# ABELIAN VARIETIES- 15 MARCH 2016

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## 1. The Poincare-Bundle as a Biextension

. All abelian varieties in this lecture are over the complex numbers. The following definition of biextensions can be found for instance in Mumford's paper 'Biextensions of Formal Groups'

**Definition 1.1.** *Let* $A, B, C$ *be abelian groups. A biextension of* $B \times C$ *by* $A$ *is a set* $G$ *along with*
*1. An action of* $A$ *on* $G$.
*2. A surjective map* $\pi : G \to B \times C$,

$$\pi(g) = (\pi_B(g), \pi_C(g))$$

*which induces a bijection* $G/A \xrightarrow{\sim} B \times C$
*3. Maps*

$$+_1 : G \times_B G \to G$$
$$+_2 : G \times_C G \to G$$

*so that the following conditions are satisfied*
*1.* $\forall b \in B$, *the fibre over* $b \times C$ *in* $G$, $G'_b := \pi_B^{-1}(b) = \pi^{-1}(b \times C)$ *is an abelian group with respect to the restriction of* $+_1$. $\pi_C$ *is a surjective homomorphism of* $G'_b$ *onto* $C$, *the kernel of* $\pi_C$ *is isomorphic to* $A$.
*2.Likewise, the fibre* $G_c$ *over* $B \times c$ *for* $c \in C$ *is an abelian group with respect to the restriction of* $+_2$. $\pi_B$ *is a surjective homomorphism of* $G_c$ *onto* $B$, *the kernel of* $\pi_B$ *is isomorphic to* $A$.
*3. Given* $x, y, u, v \in G$, *with*

$$\pi(x) = (b_1, c_1), \pi(y) = (b_1, c_2), \pi(u) = (b_2, c_1), \pi(v) = (b_2, c_2)$$

*the following compatibility relation holds*

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v)$$

*(*$G \times_B G$ *is the fibred product* $G \times_B G := \{(g_1, g_2) \in G \times G \mid \pi_B(g_1) = \pi_B(g_2)\}$, *likewise, the set* $G \times_C G := \{(g_1, g_2) \in G \times G \mid \pi_C(g_1) = \pi_C(g_2)\}$*)*

The definition of a biextension seems hard to grasp at first glance. The example of the Poincare Bundle with the zero-section removed as a bi-extension of an abelian variety and its dual by $\mathbb{C}^\times$ should be understood to put the definition in perspective. Let $X = V/\Lambda$ be an abelian variety and $\hat{X} = \Omega/\hat{\Lambda}$ be the dual abelian variety. Let $P \to X \times \hat{X}$ be the Poincare-Bundle on $X \times \hat{X}$. The example we have in mind is that of the Poincare Bundle with the zero-section removed, ie, $A = \mathbb{C}^\times$, $B = X$, $C = \hat{X}$ and $G = P/\{0\}$ with

*Date*: March 15, 2016.

$\pi : P/\{0\} \to X \times \hat{X}$ the projection map restricted to the complement of the zero section, $A$ acts by scalar multiplication. We note in passing that the trivial $\mathbb{C}^\times$ bundle on a vector space $W$, $L_0 := \mathbb{C}^\times \times W$ is an abelian group with group operation

$$(l_1, w_1) + (l_2, w_2) := (l_1 l_2, w_1 + w_2)$$

with identity $(1, 0)$ and inverse $(l, w)^{-1} = (\frac{1}{l}, -w)$. If $\Pi : L \to W/\Lambda$ is any line bundle on an abelian variety, then $\Pi^*(L/\{0\}) \simeq \mathbb{C}^\times \times W$ is an abelian group and this group structure descends to a natural group structure on $L/\{0\}$. $G_L \simeq L\{0\}$ by hypothesis, $G'_x$ is a line bundle on $\hat{X}$ with zero section removed. Points of $G \times_C G$ (resp $G \times_B G$) correspond to pairs of points $(l_1, l_2)$ on a line bundle over the abelian variety $X$ (resp $\hat{X}$), the maps $+_1$ and $+_2$ are determined so as to correspond to the group operations on the line bundles with zero section removed. The reader need not work out condition 3, it is in fact a nontrivial result which follows from Lang Duality.

. Equivalence classes of biextensions can be suitably expressed in the context of cohomology, we do not however pursue this theme any further.

## 2. Cohomologies of Line Bundles on Complex Tori

. We will now proceed to discuss the notions of characteristics of line bundles $L$ on $X$, theta-functions as sections of line-bundles and more generally describe all the cohomology groups $H^i(X, L)$. We shall then prove some vanishing theorems for cohomology and compute the alternating sums of the cohomological dimensions from which we can deduce Riemann-Roch.

. Fix $H \in NS(X)$, let $\mathrm{Pic}^H(X)$ denote the line bundles on $X$ with chern class $H$. Given a suitable decomposition of $\Lambda = \Lambda_1 \oplus \Lambda_2$ (which are in some way orthogonal) we can distinguish a line-bundle $L_0 \in \mathrm{Pic}^H(X)$. If $H$ is nondegenerate, $L \in \mathrm{Pic}^H(X)$ is a translate $L = t_c^* L_0$ and $c$ is called the characteristic of $L_0$ with respect to the decomposition of $\Lambda$. This will allow us to explicitly describe $K(L) = \ker\phi_L$. Let $E = \mathrm{Im} H$, this is a $\mathbb{Z}$ valued alternating form.

**Lemma 2.1.** *Suppose that $2g$ is the rank of the lattice $\Lambda$. There exists a $\mathbb{Z}$-basis for $\Lambda$, $U = \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ such that the matrix for $E$ wrt $U$ is*

*Proof.* Pick any basis to begin with. Since $H$ is hermitian, $E$ is skew symmetric, so the matrix for $E$ in this arbitrary basis looks like

$$\begin{pmatrix} F & A \\ -A^T & G \end{pmatrix}$$

where $F$ and $G$ are also skew symmetric. It's easy to see that we may further assume that $F = G = 0$. By row and column operations over $\mathbb{Z}$ we may reduce $A$ to a diagonal matrix, over $\mathbb{Z}$ we essentially use the fact that the gcd of two numbers can be expressed as a linear combination of these numbers. So $\exists U, V \in GL_n(\mathbb{Z})$ such that $UAV = \mathrm{diag}(d_1, d_2, \ldots, d_g)$ with $d_i$ dividing $d_{i+1}$ (note that in the case where we work over a field we may in fact insist that $V = U^{-1}$).

$$\begin{pmatrix} U & 0 \\ 0 & V^T \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

□

$(d_1, \dots, d_g)$ is uniquely determined by $E$ or $H$ or $L$.

**Definition 2.2.** *We call the tuple* $(d_1, \dots, d_g)$ *the type of* $E$ *or* $H$ *or* $L$ *and if all the* $d_i = 1$ *we call* $L$ *a principle polarization.*

We see that $K(\Lambda) = \ker\phi_L \simeq K_1 \oplus K_2$ with $K_i \simeq \oplus\mathbb{Z}/d_i\mathbb{Z}$. If all $d_i > 0$ then $H$ or $L$ or $E$ is non-degenerate.

**Definition 2.3.** *A basis* $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ *be as before giving rise to the matrix* $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ *is called a canonical or symplectic-basis for* $\Lambda$*. A sub-lattice* $\Lambda_1 \subset \Lambda$ *is called totally isotropic for* $E$ *if* $E(\lambda, \lambda') = 0\ \forall \lambda, \lambda' \in \Lambda$*.*

**Definition 2.4.** *A decomposition* $\Lambda = \Lambda_1 \oplus \Lambda_2$ *is called a decomposition for* $E$ *or* $H$ *or* $L$ *if both* $\Lambda_1$ *and* $\Lambda_2$ *are totally isotropic.*

**Definition 2.5.** *A decomposition* $V = V_1 \oplus V_2$ *of* $V$ *into real vector spaces such that* $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ *is a decomposition for* $\Lambda$ *is called the decomposition of* $V$ *for* $E$ *or* $H$ *or* $L$*.*

Let $H \in NS(X)$, $V = V_1 \oplus V_2$ a decomposition for $H$, define $\chi_0 : V \to T_1$ by

$$\chi_0(v) = e^{\pi i \operatorname{Im} H(v_1, v_2)} = e^{\pi i E(v_1, v_2)}$$

where $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. It is easily seen that for $v, w \in V$,

$$\chi_0(v + w) = \chi_0(v)\chi_0(w)e^{\pi i E(v,w)}e^{-2\pi i E(v_2, w_1)}$$

(keep in mind that $E(\lambda, \mu) \in \mathbb{Z}$ for $\lambda, \mu \in \Lambda$)

**Corollary 2.6.** $(\chi_0)_{|\Lambda}$ *is a semicharacter for* $H$

**Definition 2.7.** $L_0 := L(H, \chi_0) \in \mathit{Pic}^H(X)$ *is a distinguished element of* $\mathit{Pic}^H(X)$ *with respect to the decomposition of* $V$*.*