainbow H' then it also contains no rainbow H. Also note that it suffices to take $k=3$ here, because avoiding a monochromatic $K_{1,3}$ also avoids a monochromatic $K_{1,k}$ for every $k \geq 3$. Since $m_2(H) > 1$ is equivalent to H not being a forest, Lemma 3.1 immediately implies the required 0-statement, noting that we may apply Lemma 3.1 since H is not a triangle. $\square$

For proving the final part of Theorem 1.4, we will make use of the following fact.

Claim 3.11. *Let F be a balanced graph. For any constant $c > 0$ there exists $\zeta = \zeta(c) > 0$ such that if $p = cn^{-1/m(F)}$ then $\lim_{n\to\infty} \mathbb{P}[F \subseteq G(n,p)] \geq \zeta$.*

While the proof of this claim is an elementary exercise in studying random graphs, we include its proof for completeness. We shall use Janson's inequality.

Theorem 3.12 (Janson's inequality [1, Theorem 8.1.1])**.** *Let Ω be a finite set, let X be a random subset of Ω obtained by including each element v in Ω with some probability p_v, independently. Let $\{A_i\}_{i\in I}$ be a collection of subsets of Ω. Define μ and Δ as follows.*

$$\mu := \sum_{i\in I} \mathbb{P}\left(A_i \subseteq X\right),$$

$$\Delta := \sum_{i\neq j \text{ and } A_i\cap A_j\neq\emptyset} \mathbb{P}\left(A_i \cup A_j \subseteq X\right).$$

Then

$$\mathbb{P}\left(A_i \not\subseteq X \text{ for every } i \in I\right) \leq e^{-\mu+\Delta/2}.$$

Proof of Claim 3.11. We aim to apply Janson's inequality (Theorem 3.12), with Ω as the set of unordered pairs of elements in $[n]$, $p_v := p = cn^{-1/m(F)}$, X as a copy of $G(n,p)$, and $\{A_i\}_{i\in I}$ the collection of copies of F in the complete graph on $[n]$. Note that we may assume c is sufficiently small as subgraph inclusion is a monotone property, i.e. for any $c' > c$, $\mathbb{P}\left(F \subseteq G(n, c'n^{-1/m(F)})\right) \geq \mathbb{P}\left(F \subseteq G(n, cn^{-1/m(F)})\right)$. Defining μ and Δ as in the theorem, we have

$$\mu = \Theta(n^{v(F)}p^{e(F)}) = \Theta(c^{e(F)})$$

since F is balanced. Let $f(i)$ be the minimum number of vertices in a strict subgraph of F with i edges. By definition of $m(F)$, we have $f(i) \geq i/m(F)$. Then for every copy F' of F in K_n, there are at most $O(n^{v(F)-f(i)})$ copies F'' of F where $|E(F') \cap E(F'')| = i$. Therefore, denoting by $\mathcal{F}$ the collection of all copies of F in K_n,

$$\begin{aligned}
\Delta &= \sum_{F'\in\mathcal{F}} \sum_{\substack{F''\in\mathcal{F}\setminus\{F'\}:\\ E(F')\cap E(F'')\neq\emptyset}} \mathbb{P}\left(F' \cup F'' \subseteq G(n,p)\right) \\
&= \sum_{F'\in\mathcal{F}} \mathbb{P}\left(F' \subseteq G(n,p)\right) \sum_{\substack{F''\in\mathcal{F}\setminus\{F'\}:\\ E(F')\cap E(F'')\neq\emptyset}} \mathbb{P}\left(F'' \setminus E(F') \subseteq G(n,p)\right) \\
&\leq \mu \sum_{i\in[e(F)-1]} O(n^{v(F)-f(i)}p^{e(F)-i}) \\
&\leq \mu \sum_{i\in[e(F)-1]} O(n^{v(F)-i/m(F)}p^{e(F)-i}) = O(\mu c) < \mu,
\end{aligned}$$

since c is sufficiently small.

Applying Theorem 3.12, we get

$$\mathbb{P}\left(F \subseteq G(n,p)\right) \geq 1 - e^{-\mu+\Delta/2} = 1 - e^{-\mu/2} = 1 - e^{-\Theta(c^{e(F)})}.$$

$\square$

Proof of Theorem 1.4. The 1-statement follows from Theorem 1.2.

For the 0-statement, let $p = o(n^{-1/2})$ (notice that $m_2(H) = 2$ so this is the same as requiring $p = o(n^{-1/m_2(H)})$). We would like to show that, w.h.p., $G = G(n,p)$ can be edge-coloured without rainbow triangles and monochromatic copies of $K_{1,3}$. (Again, if there are no rainbow triangles then there are no rainbow copies of H.) By Lemma 3.4, we may assume that every triangle-connected subgraph H of G satisfies $e(H) < 2v(H)$. By using disjoint colour sets for different triangle-connected components, it suffices to show that every triangle-connected component can be coloured without rainbow triangles or monochromatic $K_{1,3}$'s, and this is indeed the case by Lemma 3.9. This completes the proof of the 0-statement.

For the final part of the theorem, we apply Claim 3.11 with F the graph defined in Lemma 3.10, noting that $m(F) = m_2(K_3) = 2$, $F \xrightarrow{\text{c−ram}} (K_{1,3}, K_3)$ and one can check that F is balanced. $\square$

4 Concluding remarks

In this paper, we closed the gap on locating the threshold for the constrained Ramsey property for (H_1, H_2) for all cases except for when $H_1 = K_{1,2}$. For graphs H_1 and H_2, let $m_{\text{c−ram}}(H_1, H_2) := \inf\{m(G) : G \xrightarrow{\text{c−ram}} (H_1, H_2)\}$. We present the full results in Table 1. Note that in [7], all threshold functions obtained are coarse thresholds. In the table, we mark all locations where the coarse threshold could be improved to a semi-sharp threshold. Note that we only consider graph H_2 with at least two edges, because if H_2 has one edge (or no edges) then every copy of H_2 is rainbow.

For a graph property $\mathcal{P}$, a function $f : \mathbb{N} \to \mathbb{R}$ is called a *sharp threshold function*, if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[G(n,p) \in \mathcal{P}] = \begin{cases} 0 & \text{if } p \leq (1-\varepsilon)f(n), \\ 1 & \text{if } p \geq (1+\varepsilon)f(n). \end{cases}$$

We are not aware of any results on sharp thresholds for the constrained Ramsey property, and it would be interesting to determine which semi-sharp thresholds can be improved to sharp. Sharp thresholds occur for some cases of the Ramsey property for graphs. Very recently, Friedgut, Kuperwasser, Samotij and Schacht [8] obtained a sharp threshold for the Ramsey property for H, whenever H is strictly 2-balanced, not a forest, and so-called *collapsible*.

For the constrained Ramsey problem for $(K_{1,2}, H_2)$, which as earlier discussed is equivalent to having the anti-Ramsey property for H_2, see Table 2 for all known results. Note that we write $m_{\text{a−ram}}(H_2) := m_{\text{c−ram}}(K_{1,2}, H_2)$.

Implicitly within our work in Section 3, we prove the following analogue of Theorem 1.6 for the constrained Ramsey property.

			1-statement		0-statement	
H_1, $e = e(H_1)$	H_2, $e = e(H_2)$	Threshold $n^{-1/f(H_1,H_2)}$	Ref	Type	Ref	Type
$K_{1,k}$, $k \geq 3$	Not forest or in $\mathcal{H}^*$ [a]	$m_2(H_2)$	1.2[10]	$\geq C$	1.3	$\leq c$
$K_{1,k}$, $k \geq 4$	In $\mathcal{H}^*$	$m_2(H_2) = 2$	1.2[10]	$\geq C$	1.3	$\leq c$
$K_{1,3}$	K_3	$m_2(H_2) = 2$	1.2[10]	$\geq C$	1.4	$\ll$
$K_{1,3}$	In $\mathcal{H}^*$ but not K_3	$m_2(H_2) = 2$	1.2[10]	$\geq C$	1.4	$\ll$[b]
Not star forest	Forest, $e \geq 3$	$m_2(H_1)$	[7]	$\gg$[b]	[16][c]	$\leq c$[d]
Star forest, not star	Forest, not short[e]	$m_2(H_2) = 1$	[7]	$\gg$[b]	[7]	$\ll$[bf]
Star forest, not star	Short forest[e]	$m_{\text{c-ram}}(H_1, H_2)$	[7]	$\gg$[b]	[7]	$\ll$
$K_{1,k}$, $k \geq 2$	Forest, $e \geq 3$	$m_{\text{c-ram}}(H_1, H_2)$	[7]	$\gg$[b]	[7]	$\ll$
Any, $e \geq 2$	$K_2 \sqcup K_2$	$m_{c-ram}(H_1, H_2) = m(H_1)$	[7]	$\gg$[b]	[7]	$\ll$
Not forest, $e \geq 2$	P_3 ('cherry')	$m_{\text{c-ram}}(H_1, H_2) = m(H_1)$	[7]	$\gg$[b]	[7]	$\ll$
Forest, k non-isolated vertices	P_3 ('cherry')	$m_{\text{c-ram}}(H_1, H_2) = \frac{k-1}{k}$	[7]	$\gg$[b]	[7]	$\ll$

Table 1: Thresholds for constrained Ramsey

[a] We write $\mathcal{H}^*$ for the class of graphs H which satisfy $m_2(H) = 2$ and the unique strictly 2-balanced graph J contained in H with $m_2(J) = 2$ is K_3.

[b] Coarse ($\gg$ / $\ll$) could potentially be improved to semi-sharp ($\geq C/ \leq c$) here.

[c] Follows from the random Ramsey threshold.

[d] With the exception of when H_1 is a path of 3 edges, since in this case the random Ramsey threshold for the 0-statement is only coarse.

[e] A *short* forest is defined to be one where all components have at most two edges, that is, a disjoint union of K_2s and P_3s.

[f] If H_1 is a matching of size 2, H_2 is a path of 3 edges, G a triangle with a path of length 2 added to each vertex of the triangle, then we have $m(G) = 1$ but $G \xrightarrow{\text{c-ram}} (H_1, H_2)$ so for this example, the 0-statement is coarse. Note that one could also take $G = C_5$ here to achieve the same conclusion.

Theorem 4.1. *Let H be a strictly 2-balanced graph with $m_2(H) > 1$ and let $k \geq 3$. To prove a semi-sharp 0-statement for the constrained Ramsey property for $(K_{1,k}, H)$ at $n^{-1/m_2(H)}$, it suffices to prove for all graphs G with $m(G) \leq m_2(H)$ that $G \not\xrightarrow{\text{c-ram}} (K_{1,k}, H)$. The same statement holds with 'semi-sharp' replaced by 'coarse' and '$m(G) \leq m_2(H)$' replaced by '$m(G) < m_2(H)$'.*

Observe by Claim 3.11, if there exists a balanced graph G with $m(G) = m_2(H)$ such that

H	Threshold $n^{-1/f(H)}$	1-statement Ref	1-statement Type	0-statement Ref	0-statement Type
Any	$\leq m_2(H)$	1.2[10]	$\geq C$		
K_k, $k \geq 5$	$m_2(H) = (k+1)/2$	1.2[10]	$\geq C$	[12, 14]	$\ll^{ab}$
C_k, $k \geq 5$	$m_2(H) = (k-1)/(k-2)$	1.2[10]	$\geq C$	[3, 14]	$\ll^{ac}$
K_3	$m_{\mathrm{a-ram}}(H) = 1$	[4]	$\gg$	[4]	$\ll$
K_4	$m_{\mathrm{a-ram}}(H) = 15/7$	[12]	$\gg^a$	[12]	$\ll$
K_4 minus edge	$m_{\mathrm{a-ram}}(H) = 3/2$	[12]	$\gg^a$	[12]	$\ll$
C_4	$m_{\mathrm{a-ram}}(H) = 4/3$	[3]	$\gg^a$	[3]	$\ll$
Forest	$m_{\mathrm{a-ram}}(H)$	[7]	$\gg^a$	[7]	$\ll$
$B_t \oplus F$, B_t book, $1 < m_2(F) < 2$	$< 2 \leq m_2(H)$	[2][d]	$\geq C$		
Strictly 2-balanced and $1 < m_2(H) < \frac{\delta(H)(\delta(H)+1)}{2\delta(H)+1}$	$m_2(H)$	1.2[10]	$\geq C$	2.12	$\leq c$

Table 2: Thresholds for anti-Ramsey

[a] Coarse ($\gg$ / $\ll$) could potentially be improved to semi-sharp ($\geq C/ \leq c$) here.

[b] $\leq c$ proved for $k \geq 19$ in [14].

[c] $\leq c$ proved for $k \geq 7$ in [14].

[d] The case $t = 1$ was proved earlier in [11].

$G \xrightarrow{\text{c-ram}} (K_{1,k}, H)$, then a coarse threshold at $n^{-1/m_2(H)}$ is the best we can hope for. This is exactly the case for the constrained Ramsey property for $(K_{1,3}, K_3)$, as seen in Theorem 1.4.

Further note that analogous results to Theorem 4.1 hold for the Ramsey property, see, e.g. [15], and the asymmetric Ramsey property (when one looks for a different graph in each different colour), see [6, 13]. That is, roughly speaking, proving a 0-statement at the natural candidate location can be reduced to solving an appropriate colouring question for not too dense graphs.

For graphs H where the colouring statement within Theorem 1.6 does not hold, it is tempting to speculate that the threshold is at $n^{-1/m_{\mathrm{a-ram}}(H)}$, as is the case when H is a copy of K_3, K_4, C_4 or any forest. An analogous result is true for the Ramsey property for graphs. Indeed, the only case for which the threshold is not $n^{-1/m_2(H)}$ is the case of stars $K_{1,k}$ which have a threshold given by the appearance of $K_{1,r(k-1)+1}$ (which whenever r-coloured gives a monochromatic copy of $K_{1,k}$). However, in the case of the Ramsey property for hypergraphs, it was shown in [9] that for each $k \geq 4$, there exists a k-uniform hypergraph H such that the threshold is not given by either the natural generalisation of $n^{-1/m_2(H)}$ nor the threshold for the appearance of a hypergraph which has the Ramsey property for H. See [9] or e.g. [6] for further discussion on the thresholds of the Ramsey property for hypergraphs.

For a graph H, finding a particular graph G with $m(G) \leq m_2(H)$ for which $G \xrightarrow{\text{a-ram}} H$ immediately gives rise to a coarse 1-statement for the anti-Ramsey property for H at $n^{-1/m(G)}$.

However, this is not how the 1-statement for $B_t \oplus F$ was proved in [2]. They instead used the appearance of lots of copies of B_{3t-2} in $G(n,p)$ at the stated threshold and the property that $B_{3t-2} \xrightarrow{\text{a-ram}} B_t$, to extend some rainbow copy of B_t to a rainbow copy of $B_t \oplus F$.

The colouring statement within Theorem 1.6 seems very hard to prove or disprove in general, even for specific H; indeed, known results about this statement and the analogous statement for other Ramsey properties make use of structural properties of H. It would be of great interest to determine the location of the threshold for the remaining open cases of the anti-Ramsey property, in particular by obtaining results on the colouring statement within Theorem 1.6.

Acknowledgements

The authors would like to thank the two anonymous referees for their careful and helpful reviews, and in particular, for a simplification of the proof of Lemma 3.3.

References

[1] N. Alon and J. H. Spencer, *The probabilistic method*, 4th ed., Wiley Publishing, 2016. 22

[2] P. Araújo, T. Martins, L. Mattos, W. Mendonça, L. Moreira, and G. O. Mota, *On the anti-Ramsey threshold for non-balanced graphs*, 2022, arXiv:2201.05106. 5, 25, 26

[3] G. F. Barros, B. P. Cavalar, G. O. Mota, and O. Parczyk, *Anti-Ramsey threshold of cycles*, Discrete Appl. Math. **323** (2022), 228–235. 5, 25

[4] B. Bollobás, *Threshold functions for small subgraphs*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 197–206. 5, 25

[5] B. Bollobás and A. Thomason, *Threshold functions*, Combinatorica **7** (1987), no. 1, 35–38. 2

[6] C. Bowtell, R. Hancock, and J. Hyde, *Proof of the Kohayakawa–Kreuter conjecture for the majority of cases*, 2023, arXiv:2307.16760. 25

[7] M. Collares, Y. Kohayakawa, C. G. Moreira, and G. O. Mota, *The threshold for the constrained Ramsey property*, 2022, arXiv:2207.05201. 3, 23, 24, 25

[8] E. Friedgut, E. Kuperwasser, W. Samotij, and M. Schacht, *Sharp thresholds for Ramsey properties*, 2022, arXiv:2207.13982. 23

[9] L. Gugelmann, R. Nenadov, Y. Person, N. Škorić, A. Steger, and H. Thomas, *Symmetric and asymmetric Ramsey properties in random hypergraphs*, Forum Math. Sigma **5** (2017), Paper No. e28, 47. 25

[10] Y. Kohayakawa, P. B. Konstadinidis, and G. O. Mota, *On an anti-Ramsey threshold for random graphs*, European J. Combin. **40** (2014), 26–41. 3, 24, 25

[11] ———, *On an anti-Ramsey threshold for sparse graphs with one triangle*, J. Graph Theory **87** (2018), no. 2, 176–187. 25

[12] Y. Kohayakawa, G. O. Mota, O. Parczyk, and J. Schnitzer, *The anti-Ramsey threshold of complete graphs*, Discrete Math. **346** (2023), no. 5, Paper No. 113343, 12. 5, 25

[13] E. Kuperwasser, W. Samotij, and Y. Wigderson, *On the Kohayakawa-Kreuter conjecture*, 2023, arXiv:2307.16611. 25

[14] R. Nenadov, Y. Person, N. Škorić, and A. Steger, *An algorithmic framework for obtaining lower bounds for random Ramsey problems*, J. Combin. Theory Ser. B **124** (2017), 1–38. 5, 6, 7, 8, 11, 25

[15] R. Nenadov and A. Steger, *A short proof of the random Ramsey theorem*, Combin. Probab. Comput. **25** (2016), no. 1, 130–144. 6, 25
[16] V. Rödl and A. Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, pp. 317–346. 3, 6, 24
[17] V. Rödl and A. Ruciński, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures Algorithms **5** (1994), no. 2, 253–270. 3
[18] V. Rödl and A. Ruciński, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942. 3
[19] V. Rödl and Z. Tuza, *Rainbow subgraphs in properly edge-colored graphs*, Random Structures Algorithms **3** (1992), no. 2, 175–182. 4