1 Nontransitive dice explained

1.1 Preface

The nontransitive dice discussed in this article can be viewed as a way to caricature complex interaction that we have to deal with daily. To gain understanding of the concepts behind those dice, an aspect of discrete mathematics, the binary relation, will be presented. While beign very high-level and abstract, it can be applied to a lot of concrete situation in almost every domain and it will help introduce transitivity and intransivity as a property of a relation. Briefs example of transitivity, antitransitivity andintransitive will be given, followed by an introduction to nontransitive dice, which is an applied example of an intransitive relation. Various situation using those dices will be analyzed mathematically and real life examples will be given.

1.1.1 Overview of relation

The relation is an abstract concept which links numbers of elements togheter. Consider a group of student and books of the library. We will define the relation $\mathfrak{R}$ as every pair of students and books suchs as the student A_i have read the book B_j. Formally, it looks like this :

R = (A,B, have read)
which yeilds the pairs
(A$_i, B_j)i, j \in \mathbb{R}$
(1)
GRAPHICS HERE

We can also define relations inside only one group, suchs as a student-student relation. For example, lets consider the following :

R = (A,A, know)
which yeilds the pairs
(A$_i, A_j)i, j \in \mathbb{R}$
(2)
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Thus, a relation from a set against a set is valid and make sens. This will be used to help define transitivity.

1.1.2 Transitivity from a mathematical point of view

In his book *Mathematiques discretes*, Kenneth H. Rosen defines the conditions for a relation to be transitive as follow :

Let A,B,C be elements of a set S.
If (A,B) & (B,C) are valid transitions in the relation R,
it implies that transition (A,C) exists.

This definition is the stronger version of antitransitivity. The differences between the two is that a intransitive relation needs all elements to be not transitive, as an antitransitive is qualified as such with only one element or more not being transitive.
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1.1.3 Transitivity from an everyday life point of view

Transitive, intransitive and antitransitive situation are part of our everyday life. For a concrete example of transitive relation, lets consider the following example : The relation is characterized by "Is in the same family as". If you are in the same family as your father, good chances are that you are from the same family as your father's father. Now for the intransitive relation, let's consider the characterization of the relation as "is father of" This relation is intransitive because your father's father cannot be in any ways your father (hopefully), which follows the properties defined before for intransitivity. For the antitransitive relation, we could consider a group of people and "is friend with" as characterization. If in this group, A is friend with B and B with C, sometimes A is friends with C, sometimes not. Transitivity is not always applied there, therefore it is antitransitive.

1.2 Nontransitive dice

The nontransitive dice were invented as a game to illustrate intransitivity. In this game, the first player offers the opponent to choose a dice from a set of different dices. After that, the first player now pick his dice. After an arbitrary but fixed number of rolls, the player who won the most rolls win. This could potentially be used as a betting game, but not one of the fairest. This is due to the fact that nontransitive dice exhibits intransitivity as we will see later. First, we will be presented a method to evaluate the odds of the set of intransitive dices presented at **??**. After that, we will map the relation between those dices to exhibits intransitivity.

1.2.1 Comparing the dices

First, a legitimate question would be "how can dices that have the same value from the sum of each faces present different odds one from another?". A unsatisfactory answer would be to use intuition while looking at the dices face. Comparing the red and green one, for example, will reveal that the red dice can only win in the cases where the roll exhibits the red dice with 9 and the green with 0 or 5 ((R:9,G:0) , (R:9, G:5)). Since the red dice have only one chance in 6 to obtain 9, and the green 5 out of 6 to obtain anything but 0, we have a strong (and correct) feeling that the green dice *weight* more than the red. The use of the word weight here is to illustrate that the green win over the red since it is *heavier*.

1.3 Evaluating the odds

We saw in the previous section that the green dice seemed heavier than the red one. There's a mathematical way to prove this, and it is quite simple assuming a little knowledge of combinatorial probability. First of all, our dice are 6 faced ones. Hence, it exists $6 * 6 = 36$ possibility for one roll. **??** illustrates the odds of that roll.

The tree here represent the possibility associated for each faces. For example, with the green dice, we have 1/6 chances to get 0 and 5/6 chances for a 5. Likewise, for the red section, if we have obtained 0 from green, there's 1/6 chance to obtain 9 and 5/6 chance to obtain 4 for the red dice. We calculate the probability of getting a 0 from green and a 9 from red as $\frac{1}{6} * \frac{1}{6}$, the probability for (G:0,R:4) as $\frac{1}{6} * \frac{5}{6}$ and so forth. In the three, we identified the winning dice with the probability number with the color of the winning dice.

While this confirms our taught that the green is *heavier* than the red dice, it also proves that two different dices with the same value for the sum of their faces dosent necesserly have the same weight. This dosen't mean that the red one will always loose against the green at each rolls, but if the number of rolls is high enough, the probability shown in **??** and the rolls outcome will certainly have a similar look.

Now lets compare the green dice against the blue one, as shown in **??**

Here we see that the blue dice win over the green one, therefore imnplying that the blue one is *heavier* than the green one. This could also mean, with some simple deduction, that the blue is heavier than the red, since the green is heavier than the latter. Let's prove this wrong with **??**.

While this clearly shows that the red dice win over the blue, this also mean that the dice weight cannot be calculated globally. It must be calculated against each dice, because as we seen, blue wins over green, green wins over red but blue looses against red.

1.3.1 Intransitivity in nontransitive dices set

While the previous graphs demonstrated the relation between the dice, we could apply the definition from **??** to characterize the relation between the dices. If the three dices were normal dice (the 1 to 6 dices), each one would have the same weight. Now, as the red weight more than the blue, the blue than the green and the green than the red, this exhibits the following relation :

?? shows that the following relation exhibits an intransitive relation. In truns, if your opponents pick the dice first and you know the relation behind the dices, and the number of rolls is sufficiently large (about 20 rolls start to exhibits the dices properties), youre chances of winning will be a lot larger.

1.4 Grime dice

The dices from **??** we studied at section **??** was a subset of a larger set known as "Grime dice". This larger sets exhibits intransivity in about the same way than the three previous dice, exept that Grime dices contains 5 different dices. This set is presented at the figure below :

1.4.1 The intransitivity in Grime dice presented

As we already know, the red, blue and green dice already form a set of intransitive dices. We see from the figure below that they still form a subset inside the Grime dice set :

We wont go in the details about how one dice beats another. It's up to the reader to apply the same method presented at **??** to mesaure the dices weight against each others.

We see that adding 2 more dices adds some pair of dices in the relation. Now, every dices beats 2 dices and is beaten by two. Choosing the right dice after your two opponent's have chosen thiers can makes the odds beats the two of them.

1.4.2 A game in a game : using a pair of dice

Grime dices hold a pretty secret. You can have two different games if you decide to change the rules. Choosing a pair of two identical dices and comparing the results of the sums of the pairs yeilds another intransitive relation similar to the first with different probabilities. This time, it is harder to guess the weight of the dices as we did before, so we go directly with a graph three that sums up the probabilities of the red dice against the pink one.

We can see that the odds have changed from one game to another. The relation diagram representing the new intransitive relation is as following :

Note however that even if the red dice beats the green one in this game, they almost weight the same. This means the number of rolls to beat the green will need to be really high.

1.5 Creating nontransitive dice

Looking back at Grime dices may makes us think that the reversing order of the relation depending on the use a single or a pair of dice is a natural property. It is not, and this behaviour is due to carefully chosen value on the dices. This section will explore how we can create such set of dice for three six-faced dice.

1.6 Conclusion

Intransitivity and transitivity is a concept that benefits to be applied to as much situations as possible. To make the right choice in an entreprise, the manager gain to identify all of the pro's and con's of each, and how it will transition in the buisness.