Math 7390 – Abelian Varieties

Farbod Shokrieh

Spring 2016

(February 4, 2016 draft)

Contents

Contributors

So far the following people have contributed to this write-up:

1. Farbod Shokrieh: Giving lectures, initial LaTeX setup, some (minimal) editing.
2. Theodore Hui: Scribing lectures: § 1,2
3. Daniel Collins: Scribing lectures: § 1,2
4. XXXXXX: Scribing lectures: § **??**
5. XXXXXX: Scribing lectures: § **??**
6. XXXXXX: Scribing lectures: § **??**
7. XXXXXX: Scribing lectures: § **??**
8. XXXXXX: Scribing lectures: § **??**
9. XXXXXX: Scribing lectures: § **??**
10. XXXXXX: Scribing lectures: § **??**

1 Introduction

1.1 About this Course

This is an introductory course on the analytic and algebraic theory of abelian (and jacobian) varieties. We will start with the classical complex-analytic case to build some intuition. Then we will discuss the general theory over other fields.

We will not follow any specific books, but the following resources were used while preparing for the lectures:

[BL04] [BLR90] [FC90] [Mil08] [Mum08] [Mum07a] [Mum07b] [Mum07c] [CS86]

1.2 What are Abelian Varieites?

Origins of the theory of abelian varieties comes from a basic question in calculus! We know we can easily compute integrals of the form

$$\int \frac{1}{\sqrt{1-x^2}}dx$$

by trigonometric substitution, but for integrals of the form

$$\int \frac{1}{\sqrt{f(x)}}dx$$

where $\deg(f) \geq 3$, it turns out to be hard. However, even though people couldn't compute these integrals, they could see that there were identities of the form

$$\int_0^a \frac{1}{\sqrt{f(x)}}dx + \int_0^b \frac{1}{\sqrt{f(x)}}dx = \int_0^{a*b} \frac{1}{\sqrt{f(x)}}dx$$

for some number $a * b$ obtained from a and b.

"Abelian Varieties are the simplest possible spaces, just tori's and thus groups" - D-Mumford.

Abelian Varieties are useful in the following areas:

- Number Theory - class field theory; rationality versus transcendence; most of the serious things we know how to do in number theory involve working with moduli of abelian varieties (Fermat's last theorem, Faltings' theorem, etc).
- Dynamical Systems - solutions to certain Hamilton systems.
- Algebraic Geometry - If we're given a variety X it's hard to understand, but we can get a handle on it by associating a canonical abelian variety $A(X)$ to it (such as Picard, Albanese, Intermediate Jacobian) and the good thing about $A(X)$ is that we can do a lot of linear algebra.
- Physics - theta functions that solve heat equations; string theory.

For the most part of this course, we will work over the base field $k = \mathbb{C}$; we see lots of very interesting ideas in this case, and we don't need any particularly hard theory to get a handle on it. One reason the theory over $\mathbb{C}$ is important is that an abelian variety A in a precise sense is just

$$A = A^{an} = \mathbb{C}^g/\Lambda$$

where $\Lambda = \pi_1(A) \simeq \mathbb{Z}^{2g}$ is a lattice, and so it is a torus. In other words, we have

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0.$$

Subtle remark: This identification makes sense in the "analytic category" but not in the algebraic category; the map $\mathbb{C}^g \to A$ is not algebraic. So we can't study abelian varieties in this way solely through algebraic methods. In the analytic setting it's "easy" to understand line bundles, theta functions, etc. by going to $\mathbb{C}^g$. (You can make sense of an analytification in nonarchimedean settings too, by using Berkovich spaces or formal schemes; this requires a lot more background but provides many important results.) Fortunately, there are some things from the complex-analytic setting which can be mimicked in the algebraic setting (e.g. the lattice Λ can be related to the Tate module) and by using those algebraic analogues you can take the complex-analytic results over $\mathbb{C}$ and try to reproduce them over other fields.

Some more advanced topics that might be covered in detail later in the class include (depending on audience interest):

- The theory over general field.
- Theta functions.
- Neron models.
- Non-archimedean uniformizations.
- Moduli and compactifications.
- Heights and metrized line bundles.
- Degenerating families.

Now, let's get to actual math. In scheme-theoretic language - for k any field, a *k-variety* is a geometrically integral k-scheme of finite type, and an *abelian variety* over k is a proper k-variety endowed with a structure of a k-group scheme. (This is the schematic definition. But we will not do it this way in this class.) We will be able to prove the following:

1.2.1 Theorem. *Abelian Varieties are automatically abelian and projective.*

Being abelian is easy to show, while being projective is much harder.

1.3 Why study Abelian Varieties?

More generally we can define an *algebraic group* G over k as a connected, smooth k-group scheme. Examples include:

- affine algebraic groups (automatically subgroups of $\mathrm{GL}_n(k)$, i.e. linear algebraic groups)
- abelian varieties (think of this as the projective case)

The following theorem says that these are the only building blocks:

1.3.1 Theorem (Chevalley's Theorem)**.** *Let G be an algebraic group over k. Suppose k is perfect. Then there exists a unique short exact sequence*

$$0 \to H \to G \to A \to 0$$

where H is linear and $A = G/H$ is abelian.

A proof can be found in B. Conrad's notes.

1.4 Some history

In the 1850s, Weierstrass studied $E = E^{an} = \mathbb{C}/\Lambda$, which is a complex group (2-dimensional torus with complex multiplication). He asked whether E^{an} is always "algebraic/algebraizable", and showed the answer is actually yes. In fact, he proved more.

1.4.1 Theorem. *$E = E^{an}$ has the structure of a smooth projective curve of genus 1. Its affine equation is given by*

$$y^2 = 4x^3 - 60G_4x - 140G_6$$

where $G_m = \sum_{\lambda\in\Lambda^*} \frac{1}{\lambda^m}$ for $m \in \mathbb{Z}$. More precisely, you can write down the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

and compute

$$y = \frac{dx}{dz} = \sum_{\lambda\in\Lambda} \frac{-2}{(2-\lambda)^3},$$

and see that the pair $(x, y) = (\wp(z), \wp'(z))$ satisfies the affine equation above. The mapping given by

$$z + \Lambda \mapsto (\wp(z), \wp'(z))$$

induces a group isomorphism $\mathbb{C}/\Lambda \simeq E$ where E is the projectivized elliptic curve.

What about higher dimensions? One direction is false: a general torus $\mathbb{C}^g/\Lambda$ will not be algebraic. However, the converse is true; a general abelian variety A will be (after analytification) of the form $\mathbb{C}^g/\Lambda$, where $g = \dim(A)$. We will prove this later, but it is not easy.

1.4.2 Theorem. *The Weierstrass parametrization gives a bijection between lattices Λ in $\mathbb{C}$, and the set of isomorphism classes of pairs (E, ω) where $E/\mathbb{C}$ are elliptic curves and ω are holomorphic differential forms.*

Note that $\omega \in H^0(E^{an}, \Omega)$ corresponds to $f(z)dz$ on $\mathbb{C}$, where f is periodic and holomorphic with Ω. Also note that the map $\mathbb{C} \to \mathbb{C}/\Omega$ is not algebraic. ($\mathbb{C}^g/\Omega$ for $g = 1$ is algebraic, while for $g \geq 2$ is usually not algebraic.) [fix]

2 Complex Tori

2.1 Some GAGA principles

Algebraic varieties vs. (complex) analytic spaces.

If X is an algebraic variety over $\mathbb{C}$, we can associate X^{an} which is a complex analytic space to it, by passing to a complex analytification; if X is locally described by some set of equations in affine space, we can pass that open set the zero locus as a subset of $\mathbb{C}^n$ (with its usual topology) and glue. Here are some facts:

- This construction is always functorial: an algebraic map $X \to Y$ can always be lifted to a holomorphic map $X^{an} \to Y^{an}$.
- X is proper/complete if and only if X^{an} is compact.
- X is smooth/connected if and only if Xan is smooth/connected.
- A complex analytic space $\mathfrak{X}$ is called **algebraic/algebraizable** if there exists a variety $X/\mathbb{C}$ such that $\mathfrak{X} \simeq X^{an}$. (Last time we explicitly showed that $\mathbb{C}/\Lambda$ is algebraic via the Weierstrass $\wp$-functions.)

2.2 Vector bundles and associated locally free sheaves

If L is a vector bundle on X, then it passes to an analytic vector bundle L^{an} on X^{an}. This is functorial in the sense that if $f : F \to G$ is a morphism of vector bundles it passes to $f^{an} : F^{an} \to G^{an}$. It is *not* true that all holomorphic vector bundles over X^{an} are algebraizable! But we have:

2.2.1 Theorem (Serre)**.**

1. *Suppose X is a proper (complete) algebraic variety over $\mathbb{C}$. If there exists a holomorphic coherent sheaf $\mathcal{F} \to X^{an}$, then there exists unique algebraic coherent sheaf F over X such that $F^{an} = \mathcal{F}$.*
2. *If there exists $\mathfrak{F} : \mathcal{F} \to \mathcal{G}$ homomorphism of holomorphic coherent sheaves on X^{an}, then there exists unique $f : F \to G$ such that $f^{an} = \mathfrak{f}$.*

Define $H^i(X, L)$ as the i-th cohomology group with values in the locally free sheaf L.

2.2.2 Theorem (Serre)**.** *Let X be a complete algebraic variety over $\mathbb{C}$ and F a coherent sheaf on X. Then the* natural maps

$$H^i(X, F) \to H^i(X^{an}, F^{an})$$

are isomorphisms of $\mathbb{C}$-vector spaces.

2.3 Complex tori

Let V be a vector space over $\mathbb{C}$, and $\Lambda \subset V$ a lattice (full rank discrete subgroup). We have Λ act naturally on V by addition; and then the quotient $X = V/\Lambda$ is a complex torus.

Some facts about a complex torus:

- it is a complex manifold.
- it inherits the structure of a complex Lie group over $\mathbb{C}$.
- it is compact (because Λ is a maximal rank lattice).
- it is an abelian complex Lie group.
- meromorphic functions on X correspond to meromorphic Λ-periodic functions on V.

Loosely speaking, an (complex analytic) *abelian variety* is a complex torus with "sufficiently many" (enough to give a closed embedding to a projective space) meromorphic functions. We will see that this is exactly what makes X algebraizable and thus an algebraic abelian variety.

2.4 Compactness implies abelian

2.4.1 Theorem. *Any connected* compact *complex Lie group X is a complex torus.*

Proof. First, X is abelian and so the commutator map $\Phi(x, y) = xyx^{-1}y^{-1}$ is continuous. Let U be any neighbourhood of the identity element 1, for $x \in X$, define open neighbourhoods $V_x, \tilde{V_x}$ such that $x \in V_x$, $1 \in \tilde{V_x}$ and $\Phi(V_x, \tilde{V_x}) \subset U$. (This can be done since $\Phi(x, 1) = 1$ and Φ is continuous.)

So we have $X = \cup_{x \in X} V_x$ and by compactness, there exist $x_1, \cdots, x_r \in X$ such that

$$X = \bigcup_{x \in \{x_1, \cdots, x_r\}} V_x.$$

Let $W = \cap_{x_1, \cdots, x_r} \tilde{V_x}$, which is a non-empty open neighbourhood of 1. So $\Phi(X, W) \subset U$. Since U is arbitrary, we have $\Phi(X, W) = 1$.

Since holomorphic functions on a compact set X which is bounded must be constant, we have $\Phi(1, y) = 1$ for all $y \in W$. Since W is open and non-empty, by connectivity,

$$\Phi(x, y) = 1$$

for all $x, y \in X$.

Then, if $\pi : V \to X$ is a universal cover, V inherits the structure of a simply connected complex Lie group and thus must be $\mathbb{C}^g$. Moreover π is homomorphic with discrete kernel, and by compactness of X the kernel must be full rank. □

Another proof can be found in B. Conrad's notes.

Remarks: Once we have $X = V/\Lambda$, we see that V is a universal cover of X. Moreover, $\Lambda = \pi(X, 0)$, and since this is already abelian it is isomorphic to $\simeq H_1(X, \mathbb{Z})$. Since X is locally isomorphic to V, we can view V as the tangent space at 0, T_0X; then the covering map $\pi : V = T_0X \to X$ is actually the exponential map.

2.5 Period matrix

Given $X = V/\Lambda$, we can associate Π a $g \times 2g$ complex matrix: fix $\{e_1, \cdots, e_g\}$ a $\mathbb{C}$-basis for V and $\{\lambda_1, \cdots, \lambda_{2g}\}$ a $\mathbb{Z}$-generator set for Λ. Define λ_{ji} such that

$$\lambda_j = \sum \lambda_{ji} e_i.$$

Then the **period matrix** of X is given by

$$\Pi := \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}.$$

Clearly, Π determines X but it depends on the choices.

Question: Given $\Pi \in M_{g\times 2g}(\mathbb{C})$, is there a complex torus such that Π is the period matrix of X?

2.5.1 Theorem. *Let* $P = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}_{2g\times 2g}$, *where* $\overline{\Pi}$ *denote the complex conjugate matrix of* Π. *Then* Π *is the period matrix for some* $\mathbb{C}^g/\Lambda$ *if and only if* P *is nonsingular.*

Proof. Π is a period matrix if and only if the columns of Π are $\mathbb{R}$-linearly independent. □

2.6 Holomorphic maps, homomorphism and isogenies

Suppose $X = V/\Lambda$ and $X' = V'/\Lambda'$ with dimensions g and g' respectively. We want to study holomorphic maps $f : X \to X'$. There are two special examples:

1. homomorphisms (holomorphic and respect group structure); and
2. translations (maps $X \to X$ by $x \mapsto x + x_0$ for some $x_0 \in X$).

The surprising thing is that that's *all*!

2.6.1 Theorem. *Suppose* $h : X \to X'$ *is a holomorphic map between complex tori. Then*

1. *there exists a unique homomorphism* $f : X \to X'$ *such that* $h = t_{h(0)} \circ f$. *That is,*
$$h(x) = f(x) + h(0)$$
for all x.
2. *There exists a unique* $\mathbb{C}$*-linear map* $F : V \to V'$ *with* $F(\Lambda) \subset \Lambda'$ *inducing* f.

Proof. Let $f := t_{-h(0)} \circ h$. Then we can lift $f \circ \pi : V \to X$ to $F : V \to V'$ where V' is the universal cover of X'. Then F is holomorphic and satisfies $F(0) = 0$. F is a $\mathbb{C}$-linear map: fix $\lambda \in \Lambda$, by construction,

$$F(v + \lambda) - F(v) \in \Lambda'$$

and so by continuity it's constant. Therefore,

$$F(v + \lambda) = F(v) + F(\lambda)$$

for all $v \in V, \lambda \in \Lambda$. We skip the remaining details. □

2.7 Hom-sets

Let $\mathrm{Hom}(X, X')$ be the set of all homomorphisms $f : X \to X'$. It is an abelian group. If $X = X'$, then we can define $\mathrm{End}(X) := \mathrm{Hom}(X, X')$. In this case, $\mathrm{End}(X)$ is actually a ring, where multiplication is given by composing endomorphisms. The above theorem gives us the following corollary:

2.7.1 Corollary. *We have injective homomorphisms:*

1. $\rho_{an} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ *given by* $f \mapsto F$*; and*
2. $\rho_{int} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$ *given by* $f \mapsto F|_\Lambda$.

Note that both of these homomorphisms respect endomorphism ring structures if $X = X'$: $\rho_*(f' \circ f) = \rho_*(f') \circ \rho_*(f)$, where $* = an, int$.

2.7.2 Theorem. $\mathrm{Hom}(X, X') \simeq \mathbb{Z}^m$ *for some* $m \le 4gg'$.

Proof. Use the second isomorphism in the corollary, since $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda' \simeq \mathbb{Z}^{2g'}$, so $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \simeq \mathbb{Z}^{4gg'}$ and $\mathrm{Hom}(X, X')$ embeds in this. □

How do these relate to period matrices? Let Π and Π' be the period matrix for X and X' respectively. If we have $f : X \to X'$, then by picking bases we get that $\rho_{an}(f) : V \to V'$ is given by some $A \in M_{g' \times g}(\mathbb{C})$ and $\rho_{int}(f) : \Lambda \to \Lambda'$ given by some $R \in M_{2g' \times 2g}(\mathbb{Z})$. Then the condition $F(\Lambda) \subset \Lambda'$ means $A\Pi = \Pi' R$. (The converse is also true: given four matrices with this property, then they correspond to a morphism between complex tori.)

What if $X = X'$? In this case, we can get

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} R$$

and thus $\rho_{int} \otimes 1 \simeq \rho_{an} \oplus \bar{\rho_{an}}$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

2.8 Kernels and Images

2.8.1 Lemma. *Given a homomorphism* $f : X \to X'$.

1. $Im(f)$ *is a complex subtorus of* X'.
2. $\ker(f)$ *is a closed subgroup of* X *with finitely many component. The connected component of* $1 = id$ *is a complex torus.*

The proof is fairly easy; for part (b) we're claiming that we have an extension

$$1 \to X_0 \to G \to \Gamma \to 1$$

with X_0 a complex torus and Γ a finite abelian group. It is a good exercise to describe Γ as a direct sum of cyclic groups in terms of Π, Π', A, R (need to compute a Smith normal form somewhere).

2.9 Isogenies

A homomorphism $f : X \to X'$ is called an **isogeny** if f is surjective with finite kernel. Equivalently, f is surjective and $\dim(X) = \dim(X')$.

2.9.1 Example (Essential example)**.** Suppose $X = V/\Lambda$ is a complex torus and $\Gamma \subset X$ is a finite subgroup. Then $X/\Gamma = V/\pi^{-1}(\Gamma)$ is a complex torus and $X \to X/\Gamma$ is an isogeny.

In fact, that's all! It is an easy exercise to show that all isogenies $X \to X/\Gamma$ over $\mathbb{C}$ are of this form. We also have the following easy lemma:

2.9.2 Lemma (Stein factorization)**.** *Any surjection $f : X \to X'$ of complex tori factors as a surjection $X \to X/(\ker f)_0$ (a quotient of X by a complex subtorus) and an isogeny $X/(\ker f)_0 \to X'$.*

For $f \in \mathrm{Hom}(X, X')$, we define $\deg(f)$ to be $|\ker f|$ if this is finite, and 0 if otherwise. It is easy to check that $\deg(f) = [\Lambda' : \rho_{int}(f)\Lambda]$. (Remark: If $X = X'$ then this index is $\deg(\rho_{int}(f))$; note that this determinant is ≥ 0 since $\rho_{int} \otimes 1 = \rho_{an} \oplus \bar{\rho}_{an}$, and is 0 if and only if the kernel is infinite.)

2.9.3 Lemma. *Supppose $f : X \to X'$ and $f' : X' \to X''$ are isogenies, then $f' \circ f$ is also an isogeny.*

Proof. $\deg(f' \circ f) = \deg(f) \cdot \deg(f')$. □

A very important example is given by the "multiplication-by-n" map: Let $n \in \mathbb{Z}^+$, define $n_X : X \to X$ by $x \mapsto nx$. Denote $X[n] := \ker(n_X)$ the set of n-torsions in A. Then we have

$$X[n] \simeq \frac{\frac{1}{n}\Lambda}{\Lambda} \simeq \frac{\Lambda}{n\Lambda} \simeq (\mathbb{Z}/n)^{2g}.$$

Therefore, n_X has degree n^{2g}, and so it is an isogeny.

2.9.4 Corollary. *Complex tori are divisible groups.*

2.9.5 Example (Tate module)**.** Let ℓ be a prime number. Define multiplication by ℓ maps $X[\ell^{n+1}] \to X[\ell^n]$. Then the *Tate module* is given by

$$T_\ell(X) = \varprojlim X[\ell^n].$$

In the case where Λ is finitely generated, $T_\ell(X)$ is actually isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and this is a subset of Λ. Note that the definition of $T_\ell(X)$ makes sense over any fields (even if we don't have Λ when we are not over $\mathbb{C}$). Here in our setting it's easy to see that a morphism $X \to X'$ is determined by the induced map $T_\ell(X) \to T_\ell(X')$. Over general fields this is much, much harder! It's the *Tate conjecture* which says that we have

$$\mathrm{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(X')) \simeq \mathrm{Hom}(X, X').$$

This conjecture was only proven over number fields by Faltings as an essential part of his proof of the Mordell conjecture.

2.10 Importance of isogenies

They are "almost isomorphisms". Namely, we have:

2.10.1 Theorem. *Let $f : X \to X'$ be an isogeny and n be the exponent of* $\ker(f)$. *(That is, $nx = 0$ for all $x \in \ker(f)$.) Then there exists an isogeny $g : X' \to X$ such that*

$$f \circ g = n_X, \;\; g \circ f = n_X.$$

Moreover, such a g is unique (up to isomorphism?).

sketch. Since n is the exponent of $\ker f$, we have $\ker(f) \subset \ker(n_X) = X[n]$. Then there exists a unique $g : X' \to X$ with $g \circ f = n_X$, defined by $g(x') := nx$ for some (all) x where $f(x) = x'$. Then use the fact that $\deg(g)\deg(f) = \deg(n_X)$ and that $\deg(f), \deg(n_X) \neq 0$, so $\deg(g) \neq 0$ to get that g is an isogeny; Then we just need to check that $g \circ f = n'_X$. □

Define $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(X, X') := \mathrm{Hom}(X, X') \otimes \mathbb{Q}$. Then the degree function extends to these via

$$\deg(rg) := r^{2g} \cdot \deg(f).$$

2.10.2 Corollary.

1. *Isogeny is an equivalence relation.*
2. *$f \in \mathrm{End}(X)$ is an isogeny if and only if it is invertible in $\mathrm{End}_{\mathbb{Q}}(X)$.*

2.11 Cohomology

We have a lot of cohomology theories (Betti, de Rham, Dolbeault, Hodge decomposition, ...).

Betti cohomology is just singular cohomology of $X(\mathbb{C})$; if $X = V/\Lambda$, then we have the following facts:

- $\Lambda = \pi_1(X_0) \simeq H_1(X, \mathbb{Z})$.
- By the universal coefficient theorem, we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.
- If $n \geq 1$, we have a map $\wedge_{i=1}^{n} H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by cup product, and this is an isomorphism (follows from Kunneth formula).
- Let $Alt^n(\Lambda, \mathbb{Z}) := \bigwedge_{i=1}^{n} \mathrm{Hom}(\Lambda, \mathbb{Z})$ be all the $\mathbb{Z}$-valued alternating n-forms. Then we have $H^n \simeq Alt^n(\Lambda, \mathbb{Z})$. This gives a very explicit way of thinking about cohomology.
- $H_n(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$.
- If we set $H^n(X, \mathbb{C}) := H^n(X, \mathbb{Z}) \otimes \mathbb{C}$, then we have

  $$H^n(X, \mathbb{C}) \simeq Alt^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge_{i=1}^{n} \mathrm{Hom}_{\mathbb{R}}(\Lambda, \mathbb{C}) \simeq \bigwedge_{i=1}^{n} H^1(X, \mathbb{C}),$$

  and the de Rham theorem tells us $H^n(X, \mathbb{C}) \simeq H_{DR}(X)$ where $H_{DR}(X)$ can be explicitly described as a complex vector space of invariant n-forms with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ with $i_1 < \cdots < i_n$.

Now, we use the $\mathbb{C}$-structure (really everthing is true for Kahler manifolds, but proofs and constructions are much more elementary for complex tori). Here we have a very nice decomposition

$$H^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(\Omega_X^p)$$

Here, $H^q(\Omega_X^p)$ is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$. In general, $H^q(\Omega_X^p)$ can be explicitly described as $\bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega}$ for $\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Set $\Omega_X^p := (\bigwedge^p \Omega) \otimes \mathcal{O}_X$.

References

[BL04] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, Second, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004. MR2062673 (2005c:14001) ↑3

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034) ↑3

[CS86] Gary Cornell and Joseph H. Silverman (eds.), *Arithmetic geometry*, Springer-Verlag, New York, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984. MR861969 (89b:14029) ↑3

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. MR1083353 (92d:14036) ↑3

[Mil08] James S. Milne, *Abelian varieties (v2.00)*, 2008. Available at http://www.jmilne.org/math/CourseNotes/av.html. ↑3

[Mum07a] David Mumford, *Tata lectures on theta. I*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition. MR2352717 (2008h:14042) ↑3

[Mum07b] David Mumford, *Tata lectures on theta. II*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura, Reprint of the 1984 original. MR2307768 (2007k:14087) ↑3

[Mum07c] David Mumford, *Tata lectures on theta. III*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. With collaboration of Madhav Nori and Peter Norman, Reprint of the 1991 original. MR2307769 (2007k:14088) ↑3

[Mum08] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. MR2514037 (2010e:14040) ↑3