Math 7390 – Abelian Varieties

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Contents

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1 Introduction

1.1 About this Course

This is an introductory course on the analytic and algebraic theory of abelian (and jacobian) varieties. We will start with the classical complex-analytic case to build some intuition. Then we will discuss the general theory over other fields.

We will not follow any specific books, but the following resources were used while preparing for the lectures:

[BL04] [BLR90] [FC90] [Mil08] [Mum08] [Mum07a] [Mum07b] [Mum07c] [CS86]

1.2 What are Abelian Varieites?

Origins of the theory of abelian varieties comes from a basic question in calculus! We know we can easily compute integrals of the form

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

by trigonometric substitution, but for integrals of the form

$$\int \frac{1}{\sqrt{f(x)}} dx$$

where $\deg(f) \geq 3$, it turns out to be hard. However, even though people couldn't compute these integrals, they could see that there were identities of the form

$$\int_0^a \frac{1}{\sqrt{f(x)}} dx + \int_0^b \frac{1}{\sqrt{f(x)}} dx = \int_0^{a*b} \frac{1}{\sqrt{f(x)}} dx$$

for some number $a * b$ obtained from a and b.

"Abelian Varieties are the simplest possible spaces, just tori's and thus groups" - D-Mumford.

Abelian Varieties are useful in the following areas:

- Number Theory - class field theory; rationality versus transcendence; most of the serious things we know how to do in number theory involve working with moduli of abelian varieties (Fermat's last theorem, Faltings' theorem, etc).
- Dynamical Systems - solutions to certain Hamilton systems.
- Algebraic Geometry - If we're given a variety X it's hard to understand, but we can get a handle on it by associating a canonical abelian variety $A(X)$ to it (such as Picard, Albanese, Intermediate Jacobian) and the good thing about $A(X)$ is that we can do a lot of linear algebra.
- Physics - theta functions that solve heat equations; string theory.

For the most part of this course, we will work over the base field $k = \mathbb{C}$; we see lots of very interesting ideas in this case, and we don't need any particularly hard theory to get a handle on it. One reason the theory over $\mathbb{C}$ is important is that an abelian variety A in a precise sense is just

$$A = A^{an} = \mathbb{C}^g / \Lambda$$

where $\Lambda = \pi_1(A) \simeq \mathbb{Z}^{2g}$ is a lattice, and so it is a torus. In other words, we have

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0.$$

Subtle remark: This identification makes sense in the "analytic category" but not in the algebraic category; the map $\mathbb{C}^g \to A$ is not algebraic. So we can't study abelian varieties in this way solely through algebraic methods. In the analytic setting it's "easy" to understand line bundles, theta functions, etc. by going to $\mathbb{C}^g$. (You can make sense of an analytification in nonarchimedean settings too, by using Berkovich spaces or formal schemes; this requires a lot more background but provides many important results.) Fortunately, there are some things from the complex-analytic setting which can be mimicked in the algebraic setting (e.g. the lattice Λ can be related to the Tate module) and by using those algebraic analogues you can take the complex-analytic results over $\mathbb{C}$ and try to reproduce them over other fields.

Some more advanced topics that might be covered in detail later in the class include (depending on audience interest):

- The theory over general field.
- Theta functions.
- Neron models.
- Non-archimedean uniformizations.
- Moduli and compactifications.
- Heights and metrized line bundles.
- Degenerating families.

Now, let's get to actual math. In scheme-theoretic language - for k any field, a *k-variety* is a geometrically integral k-scheme of finite type, and an *abelian variety* over k is a proper k-variety endowed with a structure of a k-group scheme. (This is the schematic definition. But we will not do it this way in this class.) We will be able to prove the following:

1.2.1 Theorem. *Abelian Varieties are automatically abelian and projective.*

Being abelian is easy to show, while being projective is much harder.

1.3 Why study Abelian Varieties?

More generally we can define an *algebraic group* G over k as a connected, smooth k-group scheme. Examples include:

- affine algebraic groups (automatically subgroups of $\mathrm{GL}_n(k)$, i.e. linear algebraic groups)
- abelian varieties (think of this as the projective case)

The following theorem says that these are the only building blocks:

1.3.1 Theorem (Chevalley's Theorem)**.** *Let G be an algebraic group over k. Suppose k is perfect. Then there exists a unique short exact sequence*

$$0 \to H \to G \to A \to 0$$

where H is linear and $A = G/H$ is abelian.

A proof can be found in B. Conrad's notes.

1.4 Some history

In the 1850s, Weierstrass studied $E = E^{an} = \mathbb{C}/\Lambda$, which is a complex group (2-dimensional torus with complex multiplication). He asked whether E^{an} is always "algebraic/algebraizable", and showed the answer is actually yes. In fact, he proved more.

1.4.1 Theorem. *$E = E^{an}$ has the structure of a smooth projective curve of genus 1. Its affine equation is given by*

$$y^2 = 4x^3 - 60G_4x - 140G_6$$

where $G_m = \sum_{\lambda\in\Lambda^*} \frac{1}{\lambda^m}$ for $m \in \mathbb{Z}$. More precisely, you can write down the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

and compute

$$y = \frac{dx}{dz} = \sum_{\lambda\in\Lambda} \frac{-2}{(z-\lambda)^3},$$

and see that the pair $(x, y) = (\wp(z), \wp'(z))$ satisfies the affine equation above. The mapping given by

$$z + \Lambda \mapsto (\wp(z), \wp'(z))$$

induces a group isomorphism $\mathbb{C}/\Lambda \simeq E$ where E is the projectivized elliptic curve.

What about higher dimensions? One direction is false: a general torus $\mathbb{C}^g/\Lambda$ will not be algebraic. However, the converse is true; a general abelian variety A will be (after analytification) of the form $\mathbb{C}^g/\Lambda$, where $g = \dim(A)$. We will prove this later, but it is not easy.

1.4.2 Theorem. *The Weierstrass parametrization gives a bijection between lattices Λ in $\mathbb{C}$, and the set of isomorphism classes of pairs (E, ω) where $E/\mathbb{C}$ are elliptic curves and ω are holomorphic differential forms.*

Note that $\omega \in H^0(E^{an}, \Omega)$ corresponds to $f(z)dz$ on $\mathbb{C}$, where f is periodic and holomorphic with Ω. What happens if we replace Λ with $a\Lambda$? It turns out we can say it corresponds to $(E, a\omega)$. Also note that the map $\mathbb{C} \to \mathbb{C}/\Omega$ is not algebraic. ($\mathbb{C}^g/\Omega$ for $g = 1$ is algebraic, while for $g \geq 2$ is usually not algebraic.) [farbod fix]

2 Complex Tori

2.1 Some GAGA principles

Algebraic varieties vs. (complex) analytic spaces.

If X is an algebraic variety over $\mathbb{C}$, we can associate X^{an} which is a complex analytic space to it, by passing to a complex analytification; if X is locally described by some set of equations in affine space, we can pass that open set the zero locus as a subset of $\mathbb{C}^n$ (with its usual topology) and glue. Here are some facts:

- This construction is always functorial: an algebraic map $X \to Y$ can always be lifted to a holomorphic map $X^{an} \to Y^{an}$.
- X is proper/complete if and only if X^{an} is compact.
- X is smooth/connected if and only if Xan is smooth/connected.
- A complex analytic space $\mathfrak{X}$ is called **algebraic/algebraizable** if there exists a variety $X/\mathbb{C}$ such that $\mathfrak{X} \simeq X^{an}$. (Last time we explicitly showed that $\mathbb{C}/\Lambda$ is algebraic via the Weierstrass $\wp$-functions.)

2.2 Vector bundles and associated locally free sheaves

If L is a vector bundle on X, then it passes to an analytic vector bundle L^{an} on X^{an}. This is functorial in the sense that if $f : F \to G$ is a morphism of vector bundles it passes to $f^{an} : F^{an} \to G^{an}$. It is *not* true that all holomorphic vector bundles over X^{an} are algebraizable! But we have:

2.2.1 Theorem (Serre)**.**

1. *Suppose X is a proper (complete) algebraic variety over $\mathbb{C}$. If there exists a holomorphic coherent sheaf $\mathcal{F} \to X^{an}$, then there exists unique algebraic coherent sheaf F over X such that $F^{an} = \mathcal{F}$.*
2. *If there exists $\mathfrak{F} : \mathcal{F} \to \mathcal{G}$ homomorphism of holomorphic coherent sheaves on X^{an}, then there exists unique $f : F \to G$ such that $f^{an} = \mathfrak{f}$.*

Define $H^i(X, L)$ as the i-th cohomology group with values in the locally free sheaf L.

2.2.2 Theorem (Serre)**.** *Let X be a complete algebraic variety over $\mathbb{C}$ and F a coherent sheaf on X. Then the* natural maps

$$H^i(X, F) \to H^i(X^{an}, F^{an})$$

are isomorphisms of $\mathbb{C}$-vector spaces.

2.3 Complex tori

Let V be a vector space over $\mathbb{C}$, and $\Lambda \subset V$ a lattice (full rank discrete subgroup). We have Λ act naturally on V by addition; and then the quotient $X = V/\Lambda$ is a complex torus.

Some facts about a complex torus:

- it is a complex manifold.
- it inherits the structure of a complex Lie group over $\mathbb{C}$.
- it is compact (because Λ is a maximal rank lattice).
- it is an abelian complex Lie group.
- meromorphic functions on X correspond to meromorphic Λ-periodic functions on V.

Loosely speaking, an (complex analytic) *abelian variety* is a complex torus with "sufficiently many" (enough to give a closed embedding to a projective space) meromorphic functions. We will see that this is exactly what makes X algebraizable and thus an algebraic abelian variety.

2.4 Compactness implies abelian

2.4.1 Theorem. *Any connected* compact *complex Lie group X is a complex torus.*

Proof. First, X is abelian and so the commutator map $\Phi(x,y) = xyx^{-1}y^{-1}$ is continuous. Let U be any neighbourhood of the identity element 1, for $x \in X$, define open neighbourhoods $V_x, \tilde{V_x}$ such that $x \in V_x$, $1 \in \tilde{V_x}$ and $\Phi(V_x, \tilde{V_x}) \subset U$. (This can be done since $\Phi(x,1) = 1$ and Φ is continuous.)

So we have $X = \cup_{x\in X} V_x$ and by compactness, there exist $x_1, \cdots, x_r \in X$ such that

$$X = \bigcup_{x\in\{x_1,\cdots,x_r\}} V_x.$$

Let $W = \cap_{x_1,\cdots,x_r} \tilde{V_x}$, which is a non-empty open neighbourhood of 1. So $\Phi(X, W) \subset U$. Since U is arbitrary, we have $\Phi(X, W) = 1$.

Since holomorphic functions on a compact set X which is bounded must be constant, we have $\Phi(1, y) = 1$ for all $y \in W$. Since W is open and non-empty, by connectivity,

$$\Phi(x, y) = 1$$

for all $x, y \in X$.

Then, if $\pi : V \to X$ is a universal cover, V inherits the structure of a simply connected complex Lie group and thus must be $\mathbb{C}^g$. Moreover π is homomorphic with discrete kernel, and by compactness of X the kernel must be full rank. □

Another proof can be found in B. Conrad's notes.

Remarks: Once we have $X = V/\Lambda$, we see that V is a universal cover of X. Moreover, $\Lambda = \pi(X, 0)$, and since this is already abelian it is isomorphic to $\simeq H_1(X, \mathbb{Z})$. Since X is locally isomorphic to V, we can view V as the tangent space at 0, T_0X; then the covering map $\pi : V = T_0X \to X$ is actually the exponential map.

2.5 Period matrix

Given $X = V/\Lambda$, we can associate Π a $g \times 2g$ complex matrix: fix $\{e_1, \cdots, e_g\}$ a $\mathbb{C}$-basis for V and $\{\lambda_1, \cdots, \lambda_{2g}\}$ a $\mathbb{Z}$-generator set for Λ. Define λ_{ji} such that

$$\lambda_j = \sum \lambda_{ji} e_i.$$

Then the **period matrix** of X is given by

$$\Pi := \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}.$$

Clearly, Π determines X but it depends on the choices.

Question: Given $\Pi \in M_{g\times 2g}(\mathbb{C})$, is there a complex torus such that Π is the period matrix of X?

2.5.1 Theorem. *Let* $P = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}_{2g\times 2g}$*, where* $\overline{\Pi}$ *denote the complex conjugate matrix of* Π*. Then* Π *is the period matrix for some* $\mathbb{C}^g/\Lambda$ *if and only if* P *is nonsingular.*

Proof. Π is a period matrix if and only if the columns of Π are $\mathbb{R}$-linearly independent. $\square$

2.6 Holomorphic maps, homomorphism and isogenies

Suppose $X = V/\Lambda$ and $X' = V'/\Lambda'$ with dimensions g and g' respectively. We want to study holomorphic maps $f : X \to X'$. There are two special examples:

1. homomorphisms (holomorphic and respect group structure); and
2. translations (maps $X \to X$ by $x \mapsto x + x_0$ for some $x_0 \in X$).

The surprising thing is that that's *all*!

2.6.1 Theorem. *Suppose* $h : X \to X'$ *is a holomorphic map between complex tori. Then*

1. *there exists a unique homomorphism* $f : X \to X'$ *such that* $h = t_{h(0)} \circ f$*. That is,*
$$h(x) = f(x) + h(0)$$
for all x*.*
2. *There exists a unique* $\mathbb{C}$*-linear map* $F : V \to V'$ *with* $F(\Lambda) \subset \Lambda'$ *inducing* f*.*

Proof. Let $f := t_{-h(0)} \circ h$. Then we can lift $f \circ \pi : V \to X$ to $F : V \to V'$ where V' is the universal cover of X'. Then F is holomorphic and satisfies $F(0) = 0$. F is a $\mathbb{C}$-linear map: fix $\lambda \in \Lambda$, by construction,

$$F(v + \lambda) - F(v) \in \Lambda'$$

and so by continuity it's constant. Therefore,

$$F(v + \lambda) = F(v) + F(\lambda)$$

for all $v \in V, \lambda \in \Lambda$. We skip the remaining details. $\square$

2.7 Hom-sets

Let $\mathrm{Hom}(X, X')$ be the set of all homomorphisms $f : X \to X'$. It is an abelian group. If $X = X'$, then we can define $\mathrm{End}(X) := \mathrm{Hom}(X, X')$. In this case, $\mathrm{End}(X)$ is actually a ring, where multiplication is given by composing endomorphisms. The above theorem gives us the following corollary:

2.7.1 Corollary. *We have injective homomorphisms:*

1. $\rho_{an} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ *given by* $f \mapsto F$*; and*

2. $\rho_{int} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$ *given by* $f \mapsto F|_{\Lambda}$.

Note that both of these homomorphisms respect endomorphism ring structures if $X = X'$: $\rho_*(f' \circ f) = \rho_*(f') \circ \rho_*(f)$, where $* = an, int$.

2.7.2 Theorem. $\mathrm{Hom}(X, X') \simeq \mathbb{Z}^m$ *for some* $m \leq 4gg'$.

Proof. Use the second isomorphism in the corollary, since $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda' \simeq \mathbb{Z}^{2g'}$, so $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \simeq \mathbb{Z}^{4gg'}$ and $\mathrm{Hom}(X, X')$ embeds in this. $\square$

How do these relate to period matrices? Let Π and Π' be the period matrix for X and X' respectively. If we have $f : X \to X'$, then by picking bases we get that $\rho_{an}(f) : V \to V'$ is given by some $A \in M_{g' \times g}(\mathbb{C})$ and $\rho_{int}(f) : \Lambda \to \Lambda'$ given by some $R \in M_{2g' \times 2g}(\mathbb{Z})$. Then the condition $F(\Lambda) \subset \Lambda'$ means $A\Pi = \Pi' R$. (The converse is also true: given four matrices with this property, then they correspond to a morphism between complex tori.)

What if $X = X'$? In this case, we can get

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} R$$

and thus $\rho_{int} \otimes 1 \simeq \rho_{an} \oplus \bar{\rho_{an}}$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

2.8 Kernels and Images

2.8.1 Lemma. *Given a homomorphism* $f : X \to X'$.

1. $Im(f)$ *is a complex subtorus of* X'.

2. $\ker(f)$ *is a closed subgroup of* X *with finitely many component. The connected component of* $1 = id$ *is a complex torus.*

The proof is fairly easy; for part (b) we're claiming that we have an extension

$$1 \to X_0 \to G \to \Gamma \to 1$$

with X_0 a complex torus and Γ a finite abelian group. It is a good exercise to describe Γ as a direct sum of cyclic groups in terms of Π, Π', A, R (need to compute a Smith normal form somewhere).

2.9 Isogenies

A homomorphism $f : X \to X'$ is called an **isogeny** if f is surjective with finite kernel. Equivalently, f is surjective and $\dim(X) = \dim(X')$.

2.9.1 Example (Essential example)**.** Suppose $X = V/\Lambda$ is a complex torus and $\Gamma \subset X$ is a finite subgroup. Then $X/\Gamma = V/\pi^{-1}(\Gamma)$ is a complex torus and $X \to X/\Gamma$ is an isogeny.

In fact, that's all! It is an easy exercise to show that all isogenies $X \to X/\Gamma$ over $\mathbb{C}$ are of this form. We also have the following easy lemma:

2.9.2 Lemma (Stein factorization)**.** *Any surjection $f : X \to X'$ of complex tori factors as a surjection $X \to X/(\ker f)_0$ (a quotient of X by a complex subtorus) and an isogeny $X/(\ker f)_0 \to X'$.*

Remark: Stein factorization is a special case of a very general result (in complex theory by Stein and others, and in general in EGA III): any proper $f : X \to S$ factors as a map $X \to S'$ proper with connected fibers and $S' \to S$ finite.

For $f \in \mathrm{Hom}(X, X')$, we define $\deg(f)$ to be $|\ker f|$ if this is finite, and 0 if otherwise. It is easy to check that $\deg(f) = [\Lambda' : \rho_{int}(f)\Lambda]$. (Remark: If $X = X'$ then this index is $\deg(\rho_{int}(f))$; note that this determinant is ≥ 0 since $\rho_{int} \otimes 1 = \rho_{an} \oplus \bar{\rho}_{an}$, and is 0 if and only if the kernel is infinite.)

2.9.3 Lemma. *Supppose $f : X \to X'$ and $f' : X' \to X''$ are isogenies, then $f' \circ f$ is also an isogeny.*

Proof. $\deg(f' \circ f) = \deg(f) \cdot \deg(f')$. $\square$

A very important example is given by the "multiplication-by-n" map: Let $n \in \mathbb{Z}^+$, define $n_X : X \to X$ by $x \mapsto nx$. Denote $X[n] := \ker(n_X)$ the set of n-torsions in A. Then we have

$$X[n] \simeq \frac{\frac{1}{n}\Lambda}{\Lambda} \simeq \frac{\Lambda}{n\Lambda} \simeq (\mathbb{Z}/n)^{2g}.$$

Therefore, n_X has degree n^{2g}, and so it is an isogeny.

2.9.4 Corollary. *Complex tori are divisible groups.*

2.9.5 Example (Tate module)**.** Let ℓ be a prime number. Define multiplication by ℓ maps $X[\ell^{n+1}] \to X[\ell^n]$. Then the *Tate module* is given by

$$T_\ell(X) = \varprojlim X[\ell^n].$$

In the case where Λ is finitely generated, $T_\ell(X)$ is actually isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and this is a subset of Λ. Note that the definition of $T_\ell(X)$ makes sense over any fields (even if we don't have Λ when we are not over $\mathbb{C}$). Here in our setting it's easy to see that a morphism $X \to X'$ is determined by the induced map $T_\ell(X) \to T_\ell(X')$. Over general fields this is much, much harder! It's the *Tate conjecture* which says that we have

$$\mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(X')) \simeq \mathrm{Hom}(X, X').$$

This conjecture was only proven over number fields by Faltings as an essential part of his proof of the Mordell conjecture.

2.10 Importance of isogenies

They are "almost isomorphisms". Namely, we have:

2.10.1 Theorem. *Let $f : X \to X'$ be an isogeny and n be the exponent of* $\ker(f)$. *(That is, $nx = 0$ for all $x \in \ker(f)$.) Then there exists an isogeny $g : X' \to X$ such that*

$$f \circ g = n_X, \;\; g \circ f = n_X.$$

Moreover, such a g is unique (up to isomorphism?).

sketch. Since n is the exponent of $\ker f$, we have $\ker(f) \subset \ker(n_X) = X[n]$. Then there exists a unique $g : X' \to X$ with $g \circ f = n_X$, defined by $g(x') := nx$ for some (all) x where $f(x) = x'$. Then use the fact that $\deg(g)\deg(f) = \deg(n_X)$ and that $\deg(f), \deg(n_X) \neq 0$, so $\deg(g) \neq 0$ to get that g is an isogeny; Then we just need to check that $g \circ f = n'_X$. □

Define $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(X, X') := \mathrm{Hom}(X, X') \otimes \mathbb{Q}$. Then the degree function extends to these via

$$\deg(rg) := r^{2g} \cdot \deg(f).$$

2.10.2 Corollary.

1. *Isogeny is an equivalence relation.*
2. *$f \in \mathrm{End}(X)$ is an isogeny if and only if it is invertible in $\mathrm{End}_{\mathbb{Q}}(X)$.*

2.11 Cohomology

We have a lot of cohomology theories (Betti, de Rham, Dolbeault, Hodge decomposition, ...).

Betti cohomology is just singular cohomology of $X(\mathbb{C})$; if $X = V/\Lambda$, then we have the following facts:

- $\Lambda = \pi_1(X_0) \simeq H_1(X, \mathbb{Z})$.
- By the universal coefficient theorem, we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.
- If $n \geq 1$, we have a map $\wedge_{i=1}^{n} H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by cup product, and this is an isomorphism (follows from Kunneth formula).
- Let $Alt^n(\Lambda, \mathbb{Z}) := \bigwedge_{i=1}^{n} \mathrm{Hom}(\Lambda, \mathbb{Z})$ be all the $\mathbb{Z}$-valued alternating n-forms. Then we have $H^n \simeq Alt^n(\Lambda, \mathbb{Z})$. This gives a very explicit way of thinking about cohomology.
- $H_n(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$.
- If we set $H^n(X, \mathbb{C}) := H^n(X, \mathbb{Z}) \otimes \mathbb{C}$, then we have

  $$H^n(X, \mathbb{C}) \simeq Alt^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge_{i=1}^{n} \mathrm{Hom}_{\mathbb{R}}(\Lambda, \mathbb{C}) \simeq \bigwedge_{i=1}^{n} H^1(X, \mathbb{C}),$$

  and the de Rham theorem tells us $H^n(X, \mathbb{C}) \simeq H_{DR}(X)$ where $H_{DR}(X)$ can be explicitly described as a complex vector space of invariant n-forms with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ with $i_1 < \cdots < i_n$.

Now, we use the $\mathbb{C}$-structure (really everthing is true for Kahler manifolds, but proofs and constructions are much more elementary for complex tori). Here we have a very nice decomposition

$$H^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(\Omega_X^p)$$

Here, $H^{p,q}(X) := H^q(\Omega_X^p)$ is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$. In general, $H^q(\Omega_X^p)$ can be explicitly described as $\bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega}$ for $\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Also set $\Omega_X^p := (\bigwedge^p \Omega) \otimes \mathcal{O}_X$. Right now we are not saying anything about what these vector spaces $H^{p,q}(X)$ are, but we will do that later on. We may also need ot return to this theory to prove for example vanishing results later; We will either do that, or just omit those proofs.

2.12 Sheaves on a topological space X

Let $\mathcal{O}(X)$ be the category in which objects are open subsets of X and morphisms are inclusions $V \to U$ for $V \subset U$. (It turns out that you can generalize this and allow more general things for morphisms than just inclusions; this is how you get etale cohomology and other things.) Let $\mathcal{C}$ be any other category (for instance, Sets, Abelian groups and R-modules); A **presheaf** is a contravariant functor

$$F : \mathcal{O}(X) \to \mathcal{C}.$$

(This means for each inclusion map $i : V \subset U$, we have the restriction map given by $rest_{V,U} : F(U) \to F(V)$, and this assignment is functorial.) A **sheaf** is a presheaf F with some "locality" and "gluing" properties. Assume $\mathcal{C}$ has products. Then we require for any open cover $\{U_i\}$ of $U \in \mathcal{O}(X)$, we have an exact sequence

$$0 \to F(U) \overset{rest}{\to} \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

(where the two parallel maps are "restriction to the first index" and "restriction to the second index" respectively.)

2.12.1 Example. Let $X = \mathbb{C}$. Then we have the sheaf of holomorphic functions: $F(U)$ is the set of all holomorphic functions $U \to \mathbb{C}$, and restriction maps are restriction of functions!

Morphisms of sheaves are natural transformations; this gives us the category of sheaves over X, Shf_X, in which the objects are sheaves on X and morphisms are these.

Next, a **ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$; we call $\mathcal{O}_X$ the **structure sheaf** (usually some sheaf of holomorphic functions in this class). Define a **locally ringed space** to be one such that all of the stalks $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} F(U)$ are local rings. We may want to consider sheaves of $\mathcal{O}_X$-modules, that is, each $F(U)$ is a $\mathcal{O}_X(U)$-module respecting restriction maps.

2.13 Abelian categories and cohomology

(Grothendieck's Tohoku paper) There are 4 main examples of abelian categories to keep in mind:

1. The category of abelian groups (with homomorphisms).
2. The category of R-modules for R a commutative ring (with homomorphisms).
3. The category of G-modules for G a group (with G-equivariant homomorphisms: $\phi(g \cdot m) = g \cdot \phi(m)$).
4. The category of $\mathcal{O}_X$-modules for $(X, \mathcal{O}_X)$ a ringed space (with morphisms of sheaves the morphisms).

In general there's an abstract definition of an abelian category; we are not going to say it precisely but roughly it is a category $\mathcal{A}$ in which addition of morphisms, a zero object, kernels and cokernels make sense.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor between abelian categories. If we start with an short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, we can hit it with the functor F and get maps in $\mathcal{B}$, but no guarantee for exactness. We say that F is **left exact** if for every such short exact sequence we do have

$$0 \to F(A) \to F(B) \to F(C)$$

exact.

2.13.1 Example. Let $\mathcal{A}$ be the category of R-module, $\mathcal{B}$ be the category of abelian groups and D be a fixed object. Then $F(-) = \mathrm{Hom}_R(D, -)$ is a left exact covariant functor.

Cohomology lets us study the failure of exactness of

$$0 \to F(A) \to F(B) \to F(C),$$

i.e. the failure of surjectivity of $F(B) \to F(C)$. We want to continue the above exact sequence to the right and write a long exact sequence. In general, there are many ways to do this. But if $\mathcal{A}$ has "enough injectives" (a statement that holds for the categories we care about), then there is a "canonical" and "minimal" (in the sense that there is a universal property) way to do this. There exist unique functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ with $R^0F = F$ that give us the following long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{c_1} R^1F(A) \to R^1F(B) \to R^1F(C) \xrightarrow{c_2} R^2F(A) \to \cdots$$

plus satisfying some universal properties (an "effaceable δ-functor"). These functors are called the *right derived functors*. This is the covariant version; there's a similar one for contravariant functors. In general, it is hard to compute these R^iF. However, in the examples that we are interested in, there are easier ways of computing them.

2.13.2 Example. Let $F(-) := \mathrm{Hom}_R(-, D)$ and $G(-) := \mathrm{Hom}_R(D, -)$ where D is a fixed R-module. Both F and G are left-exact functors from R-modules to abelian groups, one contravariant and one covariant. More precisely, for any short exact sequence of R-modules

$$0 \to L \to M \to N \to 0,$$

we have the following exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N).$$

In both cases, the derived functors are just the Ext groups $\mathrm{Ext}^i_R(-, D)$ and $\mathrm{Ext}^i_R(D, -)$ which give us the long exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^1_R(M, D) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N) \to \mathrm{Ext}^1_R(D, L) \to \mathrm{Ext}^1_R(D, M) \to \cdots$$

So $R^i \mathrm{Hom}_R(-, D) = \mathrm{Ext}^i_R(-, D)$ and $R^i \mathrm{Hom}_R(D, -) = \mathrm{Ext}^i_R(D, -)$ (it is a nontrivial fact that these agree!).

2.13.3 Example. In the case of G-modules, let A be an abelian group A with a G action $\phi : G \to \mathrm{Aut}(A)$, (that is, A is a $\mathbb{Z}G$-module) with morphisms respecting G-action. Let A^G be the group of $x \in A$ such that $g \cdot x = x$ for all $g \in G$. The functor we want in this case is $F(A) = A^G$ which takes G -modules to abelian groups. Note that this functor is the same as $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$, so its (left exact) derived functors are (abstractly) Ext-functors $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, -)$, but this doesn't really give us a good way to compute it. These are better known as the "group cohomology of G with coefficients in A", denoted $H^i(G, A)$.

How do we compute this? It turns out that $\mathbb{Z}$ has a very nice "standard resolution" (also called the "bar resolution"), which is given by

$$F_n = \bigotimes_{i=0}^{n} \mathbb{Z}G.$$

Using this, we get the following recipe: for G a group and A a G-module, we let $C^0(G, A) = A$ and $C^n(G, A)$ be the group of A-valued maps on $G^n = G \times \cdots \times G$ for $n \geq 1$. These C^n are called the group of n**-cochains** of G with values in A. We also define the **differential operators** $d_n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \cdots, g_{n+1}) := &\ g_1 \cdot f(g_2, \cdots, g_{n+1}) \\ &+ \Sigma_{i=1}^{n} (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \cdots, g_n). \end{aligned}$$

The cases of most interest are $n = 0, 1, 2$:

- when $n = 0$, we have $f = a \in C^0(G, A) = A$, and then we get

$$d_0(f)(g) = g \cdot a - a.$$

- when $n = 1$, then f is a function with one input and we have

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

- when $n = 2$, then f is a function with two inputs and we have

$$d_2(f)(g, h, k) = g \cdot f(h, k) - f(gh, k) + f(g, hk) - f(g, h).$$

  Amazingly, people wrote down this formula correctly before the general theory came out.

It can be shown that $d_n \circ d_{n+1} = 0$. Thus, we can define the group of n-**cocycles** as $Z^n(G, A) := \ker(d_n)$ for $n \geq 0$ and the group of n-**coboundaries** as $B^n(G, A) := \mathrm{im}(d_{n-1})$ for $n \geq 1$ (or $B^0(G, A) = 1$ when $n = 0$). Then we define the n**-th cohomology group** as

$$H^n(G, A) := Z^n(G, A)/B^n(G, A).$$

Note that $H^0(G, A) = A^G$.

2.13.4 Example. Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ a $\mathcal{O}_X$-module. Let $\Gamma(-, X)$ be the "global sections" functor, $\mathcal{F} \mapsto \mathcal{F}(X)$, from $\mathbb{O}_X$-modules to R-modules for $R = \mathcal{O}_X(X)$. This functor is left-exact (to make sense of this we need to make sure we know what exact sequences of sheaves are - defining "kernels" is easy but to define "images" we need to sheafify).

Here there is the Grothedieck (right) derived cohomology functor, $R^i\Gamma(-, X)$, which we denote $H^i(X, -)$ (we also call $H^i(X, \mathcal{F})$ the *sheaf cohomology of X with values in $\mathcal{F}$*); in particular, we have $H^0(X, -) = \Gamma(-, X)$. This is given abstractly by the theory earlier in this section; but like group cohomology, we will really need to work with sheaf cohomology, so we need to a way to compute them explicitly.

Here, we have a more concrete one, which is called the *Čech cohomology*, $\check{H}^i(X, \mathcal{F})$. In general, the two cohomologies are not equal! Fortunately they are equal in the settings we will be interested in.

The Čech cohomology is more explicitly computable, and always gives us a map $\phi : H^i(X, \mathcal{F}) \to \check{H}^i(X, \mathcal{F})$. We have:

1. For $i = 0, 1$, ϕ is an isomorphism.

2. (Grothendieck): If X is a Noetherian, separated scheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then ϕ is an isomorphism.

3. (Godement): If X is a paracompact and Hausdorff topological space, then ϕ is an isomorphism.

However, ϕ is not an isomorphism in general! Here are some counterexamples:

- In his Tôhoku paper (p.177), Grothendieck provides a counterexample where $H^2 \neq \check{H}^2$, for $X = \mathbb{A}^2$ with the Zariski topology and $\mathcal{F}$ comes from taking $\underline{\mathbb{Z}}$ and modifying it based on a space Y that is a union of two circles. This example is explicit but the proof is somehow deep!

- A recent paper of Schröer (arxiv post 1309.2524) gives a Hausdorff (not not paracompact) topological space constructed from 2-dimensional discs that is a counterexample.

2.13.5 Čech cohomology

So how do we define Čech cohomology? The idea is that if $\mathcal{U}$ is an open cover on X, the "nerve" of $\mathcal{U}$ approximates X. We define a q-simplex σ of $\mathcal{U}$ as an ordered collection of $q+1$ elements in $\mathcal{U}$ with nonempty intersection that we call $|\sigma|$. Suppose $\sigma = (U_i)$ (for $0 \le i \le q$), we define $\partial_j \sigma := (U_i)_{i \neq j}$ and then $\partial \sigma = \Sigma_{j=0}^{q} (-1)^{j+i} \partial_j \sigma$. Note that $|\sigma| = \cap U_i$.

We then define q**-cochains** of $\mathcal{U}$ with coefficients in $\mathcal{F}$ to be the set $C^q(\mathcal{U}, \mathcal{F})$ of functions $\sigma \mapsto f_\sigma \in \mathcal{F}(|\sigma|)$. Therefore, we get the **boundary maps** $C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta_q \omega)(\sigma) = \Sigma_{j=0}^{q+1} (-1)^j \mathrm{res}_{|\partial_j \sigma|, |\sigma|} \omega(\partial_j \sigma).$$

One can check that $\delta_{q+1} \circ \delta_q = 0$ and hence can define **cocycles** $Z^q(\mathcal{U}, \mathcal{F}) := \ker(\delta_q)$ and **coboundaries** $B^q(\mathcal{U}, \mathcal{F}) := Im(\delta_{q-1})$, and then the **Čech cohomology** of $\mathcal{U}$ is given by

$$\check{H}^q(\mathcal{U}, \mathcal{F}) := Z^q(\mathcal{U}, \mathcal{F}) / B^q(\mathcal{U}, \mathcal{F}).$$

So this gives us the cohomology $\check{H}^i(\mathcal{U}, \mathcal{F})$ of an open cover $\mathcal{U}$; However, we want a cohomology $\check{H}^i(X, \mathcal{F})$ associated to the whole space X! There are two ways to solve this:

1. If X has a "good" cover $\mathcal{U}$ (with all finite intersections of U_i to be contractible), then $\check{H}^i(\mathcal{U}, \mathcal{F})$ is canonical.

2. In general, we can define $\check{H}^i(\mathcal{X}, \mathcal{F})$ as $\varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$. But then we need to make sense of this direct limit. (It might be over an index set that is a proper class?) [fix farbod]

Remark: Let G be a topological group. Let $BG = K(G, 1)$ be the *Eilenberg-MacLane space*. (For example, $B\mathbb{Z} = S^1$.) Note that $\pi_1 = G$ and $\pi_n = 0$ for all $n > 1$. Then if A is a G-module, the sheaf cohomology $H^n(BG, \underline{A})$ (here $\underline{A}$ is the constant sheaf, which is the sheafification of the constant presheaf) is isomorphic to the usual CW complex cohomology $H^n(BG, A)$ and to the group cohomology $H^n(G, A)$, if A has a trivial G-action. (If A has a nontrivial G-action we can still make sense of this but the $H^n(BG, \underline{A})$ needs to be reinterpreted in terms of "local coefficient systems".)

2.14 Back to complex tori (sort of)

We want to understand line bundles on a locally ringed space $(X, \mathcal{O}_X)$. For now, we can think of it as a complex manifold or a variety. Let $\mathcal{F}$ be a sheaf. We call $\mathcal{F}$ (globally) **free** if $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X$ is the direct sum of copies of the structure sheaf; r is called the **rank** of $\mathcal{F}$. $\mathcal{F}$ is called **locally free** if there exists an open cover $\{U_i\}$ such that each $\mathcal{F}|_{U_i}$ is free. (There is a correspondence between locally free sheaves $\mathcal{F}$ of rank n and vector bundles of rank n. If $\pi : E \to X$ is a vector bundle, we get a locally free sheaf with $\mathcal{F}(U)$ being the sections of π over U; conversely, if $\mathcal{F}$ is locally free, we can construct an associated line bundle as $\coprod U_i \times \mathbb{C}^n$ modulo gluing data.)

2.14.1 Line bundles

Line bundles are vector bundles of rank 1 (equivalently, locally free sheaves of rank 1). Our goal is to give a cohomological interpretation on the set of line bundles. Let $\pi : L \to X$ be a line bundle. Let $\{U_\alpha\}$ be an open cover with trivializations given by (holomorphic) $\phi_\alpha : L|_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}$ where $L|_\alpha := \pi^{-1}[U_\alpha]$. Define the **transition functions** for L with respect to $\{\phi_\alpha\}$ as $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ which are given by

$$g_{\alpha\beta}(z) := \phi_\alpha \circ \phi_\beta^{-1}|_{L_z}$$

where $L_z := \{z\} \times \mathbb{C}$. By itself this is a linear (and hence holomorphic) map on $\{z\} \times \mathbb{C}$, which is determined by the complex number we are calling $g_{\alpha\beta}(z)$. We check that $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$ (so it is nonzero) and also $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = 1$. Rewriting this latter condition gives a cocycle condition

$$g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1.$$

To summarize, a line bundle (trivialized by an open cover $\mathcal{U}$) determines a collection of

- holomorphic functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$; which satisfy
- $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$, and
- $g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1$.

Conversely, if we are given $\{g_{\alpha\beta}\}$ satisfying these properties, then we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ as a quotient of $\coprod U_\alpha \times \mathbb{C}$ by the appropriate gluing relations.

2.14.2 Choices

If $f_\alpha \in \mathcal{O}^\times(U_\alpha)$ is a nonvanishing holomorphic function on U_α, and we construct new trivializations $\phi'_\alpha = f_\alpha \circ \phi_\alpha$, then the new transition functions $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$ give the same bundle L.

Now, the collection of $\{g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)\}$ is a Čech 1-cochain. The conditions we wrote down that it satisfies implies that it is actually a 1-cocycles. Moreover, the ambiguity mentioned above is exactly the 1-coboundaries. Therefore, we can then conclude the set of (isomorphism classes of) line bundles $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$. Moreover, this is a group homomorphism; for the group structure on line bundles coming from tensor product and the group structure naturally showing up on $H^1(X, \mathcal{O}_X^\times)$.

2.14.3 Line bundles vs "factors of automorphy"

Let X be a complex torus, and $\tilde{X} = \mathbb{C}^g$ its universal cover, with $\pi : \tilde{X} \to X$ the covering map.

2.14.4 Theorem. *There exists a canonical exact sequence*

$$0 \to H^1(\pi(X), H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)) \xrightarrow{\phi} H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times).$$

The first term in the above exact sequence is the group of *"factors of automorphy"* and the latter two are Picard groups as above. (Note that in general $\mathcal{O}_X^\times$ is the sheaf of invertible elements in our rings with respect to multiplication. In our holomorphic setting, this is equivalent to the sheaf of nonvanishing functions, because you can prove that if f is holomorphic and nonzero, then $1/f$ is holomorphic.)

2.14.5 Corollary. *If $\pi^*(L)$ is trivial, then it is completely described by a "factor of automorphy".*

2.14.6 Corollary. *If $\mathrm{Pic}(\tilde{X}) = 0$ (eg when $\tilde{X} = \mathbb{C}^g$), then ϕ is an isomorphism.*

Now, we prove the theorem above.

Proof. The 1-cochains $C^1(G, M)$ where $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \mathcal{O}_X^\times)$. Note that M contains elements like holomorphic $g : \tilde{X} \to \mathbb{C}^\times$. Therefore, for $(h : G \to M) \in C^1(G, M)$, we can naturally get holomorphic $f : G \times \tilde{X} \to \mathbb{C}^\times$. Note that G acts on $\tilde{X}$ by $g \cdot \tilde{x}$ which gives $f(\tilde{x}) \cdot g = f(g \cdot \tilde{x})$.

Also, note that the 1-cocycles satisfy

$$(h(\mu) \cdot \lambda)(h(\lambda\mu)^{-1})(h(\lambda)) = 1.$$

Therefore, we have the following cocycle condition: For $\lambda, \mu \in \pi_1(X), \tilde{x} \in \tilde{X}$,

$$f(\lambda\mu, \tilde{x}) = f(\mu, \lambda\tilde{x})f(\lambda, \tilde{x})$$

Also the 1-coboundary condition: For $h_1 \in M = C^0(G, M)$,

$$f(\lambda, \tilde{x}) = \frac{h_1(\lambda \cdot \tilde{x})}{h_1(\tilde{x})}$$

and we define $H^1(G, M) = Z^1/B^1$ like before.

Now, each $f \in Z^1(G, M)$ gives a Line Bundle on X. Let $L = \tilde{X} \times \mathbb{C} \to \tilde{X}$ be the trivial line bundle map. Then $G = \pi_1(X)$ acts on $\tilde{X}$, and so acts on L: $\lambda \cdot (\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that this action is

- free (If $g \cdot * = *$ for some g and $*$, then $g = id$.)
- properly discontinuous (For all K_1, K_2 compact subsets, $\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$ is finite.)

There is a theorem saying that complex manifolds came from complex manifolds moding out free properly discontinuous actions. Therefore, the induced $L' \to X$ is a holomorphic line bundle. And this gives a map $Z^1(G, M) \to H^1(X, \mathcal{O}_X^\times)$.

Proof of Theorem: First we will show that there is a map ϕ_1 given by

$$0 \to H^1(G, M) \xrightarrow{\phi_1} \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times)).$$

and then we will try to prove that ϕ_1 is an isomorphism.

First, for existence of ϕ_1, we can just take the above map. Or a more explicit solution for this is given by the following: Let $\pi : \tilde{X} \to X$ be the covering map. Then fix a cover $\{U_i\}_I$ such that for all $i \in I$, there exists $W_i \subset \pi_i^{-1}(U_i)$, with biholomorphic $\pi_i := \pi|_{W_i} : W_i \to U_i$. For all i, j, there exists unique $\lambda_{i,j} \in \pi_1(X)$

such that for $x \in U_i \cap U_j$, $\pi_j^{-1}(x) = \lambda_{ij} \cdot \pi_i^{-1}(x)$. It is clear that $\lambda_{ij} \cdot \lambda_{jk} = \lambda_{ik}$. Therefore, now we have a map $Z^1(G, M) \to Z^1(X, \mathcal{O}_X^\times)$ (note that $Z^1(X, \mathcal{O}_X^\times)$ maps onto $H^1(X, \mathcal{O}_X^\times)$) given by

$$f \mapsto \{g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)\}$$

where $g_{ij}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))$. It is easy to check the 1-cocycle condition

$$g_{ij}g_{jk} = g_{ik}.$$

(Explicitly, $g_{ij}(x)g_{jk}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))f(\lambda_{jk}, \pi_j^{-1}(x)) = f(\lambda_{ij}\lambda_{jk}, \pi_i^{-1}(x)) = f(\lambda_{ik}, \pi_i^{-1}(x)) = g_{ik}(x)$.)

So we have a well defined map

$$Z^1(G, M) \to \check{Z}^1(X, \mathcal{O}_X^\times) \twoheadrightarrow \check{H}^1(X, \mathcal{O}_X^\times).$$

It is easy to check that the kernel of this map contains $B^1(G, M)$, so this descends to a homomorphism $H^1(G, M) \to \check{H}^1(X, \mathcal{O}_X^\times)$. (To check this: if we have a 1-coboundary $f(\lambda, x) = h(\lambda\tilde{x})/h(x)$, this will go to a Čech 1-cocycle

$$g_{ij}(x) = \frac{h(\lambda_{ij}\pi_i^{-1}(x))}{h(\pi_i^1(x)} = \frac{h(\pi_j^{-1}(x))}{h(\pi_i^{-1}(x))}.$$

This tells us $g_{ij}(x)$ is a Čech 1-coboundary so is trivial in $\check{H}^1(X, \mathcal{O}_X^\times)$.) It is also easy to check that $Im(\phi)$ lies in $\ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Now, we want to show that ϕ_1 is an isomorphism. To do this, we want to find an inverse map. That is, given $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$, we want to find $f \in H^1(G, M)$ such that $L = \phi_1(f)$. (We want an explicit expression for this inverse map, we will use it very often when computing things.)

We know that π^*L is trivial. So we fix $\alpha : \pi^*L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$. (Note that G acts on π^*L, and hence via α, G also acts on $\tilde{X} \times \mathbb{C}$.) For all $\lambda \in G$, there exists automorphisms ϕ_λ of $\tilde{X} \times \mathbb{C}$ such that $\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that

- $f(\lambda, \tilde{x}) \in Z^1(G, M)$
- For another map $\alpha' : \pi^*L \to \tilde{X} \times \mathbb{C}$, we have

  $$\phi'_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, h(\lambda\tilde{x})f(\lambda\tilde{x})h(\tilde{x})^{-1}t)$$

  where $\alpha'\alpha^{-1}(\tilde{x}, t) = (\tilde{x}, h(\tilde{x})t)$.

□

In fact, via a similar proof, one can also prove the following:

2.14.7 Theorem. *Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \pi^*\mathcal{F})$. Then for each $n \geq 0$, there exist canonical homomorphisms $\phi_n : H^n(G, M) \to H^n(X, \mathcal{F})$. Moreover,*

- *when $n = 1$, for each $f \in Z^1(G, M)$, we have $(\phi_1 f)_{ij} = f(\lambda_{ij}, \pi_i^{-1})$.*
- *when $n = 2$, for each $f \in Z^2(G, M)$, we have $(\phi_2 f)_{ijk} = f(\lambda_{ij}, \lambda_{jk}, \pi_i^{-1})$.*

2.15 Global sections of $H^0(X, L)$

Our next goal is to describe global sections of $H^0(X, L) = \Gamma(L, X)$ in terms of $\phi_1(f) = L$, where $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Since $\phi^* L$ is trivial, if we fix a trivialization $\alpha : \pi^* L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$, this gives us an explicit cocycle $f \in Z^1(G, M)$ (giving the cohomology class $\phi_1^{-1}(L)$). Observe that there exists a canonical isomorphism between $H^0(X, L) \xrightarrow{\sim} H^0(\tilde{X}, \pi^* L)^G$. And then we have (via α) an isomorphism $H^0(\tilde{X}, \pi^* L)^G \simeq H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$, which is explicitly given by

$$\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t).$$

$H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$ then corresponds to the set $\{\vartheta : \tilde{X} \overset{hol}{\to} \mathbb{C} : \vartheta(\lambda, \tilde{x}) = f(\tilde{\lambda}, \tilde{x})\vartheta(x)\}$. So $H^0(X, L)$ is in bijection with the set of theta functions ϑ with f as its factor of automorphy. (Remark: This does depend on the trivialization and thus the specific form of f; but changing α changes f in a predictable way and thus changes the set of ϑ.) Also, one can check functoriality of everything we've done.

Now, let's get back to the case where $X = V/\Lambda$ is complex torus.

2.15.1 Proposition. *Every holomorphic line bundle on X pulls back to a trivial bundle on V.*

Sketch of proof. We need

- the *exponential sequence* of sheaves:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_V \overset{e^{2\pi i}}{\to} \mathcal{O}_V^\times \to 1$$

 ($\underline{\mathbb{Z}}$ is the constant sheaf of $\mathbb{Z}$.)

- the $\overline{\partial}$-Poincaré lemma; and
- $H^2(V, \mathbb{Z}) = 0$.

This induces the exact sequence:

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^\times) \to H^2(V, \mathbb{Z}) \to \cdots$$

First term is trivial by the $\overline{\partial}$-Poincare lemma, and last term is trivial by $H^2(V, \mathbb{Z}) = 0$. Therefore, $H^1(V, \mathcal{O}_V^\times)$ is also trivial. So, all line bundles on V are trivial. □

2.15.2 Corollary. *The above exact sequence gives an isomorphism* $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^\times) \simeq H^1(\Omega, H^0(V, \mathcal{O}_V^\times))$.

The proof can be found in Griffiths-Harris. [farbod add reference]

2.16 The exponential sequence and the first Chern class

In this section, we will talk about the exponential exact sequence, and generalize to X. We will then get a similar long exact sequence, but here $H^2(X, \mathbb{Z})$ is nontrivial, and the map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is extremely important! Our next goal is the Appell-Humbert theorem, which is actually due to Lefschetz.

On any complex manifold $(X, \mathcal{O}_X)$, the sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \overset{exp}{\to} \mathcal{O}_X^\times \to 1$$

is exact, where the map $exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ is the exponential map, given on open sets by $exp(U) : \mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ by

$$f \mapsto e^{2\pi i f}.$$

It is easy to check that this map is holomorphic with multiplicative inverse $e^{-2\pi i f}$.

Exactness of the sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times$$

is straightforward; the kernel of $exp : \mathcal{X}(U) \to \mathcal{O}_X^\times(U)$ are the locally constant $\mathbb{Z}$-valued functions, which are $\underline{\mathbb{Z}}(U)$. What about the surjectivity of $\mathcal{O}_X \to \mathcal{O}_X^\times$? This asks about the existence of logarithms. Well, taking complex logarithms are not always "globally" possible! However, they are "locally" possible; for any $f \in \mathcal{O}_X^\times(U)$ and any $x \in U$, we can find a contractible neighborhood V of x with $V \subset U$, and then $res_V(f)$ does have a logarithm. So $\mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ is surjective for small enough U, and this is enough for the sheaf map $\mathcal{O}_\mathcal{X} \to \mathcal{O}_X^\times$ to be considered surjective (the cokernel is a presheaf which sheafifies to 0). For example, $z \in \mathcal{O}^\times(\mathbb{C} - \{0\})$ has logarithm if you restrict to a small enough open set.

So now we have the long exact sequence that measures the failure of surjectivity of the exponential function, which will give us

$$0 \to H^0(X, \underline{\mathbb{Z}}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times) \to H^1(X, \underline{\mathbb{Z}}) \to \cdots$$

so surjectivity of the global exponential map is controlled by $H^1(X, \underline{\mathbb{Z}})$, a "generalized winding number". Continuing we get

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

and beyond this we won't really care.

2.16.1 The first Chern class

Note that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ and $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ in the above sequence is called the **first chern class map**, denoted by c_1. Also, $Im(c_1)$ is called the **Néron-Severi group** of X, denoted by $NS(X)$.

Remark: We can thus write this as $NS(X) \simeq \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ where $\mathrm{Pic}^0(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles L with $c_1(L) = 0$. This is just rephrasing the definition above, but we can also prove that $\mathrm{Pic}^0(X)$ is equal to the connected component of of the origin in $\mathrm{Pic}(X)$.

2.16.2 Theorem (Severi, Néron)**.** *$NS(X)$ is always a finitely generated abelian group.*

Severi proved the theorem for $k = \mathbb{C}$ and Néron proved for more general fields, but we are not going to make that precise. We define $\rho(X)$ to be the rank of $NS(X)$, which is the **Picard number** of X. We will see that $\rho(X) \leq h^{1,1}(X) = g^2$.

Recall that $H^2(X, \mathbb{Z}) \simeq H^1(X, \mathbb{Z}) \bigwedge H^1(X, \mathbb{Z}) \simeq \mathrm{Hom}(\Lambda, \mathbb{Z}) \bigwedge \mathrm{Hom}(\Lambda, \mathbb{Z})$ which we are going to denote by $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$. Note that $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ contains $\mathbb{Z}$-valued bilinear alternating 2-forms.

2.17 First Chern class for complex tori

When $X = V/\Lambda$ ($V \simeq \mathbb{C}^g$) is a complex torus, the exponential sequence gives

$$H^0(V, \mathcal{O}_V) \to H^0(V, \mathcal{O}_V^\times) \to 0$$

because $H^1(V, \mathbb{Z}) = 0$. So any non-vanishing global form f is $e^{2\pi i g}$ for some $g \in \mathcal{O}_V$. Recall that $H^1(\Lambda, H^0(V, \mathcal{O}_V^\times)) \simeq H^1(X, \mathcal{O}_X^\times)$. So given L, one has a factor of automorphy $(f : \Lambda \times V \to \mathbb{C}^\times) \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^\times))$. We assume $f = e^{2\pi i g}$ for $g : \Lambda \times V \to \mathbb{C}$.

2.17.1 Theorem. *With the canonical isomorphism discussed before, we have*

$$c_1(L) = E_L(\cdot, \cdot)$$

where

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

for $\lambda, \mu \in \Lambda$ *and* $v \in V$.

It is a good exercise to show that

- E_L is independent of v. (Using the fact that $f \in Z^1$.)
- E_L is an alternating 2-form, $\mathbb{Z}$-valued and also bilinear.

2.17.2 Theorem. *For any* $E : V \times V \to \mathbb{R}$ *which is an alternating* $\mathbb{R}$*-valued bilinear form on* $H^2(X, \mathbb{R})$*, the following are equivalent:*

1. $E = c_1(L)$ *for some* $L \in \mathrm{Pic}(X)$.
2. $E(\Lambda, \Lambda) \subset \mathbb{Z}$ *and* $E(iv, iw) = E(v, w)$ *for* $i \in V$.

These two theorems turn out to be very useful! We will skip the proofs for now.

It is an easy fact from Linear Algebra that there is a one-to-to correspondence between Hermitian forms H on V ($H : V \times V \to \mathbb{C}$ such that $H(v, w) = \overline{H(w, v)}$ for all $w, v \in CC$) and the set of alternating bilinear forms $E : V \times V \to \mathbb{R}$ satisfying $E(iv, iw) = E(v, w)$ for all $i \in V$. In fact, the correspondence maps are explicitly given by $H \mapsto Im(H)$ and $E \leftarrow E(iv, w) + iE(v, w)$.

2.17.3 Corollary. *$NS(X)$ is exactly the set of alternating bilinear forms* $E : V \times V \to \mathbb{R}$ *such that* $E(\Lambda, \Lambda) \subset \mathbb{Z}$ *and* $E(iv, iw) = E(v, w)$ *for all* $v, w, i \in V$. *By the above correspondence,* $NS(X)$ *also equals the set of Hermitian forms* $H : V \times V \to \mathbb{C}$ *such that* $ImH(\Lambda, \Lambda) \subset \mathbb{Z}$.

2.18 Appell-Humbert Theorem

In this section we are going to see an explicit description of $NS(X), \mathrm{Pic}(X), \mathrm{Pic}^0(X)$ for $X = V/\Lambda$ a complex torus. Recall that we had an explicit description of $NS(X)$ already: it corresponds to the set of Hermitian forms $H : V \times V \to \mathbb{C}$ with $Im(\Lambda, \Lambda) \subset \mathbb{Z}$. Define $\mathrm{Pic}^0(X)$ to be the collection of line bundles such that the first chern group on them are trivial. That is,

$$\mathrm{Pic}^0(X) = \ker(\mathrm{Pic}(X) \overline{c_1} \to H^2(X, \mathbb{Z})).$$

So, we have the following short exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

The Appell-Humbert theorem is going to give another (more explicit) short exact sequence that is isomorphic to the one above.

Let $T_1 = \{z \in \mathbb{C}^\times : |z| = 1\}$ be the unit circle in $\mathbb{C}$. Given a Hermitian form H in $NS(X)$, a character $\chi : \Lambda \to T_1$ is called a **semi-character** for H if for all $\lambda, \mu \in \Lambda$, we have

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i ImH(\lambda\mu)}.$$

Let $P(\Lambda)$ be the set of (H, χ) with $H \in NS(X)$ such that χ is a semi-character for H.

Remarks: Note that when $H = 0$, this is just a usual character. Also, $P(\Lambda)$ is a group under

$$(H_1, \chi_1) \circ (H_2, \chi_2) := (H_1 + H_2, \chi_1\chi_2).$$

Note that we have the following exact sequence:

$$0 \to \mathrm{Hom}(\Lambda, T_1) \to P(\Lambda) \to NS(X)$$

where the second map is given by $x \mapsto (0, x)$ and third map by $(H, \chi) \mapsto H$. And therefore we have a map $P(\Lambda) \to \mathrm{Pic}(X) \simeq H^1(\Lambda, M)$ (where $M = H^0(V, \mathcal{O}_V^\times)$) given by

$$(H, \chi) \mapsto a_{(H,\chi)}(\cdot, \cdot) + B^1(\Lambda, M)$$

where $a_{(H,\chi)}(\lambda, v) := \chi(\lambda)e^{(\pi H(\lambda,v)+\pi/2H(v,v))}$. It is easy to check that

$$a_{(H,\chi)}(\lambda, \mu, v) = a_{(H,\chi)}(\lambda, \mu + v)a_{(H,\chi)}(\mu, v).$$

This gives us an isomorphism

$$L(H, \chi) \simeq \frac{V \times \mathbb{C}}{\Lambda}$$

where Λ acts as $\lambda \circ (v, t) = (v + \lambda, a(\lambda, v)t)$. We have the following facts:

1. $P(\Lambda) \to \mathrm{Pic}(X)$ is a group homomorphism. (Easy to prove.)

2. The map given by composing $P(\Lambda) \to \mathrm{Pic}(X) \xrightarrow{c_1} NS(X)$ is the same as the "forget χ" map. In particular, the map $\alpha : P(\Lambda) \to NS(X)$ defined by $(H, \chi) \mapsto H$ is surjective (since c_1 is).

By these facts, we hence have the following theorem.

2.18.1 Theorem. *The following is a commutative diagram of short exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, T_1) \arrow[r] \arrow[d, "\alpha'"] & P(\Lambda) \arrow[r] \arrow[d, "\alpha"] & NS(X) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & NS(X) \arrow[r] & 0
\end{tikzcd}$$

where α' is the restriction of α to $\mathrm{Hom}(\Lambda, T_1)$.

Once we have shown that α' is an isomorphism in the proof, we will have a very explicit way to talk about $\mathrm{Pic}(X)$ - not only have we parametrized line bundles by (H, χ), such a pair even leads to a distinguished cocycle in the cohomology class to work with.

Proof. We already knew that the right-hand square in the above diagram commutes, and this implies that α' restricts to what we want (and the left-hand square commutes arbitrarily). So we just need to show that α' and α are isomorphisms. It suffices to show that α' is an isomorphism, since then the five-lemma implies α is.

First, we start with a diagram

$$\begin{tikzcd}
H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^\times) \arrow[r, "c_1"] \arrow[d, "="] & H^2(X,\mathbb{Z}) \\
 & H^1(X,\mathbb{C}) \arrow[u, two heads] \arrow[r, "\epsilon"] & H^1(X,\mathcal{O}_X^\times) &
\end{tikzcd}$$

Here, the surjective map is actually a projection which comes from the folllowing: By the Hodge decomposition, we have $H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \bigoplus H^{0,1}(X)$ for which by the Dolbeault cohomology is isomorphic to $H^0(X,\Omega_X) \bigoplus H^1(X,\mathcal{O}_X)$. So we have a projection from $H^1(X,\mathbb{C}) \twoheadrightarrow H^1(X,\mathcal{O}_X)$.

Next, by taking the sheaf map $\underline{\mathbb{C}}^\times \to \mathcal{O}_X^\times$ which takes a locally constant function f to the locally constant nonvanishing function $\exp(2\pi i f)$; this induces a map $\epsilon : H^1(X,\underline{\mathbb{C}}) \to H^1(X,\mathcal{O}_X^\times)$ on cohomology.

To see that the diagram commutes, note that since $\mathrm{Pic}^0(X)$ is the image of $\epsilon : H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}_X^\times) \simeq H^1(\Lambda, M))$, every element $L \in \mathrm{Pic}^0(X)$ is represented by a 1-cech cocycle (that is, an element of $Z^1(X,\mathcal{O}_X^\times))$ with constant coefficients. Therefore, $H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}_X^\times$ implies that the corresponding class in $H^1(\Lambda, M)$ is represented by a factor of automorphy (an element of $Z^1(\Lambda, M)$) with constant coefficients, that is, $f(\lambda, v)$ is independent of v.

We claim that α' is surjective. Let $L \in \mathrm{Pic}^0(X)$ with $f \in Z^1(\Lambda, M)$ with "constant coefficient" (i.e. $f(\lambda, v)$ is independent of v). Then we have

$$f(\lambda + \mu, \tilde{x}) = f(\lambda, \mu + \tilde{x}) f(\mu, \tilde{x}).$$

So f gives a homomorphism $f : \Lambda \to \mathbb{C}^\times$. Define g by $f = e^{2\pi i g}$, then obviously we have

$$g(\lambda + \mu) = g(\lambda) + g(\mu) \pmod{\mathbb{Z}}.$$

Therefore, $Im(g) : \Lambda \to \mathbb{R}$ is a homomorphism, which induces a linear map $Im(g) : V \to \mathbb{R}$. Define $\ell : V \to \mathbb{C}$ by $v \mapsto Im(g)(iv) + iIm(g)(v)$. Then $e^{2\pi i \ell(v)} \in H^0(V, \mathcal{O}_V^\times)$ for all $v \in V$. Therefore we have

$$\chi_L(\lambda, v) = f(\lambda) \cdot e^{2\pi i \ell(v) - 2\pi i \ell(v+\lambda)}.$$

is in $Z^1(\Lambda, M)$, that is, a 1-coboundary. But now $\chi_L \in \mathrm{Hom}(\Lambda, T_1)$. To check this,

$$\chi_L(\lambda, v) = e^{2\pi i g(\lambda) - 2\pi i \ell(v)} = e^{2\pi i (Reg(\lambda) - Img(i\lambda))} \in T_1$$

as $Reg(\lambda) - Img(i\lambda) \in \mathbb{R}$. Moreover, this last expression is independent of v. Therefore, f and $e^{2\pi i \ell(\cdot)}$ are homomorphisms and hence χ_L is also a homomorphism.

To show that α' is injective, suppose χ_1 and χ_2 both give $L \in \mathrm{Pic}(X)$, then we can write

$$\chi_1(\lambda) = \chi_2(\lambda) \cdot \frac{h(\lambda + v)}{h(v)}$$

where $\frac{h(\lambda+v)}{h(v)}$ is a 1-coboundary. Since $|\chi_1| = |\chi_2| = 1$,, we have $|h(\lambda+v)| = |h(v)|$ for all v, λ. Hence h is bounded. By the Liouville theorem h is constant and so $\chi_1 = \chi_2$.

$\square$

2.19 Canonical factors

Recall that in the proof of the Appell-Humbert Theorem, we expressed α as the map given by

$$(H, \chi) \mapsto a_{(H,\chi)}(\lambda, v) = \chi(\lambda) e^{\pi(H(\lambda,v)+1/2H(\lambda,\lambda))}$$

where $H = c_1(L)$. We define the **canonical factor** of (H, χ) to be $a_{(H,\chi)}$. In fact, most questions about line bundles boil down to explicit computations on these $a_{(H,\chi)}$!

First, let us see some basic properties:

1. $\chi(n\lambda) = \chi(\lambda)^n$ for all λ.
2. $a_L(\lambda, v + w) = a_L(\lambda, v) e^{\pi H(w,\lambda)}$.
3. $\frac{1}{a_L(\lambda,v)} = a_L(-\lambda, v) e^{-\pi H(\lambda,\lambda)}$.

2.20 Behaviour of line bundles under holomorphic maps

Recall that holomorphic maps are given by compositions of translations and homomorphisms.

2.20.1 Lemma. *If $t_x : X \to X$ is given by $y \mapsto x + y$ and we also have a map $L = L(H, \chi) \to X$,*

$$\begin{tikzcd} L \arrow[r, dotted, "t_x^*"] \arrow[d] & L \arrow[d] \\ X \arrow[r, "t_x"] & X \end{tikzcd}$$

then the pullback map is given by

$$t_x^* L(H, \chi) = L(H, \chi e^{2\pi i ImH(\tilde{x}, \cdot)})$$

where $\tilde{x}$ is a lift of x to the universal cover V.

2.20.2 Lemma. *If we have a map between complex tori $f : X' \to X$ where $X' = V'/\Lambda'$ and $X = V/\Lambda$. Suppose $L = L(H, \chi) \in \mathrm{Pic}(X)$, then*

$$f^* L(H, \chi) = L(f_{an}^* H, f_{Int}^* \chi).$$

2.20.3 Corollary (Theorem of squares)**.** *For $v, w \in X$ and $L \in \mathrm{Pic}(X)$, we have $t_{v+w}^* L \simeq t_v^* L \otimes t_w^* L \otimes L^{-1}$.*

Proof. For $L = L(H, \chi)$, we have $t_x^* L(H, \chi) = L(H, \chi e^{2\pi i Im(H(\lambda, v))})$. □

2.20.4 Corollary (Theorem of cubes for complex tori)**.** *Let X_1, X_2, X_3 be complex tori. Let L be a line bundle on $X_1 \times X_2 \times X_3$. If the restrictions of L on $X_1 \times X_2 \times \{0\}, X_1 \times \{0\} \times X_3$ and $\{0\} \times X_2 \times X_3$ are trivial, then L is also trivial on $X_1 \times X_2 \times X_3$.*

Remark: the theorem of cubes works if X_1, X_2, X_3 are just complete varieties. Even more, Mumford p.55 [farbod reference] says the theorem still holds if one of them is just connected.

2.20.5 Corollary. *For $n \in \mathbb{Z}$, let $n_X : X \to X$ be the multiplication-by-n map $x \mapsto nx$. If $L \in \mathrm{Pic}(X)$, then*

$$n_X^* L = L^{\frac{n^2+n}{2}} \bigotimes (-1)^* L^{\frac{n^2-n}{2}}$$

where L^k is the tensor product $L \bigotimes \cdots \bigotimes L$ of k copies of L.

Proof. Write $L = L(H, \chi)$. Then

$$L^{\frac{n^2+n}{2}} \bigotimes (-1)^* L^{\frac{n^2-n}{2}} = L(\frac{n^2+n}{2} H + (-1)^* \frac{n^2-n}{2} H, \chi^{\frac{n^2+n}{2}} \times (-1)^* \chi^{\frac{n^2-n}{2}}).$$

Since $(-1)^* H(u, v) = H(-u, -v) = H(u, v)$ and $(-1)^* \chi(\lambda) = \chi(-\lambda) = \frac{1}{\chi(\lambda)}$, this is equal to

$$L(n^2 H, \chi^n) = L(n^* H, n^* \chi(\cdot)) = n^* L(H, \chi).$$

□

A line bundle L is called **symmetric** if $(-1)^* L = L$.

2.20.6 Corollary. *If L is symmetric, then $n_X^* L = L^{n^2}$.*

2.20.7 Lemma. *$L = L(H, \chi)$ is symmetric if and only if $\chi(\lambda) = \pm 1$ for all $\lambda \in \Lambda$.*

Proof. Since $(-1)^* L(H, \chi) = L(H, \frac{1}{\chi})$, we have $\chi^2(\cdot) = 1$. □

2.21 Dual complex tori

Our goal of this section is to show that any complex torus $X = V/\Lambda$ has a dual $\hat{X}$ with functorial properties. Moreover, given $L \in \mathrm{Pic}(X)$, one has $\varphi_L : X \to \hat{X}$. We will also talk about $X \times \hat{X}$, Poincaré bundles and biextensions.

First of all, the Appell-Humbert theorem tells us that we have an isomorphism

$$\mathrm{Hom}(\Lambda, T_1) \simeq \mathrm{Pic}^0(X).$$

But in the case of complex tori, we have $\mathrm{Hom}(\Lambda, T_1) \simeq \mathrm{Hom}(\mathbb{Z}^{2g}, S^1) \simeq (\mathbb{R}/\mathbb{Z})^{2g}$. Is $\mathrm{Pic}^0(X)$ naturally a complex torus? The answer is yes! This is because we have a canonical isomorphism $\mathrm{Pic}^0(X) \xleftarrow{\sim} \hat{X}$ from the dual complex torus.

Recall that the cotangent space at $\underline{0}$ is given by

$$\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \{\ell : V \to \mathbb{C} :\ \ell(av + bw) = a\ell(v) + b\ell(w),\ \forall a, b \in \mathbb{C}, v, w \in V\}$$

where V is the tangent space at 0. We have a conjugate of Ω given by

$$\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) = \{\ell : V \to \mathbb{C} :\ \ell(av + bw) = \overline{a}\ell(v) + \overline{b}\ell(w),\ \forall a, b \in \mathbb{C}, v, w \in V\}.$$

Then we claim that we have a functional

$$\overline{\Omega} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}).$$

Note that $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \simeq \mathbb{R}^{2g}$. It is not hard to see that the inverse maps are given by $\ell \mapsto K = Im(\ell)$ and $\ell \leftarrow\!\shortmid K$ where $\ell(v) := -K(iv) + iK(v)$. It follows that

$$\langle \cdot, \cdot \rangle : \overline{\Omega} \times V \to \mathbb{R}$$

defined by

$$\langle \ell, v \rangle = Im\ell(v)$$

is $\mathbb{R}$-bilinear and is non-degenerate. By non-degenerate, we mean

- If $\langle \ell, v \rangle = 0$ for all ℓ, then $v = 0$; and
- If $\langle \ell, v \rangle = 0$ for all v, then $\ell = 0$.

Therefore,

$$\hat{\Lambda} := \{\ell \in \overline{\Omega} :\ \langle \ell, \lambda \rangle \in \mathbb{Z}\ \forall \lambda \in \Lambda\}$$

is a lattice (full rank, i.e. $\simeq \mathbb{Z}^{2g}$) in $\overline{\Omega}$. We then define

$$\hat{X} = \hat{\Omega}/\hat{\Lambda}$$

to be the **"dual"** of X.

2.21.1 Theorem. *There is a canonical isomorphism*

$$\hat{X} \xrightarrow{\sim} \mathrm{Pic}^0(X)$$

induced by $\overline{\Omega} \to \mathrm{Hom}(\Lambda, T_1)$ *which maps* $\ell \mapsto e^{2\pi i Im\ell(\cdot)}$ *where* $Im\ell(\cdot) = \langle \ell, \cdot \rangle$.

Proof. By non-degeneracy, the map is surjective. Also the kernel by definition is just $\hat{\Lambda}$. □

2.21.2 Corollary. $\mathrm{Pic}^0(X) = \hat{X}$ *is a complex torus.*

2.22 Basic functorial properties of $\hat{X}$

Here are some of the basic functorial properties of $\hat{X}$:

1. For $f : X_1 \to X_2$ homomorphism with $f_{an} : V_1 \to V_2$, then $f^*_{an} : \overline{\Omega_2} \to \overline{\Omega_1}$ satisfies

$$f^*_{an}(\hat{\Lambda}_2) \subset \hat{\Lambda}_1.$$

So we get a homomorphism $\hat{f} : \hat{X}_2 \to \hat{X}_1$. Moreover,

$$\begin{tikzcd}
\hat{X}_2 \arrow[r, "="] \arrow[d, "\hat{f}"] & \mathrm{Pic}^0(X_2) \arrow[r, "="] \arrow[d, "f^*"] & \mathrm{Hom}(\Lambda_2, T_1) \arrow[d, "f^*_{int}"] \\
\hat{X}_1 \arrow[r, "="] & \mathrm{Pic}^0(X_1) \arrow[r, "="] & \mathrm{Hom}(\Lambda_1, T_1)
\end{tikzcd}$$

commutes.

2. If $f : X_1 \to X_2, g : X_2 \to X_3$ are homomorphisms, then

$$\widehat{g \circ f} = \hat{f} \circ \hat{g}.$$

Also, $\hat{Id_X} = Id_{\hat{X}}$ with $\hat{\hat{X}} = X$ and hence $\hat{\hat{f}} = f$.

As a consequence, we see that $\hat{(\cdot)}$ is a contravariant functor on the category of complex tori.

2.22.1 Lemma. *The functor $\hat{(\cdot)}$ is exact. That is, for all short exact sequences*

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

we have

$$0 \to \hat{X}_3 \to \hat{X}_2 \to \hat{X}_1 \to 0.$$

Proof. An easy application of the snake lemma gives us the short exact sequence

$$0 \to \Lambda_1 \to \Lambda_2 \to \Lambda_3 \to 0.$$

Since Λ_3 is a projective (free) $\mathbb{Z}$-module, we therefore have

$$0 \to \mathrm{Hom}(\Lambda_3, T_1) \to \mathrm{Hom}(\Lambda_2, T_1) \to \mathrm{Hom}(\Lambda_1, T_1) \to 0.$$

Lastly, $\mathrm{Hom}(\Lambda_i, T_1) \simeq \mathrm{Pic}^0(X_i) \simeq \hat{X}_i$. □

2.23 Isogenies and duality

If $f : X_1 \to X_2$ is an isogeny, what can we say about $\hat{f}$?

2.23.1 Proposition. *Let $f : X_1 \to X_2$ be an isogeny, with dual homomorphism $\hat{f} = \hat{X}_2 \to \hat{X}_1$. Then*

(a) $\hat{f}$ *is an isogeny.*

(b) $\ker(\hat{f}) = \mathrm{Hom}(\ker(f), T_1)$.

(c) $\deg(\hat{f}) = \deg(f)$.

Sketch of proof. We have $\ker(\hat{f}) \simeq \ker(\mathrm{Hom}(\Lambda_2, T_1) \overset{f^*_{int}}{\to}) \, \mathrm{Hom}(\Lambda_1, T_1) \simeq \ker(\mathrm{Hom}(\Lambda_2/f^*_{int}(\Lambda$
But $\Lambda_2/f^*_{int}(\Lambda_1) \simeq \ker(f)$. □

2.24 Line bundels v.s. duality

Do line bundles descend under isogeny?

2.24.1 Proposition. *Suppose $f : X_1 \to X_2$ is an isogeny. Let $L \in \mathrm{Pic}(X_1)$. As before, we write $L = L(h, \chi)$. The following are equivalent:*

1. *$L = f^*M$ for some $M \in \mathrm{Pic}(X_2)$.*
2. *The image of $H(f_{an}^{-1}\Lambda_2, f_{an}^{-1}\Lambda_2)$ is contained in $\mathbb{Z}$.*

Proof. First statement implies the second follows from the pull-back formula above. Now, assume the image of $H(f_{an}^{-1}\Lambda_2, f_{an}^{-1}\Lambda_2)$ is contained in $\mathbb{Z}$, then $H_1 := (f_{an}^{-1})^*H \in NS(X_2)$. So there exists $\tilde{M} \in \mathrm{Pic}(X_2)$ with $c_1(\tilde{M}) = H_1$. By the pull-back formula, $c_1(f^*\tilde{M}) = H$. Then $c_1(L \bigotimes (f^*\tilde{M})^{-1}) = 0$. Since $\hat{f} : \mathrm{Pic}^0(X_2) \to \mathrm{Pic}^0(X_1)$ is surjective, there exists $N \in \mathrm{Pic}^0(X_2)$ such that $f^*N = L \bigotimes (f^*\tilde{M})^{-1}$. Now, $M = N \bigotimes \tilde{M}$ does the job! □

So far, what we have for duality is very basic. But in the next section, we will see something highly nontrivial.

2.25 Line bundles and maps $X \to \tilde{X}$

Let $L \in \mathrm{Pic}(X)$. We write the map $\varphi_L : X \to \tilde{X} = \mathrm{Pic}^0(X)$ which is given by

$$x \mapsto t_x^*L \bigotimes L^{-1}.$$

Here are some facts about this φ_L:

1. First of all, φ_L is well defined, since $c_1(t_x^*L \bigotimes L^{-1}) = 0$.
2. φ_L is a group homomorphism. To see this, recall that by the theorem of squares, we have
$$f_{x+y}^*L \bigotimes L^{-1} \simeq t_x^*L \bigotimes L^{-1} \bigotimes t_y^*L \otimes L.$$
3. φ_L has the analytic representation $\varphi_H : V \to \overline{\Omega}$ given by
$$v \mapsto H(v, \cdot).$$
(This follows from the fact that $\varphi_L(x) = L(0, e^{2\pi i Im(H(v,\cdot))})$.)
4. φ_L only depends on $c_1(L) = H$.
5. $\varphi_{L \bigotimes M} = \varphi_L + \varphi_M$.
6. For any $L \in \mathrm{Pic}(X)$, the diagram
$$\begin{tikzcd} X \arrow[r, "\varphi_L"] & \hat{X} \arrow[d, "\hat{f}"] \\ Y \arrow[u, "f"] \arrow[r, "\varphi_{f^*(L)}"'] & \hat{Y} \end{tikzcd}$$
commutes.

2.26 Kernel of φ_L

Define $K(L)$ to be $\ker(\varphi_L : X \to \tilde{X})$. Here are some basic properties of $K(L)$:

1. Let $\Lambda(L) := \{v \in V : \ Im(H(v,\lambda)) \in \mathbb{Z} \ \forall \lambda \in \Lambda\} = \varphi_H^{-1}(\hat{\Lambda})$. Then
$$K(L) \simeq \Lambda(L)/\Lambda.$$
Note that hence $K(L)$ only depends on $H = c_1(L)$.

2. $K(L \bigotimes P) \simeq K(L)$ for $P \in \mathrm{Pic}^0(X)$.

3. $K(L) = X$ if $L \in \mathrm{Pic}^0(X)$.

4. $K(L^n) = n_X^{-1} K(L)$ where $L^n = L \bigotimes \cdots \bigotimes L$ (n-copies) and n_X^{-1} is the inverse isogeny.

5. $K(L) = n_X K(L^n)$ if $n \neq 0$.

For the last two facts, recall that for $L = L(H,\chi)$, we have $L^n = L(nH,\chi^n)$. Hence, $L_1 \bigotimes L_2 = L_1 \bigotimes L_2(H_1{+}H_2, \chi_1\chi_2)$. Also, $\Lambda(L^n) = \{v \in V : \ Im(H(nv,\lambda)) \in \mathbb{Z}, \ \forall \lambda \in \Lambda\} = \{\frac{1}{n}v \in V : \ v \in \Lambda(L)\}$.

Recall that $H(\cdot,\cdot)$ is called **non-degenerate** if

- $H(v,w) = 0$ for all v implies $w = 0$; and
- $H(v,w) = 0$ for all w implies $v = 0$.

We say that $L \in \mathrm{Pic}(X)$ is **non-degenerate** if $c_1(L) = H$ is non-degenerate. (Equivalently, the associated alternating form $Im(H)$ is non-degenerate.)

2.26.1 Lemma.

1. *L is non-degenerate if and only if $K(L)$ is finite.*

2. $\deg(\varphi_L) = \det(Im(H)) = [\Lambda(L) : L]$.

2.27 Poincaré bundle:

We have seen that for X a complex torus,

(a) a point in $\hat{X} = \mathrm{Pic}^0(X)$ gives a line bundle on X;

(b) a point on $X = \hat{\hat{X}} = \mathrm{Pic}^0(\hat{X})$ gives a line bundle on $\hat{X}$.

So, does there exist a (universal) line bundle on $X \times \hat{X}$ such that (a) and (b) are "shadows" of that line bundle? The answer is yes!

A **Poincaré bundle** is a holomorphic line bundle $\mathcal{P}$ on $X \times \hat{X}$ satisfying

1. $\mathcal{P}|_{X \times \{L\}} \simeq L$;

2. $\mathcal{P}|_{\{0\} \times \hat{X}}$ is the trivial line bundle on $\hat{X}$.

2.27.1 Theorem. *There exists a Poincaré bundle on $X \times \hat{X}$ uniquely determined up to isomorphism.*

Proof. First note that we have

$$X \times \hat{X} \simeq V \times \overline{\Omega}/\Lambda \times \hat{\Lambda}.$$

For the existence, we define $H : (V \times \overline{\Omega}) \times (V \times \overline{\Omega}) \to \mathbb{C}$ by

$$((v_1, \ell_1), (v_2, \ell_2)) \mapsto \overline{\ell_2(v_1)} + \ell_1(v_2)$$

which is in fact a Hermitian form. (In particular, it is non-degenerate!) Note that

$$Im(H(\Lambda \times \hat{\Lambda}, \Lambda \times \hat{\Lambda})) \subset \mathbb{Z}.$$

So there exist $\mathcal{L} = L(H, \chi)$ for semi-characters χ. We then define $\chi_0 : \Lambda \times \hat{\Lambda} \to T_1$ by

$$(\lambda, \ell_0) \mapsto e^{\pi i Im(\ell_0(\lambda))}$$

which is indeed a semi-character for H.

Now, we claim that $\mathcal{P} = L(H, \chi_0)$ is a Poincaré bundle. To see this, we consider the associated canonical factor

$$a_{\mathcal{P}}((\lambda, \ell_0), (v, \ell)) = \chi((\lambda, \ell_0)) \cdot e^{\pi(H((\lambda,\ell_0),(v,\ell)) + \frac{1}{2}H((\lambda,\ell_0),(\lambda,\ell_0)))}.$$

Now, what is left is just the checkings:

1. For any $L \in \hat{X} = \mathrm{Pic}^0(X)$, we have $L = L(0, e^{2\pi i Im(\ell(\cdot))})$ for some $\ell \in \overline{\Omega}$. Note that
$$a_L(\lambda, v) = e^{2\pi i Im(\ell(\lambda))}.$$
Therefore, $\mathcal{P}|_{X \times \{L\}}$ corresponds to $a_{\mathcal{P}}|_{(\Lambda,0)\times(V\times\{\ell\})}$ but
$$a_{\mathcal{P}}((\lambda, 0), (v, \ell)) = e^{\pi \ell(\lambda)}.$$
Now, multiplication by the 1-coboundary $e^{\pi\overline{\ell(v)}}/e^{\pi\overline{\ell(v+\lambda)}}$ takes $a_{\mathcal{P}}((\lambda, 0), (v, \ell))$ to $a_L(\lambda, v)$. Therefore, $\mathcal{P}|_{X\times\{L\}} \simeq L$.

2. $\mathcal{P}_{\{0\}\times\hat{X}}$ has $a_{\mathcal{P}}((0, \ell_0), (0, \ell)) = 1$ as 1-cocycle. So it is trivial.

Uniqueness of the Poincaré bundle follows from the *Seesaw Principle*: Let X, Y be compact complex manifolds and $\mathcal{L}$ be a holomorphic line bundle on $X \times Y$. If $L|_{X\times\{z\}}$ is trivial for all $z \in \mathcal{U}$ where $\mathcal{U}$ is an open dense subset of Y, and if $L|_{\{x_0\}\times Y}$ is trivial for some $x_0 \in X$, then L is trivial.

□

Here are some remarks:

- Poincaré line bundles are non-degenerate.
- Let T be any *normal* complex analytic space and X be a complex torus. If L is a line bundle on $X \times T$ such that
    1. $L|_{X\times\{t\}} \in \mathrm{Pic}^0(X)$ for all $t \in T$ (If T is connected, it suffices to check that there exists $t \in T$ such that this is true); and
    2. $L|_{\{0\}\times T}$ is trivial;

  then there exists a unique holomorphic $\psi : T \to \hat{X}$ such that $L \simeq (id \times \psi)^*\mathcal{P}$. That is, $\mathcal{P}$ factors through ψ. (The proof of this uses *Zariski's main theorem* and a more general Seesaw Principle.)

2.28 A few applications of Poincaré bundles

Let L_1, L_2 be line bundles on X. We say that L_1 is **analytically equivalent** to L_2, denoted by $L_1 \sim_{an} L_2$ if there exist a connected complex analytic space T, a line bundle L on $X \times T$ and $t_1, t_2 \in T$ such that

$$L|_{X\times\{t_i\}} \simeq L_i.$$

2.28.1 Proposition. *Let L_1, L_2 be line bundles on a complex torus $X = V/\Lambda$. The following are equivalent:*

1. $L_1 \sim_{an} L_2$.
2. $L_1 \bigotimes L_2^{-1} \in \mathrm{Pic}^0(X)$.
3. $\varphi_{L_1} = \varphi_{L_2}$.
4. $c_1(L_1) = c_1(L_2)$.

For a complex torus X, we use $\mathrm{Pic}^H(X)$ to denote the classes in $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

2.28.2 Corollary. *Let $X = V/\Lambda$ be a complex torus. Then there is a correspondence between analytic equivalence classes of X and $\mathrm{Pic}^H(X)$.*

Now we prove the proposition.

Proof. We already know that (2), (3) and (4) implies one and other. To show that (2) implies (1), assume that $L_1 \bigotimes L_2^{-1} \in \mathrm{Pic}^0(X)$ and $p : X \times \hat{X} \to X$ is the projection map. Then $L = p^*L_2 \bigotimes \mathcal{P}$ is a line bundle on $X \times \hat{X}$ such that $L|_{X\times\{0\}} \simeq L_2$ and $L|_{X\times\{L_1 \bigotimes L_2^{-1}\}} \simeq L_1$.

Now to show that (1) implies (4), suppose $L_1 \sim_{an} L_2$, that is, there exist complex analytic space T, a line bundle L on $X \times T$ such that $L|_{X\times\{t_i\}} \simeq L_i$. Consider $T \to H^2(X, \mathbb{Z})$ given by $t \mapsto c_1(L|_{X\times\{t\}})$ is continuous and the image is discrete and so it is a constant map. This implies that $c_1(L_1) = c_1(L_2)$.

□

2.28.3 Lemma. *Let $L, L' \in \mathrm{Pic}(X)$. Suppose L is non-degenerate, then $L \sim_{an} L'$ if and only if there exists $x \in X$ such that $L' \simeq t_x^*L$.*

Proof. The "if" direction is always true. To show the "only if" direction, $L \sim_{an} L'$ is equivalent to saying that $L' \bigotimes L^{-1} \in \mathrm{Pic}^0(X)$. But since L is non-degenerate, $\varphi_L : X \to \mathrm{Pic}^0(X)$ defined by $x \mapsto t_x^*L \bigotimes L^{-1}$ is surjective. So there exists $x \in X$ such that $t_x^*L \bigotimes L^{-1} \simeq L' \bigotimes L^{-1}$. □

Given a homomorphism $f : X \to \hat{X}$, does there exist a line bundle L such that $f = \varphi_L$?

2.28.4 Theorem. *Let $X = V\Lambda$ be a complex torus and $f : X \to \hat{X}$ a homomorphism with $f_{an} : V \to \overline{\Omega}$. The following are equivalent:*

1. $f = \varphi_L$ *for* $L \in \mathrm{Pic}(X)$.

2. $F : V \times V \to \mathbb{C}$ *defined by*

$$(v, w) \mapsto f_{an}(v)(w)$$

is Hermitian.

The proof to this theorem is pretty elementary, straight-forward and uses the following lemma:

2.28.5 Lemma. *Let* $M \in \text{Pic}(X)$ *and* $n \in \mathbb{Z}$. *Then the following are equivalent:*

1. $M = L^n$ *for some* $L \in \text{Pic}(X)$.

2. $X[n] \subset K(M)$ *where* $K(M)$ *is the kernel of* $\varphi_M : X \to \hat{X}$.

2.29 The Poincare-Bundle as a Biextension

A Poincaré bundle $\mathcal{P}$ is a biextension of $X \times \hat{X}$ by $\mathbb{C}^\times$. Then we will define an object $Bi - ext(B \times C, A)$.

All abelian varieties in this lecture are over the complex numbers. The following definition of biextensions can be found for instance in Mumford's paper 'Biextensions of Formal Groups'

2.29.1 Definition. Let A, B, C be abelian groups. A biextension of $B \times C$ by A is a set G along with
1. An action of A on G.
2. A surjective map $\pi : G \to B \times C$,

$$\pi(g) = (\pi_B(g), \pi_C(g))$$

which induces a bijection $G/A \xrightarrow{\sim} B \times C$
3. Maps

$$+_1 : G \times_B G \to G$$

$$+_2 : G \times_C G \to G$$

so that the following conditions are satisfied
1. $\forall b \in B$, the fibre over $b \times C$ in G, $G'_b := \pi_B^{-1}(b) = \pi^{-1}(b \times C)$ is an abelian group with respect to the restriction of $+_1$. π_C is a surjective homomorphism of G'_b onto C, the kernel of π_C is isomorphic to A.
2.Likewise, the fibre G_c over $B \times c$ for $c \in C$ is an abelian group with respect to the restriction of $+_2$. π_B is a surjective homomorphism of G_c onto B, the kernel of π_B is isomorphic to A.
3. Given $x, y, u, v \in G$, with

$$\pi(x) = (b_1, c_1), \pi(y) = (b_1, c_2), \pi(u) = (b_2, c_1), \pi(v) = (b_2, c_2)$$

the following compatibility relation holds

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v)$$

($G \times_B G$ is the fibred product $G \times_B G := \{(g_1, g_2) \in G \times G \mid \pi_B(g_1) = \pi_B(g_2)\}$, likewise, the set $G \times_C G := \{(g_1, g_2) \in G \times G \mid \pi_C(g_1) = \pi_C(g_2)\}$)

The definition of a biextension seems hard to grasp at first glance. The example of the Poincare Bundle with the zero-section removed as a bi-extension of an abelian variety and its dual by $\mathbb{C}^\times$ should be understood to put the definition in perspective. Let $X = V/\Lambda$ be an abelian variety and $\hat{X} = \Omega/\hat{\Lambda}$ be the dual abelian variety. Let $P \to X \times \hat{X}$ be the Poincare-Bundle on $X \times \hat{X}$. The example we have in mind is that of the Poincare Bundle with the zero-section removed, ie, $A = \mathbb{C}^\times$, $B = X$, $C = \hat{X}$ and $G = P/\{0\}$ with $\pi : P/\{0\} \to X \times \hat{X}$ the projection map restricted to the complement of the zero section, A acts by scalar multiplication. We note in passing that the trivial $\mathbb{C}^\times$ bundle on a vector space W, $L_0 := \mathbb{C}^\times \times W$ is an abelian group with group operation

$$(l_1, w_1) + (l_2, w_2) := (l_1 l_2, w_1 + w_2)$$

with identity $(1, 0)$ and inverse $(l, w)^{-1} = (\frac{1}{l}, -w)$. If $\Pi : L \to W/\Lambda$ is any line bundle on an abelian variety, then $\Pi^*(L/\{0\}) \simeq \mathbb{C}^\times \times W$ is an abelian group and this group structure descends to a natural group structure on $L/\{0\}$. $G_L \simeq L\{0\}$ by hypothesis, G'_x is a line bundle on $\hat{X}$ with zero section removed. Points of $G \times_C G$ (resp $G \times_B G$) correspond to pairs of points (l_1, l_2) on a line bundle over the abelian variety X (resp $\hat{X}$), the maps $+_1$ and $+_2$ are determined so as to correspond to the group operations on the line bundles with zero section removed. The reader need not work out condition 3, it is in fact a nontrivial result which follows from Lang Duality.

Equivalence classes of biextensions can be suitably expressed in the context of cohomology, we do not however pursue this theme any further.

2.30 Cohomologies of Line Bundles on Complex Tori

We will now proceed to discuss the notions of characteristics of line bundles L on X, theta-functions as sections of line-bundles and more generally describe all the cohomology groups $H^i(X, L)$. We shall then prove some vanishing theorems for cohomology and compute the alternating sums of the cohomological dimensions from which we can deduce Riemann-Roch.

Fix $H \in NS(X)$, let $\mathrm{Pic}^H(X)$ denote the line bundles on X with chern class H. Given a suitable decomposition of $\Lambda = \Lambda_1 \oplus \Lambda_2$ (which are in some way orthogonal) we can distinguish a line-bundle $L_0 \in Pic^H(X)$. If H is nondegenerate, $L \in Pic^H(X)$ is a translate $L = t_c^* L_0$ and c is called the characteristic of L_0 with respect to the decomposition of Λ. This will allow us to explicitly describe $K(L) = \ker \phi_L$. Let $E = \mathrm{Im} H$, this is a $\mathbb{Z}$ valued alternating form.

2.30.1 Lemma. *Suppose that $2g$ is the rank of the lattice Λ. There exists a $\mathbb{Z}$-basis for Λ, $U = \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ such that the matrix for E wrt U is*

Proof. Pick any basis to begin with. Since H is hermitian, E is skew symmetric, so the matrix for E in this arbitrary basis looks like

$$\begin{pmatrix} F & A \\ -A^T & G \end{pmatrix}$$

where F and G are also skew symmetric. It's easy to see that we may further assume that $F = G = 0$. By row and column operations over $\mathbb{Z}$ we may reduce A to a diagonal matrix, over $\mathbb{Z}$ we essentially use the fact that the gcd of two numbers can be expressed as a linear combination of these numbers. So $\exists U, V \in GL_n(\mathbb{Z})$ such that $UAV = \mathrm{diag}(d_1, d_2, \ldots, d_g)$ with d_i dividing d_{i+1} (note that in the case where we work over a field we may in fact insist that $V = U^{-1}$).

$$\begin{pmatrix} U & 0 \\ 0 & V^T \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

□

$(d_1, \ldots, d_g)$ is uniquely determined by E or H or L.

2.30.2 Definition. We call the tuple $(d_1, \ldots, d_g)$ the type of E or H or L and if all the $d_i = 1$ we call L a principle polarization.

We see that $K(\Lambda) = \ker\phi_L \simeq K_1 \oplus K_2$ with $K_i \simeq \oplus\mathbb{Z}/d_i\mathbb{Z}$. If all $d_i > 0$ then H or L or E is non-degenerate.

2.30.3 Definition. A basis $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ be as before giving rise to the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ is called a canonical or symplectic-basis for Λ. A sub-lattice $\Lambda_1 \subset \Lambda$ is called totally isotropic for E if $E(\lambda, \lambda') = 0\ \forall \lambda, \lambda' \in \Lambda$.

2.30.4 Definition. A decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ is called a decomposition for E or H or L if both Λ_1 and Λ_2 are totally isotropic.

2.30.5 Definition. A decomposition $V = V_1 \oplus V_2$ of V into real vector spaces such that $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a decomposition for Λ is called the decomposition of V for E or H or L.

Let $H \in NS(X)$, $V = V_1 \oplus V_2$ a decomposition for H, define $\chi_0 : V \to T_1$ by

$$\chi_0(v) = e^{\pi i ImH(v_1, v_2)} = e^{\pi i E(v_1, v_2)}$$

where $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. It is easily seen that for $v, w \in V$,

$$\chi_0(v + w) = \chi_0(v)\chi_0(w)e^{\pi i E(v,w)}e^{-2\pi i E(v_2, w_1)}$$

(keep in mind that $E(\lambda, \mu) \in \mathbb{Z}$ for $\lambda, \mu \in \Lambda$)

2.30.6 Corollary. $(\chi_0)_{|\Lambda}$ *is a semicharacter for* H

2.30.7 Definition. $L_0 := L(H, \chi_0) \in \mathrm{Pic}^H(X)$ is a distinguished element of $\mathrm{Pic}^H(X)$ with respect to the decomposition of V.

3 Appendix

3.1 Poincaré lemmas, De Rham cohomology, Dolbeault cohomology

If we are over $\mathbb{R}$ we have the classical Poincaré lemma, which lets us get the connection between De Rham cohomology and singular cohomology. If we're over $\mathbb{C}$, we have a $\bar{\partial}$-Poincaré lemma which tells us something about Dolbeault cohomology. We will say things about these, and then a bit more about Hodge theory (and what is the major simplification for abelian varieties versus Kalher manifolds).

3.1.1 Classical Poincaré lemma and De Rham cohomology

Let us look at $\mathbb{C}^n$, which is $\mathbb{R}^{2n}$ as a real manifold (or more generally any smooth manifold M). Let $T_0^*(\mathbb{C}^n)$ (or $T_z^*(M)$) be the cotangent space, and pick the usual basis $\{dx_j, dy_j\}_{j=1}^n$. Over $\mathbb{C}$ we will also want to work with the complex basis $\{dz_j, d\bar{z}_j\}_{j=1}^n$ with $dz_j = dx_j + i dy_j$ and $d\bar{z}_j = dx_j - i dy_j$. For the tangent space $T_0\mathbb{C}^n$ or $T_z M$, let $\{\partial/\partial z_j, \partial/\partial \bar{z}_j\}$ denote the dual basis. One can easily check that

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right).$$

Remark: Over $\mathbb{C}$, the Cauchy-Riemann equations tell us that if $f \in C^\infty(U)$ then f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$.

Given $f \in C^\infty(U)$ with $U \subset \mathbb{C}^n$, define the **total differential** as

$$df = \partial f + \bar{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_i + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

3.1.2 Theorem. *For $f \in C^\infty(U)$ with $U \subset \mathbb{C}^n$, f is holomorphic if and only if $\partial f = 0$.*

A **differential form** of degree k is a C^∞-global section of $\bigwedge^k T^* = \Omega^k$. At each $p \in M$, it is an alternating multilinear form $T_p(M) \times \cdots \times T_p(M) \to \mathbb{R}$. The set of k-forms on M can be denoted as $\Omega^k(M)$, $A^k(M)$, $\Gamma(\Omega^k, M)$ or $H^0(M, \Omega^k)$. The differential maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ can be defined explicitly on coordinates, or alternatively characterized as the unique collection of $\mathbb{R}$-linear maps satisfying:

1. $\forall f \in \Omega^0(M) = C^\infty(M)$, df is the total differential of f, and $d(df) = 0$;
2. We have $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$.

From these one can write down d in local coordinate as $d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$, and verify that for any differential form α, $dd\alpha = 0$. Hence we have the de Rham cochain complex:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots$$

whose cohomology

$$H^p_{dR}(M) = \frac{\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{Im(d : \Omega^{p-1}(M) \to \Omega^p(M))}$$

is called the **de Rham cohomology**. Elements of this kernel are called **closed** and elements of the image are called **exact**. (Example to keep in mind: If S^1 is the circle we have a closed form $d\theta$ which is not exact.) Note that $\Omega^k(M) = 0$ for $k > \dim M$.

3.1.3 Lemma (Poincaré Lemma). *If $U \subset \mathbb{R}^n$ is contractible, then $H^p_{dR}(U) = 0$ for $p \geq 1$.*

This shows that $0 \to \underline{\mathbb{R}} \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$ is an exact sequence of sheaves. As a consequence, we have:

3.1.4 Theorem (de Rham theorem). *If M is a smooth manifold, then $\check{H}^p(M, \underline{\mathbb{R}})$ (which is our usual singular cohomology) is isomorphic to $H^p_{dR}(M)$.*

The idea of the proof is to break the above exact sequence up into short exact sequences in the standard way, then use the corresponding long exact sequences on cohomology.

3.1.5 $\overline{\partial}$ Dolbeault cohomology

Now we want to start using the complex structure on a complex manifold. Using holomorphic charts, we locally have differentials $\{dz_i, d\overline{z}_i\}$ spanning the cotangent space, with dual elements $\{\partial/\partial z_i, \partial/\partial \overline{z}_i\}$ spanning the tangent space. Using these we have the following splitting

$$T^*_p(M) = {T^*}'_p(M) + {T^*}''_p(M),$$

where $T^{*\prime}$ is spanned by dz_j and $T^{*\prime\prime}$ is spanned by $\overline{dz_j}$. This gives a direct sum decomposition

$$\bigwedge^n T^*_p(M) = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right).$$

with the (p,q) part spanned by things of the form $dz_I \wedge d\overline{z}_J$ with $|I| = p$ and $|J| = q$. Using this decomposition, define a sheaf $\Omega^{p,q}$ of $C^\infty(p,q)$ forms: we set

$$\Omega^{p,q}(M) = \left\{\varphi \in \Omega^n(M) : \forall z, \varphi(z) \in \left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right)\right\}.$$

It is also denoted by $A^{p,q}(M)$, $\Gamma(\Omega^{p,q}, M)$ or $H^0(M, \Omega^{p,q})$. This gives a filtration $\Omega^n(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M)$.

What happens when we apply d to something in $\Omega^{p,q}(M)$? We can see it maps into $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ because $\varphi(z)$ is in

$$\left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right) \wedge T^*_z(M).$$

The decomposition of Ω^n induces a decomposition $d = \partial + \overline{\partial}$, where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. By writing them down in local coordinates one can check that $\overline{\partial}^2 = 0$, hence there is a cochain complex:

$$0 \to \Omega^{p,0}(M) \xrightarrow{\partial} \Omega^{p,1}(M) \xrightarrow{\partial} \Omega^{p,2}(M) \cdots$$

and we define the **Dolbeault cohomology** as

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{Im(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}$$

Now, we want to relate this to some other sheaf cohomology; our exact-seqeunce-based proof of the de Rham theorem suggests what we need. The *$\overline{\partial}$-Poincaré lemma* says that on a polydisc D in $\mathbb{C}^n$ (a product of discs in $\mathbb{C}$), $H^{p,q}_{\overline{\partial}}(D) = 0$ for all $q \geq 1$. (Poincaré was studying the problem that if $g \in C^\infty(D)$ for $D \subset \mathbb{C}$ then he wanted to find f with $\partial f \partial \overline{z} = g$ which can be solved on a slightly smaller disc.)

Hence, as an analogue of the de Rham theorem, by letting Ω^p_{hol} be the sheaf of holomorphic p-forms, Poincaré lemma tells us that we have an exact sequence

$$0 \to \Omega^p_{\text{hol}} \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \cdots$$

with $\Omega^p_{hol} \to \Omega^{p,0}$ the inclusion and the other maps $\overline{\partial}$. Note that Ω^p_{hol} is indeed the kernel of $\overline{\partial}$ on Ω^p_{hol}. By the same argument from our sheaf proof of the de Rham theorem we can prove the Dolbeault theorem:

3.1.6 Theorem (Dolbeault theorem)**.**

$$H^{p,q}_{\overline{\partial}}(M) \simeq H^q(M, \Omega^p_{hol})$$

for $p, q \geq 0$.

3.1.7 Example.

- When $q \geq \dim(M)$, $H^{0,q}(M) = H^q(M, \mathcal{O}_M) = 0$. (So if $q \geq \dim(M)$ both are zero.)
- When $q \geq 1$, $H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0$ by the above example and the $\overline{\partial}$-Poincaré lemma.
- When D is a polydisc in $\mathbb{C}^n$, $H^{p,0}_{\overline{\partial}}(D, \Omega^p_D) = H^0(D, \mathcal{O}_D) \otimes \Omega^p_D$ is usually nontrivial, so the $q > 1$ hypothesis in the Poincaré lemma matters!

3.2 Hodge decomposition

In this section, we will talk about the decomposition on complex tori. Assume $X = V/\Lambda$ is a complex torus. Let $e_1, \ldots e_g$ be a complex basis of V and $v_1, \cdots, v_g$ the corresponding coordinate functions. Then $H^n(X, \mathbb{C})$ is isomorphic to the set

$$IF^n(X) = \bigoplus_{p+q=n} IF^{p,q}(X)$$

where $IF^{p,q}(X)$ (the "invariant forms") are the things of the form

$$\sum_{|I|=p, |J|=q} a_{IJ} dv_I \wedge d\overline{v}_J$$

for $a_{IJ} \in \mathbb{C}$. Note that because we are on a torus, it is easy to write down this decomposition.

For a more general (compact) X, we have the following more general theory:

3.2.1 Theorem. *If X is a compact complex manifold with a "nice" metric, then we have*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X)$ is something that is isomorphic to $H^{p,q}_{\overline{\partial}}(X)$ and is isomorphic to the space ot harmonic forms. Also we have

$$H^{p,q} = \overline{H^{q,p}}.$$

In the above theorem, "nice" metric means a Euclidean metric or a "degree 2 approximation" of one, i.e. Kähler metric. For example, $X = V/\Lambda$ has the Euclidean metric and any complex projective variety has a Kähler metric (since $\mathbb{P}^n$ has a Fubini-Study metric). For our situation of $X = V\Lambda$, one can show the $IF^{p,q}(X)$ we wrote down is isomorphic to $H^{p,q}(X)$ is isomorphic to $H^{p,q}_{\overline{\delta}}(X) \simeq H^{p+q}_d(X) \simeq H^q(X,\Omega^p)$. (These are all hard; once we have shown these, we can then show that in the case of a complex torus, the harmonic things are just hte invariant forms and we recover what we had above.)

What is in the background of these isomorphisms? (We only consider the case $X = V/\Lambda$, but the following also contains all of the ideas we need in general.) We start with our Euclidean metric $ds^2 = \Sigma dv_i \otimes d\overline{v}_i$. This has an associated $(1,1)$-form

$$\omega = -\frac{1}{2}Im(ds^2) = \frac{i}{2}\sum_{i=1}^{g} dv_i \wedge d\overline{v}_i.$$

Then we get a volume form

$$dv = \frac{1}{g!}\bigwedge^g \omega = (-1)^{\binom{g}{2}}\left(\frac{i}{2}\right)^g (dv_1 \wedge d\overline{v}_1 \wedge dv_2 \wedge \cdots).$$

Now that we have a volume form we can define an inner product on $\Omega^{p,q}(M)$ (but not complete, so not a Hilbert space) by

$$(\varphi,\psi) = \sum_{|I|=p,|J|=q}\int_X \varphi_{IJ}\overline{\psi}_{IJ}dv$$

for $\varphi = \sum \varphi_{IJ} dv_I \wedge \overline{v}_J$ and similarly for ψ. This makes it into a pre-Hilbert space (i.e. a non-complete inner product space) and we can then define an adjoint map $\overline{\delta}$ of $\overline{\partial}$ satisfying $(\varphi, \overline{\partial}\psi) = (p\overline{\varphi}, \psi)$. Then the **Laplace-Beltrami operator** is given by

$$\Delta = \overline{\partial\delta} + \overline{\delta\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M).$$

Note that Δ can also be written as $(\overline{\delta} + \overline{\partial})^2$. In our situation in coordinates we can compute

$$\Delta(\varphi dv_I \wedge dv_J) = -\sum_i \frac{\partial^2\varphi}{\partial v_i \partial\overline{v}_i}(dv_I \wedge dv_J),$$

and so this is really the usual Laplacian.

The main point of this formal setup is that we can very easily show that a closed form ψ whose "norm" (ψ,ψ) is minimal in its class (in the de Rham or Dolbeault cohomology) is the unique solution in that class to $\overline{\delta}\psi = 0$. Note that

$\overline{\partial}\psi = 0$ and $\overline{\delta}\psi = 0$. Therefore $\Delta\psi = 0$ and ψ is Harmonic. (The converse is easy to prove too.) Therefore, we conclude that elements of $H^{p,q}(X)$, which are Harmonic (p,q)-forms, are unique representatives for classes in $H^{p,q}_{\overline{\partial}}(X)$. In order to understand these Harmonic forms, we need to solve partial differential equations. And there are two operators that can help solving these equations:

1. $H : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$ which is defined by

$$\varphi dv_I \wedge d\overline{v}_J \mapsto \left(\frac{1}{\mathrm{vol}(X)}\int_X \varphi dV\right) dv_I \wedge d\overline{v}_J.$$

   This projects any form onto an invariant thing (certainly satisfies $H^2 = H$) and in the abelian variety cse this projects onto $IF^{p,q}(M)$.

2. The second operator is G, a Green's function for Δ such that it is an inverse to the extent we can have: $\Delta G = G\Delta = 1 - H$ and $HG = GH = 0$. In general, such a G is hard to find! (For general Kähler manifolds, one can use the Sobolev lemma. See Griffiths and Harris Chapter 0. [farbod cite]) For complex tori, we are okay because such a G has a formula given in terms of Fourier analysis: Let φ be a function on X, then we can lift it to $\tilde{\varphi}$ a periodic function on $V \simeq \mathbb{C}^g$. For such a $\tilde{\varphi}$, we have a fourier expansion and $G\tilde{\varphi}$ is just the same expression with renormalized coefficients.

   Once we have this, we can then prove the Hodge theorem, that every class has a unique harmonic representative (ultimately the hard part of the theory is showing that we have enough harmonic forms!) because for a form φ, we can explicitly write down

$$\varphi = H\varphi + \overline{\delta\partial}\varphi + \overline{\partial\delta}\varphi.$$

   as our Hodge decomposition.

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