Prof. S.M. Tobias
Jan 2009

VECTOR CALCULUS: USEFUL STUFF

Revision of Basic Vectors

A *scalar* is a physical quantity with magnitude only

A *vector* is a physical quantity with magnitude and direction

A *unit vector* has magnitude one.

In Cartesian coordinates $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1, a_2, a_3)$

Magnitude: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

The position vector $\mathbf{r} = (x, y, z)$

The *dot product* (scalar product)

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

is a scalar

The *cross product* (vector product) $\mathbf{a} \times \mathbf{b}$ is a vector with magnitude $|\mathbf{a}||\mathbf{b}| \cos\theta$ and a direction perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in a right-handed sense.

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3$$

The *scalar triple product* $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is a scalar

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a}$$

The *vector triple product* is a vector

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Vector Algebra and Suffix Notation

The rules of suffix notation:

(1) Any suffix may appear *once* or *twice* in any term in an equation

(2) A suffix that appears just once is called a *free suffix*.
The free suffices must be the same on both sides of the equation.
Free suffices take the values 1, 2 and 3

(3) A suffix that appears twice is called a *dummy suffix*.
Summation Convention: Dummy Suffices are summed over from 1 to 3
The name of a dummy suffix is not important.

$$\mathbf{a} \cdot \mathbf{b} = a_jb_j = a_lb_l = a_pb_p = a_1b_1 + a_2b_2 + a_3b_3$$

(4) The Kronecker Delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

or

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Kronecker Delta is *symmetric* $\delta_{ij} = \delta_{ji}$ and $\delta_{ij}a_j = a_i$

(5) The Alternating Tensor:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i,\ j \text{ or } k \text{ are equal,} \\ 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (3,2,1) \text{ or } (2,1,3) \end{cases}$$

The Alternating Tensor is *antisymmetric*:

$$\epsilon_{ijk} = -\epsilon_{jik}$$

The Alternating Tensor is invariant under cyclic permutations of the indices:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon kij$$

(6) The vector product:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}\, a_jb_k$$

(7) The relation between δ_{ij} and ϵ_{ijk}:

$$\epsilon_{ijk}\, \epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Vector Differentiation

In all of the below formulae we are considering the vector $\boldsymbol{F} = (F_1, F_2, F_3)$

Basic Vector Differentiation

(1) If $\boldsymbol{F} = \boldsymbol{F}(t)$ then

$$\frac{d\boldsymbol{F}}{dt} = \left(\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt}\right)$$

(2) The unit tangent to the curve $\mathbf{x} = \boldsymbol{\psi}(t)$ is given by

$$\frac{d\mathbf{x}/dt}{|d\mathbf{x}/dt|}$$

Grad, Div and Curl

(3) The *gradient* of a scalar field $f(x, y, z)$ $(= f(x_1, x_2, x_3))$ is given by

$$\text{grad} f = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

(4) $\boldsymbol{\nabla} f$ is the vector field with a direction perpendicular to the isosurfaces of f with a magnitude equal to the rate of change of f in that direction.

(5) The *directional derivative* of f in the direction of a unit vector $\hat{\boldsymbol{u}}$ is $(\boldsymbol{\nabla} f) \cdot \hat{\boldsymbol{u}}$

(6) $\boldsymbol{\nabla}$ pronounced *del* or *nabla* is a *vector differential operator.* It is possible to study the 'algebra of $\boldsymbol{\nabla}$'.

(7) The *divergence* of a vector field $\boldsymbol{F}$ is given by

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

(8) A vector field $\boldsymbol{F}$ is *solenoidal* if $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 0$ everywhere.

(9) The *curl* of a vector field $\boldsymbol{F}$ is given by

$$\begin{aligned}
\text{curl}\, \boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{e}_3 \\
&= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right)\mathbf{e}_3 \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}
\end{aligned}$$

(10) A vector field $\boldsymbol{F}$ is *irrotational* if $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ everywhere.

(11) $(\boldsymbol{F}\cdot\boldsymbol{\nabla})$ is a vector differential operator which can act on a scalar or a vector

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla})\, f = F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}$$

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} = ((\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_1\,,\, (\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_2\,,\, (\boldsymbol{F}\cdot\boldsymbol{\nabla})\, G_3)$$

(12) The *Laplacian* operator $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ can act on a scalar or a vector.

Grad, Div and Curl and suffix notation

In suffix notation

$$\mathbf{r} = (x, y, z) = x_i$$

$$\text{grad} f = (\boldsymbol{\nabla} f)_i = \frac{\partial f}{\partial x_i}$$

$$(\boldsymbol{\nabla})_i = \frac{\partial}{\partial x_i}$$

$$\text{div}\, \boldsymbol{F} = \boldsymbol{\nabla}\cdot\boldsymbol{F} = \frac{\partial F_j}{\partial x_j}$$

$$(\text{curl}\, \boldsymbol{F})_i = (\boldsymbol{\nabla}\times\boldsymbol{F})_i = \epsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$(\boldsymbol{F}\cdot\boldsymbol{\nabla}) = F_j\frac{\partial}{\partial x_j}$$

Note: Here you cannot move the $\dfrac{\partial}{\partial x_j}$ around as it acts on everything that follows it.

These can all be used to prove the vector differential identities.

Vector Identities

Here are some simple vector identities that can all be proved with suffix notation.

If $\boldsymbol{F}$ and $\boldsymbol{G}$ are vector fields and φ and ψ are scalar fields then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi) = \nabla^2\varphi$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) = \boldsymbol{0}$$

$$\boldsymbol{\nabla}(\varphi\psi) = \varphi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla}\varphi$$

$$\boldsymbol{\nabla}\times(\varphi\boldsymbol{F}) = \varphi\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla}\varphi\times\boldsymbol{F}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}$$

$$\boldsymbol{\nabla}(\boldsymbol{F}\cdot\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$$

$$\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{F}(\boldsymbol{\nabla}\cdot\boldsymbol{G}) - \boldsymbol{G}(\boldsymbol{\nabla}\cdot\boldsymbol{F}) + (\boldsymbol{G}\cdot\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{F}\cdot\boldsymbol{\nabla})\boldsymbol{G}$$

Vector Integral Theorems

Alternative Definitions of divergence and curl

(1) An alternative definition of *divergence* is given by

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\oint\!\!\!\oint_{\delta S}\boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where δV is a small volume bounded by a surface δS which has outward-pointing normal $\boldsymbol{n}$.

(2) An alternative definition of *curl* is given by

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\times\boldsymbol{F} = \lim_{\delta S\to 0}\frac{1}{\delta S}\oint_{\delta C}\boldsymbol{F}\cdot d\boldsymbol{r},$$

where δS is a small open surface bounded by a curve δC which is oriented in a right-handed sense.

Physical Interpretation of divergence and curl

(3) The divergence of a vector field gives a measure of how much expansion and contraction there is in the field.

(4) The curl of a vector field gives a measure of how much rotation or twist there is in the field.

The Divergence and Stokes' Theorems

(5) The *divergence theorem* states that

$$\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{F} = \oint\!\!\!\oint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS,$$

where S is the closed surface enclosing the volume V and $\boldsymbol{n}$ is the outward-pointing normal from the surface.

(6) *Stokes' theorem* states that

$$\iint_S \boldsymbol{\nabla}\times\boldsymbol{F}\cdot\boldsymbol{n}\,dS = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r},$$

where C is the closed curve enclosing the open surface S and $\boldsymbol{n}$ is the normal from the surface.

Conservative Vector fields, line integrals and exact differentials

(7) The following 5 statements are equivalent in a simply-connected domain:

(i) $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ at each point in the domain.

(ii) $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ around every closed curve in the region.

(iii) $\int_P^Q \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of the path of integration from P to Q.

(iv) $\boldsymbol{F}\cdot d\boldsymbol{r}$ is an exact differential.

(v) $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar ϕ which is single-valued in the region.

(8) If $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 0$ then $\boldsymbol{F} = \boldsymbol{\nabla}\times\boldsymbol{A}$ for some $\boldsymbol{A}$. (This *vector potential* $\boldsymbol{A}$ is not unique.)