COMPLEX ANALYSIS CHEAT SHEET CONT'D

ALEX NELSON

Contents

1. Laplace Transform

Definition 1. (Laplace Transform) The **Laplace Transform** of a function $f(t)$ (for $t \geq 0$) is the function $\widetilde{f}(z)$ defined by

$$\widetilde{f}(z) = \mathcal{L}\{f\}(z) = \int_0^\infty e^{-zt} f(t)dt \tag{1}$$

where z is complex.

Proposition 1. (Asymptotic Behavior of Laplace Transform) Suppose g is analytic in a region containing the positive real axis and is bounded on the positve real axis. Let the Taylor series for g centered at 0 be

$$\sum_{n=0}^\infty a_n z^n \tag{2}$$

and let

$$\widetilde{g}(z) = \int_0^\infty e^{-zt} g(t)dt. \tag{3}$$

Then

$$\widetilde{g}(z) \sim \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{2a_2}{z^3} + \cdots + \frac{n!a_n}{z^{n+1}} + \cdots \tag{4}$$

as $z \to \infty$, $\arg(z) = 0$.

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Proposition 2. Suppose g is infinitely differnetiable on the positive real axis and that g and each of its derivatives are of exponential order. That is, there are constants A_n and B_n such that

$$|g^{(n)}(t)| \leq A_n e^{B_n t} \tag{5}$$

for $t \geq 0$. Let

$$\widetilde{g}(z) = \int_0^\infty e^{-zt} g(t)dt. \tag{6}$$

Then

$$\widetilde{g}(z) \sim \frac{g(0)}{z} + \frac{g'(0)}{z^2} + \frac{g''(0)}{z^3} + \cdots + \frac{g^{(n)}(0)}{z^{n+1}} + \cdots \tag{7}$$

as $z \to \infty$, $\arg(z) = 0$.

Theorem 2. (Convergence Theorem for Laplace Transform) Assume

$$f : (0, \infty) \to \mathbb{C} \tag{8}$$

is of exponential order and let

$$\widetilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt. \tag{9}$$

There exists a unique number σ, $-\infty \leq \sigma < \infty$ such that this integral converges if $\mathrm{Re}(z) > \sigma$ and diverges if $\mathrm{Re}(z) < \sigma$. Furthermore if $\widetilde{f}$ is analytic on the set

$$A = \{z | \mathrm{Re}(z) > \sigma\} \tag{10}$$

and we have

$$\frac{d}{dz}\widetilde{f}(z) = -\int_0^\infty te^{-zt} f(t)dt \tag{11}$$

for $\mathrm{Re}(z) > \sigma$. The number σ is called the "**Abscissa of Convergence**" and if we define the number ρ by

define ρ

$$\rho = \inf\{B \in \mathbb{R} | \text{there exists an } A > 0 \text{ such that } |f(t)| \leq Ae^{Bt}\} \tag{12}$$

then $\sigma \leq \rho$.

Theorem 3. (Laplace Transforms) Suppose that the functions f and h are continuous and that $\widetilde{f}(z) = \widetilde{h}(z)$ for $\mathrm{Re}(z) > \gamma_0$ for some γ_0. Then $f(t) = h(t)$ for all $t \in (0, \infty)$.

Proposition 3. Let $f(t)($ be continuous on $(0, \infty)$ and piecewise C^1. Then for $\mathrm{Re}(z) > \rho$

$$\widetilde{\left(\frac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0). \tag{13}$$

Proposition 4. Let

$$g(t) = \int_0^t f(\tau)d\tau \tag{14}$$

Then for $\mathrm{Re}(z) > \max[0, \rho(f)]$,

$$\widetilde{g}(z) = \frac{\widetilde{f}(z)}{z}. \tag{15}$$

Theorem 4. (First Shifting Theorem)Fix $a \in \mathbb{C}$ and let $g(t) = e^{-at}f(t)$. Then for $\mathrm{Re}(z) > \sigma(f) - \mathrm{Re}(a)$, we have

$$\widetilde{g}(z) = \widetilde{f}(z+a). \tag{16}$$

Theorem 5. (Second Shifting Theorem)Let $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 1$ be the **Step Function** or **Heaviside Step Function**. Let $a \geq 0$ and let $g(t) = f(t-a)H(t-a)$; that is, $g(t) = 0$ if $t < a$ while $g(t) = f(t-a)$ if $t \geq a$. Then for $\mathrm{Re}(z) > 0$ we have

$$\widetilde{g}(z) = e^{-az}\widetilde{f}(z). \tag{17}$$

Definition 6. (Convolution)The "**Convolution**" of two functions $f(t)$ and $g(t)$ is defined for $t \geq 0$ by

$$(f * g)(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau \tag{18}$$

where we set $f(t) = 0$ if $t < 0$.

Theorem 7. (Convolution Theorem)The equalities

$$(f * g)(t) = (g * f)(t) \tag{19}$$

whenever $\mathrm{Re}(z) > \max[\rho(f), \rho(g)]$.

1.1. **Table of Properties of the Laplace Transform.** Let $u(t)$ be the Heaviside step function.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{20}$$

where δ is the delta function we all know and love.

Linearity	$af(t) + bg(t)$	$a\widetilde{f}(z) + b\widetilde{g}(z)$
Frequency Differentiation	$tf(t)$	$-\widetilde{f}'(z)$
Frequency Differentiation	$t^n f(t)$	$(-1)^n \widetilde{f}^n(z)$
Differentiation	$f'(t)$	$z\widetilde{f}(z) - f(0)$
Differentiation	$f''(t)$	$z^2\widetilde{f}(z) - zf(0) - f'(0)$
Differentiation	$f^{(n)}(t)$	$z^n\widetilde{f}(z) - z^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Frequency Integration	$f(t)/t$	$\int_z^\infty \widetilde{f}(\omega)d\omega$
Integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\widetilde{f}(z)/z$
Scaling	$f(at)$	$\widetilde{f}(z/a)/\lvert a\rvert$
Frequency Shifting	$e^{at}f(t)$	$\widetilde{f}(z-a)$
Time shifting	$f(t-a)u(t-a)$	$e^{-az}\widetilde{f}(z)$
Convolution	$(f * g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
Periodic Function	$f(t)$	$(\int_0^T e^{-zt}f(t)dt)/(1-e^{-Tz})$

1.2. **List of Properties of the Laplace Transform. Definition**

$$\widetilde{f}(z) = \int_0^\infty e^{-zt}f(t)dt. \tag{21}$$

(1) $\widetilde{g}(z) = -\dfrac{d}{dz}\widetilde{f}(z)$ where $g(t) = tf(t)$.
(2) $\mathcal{L}\{af + bg\} = a\widetilde{f} + b\widetilde{g}$
(3) $\widetilde{\left(\dfrac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0)$.

2. Gamma Function

So for n a positive integer, we have

$$\Gamma(n) = (n-1)! \tag{22}$$

2.1. **List of Properties of the Gamma Function.** Remember that it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{23}$$

or equivalently as an infinite product

$$\Gamma(z) = \frac{1}{ze^{\gamma z}\left[\prod_{n=1}^\infty \left(1 + \frac{z}{n}\right)e^{-z/n}\right]} \tag{24}$$

where

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)\right)\\ &\approx 0.57721566490153286060651209008 2\end{aligned} \tag{25}$$

It has the following properties:

(1) Γ is meromorphic with simple poles at $0, -1, -2, \dots$
(2) $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \dots$
(3) $\Gamma(n+1) = n!$ for $n = 0, 1, \dots$
(4) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$
(5) $\Gamma(z) \neq 0$ for all z
(6) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{1 \cdot 3 \cdot (\cdots) \cdot (2n-1)}{2n}\sqrt{\pi}$
(7) $\Gamma(z) = \dfrac{1}{z}\displaystyle\prod_{n=1}^\infty \left[\left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1}\right]$
(8) $\Gamma(z) = \displaystyle\lim_{n\to\infty} \frac{n!n^z}{z(z+1)(\cdots)(z+n)}$
(9) $\Gamma(z)\Gamma\left(z + \dfrac{1}{n}\right)(\cdots)\Gamma\left(z + \dfrac{n-1}{n}\right) = (2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz)$
(10) $2^{2z-1}\Gamma(z)\Gamma\left(z + \dfrac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$
(11) The residue of $\Gamma(z)$ at $z = -m$ is equals $(-1)^m/m!$

(12) (Euler's Integral) $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}dt$ for $\mathrm{Re}(z) > 0$. The convergence is uniform and absolute for $-\pi/2 + \delta \le \arg(z) \le \pi/2 + \delta$ ($\delta > 0$) and for $\varepsilon \le |z| \le R$ where $0 < \varepsilon < R$.

(13) $\displaystyle\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right) dt.$

(14) $\pi^{-z/2}\,\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\,\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)$. (Where $\zeta(s)$ is the Riemann zeta function)

(15) $\zeta(z)\,\Gamma(z) = \displaystyle\int_0^\infty \frac{u^{z-1}}{e^u - 1}\,du$ which holds for $\mathrm{Re}(z) > 1$.

3. Zeta Function

The definition for the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{26}$$

It is holomorphic everwhere except for a simple pole at $s = 1$ with residue 1.

For any positive even integer $2n$, we have

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!} \tag{27}$$

where B_{2n} is a Bernoulli number, and for negative integers we have

$$\zeta(-n) = \frac{-B_{n+1}}{n+1} \tag{28}$$

for $n \ge 1$.

Let

$$f(x) = \frac{x}{e^x - 1} \tag{29}$$

then the Bernoulli numbers may be found from

$$B_n = \lim_{x\to 0}\frac{d^n}{dx^n}\frac{x}{(e^x - 1)}. \tag{30}$$

Observe that for $n = 1$

$$f'(x) = \left(\frac{1}{e^x - 1}\right)\left(1 - \frac{f(x)}{e^x - 1}\right) \tag{31}$$

and now observe that

$$\frac{d}{dx}\left(\frac{1}{e^x - 1}\right) = -e^x\left(\frac{1}{e^x - 1}\right)^2 \tag{32}$$

and we can use the product rule to find all of our favorite Bernoulli numbers.

We have a table of the first few Bernoulli numbers:

n	B_n
0	1
1	-1/2
2	1/6
4	-1/30
6	1/42
8	-1/30
10	5/66≈ 0.07575757576
12	-691/2730≈-0.25311355311
14	7/6
16	-3617/510≈ -7.09125686275
18	43867/798≈ 54.9711779448

The zeta function satisfies the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\,\sin\left(\frac{\pi s}{2}\right)\,\Gamma(1-s)\,\zeta(1-s), \tag{33}$$

valid for all $s \in \mathbb{C}$. An equivalent relationship may be expressed as a sum

$$\zeta(s)(1 - 2^{1-s}) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}. \tag{34}$$

3.1. Mellin Transform.

The Mellin transform of a function $f(x)$ is defined as

$$\int_0^\infty f(x)x^{s-1}\,dx, \tag{35}$$

when defined. We can relate the zeta function to one million and one things this way, we have

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{\exp(x) - 1}dx, \tag{36}$$

where Γ is our favorite gamma function, and

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i\oint_C \frac{(-x)^{s-1}}{\exp(x) - 1}dx \tag{37}$$

for all s where the contour C begins and ends at $+\infty$ and circles the origin once.

3.2. Laurent Series.

Since the zeta function has a single simple pole at $s = 1$ we can expand it around the singular point. The series is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\,(s-1)^n. \tag{38}$$

where γ_n are the Stieltjes constants, defined by the limit

$$\gamma_n = \lim_{m\to\infty}\left(\left(\sum_{k=1}^{m}\frac{(\log k)^n}{k}\right) - \frac{(\log m)^{n+1}}{n+1}\right). \tag{39}$$

where the constant $n = 0$ term in the Laurent series is just γ_0 the Euler-Mascheroni constant.

References

[1] J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*. W. H. Freeman, third ed., 1998.

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