Math 7390 – Abelian Varieties

Farbod Shokrieh

Spring 2016

(June 3, 2016 draft)

Contents

Contributors

So far the following people have contributed to this write-up:

1. Farbod Shokrieh: Giving lectures, initial LaTeX setup, some (minimal) editing.
2. Theodore Hui: Scribing lectures: § 1,2,3,4
3. Daniel Collins: Scribing lectures: § 1,2,3,4
4. Pak Hin, Li: Proofreading: § 2,4
5. Chenxi Wu: Scribing lectures: § 4
6. Anwesh Ray: Scribing lectures: § 2.29
7. Karl Thomas Bøaøath Sjöblom: Scribing lectures: § 4
8. Rakvi: Scribing lectures: § 4
9. XXXXXX: Scribing lectures: § **??**
10. XXXXXX: Scribing lectures: § **??**

1 Introduction

1.1 About this Course

This is an introductory course on the analytic and algebraic theory of abelian (and jacobian) varieties. We will start with the classical complex-analytic case to build some intuition. Then we will discuss the general theory over other fields.

We will not follow any specific books, but the following resources were used while preparing for the lectures:

[BL04] [BLR90] [FC90] [Mil08] [Mum08] [Mum07a] [Mum07b] [Mum07c] [CS86]

1.2 What are Abelian Varieites?

Origins of the theory of abelian varieties comes from a basic question in calculus! We know we can easily compute integrals of the form

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

by trigonometric substitution, but for integrals of the form

$$\int \frac{1}{\sqrt{f(x)}} dx$$

where $\deg(f) \geq 3$, it turns out to be hard. However, even though people couldn't compute these integrals, they could see that there were identities of the form

$$\int_0^a \frac{1}{\sqrt{f(x)}} dx + \int_0^b \frac{1}{\sqrt{f(x)}} dx = \int_0^{a*b} \frac{1}{\sqrt{f(x)}} dx$$

for some number $a * b$ obtained from a and b.

"Abelian Varieties are the simplest possible spaces, just tori's and thus groups" - D-Mumford.

Abelian Varieties are useful in the following areas:

- Number Theory - class field theory; rationality versus transcendence; most of the serious things we know how to do in number theory involve working with moduli of abelian varieties (Fermat's last theorem, Faltings' theorem, etc).
- Dynamical Systems - solutions to certain Hamilton systems.
- Algebraic Geometry - If we're given a variety X it's hard to understand, but we can get a handle on it by associating a canonical abelian variety $A(X)$ to it (such as Picard, Albanese, Intermediate Jacobian) and the good thing about $A(X)$ is that we can do a lot of linear algebra.
- Physics - theta functions that solve heat equations; string theory.

For the most part of this course, we will work over the base field $k = \mathbb{C}$; we see lots of very interesting ideas in this case, and we don't need any particularly hard theory to get a handle on it. One reason the theory over $\mathbb{C}$ is important is that an abelian variety A in a precise sense is just

$$A = A^{an} = \mathbb{C}^g/\Lambda$$

where $\Lambda = \pi_1(A) \simeq \mathbb{Z}^{2g}$ is a lattice, and so it is a torus. In other words, we have

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0.$$

Subtle remark: This identification makes sense in the "analytic category" but not in the algebraic category; the map $\mathbb{C}^g \to A$ is not algebraic. So we can't study abelian varieties in this way solely through algebraic methods. In the analytic setting it's "easy" to understand line bundles, theta functions, etc. by going to $\mathbb{C}^g$. (You can make sense of an analytification in nonarchimedean settings too, by using Berkovich spaces or formal schemes; this requires a lot more background but provides many important results.) Fortunately, there are some things from the complex-analytic setting which can be mimicked in the algebraic setting (e.g. the lattice Λ can be related to the Tate module) and by using those algebraic analogues you can take the complex-analytic results over $\mathbb{C}$ and try to reproduce them over other fields.

Some more advanced topics that might be covered in detail later in the class include (depending on audience interest):

- The theory over general field.
- Theta functions.
- Neron models.
- Non-archimedean uniformizations.
- Moduli and compactifications.
- Heights and metrized line bundles.
- Degenerating families.

Now, let's get to actual math. In scheme-theoretic language - for k any field, a *k-variety* is a geometrically integral k-scheme of finite type, and an *abelian variety* over k is a proper k-variety endowed with a structure of a k-group scheme. (This is the schematic definition. But we will not do it this way in this class.) We will be able to prove the following:

1.2.1 Theorem. *Abelian Varieties are automatically abelian and projective.*

Being abelian is easy to show, while being projective is much harder.

1.3 Why study Abelian Varieties?

More generally we can define an *algebraic group G* over k as a connected, smooth k-group scheme. Examples include:

- affine algebraic groups (automatically subgroups of $\mathrm{GL}_n(k)$, i.e. linear algebraic groups)
- abelian varieties (think of this as the projective case)

The following theorem says that these are the only building blocks:

1.3.1 Theorem (Chevalley's Theorem). *Let G be an algebraic group over k. Suppose k is perfect. Then there exists a unique short exact sequence*

$$0 \to H \to G \to A \to 0$$

where H is linear and $A = G/H$ is abelian.

A proof can be found in B. Conrad's notes.

1.4 Some history

In the 1850s, Weierstrass studied $E = E^{an} = \mathbb{C}/\Lambda$, which is a complex group (2-dimensional torus with complex multiplication). He asked whether E^{an} is always "algebraic/algebraizable", and showed the answer is actually yes. In fact, he proved more.

1.4.1 Theorem. *$E = E^{an}$ has the structure of a smooth projective curve of genus 1. Its affine equation is given by*

$$y^2 = 4x^3 - 60G_4x - 140G_6$$

where $G_m = \sum_{\lambda\in\Lambda^*} \frac{1}{\lambda^m}$ for $m \in \mathbb{Z}$. More precisely, you can write down the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

and compute

$$y = \frac{dx}{dz} = \sum_{\lambda\in\Lambda} \frac{-2}{(z-\lambda)^3},$$

and see that the pair $(x, y) = (\wp(z), \wp'(z))$ satisfies the affine equation above. The mapping given by

$$z + \Lambda \mapsto (\wp(z), \wp'(z))$$

induces a group isomorphism $\mathbb{C}/\Lambda \simeq E$ where E is the projectivized elliptic curve.

What about higher dimensions? One direction is false: a general torus $\mathbb{C}^g/\Lambda$ will not be algebraic. However, the converse is true; a general abelian variety A will be (after analytification) of the form $\mathbb{C}^g/\Lambda$, where $g = \dim(A)$. We will prove this later, but it is not easy.

1.4.2 Theorem. *The Weierstrass parametrization gives a bijection between lattices Λ in $\mathbb{C}$, and the set of isomorphism classes of pairs (E, ω) where $E/\mathbb{C}$ are elliptic curves and ω are holomorphic differential forms.*

Note that $\omega \in H^0(E^{an}, \Omega)$ corresponds to $f(z)dz$ on $\mathbb{C}$, where f is periodic and holomorphic with Ω. What happens if we replace Λ with $a\Lambda$? It turns out we can say it corresponds to $(E, a\omega)$. Also note that the map $\mathbb{C} \to \mathbb{C}/\Omega$ is not algebraic. ($\mathbb{C}^g/\Omega$ for $g = 1$ is algebraic, while for $g \geq 2$ is usually not algebraic.) [farbod fix]

2 Complex Tori

2.1 Some GAGA principles

Algebraic varieties vs. (complex) analytic spaces.

If X is an algebraic variety over $\mathbb{C}$, we can associate X^{an} which is a complex analytic space to it, by passing to a complex analytification; if X is locally described by some set of equations in affine space, we can pass that open set the zero locus as a subset of $\mathbb{C}^n$ (with its usual topology) and glue. Here are some facts:

- This construction is always functorial: an algebraic map $X \to Y$ can always be lifted to a holomorphic map $X^{an} \to Y^{an}$.
- X is proper/complete if and only if X^{an} is compact.
- X is smooth/connected if and only if Xan is smooth/connected.
- A complex analytic space $\mathfrak{X}$ is called **algebraic/algebraizable** if there exists a variety $X/\mathbb{C}$ such that $\mathfrak{X} \simeq X^{an}$. (Last time we explicitly showed that $\mathbb{C}/\Lambda$ is algebraic via the Weierstrass $\wp$-functions.)

2.2 Vector bundles and associated locally free sheaves

If L is a vector bundle on X, then it passes to an analytic vector bundle L^{an} on X^{an}. This is functorial in the sense that if $f : F \to G$ is a morphism of vector bundles it passes to $f^{an} : F^{an} \to G^{an}$. It is *not* true that all holomorphic vector bundles over X^{an} are algebraizable! But we have:

2.2.1 Theorem (Serre)**.**

1. *Suppose X is a proper (complete) algebraic variety over $\mathbb{C}$. If there exists a holomorphic coherent sheaf $\mathcal{F} \to X^{an}$, then there exists unique algebraic coherent sheaf F over X such that $F^{an} = \mathcal{F}$.*
2. *If there exists $\mathfrak{F} : \mathcal{F} \to \mathcal{G}$ homomorphism of holomorphic coherent sheaves on X^{an}, then there exists unique $f : F \to G$ such that $f^{an} = \mathfrak{f}$.*

Define $H^i(X, L)$ as the i-th cohomology group with values in the locally free sheaf L.

2.2.2 Theorem (Serre)**.** *Let X be a complete algebraic variety over $\mathbb{C}$ and F a coherent sheaf on X. Then the* natural maps

$$H^i(X, F) \to H^i(X^{an}, F^{an})$$

are isomorphisms of $\mathbb{C}$-vector spaces.

2.3 Complex tori

Let V be a vector space over $\mathbb{C}$, and $\Lambda \subset V$ a lattice (full rank discrete subgroup). We have Λ act naturally on V by addition; and then the quotient $X = V/\Lambda$ is a complex torus.

Some facts about a complex torus:

- it is a complex manifold.
- it inherits the structure of a complex Lie group over $\mathbb{C}$.
- it is compact (because Λ is a maximal rank lattice).
- it is an abelian complex Lie group.
- meromorphic functions on X correspond to meromorphic Λ-periodic functions on V.

Loosely speaking, an (complex analytic) *abelian variety* is a complex torus with "sufficiently many" (enough to give a closed embedding to a projective space) meromorphic functions. We will see that this is exactly what makes X algebraizable and thus an algebraic abelian variety.

2.4 Compactness implies abelian

2.4.1 Theorem. *Any connected* compact *complex Lie group X is a complex torus.*

Proof. First, X is abelian and so the commutator map $\Phi(x, y) = xyx^{-1}y^{-1}$ is continuous. Let U be any neighbourhood of the identity element 1, for $x \in X$, define open neighbourhoods $V_x, \tilde{V_x}$ such that $x \in V_x$, $1 \in \tilde{V_x}$ and $\Phi(V_x, \tilde{V_x}) \subset U$. (This can be done since $\Phi(x, 1) = 1$ and Φ is continuous.)

So we have $X = \cup_{x \in X} V_x$ and by compactness, there exist $x_1, \cdots, x_r \in X$ such that

$$X = \bigcup_{x \in \{x_1, \cdots, x_r\}} V_x.$$

Let $W = \cap_{x_1, \cdots, x_r} \tilde{V_x}$, which is a non-empty open neighbourhood of 1. So $\Phi(X, W) \subset U$. Since U is arbitrary, we have $\Phi(X, W) = 1$.

Since holomorphic functions on a compact set X which is bounded must be constant, we have $\Phi(1, y) = 1$ for all $y \in W$. Since W is open and non-empty, by connectivity,

$$\Phi(x, y) = 1$$

for all $x, y \in X$.

Then, if $\pi : V \to X$ is a universal cover, V inherits the structure of a simply connected complex Lie group and thus must be $\mathbb{C}^g$. Moreover π is homomorphic with discrete kernel, and by compactness of X the kernel must be full rank. □

Another proof can be found in B. Conrad's notes.

Remarks: Once we have $X = V/\Lambda$, we see that V is a universal cover of X. Moreover, $\Lambda = \pi(X, 0)$, and since this is already abelian it is isomorphic to $\simeq H_1(X, \mathbb{Z})$. Since X is locally isomorphic to V, we can view V as the tangent space at 0, T_0X; then the covering map $\pi : V = T_0X \to X$ is actually the exponential map.

2.5 Period matrix

Given $X = V/\Lambda$, we can associate Π a $g \times 2g$ complex matrix: fix $\{e_1, \cdots, e_g\}$ a $\mathbb{C}$-basis for V and $\{\lambda_1, \cdots, \lambda_{2g}\}$ a $\mathbb{Z}$-generator set for Λ. Define λ_{ji} such that

$$\lambda_j = \sum \lambda_{ji} e_i.$$

Then the **period matrix** of X is given by

$$\Pi := \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}.$$

Clearly, Π determines X but it depends on the choices.

Question: Given $\Pi \in M_{g\times 2g}(\mathbb{C})$, is there a complex torus such that Π is the period matrix of X?

2.5.1 Theorem. *Let* $P = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}_{2g\times 2g}$*, where* $\overline{\Pi}$ *denote the complex conjugate matrix of* Π*. Then* Π *is the period matrix for some* $\mathbb{C}^g/\Lambda$ *if and only if* P *is nonsingular.*

Proof. Π is a period matrix if and only if the columns of Π are $\mathbb{R}$-linearly independent. □

2.6 Holomorphic maps, homomorphism and isogenies

Suppose $X = V/\Lambda$ and $X' = V'/\Lambda'$ with dimensions g and g' respectively. We want to study holomorphic maps $f : X \to X'$. There are two special examples:

1. homomorphisms (holomorphic and respect group structure); and
2. translations (maps $X \to X$ by $x \mapsto x + x_0$ for some $x_0 \in X$).

The surprising thing is that that's *all*!

2.6.1 Theorem. *Suppose* $h : X \to X'$ *is a holomorphic map between complex tori. Then*

1. *there exists a unique homomorphism* $f : X \to X'$ *such that* $h = t_{h(0)} \circ f$*. That is,*

$$h(x) = f(x) + h(0)$$

   for all x*.*

2. *There exists a unique* $\mathbb{C}$*-linear map* $F : V \to V'$ *with* $F(\Lambda) \subset \Lambda'$ *inducing* f*.*

Proof. Let $f := t_{-h(0)} \circ h$. Then we can lift $f \circ \pi : V \to X$ to $F : V \to V'$ where V' is the universal cover of X'. Then F is holomorphic and satisfies $F(0) = 0$. F is a $\mathbb{C}$-linear map: fix $\lambda \in \Lambda$, by construction,

$$F(v + \lambda) - F(v) \in \Lambda'$$

and so by continuity it's constant. Therefore,

$$F(v + \lambda) = F(v) + F(\lambda)$$

for all $v \in V, \lambda \in \Lambda$. We skip the remaining details. □

2.7 Hom-sets

Let $\mathrm{Hom}(X, X')$ be the set of all homomorphisms $f : X \to X'$. It is an abelian group. If $X = X'$, then we can define $\mathrm{End}(X) := \mathrm{Hom}(X, X')$. In this case, $\mathrm{End}(X)$ is actually a ring, where multiplication is given by composing endomorphisms. The above theorem gives us the following corollary:

2.7.1 Corollary. *We have injective homomorphisms:*

1. $\rho_{an} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ *given by* $f \mapsto F$*; and*

2. $\rho_{int} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$ *given by* $f \mapsto F|_{\Lambda}$.

Note that both of these homomorphisms respect endomorphism ring structures if $X = X'$: $\rho_*(f' \circ f) = \rho_*(f') \circ \rho_*(f)$, where $* = an, int$.

2.7.2 Theorem. $\mathrm{Hom}(X, X') \simeq \mathbb{Z}^m$ *for some* $m \leq 4gg'$.

Proof. Use the second isomorphism in the corollary, since $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda' \simeq \mathbb{Z}^{2g'}$, so $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \simeq \mathbb{Z}^{4gg'}$ and $\mathrm{Hom}(X, X')$ embeds in this. $\square$

How do these relate to period matrices? Let Π and Π' be the period matrix for X and X' respectively. If we have $f : X \to X'$, then by picking bases we get that $\rho_{an}(f) : V \to V'$ is given by some $A \in M_{g' \times g}(\mathbb{C})$ and $\rho_{int}(f) : \Lambda \to \Lambda'$ given by some $R \in M_{2g' \times 2g}(\mathbb{Z})$. Then the condition $F(\Lambda) \subset \Lambda'$ means $A\Pi = \Pi' R$. (The converse is also true: given four matrices with this property, then they correspond to a morphism between complex tori.)

What if $X = X'$? In this case, we can get

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} R$$

and thus $\rho_{int} \otimes 1 \simeq \rho_{an} \oplus \bar{\rho_{an}}$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

2.8 Kernels and Images

2.8.1 Lemma. *Given a homomorphism* $f : X \to X'$.

1. $Im(f)$ *is a complex subtorus of* X'.

2. $\ker(f)$ *is a closed subgroup of* X *with finitely many component. The connected component of* $1 = id$ *is a complex torus.*

The proof is fairly easy; for part (b) we're claiming that we have an extension

$$1 \to X_0 \to G \to \Gamma \to 1$$

with X_0 a complex torus and Γ a finite abelian group. It is a good exercise to describe Γ as a direct sum of cyclic groups in terms of Π, Π', A, R (need to compute a Smith normal form somewhere).

2.9 Isogenies

A homomorphism $f : X \to X'$ is called an **isogeny** if f is surjective with finite kernel. Equivalently, f is surjective and $\dim(X) = \dim(X')$.

2.9.1 Example (Essential example)**.** Suppose $X = V/\Lambda$ is a complex torus and $\Gamma \subset X$ is a finite subgroup. Then $X/\Gamma = V/\pi^{-1}(\Gamma)$ is a complex torus and $X \to X/\Gamma$ is an isogeny.

In fact, that's all! It is an easy exercise to show that all isogenies $X \to X/\Gamma$ over $\mathbb{C}$ are of this form. We also have the following easy lemma:

2.9.2 Lemma (Stein factorization)**.** *Any surjection $f : X \to X'$ of complex tori factors as a surjection $X \to X/(\ker f)_0$ (a quotient of X by a complex subtorus) and an isogeny $X/(\ker f)_0 \to X'$.*

Remark: Stein factorization is a special case of a very general result (in complex theory by Stein and others, and in general in EGA III): any proper $f : X \to S$ factors as a map $X \to S'$ proper with connected fibers and $S' \to S$ finite.

For $f \in \operatorname{Hom}(X, X')$, we define $\deg(f)$ to be $|\ker f|$ if this is finite, and 0 if otherwise. It is easy to check that $\deg(f) = [\Lambda' : \rho_{int}(f)\Lambda]$. (Remark: If $X = X'$ then this index is $\deg(\rho_{int}(f))$; note that this determinant is ≥ 0 since $\rho_{int} \otimes 1 = \rho_{an} \oplus \bar{\rho}_{an}$, and is 0 if and only if the kernel is infinite.)

2.9.3 Lemma. *Supppose $f : X \to X'$ and $f' : X' \to X''$ are isogenies, then $f' \circ f$ is also an isogeny.*

Proof. $\deg(f' \circ f) = \deg(f) \cdot \deg(f')$. $\square$

A very important example is given by the "multiplication-by-n" map: Let $n \in \mathbb{Z}^+$, define $n_X : X \to X$ by $x \mapsto nx$. Denote $X[n] := \ker(n_X)$ the set of n-torsions in A. Then we have

$$X[n] \simeq \frac{\frac{1}{n}\Lambda}{\Lambda} \simeq \frac{\Lambda}{n\Lambda} \simeq (\mathbb{Z}/n)^{2g}.$$

Therefore, n_X has degree n^{2g}, and so it is an isogeny.

2.9.4 Corollary. *Complex tori are divisible groups.*

2.9.5 Example (Tate module)**.** Let ℓ be a prime number. Define multiplication by ℓ maps $X[\ell^{n+1}] \to X[\ell^n]$. Then the *Tate module* is given by

$$T_\ell(X) = \varprojlim X[\ell^n].$$

In the case where Λ is finitely generated, $T_\ell(X)$ is actually isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and this is a subset of Λ. Note that the definition of $T_\ell(X)$ makes sense over any fields (even if we don't have Λ when we are not over $\mathbb{C}$). Here in our setting it's easy to see that a morphism $X \to X'$ is determined by the induced map $T_\ell(X) \to T_\ell(X')$. Over general fields this is much, much harder! It's the *Tate conjecture* which says that we have

$$\operatorname{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(X')) \simeq \operatorname{Hom}(X, X').$$

This conjecture was only proven over number fields by Faltings as an essential part of his proof of the Mordell conjecture.

2.10 Importance of isogenies

They are “almost isomorphisms”. Namely, we have:

2.10.1 Theorem. *Let $f : X \to X'$ be an isogeny and n be the exponent of* $\ker(f)$*. (That is, $nx = 0$ for all $x \in \ker(f)$.) Then there exists an isogeny $g : X' \to X$ such that*

$$f \circ g = n_X, \ \ g \circ f = n_X.$$

Moreover, such a g is unique (up to isomorphism?).

sketch. Since n is the exponent of $\ker f$, we have $\ker(f) \subset \ker(n_X) = X[n]$. Then there exists a unique $g : X' \to X$ with $g \circ f = n_X$, defined by $g(x') := nx$ for some (all) x where $f(x) = x'$. Then use the fact that $\deg(g)\deg(f) = \deg(n_X)$ and that $\deg(f), \deg(n_X) \neq 0$, so $\deg(g) \neq 0$ to get that g is an isogeny; Then we just need to check that $g \circ f = n'_X$. □

Define $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(X, X') := \mathrm{Hom}(X, X') \otimes \mathbb{Q}$. Then the degree function extends to these via

$$\deg(rg) := r^{2g} \cdot \deg(f).$$

2.10.2 Corollary.

1. *Isogeny is an equivalence relation.*

2. *$f \in \mathrm{End}(X)$ is an isogeny if and only if it is invertible in $\mathrm{End}_{\mathbb{Q}}(X)$.*

2.11 Cohomology

We have a lot of cohomology theories (Betti, de Rham, Dolbeault, Hodge decomposition, ...).

Betti cohomology is just singular cohomology of $X(\mathbb{C})$; if $X = V/\Lambda$, then we have the following facts:

- $\Lambda = \pi_1(X_0) \simeq H_1(X, \mathbb{Z})$.

- By the universal coefficient theorem, we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.

- If $n \geq 1$, we have a map $\wedge_{i=1}^{n} H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by cup product, and this is an isomorphism (follows from Kunneth formula).

- Let $Alt^n(\Lambda, \mathbb{Z}) := \bigwedge_{i=1}^{n} \mathrm{Hom}(\Lambda, \mathbb{Z})$ be all the $\mathbb{Z}$-valued alternating n-forms. Then we have $H^n \simeq Alt^n(\Lambda, \mathbb{Z})$. This gives a very explicit way of thinking about cohomology.

- $H_n(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$.

- If we set $H^n(X, \mathbb{C}) := H^n(X, \mathbb{Z}) \otimes \mathbb{C}$, then we have

$$H^n(X, \mathbb{C}) \simeq Alt^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge_{i=1}^{n} \mathrm{Hom}_{\mathbb{R}}(\Lambda, \mathbb{C}) \simeq \bigwedge_{i=1}^{n} H^1(X, \mathbb{C}),$$

  and the de Rham theorem tells us $H^n(X, \mathbb{C}) \simeq H_{DR}(X)$ where $H_{DR}(X)$ can be explicitly described as a complex vector space of invariant n-forms with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ with $i_1 < \cdots < i_n$.

Now, we use the $\mathbb{C}$-structure (really everthing is true for Kahler manifolds, but proofs and constructions are much more elementary for complex tori). Here we have a very nice decomposition

$$H^n(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(\Omega_X^p)$$

Here, $H^{p,q}(X) := H^q(\Omega_X^p)$ is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$. In general, $H^q(\Omega_X^p)$ can be explicitly described as $\bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega}$ for $\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Also set $\Omega_X^p := (\bigwedge^p \Omega) \otimes \mathcal{O}_X$. Right now we are not saying anything about what these vector spaces $H^{p,q}(X)$ are, but we will do that later on. We may also need ot return to this theory to prove for example vanishing results later; We will either do that, or just omit those proofs.

2.12 Sheaves on a topological space X

Let $\mathcal{O}(X)$ be the category in which objects are open subsets of X and morphisms are inclusions $V \to U$ for $V \subset U$. (It turns out that you can generalize this and allow more general things for morphisms than just inclusions; this is how you get etale cohomology and other things.) Let $\mathcal{C}$ be any other category (for instance, Sets, Abelian groups and R-modules); A **presheaf** is a contravariant functor

$$F : \mathcal{O}(X) \to \mathcal{C}.$$

(This means for each inclusion map $i : V \subset U$, we have the restriction map given by $rest_{V,U} : F(U) \to F(V)$, and this assignment is functorial.) A **sheaf** is a presheaf F with some "locality" and "gluing" properties. Assume $\mathcal{C}$ has products. Then we require for any open cover $\{U_i\}$ of $U \in \mathcal{O}(X)$, we have an exact sequence

$$0 \to F(U) \overset{rest}{\to} \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

(where the two parallel maps are "restriction to the first index" and "restriction to the second index" respectively.)

2.12.1 Example. Let $X = \mathbb{C}$. Then we have the sheaf of holomorphic functions: $F(U)$ is the set of all holomorphic functions $U \to \mathbb{C}$, and restriction maps are restriction of functions!

Morphisms of sheaves are natural transformations; this gives us the category of sheaves over X, Shf_X, in which the objects are sheaves on X and morphisms are these.

Next, a **ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$; we call $\mathcal{O}_X$ the **structure sheaf** (usually some sheaf of holomorphic functions in this class). Define a **locally ringed space** to be one such that all of the stalks $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} F(U)$ are local rings. We may want to consider sheaves of $\mathcal{O}_X$-modules, that is, each $F(U)$ is a $\mathcal{O}_X(U)$-module respecting restriction maps.

2.13 Abelian categories and cohomology

(Grothendieck's Tohoku paper) There are 4 main examples of abelian categories to keep in mind:

1. The category of abelian groups (with homomorphisms).
2. The category of R-modules for R a commutative ring (with homomorphisms).
3. The category of G-modules for G a group (with G-equivariant homomorphisms: $\phi(g \cdot m) = g \cdot \phi(m)$).
4. The category of $\mathcal{O}_X$-modules for $(X, \mathcal{O}_X)$ a ringed space (with morphisms of sheaves the morphisms).

In general there's an abstract definition of an abelian category; we are not going to say it precisely but roughly it is a category $\mathcal{A}$ in which addition of morphisms, a zero object, kernels and cokernels make sense.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor between abelian categories. If we start with an short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, we can hit it with the functor F and get maps in $\mathcal{B}$, but no guarantee for exactness. We say that F is **left exact** if for every such short exact sequence we do have

$$0 \to F(A) \to F(B) \to F(C)$$

exact.

2.13.1 Example. Let $\mathcal{A}$ be the category of R-module, $\mathcal{B}$ be the category of abelian groups and D be a fixed object. Then $F(-) = \mathrm{Hom}_R(D, -)$ is a left exact covariant functor.

Cohomology lets us study the failure of exactness of

$$0 \to F(A) \to F(B) \to F(C),$$

i.e. the failure of surjectivity of $F(B) \to F(C)$. We want to continue the above exact sequence to the right and write a long exact sequence. In general, there are many ways to do this. But if $\mathcal{A}$ has "enough injectives" (a statement that holds for the categories we care about), then there is a "canonical" and "minimal" (in the sense that there is a universal property) way to do this. There exist unique functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ with $R^0F = F$ that give us the following long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{c_1} R^1F(A) \to R^1F(B) \to R^1F(C) \xrightarrow{c_2} R^2F(A) \to \cdots$$

plus satisfying some universal properties (an "effaceable δ-functor"). These functors are called the *right derived functors*. This is the covariant version; there's a similar one for contravariant functors. In general, it is hard to compute these R^iF. However, in the examples that we are interested in, there are easier ways of computing them.

2.13.2 Example. Let $F(-) := \mathrm{Hom}_R(-, D)$ and $G(-) := \mathrm{Hom}_R(D, -)$ where D is a fixed R-module. Both F and G are left-exact functors from R-modules to abelian groups, one contravariant and one covariant. More precisely, for any short exact sequence of R-modules

$$0 \to L \to M \to N \to 0,$$

we have the following exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N).$$

In both cases, the derived functors are just the Ext groups $\mathrm{Ext}^i_R(-, D)$ and $\mathrm{Ext}^i_R(D, -)$ which give us the long exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^1_R(M, D) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N) \to \mathrm{Ext}^1_R(D, L) \to \mathrm{Ext}^1_R(D, M) \to \cdots$$

So $R^i \mathrm{Hom}_R(-, D) = \mathrm{Ext}^i_R(-, D)$ and $R^i \mathrm{Hom}_R(D, -) = \mathrm{Ext}^i_R(D, -)$ (it is a nontrivial fact that these agree!).

2.13.3 Example. In the case of G-modules, let A be an abelian group A with a G action $\phi : G \to \mathrm{Aut}(A)$, (that is, A is a $\mathbb{Z}G$-module) with morphisms respecting G-action. Let A^G be the group of $x \in A$ such that $g \cdot x = x$ for all $g \in G$. The functor we want in this case is $F(A) = A^G$ which takes G -modules to abelian groups. Note that this functor is the same as $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$, so its (left exact) derived functors are (abstractly) Ext-functors $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, -)$, but this doesn't really give us a good way to compute it. These are better known as the "group cohomology of G with coefficients in A", denoted $H^i(G, A)$.

How do we compute this? It turns out that $\mathbb{Z}$ has a very nice "standard resolution" (also called the "bar resolution"), which is given by

$$F_n = \bigotimes_{i=0}^{n} \mathbb{Z}G.$$

Using this, we get the following recipe: for G a group and A a G-module, we let $C^0(G, A) = A$ and $C^n(G, A)$ be the group of A-valued maps on $G^n = G \times \cdots \times G$ for $n \geq 1$. These C^n are called the group of n**-cochains** of G with values in A. We also define the **differential operators** $d_n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \cdots, g_{n+1}) := {} & g_1 \cdot f(g_2, \cdots, g_{n+1}) \\ & + \Sigma_{i=1}^n (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \cdots, g_n). \end{aligned}$$

The cases of most interest are $n = 0, 1, 2$:

- when $n = 0$, we have $f = a \in C^0(G, A) = A$, and then we get

$$d_0(f)(g) = g \cdot a - a.$$

- when $n = 1$, then f is a function with one input and we have

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

- when $n = 2$, then f is a function with two inputs and we have

$$d_2(f)(g,h,k) = g \cdot f(h,k) - f(gh,k) + f(g,hk) - f(g,h).$$

  Amazingly, people wrote down this formula correctly before the general theory came out.

It can be shown that $d_n \circ d_{n+1} = 0$. Thus, we can define the group of n-**cocycles** as $Z^n(G,A) := \ker(d_n)$ for $n \geq 0$ and the group of n-**coboundaries** as $B^n(G,A) := \mathrm{im}(d_{n-1})$ for $n \geq 1$ (or $B^0(G,A) = 1$ when $n = 0$). Then we define the n**-th cohomology group** as

$$H^n(G,A) := Z^n(G,A)/B^n(G,A).$$

Note that $H^0(G,A) = A^G$.

2.13.4 Example. Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ a $\mathcal{O}_X$-module. Let $\Gamma(-,X)$ be the "global sections" functor, $\mathcal{F} \mapsto \mathcal{F}(X)$, from $\mathbb{O}_X$-modules to R-modules for $R = \mathcal{O}_X(X)$. This functor is left-exact (to make sense of this we need to make sure we know what exact sequences of sheaves are - defining "kernels" is easy but to define "images" we need to sheafify).

Here there is the Grothedieck (right) derived cohomology functor, $R^i\Gamma(-,X)$, which we denote $H^i(X,-)$ (we also call $H^i(X,\mathcal{F})$ the *sheaf cohomology of X with values in $\mathcal{F}$*); in particular, we have $H^0(X,-) = \Gamma(-,X)$. This is given abstractly by the theory earlier in this section; but like group cohomology, we will really need to work with sheaf cohomology, so we need to a way to compute them explicitly.

Here, we have a more concrete one, which is called the *Čech cohomology*, $\check{H}^i(X,\mathcal{F})$. In general, the two cohomologies are not equal! Fortunately they are equal in the settings we will be interested in.

The Čech cohomology is more explicitly computable, and always gives us a map $\phi : H^i(X,\mathcal{F}) \to \check{H}^i(X,\mathcal{F})$. We have:

1. For $i = 0, 1$, ϕ is an isomorphism.

2. (Grothendieck): If X is a Noetherian, separated scheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then ϕ is an isomorphism.

3. (Godement): If X is a paracompact and Hausdorff topological space, then ϕ is an isomorphism.

However, ϕ is not an isomorphism in general! Here are some counterexamples:

- In his Tôhoku paper (p.177), Grothendieck provides a counterexample where $H^2 \neq \check{H}^2$, for $X = \mathbb{A}^2$ with the Zariski topology and $\mathcal{F}$ comes from taking $\underline{\mathbb{Z}}$ and modifying it based on a space Y that is a union of two circles. This example is explicit but the proof is somehow deep!

- A recent paper of Schröer (arxiv post 1309.2524) gives a Hausdorff (not not paracompact) topological space constructed from 2-dimensional discs that is a counterexample.

2.13.5 Čech cohomology

So how do we define Čech cohomology? The idea is that if $\mathcal{U}$ is an open cover on X, the "nerve" of $\mathcal{U}$ approximates X. We define a q-simplex σ of $\mathcal{U}$ as an ordered collection of $q+1$ elements in $\mathcal{U}$ with nonempty intersection that we call $|\sigma|$. Suppose $\sigma = (U_i)$ (for $0 \leq i \leq q$), we define $\partial_j \sigma := (U_i)_{i \neq j}$ and then $\partial \sigma = \Sigma_{j=0}^{q} (-1)^{j+i} \partial_j \sigma$. Note that $|\sigma| = \cap U_i$.

We then define q**-cochains** of $\mathcal{U}$ with coefficients in $\mathcal{F}$ to be the set $C^q(\mathcal{U}, \mathcal{F})$ of functions $\sigma \mapsto f_\sigma \in \mathcal{F}(|\sigma|)$. Therefore, we get the **boundary maps** $C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta_q \omega)(\sigma) = \Sigma_{j=0}^{q+1} (-1)^j \mathrm{res}_{|\partial_j \sigma|, |\sigma|} \omega(\partial_j \sigma).$$

One can check that $\delta_{q+1} \circ \delta_q = 0$ and hence can define **cocycles** $Z^q(\mathcal{U}, \mathcal{F}) := \ker(\delta_q)$ and **coboundaries** $B^q(\mathcal{U}, \mathcal{F}) := Im(\delta_{q-1})$, and then the **Čech cohomology** of $\mathcal{U}$ is given by

$$\check{H}^q(\mathcal{U}, \mathcal{F}) := Z^q(\mathcal{U}, \mathcal{F}) / B^q(\mathcal{U}, \mathcal{F}).$$

So this gives us the cohomology $\check{H}^i(\mathcal{U}, \mathcal{F})$ of an open cover $\mathcal{U}$; However, we want a cohomology $\check{H}^i(X, \mathcal{F})$ associated to the whole space X! There are two ways to solve this:

1. If X has a "good" cover $\mathcal{U}$ (with all finite intersections of U_i to be contractible), then $\check{H}^i(\mathcal{U}, \mathcal{F})$ is canonical.

2. In general, we can define $\check{H}^i(\mathcal{X}, \mathcal{F})$ as $\varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$. But then we need to make sense of this direct limit. (It might be over an index set that is a proper class?) [fix farbod]

Remark: Let G be a topological group. Let $BG = K(G, 1)$ be the *Eilenberg-MacLane space.* (For example, $B\mathbb{Z} = S^1$.) Note that $\pi_1 = G$ and $\pi_n = 0$ for all $n > 1$. Then if A is a G-module, the sheaf cohomology $H^n(BG, \underline{A})$ (here $\underline{A}$ is the constant sheaf, which is the sheafification of the constant presheaf) is isomorphic to the usual CW complex cohomology $H^n(BG, A)$ and to the group cohomology $H^n(G, A)$, if A has a trivial G-action. (If A has a nontrivial G-action we can still make sense of this but the $H^n(BG, \underline{A})$ needs to be reinterpreted in terms of "local coefficient systems".)

2.14 Back to complex tori (sort of)

We want to understand line bundles on a locally ringed space $(X, \mathcal{O}_X)$. For now, we can think of it as a complex manifold or a variety. Let $\mathcal{F}$ be a sheaf. We call $\mathcal{F}$ (globally) **free** if $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X$ is the direct sum of copies of the structure sheaf; r is called the **rank** of $\mathcal{F}$. $\mathcal{F}$ is called **locally free** if there exists an open cover $\{U_i\}$ such that each $\mathcal{F}|_{U_i}$ is free. (There is a correspondence between locally free sheaves $\mathcal{F}$ of rank n and vector bundles of rank n. If $\pi : E \to X$ is a vector bundle, we get a locally free sheaf with $\mathcal{F}(U)$ being the sections of π over U; conversely, if $\mathcal{F}$ is locally free, we can construct an associated line bundle as $\coprod U_i \times \mathbb{C}^n$ modulo gluing data.)

2.14.1 Line bundles

Line bundles are vector bundles of rank 1 (equivalently, locally free sheaves of rank 1). Our goal is to give a cohomological interpretation on the set of line bundles. Let $\pi : L \to X$ be a line bundle. Let $\{U_\alpha\}$ be an open cover with trivializations given by (holomorphic) $\phi_\alpha : L|_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}$ where $L|_\alpha := \pi^{-1}[U_\alpha]$. Define the **transition functions** for L with respect to $\{\phi_\alpha\}$ as $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ which are given by

$$g_{\alpha\beta}(z) := \phi_\alpha \circ \phi_\beta^{-1}|_{L_z}$$

where $L_z := \{z\} \times \mathbb{C}$. By itself this is a linear (and hence holomorphic) map on $\{z\} \times \mathbb{C}$, which is determined by the complex number we are calling $g_{\alpha\beta}(z)$. We check that $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$ (so it is nonzero) and also $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = 1$. Rewriting this latter condition gives a cocycle condition

$$g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1.$$

To summarize, a line bundle (trivialized by an open cover $\mathcal{U}$) determines a collection of

- holomorphic functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$; which satisfy
- $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$, and
- $g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1$.

Conversely, if we are given $\{g_{\alpha\beta}\}$ satisfying these properties, then we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ as a quotient of $\coprod U_\alpha \times \mathbb{C}$ by the appropriate gluing relations.

2.14.2 Choices

If $f_\alpha \in \mathcal{O}^\times(U_\alpha)$ is a nonvanishing holomorphic function on U_α, and we construct new trivializations $\phi'_\alpha = f_\alpha \circ \phi_\alpha$, then the new transition functions $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$ give the same bundle L.

Now, the collection of $\{g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)\}$ is a Čech 1-cochain. The conditions we wrote down that it satisfies implies that it is actually a 1-cocycles. Moreover, the ambiguity mentioned above is exactly the 1-coboundaries. Therefore, we can then conclude the set of (isomorphism classes of) line bundles $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$. Moreover, this is a group homomorphism; for the group structure on line bundles coming from tensor product and the group structure naturally showing up on $H^1(X, \mathcal{O}_X^\times)$.

2.14.3 Line bundles vs "factors of automorphy"

Let X be a complex torus, and $\tilde{X} = \mathbb{C}^g$ its universal cover, with $\pi : \tilde{X} \to X$ the covering map.

2.14.4 Theorem. *There exists a canonical exact sequence*

$$0 \to H^1(\pi(X), H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)) \xrightarrow{\phi} H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times).$$

The first term in the above exact sequence is the group of *"factors of automorphy"* and the latter two are Picard groups as above. (Note that in general $\mathcal{O}_X^\times$ is the sheaf of invertible elements in our rings with respect to multiplication. In our holomorphic setting, this is equivalent to the sheaf of nonvanishing functions, because you can prove that if f is holomorphic and nonzero, then $1/f$ is holomorphic.)

2.14.5 Corollary. *If $\pi^*(L)$ is trivial, then it is completely described by a "factor of automorphy".*

2.14.6 Corollary. *If $\mathrm{Pic}(\tilde{X}) = 0$ (eg when $\tilde{X} = \mathbb{C}^g$), then ϕ is an isomorphism.*

Now, we prove the theorem above.

Proof. The 1-cochains $C^1(G, M)$ where $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)$. Note that M contains elements like holomorphic $g : \tilde{X} \to \mathbb{C}^\times$. Therefore, for $(h : G \to M) \in C^1(G, M)$, we can naturally get holomorphic $f : G \times \tilde{X} \to \mathbb{C}^\times$. Note that G acts on $\tilde{X}$ by $g \cdot \tilde{x}$ which gives $f(\tilde{x}) \cdot g = f(g \cdot \tilde{x})$.

Also, note that the 1-cocycles satisfy

$$(h(\mu) \cdot \lambda)(h(\lambda\mu)^{-1})(h(\lambda)) = 1.$$

Therefore, we have the following cocycle condition: For $\lambda, \mu \in \pi_1(X), \tilde{x} \in \tilde{X}$,

$$f(\lambda\mu, \tilde{x}) = f(\mu, \lambda\tilde{x}) f(\lambda, \tilde{x})$$

Also the 1-coboundary condition: For $h_1 \in M = C^0(G, M)$,

$$f(\lambda, \tilde{x}) = \frac{h_1(\lambda \cdot \tilde{x})}{h_1(\tilde{x})}$$

and we define $H^1(G, M) = Z^1/B^1$ like before.

Now, each $f \in Z^1(G, M)$ gives a Line Bundle on X. Let $L = \tilde{X} \times \mathbb{C} \to \tilde{X}$ be the trivial line bundle map. Then $G = \pi_1(X)$ acts on $\tilde{X}$, and so acts on L: $\lambda \cdot (\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that this action is

- free (If $g \cdot * = *$ for some g and $*$, then $g = id$.)
- properly discontinuous (For all K_1, K_2 compact subsets, $\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$ is finite.)

There is a theorem saying that complex manifolds came from complex manifolds moding out free properly discontinuous actions. Therefore, the induced $L' \to X$ is a holomorphic line bundle. And this gives a map $Z^1(G, M) \to H^1(X, \mathcal{O}_X^\times)$.

Proof of Theorem: First we will show that there is a map ϕ_1 given by

$$0 \to H^1(G, M) \xrightarrow{\phi_1} \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)).$$

and then we will try to prove that ϕ_1 is an isomorphism.

First, for existence of ϕ_1, we can just take the above map. Or a more explicit solution for this is given by the following: Let $\pi : \tilde{X} \to X$ be the covering map. Then fix a cover $\{U_i\}_I$ such that for all $i \in I$, there exists $W_i \subset \pi_i^{-1}(U_i)$, with biholomorphic $\pi_i := \pi|_{W_i} : W_i \to U_i$. For all i, j, there exists unique $\lambda_{i,j} \in \pi_1(X)$

such that for $x \in U_i \cap U_j$, $\pi_j^{-1}(x) = \lambda_{ij} \cdot \pi_i^{-1}(x)$. It is clear that $\lambda_{ij} \cdot \lambda_{jk} = \lambda_{ik}$. Therefore, now we have a map $Z^1(G, M) \to Z^1(X, \mathcal{O}_X^\times)$ (note that $Z^1(X, \mathcal{O}_X^\times)$ maps onto $H^1(X, \mathcal{O}_X^\times)$) given by

$$f \mapsto \{g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)\}$$

where $g_{ij}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))$. It is easy to check the 1-cocycle condition

$$g_{ij}g_{jk} = g_{ik}.$$

(Explicitly, $g_{ij}(x)g_{jk}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))f(\lambda_{jk}, \pi_j^{-1}(x)) = f(\lambda_{ij}\lambda_{jk}, \pi_i^{-1}(x)) = f(\lambda_{ik}, \pi_i^{-1}(x)) = g_{ik}(x)$.)

So we have a well defined map

$$Z^1(G, M) \to \check{Z}^1(X, \mathcal{O}_X^\times) \twoheadrightarrow \check{H}^1(X, \mathcal{O}_X^\times).$$

It is easy to check that the kernel of this map contains $B^1(G, M)$, so this descends to a homomorphism $H^1(G, M) \to \check{H}^1(X, \mathcal{O}_X^\times)$. (To check this: if we have a 1-coboundary $f(\lambda, x) = h(\lambda\tilde{x})/h(x)$, this will go to a Čech 1-cocycle

$$g_{ij}(x) = \frac{h(\lambda_{ij}\pi_i^{-1}(x))}{h(\pi_i^1(x)} = \frac{h(\pi_j^{-1}(x))}{h(\pi_i^{-1}(x))}.$$

This tells us $g_{ij}(x)$ is a Čech 1-coboundary so is trivial in $\check{H}^1(X, \mathcal{O}_X^\times)$.) It is also easy to check that $Im(\phi)$ lies in $\ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Now, we want to show that ϕ_1 is an isomorphism. To do this, we want to find an inverse map. That is, given $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$, we want to find $f \in H^1(G, M)$ such that $L = \phi_1(f)$. (We want an explicit expression for this inverse map, we will use it very often when computing things.)

We know that $\pi^* L$ is trivial. So we fix $\alpha : \pi^* L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$. (Note that G acts on $\pi^* L$, and hence via α, G also acts on $\tilde{X} \times \mathbb{C}$.) For all $\lambda \in G$, there exists automorphisms ϕ_λ of $\tilde{X} \times \mathbb{C}$ such that $\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that

- $f(\lambda, \tilde{x}) \in Z^1(G, M)$
- For another map $\alpha' : \pi^* L \to \tilde{X} \times \mathbb{C}$, we have

  $$\phi'_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, h(\lambda\tilde{x})f(\lambda\tilde{x})h(\tilde{x})^{-1}t)$$

  where $\alpha'\alpha^{-1}(\tilde{x}, t) = (\tilde{x}, h(\tilde{x})t)$.

□

In fact, via a similar proof, one can also prove the following:

2.14.7 Theorem. *Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \pi^*\mathcal{F})$. Then for each $n \geq 0$, there exist canonical homomorphisms $\phi_n : H^n(G, M) \to H^n(X, \mathcal{F})$. Moreover,*

- *when $n = 1$, for each $f \in Z^1(G, M)$, we have $(\phi_1 f)_{ij} = f(\lambda_{ij}, \pi_i^{-1})$.*
- *when $n = 2$, for each $f \in Z^2(G, M)$, we have $(\phi_2 f)_{ijk} = f(\lambda_{ij}, \lambda_{jk}, \pi_i^{-1})$.*

2.15 Global sections of $H^0(X, L)$

Our next goal is to describe global sections of $H^0(X, L) = \Gamma(L, X)$ in terms of $\phi_1(f) = L$, where $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Since $\phi^* L$ is trivial, if we fix a trivialization $\alpha : \pi^* L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$, this gives us an explicit cocycle $f \in Z^1(G, M)$ (giving the cohomology class $\phi_1^{-1}(L)$). Observe that there exists a canonical isomorphism between $H^0(X, L) \xrightarrow{\sim} H^0(\tilde{X}, \pi^* L)^G$. And then we have (via α) an isomorphism $H^0(\tilde{X}, \pi^* L)^G \simeq H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$, which is explicitly given by

$$\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t).$$

$H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$ then corresponds to the set $\{\vartheta : \tilde{X} \overset{hol}{\to} \mathbb{C} : \vartheta(\lambda, \tilde{x}) = f(\tilde{\lambda}, \tilde{x})\vartheta(x)\}$. So $H^0(X, L)$ is in bijection with the set of theta functions ϑ with f as its factor of automorphy. (Remark: This does depend on the trivialization and thus the specific form of f; but changing α changes f in a predictable way and thus changes the set of ϑ.) Also, one can check functoriality of everything we've done.

Now, let's get back to the case where $X = V/\Lambda$ is complex torus.

2.15.1 Proposition. *Every holomorphic line bundle on X pulls back to a trivial bundle on V.*

Sketch of proof. We need

- the *exponential sequence* of sheaves:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_V \overset{e^{2\pi i}}{\to} \mathcal{O}_V^\times \to 1$$

  ($\underline{\mathbb{Z}}$ is the constant sheaf of $\mathbb{Z}$.)

- the $\overline{\partial}$-Poincaré lemma; and
- $H^2(V, \mathbb{Z}) = 0$.

This induces the exact sequence:

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^\times) \to H^2(V, \mathbb{Z}) \to \cdots$$

First term is trivial by the $\overline{\partial}$-Poincare lemma, and last term is trivial by $H^2(V, \mathbb{Z}) = 0$. Therefore, $H^1(V, \mathcal{O}_V^\times)$ is also trivial. So, all line bundles on V are trivial. □

2.15.2 Corollary. *The above exact sequence gives an isomorphism* $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^\times) \simeq H^1(\Omega, H^0(V, \mathcal{O}_V^\times))$.

The proof can be found in Griffiths-Harris. [farbod add reference]

2.16 The exponential sequence and the first Chern class

In this section, we will talk about the exponential exact sequence, and generalize to X. We will then get a similar long exact sequence, but here $H^2(X, \mathbb{Z})$ is nontrivial, and the map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is extremely important! Our next goal is the Appell-Humbert theorem, which is actually due to Lefschetz.

On any complex manifold $(X, \mathcal{O}_X)$, the sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \overset{exp}{\to} \mathcal{O}_X^\times \to 1$$

is exact, where the map $exp : \mathcal{O}_X \to \mathcal{O}_X^\times$ is the exponential map, given on open sets by $exp(U) : \mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ by

$$f \mapsto e^{2\pi i f}.$$

It is easy to check that this map is holomorphic with multiplicative inverse $e^{-2\pi i f}$.

Exactness of the sequence

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times$$

is straightforward; the kernel of $exp : \mathcal{X}(U) \to \mathcal{O}_X^\times(U)$ are the locally constant $\mathbb{Z}$-valued functions, which are $\underline{\mathbb{Z}}(U)$. What about the surjectivity of $\mathcal{O}_X \to \mathcal{O}_X^\times$? This asks about the existence of logarithms. Well, taking complex logarithms are not always "globally" possible! However, they are "locally" possible; for any $f \in \mathcal{O}_X^\times(U)$ and any $x \in U$, we can find a contractible neighborhood V of x with $V \subset U$, and then $res_V(f)$ does have a logarithm. So $\mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ is surjective for small enough U, and this is enough for the sheaf map $\mathcal{O}_\mathcal{X} \to \mathcal{O}_X^\times$ to be considered surjective (the cokernel is a presheaf which sheafifies to 0). For example, $z \in \mathcal{O}^\times(\mathbb{C} - \{0\})$ has logarithm if you restrict to a small enough open set.

So now we have the long exact sequence that measures the failure of surjectivity of the exponential function, which will give us

$$0 \to H^0(X, \underline{\mathbb{Z}}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times) \to H^1(X, \underline{\mathbb{Z}}) \to \cdots$$

so surjectivity of the global exponential map is controlled by $H^1(X, \underline{\mathbb{Z}})$, a "generalized winding number". Continuing we get

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

and beyond this we won't really care.

2.16.1 The first Chern class

Note that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ and $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ in the above sequence is called the **first chern class map**, denoted by c_1. Also, $Im(c_1)$ is called the **Néron-Severi group** of X, denoted by $NS(X)$.

Remark: We can thus write this as $NS(X) \simeq \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ where $\mathrm{Pic}^0(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles L with $c_1(L) = 0$. This is just rephrasing the definition above, but we can also prove that $\mathrm{Pic}^0(X)$ is equal to the connected component of of the origin in $\mathrm{Pic}(X)$.

2.16.2 Theorem (Severi, Néron)**.** *$NS(X)$ is always a finitely generated abelian group.*

Severi proved the theorem for $k = \mathbb{C}$ and Néron proved for more general fields, but we are not going to make that precise. We define $\rho(X)$ to be the rank of $NS(X)$, which is the **Picard number** of X. We will see that $\rho(X) \leq h^{1,1}(X) = g^2$.

Recall that $H^2(X, \mathbb{Z}) \simeq H^1(X, \mathbb{Z}) \bigwedge H^1(X, \mathbb{Z}) \simeq \mathrm{Hom}(\Lambda, \mathbb{Z}) \bigwedge \mathrm{Hom}(\Lambda, \mathbb{Z})$ which we are going to denote by $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$. Note that $\mathrm{Alt}^2(\Lambda, \mathbb{Z})$ contains $\mathbb{Z}$-valued bilinear alternating 2-forms.

2.17 First Chern class for complex tori

When $X = V/\Lambda$ ($V \simeq \mathbb{C}^g$) is a complex torus, the exponential sequence gives

$$H^0(V, \mathcal{O}_V) \to H^0(V, \mathcal{O}_V^\times) \to 0$$

because $H^1(V, \mathbb{Z}) = 0$. So any non-vanishing global form f is $e^{2\pi i g}$ for some $g \in \mathcal{O}_V$. Recall that $H^1(\Lambda, H^0(V, \mathcal{O}_V^\times)) \simeq H^1(X, \mathcal{O}_X^\times)$. So given L, one has a factor of automorphy $(f : \Lambda \times V \to \mathbb{C}^\times) \in Z^1(\Lambda, H^0(V, \mathcal{O}_V^\times))$. We assume $f = e^{2\pi i g}$ for $g : \Lambda \times V \to \mathbb{C}$.

2.17.1 Theorem. *With the canonical isomorphism discussed before, we have*

$$c_1(L) = E_L(\cdot, \cdot)$$

where

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

for $\lambda, \mu \in \Lambda$ and $v \in V$.

It is a good exercise to show that

- E_L is independent of v. (Using the fact that $f \in Z^1$.)
- E_L is an alternating 2-form, $\mathbb{Z}$-valued and also bilinear.

2.17.2 Theorem. *For any $E : V \times V \to \mathbb{R}$ which is an alternating $\mathbb{R}$-valued bilinear form on $H^2(X, \mathbb{R})$, the following are equivalent:*

1. *$E = c_1(L)$ for some $L \in \mathrm{Pic}(X)$.*
2. *$E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for $v, w \in V$.*

These two theorems turn out to be very useful! We will skip the proofs for now.

It is an easy fact from Linear Algebra that there is a one-to-to correspondence between Hermitian forms H on V ($H : V \times V \to \mathbb{C}$ such that $H(v, w) = \overline{H(w, v)}$ for all $w, v \in CC$) and the set of alternating bilinear forms $E : V \times V \to \mathbb{R}$ satisfying $E(iv, iw) = E(v, w)$ for all $i \in V$. In fact, the correspondence maps are explicitly given by $H \mapsto Im(H)$ and $E \leftarrow E(iv, w) + iE(v, w)$.

2.17.3 Corollary. *$NS(X)$ is exactly the set of alternating bilinear forms $E : V \times V \to \mathbb{R}$ such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all $v, w, i \in V$. By the above correspondence, $NS(X)$ also equals the set of Hermitian forms $H : V \times V \to \mathbb{C}$ such that $ImH(\Lambda, \Lambda) \subset \mathbb{Z}$.*

2.18 Appell-Humbert Theorem

In this section we are going to see an explicit description of $NS(X), \mathrm{Pic}(X), \mathrm{Pic}^0(X)$ for $X = V/\Lambda$ a complex torus. Recall that we had an explicit description of $NS(X)$ already: it corresponds to the set of Hermitian forms $H : V \times V \to \mathbb{C}$ with $Im(\Lambda, \Lambda) \subset \mathbb{Z}$. Define $\mathrm{Pic}^0(X)$ to be the collection of line bundles such that the first chern group on them are trivial. That is,

$$\mathrm{Pic}^0(X) = \ker(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z})).$$

It is easy to see

$$\mathrm{Pic}^0(X) \simeq H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

Clearly, we have the following short exact sequence:

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

The Appell-Humbert theorem is going to give another (more explicit) short exact sequence that is isomorphic to the one above.

Let $T_1 = \{z \in \mathbb{C}^\times : |z| = 1\}$ be the unit circle in $\mathbb{C}$. Given a Hermitian form H in $NS(X)$, a character $\chi : \Lambda \to T_1$ is called a **semi-character** for H if for all $\lambda, \mu \in \Lambda$, we have

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i ImH(\lambda\mu)}.$$

Let $P(\Lambda)$ be the set of (H, χ) with $H \in NS(X)$ such that χ is a semi-character for H.

Remarks: Note that when $H = 0$, this is just a usual character. Also, $P(\Lambda)$ is a group under

$$(H_1, \chi_1) \circ (H_2, \chi_2) := (H_1 + H_2, \chi_1\chi_2).$$

Note that we have the following exact sequence:

$$0 \to \mathrm{Hom}(\Lambda, T_1) \to P(\Lambda) \to NS(X)$$

where the second map is given by $x \mapsto (0, x)$ and third map by $(H, \chi) \mapsto H$. And therefore we have a map $P(\Lambda) \to \mathrm{Pic}(X) \simeq H^1(\Lambda, M)$ (where $M = H^0(V, \mathcal{O}_V^\times)$) given by

$$(H, \chi) \mapsto a_{(H,\chi)}(\cdot, \cdot) + B^1(\Lambda, M)$$

where $a_{(H,\chi)}(\lambda, v) := \chi(\lambda)e^{(\pi H(\lambda, v) + \pi/2H(v,v))}$. It is easy to check that

$$a_{(H,\chi)}(\lambda, \mu, v) = a_{(H,\chi)}(\lambda, \mu + v)a_{(H,\chi)}(\mu, v).$$

This gives us an isomorphism

$$L(H, \chi) \simeq \frac{V \times \mathbb{C}}{\Lambda}$$

where Λ acts as $\lambda \circ (v, t) = (v + \lambda, a(\lambda, v)t)$. We have the following facts:

1. $P(\Lambda) \to \mathrm{Pic}(X)$ is a group homomorphism. (Easy to prove.)

2. The map given by composing $P(\Lambda) \to \mathrm{Pic}(X) \overset{c_1}{\to} NS(X)$ is the same as the "forget χ" map. In particular, the map $\alpha : P(\Lambda) \to NS(X)$ defined by $(H, \chi) \mapsto H$ is surjective (since c_1 is).

By these facts, we hence have the following theorem.

2.18.1 Theorem. *The following is a commutative diagram of short exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, T_1) \arrow[r] \arrow[d, "\alpha'"] & P(\Lambda) \arrow[r] \arrow[d, "\alpha"] & NS(X) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & NS(X) \arrow[r] & 0
\end{tikzcd}$$

where α' is the restriction of α to $\mathrm{Hom}(\Lambda, T_1)$.

Once we have shown that α' is an isomorphism in the proof, we will have a very explicit way to talk about $\mathrm{Pic}(X)$ - not only have we parametrized line bundles by (H, χ), such a pair even leads to a distinguished cocycle in the cohomology class to work with.

Proof. We already knew that the right-hand square in the above diagram commutes, and this implies that α' restricts to what we want (and the left-hand square commutes arbitrarily). So we just need to show that α' and α are isomorphisms. It suffices to show that α' is an isomorphism, since then the five-lemma implies α is.

First, we start with a diagram

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^\times) \arrow[r, "c_1"] \arrow[d, "="] & H^2(X,\mathbb{Z}) \\ & H^1(X,\mathbb{C}) \arrow[u, two heads] \arrow[r, "\epsilon"] & H^1(X,\mathcal{O}_X^\times) & \end{tikzcd}$$

Here, the surjective map is actually a projection which comes from the folllowing: By the Hodge decomposition, we have $H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \bigoplus H^{0,1}(X)$ for which by the Dolbeault cohomology is isomorphic to $H^0(X, \Omega_X) \bigoplus H^1(X, \mathcal{O}_X)$. So we have a projection from $H^1(X,\mathbb{C}) \twoheadrightarrow H^1(X,\mathcal{O}_X)$.

Next, by taking the sheaf map $\underline{\mathbb{C}}^\times \to \mathcal{O}_X^\times$ which takes a locally constant function f to the locally constant nonvanishing function $\exp(2\pi i f)$; this induces a map $\epsilon : H^1(X, \underline{\mathbb{C}}) \to H^1(X, \mathcal{O}_X^\times)$ on cohomology.

To see that the diagram commutes, note that since $\mathrm{Pic}^0(X)$ is the image of $\epsilon : H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}_X^\times) \simeq H^1(\Lambda, M))$, every element $L \in \mathrm{Pic}^0(X)$ is represented by a 1-cech cocycle (that is, an element of $Z^1(X, \mathcal{O}_X^\times))$ with constant coefficients. Therefore, $H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}_X^\times$ implies that the corresponding class in $H^1(\Lambda, M)$ is represented by a factor of automorphy (an element of $Z^1(\Lambda, M)$) with constant coefficients, that is, $f(\lambda, v)$ is independent of v.

We claim that α' is surjective. Let $L \in \mathrm{Pic}^0(X)$ with $f \in Z^1(\Lambda, M)$ with "constant coefficient" (i.e. $f(\lambda, v)$ is independent of v). Then we have

$$f(\lambda + \mu, \tilde{x}) = f(\lambda, \mu + \tilde{x}) f(\mu, \tilde{x}).$$

So f gives a homomorphism $f : \Lambda \to \mathbb{C}^\times$. Define g by $f = e^{2\pi i g}$, then obviously we have

$$g(\lambda + \mu) = g(\lambda) + g(\mu) \pmod{\mathbb{Z}}.$$

Therefore, $Im(g) : \Lambda \to \mathbb{R}$ is a homomorphism, which induces a linear map $Im(g) : V \to \mathbb{R}$. Define $\ell : V \to \mathbb{C}$ by $v \mapsto Im(g)(iv) + iIm(g)(v)$. Then $e^{2\pi i \ell(v)} \in H^0(V, \mathcal{O}_V^\times)$ for all $v \in V$. Therefore we have

$$\chi_L(\lambda, v) = f(\lambda) \cdot e^{2\pi i \ell(v) - 2\pi i \ell(v+\lambda)}.$$

is in $Z^1(\Lambda, M)$, that is, a 1-coboundary. But now $\chi_L \in \mathrm{Hom}(\Lambda, T_1)$. To check this,

$$\chi_L(\lambda, v) = e^{2\pi i g(\lambda) - 2\pi i \ell(v)} = e^{2\pi i (Reg(\lambda) - Img(i\lambda))} \in T_1$$

as $Reg(\lambda) - Img(i\lambda) \in \mathbb{R}$. Moreover, this last expression is independent of v. Therefore, f and $e^{2\pi i \ell(\cdot)}$ are homomorphisms and hence χ_L is also a homomorphism.

To show that α' is injective, suppose χ_1 and χ_2 both give $L \in \mathrm{Pic}(X)$, then we can write

$$\chi_1(\lambda) = \chi_2(\lambda) \cdot \frac{h(\lambda + v)}{h(v)}$$

where $\frac{h(\lambda+v)}{h(v)}$ is a 1-coboundary. Since $|\chi_1| = |\chi_2| = 1$,, we have $|h(\lambda+v)| = |h(v)|$ for all v, λ. Hence h is bounded. By the Liouville theorem h is constant and so $\chi_1 = \chi_2$.

□

2.19 Canonical factors

Recall that in the proof of the Appell-Humbert Theorem, we expressed α as the map given by

$$(H, \chi) \mapsto a_{(H,\chi)}(\lambda, v) = \chi(\lambda) e^{\pi(H(\lambda,v)+1/2H(\lambda,\lambda))}$$

where $H = c_1(L)$. We define the **canonical factor** of (H, χ) to be $a_{(H,\chi)}$. In fact, most questions about line bundles boil down to explicit computations on these $a_{(H,\chi)}$!

First, let us see some basic properties:

1. $\chi(n\lambda) = \chi(\lambda)^n$ for all λ.
2. $a_L(\lambda, v + w) = a_L(\lambda, v) e^{\pi H(\lambda,w)}$.
3. $\frac{1}{a_L(\lambda,v)} = a_L(-\lambda, v) e^{-\pi H(\lambda,\lambda)}$.

2.20 Behaviour of line bundles under holomorphic maps

Recall that holomorphic maps are given by compositions of translations and homomorphisms.

2.20.1 Lemma. *If $t_x : X \to X$ is given by $y \mapsto x + y$ and we also have a map $L = L(H, \chi) \to X$,*

$$\begin{tikzcd} L \arrow[r, dotted, "t_x^*"] \arrow[d] & L \arrow[d] \\ X \arrow[r, "t_x"] & X \end{tikzcd}$$

then the pullback map is given by

$$t_x^* L(H, \chi) = L(H, \chi e^{2\pi i Im H(\tilde{x}, \cdot)})$$

where $\tilde{x}$ is a lift of x to the universal cover V.

2.20.2 Lemma. *If we have a map between complex tori $f : X' \to X$ where $X' = V'/\Lambda'$ and $X = V/\Lambda$. Suppose $L = L(H, \chi) \in \mathrm{Pic}(X)$, then*

$$f^* L(H, \chi) = L(f_{an}^* H, f_{Int}^* \chi).$$

2.20.3 Corollary (Theorem of squares)**.** *For $v, w \in X$ and $L \in \mathrm{Pic}(X)$, we have $t^*_{v+w}L \simeq t^*_v L \otimes t^*_w L \otimes L^{-1}$.*

Proof. For $L = L(H, \chi)$, we have $t^*_x L(H, \chi) = L(H, \chi e^{2\pi i Im(H(\lambda,v))})$. □

2.20.4 Corollary (Theorem of cubes for complex tori)**.** *Let X_1, X_2, X_3 be complex tori. Let L be a line bundle on $X_1 \times X_2 \times X_3$. If the restrictions of L on $X_1 \times X_2 \times \{0\}, X_1 \times \{0\} \times X_3$ and $\{0\} \times X_2 \times X_3$ are trivial, then L is also trivial on $X_1 \times X_2 \times X_3$.*

Remark: the theorem of cubes works if X_1, X_2, X_3 are just complete varieties. Even more, Mumford p.55 [farbod reference] says the theorem still holds if one of them is just connected.

2.20.5 Corollary. *For $n \in \mathbb{Z}$, let $n_X : X \to X$ be the multiplication-by-n map $x \mapsto nx$. If $L \in \mathrm{Pic}(X)$, then*

$$n^*_X L = L^{\frac{n^2+n}{2}} \bigotimes (-1)^* L^{\frac{n^2-n}{2}}$$

where L^k is the tensor product $L \bigotimes \cdots \bigotimes L$ of k copies of L.

Proof. Write $L = L(H, \chi)$. Then

$$L^{\frac{n^2+n}{2}} \bigotimes (-1)^* L^{\frac{n^2-n}{2}} = L(\frac{n^2+n}{2} H + (-1)^* \frac{n^2-n}{2} H, \chi^{\frac{n^2+n}{2}} \times (-1)^* \chi^{\frac{n^2-n}{2}}).$$

Since $(-1)^* H(u, v) = H(-u, -v) = H(u, v)$ and $(-1)^* \chi(\lambda) = \chi(-\lambda) = \frac{1}{\chi(\lambda)}$, this is equal to

$$L(n^2 H, \chi^n) = L(n^* H, n^* \chi(\cdot)) = n^* L(H, \chi).$$

□

A line bundle L is called **symmetric** if $(-1)^* L = L$.

2.20.6 Corollary. *If L is symmetric, then $n^*_X L = L^{n^2}$.*

2.20.7 Lemma. *$L = L(H, \chi)$ is symmetric if and only if $\chi(\lambda) = \pm 1$ for all $\lambda \in \Lambda$.*

Proof. Since $(-1)^* L(H, \chi) = L(H, \frac{1}{\chi})$, we have $\chi^2(\cdot) = 1$. □

2.21 Dual complex tori

Our goal of this section is to show that any complex torus $X = V/\Lambda$ has a dual $\hat{X}$ with functorial properties. Moreover, given $L \in \mathrm{Pic}(X)$, one has $\varphi_L : X \to \hat{X}$. We will also talk about $X \times \hat{X}$, Poincaré bundles and biextensions.

First of all, the Appell-Humbert theorem tells us that we have an isomorphism

$$\mathrm{Hom}(\Lambda, T_1) \simeq \mathrm{Pic}^0(X).$$

But in the case of complex tori, we have $\mathrm{Hom}(\Lambda, T_1) \simeq \mathrm{Hom}(\mathbb{Z}^{2g}, S^1) \simeq (\mathbb{R}/\mathbb{Z})^{2g}$. Is $\mathrm{Pic}^0(X)$ naturally a complex torus? The answer is yes! This is because we have a canonical isomorphism $\mathrm{Pic}^0(X) \xleftarrow{\sim} \hat{X}$ from the dual complex torus.

Recall that the cotangent space at $\underline{0}$ is given by

$$\Omega = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \{\ell : V \to \mathbb{C} :\ \ell(av + bw) = a\ell(v) + b\ell(w),\ \forall a, b \in \mathbb{C}, v, w \in V\}$$

where V is the tangent space at 0. We have a conjugate of Ω given by

$$\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) = \{\ell : V \to \mathbb{C} :\ \ell(av + bw) = \overline{a}\ell(v) + \overline{b}\ell(w),\ \forall a, b \in \mathbb{C}, v, w \in V\}.$$

Then we claim that we have a functional

$$\overline{\Omega} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}).$$

Note that $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \simeq \mathbb{R}^{2g}$. It is not hard to see that the inverse maps are given by $\ell \mapsto K = Im(\ell)$ and $\ell \leftarrow\!\shortmid K$ where $\ell(v) := -K(iv) + iK(v)$. It follows that

$$\langle \cdot, \cdot \rangle : \overline{\Omega} \times V \to \mathbb{R}$$

defined by

$$\langle \ell, v \rangle = Im\ell(v)$$

is $\mathbb{R}$-bilinear and is non-degenerate. By non-degenerate, we mean

- If $\langle \ell, v \rangle = 0$ for all ℓ, then $v = 0$; and
- If $\langle \ell, v \rangle = 0$ for all v, then $\ell = 0$.

Therefore,

$$\hat{\Lambda} := \{\ell \in \overline{\Omega} :\ \langle \ell, \lambda \rangle \in \mathbb{Z}\ \forall \lambda \in \Lambda\}$$

is a lattice (full rank, i.e. $\simeq \mathbb{Z}^{2g}$) in $\overline{\Omega}$. We then define

$$\hat{X} = \hat{\Omega}/\hat{\Lambda}$$

to be the **"dual"** of X.

2.21.1 Theorem. *There is a canonical isomorphism*

$$\hat{X} \xrightarrow{\sim} \mathrm{Pic}^0(X)$$

induced by $\overline{\Omega} \to \mathrm{Hom}(\Lambda, T_1)$ *which maps* $\ell \mapsto e^{2\pi i Im\ell(\cdot)}$ *where* $Im\ell(\cdot) = \langle \ell, \cdot \rangle$.

Proof. By non-degeneracy, the map is surjective. Also the kernel by definition is just $\hat{\Lambda}$. □

2.21.2 Corollary. $\mathrm{Pic}^0(X) = \hat{X}$ *is a complex torus.*

2.22 Basic functorial properties of $\hat{X}$

Here are some of the basic functorial properties of $\hat{X}$:

1. For $f : X_1 \to X_2$ homomorphism with $f_{an} : V_1 \to V_2$, then $f_{an}^* : \overline{\Omega_2} \to \overline{\Omega_1}$ satisfies

$$f_{an}^*(\hat{\Lambda}_2) \subset \hat{\Lambda}_1.$$

   So we get a homomorphism $\hat{f} : \hat{X}_2 \to \hat{X}_1$. Moreover,

$$
\begin{tikzcd}
\hat{X}_2 \arrow[r, "="] \arrow[d, "\hat{f}"] & \mathrm{Pic}^0(X_2) \arrow[r, "="] \arrow[d, "f^*"] & \mathrm{Hom}(\Lambda_2, T_1) \arrow[d, "f^*_{int}"] \\
\hat{X}_1 \arrow[r, "="] & \mathrm{Pic}^0(X_1) \arrow[r, "="] & \mathrm{Hom}(\Lambda_1, T_1)
\end{tikzcd}
$$

commutes.

2. If $f : X_1 \to X_2, g : X_2 \to X_3$ are homomorphisms, then

$$\widehat{g \circ f} = \hat{f} \circ \hat{g}.$$

Also, $\hat{Id_X} = Id_{\hat{X}}$ with $\hat{\hat{X}} = X$ and hence $\hat{\hat{f}} = f$.

As a consequence, we see that $\hat{(\cdot)}$ is a contravariant functor on the category of complex tori.

2.22.1 Lemma. *The functor $\hat{(\cdot)}$ is exact. That is, for all short exact sequences*

$$0 \to X_1 \to X_2 \to X_3 \to 0,$$

we have

$$0 \to \hat{X}_3 \to \hat{X}_2 \to \hat{X}_1 \to 0.$$

Proof. An easy application of the snake lemma gives us the short exact sequence

$$0 \to \Lambda_1 \to \Lambda_2 \to \Lambda_3 \to 0.$$

Since Λ_3 is a projective (free) $\mathbb{Z}$-module, we therefore have

$$0 \to \mathrm{Hom}(\Lambda_3, T_1) \to \mathrm{Hom}(\Lambda_2, T_1) \to \mathrm{Hom}(\Lambda_1, T_1) \to 0.$$

Lastly, $\mathrm{Hom}(\Lambda_i, T_1) \simeq \mathrm{Pic}^0(X_i) \simeq \hat{X}_i$. □

2.23 Isogenies and duality

If $f : X_1 \to X_2$ is an isogeny, what can we say about $\hat{f}$?

2.23.1 Proposition. *Let $f : X_1 \to X_2$ be an isogeny, with dual homomorphism $\hat{f} = \hat{X}_2 \to \hat{X}_1$. Then*

(a) *$\hat{f}$ is an isogeny.*

(b) $\ker(\hat{f}) = \mathrm{Hom}(\ker(f), T_1)$.

(c) $\deg(\hat{f}) = \deg(f)$.

Sketch of proof. We have $\ker(\hat{f}) \simeq \ker(\mathrm{Hom}(\Lambda_2, T_1) \xrightarrow{f^*_{int}}) \mathrm{Hom}(\Lambda_1, T_1) \simeq \ker(\mathrm{Hom}(\Lambda_2/f^*_{int}(\Lambda$
But $\Lambda_2/f^*_{int}(\Lambda_1) \simeq \ker(f)$. □

2.24 Line bundels v.s. duality

Do line bundles descend under isogeny?

2.24.1 Proposition. *Suppose $f : X_1 \to X_2$ is an isogeny. Let $L \in \mathrm{Pic}(X_1)$. As before, we write $L = L(h, \chi)$. The following are equivalent:*

1. *$L = f^*M$ for some $M \in \mathrm{Pic}(X_2)$.*
2. *The image of $H(f_{an}^{-1}\Lambda_2, f_{an}^{-1}\Lambda_2)$ is contained in $\mathbb{Z}$.*

Proof. First statement implies the second follows from the pull-back formula above. Now, assume the image of $H(f_{an}^{-1}\Lambda_2, f_{an}^{-1}\Lambda_2)$ is contained in $\mathbb{Z}$, then $H_1 := (f_{an}^{-1})^*H \in NS(X_2)$. So there exists $\tilde{M} \in \mathrm{Pic}(X_2)$ with $c_1(\tilde{M}) = H_1$. By the pull-back formula, $c_1(f^*\tilde{M}) = H$. Then $c_1(L \bigotimes (f^*\tilde{M})^{-1}) = 0$. Since $\hat{f} : \mathrm{Pic}^0(X_2) \to \mathrm{Pic}^0(X_1)$ is surjective, there exists $N \in \mathrm{Pic}^0(X_2)$ such that $f^*N = L \bigotimes (f^*\tilde{M})^{-1}$. Now, $M = N \bigotimes \tilde{M}$ does the job! □

So far, what we have for duality is very basic. But in the next section, we will see something highly nontrivial.

2.25 Line bundles and maps $X \to \tilde{X}$

Let $L \in \mathrm{Pic}(X)$. We write the map $\varphi_L : X \to \tilde{X} = \mathrm{Pic}^0(X)$ which is given by

$$x \mapsto t_x^*L \bigotimes L^{-1}.$$

Here are some facts about this φ_L:

1. First of all, φ_L is well defined, since $c_1(t_x^*L \bigotimes L^{-1}) = 0$.
2. φ_L is a group homomorphism. To see this, recall that by the theorem of squares, we have
$$f_{x+y}^*L \bigotimes L^{-1} \simeq t_x^*L \bigotimes L^{-1} \bigotimes t_y^*L \otimes L.$$
3. φ_L has the analytic representation $\varphi_H : V \to \overline{\Omega}$ given by
$$v \mapsto H(v, \cdot).$$
(This follows from the fact that $\varphi_L(x) = L(0, e^{2\pi i Im(H(v,\cdot))})$.)
4. φ_L only depends on $c_1(L) = H$.
5. $\varphi_{L \bigotimes M} = \varphi_L + \varphi_M$.
6. For any $L \in \mathrm{Pic}(X)$, the diagram
$$\begin{tikzcd} X \arrow[r, "\varphi_L"] & \hat{X} \arrow[d, "\hat{f}"] \\ Y \arrow[u, "f"] \arrow[r, "\varphi_{f^*(L)}"'] & \hat{Y} \end{tikzcd}$$
commutes.

2.26 Kernel of φ_L

Define $K(L)$ to be $\ker(\varphi_L : X \to \tilde{X})$. Here are some basic properties of $K(L)$:

1. Let $\Lambda(L) := \{v \in V : \ Im(H(v,\lambda)) \in \mathbb{Z}\ \forall \lambda \in \Lambda\} = \varphi_H^{-1}(\hat{\Lambda})$. Then

$$K(L) \simeq \Lambda(L)/\Lambda.$$

   Note that hence $K(L)$ only depends on $H = c_1(L)$.

2. $K(L \bigotimes P) \simeq K(L)$ for $P \in \mathrm{Pic}^0(X)$.

3. $K(L) = X$ if $L \in \mathrm{Pic}^0(X)$.

4. $K(L^n) = n_X^{-1} K(L)$ where $L^n = L \bigotimes \cdots \bigotimes L$ (n-copies) and n_X^{-1} is the inverse isogeny.

5. $K(L) = n_X K(L^n)$ if $n \neq 0$.

For the last two facts, recall that for $L = L(H, \chi)$, we have $L^n = L(nH, \chi^n)$. Hence, $L_1 \bigotimes L_2 = L_1 \bigotimes L_2(H_1+H_2, \chi_1\chi_2)$. Also, $\Lambda(L^n) = \{v \in V : \ Im(H(nv,\lambda)) \in \mathbb{Z},\ \forall \lambda \in \Lambda\} = \{\frac{1}{n}v \in V : \ v \in \Lambda(L)\}$.

Recall that $H(\cdot,\cdot)$ is called **non-degenerate** if

- $H(v,w) = 0$ for all v implies $w = 0$; and
- $H(v,w) = 0$ for all w implies $v = 0$.

We say that $L \in \mathrm{Pic}(X)$ is **non-degenerate** if $c_1(L) = H$ is non-degenerate. (Equivalently, the associated alternating form $Im(H)$ is non-degenerate.)

2.26.1 Lemma.

1. *L is non-degenerate if and only if $K(L)$ is finite.*

2. $\deg(\varphi_L) = \det(Im(H)) = [\Lambda(L) : L]$.

2.27 Poincaré bundle:

We have seen that for X a complex torus,

(a) a point in $\hat{X} = \mathrm{Pic}^0(X)$ gives a line bundle on X;

(b) a point on $X = \hat{\hat{X}} = \mathrm{Pic}^0(\hat{X})$ gives a line bundle on $\hat{X}$.

So, does there exist a (universal) line bundle on $X \times \hat{X}$ such that (a) and (b) are "shadows" of that line bundle? The answer is yes!

A **Poincaré bundle** is a holomorphic line bundle $\mathcal{P}$ on $X \times \hat{X}$ satisfying

1. $\mathcal{P}|_{X \times \{L\}} \simeq L$;

2. $\mathcal{P}|_{\{0\} \times \hat{X}}$ is the trivial line bundle on $\hat{X}$.

2.27.1 Theorem. *There exists a Poincaré bundle on $X \times \hat{X}$ uniquely determined up to isomorphism.*

Proof. First note that we have

$$X \times \hat{X} \simeq V \times \overline{\Omega}/\Lambda \times \hat{\Lambda}.$$

For the existence, we define $H : (V \times \overline{\Omega}) \times (V \times \overline{\Omega}) \to \mathbb{C}$ by

$$((v_1, \ell_1), (v_2, \ell_2)) \mapsto \overline{\ell_2(v_1)} + \ell_1(v_2)$$

which is in fact a Hermitian form. (In particular, it is non-degenerate!) Note that

$$Im(H(\Lambda \times \hat{\Lambda}, \Lambda \times \hat{\Lambda})) \subset \mathbb{Z}.$$

So there exist $\mathcal{L} = L(H, \chi)$ for semi-characters χ. We then define $\chi_0 : \Lambda \times \hat{\Lambda} \to T_1$ by

$$(\lambda, \ell_0) \mapsto e^{\pi i Im(\ell_0(\lambda))}$$

which is indeed a semi-character for H.

Now, we claim that $\mathcal{P} = L(H, \chi_0)$ is a Poincaré bundle. To see this, we consider the associated canonical factor

$$a_{\mathcal{P}}((\lambda, \ell_0), (v, \ell)) = \chi((\lambda, \ell_0)) \cdot e^{\pi(H((\lambda,\ell_0),(v,\ell)) + \frac{1}{2}H((\lambda,\ell_0),(\lambda,\ell_0)))}.$$

Now, what is left is just the checkings:

1. For any $L \in \hat{X} = \mathrm{Pic}^0(X)$, we have $L = L(0, e^{2\pi i Im(\ell(\cdot))})$ for some $\ell \in \overline{\Omega}$. Note that

   $$a_L(\lambda, v) = e^{2\pi i Im(\ell(\lambda))}.$$

   Therefore, $\mathcal{P}|_{X \times \{L\}}$ corresponds to $a_{\mathcal{P}}|_{(\Lambda,0)\times(V\times\{\ell\})}$ but

   $$a_{\mathcal{P}}((\lambda, 0), (v, \ell)) = e^{\pi \ell(\lambda)}.$$

   Now, multiplication by the 1-coboundary $e^{\pi\overline{\ell(v)}}/e^{\pi\overline{\ell(v+\lambda)}}$ takes $a_{\mathcal{P}}((\lambda, 0), (v, \ell))$ to $a_L(\lambda, v)$. Therefore, $\mathcal{P}|_{X\times\{L\}} \simeq L$.

2. $\mathcal{P}_{\{0\}\times\hat{X}}$ has $a_{\mathcal{P}}((0, \ell_0), (0, \ell)) = 1$ as 1-cocycle. So it is trivial.

Uniqueness of the Poincaré bundle follows from the *Seesaw Principle*: Let X, Y be compact complex manifolds and $\mathcal{L}$ be a holomorphic line bundle on $X \times Y$. If $L|_{X\times\{z\}}$ is trivial for all $z \in \mathcal{U}$ where $\mathcal{U}$ is an open dense subset of Y, and if $L|_{\{x_0\}\times Y}$ is trivial for some $x_0 \in X$, then L is trivial.

□

Here are some remarks:

- Poincaré line bundles are non-degenerate.
- Let T be any *normal* complex analytic space and X be a complex torus. If L is a line bundle on $X \times T$ such that
  1. $L|_{X\times\{t\}} \in \mathrm{Pic}^0(X)$ for all $t \in T$ (If T is connected, it suffices to check that there exists $t \in T$ such that this is true); and
  2. $L|_{\{0\}\times T}$ is trivial;

  then there exists a unique holomorphic $\psi : T \to \hat{X}$ such that $L \simeq (id \times \psi)^*\mathcal{P}$. That is, $\mathcal{P}$ factors through ψ. (The proof of this uses *Zariski's main theorem* and a more general Seesaw Principle.)

2.28 A few applications of Poincaré bundles

Let L_1, L_2 be line bundles on X. We say that L_1 is **analytically equivalent** to L_2, denoted by $L_1 \sim_{an} L_2$ if there exist a connected complex analytic space T, a line bundle L on $X \times T$ and $t_1, t_2 \in T$ such that

$$L|_{X\times\{t_i\}} \simeq L_i.$$

2.28.1 Proposition. *Let L_1, L_2 be line bundles on a complex torus $X = V/\Lambda$. The following are equivalent:*

1. $L_1 \sim_{an} L_2$.

2. $L_1 \bigotimes L_2^{-1} \in \mathrm{Pic}^0(X)$.

3. $\varphi_{L_1} = \varphi_{L_2}$.

4. $c_1(L_1) = c_1(L_2)$.

For a complex torus X, we use $\mathrm{Pic}^H(X)$ to denote the classes in $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

2.28.2 Corollary. *Let $X = V/\Lambda$ be a complex torus. Then there is a correspondence between analytic equivalence classes of X and* $\mathrm{Pic}^H(X)$.

Now we prove the proposition.

Proof. We already know that (2), (3) and (4) implies one and other. To show that (2) implies (1), assume that $L_1 \bigotimes L_2^{-1} \in \mathrm{Pic}^0(X)$ and $p : X \times \hat{X} \to X$ is the projection map. Then $L = p^* L_2 \bigotimes \mathcal{P}$ is a line bundle on $X \times \hat{X}$ such that $L|_{X\times\{0\}} \simeq L_2$ and $L|_{X\times\{L_1 \bigotimes L_2^{-1}\}} \simeq L_1$.

Now to show that (1) implies (4), suppose $L_1 \sim_{an} L_2$, that is, there exist complex analytic space T, a line bundle L on $X \times T$ such that $L|_{X\times\{t_i\}} \simeq L_i$. Consider $T \to H^2(X, \mathbb{Z})$ given by $t \mapsto c_1(L|_{X\times\{t\}})$ is continuous and the image is discrete and so it is a constant map. This implies that $c_1(L_1) = c_1(L_2)$. □

2.28.3 Lemma. *Let $L, L' \in \mathrm{Pic}(X)$. Suppose L is non-degenerate, then $L \sim_{an} L'$ if and only if there exists $x \in X$ such that $L' \simeq t_x^* L$.*

Proof. The "if" direction is always true. To show the "only if" direction, $L \sim_{an} L'$ is equivalent to saying that $L' \bigotimes L^{-1} \in \mathrm{Pic}^0(X)$. But since L is non-degenerate, $\varphi_L : X \to \mathrm{Pic}^0(X)$ defined by $x \mapsto t_x^* L \bigotimes L^{-1}$ is surjective. So there exists $x \in X$ such that $t_x^* L \bigotimes L^{-1} \simeq L' \bigotimes L^{-1}$. □

Given a homomorphism $f : X \to \hat{X}$, does there exist a line bundle L such that $f = \varphi_L$?

2.28.4 Theorem. *Let $X = V\Lambda$ be a complex torus and $f : X \to \hat{X}$ a homomorphism with $f_{an} : V \to \overline{\Omega}$. The following are equivalent:*

1. $f = \varphi_L$ *for* $L \in \mathrm{Pic}(X)$.

2. $F : V \times V \to \mathbb{C}$ *defined by*

$$(v, w) \mapsto f_{an}(v)(w)$$

is Hermitian.

The proof to this theorem is pretty elementary, straight-forward and uses the following lemma:

2.28.5 Lemma. *Let* $M \in \mathrm{Pic}(X)$ *and* $n \in \mathbb{Z}$. *Then the following are equivalent:*

1. $M = L^n$ *for some* $L \in \mathrm{Pic}(X)$.

2. $X[n] \subset K(M)$ *where* $K(M)$ *is the kernel of* $\varphi_M : X \to \hat{X}$.

2.29 The Poincare-Bundle as a Biextension

A Poincaré bundle $\mathcal{P}$ is a biextension of $X \times \hat{X}$ by $\mathbb{C}^\times$. Then we will define an object $Bi - ext(B \times C, A)$.

All abelian varieties in this lecture are over the complex numbers. The following definition of biextensions can be found for instance in Mumford's paper 'Biextensions of Formal Groups'

2.29.1 Definition. Let A, B, C be abelian groups. A biextension of $B \times C$ by A is a set G along with
1. An action of A on G.
2. A surjective map $\pi : G \to B \times C$,

$$\pi(g) = (\pi_B(g), \pi_C(g))$$

which induces a bijection $G/A \xrightarrow{\sim} B \times C$
3. Maps

$$+_1 : G \times_B G \to G$$

$$+_2 : G \times_C G \to G$$

so that the following conditions are satisfied
1. $\forall b \in B$, the fibre over $b \times C$ in G, $G'_b := \pi_B^{-1}(b) = \pi^{-1}(b \times C)$ is an abelian group with respect to the restriction of $+_1$. π_C is a surjective homomorphism of G'_b onto C, the kernel of π_C is isomorphic to A.
2.Likewise, the fibre G_c over $B \times c$ for $c \in C$ is an abelian group with respect to the restriction of $+_2$. π_B is a surjective homomorphism of G_c onto B, the kernel of π_B is isomorphic to A.
3. Given $x, y, u, v \in G$, with

$$\pi(x) = (b_1, c_1), \pi(y) = (b_1, c_2), \pi(u) = (b_2, c_1), \pi(v) = (b_2, c_2)$$

the following compatibility relation holds

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v)$$

($G \times_B G$ is the fibred product $G \times_B G := \{(g_1, g_2) \in G \times G \mid \pi_B(g_1) = \pi_B(g_2)\}$, likewise, the set $G \times_C G := \{(g_1, g_2) \in G \times G \mid \pi_C(g_1) = \pi_C(g_2)\}$)

The definition of a biextension seems hard to grasp at first glance. The example of the Poincare Bundle with the zero-section removed as a bi-extension of an abelian variety and its dual by $\mathbb{C}^\times$ should be understood to put the definition in perspective. Let $X = V/\Lambda$ be an abelian variety and $\hat{X} = \Omega/\hat{\Lambda}$ be the dual abelian variety. Let $P \to X \times \hat{X}$ be the Poincare-Bundle on $X \times \hat{X}$. The example we have in mind is that of the Poincare Bundle with the zero-section removed, ie, $A = \mathbb{C}^\times$, $B = X$, $C = \hat{X}$ and $G = P/\{0\}$ with $\pi : P/\{0\} \to X \times \hat{X}$ the projection map restricted to the complement of the zero section, A acts by scalar multiplication. We note in passing that the trivial $\mathbb{C}^\times$ bundle on a vector space W, $L_0 := \mathbb{C}^\times \times W$ is an abelian group with group operation

$$(l_1, w_1) + (l_2, w_2) := (l_1 l_2, w_1 + w_2)$$

with identity $(1, 0)$ and inverse $(l, w)^{-1} = (\frac{1}{l}, -w)$. If $\Pi : L \to W/\Lambda$ is any line bundle on an abelian variety, then $\Pi^*(L/\{0\}) \simeq \mathbb{C}^\times \times W$ is an abelian group and this group structure descends to a natural group structure on $L/\{0\}$. $G_L \simeq L\{0\}$ by hypothesis, G'_x is a line bundle on $\hat{X}$ with zero section removed. Points of $G \times_C G$ (resp $G \times_B G$) correspond to pairs of points (l_1, l_2) on a line bundle over the abelian variety X (resp $\hat{X}$), the maps $+_1$ and $+_2$ are determined so as to correspond to the group operations on the line bundles with zero section removed. The reader need not work out condition 3, it is in fact a nontrivial result which follows from Lang Duality.

Equivalence classes of biextensions can be suitably expressed in the context of cohomology, we do not however pursue this theme any further.

2.30 Cohomologies of Line Bundles on Complex Tori

We will now proceed to discuss the notions of characteristics of line bundles L on X, theta-functions as sections of line-bundles and more generally describe all the cohomology groups $H^i(X, L)$. We shall then prove some vanishing theorems for cohomology and compute the alternating sums of the cohomological dimensions from which we can deduce Riemann-Roch.

Fix $H \in NS(X)$, let $\mathrm{Pic}^H(X)$ denote the line bundles on X with chern class H. Given a suitable decomposition of $\Lambda = \Lambda_1 \oplus \Lambda_2$ (which are in some way orthogonal) we can distinguish a line-bundle $L_0 \in Pic^H(X)$. If H is nondegenerate, $L \in Pic^H(X)$ is a translate $L = t_c^* L_0$ and c is called the characteristic of L_0 with respect to the decomposition of Λ. This will allow us to explicitly describe $K(L) = \ker \phi_L$. Let $E = \mathrm{Im} H$, this is a $\mathbb{Z}$ valued alternating form.

2.30.1 Lemma. *Suppose that $2g$ is the rank of the lattice Λ. There exists a $\mathbb{Z}$-basis for Λ, $U = \{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ such that the matrix for E wrt U is*

Proof. Pick any basis to begin with. Since H is hermitian, E is skew symmetric, so the matrix for E in this arbitrary basis looks like

$$\begin{pmatrix} F & A \\ -A^T & G \end{pmatrix}$$

where F and G are also skew symmetric. It's easy to see that we may further assume that $F = G = 0$. By row and column operations over $\mathbb{Z}$ we may reduce A to a diagonal matrix, over $\mathbb{Z}$ we essentially use the fact that the gcd of two numbers can be expressed as a linear combination of these numbers. So $\exists U, V \in GL_n(\mathbb{Z})$ such that $UAV = \mathrm{diag}(d_1, d_2, \dots, d_g)$ with d_i dividing d_{i+1} (note that in the case where we work over a field we may in fact insist that $V = U^{-1}$).

$$\begin{pmatrix} U & 0 \\ 0 & V^T \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

□

$(d_1, \dots, d_g)$ is uniquely determined by E or H or L.

2.30.2 Definition. We call the tuple $(d_1, \dots, d_g)$ the type of E or H or L and if all the $d_i = 1$ we call L a principle polarization.

We see that $K(\Lambda) = \ker\phi_L \simeq K_1 \oplus K_2$ with $K_i \simeq \oplus\mathbb{Z}/d_i\mathbb{Z}$. If all $d_i > 0$ then H or L or E is non-degenerate.

2.30.3 Definition. A basis $\{\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g\}$ be as before giving rise to the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ is called a canonical or symplectic-basis for Λ. A sub-lattice $\Lambda_1 \subset \Lambda$ is called totally isotropic for E if $E(\lambda, \lambda') = 0\ \forall \lambda, \lambda' \in \Lambda$.

2.30.4 Definition. A decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ is called a decomposition for E or H or L if both Λ_1 and Λ_2 are totally isotropic.

2.30.5 Definition. A decomposition $V = V_1 \oplus V_2$ of V into real vector spaces such that $(V_1 \cap \Lambda) \oplus (V_2 \cap \Lambda)$ is a decomposition for Λ is called the decomposition of V for E or H or L.

Let $H \in NS(X)$, $V = V_1 \oplus V_2$ a decomposition for H, define $\chi_0 : V \to T_1$ by

$$\chi_0(v) = e^{\pi i ImH(v_1, v_2)} = e^{\pi i E(v_1, v_2)}$$

where $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. It is easily seen that for $v, w \in V$,

$$\chi_0(v + w) = \chi_0(v)\chi_0(w)e^{\pi i E(v,w)}e^{-2\pi i E(v_2, w_1)}$$

(keep in mind that $E(\lambda, \mu) \in \mathbb{Z}$ for $\lambda, \mu \in \Lambda$)

2.30.6 Corollary. $(\chi_0)_{|\Lambda}$ *is a semicharacter for* H

2.30.7 Definition. $L_0 := L(H, \chi_0) \in \mathrm{Pic}^H(X)$ is a distinguished element of $\mathrm{Pic}^H(X)$ with respect to the decomposition of V.

Throughout this lecture $X = V/\lambda$ is a complex torus.

2.30.8 Lemma. *Let* $V = V_1 \oplus V_2$ *be a decomposition for* $H \in NS(X)$*, assume that* H *is non-degenerate*

1. $L_0 = L(H, \chi_0)$ *is the unique element in* $Pic^H(X)$ *where the semi character* χ_0 *is trivial on* $\Lambda_1 = V_1 \cap \Lambda$ *and* $\Lambda_2 = V_2 \cap \Lambda$.

2. *For $L \in \mathrm{Pic}^H(X)$, there is a $c \in V$ uniquely determined up to translation by $\Lambda(L)$ such that if $\bar{c}$ denotes the class of c modulo $\Lambda(L)$,*

$$L \simeq t_{\bar{c}}^* L_0$$

Proof. 1. We saw last time that χ_0 restricted to λ was a character, since $\chi_0(\lambda) := e^{\pi i E(\lambda_1, \lambda_2)}$ and each V_i is totally isotropic, it follows that χ_0 is trivial on both Λ_1 and Λ_2.

2. Certainly, since L and L_0 have the same Chern class H it follows that there is a $c \in V$ such that $L \simeq t_c^* L_0$, c modulo Λ is thus uniquely determined upto the kernel of $\phi_L = \Lambda(L)/\Lambda$, and thus $c \in V$ is uniquely determined up to $\Lambda(L)$.

□

2.30.9 Definition. Let L be a non-degenerate line bundle. c modulo $\Lambda(L)$ is called the class of the line bundle L with respect to the decomposition of $V = V_1 \oplus V_2$ into totally isotropic spaces for H.

We shall discuss proceed to describe the global sections of line-bundles. In doing so, we need to extend the canonical factor to $V \times V$. Recall that the canonical factor associated to a line-bundle is defined by the equation

$$a_L(\lambda, v) := \chi(\lambda) e^{H(v,\lambda) + \frac{1}{2} H(\lambda,\lambda)}$$

2.30.10 Definition. Let L be a non-degenerate line bundle with characteristic c. We extend the canonical factor a_L to a function on $V \times V$ using the characteristic of L,

$$a_L(u, v) := \chi_0(u) e^{2\pi i E(c,u)} e^{\pi(H(u,v) + \frac{1}{2} H(u,u))}$$

Note that the term $\chi_0(u) e^{2\pi i E(c,u)}$ restricts to χ on $\lambda \times \lambda$ since c is the characteristic and translation by c changes the character by $e^{2\pi i E(c,u)}$.

2.30.11 Proposition. *Let L be a line bundle with $H = c_1(L)$ non degenerate, let $V = V_1 \oplus V_2$ be a decomposition for V into isotropic subspaces with respect to L, for $u, v, w \in V$,*

1. $a_L(u, v + w) = a_L(u, v) e^{\pi H(w,u)}$

2. $a_L(u + v, w) = a_L(u, v + w) a_L(v, w) e^{2\pi i E(u_1, u_2)}$

3. $a_L(u, v)^{-1} = a_L(-u, v) \chi_0(u)^{-2} e^{-\pi H(u,u)}$

4. *If $L' = t_w^* L$ then $a_{L'}(u, v) = a_L(u, v) e^{2\pi i E(w,u)}$*

Proof. Excercise □

2.30.12 Lemma. *Let L be a non-degenerate line bundle and $V = V_1 \oplus V_2$ decomposition of V*

1. $\Lambda(L) = \Lambda(L)_1 \oplus \Lambda(L)_2$ *where* $\Lambda(L)_i := V_i \cap \Lambda(L)$

2. *$K(L) = K_1 \oplus K_2$ where $K_i = \Lambda(L)_i/\Lambda_i \simeq \oplus_{j=1}^{g} \mathbb{Z}/d_i\mathbb{Z}$ where $(d_1, \ldots, d_g)$ is the type of L.*

Proof. (2) follows from (1) it suffices to prove (1). The containment $\Lambda(L) \supseteq \Lambda(L)_1 \oplus \Lambda(L)_2$ is clear since we are using a decomposition of V into isotropic spaces. $v \in \Lambda(L)$, $v = v_1 + v_2$ a decomposition for v we need to show that $v_1, v_2 \in \Lambda(L)$, indeed we need only show that $v_1 \in \Lambda(L)$. Note that $E(v_1, \lambda_1) = E(v_2, \lambda_2) = 0$. Hence,

$$E(v_1, \lambda) = E(v_1, \lambda_2) = E(v, \lambda_2) \in \mathbb{Z}$$

thus $v_1 \in \Lambda(L)$. □

Now define two intermediate lattices between Λ and $\Lambda(L)$, namely, $\Lambda' = \Lambda(L)_1 \oplus \Lambda_2$ and $\Lambda'' = \Lambda_1 \oplus \Lambda(L)_2$, let $X' = V/\Lambda' = X/K_1$ and $X'' = V/\Lambda'' = X/K_2$ (with K_i defined in Lemma 1.5). Let $P_i : X \to X/K_i =: X_i$ be the quotient isogenies. The restrictions of a_L to $\Lambda' \times V$ and $\Lambda'' \times V$ are $1 -$ *cocycles* defining line bundles $M_1 \to X_1$ and $M_2 \to X_2$ which pull back to L under P_1 and P_2 to the line bundle L on X. Moreover since P_i comes from taking a v modulo Λ to v Λ' or Λ'', the characteristic of M_i is the same as the characteristic of L.

2.31 Theta Functions

Assume H is positive definite, we shall describe a basis for $H^0(X, L)$. Let $H = c_1(L)$ be the chern class of L. Let $X = V/\Lambda$ and $\pi : V \to X$ the exponential map. From previous lectures recall that the global sections may be described as functions $\vartheta : V \to \mathbb{C}$ with automorphy factor $f \in Z^1(\Lambda, H^0(V, O_V^*))$, ie, satisfying

$$\vartheta(v + \lambda) = f(v, \lambda)\vartheta(v)$$

Note that changing f by a co-boundary amounts to replacing the space space with an isomorphic vector space.

2.32 Classical Factors of Automorphy

We are trying to get a more suitable factor replacing H which is assumed to be positive definite. Let $L(H, \chi)$, $V = V_1 \oplus V_2$ a decomposition for L, also $E = ImH$.

2.32.1 Lemma. *$V = V_2 \oplus iV_2$ and so V_2 generates V as a $\mathbb{C}$ vector space.*

Proof. $U = V_2 \cap iV_2$ is a $\mathbb{C}$ subspace. For all $v, w \in U$, we have that

$$H(v, w) = E(iv, w) + iE(v, w) = 0$$

since iv is also in $U \subseteq V_2$. H is positive definite and so $U = 0$. Since V_2 has dimension g so does iV_2 and so $V = V_2 \oplus iV_2$. □

Since $E = 0$ on V_2, then $H = \mathrm{Re}H = E(i., .)$ is a symmetric real valued form on V_2.

Let $B : V \times V \to \mathbb{C}$ be the $\mathbb{C}$ bilinear extension of $H_{|V_2 \times V_2}$, note that since $V = V_2 \oplus iV_2$ a unique $\mathbb{C}$ bilinear extension B exists. $H - B$ is then a hermitian form, it satisfies the properties

1. $(H-B)(v,w) = \begin{cases} 0 & \text{if } w \in V_2 \\ 2iE(v,w) & \text{if } v \in V_2 \end{cases}$

2. $\mathrm{Re}(H-B)$ is positive definite on V_1

2.32.2 Definition. The classical factor of automorphy $e_L : \Lambda \times V \to \mathbb{C}^{\times}$ is defined as

$$e_L(\lambda, v) = \chi(\lambda) e^{\pi((H-B)(\lambda,v) - \frac{1}{2}(H-B)(\lambda,\lambda))} = a_L(\lambda, v)\left(\frac{e^{\frac{\pi}{2}B(v,v)}}{e^{\frac{\pi}{2}B(v+\lambda,v+\lambda)}}\right)$$

2.32.3 Definition. Classical theta functions are the functions with automorphy factor e_L.

1. When the type is $(1, 1, \ldots, 1)$ the polarization is said to be principal as noted earlier. The classical theta function associated to a principal polarization is unique up to scalar multiple. These were studied by Riemann, so they really are classical.

2. $e_L(\lambda, \lambda_2) = 1$ allows us expand theta functions in fourier series.

2.33 Global sections of a positive definite line bundle

2.33.1 Definition. Let L be a line bundle with $E := \mathrm{Im} c_1(L)$ and type $(d_1, \ldots, d_g)$. The Pfaffian of E, $Pf(E) := d_1 \ldots d_g$. This is indeed $\sqrt{\det E}$.

Later we will see that If L is a positive definite line-bundle on X, then, $h^0(X, L) = Pf(E)$.

2.33.2 Lemma. *$H^0(X, L) \simeq H^0(X, L_0)$.*

Proof. The idea is to present H^0 with classical theta functions. We claim that $H^0(X, L) \to H^0(X, L_0)$ by $\vartheta(\cdot) \mapsto e^{\pi(H-B)(\cdot, C)}\vartheta(\cdot - C) = \tilde{\vartheta}(\cdot)$ is an isomorphism of $\mathbb{C}$-vector spaces. Once we have that, then we only need to show $\tilde{\vartheta}(\cdot)$ is a theta function for the 1-cocycle $e_{L_0}(\cdot, \cdot)$. This is an easy computation, and the statement follows from the fact that if $L = t_\epsilon^* L^0$ then $e_L(\lambda, v) = e_{L_0}(\lambda, v) e^{2\pi i E(C,\lambda)}$ and 2.30.11. $\square$

2.33.3 Lemma. *$h^0(X, L_0) \leq Pf(E)$.*

Proof. Let $\vartheta \in H^0(X, L_0)$ be a classical theta function. That is $\vartheta(v + \lambda) = e_{L_0}(\lambda, v)\vartheta(v)$ for all $\lambda \in \Lambda$ and $v \in V$. If $\lambda_2 \in \Lambda_2$ (where $\Lambda = \Lambda_1 \bigoplus \Lambda_2$), then $e_{L_0}(\lambda_2, v) = 1$, and so $\vartheta(v + \lambda_2) = \vartheta(v)$ for all $\lambda_2 \in \Lambda_2, v \in V$. And therefore, we have a Fourier series! The Fourier series for ϑ is of the following form:

$$\vartheta(v) = \sum_{\lambda \in \Lambda(L)_1} \alpha_\lambda e^{\pi(H-B)(v,\lambda)}.$$

One way to get this is by properties of $H-B$ from earlier discussions. Alternatively, one can use the fact that there exist suitable coordinates $V=(v_1,\cdots,v_g)^T$ and $\lambda=(\lambda_1,\cdots,\lambda_g)^T$ such that $e^{\pi(H-B)(v,\lambda)}=e^{-2\pi i v^T\lambda}$. Also note that for $\lambda_1\in\Lambda_1$, we have $\vartheta(v+\lambda_1)=e_{L_0}(\lambda_1,v)\vartheta(v)$. Therefore, $\alpha_{\lambda-\lambda_1}=\alpha_\lambda e_{L_0}(\lambda_1,0)^{-1}e^{\pi(H-B)(\lambda_1,\lambda)}$ for all $\lambda_1\in\Lambda_1,\lambda\in\Lambda(L)_1$. Now if we fix a_λ for $\lambda\in\Lambda(L)_1$ varying over representatives of $\Lambda(L)_1/\Lambda_1=K(L)_1$, then all other a_λ are fixed by our above expression of $\alpha_{\lambda-\lambda_1}$. Therefore, $h^0\leq|K(L)_1|=d_1\cdots d_g$. □

Remark: When $X=V/\Lambda$ is a complex torus and $H\in NS(X)$ is positive definite of type $(d_1,\cdots,d_g)$ with symplectic basis $\{\lambda_1,\cdots,\lambda_g;\mu_1,\cdots,\mu_g\}$ for Λ for $Im(H)=E$. Then $D=diag(d_1,\cdots,d_g)$ with $d_i>0$ and $V=V_1\oplus V_2=\langle\lambda_i\rangle\bigoplus\langle\mu_i\rangle$. Define $e_i=\frac{1}{d_i}\mu_i$. We have seen that $\{e_i:\ 1\leq i\leq g\}$ is a $\mathbb{C}$-vector space basis. Consider the period matrix with respect to $\{e_i\}$ for V and $\{\lambda_i,\mu_i\}$ for Λ such that $\Pi=(Z|D)_{g\times 2g}$ for some $Z\in M_{g\times g}(\mathbb{C})$. One can check that

1. $Z=Z^T$ and $Im(Z)>0$. That is, Z is in *Siegel's upper half-space of degree g*.
2. with respect to $\{e_1,\cdots,e_g\}$ basis for V:
   - $(ImZ)^{-1}$ is the matrix for H.
   - $B(v,w)=v^T(ImZ)^{-1}w$.
   - $(H-B)(v,w)=-2iv^Tw_1$ where $w=w_1+w_2\in V_1\bigoplus V_2$.

Next, we want to find an explicit basis for $H^0(X,L)$. Suppose $L\in\mathrm{Pic}(X)$ with $char(L)=c$ with respect to $V_1\bigoplus V_2$. Our goal is to express one ϑ very explicitly, then show that we can produce a lot more by twisting. We define $\vartheta^c:V\to\mathbb{C}$ by

$$v\mapsto e^{-\pi(H(v,c)+1/2H(c,c)+1/2B(v+c,v+c))}\times\sum_{\lambda\in\Lambda_1}e^{\pi((H-B)(v+c,\lambda)-1/2(H-B)(\lambda,\lambda))}.$$

2.33.4 Theorem. *ϑ^c is a canonical theta function for $L=t_{\mathbb{C}}^*L_0$.*

Proof. We sketch the proof.

- ϑ^c is holomorphic on V. To see this, we need to show that $f(v)=\sum_{\lambda\in\Lambda_1}|e^{\pi((H-B)(v+c,\lambda}$ converges uniformly on every compact subset of V_1. This can be achieved by fixing a norm $||\cdot||:V\to\mathbb{R}$ such that $||\Lambda||\subset\mathbb{Z}$. Since $Re(H-B)$ is positive definite on V_1 and V_1 is discrete, there exists $R>0$ such that $|e^{1/2(H-B)(\lambda,\lambda))}|\geq e^{-R||\lambda||^2}$. Also, for all $r>0$, there exists $R'>0$ such that if $||v||<r$, then $|e^{\pi(H-B)(v,\lambda))}|<e^{R'||\lambda||}$. So
$$f(v)\leq\sum_{\lambda\in\Lambda_1}(e^{R'||\lambda||}/e^{R||\lambda||^2})<\infty.$$
- We need to check the equation of the automorphy. That is, we want to show that $\vartheta^c(v+\lambda)=a_L(\lambda,v)\vartheta^c(v)$ for all $(\lambda,v)\in\Lambda\times V$. For $c=0$, this boils down to 2.30.11. For $c\neq 0$, it follows from
$$\vartheta^c(v)=e^{-\pi(H(v,c))+1/2H(c,c)}\vartheta^0(v+c).$$

□

Remark: The proof actually shows that ϑ^c is a canonical theta function for M_2 (descent of L to $V/(\Lambda_1 \bigoplus \Lambda(L)_2)$).

So far we constructed *one* canonical theta for L, that is, an element of $H^0(X, L)$. We can get more as follows: for $\overline{\omega} \in K(L) = \Lambda(L)/\Lambda$, define

$$\vartheta^c_{\overline{\omega}}(\cdot) := a_L(\omega, \cdot)^{-1}\vartheta^c(\cdot + \omega)$$

where $\omega \in \Lambda(L)$ is any lift of $\overline{\omega}$. It is easy to check that this definition is independent of the choice of ω.

2.33.5 Lemma. *For all $\overline{\omega} \in K(L)$, $\vartheta^c_{\overline{\omega}}$ is a* canonical *theta function for L.*

Proof. It follows from 2.30.11. □

2.33.6 Theorem. *Let $L = L(H, \chi)$ be a positive definite line bundle on X and $C = char(L)$ be the characteristic with respect to $V = V_1 \bigoplus V_2$ for L. The set of canonical theta functions $\{\vartheta^c_{\overline{\omega}} : \overline{\omega} \in K(L)_1\}$ is a basis of canonical theta function for the $\mathbb{C}$-vector space $H^0(X, L)$.*

Proof. First, we already know $h^0(X, L) \leq Pf(E)$. So it suffices to show that $\vartheta^c_{\overline{\omega}}$ is independent of $(\#K(L)_1 = Pf(E))$.

Next, the expression from earlier generalizes to $\vartheta^c_{\overline{\omega}}(v) = e^{-\pi(H(v,c)+1/2H(c,c))}\theta^0_{\overline{\omega}}(v+c)$. So we may further assume that $c = 0$.

Furthermore, by $e_L(\lambda, v) = a_L(\lambda, v)\frac{e^{\pi B(v,v)}}{e^{\pi}B(v+\lambda,v+\lambda)}$. Therefore, we see that $\vartheta^c_{\overline{\omega}}(v) = e^{-\pi/2B(v,v)}\vartheta^c_{\overline{\omega}}(v)$ is classical. So it suffices to show that $\{\vartheta^c_{\overline{\omega}}(\cdot) : \overline{\omega} \in K(K)_1\}_{c=0}$ with is independent.

Let $\omega_1, \cdots, \omega_N \in \Lambda(L)_1$ be representations for $K(L)_1$. We will show that $\vartheta^0_{\overline{\omega_1}}(\cdot) = e^{-\pi/2B(\cdot,\cdot)}\vartheta^0_{\overline{\omega_i}}$ with $1 \leq i \leq N$ are linearly independent.

Note that

$$\vartheta^0_{\overline{\omega_i}} = e^{-\pi/2B(v,v)}\vartheta^0_{\overline{\omega_i}}(v) = \cdots = \sum_{\lambda \in \Lambda_1 - \omega_1} e^{-\pi/2(H-B)(\lambda,\lambda)}e^{\pi(H-B)(v,\lambda)}.$$

Recall that a general Fourier series is of the form

$$\sum_{\lambda \in \Lambda(L)_1} \alpha_\lambda e^{\pi(H-B)(v,\lambda)}.$$

Note that $\vartheta^0_{\overline{\omega_i}}(\cdot)$ only have coefficients in the coset $-\omega_i + \Lambda_i$ (in $\Lambda(L)_1/\Lambda_1$) and cosets are disjoint. So they are independent. □

2.33.7 Corollary. $h^0(L) = d_1 \cdots d_g$.

2.33.8 Corollary. *For any $L \in L(H, \chi)$ and $L' \in L(H, \chi')$ with $c = char(L)$ and $c' = char(L)$, define $\tau : V \to \mathbb{C}^\times$ by*

$$v \mapsto e^{\pi i ImH(c',c) - \pi H(v,c-c') - \pi/2H(c'-c,c'-c)},$$

then the map $H^0(X, L) \to H^0(X, L')$ by $\vartheta \mapsto \tau t^\times_{c'-c}\vartheta$ is an isomorphism. Moreover, it is diagonal with respect to the bases given above.

2.34 Positive semi-definite line bundles

Recall that if L is degenerate, then $K(L) = \ker(\varphi_L)$ is not finite. Equivalently, $\Lambda(L)/\Lambda$ is infinite, where $\Lambda(L) = \{v \in V : \; ImH(v, \lambda) \in \mathbb{Z} \; \forall \lambda \in \Lambda\}$. Therefore, they can not be lattices of the same rank. Let $\Lambda(L)_0$ be the connected component of $\Lambda(L)$ containing 0,m which is also a subspace of V. By an earlier fact that $\varphi_L^{an}(v) = H(v, \cdot)$, we have $\Lambda(L)_0 = \{v \in V : H(v, x) = 0 \; \forall x \in V\}$. Therefore $K(L)_0 = \Lambda(L)_0/\Lambda(L)_0 \cap \Lambda$ is a subtorus of X.

Now define $\overline{X} = X/K(L)_0 = \overline{V}/\overline{\Lambda}$ where $\overline{V} = V/\Lambda(L)_0$ and $\overline{\Lambda} = \Lambda/(\Lambda(L)_0 \cap \Lambda)$. Let $p : X \to \overline{X}$ be the projection map.

2.34.1 Lemma.

1. *There exists $\overline{L}$ on $\overline{X}$ with $L \simeq p^*\overline{L}$ if and only if $L|_{K(L)_0}$ is trivial.*

2. *If $\overline{L}$ exists, then $h^0(X, L) = h^0(\overline{X}, \overline{L})$ and $\overline{L}$ is non-degenerate.*

2.34.2 Theorem. *If $L = L(H, \chi)$ is positive semi-definite on X, then $h^0(X, L) = Pf_{reduced}(E)$ if $L|_{K(L)_0}$ is trivial and 0 otherwise, where $Pf_{reduced}(E) = 1$ if all $d_i = 0$ and $\prod_{d_i \neq 0} d_i$ otherwise.*

Now we will go into some deeper results. For $L \in \mathrm{Pic}(X)$ on $X = V/\Lambda$ a g-dimensional complex torus with $H = c_1(L)$. Assume H has r positive and s negative eigenvalues. Note then that $r + s \leq g$. Our goal is to prove the following theorem:

2.35 All cohomologies and analytic Riemann-Roch

2.35.1 Theorem (Mumford-Kempf-Deligne,$\cdots$)**.**

1. *$H^q(X, L) = 0$ if $q < s$ or $q > g - r$.*

2. *$h^q(X, L) = \binom{g-s-r}{q-s} h^s(x, L)$ for $s \leq q \leq g - r$.*

3. *$h^s(X, L) = Pf_{reduced}(E)$ for $L|_{K(L)_0}$ trivial and 0 otherwise.*

2.35.2 Corollary.

1. *If L is non-degenerate, then $r + s = g$. So cohomology is non-trivial only for $q = s = g - r$. We call such q the* index *of L and denote it by $i(L)$.*

2. *If $h^s \neq 0$, then $H^q(X, L)$ are nontrivial for all $s \leq q \leq g - r$.*

3. *If L is positive-definite, then $r = g$ and $s = 0$ and so only H^0 is non-trivial.*

4. *If L is positive-semi-definite and if $L|_{K(L)_0}$ is trivial, then $f < g$ and $s = 0$ and so $H^0, \cdots, H^{g-r}$ are all non-trivial.*

Recall that the Riemann-Roch Theorem is a statement relating algebraic objects

$$\chi(X, L) := \sum_{i=s}^{g-r} (-1)^i h^i(X, L)$$

to topological objects. Now we state a deeper result, by Deligne.

2.35.3 Corollary (Analytic Riemann-Roch theorem for complex tori)**.** *As usual, define*

$$\chi(X,L) := \sum_{i=s}^{g-r} (-1)^i h^i(X,L).$$

Assume $c_i(L) = H$ has s negative eigenvalues, then

$$\chi(X,L) = (-1)^s Pf(E)$$

where $E = Im(H)$ and $Pf(E) = d_1 \cdots d_g$.

Proof. If $L_{K(L)_0}$ is non-trivial, then there is nothing to prove. We may hence assume that

$$\chi(X,L) = \sum_{q=s}^{g-r} (-1)^q \binom{g-r-s}{q-s} Pf_{reduced}(E).$$

By putting $N = g-r-s$, since $(1-1)^N = \sum_{i=0}^{N} (-1)^i \binom{N}{i}$ equals to 1 if $N = 0$ and 0 otherwise, we hence have $\chi(X,L) = (-1)^s Pf(E)$.

□

Remarks:

1. $\deg(\varphi_L) = \det(E) = (Pf(E))^2 = \chi(X,L)^2$.
2. It is useful (for instance to get an algebraic/geometric statement) to use $\frac{1}{g!}(L^g)$ instead of $(-1)^s Pf(E)$. (Later we will be able to express $\frac{1}{g!}(L^g)$ in terms of $\bigwedge^g c_1(L)$.)
3. For $f : X' \to X$ and $L \in \mathrm{Pic}(X)$, we have $\chi(X', f^*L) = (\deg(f))\chi(X,L)$.
4. It is a good exercise to show that for a Poincaré bundle $\mathcal{P}$ on $X \times \hat{X}$, we have $h^q(X \times \hat{X}, \mathcal{P}) = \mathbb{C}$ for $q = g$ and 0 otherwise. (Hint: we have given the first chern class $c_i(\mathcal{P})$ explicitly.)

2.36 Vanishing theorem of Mumford and Kempf

Recall that we have by $\overline{\partial}$-Poincaré an exact sequence of sheaves:

$$0 \to \Omega^p_{hol} \to \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots$$

and in particular when $p = 0$, we have

$$0 \to \mathcal{O}_X \hookrightarrow \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \cdots$$

Since L is locally free, we also get exact sequence of sheaves

$$0 \to L \to \Omega^{0,0}(L) \xrightarrow{\overline{\partial}} \Omega^{0,1}(L) \to \cdots$$

with $\Omega^{0,q}(L) = \Omega^{0,q} \bigotimes_{\mathcal{O}_X} L$ which is called the *Dolbeault resolution.*

By tensoring with the *locally free* sheaf L, we get an exact sequence:

When $p = 0$:

$$0 \to L \to \Omega^{0,0}(L) \overset{\bar{\partial}=\bar{\partial}\otimes id}{\to} \Omega^{0,1}(L) \to \cdots$$

where $\Omega^{p,q}(L) := \Omega^{p,q} \bigotimes L$ is the sheaf of C^∞-forms of type (p,q) with values in L. Now, we apply $\Gamma(X,-)$ to get the complex

$$0 \to H^0(X,L) \to \Gamma(X,\Omega^{0,0}(L)) \to \Gamma(X,\Omega^{0,1}(L)) \to \cdots$$

and therefore we get cohomology groups $H^{0,q}_{\bar{\partial}}(X,L)$. By Dolbeault theorem, this is isomorphic to $H^q(X,L)$. Moreover, we have a Hodge theorem stating $H^{0,q}_{\bar{\partial}}(X,L)$ is isomorphic to $H^q(L)$ which is the subvector space $\ker \Delta$ of $\Gamma(X,\Omega^{0,q}(L))$, i.e. "Harmonic forms with values in L with respect to the Laplacian Δ".

Question: What is Δ? We need $\Delta = \bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}$. What is $\bar{\delta}$? We want $\bar{\delta}$ to be the "adjoint" of $\bar{\delta}$, with respect to a global "inner product" on $\Gamma(X,\Omega^{0,q}(L))$. Recall previously we have a metric ds^2 on X which gives rise to a $(1,1)$-form ω and hence a (n,n)-form $dv = \frac{1}{g!}\bigwedge^g \omega$ which is a volume form. Therefore, we have an inner product

$$(\varphi,\psi) = \sum \int \varphi_{IJ}\psi_{IJ}dv.$$

However, this is not enough - we need to fix some "metric data" on L.

2.36.1 Definition. A **Hermitian metric** on a line bundle L is a positive definite Hermitian form on each fibre L_x depending smoothly on $x \in X$. In other words, $h \in \Gamma_{sm}(X,(L \bigotimes L)^v)$ such that

$$h_P(\eta,\bar{\xi}) = \overline{h_P(\xi,\bar{\eta})}$$

and $h_P(\xi,\bar{\xi}) > 0$ if $\xi \neq 0$ for all $\xi,\eta \in L_P$.

Irrelevant remark: If (L,h) is Hermitian line bundle, then the first chern class $c_1(L,h)$ gives a *curvature form* which is also a $(1,1)$-form on X. It turns out that $[c_1(L,h)] = c_1(L)$.

2.37 Construction on $L \to X$

Let $L = (H,\chi)$ be a Hermitian line bundle and X be our complex torus. If we consider the $(0,0)$ case as in above, then notice that

$$\Gamma(X,\Omega^{0,0}(L)) = \{f : V \to \mathbb{C} :\ f \in C^\infty,\ f(x+\lambda) = a_L(\lambda,v)f(v)\ \forall(v,\lambda) \in V \times \Lambda\}$$

are what we call "smooth" theta functions. For $f,g \in \Gamma(X,\Omega^{0,0}(L))$, define $h = \langle f,g\rangle$ by

$$h(v) = \langle f,g\rangle(v) = f(v)\overline{g(v)}e^{-\pi H(v,v)}.$$

It is easy to check that $h(v+\lambda) = h(v)$ and that h is C^∞, so h gives an element of $\Gamma(X,\Omega^{0,0})$. Therefore, $\langle\cdot,\cdot\rangle : \Gamma(X,\Omega^{0,0}(L)) \times \Gamma(X,\Omega^{0,0}(L)) \to \Gamma(X,\Omega^{0,0}))$ defines a *Hermitian metric.*

Recall that a *Kähler metric* ds^2 on $V/\Lambda = X$ is given by the following: fix $\{e_1,\cdots,e_g\}$ a basis for V such that $H = c_1(L)$ with respect to the basis is diagonal. (Hermitian matrices are diagonalizable.) We let $h_1,\cdots,h_g$ diagonal entries of the matrix, that is, let $\{v_1, \cdots, v_g\}$ be the coordinate functions corresponding to $\{e_1, \cdots, e_g\}$. Then $H(v,w) = \sum h_i v_i w_i$.

If we fix $K_1, \cdots, K_g \in \mathbb{R}^{>0}$, then

$$ds^2 = \sum_{i=1}^{g} K_i dv_i \otimes d\overline{v_i}$$

defines a Kähler metric and so the associated $(1,1)$ form $\omega = 1\frac{1}{2}Im(ds^2)$ is closed $(d\omega = 0)$ and given by

$$\omega = \frac{i}{2}\sum_{j=1}^{g} K_j dv_j \bigwedge d\overline{v_j}$$

and hence we have a volume form

$$dv = \frac{1}{g!}\bigwedge^{g}\omega = (i/2)^g(\prod K_i) dv_1 \wedge d\overline{v_1} \wedge dv_2 \wedge \cdots$$

which is also a (g,g)-form. Now, we define

$$(\cdot,\cdot) : \Gamma(X, \Omega^{0,0}(L)) \times \Gamma(X, \Omega^{0,0}(L)) \to \mathbb{C}$$

by

$$(f,g) = \int_X \langle f, g\rangle dv$$

2.38 The $(0,q)$ case

Let $\omega = \sum_{|I|=q} \varphi_I d\overline{v_I}, \omega' = \sum_{|I|=q} \psi_I d\overline{v_I} \in \Gamma(X, \Omega^{0,q}(L))$ for $\varphi_I \in \Gamma(X, \Omega^{0,0}(L))$. Define

$$(\omega, \omega') = \sum_{|I|=q} K^{-I}(\varphi_I, \psi_I)$$

where $K^{-I} = \prod_{i\in I} K_i^{-1}$. Recall the notations $\partial_i = \frac{\partial}{\partial v_i}$ and $\overline{\partial_i} = \frac{\partial}{\partial \overline{v_i}}$. Also recall that $\Gamma(X, \Omega^{0,0}(L)) = \{f : V \to \mathbb{C} : f(v+\lambda) = a_L(\lambda, v) f(v)\}$. Since $a_L(\lambda, v)$ is holomorphic, so $\overline{\partial_i} a_L = 0$ and so $\overline{\partial_i}$ is a linear operator on $\Gamma(X, \Omega^{0,0}(L))$. So there exists $\overline{\partial} : \Gamma(X, \Omega^{0,q}(L)) \to \Gamma(X, \Omega^{0,q+1}(L))$ given by

$$\overline{\partial}(\varphi d\overline{v_I}) = \sum_i (\overline{\partial_i}\varphi) d\overline{v_i} \wedge d\overline{v_I}.$$

Note that this is the map on the complex we had. Define $\overline{\delta_i}$ the adjoint of $\overline{\partial_i}$ and $\overline{\delta}$ the adjoint of ∂ with respect to $(\cdot,\cdot)$. Explicitly, we have the following lemma.

2.38.1 Lemma. *Let $\varphi \in \Gamma(X, \Omega^{0,0}(L))$.*

(a) $\overline{\delta_i}\varphi = -\partial_i\varphi + \pi h_i \overline{v_i}\varphi$

(b) $\overline{\delta}(\varphi d\overline{v_J}) = \sum_{i=1}^{q+1}(-1)^{i-1}\frac{1}{K_{j_i}}(\overline{\delta_{j_i}}\varphi) d\overline{v_{J-j_i}}$ *where* $J = (j_1 < \cdots < j_{q+1})$.

Proof. (b) follows from (a). To show (a), we need to prove that for all $\varphi, \psi \in \Gamma(X, \Omega^{0,0}(L))$, we have

$$(\overline{\partial_i}\varphi, \psi) = (\varphi, -\partial_i\psi + \pi h_i \overline{v_i}\psi)$$

but $\langle \overline{\partial_i}\varphi, \psi\rangle - \langle \varphi, -\partial_i\psi + \pi h_i\overline{v_i}\psi\rangle = \overline{\partial_i}\langle\varphi,\psi\rangle$ but $\int_X \overline{\partial_i}\langle\varphi,\psi\rangle = (i/2)^g(\prod K_i)\int_X d(\langle\varphi,\psi\rangle dv_1 \wedge d\overline{v_1} \wedge dv_2 \wedge \cdots) = 0$ by Stokes theorem. ☐

2.38.2 Lemma. *Let* $\Delta = \overline{\partial\delta} + \overline{\delta\partial}$ *and* $\varphi d\overline{v_I} \in \Gamma(X, \Omega^{0,q}(L))$ *where* $I = (i_1 < \cdots < i_q)$ *and* $\varphi \in \Gamma(X, \Omega^{0,0}(L))$. *Then*

$$\Delta(\varphi d\overline{v_I}) = \sum_{i=1}^{g} \frac{1}{K_i}\overline{\delta_i\partial_i}\varphi d\overline{v_I} + \pi\sum_{j=1}^{q}\frac{1}{K_{i_j}}h_{i_j}\varphi d\overline{v_I}.$$

The proof is just computations.

2.38.3 Corollary. Δ *acts on subvector space of "monomials"* $A_I = \{\varphi_I d\overline{v_I}\}$ *(for a fix I) of* $\Gamma(X, \Omega^{0,q}(L))$. *So* $\ker\Delta = H^q(L) = \bigoplus_{|I|=q}\ker\Delta|_{A_I} =: \bigoplus_{|I|=q} H^q_I(L)$.

Now, we fix "nice" h_i, k_i so that the computations would be easier.

1. (after permuting and scaling $\{e_1, \cdots, e_g\}$) we may assume $h_i = 1$ for $1 \le i \le r$, $h_i = -1$ for $r+1 \le i \le r+s$ and $h_i = 0$ for $r+s < i \le g$.
2. $K_i = \frac{1}{s+1}$ for $i \le r$ and $K_i = 0$ for $i > r$.

Given a multi-index set $I = \{i_1 < \cdots < i_q\}$, we write

$$R_I = |I \cap \{1, \cdots, r\}|$$

and

$$S_I = |I \cap \{r+1, \cdots, r+s\}|.$$

Below we give the key proposition.

2.38.4 Proposition. *With* K_i *and* h_i *as above, for any* $pd\overline{v_I} \in \Gamma(X, \Omega^{0,q}(L))$, *we have*

$$(\Delta(\varphi_I d\overline{v_I}), \varphi_I d\overline{v_I}) \ge \pi((s+1)R_I - S_I)(\varphi d\overline{v_I}, \varphi d\overline{v_I}).$$

Proof. Note that

$$(\Delta(\varphi_I d\overline{v_I}), \varphi_I d\overline{v_I}) = \sum_{i=1}^{g}\frac{1}{K_i}(\overline{\delta_i\partial_i}\varphi d\overline{v_I}, \varphi d\overline{v_I}) + \pi(\sum_{j=1}^{q}\frac{h_{i_j}}{k_{i_j}})(\varphi d\overline{v_I}, \varphi d\overline{v_I}).$$

Since $(\overline{\delta_i\partial_i}\varphi d\overline{v_I}, \varphi d\overline{v_I}) \ge 0$ and $(\sum_{j=1}^{q}\frac{h_{i_j}}{k_{i_j}})(\varphi d\overline{v_I}, \varphi d\overline{v_I}) = (s+1)R_I - S_I$, the result follows. ☐

2.38.5 Corollary. $H^q_I(L) := \ker\Delta|_{\{\varphi_I d\overline{v_I}\}} = 0$ *if* $R_I > 0$.

Proof. Let $\varphi d\overline{v_I} \in H^q(L)$, i.e. $\Delta(\varphi d\overline{v_I}) = 0$. Therefore the above proposition implies that $0 \ge \pi((s+1)R_I - S_I)(\varphi d\overline{v_I}, \varphi d\overline{v_I}) \ge 0$. Since $R_I \ge 1$ and $S_I \le s$, so $(\varphi d\overline{v_I}, \varphi d\overline{v_I}) = 0$. Therefore, $\varphi d\overline{v_I} = 0$. ☐

2.38.6 Theorem (Mumford-Kempf). *Let $X = V/\Lambda$ be a complex torus. Suppose $H = c_i(L)$ has r positive and s negative eigenvalues, then $H^q(X,L) = 0$ if $q > g-r$ or $q < s$.*

Proof. Let $q > g-r$. Then for any I, $|I| = q$ and so we have $R_I > 0$. So $H^q_I(L) = 0$ but $H^q(L) = \bigoplus_{|I|=q} H^q_I(L) = 0$ but $H^q(X,L) = H^q(L) = 0$ by Hodge.

For $q < s$ we use Serre duality: If X is a smooth compact complex manifold and L is a holomorphic line (or vector) bundle over X and K is a canonical bundle (which is a line bundle), then there is a canonical line bundle isomorphism $H^q(X,L) \simeq H^{g-q}(X, K \bigotimes L^v)^\times$. In our situation, X is a complex torus and $K = \Omega^g$ which is free of rank $\binom{g}{g} = 1 \simeq \mathcal{O}_X$ and $L^v \simeq L^{-1}$. Therefore, by Serre duality, we have

$$H^q(X,L) \simeq H^{g-q}(X, L^{-1})^\times.$$

Note that $c_1(L^{-1}) = -c_1(L)$ has s positive eigenvalues. Since $q < s$, $g - q > g - s$ and therefore $H^q(X,L) = 0$. $\square$

Remark: Coherent duality of Grothendieck gives a more general version of Serre duality.

2.39 $q = s, s+1, ..., g-r$

2.39.1 Theorem. *For $s \leq q \leq g-r$, we have*

$$h^q(X,L) = \dim H^q(X,L) = \binom{g-r-s}{q-s} h^s(X,L).$$

Indeed,

$$H^q(X,L) \simeq \bigoplus_{i=1}^{\binom{g-r-s}{q-s}} H^s(X,L)$$

Proof. First, we claim that $H^q(L) = \bigoplus_{|I|=q, R_I=0, S_I=s} H^q_I(L)$. Really, we only need to show that $S_I = s$. To show this, we may assume that $R_i = 0$ from above. We know that $S_I \leq s$. Note that

$$\Delta(\varphi d\overline{v_I}) = \psi d\overline{v_I}$$

where $\psi = \sum_{j=1}^g \frac{1}{k_j} \overline{\delta_j \partial_j} \varphi - (\pi \sum_{j=1}^q \frac{h_{i_j}}{K_{i_j}})\varphi$. We can write $\pi S_I = (\pi \sum_{j=1}^q \frac{h_{i_j}}{K_{i_j}})\varphi$. If we pick any $J = I \cap \{r+1, \cdots, r+s\}$ then $R_J = 0$ and $S_J = S_I$, and hence $\Delta(\varphi d\overline{v_J}) = \Delta(\varphi d\overline{v_I})$. Moreover, the map $H^q_I(L) \to H^{S_I}_J(L)$ given by $\varphi d\overline{v_I} \mapsto \varphi d\overline{v_J}$ is an isomorphism! Since $S_I < s$, by the vanishing theorem, we have $H^{S_I}_J(L) = 0$ and so $S_I = s$.

Next, we claim that $H^q(L) \simeq H^s(L)^{\binom{g-r-s}{q-s}}$. Let $J = (r+1 < \cdots < r+s)$. Note then that $R_J = 0$ and $S_J = s$. By the first claim, we have $H^q(L) = \bigoplus_{|I|=q, R_I=0, S_I=s} H^q_I(L)$ which is isomorphic to $H^s_J(L)$. By picking q numbers from $\{1, \cdots, g\}$ such that all of $\{r+1, \cdots, r+s\}$ are being picked but not picking any of $\{1, \cdots, r\}$, there are exactly $\binom{g-r-s}{q-s}$ ways and hence the theorem. $\square$

2.40 $q = s$

In the last section we have reduced every to the case $q = s$ and so we can focus on this for now. Note that if $s = 0$, then this is a positive semi-definite case and therefore we are done!

If $s > 0$, it then suffices to find another complex torus X', L' such that L' is positive semi-definite on X' and $H^s(X, L) \simeq H^0(X', L')$. Here we introduce *Wirtinger's trick.* The idea is o change the "complex structure". Recall that the connected component $\Lambda(L)_0$ of $\Lambda(L)$ is a radical of H. Also we have fixed a basis $\{e_1, \cdots, e_g\}$ two sections earlier such that H is diagonal matrix with eigenvalues $1, -1, 0$. Note that $\Lambda(L)_0 = \text{span}\{e_{r+s+1}, \cdots, e_g\}$. Define $V_+ = \text{span}\{e_1, \cdots, e_r\}$ and $V_- = \text{span}\{e_{r+1}, \cdots, e_{r+s}\}$. Then as $\mathbb{C}$ vector spaces, we have

$$\mathbb{C}^g \simeq V = V_+ \bigoplus V_- \bigoplus V_0.$$

Let $\mathbb{R}^{2g} \simeq W$ be the underlying $\mathbb{R}$-vector space. Let j be the complex structure on w corresponding to V. In particular, $j : W \to W$ given by $v \mapsto iv$ is $\mathbb{R}$-linear. We therefore have

$$\mathbb{R}^g \simeq W = W_+ \bigoplus W_- \bigoplus W_0.$$

induced by above. We define $j'(w) := j(w)$ if $w \in W_+ \bigoplus W_0$ and $j'(w) := -j(w)$ if $w \in W_-$. Now let $V' = (W, j')$ be a complex vector space and note that $\Lambda \subset V'$ is obviously a lattice. Let

$$X' := V'/\Lambda$$

which is a new complex torus. Recall that for $H = c_i(L)$, we have

$$H(v, w) = E(j(v), w) + iE(v, w)$$

where $E = Im(H)$ and $L = L(H, X)$. We now define

$$H'(v, w) := E(j'(v), w) + iE(v, w).$$

One can check that $Im(H'(\Lambda, \Lambda)) \subset \mathbb{Z}$. Since $Im(H) = Im(H')$, X works for H' as well. We can then define

$$L' := L(H', X).$$

2.40.1 Theorem. *$H^s(X, L) \xrightarrow{\sim} H^0(X', L')$ and L' positive semi-definite.*

Proof. The idea is to let $f : W \to \mathbb{C}^\times$ which is given by $w \mapsto e^{\pi H'(w_-, w_+)}$ where $w = w_+ + w_- + w_0$. Then one can consider the map $\{\varphi d\overline{v_J} : \varphi \in \Gamma(X, \Omega^{0,0}(L))\} \to \Gamma(X', \Omega^{0,0}(L'))$ given by mapping $\varphi d\overline{v_J}$ which is a $(0, s)$-form on V to φf which is a C^∞ function on V'. Then this induces the above isomorphism. $\square$

2.41 Intersection of line bundles and Geometric Riemann-Roch

2.41.1 Definition. For line bundles $L_1, \cdots, L_g \in \text{Pic}(X)$ where X is a g dimensional abelian variety, the **intersection number** of $L_1, \cdots, L_g \in \text{Pic}(X)$ is given by

$$(L_1, \cdots, L_g) := \int_X c_1(L_1) \bigwedge c_1(L_2) \bigwedge \cdots \bigwedge c_1(L_g).$$

Remarks:

1. $c_1(L) \in H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \simeq H^{2,0} \bigoplus H^{1,1} \bigoplus H^{0,2}$. In fact, we have $c_1(L) \in H^2(X,\mathbb{Z}) \cap H^{1,1}$.

2. Note that $c_1(L) \bigwedge c_1(L_2) \bigwedge \cdots \bigwedge c_1(L_g)$ is a (g,g)-form which can be viewed as a volume form, and so the integral makes sense.

3. $c_1(L) \in H^2(X,\mathbb{Z})$ is Poincaré dual to $\{D\} \in H_{2g-2}(X,\mathbb{Z})$, for any divisor D associated to L. (Also $\{D\} \in H_{2g-2}(X,\mathbb{Z})$ makes sense because it is "triangulizable".) So $(L_1, \cdots, L_g)$ can be thought of as *intersection* (in the topological sense) of associated divisor.

2.41.2 Definition. For $L \in \mathrm{Pic}(X)$, the **self intersection** of L is defined by

$$(L^g) := (L, \cdots, L) = \int_X \bigwedge^g c_1(L).$$

2.41.3 Theorem. *If $L \in \mathrm{Pic}(X)$ is type $(d_1, \cdots, d_g)$ where X is a complex torus, and H has s negative eigenvalues, then*

$$\frac{1}{g!}(L^g) = (-1)^s Pf(L)$$

where $Pf(L) = d_1 \cdots d_g$.

2.41.4 Corollary.

$$\chi(X,L) = \frac{1}{g!}(L^g).$$

2.41.5 Lemma. *If L is type $(d_1, \cdots, d_g)$. Let $\{\lambda_1, \cdots, \lambda_g; \mu_1, \cdots, \mu_g\}$ be a symplectic basis for $E = Im(H)$ with corresponding coordinate functions given by $\{x_1, \cdots, x_g; y_1, \cdots, y_g\}$, then*

$$c_1(L) = -\sum_{j=1}^{g} d_j dx_j \wedge dy_j$$

as a $(1,1)$-form.

The proof follows from the definitions.

Remark: If L is degenerate, then there exists $d_i = 0$ and so $\bigwedge^g c_1(L) = 0$ and $Pf(L) = 0$. Therefore, in this case $\frac{1}{g!}(L^g) = 0$. So we may assume that L is non-degenerate. In the non-degenerate case, note that $s = g - r$ which is the index of L and $\chi(X,L) = h^s(X,L)$.

2.41.6 Lemma. $\int_X \bigwedge_{j=1}^g dx_j \wedge dy_j = (-1)^{g+s}$.

Remark: This lemma can be thought of as the orientation associated to the symplectic basis is ± 1 depending on s.

Proof. We have seen that $\{\mu_1, \cdots, \mu_g\}$ is a $\mathbb{C}$-vector space basis for V with $\{v_1, \cdots, g_g\}$ the corresponding coordinate functions. Then $(1/2)^g \bigwedge_{j=1}^g dv_j \wedge d\overline{v_j}$ gives the natural *positive* orientation. Let Π be the period matrix with respect to

$\{\lambda_1, \cdots, \lambda_g; \mu_1, \cdots, \mu_g\}$. Note that $\{\lambda_1, \cdots, \lambda_g; \mu_1, \cdots, \mu_g\}$ is a $\mathbb{Z}$-basis for Λ and $\{\mu_1, \cdots, \mu_g\}$ is a $\mathbb{C}$-basis for V. Write $\Pi = (Z|I)$. Then we have

$$(i/2)^g \bigwedge_{j=1}^{g} dv_j \wedge d\overline{v_j} = (-1)^g \det(Im(Z)) \bigwedge_{j=1}^{g} dx_j \wedge dy_j.$$

It then suffices to show that $(-1)^s \det(Im(Z)) > 0$.

Let $y \in M_{g\times g}(\mathbb{C})$ be the matrix of H with respect to the $\mathbb{C}$-basis $\{\mu_1, \cdots, \mu_g\}$ of $V \simeq \mathbb{C}^g$, then the matrix of H with respect to $\{\lambda_1, \cdots, \lambda_g; \mu_1, \cdots, \mu_g\}$ of $V \simeq \mathbb{R}^{2g}$ is

$$\Pi^t Y \overline{\Pi} = \begin{pmatrix} Z^t \\ I \end{pmatrix} Y(\overline{Z}|I) = \begin{pmatrix} Z^t Y \overline{Z} & Z^t Y \\ Y\overline{Z} & Y \end{pmatrix}.$$

Therefore, the matrix of E with respect to $\{\lambda_1, \cdots, \lambda_g; \mu_1, \cdots, \mu_g\}$ is just $Im(H) = Im \begin{pmatrix} Z^t Y \overline{Z} & Z^t Y \\ Y\overline{Z} & Y \end{pmatrix}$. On the other hand, the matrix is also given by $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$. This implies that Y is real, and $D = (ImZ^t)Y$. Therefore, the result follows. $\square$

Note that $\bigwedge_{i=1}^{g} c_1(L) = (-1)^g (g!)(d_1 \cdots d_g) \bigwedge_{i=1}^{g} dx_j \wedge dy_j$. Integrating this expression gives the following corollary:

2.41.7 Corollary. $(L^g) = \int_X \bigwedge_{i=1}^{g} c_1(L) = (-1)^s g! Pf(L)$.

The proof is done with Algebraic/Geometric Riemann Roch.

2.41.8 Corollary. *Let $f : X' \to X$ be a surjective homomorphism of complex tori. For any $L \in \mathrm{Pic}(X)$, we have*

$$\chi(X', f^*L) = (\deg f)(\chi(X, L)).$$

Proof. If f is not an isogeny, then we have seen that f^*L is degenerate and the statement follows trivially. If f is an isogeny, then $\deg(f) = [\Lambda : f_{Int}(\Lambda')]$ and

$$\int_{X'} \bigwedge_{i=1}^{g} c_1(f^*L) = (\deg f) \int_X \Lambda^g c_1(L).$$

Note that equality sign follows from the surjectivity assumption (so that the "change of variable" is okay). Now the rest is done by Geometric Riemann-Roch. $\square$

3 Abelian varieties

3.1 Definition and basic properties

3.1.1 Definition. $X = V/\Lambda$ is called an **abelian variety** if it admits a positive definite line bundle, i.e. there exists $H \in NS(X)$ such that H is positive definite.

Remark: We will see this agrees with our initial definition, i.e. such complex tori are "algebraic/algebraizable". (Hint: If $f : X \hookrightarrow \mathbb{P}^N$, then $f^*\mathcal{O}(1)$ must be positive definite.)

3.1.2 Definition.

1. A **polarization** on X is $c_1(L)$ where L is positive definite.
2. The **type** of a polarization is defined to be the type $(d_1, \cdots, d_g)$ of L (or $c_1(L)$.)
3. A polarization is called **principal** if its type is $(1, \cdots, 1)$.
4. For (X, L) where X is a complex torus and $L \in \mathrm{Pic}(X)$ is positive definite and $H = c_1(L)$, the pair (X, H) is called a **polarized abelian variety** (P.A.V.).
5. A **homomorphism** of polarized abelian varieties is a map $f : (Y, M) \to (X, L)$ such that $f : Y \to X$ is a homomorphism of complex tori and $f^*c_1(L) = c_1(M)$. (We have seen that $c_1(f^*L) = f^*c_1(L)$.)

Remark: $f^*L \sim_{an} M$. This forces any homomorphism f to have a finite kernel. (Otherwise f^*L would be degenerate and so would M be degenerate.) Conversely, if $f : Y \to X$ is a homomorphism of complex tori with finite kernel and $L \in \mathrm{Pic}(X)$ is positive definite, then $f^*L \in \mathrm{Pic}(Y)$ is positive definite. We call this the "induced" polarization.

3.1.3 Corollary.

1. *Any complex subtorus of an abelian variety is an abelian variety.*
2. *Let X be an abelian variety and Y be a complex torus. If X is isogenous to Y, then Y is also an abelian variety.*
3. *For any abelian variety X, $\hat{X} = \mathrm{Pic}^0(X)$ is also an abelian variety.*

Proof. (3) follows from the fact that any non-degenerate $\varphi_L : X \to \hat{X}$ has finite kernel. □

3.1.4 Lemma. *Every polarization is "induced by" a principal polarization via an isogeny.*

Proof. Let (X, L) be a polarized abelian variety of type $(d_1, \cdots, d_g)$. Recall [insert label] we have an isogeny $p_1 : X \to X_1$ where $X_1 = V/(\Lambda_1 \bigoplus \Lambda_2)$ and we have $M_1 \in \mathrm{Pic}(X_1)$ with $P^*M_1 \simeq L$. Clearly M_1 is positive definite.

We claim that $\mathrm{type}(M_1) = (1, \cdots, 1)$. We need Riemann-Roch to show this:

$$d_1 \cdots d_g = \chi(L) = (\deg p_1)\chi(M_1) = d_1 \cdots d_g \chi(M_1)$$

which implies that $\chi(M_1) = 1 = (-1)^s Pf(M_1) = Pf(M_1)$. □

3.1.5 Example. When $X = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}\omega_1 \bigoplus \mathbb{Z}\omega_2 \simeq \mathbb{Z}^2$. (Without loss of generality, we may assume that $\omega_1 = 1, \omega_2 = \tau$ with $Im\tau > 0$.) Define $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$(v, w) \mapsto \frac{v\overline{w}}{Im(\overline{\omega_1}\omega_2)}.$$

H is obviously positive definite. It is easy to show that H is Hermitian and that $Im(H(\Lambda, \Lambda)) \subset \mathbb{Z}$. (That is, H is a Riemann form.) Hence, $\mathbb{C}/\Lambda$ is an abelian variety.

(Remark: $\mathbb{C}^g/\Lambda$ where $g > 1$ is usually not an abelian variety.)

Recall the theorem of the square says that for $\bar{v}\bar{w} \in X$, we have

$$t^*_{\bar{v}+\bar{w}}L \simeq t^*_{\bar{v}}L \bigotimes t^*_{\bar{w}}L \bigotimes L^{-1}.$$

Note that $t^*_{\bar{w}}L \otimes L^{-1}$ can be re-written as $t^*_x L \otimes t^*_{-x}L \simeq L^2$, hence we have

$$t^*_{\bar{v}}L \bigotimes t^*_{\bar{w}}L \bigotimes t^*_{-\bar{v}-\bar{w}}L \simeq L^3.$$

This generalizes to the following lemma:

3.1.6 Lemma. *For $\bar{v_1}, \cdots, \bar{v_n} \in X$, if $\sum_{i=1}^n \bar{v_i} = 0$, then $\bigotimes_{i=1}^n t^*_{v_i}L \simeq L^n$.*

Proof. Since $L = L(H, \chi)$, $\bigotimes_{i=1}^n t^*_{v_i}L = L(nH, \prod \chi e^{2\pi i Im H(v_i, \cdot)}) = L(nH, \chi^n) \simeq L^n$. $\square$

Remarks:

1. I will give an interpretation in terms of divisors.
2. One can use this to give an explicit basis for $H^0(X, L^n)$ in terms of basis for $H^0(X, L)$. For instance, if (X, L) is of type $(1, \cdots, 1)$, then $H^0(X, L)$ is one dimensional.

3.2 Riemann relations

We now turn to the following question: What does it mean, in terms of a period matrix Π, for X to be an abelian variety? Let $X = V/\Lambda$. Let $\{e_1, \cdots, e_g\}$ be a $\mathbb{C}$-basis for V and $\{\lambda_1, \cdots, \lambda_{2g}\}$ be a $\mathbb{Z}$-basis for Λ. Then we have

$$X = \frac{\mathbb{C}^g}{\Pi\mathbb{Z}^{2g}}$$

where Π is the $g \times 2g$ period matrix.

3.2.1 Theorem (Riemann relations)**.** *X is an abelian variety if and only if there exists a non-degenerate antisymmetric (i.e. alternating) matrix $A \in M_{2g\times 2g}(\mathbb{Z})$ such that*

1. *(Hermitian condition) $\Pi A^{-1}\Pi^t = 0$; and*
2. *(positive definite condition) $i\Pi A^{-1}\overline{\Pi}^t > 0$.*

Proof. We sketch the proof. Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be an arbitrary non-degenerate alternating form on Λ (in $NS(X)$). Let A be its matrix with respect to λ_i's: $a_{ij} = E(\lambda_i, \lambda_j)$. Define $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ by

$$H(u,v) := E(iu,v) + iE(u,v) \in \mathbb{R}^{2g}$$

where E is the natural extension to $\mathbb{R}^{2g}$. Define

$$\mathcal{J} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1}_{2g\times 2g} \begin{pmatrix} iIg & \\ & -iI_g \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}.$$

Such a construction is useful because $i\Pi = \Pi\mathcal{J}$. Note also that $E(\Pi x, \Pi y) = x^t Ay$ for $x, y \in \mathbb{Z}^{2g}$.

We claim that A is Hermitian if and only if $\Pi A^{-1}\Pi^t = 0$. We have seen that H is Hermitian if and only if $E(iu, iv) = E(u,v)$ for all $u, v \in \mathbb{C}^g \simeq \mathbb{R}^{2g}$. So H is Hermitian if and only if $\mathcal{J}^t A \mathcal{J} = A$.

Next, we claim that assume $\Pi A^{-1}\Pi^t = 0$, then the matrix of H with respect to $\{e_1, \cdots, e_g\}$ is $2i(\Pi A^{-1}\overline{\Pi}^t)^{-1}$. Therefore, $H(u,v) = E(iu,v) + iE(u,v)$ is positive definite. ☐

Remark: If we assume that $\{\lambda_i\}$ is symplectic, and if we write $A = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ and $\Pi = ((\Pi_1)_{g\times g}|(\Pi_2)_{g\times g})$. Then the Riemann relations can be stated as

1. $\Pi_2 D^{-1}\Pi_1^t - \Pi_1 D^{-1}\Pi_2^t = 0$.
2. $i(\Pi_2 D^{-1}\overline{\Pi_1}^t - \Pi D^{-1}\overline{\Pi}^t) > 0$.

3.3 Divisors and Maps to $\mathbb{P}^n$ for abelian varieties

Let $X = V/\Lambda$ be an abelian variety of dimension g with (positive definite) polarization $L \in \mathrm{Pic}(X)$. Define the rational (meromorphic?) map $\Psi_L : X \dashrightarrow \mathbb{P}^N$ by

$$x \mapsto [\sigma_0(x) : \cdots : \sigma_N(x)]$$

whenever there exists i such that $\sigma_i(x) \neq 0$ where $\mathrm{span}_{\mathbb{C}}\{\sigma_0, \cdots, \sigma_N\} = H^0(X, L)$.

By fixing a factor of automorphy, we may identify $H^0(X,L)$ with theta functions: fix a basis $\vartheta_0, \cdots, \vartheta_N$ for $H^0(X,L)$ and write $\Psi_L(\overline{v}) = [\vartheta_0(v) : \cdots : \vartheta_N(v)]$ where $\overline{v} = v$ modulo Λ.

The big question is: when is Ψ_L an embedding? Our goal is to show the following theorem.

3.3.1 Theorem (Lefschetz)**.** *If L is of the type $(d_1, \cdots, d_g)$ and $d_i \geq 3$, then Ψ_L is an embedding.*

3.3.2 Example. It is a good exercise to show that if L is positive definite and of the type $(d_1, \cdots, d_g)$, then

$$\begin{aligned} N + 1 &= \dim H^0(X, L^{\otimes k}) = \chi(L^{\otimes k}) = Pf(L^{\otimes k}) \\ &= Pf(kc_1(L)) = (kd_1)(kd_2)\cdots(kd_g) = k^g(d_1 d_2 \cdots d_g). \end{aligned}$$

Now assume L is principal (i.e. of type $(1, \cdots, 1)$), so $\dim H^0(X, L^k) = k^g$. This gives that $N = k^g - 1$. So for example when $k = 3$, $N = 3^g - 1$. When $g = 1$, we can embed elliptic curves inside $\mathbb{P}^2$ which says that the quantity $N = k^g - 1$ is sharp in some sense.

If L is principally polarized, then $\dim H^0(X, L) = 1$, so there exists a unique (up to scaling) theta function θ generating $H^0(X, L)$, which is called a *Riemann's theta function.*

It is a good exercise to give $\vartheta_0, \cdots, \vartheta_N$ a basis for $H^0(X, L^k)$ in terms of Riemann's theta function. (Hint: Recall that $L^{\otimes k} \simeq \bigotimes_{i=1}^k t_{\bar{v}_1}^* L$ and $\sum_{i=1}^k \bar{v}_i = 0$. So one needs to take appropriate shifts of θ (by, say $(\frac{1}{k}\Lambda/\Lambda)$) and appropriate products to get a basis.)

3.4 Basic properties of divisors on Abelian Varieties

we said that if X is an abelian variety and L is a positive-definite line bundle and $\vartheta_0, \cdots, \vartheta_N$ are a basis of theta-functions for $H^0(X, L)$, then we have a rational map $\Psi_L : X \dashrightarrow \mathbb{P}^N$ by

$$\bar{p} \mapsto [\vartheta_0(p) : \cdots : \vartheta_N(p)]$$

. We want to study when this is defined everywhere.

3.4.1 Lemma. *Let X be a complex torus. Suppose $\bar{v_1}, \cdots, \bar{v_n} \in X$ such that*

$$\sum \bar{v_i} = 0.$$

Suppose $D \in |L|$ is an effective divisor, then

$$\sum_{i=1}^n t_{\bar{v_i}}^* D \sim nD.$$

Proof. $\bigotimes_{i=1}^n t_{\bar{v_i}}^* L \simeq L^{\otimes n}$. □

3.4.2 Remark.

1. $t_x^* D = D - x$.
2. $y \in t_x^* D \Leftrightarrow x + y \in D \Leftrightarrow x \in t_y^* D$.

3.4.3 Proposition. *Let X be a complex torus. Let L be a positive definite line bundle of type $(d_1, \cdots, d_g)$. If $d_1 \geq 2$, then Ψ_L is holomorphic (i.e. $|L|$ is base-point free).*

3.4.4 Example. If L is principal, then L^2 is base point free.

3.4.5 Remark. Why do we have 2 in the above example? This is because of the Theorem of the square!

3.4.6 Remark. Recall that $L \simeq L_1^n$ for some L_1 if and only if $X[n] \subset K(L) = (\bigotimes \mathbb{Z}/d_i\mathbb{Z})^2$ where $K(L) = \ker(\varphi_L : X \to \hat{X})$. It follows that $L = L_1^{d_1}$ where $d_1 \geq 2$.

Now we will give the proof of the above proposition.

Proof. (In the proof, we use the notation $u \in D$ to denote $u \in Supp(D)$.) Let $x \in X$. We need to show that there exists $E \in |L|$ not containing x (i.e. $x \notin Supp(E)$). By the above remark, $L = M^{d_1}$ for some M. Then M is also positive definite (because L is). So there exists $D \in |M|$. (Note that $\dim H^0(X, M) \geq 1$ and so $|D| \neq \emptyset$.) Consider $t_x^* D = D - x$. We choose $x_1, \cdots, x_{d_1-1} \notin t_x^* D$. Let $x_{d_1} = -\sum_{i=1}^{d_1-1} x_i$ be such that $x_{d_1} \notin t_x^* D$ by continuity of summation. Therefore, $x \notin t_{x_i}^* D$ for all $1 \leq i \leq d_1$. So $x \notin \sum_{i=1}^{d_1} t_{x_i}^* D$ and hence $x \notin \cup Supp(t_{x_i}^* D)$. But $\sum_{i=1}^{d_1} t_{x_i}^* D \sim d_1 D \in |L|$.

□

3.4.7 Definition. $D = \sum a_i Y_i$ is called **reduced** if all nonzero $a_i = 1$.

3.4.8 Lemma. *Let $L \in \mathrm{Pic}(X)$ where X is a complex torus. Suppose $|L| \neq \emptyset$ and L is positive definite (i.e. L is positive semi-definite with $L|_{K(L)_0}$ trivial). A "general member" of $|L|$ is reduced. More precisely, if $D = nE + F \in |L|$ with $E > 0, F \geq 0$ and $n \geq 2$, then D is reduced.*

Proof. Assume $D = nE + F \in |L|$ with $E > 0, F \geq 0$ and $n \geq 2$. But $nE \sim \sum_{i=1}^{n} t_{x_i}^* E$ for $\sum x_i = 0$. Since we may pick $x_1, \cdots, x_{n-1}$ arbitrarily, we are done. □

3.4.9 Lemma. *Suppose $L \in \mathrm{Pic}(X)$ is positive definite, then there exists an open dense $\mathcal{U} \subset |L| \simeq \mathbb{P}^N$ such that if $D \in \mathcal{U}$, then $t_x^* D = D$ only holds for $x = 0$.*

3.4.10 Remark. Let D be an effective divisor. Define $H(D) = \{x \in X : t_x^* D = D\}$. The above lemma says that for "almost all" D, we have $H(D) = \{0\}$. Also, we have $H(D)$ is Zariski closed. Moreover, if $L = \mathcal{L}(D) = \mathcal{O}_X(L)$, then the following are equivalent:

1. L is positive definite.
2. $K(L)$ is finite.
3. $H(D)$ is finite.

The proof of the lemma uses $K(L)$ is finite: if there exists $x \neq 0$ such that $t_x^* D = D$, then $x \in K(L)$ and $G := \langle x \rangle$ is finite. Then the idea is to consider the projection map $X \to X/G$ which is of degree $|G| \geq 2$ and we will omit the rest of the proof.

3.5 Decomposition of Polarized abelian variety

Let (X, L) be a polarized abelian variety. We want to decompose it to "irreducible" polarized abelian varieties. One benefit of doing so is for studying $X \to \mathbb{P}^N$. (If $d_1 = 2$, then Ψ_L is sometimes an embedding. If it is not an embedding, then on irreducibles Ψ_L factors through the *Kummer* variety $X/\langle -1 \rangle$.) Let $|L| = |M| + F_1 + \cdots + F_r$ where $|M|$ is the "moving part" and $F_1 + \cdots + F_r$ is the "fixed component". Since generating elements are reduced ($F_i \neq F_j$ for $i \neq j$), one can consider M and $N_i = \mathcal{L}(F_i)$ for $1 \leq i \leq r$ and therefore we have $h^0(X, M) > 1$ and $h^0(X, N_i) = 1$. So we have $M|_{K(M)_0}$ is trivial and $N_i|_{K(N_i)_0}$ is trivial and hence $P_M : X \to X_M := X/K(M)_0$ and $P_{N_i} : X \to X_{N_i} := X/K(N_i)_0$ and so there exists descents $\overline{M}$ on X_M and $\overline{N_i}$ on X_{N_i}. So we get polarized abelian varieties $(X_M, \overline{M}), (X_{N_i}, \overline{N_i})_{1 \leq i \leq r}$.

3.5.1 Theorem. *The homomorphism*

$$P = (P_M, P_{N_1}, \cdots, P_{N_r}) : X \to X_m \times X_{N_1} \times \cdots \times X_{N_r}$$

induces an isomorphism of polarized abelian varieties

$$P : (X, L) \to (X_M \times X_{N_1} \times \cdots \times X_{N_r}, q_M^* \overline{M} \bigotimes \cdots \bigotimes q_{N_r}^* \overline{N_r})$$

which we informally write

$$(X, L) \simeq (X_M, \overline{M}) \times \cdots \times (X_{N_r}, \overline{N_r}).$$

Here $q_M : X_M \times X_{N_1} \times \cdots \times X_{N_r} \to X_M$ *and* $q_{N_i} : X_M \times X_{N_1} \times \cdots \times X_{N_r} \to X_{N_i}$ *denote the natural projection maps.*

For the proof, one uses new inequalities for $(L_1, \cdots, L_g)$, Riemann Roch, and other tools that we have seen so far. Since this is not used in Lefschetz theorem, I will skip the proof.

3.6 Gauss map

Let (X, L) be a polarized abelian variety of dimension g. Take $D \in |L|$ such that D reduced (the theorem of squares guarantees existence; in fact, one can pick any general member). Let D_{sm} be the smooth locus of $supp(D)$ which has dimension $g-1$. For $\bar{\omega} \in D_{sm}$, the tangent $T_{D,\bar{\omega}}$ to D at $\bar{\omega}$ is a $(g-1)$-dimensional $\mathbb{C}$-vector space. Let $W_{\bar{\omega}}$ be the translation of $T_{D,\bar{\omega}}$ (via $t_{-\bar{\omega}}$) to the origin; it is a $(g-1)$-dimensional subspace of the g-dimensional tangent space $T_{x,0} = V$ at 0. Then we define the **Gauss map** $G : D_{sm} \to \mathbb{P}(V^*)$ by

$$\bar{\omega} \mapsto W_{\bar{\omega}}^{\perp}.$$

Explicitly, we can identify $H^0(X, L)$ with the space of theta functions. Then the covering map $\pi : V \to X = V/\Lambda$ satisfies

$$\pi^* D = div(\vartheta)$$

for some holomorphic function ϑ on the universal cover. (Note that $div(\vartheta)$ is periodic with respect to Λ because $\vartheta(v + \lambda) = a_L(\lambda, v)\vartheta(v)$. So the "zero set" is periodic.) Let $\{v_1, \cdots, v_g\}$ be coordinate functions with respect to some basis. Let $\bar{\omega} = \omega + \Lambda$. Then

$$T_{D,\bar{\omega}} = \{(v_1, \cdots, v_g) : \sum_{i=1}^{g} \frac{\partial \vartheta}{\partial v_i}(\omega)(v_i - \omega_i) = 0\}.$$

So the dual to $T_{D,\bar{\omega}}$ is $\left(\frac{\partial \vartheta}{\partial v_i}(\omega)\right)_{1 \le i \le g}$. So the Gauss map maybe identified with $G : D_{sm} \to \mathbb{P}^{g-1}$ defined by

$$\bar{\omega} \mapsto \left(\frac{\partial \vartheta}{\partial v_i}(\omega)\right)_{1 \le i \le g}.$$

It is easy to check that G is holomorphic and independent of the choice of factors of automorphy.

3.6.1 Proposition. *Let L be positive definite and $D \in |L|$ be reduced. Then $Im(G : D_{sm} \to \mathbb{P}(V^*) \simeq \mathbb{P}^{g-1}$ is* not *contained in a hyperplane.*

Proof. Assume not; i.e. assume there exists some nonzero $t \in V$ contained in all tangent spaces $T_{D,\bar{\omega}}$ for $\bar{\omega} \in D_{sm}$. Choose basis for V such that $t = (1, 0, \cdots, 0)$. Fix ϑ canonical: $a_L = a_{L(H,\chi)}$. So $\frac{\partial \vartheta}{\partial v_i}(\omega) = 0$ for all $\omega \in V$ for which $\vartheta(\omega) = 0$ (because it is true for an open dense subset).

Let $f = \frac{1}{\vartheta} \cdot \frac{\partial \vartheta}{\partial v_i}$. Since D is reduced, f is holomorphic on V. From $\vartheta(v + \lambda) = a_{L(H,\chi)}\vartheta(v)$ we obtain

$$f(v + \lambda) = f(v) + \pi H(t, \lambda)$$

for all $v \in V$ and $\lambda \in \Lambda$. Since df is Λ-periodic, it is a pullback of a holomorphic differential form on X. Hence

$$df = \sum \alpha_i dv_i$$

for some $\alpha_i \in \mathbb{C}$. So $f = \sum \alpha_i v_i + C$. By $f(v + \lambda) = f(v) + \pi H(t, \lambda)$, we obtain

$$\sum \alpha_i \lambda_i = \pi H(t, \lambda)$$

where $\lambda_1, \cdots, \lambda_g$ are coordinates of λ. So $f(v) = \pi H(t, v) + C$. Note that f is holomorphic, but $H(t, \cdot)$ is anti-$\mathbb{C}$-linear (as H is Hermitian). Since H is non-degenerate, we must have $t = 0$ and this is a contradiction. $\square$

3.7 Projective embedding and Lefschetz theorem

Let (X, L) be a polarized abelian variety of type $(d_1, \cdots, d_g)$. Let $\Psi_L : X \to \mathbb{P}^N$ be the associated map. We have seen that $d_1 \geq 2$ and Ψ_L is holomorphic.

3.7.1 Theorem (Lefschetz)**.** *If $d_1 \geq 3$, then Ψ_L is an embedding.*

Proof. Recall that we have $\Psi_L : X \to \mathbb{P}(W^*)$ which is defined by $P \mapsto \{D \in |W| : P \in D\}$. We need to show that Ψ_L is injective and for all $x \in X$, $d\Psi_{L,x}$ is injective.

Assume $\Psi_L(y_1) = \Psi_L(y_2)$ for $y_1, y_2 \in X$; i.e. for any $D \in |L|$, we have $y_1 \in D$ if and only if $y_2 \in D$. Since $X[d_1] \subset K(L) \simeq (\bigoplus \mathbb{Z}/d_i)^2$, we know $L = M^{d_1}$ for some M positive definite. For a general member $D_M \in |M|$, we have $-D_M$ is reduced and $-t_x^* D_M = D_M$ only for $x = 0$. Pick any point $x_1 \in t_{y_1}^* D_M$. Then pick $x_2, \cdots, x_{d_1} \in X$ (where $d_1 \geq 3$) with $\sum_1^{d_1} x_i = 0$ such that $y_2 \notin t_{x_i}^* D_M$ for $2 \leq i \leq d_1$. (This can be done because summation is continuous and $d_1 \geq 3$.) Let

$$E = \sum_{i=1}^{d_1} t_{x_i}^* D_M \in |M^{d_1}| = |L|.$$

Since $y_1 \in t_{x_1}^* D_M$ we have $y_1 \in E$. By assumption, $y_2 \in E$ but $y_2 \notin t_{x_i}^* D_M$ for $2 \leq i \leq d_1$. So $y_2 \in t_{x_1}^* D_M$ and hence $x_1 \in t_{y_2}^* D_M$. Therefore, $t_{y_1}^* D_M \subset t_{y_2}^* D_M$ and by symmetry we also have $t_{y_1}^* D_M \supset t_{y_2}^* D_M$. Since D_M is reduced, we get $t_{y_1}^* D_M = t_{y_2}^* D_M$ as divisors. So $t_{y_1 - y_2}^* D_M = d_M$ and thus $y_1 - y_2 = 0$. Therefore, Ψ_L is injective.

Next, let $0 \neq t \in T_{X,x}$ be the tangent vector at $x \in X$. We need to show that there exist $D \in |L|$ and $x \in D$ such that t is *not* tangent to D at x. Assume not, i.e. t is tangent to D at x for all $D \in |L|$ with $x \in D$. Pick $D_M \in |M|$ generic and so it is reduced. Pick $x_1 \in t_x^* D_M$ and choose (as above) $x_2, \cdots, x_{d_1} \in X$ such that $\sum_1^{d_1} x_i = 0$ and $x \notin t_{x_i}^* D_M$ for $2 \leq i \leq d_1$. Let $E' = \sum_{i=1}^{d_1} t_{x_i}^* D_M \in |L|$. Note that $x \in E'$. By assumption, t is tangent to E' at x. So t is tangent to $t_{x_1}^* D_M$ at x. Since this is true for all $x_1 \in t_x^* D_M$, it follows that t is tangent to D_M at all points $u \in D_M$. (This is because for $u \in D_M$, $0 \in D_M - u = t_u^* D_M$ and hence $x \in t_{u-x}^* D_M$.) This implies that the image of the Gauss map is contained in a hyperplane. □

3.7.2 Definition. A line bundle is called **very ample** if Ψ_L is an embedding. A line bundle is called **ample** if $L^{\bigotimes n}$ is very ample for some $n \geq 1$.

3.7.3 Proposition. *Let X be a complex torus and $L \in \mathrm{Pic}(X)$. Then the following are equivalent:*

1. *L is ample.*
2. *L is positive definite.*
3. *$H^0(X, L) \neq 0$ and $K(L)$ is finite.*
4. *$H^0(X, L) \neq 0$ and $(L^g) > 0$.*

3.7.4 Corollary. *If $L \in \mathrm{Pic}(X)$ is ample, then L^3 is very ample.*

Proof. Since L is ample, by the above proposition, L is positive definite and hence of type $(d_1, \cdots, d_g)$ with $d_1 \geq 1$. Hence L^3 is of type $(3d_1, \cdots, 3d_g)$ with $3d_1 \geq 3$. □

3.7.5 Theorem. *Let X be a complex torus. The following are equivalent:*

1. *X is an abelian variety (ie. has positive definite L).*
2. *X admits the structure of a projective algebraic variety.*

Proof. Chow's theorem (GAGA) says if Y is a complete algebraic variety and Z is a closed analytic subset of Y^{an}, then there exists an algebraic subvariety Z of Y such that $Z^{an} = Z$. □

3.8 L^2

Let $(X, L) \simeq (X_M, M) \times (X_{N_1}, \overline{N_1}) \times \cdots \times (X_{N_r}, \overline{N_r})$. If suffices to understand $(X_M, \overline{M})$ and $(X_{N_i}, \overline{N_i})$. And thus it suffices to understand

1. $L = M$. (no fixed component)
2. $L = N_i$. (irreducible principal polarization)

Let $K_x := X / \langle (-1)_x \rangle$ be a *Kummer variety* which is an algebraic variety with 2^{2g} singular points.

3.8.1 Theorem. *When $L = M$, L^2 gives an embedding. When $L = N_i$, if L is symmetric, then $\Psi : K_X \to \mathbb{P}^{2^g - 1}$ is an embedding.*

4 Appendix

4.1 Poincaré lemmas, De Rham cohomology, Dolbeault cohomology

If we are over $\mathbb{R}$ we have the classical Poincaré lemma, which lets us get the connection between De Rham cohomology and singular cohomology. If we're over $\mathbb{C}$, we have a $\bar{\partial}$-Poincaré lemma which tells us something about Dolbeault cohomology. We will say things about these, and then a bit more about Hodge theory (and what is the major simplification for abelian varieties versus Kalher manifolds).

4.1.1 Classical Poincaré lemma and De Rham cohomology

Let us look at $\mathbb{C}^n$, which is $\mathbb{R}^{2n}$ as a real manifold (or more generally any smooth manifold M). Let $T_0^*(\mathbb{C}^n)$ (or $T_z^*(M)$) be the cotangent space, and pick the usual basis $\{dx_j, dy_j\}_{j=1}^n$. Over $\mathbb{C}$ we will also want to work with the complex basis $\{dz_j, d\bar{z}_j\}_{j=1}^n$ with $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - idy_j$. For the tangent space $T_0\mathbb{C}^n$ or T_zM, let $\{\partial/\partial z_j, \partial/\partial \bar{z}_j\}$ denote the dual basis. One can easily check that

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right).$$

Remark: Over $\mathbb{C}$, the Cauchy-Riemann equations tell us that if $f \in C^\infty(U)$ then f is holomorphic if and only if $\partial f/\partial \bar{z} = 0$.

Given $f \in C^\infty(U)$ with $U \subset \mathbb{C}^n$, define the **total differential** as

$$df = \partial f + \bar{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_i + \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

4.1.2 Theorem. *For $f \in C^\infty(U)$ with $U \subset \mathbb{C}^n$, f is holomorphic if and only if $\partial f = 0$.*

A **differential form** of degree k is a C^∞-global section of $\bigwedge^k T^* = \Omega^k$. At each $p \in M$, it is an alternating multilinear form $T_p(M) \times \cdots \times T_p(M) \to \mathbb{R}$. The set of k-forms on M can be denoted as $\Omega^k(M)$, $A^k(M)$, $\Gamma(\Omega^k, M)$ or $H^0(M, \Omega^k)$. The differential maps $d : \Omega^k(M) \to \Omega^{k+1}(M)$ can be defined explicitly on coordinates, or alternatively characterized as the unique collection of $\mathbb{R}$-linear maps satisfying:

1. $\forall f \in \Omega^0(M) = C^\infty(M)$, df is the total differential of f, and $d(df) = 0$;
2. We have $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta$.

From these one can write down d in local coordinate as $d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge (dx_{i_1} \wedge \cdots \wedge dx_{i_k})$, and verify that for any differential form α, $dd\alpha = 0$. Hence we have the de Rham cochain complex:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots$$

whose cohomology

$$H^p_{dR}(M) = \frac{\ker(d : \Omega^p(M) \to \Omega^{p+1}(M))}{Im(d : \Omega^{p-1}(M) \to \Omega^p(M))}$$

is called the **de Rham cohomology**. Elements of this kernel are called **closed** and elements of the image are called **exact**. (Example to keep in mind: If S^1 is the circle we have a closed form $d\theta$ which is not exact.) Note that $\Omega^k(M) = 0$ for $k > \dim M$.

4.1.3 Lemma (Poincaré Lemma). *If $U \subset \mathbb{R}^n$ is contractible, then $H^p_{dR}(U) = 0$ for $p \geq 1$.*

This shows that $0 \to \underline{\mathbb{R}} \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots$ is an exact sequence of sheaves. As a consequence, we have:

4.1.4 Theorem (de Rham theorem). *If M is a smooth manifold, then $\check{H}^p(M, \underline{\mathbb{R}})$ (which is our usual singular cohomology) is isomorphic to $H^p_{dR}(M)$.*

The idea of the proof is to break the above exact sequence up into short exact sequences in the standard way, then use the corresponding long exact sequences on cohomology.

4.1.5 $\overline{\partial}$ Dolbeault cohomology

Now we want to start using the complex structure on a complex manifold. Using holomorphic charts, we locally have differentials $\{dz_i, d\overline{z}_i\}$ spanning the cotangent space, with dual elements $\{\partial/\partial z_i, \partial/\partial \overline{z}_i\}$ spanning the tangent space. Using these we have the following splitting

$$T^*_p(M) = {T^*}'_p(M) + {T^*}''_p(M),$$

where $T^{*\prime}$ is spanned by dz_j and $T^{*\prime\prime}$ is spanned by $\overline{dz_j}$. This gives a direct sum decomposition

$$\bigwedge^n T^*_p(M) = \bigoplus_{p+q=n} \left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right).$$

with the (p, q) part spanned by things of the form $dz_I \wedge d\overline{z}_J$ with $|I| = p$ and $|J| = q$. Using this decomposition, define a sheaf $\Omega^{p,q}$ of $C^\infty(p, q)$ forms: we set

$$\Omega^{p,q}(M) = \left\{\varphi \in \Omega^n(M) : \forall z, \varphi(z) \in \left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right)\right\}.$$

It is also denoted by $A^{p,q}(M)$, $\Gamma(\Omega^{p,q}, M)$ or $H^0(M, \Omega^{p,q})$. This gives a filtration $\Omega^n(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M)$.

What happens when we apply d to something in $\Omega^{p,q}(M)$? We can see it maps into $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ because $\varphi(z)$ is in

$$\left(\bigwedge^p T^*_p(M)'\right) \otimes \left(\bigwedge^p T^*_p(M)''\right) \wedge T^*_z(M).$$

The decomposition of Ω^n induces a decomposition $d = \partial + \overline{\partial}$, where $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$. By writing them down in local coordinates one can check that $\overline{\partial}^2 = 0$, hence there is a cochain complex:

$$0 \to \Omega^{p,0}(M) \xrightarrow{\partial} \Omega^{p,1}(M) \xrightarrow{\partial} \Omega^{p,2}(M) \cdots$$

and we define the **Dolbeault cohomology** as

$$H^{p,q}_{\overline{\partial}}(M) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{Im(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))}$$

Now, we want to relate this to some other sheaf cohomology; our exact-seqeunce-based proof of the de Rham theorem suggests what we need. The $\overline{\partial}$-*Poincaré lemma* says that on a polydisc D in $\mathbb{C}^n$ (a product of discs in $\mathbb{C}$), $H^{p,q}_{\overline{\partial}}(D) = 0$ for all $q \geq 1$. (Poincaré was studying the problem that if $g \in C^\infty(D)$ for $D \subset \mathbb{C}$ then he wanted to find f with $\partial f \partial \overline{z} = g$ which can be solved on a slightly smaller disc.)

Hence, as an analogue of the de Rham theorem, by letting Ω^p_{hol} be the sheaf of holomorphic p-forms, Poincaré lemma tells us that we have an exact sequence

$$0 \to \Omega^p_{\text{hol}} \hookrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \dots$$

with $\Omega^p_{hol} \to \Omega^{p,0}$ the inclusion and the other maps $\overline{\partial}$. Note that Ω^p_{hol} is indeed the kernel of $\overline{\partial}$ on Ω^p_{hol}. By the same argument from our sheaf proof of the de Rham theorem we can prove the Dolbeault theorem:

4.1.6 Theorem (Dolbeault theorem)**.**

$$H^{p,q}_{\overline{\partial}}(M) \simeq H^q(M, \Omega^p_{hol})$$

for $p, q \geq 0$.

4.1.7 Example.

- When $q \geq \dim(M)$, $H^{0,q}(M) = H^q(M, \mathcal{O}_M) = 0$. (So if $q \geq \dim(M)$ both are zero.)
- When $q \geq 1$, $H^q(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = 0$ by the above example and the $\overline{\partial}$-Poincaré lemma.
- When D is a polydisc in $\mathbb{C}^n$, $H^{p,0}_{\overline{\partial}}(D, \Omega^p_D) = H^0(D, \mathcal{O}_D) \otimes \Omega^p_D$ is usually nontrivial, so the $q > 1$ hypothesis in the Poincaré lemma matters!

4.2 Hodge decomposition

In this section, we will talk about the decomposition on complex tori. Assume $X = V/\Lambda$ is a complex torus. Let $e_1, \dots e_g$ be a complex basis of V and $v_1, \cdots, v_g$ the corresponding coordinate functions. Then $H^n(X, \mathbb{C})$ is isomorphic to the set

$$IF^n(X) = \bigoplus_{p+q=n} IF^{p,q}(X)$$

where $IF^{p,q}(X)$ (the "invariant forms") are the things of the form

$$\sum_{|I|=p,|J|=q} a_{IJ} dv_I \wedge d\overline{v}_J$$

for $a_{IJ} \in \mathbb{C}$. Note that because we are on a torus, it is easy to write down this decomposition.

For a more general (compact) X, we have the following more general theory:

4.2.1 Theorem. *If X is a compact complex manifold with a "nice" metric, then we have*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q}(X)$ is something that is isomorphic to $H^{p,q}_{\overline{\partial}}(X)$ and is isomorphic to the space ot harmonic forms. Also we have

$$H^{p,q} = \overline{H^{q,p}}.$$

In the above theorem, "nice" metric means a Euclidean metric or a "degree 2 approximation" of one, i.e. Kähler metric. For example, $X = V/\Lambda$ has the Euclidean metric and any complex projective variety has a Kähler metric (since $\mathbb{P}^n$ has a Fubini-Study metric). For our situation of $X = V/\Lambda$, one can show the $IF^{p,q}(X)$ we wrote down is isomorphic to $H^{p,q}(X)$ is isomorphic to $H^{p,q}_{\overline{\delta}}(X) \simeq H^{p+q}_d(X) \simeq H^q(X,\Omega^p)$. (These are all hard; once we have shown these, we can then show that in the case of a complex torus, the harmonic things are just hte invariant forms and we recover what we had above.)

What is in the background of these isomorphisms? (We only consider the case $X = V/\Lambda$, but the following also contains all of the ideas we need in general.) We start with our Euclidean metric $ds^2 = \Sigma dv_i \otimes d\overline{v}_i$. This has an associated $(1,1)$-form

$$\omega = -\frac{1}{2}Im(ds^2) = \frac{i}{2}\sum_{i=1}^{g} dv_i \wedge d\overline{v}_i.$$

Then we get a volume form

$$dv = \frac{1}{g!}\bigwedge^{g}\omega = (-1)^{\binom{g}{2}}\left(\frac{i}{2}\right)^{g}(dv_1 \wedge d\overline{v}_1 \wedge dv_2 \wedge \cdots).$$

Now that we have a volume form we can define an inner product on $\Omega^{p,q}(M)$ (but not complete, so not a Hilbert space) by

$$(\varphi,\psi) = \sum_{|I|=p,|J|=q}\int_X \varphi_{IJ}\overline{\psi}_{IJ}dv$$

for $\varphi = \sum \varphi_{IJ} dv_I \wedge \overline{v}_J$ and similarly for ψ. This makes it into a pre-Hilbert space (i.e. a non-complete inner product space) and we can then define an adjoint map $\overline{\delta}$ of $\overline{\partial}$ satisfying $(\varphi, \overline{\partial}\psi) = (p\overline{\varphi}, \psi)$. Then the **Laplace-Beltrami operator** is given by

$$\Delta = \overline{\partial\delta} + \overline{\delta\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M).$$

Note that Δ can also be written as $(\overline{\delta} + \overline{\partial})^2$. In our situation in coordinates we can compute

$$\Delta(\varphi dv_I \wedge dv_J) = -\sum_i \frac{\partial^2\varphi}{\partial v_i \partial\overline{v}_i}(dv_I \wedge dv_J),$$

and so this is really the usual Laplacian.

The main point of this formal setup is that we can very easily show that a closed form ψ whose "norm" (ψ,ψ) is minimal in its class (in the de Rham or Dolbeault cohomology) is the unique solution in that class to $\overline{\delta}\psi = 0$. Note that

$\overline{\partial}\psi = 0$ and $\overline{\delta}\psi = 0$. Therefore $\Delta\psi = 0$ and ψ is Harmonic. (The converse is easy to prove too.) Therefore, we conclude that elements of $H^{p,q}(X)$, which are Harmonic (p,q)-forms, are unique representatives for classes in $H^{p,q}_{\overline{\partial}}(X)$. In order to understand these Harmonic forms, we need to solve partial differential equations. And there are two operators that can help solving these equations:

1. $H : \Omega^{p,q}(M) \to \Omega^{p,q}(M)$ which is defined by
$$\varphi dv_I \wedge d\overline{v}_J \mapsto \left(\frac{1}{\mathrm{vol}(X)} \int_X \varphi dV\right) dv_I \wedge d\overline{v}_J.$$
This projects any form onto an invariant thing (certainly satisfies $H^2 = H$) and in the abelian variety cse this projects onto $IF^{p,q}(M)$.

2. The second operator is G, a Green's function for Δ such that it is an inverse to the extent we can have: $\Delta G = G\Delta = 1 - H$ and $HG = GH = 0$. In general, such a G is hard to find! (For general Kähler manifolds, one can use the Sobolev lemma. See Griffiths and Harris Chapter 0. [farbod cite]) For complex tori, we are okay because such a G has a formula given in terms of Fourier analysis: Let φ be a function on X, then we can lift it to $\tilde{\varphi}$ a periodic function on $V \simeq \mathbb{C}^g$. For such a $\tilde{\varphi}$, we have a fourier expansion and $G\tilde{\varphi}$ is just the same expression with renormalized coefficients.

   Once we have this, we can then prove the Hodge theorem, that every class has a unique harmonic representative (ultimately the hard part of the theory is showing that we have enough harmonic forms!) because for a form φ, we can explicitly write down
$$\varphi = H\varphi + \overline{\delta\partial}\varphi + \overline{\partial\delta}\varphi.$$
as our Hodge decomposition.

4.3 Divisors v.s. line bundles

Let $(X, \mathcal{O}_X)$ ($\mathcal{O}_X$ is the holomorphic structural sheaf of X) be a complex g-dimensional manifold (which is not necessarily compact). Everything has algebraic analogue as I will remark. (Refer to chapter 2 and also §§4 – 6 in Hartshorne.)

Let $\mathcal{K}_X$ be the sheaf (of rings) of meromorphic functions. Algebraically, $\mathcal{K}_X(\mathcal{U})$ is $\mathcal{O}_X(\mathcal{U})$ localized by the multiplicative system of "non-zero divisors". This defines a presheaf that will be sheaffified. If X is "integral", then $\mathcal{K}_X(\mathcal{U})$ is simply the fraction field of $\mathcal{O}_X(\mathcal{U})$.

We have an exact sequence of sheaves
$$0 \to \mathcal{O}_X^\times \xrightarrow{i} \mathcal{K}_X^\times \xrightarrow{j} \mathcal{K}_X^\times/\mathcal{O}_X^\times \to 0.$$
where $\mathcal{K}_X^\times$ is the sheaf of multiplicative subgroups of $\mathcal{K}_X$. The associated long exact sequence of cohomology is then given by
$$H^0(X, \mathcal{K}_{\mathcal{X}}^\times) \xrightarrow{j_*} H^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^\times) \to \cdots$$

If X is a submanifold of $\mathbb{P}^N$ (algebraically, if X is projective), then $H^1(X, \mathcal{K}_X^*) = 0$. So δ induces an isomorphism $\mathrm{Pic}(X) \simeq \mathrm{CDiv}(X)/\,\mathrm{CPrin}(X)$.

We are going to make all of these objects very explicit.

4.3.1 Pic(X)

Let L be a line bundle, there exists an open cover $\langle U_\alpha \rangle$ of X with trivialization $\{\phi_\alpha : L(U_\alpha) \to U_\alpha \times \mathbf{C}\}$. The associated transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}\}$ are defined by

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{C} &\to (U_\alpha \cap U_\beta) \times \mathbf{C} \\ (z, \cdot) &\mapsto (z, g_{\alpha\beta}(\cdot)) \end{aligned}$$

and satisfy that $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}^*\}$ are holomorphic, and $g_{\alpha\beta} g_{\beta\alpha} = 1$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$, so $\{g_{\alpha\beta}\}$ is a 1-Čhech-cocycle.

Changing ϕ_α to $\phi'_\alpha = f_\alpha \phi_\alpha$ gives a 1-coboundary $g'_{\alpha\beta} = (f_\alpha / f_\beta) g_{\alpha\beta}$.

Conversely, $\{g_{\alpha\beta}\}$ gives a line bundle $\bigsqcup U_\alpha \times \mathbf{C} / \sim$

4.3.2 CDiv(X)

$\mathrm{CDiv}(X) = H^0(X, \mathcal{K}_X^* / \mathscr{O}_X^*)$ (this makes sense for any scheme). A global section of $\mathcal{K}_X^* / \mathscr{O}_X^*$ (i.e., a Cartier divisor) can be presented by

1. An open cover $\{U_\alpha\}$ of X.
2. Meromorphic functions f_α on U_α (not identically zero). (**Remark:** Changing f_α to $s_\alpha f_\alpha$ for $s_\alpha \in \mathscr{O}_X^*$ is ok!) such that for all α and β, $f_\alpha / f_\beta \in \mathscr{O}_X^*(U_\alpha \cap U_\beta)$.

We write $D = (\{U_\alpha\}, \{f_\alpha\})$.

4.3.3 CPrin(X)

$\mathrm{CPrin}(X) = \Im(H^0(X, \mathcal{K}_X^*) \xrightarrow{j_*} \mathrm{CDiv}(X))$, so $D \in \mathrm{CPrin}(X)$ if $D = (\{U_\alpha\}, \{f_\alpha\})$, where $f_\alpha = f|_{U_\alpha}$ for some $f \in H^0(X, \mathcal{K}_X^*)$.

4.3.4 Definition. $D_1 = (\{U_\alpha^{(1)}\}, \{f_\alpha^{(1)}\})$ and $D_2 = (\{U_\alpha^{(2)}\}, \{f_\alpha^{(2)}\})$ are linearly equivalent, $D_1 \sim D_2$, if $D_1 - D_2 := (\{U_\alpha^{(1)} \cap U_{\alpha'}^{(2)}\}, \{f_\alpha^{(1)} / f_{\alpha'}^{(2)}\}) \in \mathrm{CPrin}(X)$.

4.3.5 Weil Divisors

In nice situations, for example,

- Complex manifold
- Noetherian, integral, separated, regular in codimension 1 (if $\dim \mathscr{O}_{X,p} = 1$, then $\mathscr{O}_{X,p}$ is *regular*, i.e., $\dim(\mathfrak{m}_p / \mathfrak{m}_p^2) = 1$.

Let Y be a codimension 1 analytic submanifold of X.

4.3.6 Fact. Then Y is an "analytic hypersurface", i.e., for any $p \in Y$, there is an open set U containing p such that $U \cap Y$ is the zero set of some holomorphic function g.

$Y = Y_1 \cup \cdots \cup Y_m$, where each Y_i is the closure of a connected component of $Y \setminus Y_{\mathrm{sing}}$.

We say that Y is irreducible if $m = 1$.

4.3.7 Definition. A *Weil divisor* D on X is a "locally finite" formal linear combination $D = \sum_i a_i Y_i$, $a_i \in \mathbf{Z}$, where the *prime divisors* Y_i are irreducible, analytic codimension 1 submanifolds. (Algebraically, finite sum, prime divisors are closed integral (implies irreducible) codimension 1 subschemes.

"locally finite" means that for every $p \in X$, there is an open set U containing p such that only finitely many of the intersections $U \cap Y_i$ are nonempty.

If X is compact, locally finite is the same as the sum being finite.

$\mathrm{WDiv}(X) = \{\sum_i a_i Y_i \text{ locally finite, } a_i \in \mathbf{Z}\}$

We will now answer two questions:

1. What is $\mathrm{WPrin}(X)$?
2. How does WDiv compare to CDiv?

4.3.8 Answer 1

Let Y be a prime divisor.

In the analytic case, for $p \in Y$, given f holomorphic, we say that the order of f along Y at p is n if in $\mathscr{O}_{X,p}$, $f = g^n h$, where g is a local equation for Y and n is the largest such number.

4.3.9 Fact. The order is independent of $p \in Y$, and let $\mathrm{ord}_Y f$ be this number.

For meromorphic f, locally $f = g/h$, and we define $\mathrm{ord}_Y f = \mathrm{ord}_Y g - \mathrm{ord}_Y h$.

In the algebraic case, lLet $\eta \in Y$ be a generic point (e.g., $\eta = (0) \in \{(0), (p)\} = \operatorname{Spec} \mathbf{Z}$. $\mathscr{O}_\eta$ is DVR, so we define $\mathrm{ord}_Y f = \mathrm{Val}_\eta(f)$.

Given f, we define $\mathrm{Div}(f) = \sum_Y \mathrm{ord}_Y(f) Y$. The collection of divisors is $\mathrm{WPrin}(X)$.

4.3.10 Fact. The above sum is locally finite in the analytic case and finite in the algebraic case.

If $\mathrm{ord}_Y(f) = n > 0$, we say that f has a zero with multiplicity n along Y, similarily, if $n < 0$, f has a pole with multiplicity n along Y.

4.3.11 Answer 2

In our situation (complex manifolds or integral, Noetherian, separable, regular in codimension 1 and locally factorial), there exists an isomorphism CDiv $\to$ WDiv respecting principal divisors.

Given $D = (\{U_\alpha\}, \{f_\alpha\}) \in \mathrm{CDiv}$, since $f_\alpha / f_\beta \in \mathscr{O}_X^*(U_\alpha \cap U_\beta)$, for all prime divisors Y, we get $\mathrm{ord}_Y(f_\alpha) = \mathrm{ord}_Y(f_\beta)$, so we can define $D' = \sum_{Y \cap U_\alpha \neq \emptyset} \mathrm{ord}_Y(f_\alpha) Y \in \mathrm{WDiv}(X)$.

Conversely, given $D = \sum_i a_i Y_i \in \mathrm{WDiv}(X)$, we fix $\{U_\alpha\}$ such that Y_i has a local defining function $g_{i\alpha} \in \mathscr{O}(U_\alpha)$, and let $f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathcal{K}_X(U_\alpha)$, and let $D' = (\{U_\alpha\}, \{f_\alpha\}) \in \mathrm{CDiv}(X)$.

Now we have set up a correspondence between Cartier divisors $CDiv(X)$ and Weil divisors $WDiv(X)$.

4.3.12 Lemma. *These maps (corresponding to $Cdiv(X) \leftrightarrow WDiv(X)$) are homomorphisms and inverses of each other, respecting $WPrin$ and $CPrin$.*

Note that in "nice situations", we will talk about $Div(X) := CDiv(X)$ and we can identify them as Weil divisors also. Then, $Prin(X) = CPrin(X)$, can be identified with $WPrin(X)$.

4.3.13 Remark. If X is a Riemann surface, then the points are prime divisors. So $Div(X)$ is huge, but in general $Div(X)$ might even be empty. However, if $i : X \hookrightarrow \mathbb{P}^N$, then we obtain lots of divisors by intersecting $i(X)$ with hyperplanes. Converse is also true, if one has enough divisors then $\exists\ X \hookrightarrow \mathbb{P}^N$ (Kodaira embedding).

Recall that we have a connecting homomorphism $\delta : \dfrac{CDiv(X)}{CPrin(X)} \to Pic(X) = H^1(X, \mathcal{O}_X^*)$. We will describe this map explicitly in next section, i.e. going from a divisor to a line bundle.

4.3.14 Divisors vs Line bundles

Consider a map $\delta : Div(X) \to Pic(X)$ denoted by $D \mapsto \mathcal{L}(D)$ or $(\mathcal{O}_X(D))$. It can be described explicitly as follows. Given a Cartier divisor, (U_α, f_α) define $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$.

It is easy to check the following facts:

a. $g_{\alpha\beta}$ is a Čech 1-cocycle.

b. The map is well-defined because changing f_α (for the same divisor D) gives us a 1-coboundary condition.

c. The map is a homomorphism and coincides with the connecting homomorphism δ.

d. If D is principal, then $g_{\alpha\beta} = 1$ and $\mathcal{L}(D) \simeq \mathcal{O}_X$, i.e. is trivial.

e. Conversely, if $\mathcal{L}(D)$ is trivial then

$$g_{\alpha\beta} = \tfrac{h_\alpha}{h_\beta},\ h_\alpha \in \mathcal{O}_X(U_\alpha)$$
$$\begin{aligned} g_{\alpha\beta} &= \tfrac{f_\alpha}{f_\beta} \\ f_\alpha h_\alpha^{-1} &= f_\beta h_\beta^{-1} \end{aligned}$$

So, if we define $f = f_\alpha h_\alpha^{-1}$ on U_α then f extends to a global meromorphic function. Hence, D is principal divisor.

Observe that by (d) and (e) we get kernel $\delta = Prin(X)$.

4.3.15 Definition. Two divisors D and D' are called linearly equivalent, $D \sim D'$ if

$$D - D' = div(f)$$

for some $f \in H^0(X, K_X^*)$.

4.3.16 Remark. a. If $D \sim D'$, then $\mathcal{L}(D) = \mathcal{L}(D')$.

b. $\mathcal{L}(.)$ is functorial.

c. If $f: X \to X'$ is a morphism then we can define a pullback $f^*: \mathrm{Div}(X) \to \mathrm{Div}(X')$ as follows. If $D = (U_\alpha, f_\alpha) \in \mathrm{Div}(X)$ then it maps to $(f^{-1}(U_\alpha), f_\alpha \circ f)$. It makes sense if $f(X') \not\subset D$ as Weil Divisor. In terms of Weil divisors $\sum a_i Y_i \mapsto \sum a_i f^*(Y)$, $f^*(Y)$ lies over hypersurface $f^{-1}(Y)$ but perhaps with multiplicity.

d. $f^*(\mathcal{L}(D)) = \mathcal{L}(f^*(D))$

4.3.17 Divisors of Sections

For a global meromorphic function $f \in H^0(X, K_X^*)$, we defined $div(f)$. We now want to define a notion of $div(s)$ for a global meromorphic section of an arbitrary line bundle L $s \in H^0(X, L \otimes_{\mathcal{O}_X} K_X)$.

The following theorem explains well the reason behind studying $div(s)$.

4.3.18 Theorem. *Let L be a line divisor. Then, $L = \mathcal{L}(div(S))$.*

4.3.19 Corollary. $L \in Im\left(\delta : \dfrac{Div(X)}{Prin(X)} \to Pic(X)\right) \Leftrightarrow$ *L has a non-zero global meromorphic section.*

Let L be a line bundle on X. Fix $\{U_\alpha\}$ with trivializations $\phi_\alpha : L(U_\alpha) \to U_\alpha \times \mathbb{C}$ and transition functions $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$.

Note that:

a. A holomorphic (respectively, meromorphic) section of L over U open is given by a collection of functions $s_\alpha \in \mathcal{O}_X(U \cap U_\alpha)$ (respectively, $K_X(U \cap U_\alpha)$) satisfying $s_\alpha = g_{\alpha\beta} s_\beta$ on $U \cap U_\alpha \cap U_\beta$.

b. If s and s' are global meromorphic sections then $\dfrac{s}{s'}$ is a global meromorphic function. One consequence of this is if $D = div(s)$ and $D' = div(s')$ then $D - D' = div\left(\dfrac{s}{s'}\right) = div(f)$, which is a principal divisor hence, D and D' are linearly equivalent so, $\mathcal{L}(D) = \mathcal{L}(D')$.

Let s be a meromorphic section of L given by s_α. Then $\dfrac{s_\alpha}{s_\beta} = g_{\alpha\beta} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$. So, $ord_Y(s_\alpha) = ord_Y(s_\beta)$.

Therefore, we can define $div(s) = \sum_{U_\alpha \cap Y \neq \emptyset} ord(s_\alpha) Y$.

4.3.20 Example. Let $D = (U_\alpha, f_\alpha) \in Div(X)$. Then f_α gives a global section s for $\mathcal{L}(D)$. Also, $div(s)$ is the element of $WDiv(X)$. The theorem and corollary from before hold in this case.

4.3.21 Remark. D is called effective divisor if one of the following equivalent condition holds:

a. $D = \sum a_i Y_i$ $a_i \geq 0$

b. $D = (U_\alpha, f_\alpha),\ f_\alpha \in \mathcal{O}_X(U_\alpha)$

c. $D = div(s)$, where s is a global holomorphic section

4.3.22 Another interpretation of global holomorphic sections ($H^0(X,L)$) and effective divisors

Let $D = \sum a_i Y_i \in Div(X)$

4.3.23 Definition. a. $R(D) = \{f \in H^0(X, K_X^*) \mid div(f) + D \geq 0\}$. A function $f \in R(D)$ is holomorphic on $X \setminus \cup Y_i$ and $ord_{Y_i}(f) \geq -a_i$

b. The complete linear system of D, $|D| = \{E \in Div(X) \mid E \sim D, E \geq 0\}$

Let L be a line bundle, define $|L| := |D|$, if there exists a divisor D such that $\mathcal{L}(D) = L$.

4.3.24 Lemma. *Let $D = div(s_0)$ for a global meromorphic section s_0 of L. Then*

$$\begin{aligned} R(D) &\to H^0(X, L) \\ f &\mapsto f s_0 \end{aligned}$$

is an isomorphism.

Proof. The map is well defined because $div(fs_0) = div(f) + div(s_0) = div(f) + D \geq 0$. So, fs_0 is holomorphic. It is easy to see that the map is injective. Let s be a global holomorphic section. Define $f_s := \dfrac{s}{s_0}$, we have shown before that f_s is a global meromorphic function and $div(f_s) = div(s) - div(s_0) \geq -D$. So, $f_s \in R(D)$. Hence, the map is surjective. □

4.3.25 $R(D)$ vs $|D|$

Let $E \in |D|$, then $\exists\ f \in R(D)$ such that $E = D + div(f)$. If X is compact, any two such f are related by a multiplicative constant. This is because, if $E = D + div(f) = D + div(h)$ then $div\left(\dfrac{f}{h}\right) = 0$. Compactness implies that $\dfrac{f}{g}$ is constant.

4.3.26 Corollary. $|D| = \mathbb{P}(R(D)) \simeq \mathbb{P}(H^0(X, L))$. *Since, $R(D)$ is a vector space, $|D|$ is projective space.*

4.3.27 linear system

Recall that $|D| = \{E \geq 0 : \ E \sim D\}$ is called a **complete linear system** of D. Here we introduce a more general notion.

4.3.28 Definition. A **linear system** is a family of effective divisor $\mathcal{E} = \{D_\lambda\}_{\lambda \in I}$ corresponding to a linear subspace of $\mathbb{P}^N \simeq \mathbb{P}(H^0(X, L))$ for some L, i.e. $I \simeq \mathbb{P}^n$ for some $n \leq N$. That is, fix a subspace $V \subset H^0(X, L)$ and $\mathcal{E} = \mathbb{P}(V)$.

4.3.29 Definition. The **dimension** (or **rank**) of $\mathcal{E} = \{D_\lambda\}_{\lambda \in I \simeq \mathbb{P}^n}$ is the projective dimension n.

For example, $\dim |D| = \dim H^0(X, L) - 1$. We also have the following names:

- a **pencil** is a linear system of dimension 1
- a **net** is a linear system of dimension 2

- a **web** is a linear system of dimension 3

4.3.30 Definition. Let $\mathcal{E} = \{D_\lambda\}_{\lambda \in I} \subset |D|$ be a linear system. The **base locus** of $\mathcal{E}$ is defined to be

$$\cap_{\lambda \in I} D_\lambda := \{p \in X : \; p \in D_\lambda \; \forall \lambda \in I\} = \cap Supp D_\lambda.$$

Here $p \in D$ for $D = \sum a_i Y_i$ means $p \in \mathrm{Supp}(D) := \cup Y_i$.

We call a linear system **base point free** if its base locus is empty. Any divisor $F \subset \cap_{\lambda \in I} D_\lambda$ is called a **fixed component** of $\mathcal{E}$.

4.3.31 Theorem (Bertini)**.** *If D is a "generic" element of a linear system $\mathcal{E}$, then D is smooth away from the base locus.*

4.3.32 Maps to $\mathbb{P}^N$

4.3.33 Definition. Let $\mathcal{E} = \mathbb{P}(W)$ where X is a compact complex manifold W is a subspace of $H^0(X, L)$. We say that $\mathcal{E}$ is **base point free** if not all $s \in W$ (global holomorphic sections of L) vanish at any point $P \in X$. That is, there does not exist $P \in X$ such that for all $s \in W$ we have $s(P) = 0$.

Let X be a compact complex manifold. We have a one-to-one correspondence between non-degenerate maps $f : X \to \mathbb{P}^N$ modulo projective transformations and line bundles $L \in \mathrm{Pic}(X)$ with subspace $W \subset H^0(X, L)$ such that the linear system $\mathcal{E} = \mathbb{P}(N)$ is base-point free.

We are going to give explicit maps that lead to the above correspondence. For a given line bundle $L \in \mathrm{Pic}(X)$ and $W \subset H^0(X, L)$, there are three versions of defining maps from X to $\mathbb{P}^N$ which we will list them all below:

1. $f : X \to \mathbb{P}(W^*)$ given by
$$P \mapsto \{s \in W : s(P) = 0\}/\text{scaling}.$$
We denote $\{s \in W : s(P) = 0\}$ by $\tilde{H_P}$.

2. $f : X \to \mathbb{P}(W^*)$ given by
$$P \mapsto \{D \in \mathcal{E} : \; P \in D\}.$$
We denote $\{D \in \mathcal{E} : \; P \in D\}$ by H_P. Note that H_P is already scaled.

3. Let $s_0, \cdots, s_n$ be a basis for W. Then $f : X \to \mathbb{P}^n$ is given by
$$P \mapsto [s_0(P) : s_1(P) : \cdots : s_n(P)].$$
Note that this is an explicit map.

Now, suppose we are given $f : X \to \mathbb{P}^n$, we let $W = f^*(H^0(\mathbb{P}^n, \mathcal{L}(H)))$ where H is any hyperplane. Then $W \subset H^0(X, L)$ where $L = f^*(\mathcal{L}(H))$. It is easy to show that this construction is base point free.

Remark: The choice of $\{s_0, \cdots, s_n\}$ for W corresponds to the choice of $\{x_0, \cdots, x_n\}$ for $\mathbb{P}^n$. Also, note that $\deg(f(X)) = \int_X \bigwedge^n c_1(L) = (L^n) =: (D^n)$.

References

[BL04] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, Second, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004. MR2062673 (2005c:14001) ↑4

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034) ↑4

[CS86] Gary Cornell and Joseph H. Silverman (eds.), *Arithmetic geometry*, Springer-Verlag, New York, 1986. Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984. MR861969 (89b:14029) ↑4

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990. With an appendix by David Mumford. MR1083353 (92d:14036) ↑4

[Mil08] James S. Milne, *Abelian varieties (v2.00)*, 2008. Available at http://www.jmilne.org/math/CourseNotes/av.html. ↑4

[Mum07a] David Mumford, *Tata lectures on theta. I*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition. MR2352717 (2008h:14042) ↑4

[Mum07b] David Mumford, *Tata lectures on theta. II*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Jacobian theta functions and differential equations, With the collaboration of C. Musili, M. Nori, E. Previato, M. Stillman and H. Umemura, Reprint of the 1984 original. MR2307768 (2007k:14087) ↑4

[Mum07c] David Mumford, *Tata lectures on theta. III*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. With collaboration of Madhav Nori and Peter Norman, Reprint of the 1991 original. MR2307769 (2007k:14088) ↑4

[Mum08] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. MR2514037 (2010e:14040) ↑4