Math 7390 – Abelian Varieties

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Contents

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1 Introduction

1.1 About this Course

This is an introductory course on the analytic and algebraic theory of abelian (and jacobian) varieties. We will start with the classical complex-analytic case to build some intuition. Then we will discuss the general theory over other fields.

We will not follow any specific books, but the following resources were used while preparing for the lectures:

[BL04] [BLR90] [FC90] [Mil08] [Mum08] [Mum07a] [Mum07b] [Mum07c] [CS86]

1.2 What are Abelian Varieites?

Origins of the theory of abelian varieties comes from a basic question in calculus! We know we can easily compute integrals of the form

$$\int \frac{1}{\sqrt{1-x^2}}dx$$

by trigonometric substitution, but for integrals of the form

$$\int \frac{1}{\sqrt{f(x)}}dx$$

where $\deg(f) \geq 3$, it turns out to be hard. However, even though people couldn't compute these integrals, they could see that there were identities of the form

$$\int_0^a \frac{1}{\sqrt{f(x)}}dx + \int_0^b \frac{1}{\sqrt{f(x)}}dx = \int_0^{a*b} \frac{1}{\sqrt{f(x)}}dx$$

for some number $a * b$ obtained from a and b.

"Abelian Varieties are the simplest possible spaces, just tori's and thus groups" - D-Mumford.

Abelian Varieties are useful in the following areas:

- Number Theory - class field theory; rationality versus transcendence; most of the serious things we know how to do in number theory involve working with moduli of abelian varieties (Fermat's last theorem, Faltings' theorem, etc).
- Dynamical Systems - solutions to certain Hamilton systems.
- Algebraic Geometry - If we're given a variety X it's hard to understand, but we can get a handle on it by associating a canonical abelian variety $A(X)$ to it (such as Picard, Albanese, Intermediate Jacobian) and the good thing about $A(X)$ is that we can do a lot of linear algebra.
- Physics - theta functions that solve heat equations; string theory.

For the most part of this course, we will work over the base field $k = \mathbb{C}$; we see lots of very interesting ideas in this case, and we don't need any particularly hard theory to get a handle on it. One reason the theory over $\mathbb{C}$ is important is that an abelian variety A in a precise sense is just

$$A = A^{an} = \mathbb{C}^g/\Lambda$$

where $\Lambda = \pi_1(A) \simeq \mathbb{Z}^{2g}$ is a lattice, and so it is a torus. In other words, we have

$$0 \to \Lambda \to \mathbb{C}^g \to A \to 0.$$

Subtle remark: This identification makes sense in the "analytic category" but not in the algebraic category; the map $\mathbb{C}^g \to A$ is not algebraic. So we can't study abelian varieties in this way solely through algebraic methods. In the analytic setting it's "easy" to understand line bundles, theta functions, etc. by going to $\mathbb{C}^g$. (You can make sense of an analytification in nonarchimedean settings too, by using Berkovich spaces or formal schemes; this requires a lot more background but provides many important results.) Fortunately, there are some things from the complex-analytic setting which can be mimicked in the algebraic setting (e.g. the lattice Λ can be related to the Tate module) and by using those algebraic analogues you can take the complex-analytic results over $\mathbb{C}$ and try to reproduce them over other fields.

Some more advanced topics that might be covered in detail later in the class include (depending on audience interest):

- The theory over general field.
- Theta functions.
- Neron models.
- Non-archimedean uniformizations.
- Moduli and compactifications.
- Heights and metrized line bundles.
- Degenerating families.

Now, let's get to actual math. In scheme-theoretic language - for k any field, a *k-variety* is a geometrically integral k-scheme of finite type, and an *abelian variety* over k is a proper k-variety endowed with a structure of a k-group scheme. (This is the schematic definition. But we will not do it this way in this class.) We will be able to prove the following:

1.2.1 Theorem. *Abelian Varieties are automatically abelian and projective.*

Being abelian is easy to show, while being projective is much harder.

1.3 Why study Abelian Varieties?

More generally we can define an *algebraic group G* over k as a connected, smooth k-group scheme. Examples include:

- affine algebraic groups (automatically subgroups of $\mathrm{GL}_n(k)$, i.e. linear algebraic groups)
- abelian varieties (think of this as the projective case)

The following theorem says that these are the only building blocks:

1.3.1 Theorem (Chevalley's Theorem)**.** *Let G be an algebraic group over k. Suppose k is perfect. Then there exists a unique short exact sequence*

$$0 \to H \to G \to A \to 0$$

where H is linear and $A = G/H$ is abelian.

A proof can be found in B. Conrad's notes.

1.4 Some history

In the 1850s, Weierstrass studied $E = E^{an} = \mathbb{C}/\Lambda$, which is a complex group (2-dimensional torus with complex multiplication). He asked whether E^{an} is always "algebraic/algebraizable", and showed the answer is actually yes. In fact, he proved more.

1.4.1 Theorem. *$E = E^{an}$ has the structure of a smooth projective curve of genus 1. Its affine equation is given by*

$$y^2 = 4x^3 - 60G_4x - 140G_6$$

where $G_m = \sum_{\lambda \in \Lambda^*} \frac{1}{\lambda^m}$ for $m \in \mathbb{Z}$. More precisely, you can write down the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda^*} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

and compute

$$y = \frac{dx}{dz} = \sum_{\lambda \in \Lambda} \frac{-2}{(z-\lambda)^3},$$

and see that the pair $(x, y) = (\wp(z), \wp'(z))$ satisfies the affine equation above. The mapping given by

$$z + \Lambda \mapsto (\wp(z), \wp'(z))$$

induces a group isomorphism $\mathbb{C}/\Lambda \simeq E$ where E is the projectivized elliptic curve.

What about higher dimensions? One direction is false: a general torus $\mathbb{C}^g/\Lambda$ will not be algebraic. However, the converse is true; a general abelian variety A will be (after analytification) of the form $\mathbb{C}^g/\Lambda$, where $g = \dim(A)$. We will prove this later, but it is not easy.

1.4.2 Theorem. *The Weierstrass parametrization gives a bijection between lattices Λ in $\mathbb{C}$, and the set of isomorphism classes of pairs (E, ω) where $E/\mathbb{C}$ are elliptic curves and ω are holomorphic differential forms.*

Note that $\omega \in H^0(E^{an}, \Omega)$ corresponds to $f(z)dz$ on $\mathbb{C}$, where f is periodic and holomorphic with Ω. What happens if we replace Λ with $a\Lambda$? It turns out we can say it corresponds to $(E, a\omega)$. Also note that the map $\mathbb{C} \to \mathbb{C}/\Omega$ is not algebraic. ($\mathbb{C}^g/\Omega$ for $g = 1$ is algebraic, while for $g \geq 2$ is usually not algebraic.) [farbod fix]

2 Complex Tori

2.1 Some GAGA principles

Algebraic varieties vs. (complex) analytic spaces.

If X is an algebraic variety over $\mathbb{C}$, we can associate X^{an} which is a complex analytic space to it, by passing to a complex analytification; if X is locally described by some set of equations in affine space, we can pass that open set the zero locus as a subset of $\mathbb{C}^n$ (with its usual topology) and glue. Here are some facts:

- This construction is always functorial: an algebraic map $X \to Y$ can always be lifted to a holomorphic map $X^{an} \to Y^{an}$.
- X is proper/complete if and only if X^{an} is compact.
- X is smooth/connected if and only if Xan is smooth/connected.
- A complex analytic space $\mathfrak{X}$ is called **algebraic/algebraizable** if there exists a variety $X/\mathbb{C}$ such that $\mathfrak{X} \simeq X^{an}$. (Last time we explicitly showed that $\mathbb{C}/\Lambda$ is algebraic via the Weierstrass $\wp$-functions.)

2.2 Vector bundles and associated locally free sheaves

If L is a vector bundle on X, then it passes to an analytic vector bundle L^{an} on X^{an}. This is functorial in the sense that if $f : F \to G$ is a morphism of vector bundles it passes to $f^{an} : F^{an} \to G^{an}$. It is *not* true that all holomorphic vector bundles over X^{an} are algebraizable! But we have:

2.2.1 Theorem (Serre)**.**

1. *Suppose X is a proper (complete) algebraic variety over $\mathbb{C}$. If there exists a holomorphic coherent sheaf $\mathcal{F} \to X^{an}$, then there exists unique algebraic coherent sheaf F over X such that $F^{an} = \mathcal{F}$.*
2. *If there exists $\mathfrak{F} : \mathcal{F} \to \mathcal{G}$ homomorphism of holomorphic coherent sheaves on X^{an}, then there exists unique $f : F \to G$ such that $f^{an} = \mathfrak{f}$.*

Define $H^i(X, L)$ as the i-th cohomology group with values in the locally free sheaf L.

2.2.2 Theorem (Serre)**.** *Let X be a complete algebraic variety over $\mathbb{C}$ and F a coherent sheaf on X. Then the* natural maps

$$H^i(X, F) \to H^i(X^{an}, F^{an})$$

are isomorphisms of $\mathbb{C}$-vector spaces.

2.3 Complex tori

Let V be a vector space over $\mathbb{C}$, and $\Lambda \subset V$ a lattice (full rank discrete subgroup). We have Λ act naturally on V by addition; and then the quotient $X = V/\Lambda$ is a complex torus.

Some facts about a complex torus:

- it is a complex manifold.
- it inherits the structure of a complex Lie group over $\mathbb{C}$.
- it is compact (because Λ is a maximal rank lattice).
- it is an abelian complex Lie group.
- meromorphic functions on X correspond to meromorphic Λ-periodic functions on V.

Loosely speaking, an (complex analytic) *abelian variety* is a complex torus with "sufficiently many" (enough to give a closed embedding to a projective space) meromorphic functions. We will see that this is exactly what makes X algebraizable and thus an algebraic abelian variety.

2.4 Compactness implies abelian

2.4.1 Theorem. *Any connected* compact *complex Lie group X is a complex torus.*

Proof. First, X is abelian and so the commutator map $\Phi(x, y) = xyx^{-1}y^{-1}$ is continuous. Let U be any neighbourhood of the identity element 1, for $x \in X$, define open neighbourhoods $V_x, \tilde{V_x}$ such that $x \in V_x$, $1 \in \tilde{V_x}$ and $\Phi(V_x, \tilde{V_x}) \subset U$. (This can be done since $\Phi(x, 1) = 1$ and Φ is continuous.)

So we have $X = \cup_{x \in X} V_x$ and by compactness, there exist $x_1, \cdots, x_r \in X$ such that

$$X = \bigcup_{x \in \{x_1, \cdots, x_r\}} V_x.$$

Let $W = \cap_{x_1, \cdots, x_r} \tilde{V_x}$, which is a non-empty open neighbourhood of 1. So $\Phi(X, W) \subset U$. Since U is arbitrary, we have $\Phi(X, W) = 1$.

Since holomorphic functions on a compact set X which is bounded must be constant, we have $\Phi(1, y) = 1$ for all $y \in W$. Since W is open and non-empty, by connectivity,

$$\Phi(x, y) = 1$$

for all $x, y \in X$.

Then, if $\pi : V \to X$ is a universal cover, V inherits the structure of a simply connected complex Lie group and thus must be $\mathbb{C}^g$. Moreover π is homomorphic with discrete kernel, and by compactness of X the kernel must be full rank. $\square$

Another proof can be found in B. Conrad's notes.

Remarks: Once we have $X = V/\Lambda$, we see that V is a universal cover of X. Moreover, $\Lambda = \pi(X, 0)$, and since this is already abelian it is isomorphic to $\simeq H_1(X, \mathbb{Z})$. Since X is locally isomorphic to V, we can view V as the tangent space at 0, T_0X; then the covering map $\pi : V = T_0X \to X$ is actually the exponential map.

2.5 Period matrix

Given $X = V/\Lambda$, we can associate Π a $g \times 2g$ complex matrix: fix $\{e_1, \cdots, e_g\}$ a $\mathbb{C}$-basis for V and $\{\lambda_1, \cdots, \lambda_{2g}\}$ a $\mathbb{Z}$-generator set for Λ. Define λ_{ji} such that

$$\lambda_j = \sum \lambda_{ji} e_i.$$

Then the **period matrix** of X is given by

$$\Pi := \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2g} \\ \vdots & \ddots & \vdots \\ \lambda_{g,1} & \cdots & \lambda_{g,2g} \end{pmatrix}.$$

Clearly, Π determines X but it depends on the choices.

Question: Given $\Pi \in M_{g\times 2g}(\mathbb{C})$, is there a complex torus such that Π is the period matrix of X?

2.5.1 Theorem. *Let* $P = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}_{2g\times 2g}$*, where* $\overline{\Pi}$ *denote the complex conjugate matrix of* Π*. Then* Π *is the period matrix for some* $\mathbb{C}^g/\Lambda$ *if and only if* P *is nonsingular.*

Proof. Π is a period matrix if and only if the columns of Π are $\mathbb{R}$-linearly independent. $\square$

2.6 Holomorphic maps, homomorphism and isogenies

Suppose $X = V/\Lambda$ and $X' = V'/\Lambda'$ with dimensions g and g' respectively. We want to study holomorphic maps $f : X \to X'$. There are two special examples:

1. homomorphisms (holomorphic and respect group structure); and
2. translations (maps $X \to X$ by $x \mapsto x + x_0$ for some $x_0 \in X$).

The surprising thing is that that's *all*!

2.6.1 Theorem. *Suppose* $h : X \to X'$ *is a holomorphic map between complex tori. Then*

1. *there exists a unique homomorphism* $f : X \to X'$ *such that* $h = t_{h(0)} \circ f$*. That is,*
$$h(x) = f(x) + h(0)$$
for all x*.*
2. *There exists a unique* $\mathbb{C}$*-linear map* $F : V \to V'$ *with* $F(\Lambda) \subset \Lambda'$ *inducing* f*.*

Proof. Let $f := t_{-h(0)} \circ h$. Then we can lift $f \circ \pi : V \to X$ to $F : V \to V'$ where V' is the universal cover of X'. Then F is holomorphic and satisfies $F(0) = 0$. F is a $\mathbb{C}$-linear map: fix $\lambda \in \Lambda$, by construction,

$$F(v + \lambda) - F(v) \in \Lambda'$$

and so by continuity it's constant. Therefore,

$$F(v + \lambda) = F(v) + F(\lambda)$$

for all $v \in V, \lambda \in \Lambda$. We skip the remaining details. $\square$

2.7 Hom-sets

Let $\mathrm{Hom}(X, X')$ be the set of all homomorphisms $f : X \to X'$. It is an abelian group. If $X = X'$, then we can define $\mathrm{End}(X) := \mathrm{Hom}(X, X')$. In this case, $\mathrm{End}(X)$ is actually a ring, where multiplication is given by composing endomorphisms. The above theorem gives us the following corollary:

2.7.1 Corollary. *We have injective homomorphisms:*

1. $\rho_{an} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ *given by* $f \mapsto F$*; and*
2. $\rho_{int} : \mathrm{Hom}(X, X') \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda')$ *given by* $f \mapsto F|_\Lambda$.

Note that both of these homomorphisms respect endomorphism ring structures if $X = X'$: $\rho_*(f' \circ f) = \rho_*(f') \circ \rho_*(f)$, where $* = an, int$.

2.7.2 Theorem. $\mathrm{Hom}(X, X') \simeq \mathbb{Z}^m$ *for some* $m \leq 4gg'$.

Proof. Use the second isomorphism in the corollary, since $\Lambda \simeq \mathbb{Z}^{2g}$ and $\Lambda' \simeq \mathbb{Z}^{2g'}$, so $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \Lambda') \simeq \mathbb{Z}^{4gg'}$ and $\mathrm{Hom}(X, X')$ embeds in this. □

How do these relate to period matrices? Let Π and Π' be the period matrix for X and X' respectively. If we have $f : X \to X'$, then by picking bases we get that $\rho_{an}(f) : V \to V'$ is given by some $A \in M_{g' \times g}(\mathbb{C})$ and $\rho_{int}(f) : \Lambda \to \Lambda'$ given by some $R \in M_{2g' \times 2g}(\mathbb{Z})$. Then the condition $F(\Lambda) \subset \Lambda'$ means $A\Pi = \Pi' R$. (The converse is also true: given four matrices with this property, then they correspond to a morphism between complex tori.)

What if $X = X'$? In this case, we can get

$$\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} R$$

and thus $\rho_{int} \otimes 1 \simeq \rho_{an} \oplus \bar{\rho_{an}}$ in $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

2.8 Kernels and Images

2.8.1 Lemma. *Given a homomorphism* $f : X \to X'$.

1. $Im(f)$ *is a complex subtorus of* X'.
2. $\ker(f)$ *is a closed subgroup of* X *with finitely many component. The connected component of* $1 = id$ *is a complex torus.*

The proof is fairly easy; for part (b) we're claiming that we have an extension

$$1 \to X_0 \to G \to \Gamma \to 1$$

with X_0 a complex torus and Γ a finite abelian group. It is a good exercise to describe Γ as a direct sum of cyclic groups in terms of Π, Π', A, R (need to compute a Smith normal form somewhere).

2.9 Isogenies

A homomorphism $f : X \to X'$ is called an **isogeny** if f is surjective with finite kernel. Equivalently, f is surjective and $\dim(X) = \dim(X')$.

2.9.1 Example (Essential example)**.** Suppose $X = V/\Lambda$ is a complex torus and $\Gamma \subset X$ is a finite subgroup. Then $X/\Gamma = V/\pi^{-1}(\Gamma)$ is a complex torus and $X \to X/\Gamma$ is an isogeny.

In fact, that's all! It is an easy exercise to show that all isogenies $X \to X/\Gamma$ over $\mathbb{C}$ are of this form. We also have the following easy lemma:

2.9.2 Lemma (Stein factorization)**.** *Any surjection $f : X \to X'$ of complex tori factors as a surjection $X \to X/(\ker f)_0$ (a quotient of X by a complex subtorus) and an isogeny $X/(\ker f)_0 \to X'$.*

Remark: Stein factorization is a special case of a very general result (in complex theory by Stein and others, and in general in EGA III): any proper $f : X \to S$ factors as a map $X \to S'$ proper with connected fibers and $S' \to S$ finite.

For $f \in \operatorname{Hom}(X, X')$, we define $\deg(f)$ to be $|\ker f|$ if this is finite, and 0 if otherwise. It is easy to check that $\deg(f) = [\Lambda' : \rho_{int}(f)\Lambda]$. (Remark: If $X = X'$ then this index is $\deg(\rho_{int}(f))$; note that this determinant is ≥ 0 since $\rho_{int} \otimes 1 = \rho_{an} \oplus \bar{\rho}_{an}$, and is 0 if and only if the kernel is infinite.)

2.9.3 Lemma. *Supppose $f : X \to X'$ and $f' : X' \to X''$ are isogenies, then $f' \circ f$ is also an isogeny.*

Proof. $\deg(f' \circ f) = \deg(f) \cdot \deg(f')$. □

A very important example is given by the "multiplication-by-n" map: Let $n \in \mathbb{Z}^+$, define $n_X : X \to X$ by $x \mapsto nx$. Denote $X[n] := \ker(n_X)$ the set of n-torsions in A. Then we have

$$X[n] \simeq \frac{\frac{1}{n}\Lambda}{\Lambda} \simeq \frac{\Lambda}{n\Lambda} \simeq (\mathbb{Z}/n)^{2g}.$$

Therefore, n_X has degree n^{2g}, and so it is an isogeny.

2.9.4 Corollary. *Complex tori are divisible groups.*

2.9.5 Example (Tate module)**.** Let ℓ be a prime number. Define multiplication by ℓ maps $X[\ell^{n+1}] \to X[\ell^n]$. Then the *Tate module* is given by

$$T_\ell(X) = \varprojlim X[\ell^n].$$

In the case where Λ is finitely generated, $T_\ell(X)$ is actually isomorphic to $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and this is a subset of Λ. Note that the definition of $T_\ell(X)$ makes sense over any fields (even if we don't have Λ when we are not over $\mathbb{C}$). Here in our setting it's easy to see that a morphism $X \to X'$ is determined by the induced map $T_\ell(X) \to T_\ell(X')$. Over general fields this is much, much harder! It's the *Tate conjecture* which says that we have

$$\operatorname{Hom}_{\mathrm{Gal}}(T_\ell(X), T_\ell(X')) \simeq \operatorname{Hom}(X, X').$$

This conjecture was only proven over number fields by Faltings as an essential part of his proof of the Mordell conjecture.

2.10 Importance of isogenies

They are "almost isomorphisms". Namely, we have:

2.10.1 Theorem. *Let $f : X \to X'$ be an isogeny and n be the exponent of* $\ker(f)$*. (That is, $nx = 0$ for all $x \in \ker(f)$.) Then there exists an isogeny $g : X' \to X$ such that*

$$f \circ g = n_X, \ \ g \circ f = n_X.$$

Moreover, such a g is unique (up to isomorphism?).

sketch. Since n is the exponent of $\ker f$, we have $\ker(f) \subset \ker(n_X) = X[n]$. Then there exists a unique $g : X' \to X$ with $g \circ f = n_X$, defined by $g(x') := nx$ for some (all) x where $f(x) = x'$. Then use the fact that $\deg(g)\deg(f) = \deg(n_X)$ and that $\deg(f), \deg(n_X) \neq 0$, so $\deg(g) \neq 0$ to get that g is an isogeny; Then we just need to check that $g \circ f = n'_X$. ☐

Define $\mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes \mathbb{Q}$ and $\mathrm{Hom}_{\mathbb{Q}}(X, X') := \mathrm{Hom}(X, X') \otimes \mathbb{Q}$. Then the degree function extends to these via

$$\deg(rg) := r^{2g} \cdot \deg(f).$$

2.10.2 Corollary.

1. *Isogeny is an equivalence relation.*
2. *$f \in \mathrm{End}(X)$ is an isogeny if and only if it is invertible in $\mathrm{End}_{\mathbb{Q}}(X)$.*

2.11 Cohomology

We have a lot of cohomology theories (Betti, de Rham, Dolbeault, Hodge decomposition, ...).

Betti cohomology is just singular cohomology of $X(\mathbb{C})$; if $X = V/\Lambda$, then we have the following facts:

- $\Lambda = \pi_1(X_0) \simeq H_1(X, \mathbb{Z})$.
- By the universal coefficient theorem, we have $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.
- If $n \geq 1$, we have a map $\wedge_{i=1}^n H^1(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ induced by cup product, and this is an isomorphism (follows from Kunneth formula).
- Let $Alt^n(\Lambda, \mathbb{Z}) := \bigwedge_{i=1}^n \mathrm{Hom}(\Lambda, \mathbb{Z})$ be all the $\mathbb{Z}$-valued alternating n-forms. Then we have $H^n \simeq Alt^n(\Lambda, \mathbb{Z})$. This gives a very explicit way of thinking about cohomology.
- $H_n(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$ are free $\mathbb{Z}$-modules of rank $\binom{2g}{n}$.
- If we set $H^n(X, \mathbb{C}) := H^n(X, \mathbb{Z}) \otimes \mathbb{C}$, then we have

$$H^n(X, \mathbb{C}) \simeq Alt^n_{\mathbb{R}}(V, \mathbb{C}) = \bigwedge_{i=1}^{n} \mathrm{Hom}_{\mathbb{R}}(\Lambda, \mathbb{C}) \simeq \bigwedge_{i=1}^{n} H^1(X, \mathbb{C}),$$

  and the de Rham theorem tells us $H^n(X, \mathbb{C}) \simeq H_{DR}(X)$ where $H_{DR}(X)$ can be explicitly described as a complex vector space of invariant n-forms with basis $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$ with $i_1 < \cdots < i_n$.

Now, we use the $\mathbb{C}$-structure (really everthing is true for Kahler manifolds, but proofs and constructions are much more elementary for complex tori). Here we have a very nice decomposition

$$H^n(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(\Omega_X^p)$$

Here, $H^{p,q}(X) := H^q(\Omega_X^p)$ is isomorphic to the Dolbeault cohomology $H^{p,q}(X)$. In general, $H^q(\Omega_X^p)$ can be explicitly described as $\bigwedge^p \Omega \otimes \bigwedge^q \overline{\Omega}$ for $\Omega = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{\Omega} = \mathrm{Hom}_{\overline{\mathbb{C}}}(V,\mathbb{C})$. Also set $\Omega_X^p := (\bigwedge^p \Omega) \otimes \mathcal{O}_X$. Right now we are not saying anything about what these vector spaces $H^{p,q}(X)$ are, but we will do that later on. We may also need ot return to this theory to prove for example vanishing results later; We will either do that, or just omit those proofs.

2.12 Sheaves on a topological space X

Let $\mathcal{O}(X)$ be the category in which objects are open subsets of X and morphisms are inclusions $V \to U$ for $V \subset U$. (It turns out that you can generalize this and allow more general things for morphisms than just inclusions; this is how you get etale cohomology and other things.) Let $\mathcal{C}$ be any other category (for instance, Sets, Abelian groups and R-modules); A **presheaf** is a contravariant functor

$$F : \mathcal{O}(X) \to \mathcal{C}.$$

(This means for each inclusion map $i : V \subset U$, we have the restriction map given by $rest_{V,U} : F(U) \to F(V)$, and this assignment is functorial.) A **sheaf** is a presheaf F with some "locality" and "gluing" properties. Assume $\mathcal{C}$ has products. Then we require for any open cover $\{U_i\}$ of $U \in \mathcal{O}(X)$, we have an exact sequence

$$0 \to F(U) \stackrel{rest}{\to} \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

(where the two parallel maps are "restriction to the first index" and "restriction to the second index" respectively.)

2.12.1 Example. Let $X = \mathbb{C}$. Then we have the sheaf of holomorphic functions: $F(U)$ is the set of all holomorphic functions $U \to \mathbb{C}$, and restriction maps are restriction of functions!

Morphisms of sheaves are natural transformations; this gives us the category of sheaves over X, Shf_X, in which the objects are sheaves on X and morphisms are these.

Next, a **ringed space** is a topological space X together with a sheaf of rings $\mathcal{O}_X$; we call $\mathcal{O}_X$ the **structure sheaf** (usually some sheaf of holomorphic functions in this class). Define a **locally ringed space** to be one such that all of the stalks $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} F(U)$ are local rings. We may want to consider sheaves of $\mathcal{O}_X$-modules, that is, each $F(U)$ is a $\mathcal{O}_X(U)$-module respecting restriction maps.

2.13 Abelian categories and cohomology

(Grothendieck's Tohoku paper) There are 4 main examples of abelian categories to keep in mind:

1. The category of abelian groups (with homomorphisms).
2. The category of R-modules for R a commutative ring (with homomorphisms).
3. The category of G-modules for G a group (with G-equivariant homomorphisms: $\phi(g \cdot m) = g \cdot \phi(m)$).
4. The category of $\mathcal{O}_X$-modules for $(X, \mathcal{O}_X)$ a ringed space (with morphisms of sheaves the morphisms).

In general there's an abstract definition of an abelian category; we are not going to say it precisely but roughly it is a category $\mathcal{A}$ in which addition of morphisms, a zero object, kernels and cokernels make sense.

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a covariant functor between abelian categories. If we start with an short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, we can hit it with the functor F and get maps in $\mathcal{B}$, but no guarantee for exactness. We say that F is **left exact** if for every such short exact sequence we do have

$$0 \to F(A) \to F(B) \to F(C)$$

exact.

2.13.1 Example. Let $\mathcal{A}$ be the category of R-module, $\mathcal{B}$ be the category of abelian groups and D be a fixed object. Then $F(-) = \mathrm{Hom}_R(D, -)$ is a left exact covariant functor.

Cohomology lets us study the failure of exactness of

$$0 \to F(A) \to F(B) \to F(C),$$

i.e. the failure of surjectivity of $F(B) \to F(C)$. We want to continue the above exact sequence to the right and write a long exact sequence. In general, there are many ways to do this. But if $\mathcal{A}$ has "enough injectives" (a statement that holds for the categories we care about), then there is a "canonical" and "minimal" (in the sense that there is a universal property) way to do this. There exist unique functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ with $R^0F = F$ that give us the following long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \xrightarrow{c_1} R^1F(A) \to R^1F(B) \to R^1F(C) \xrightarrow{c_2} R^2F(A) \to \cdots$$

plus satisfying some universal properties (an "effaceable δ-functor"). These functors are called the *right derived functors.* This is the covariant version; there's a similar one for contravariant functors. In general, it is hard to compute these R^iF. However, in the examples that we are interested in, there are easier ways of computing them.

2.13.2 Example. Let $F(-) := \mathrm{Hom}_R(-, D)$ and $G(-) := \mathrm{Hom}_R(D, -)$ where D is a fixed R-module. Both F and G are left-exact functors from R-modules to abelian groups, one contravariant and one covariant. More precisely, for any short exact sequence of R-modules

$$0 \to L \to M \to N \to 0,$$

we have the following exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D)$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N).$$

In both cases, the derived functors are just the Ext groups $\mathrm{Ext}^i_R(-, D)$ and $\mathrm{Ext}^i_R(D, -)$ which give us the long exact sequences

$$0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(L, D) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^1_R(M, D) \to \cdots$$

and

$$0 \to \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M) \to \mathrm{Hom}_R(D, N) \to \mathrm{Ext}^1_R(D, L) \to \mathrm{Ext}^1_R(D, M) \to \cdots$$

So $R^i \mathrm{Hom}_R(-, D) = \mathrm{Ext}^i_R(-, D)$ and $R^i \mathrm{Hom}_R(D, -) = \mathrm{Ext}^i_R(D, -)$ (it is a nontrivial fact that these agree!).

2.13.3 Example. In the case of G-modules, let A be an abelian group A with a G action $\phi : G \to \mathrm{Aut}(A)$, (that is, A is a $\mathbb{Z}G$-module) with morphisms respecting G-action. Let A^G be the group of $x \in A$ such that $g \cdot x = x$ for all $g \in G$. The functor we want in this case is $F(A) = A^G$ which takes G -modules to abelian groups. Note that this functor is the same as $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$, so its (left exact) derived functors are (abstractly) Ext-functors $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, -)$, but this doesn't really give us a good way to compute it. These are better known as the "group cohomology of G with coefficients in A", denoted $H^i(G, A)$.

How do we compute this? It turns out that $\mathbb{Z}$ has a very nice "standard resolution" (also called the "bar resolution"), which is given by

$$F_n = \bigotimes_{i=0}^{n} \mathbb{Z}G.$$

Using this, we get the following recipe: for G a group and A a G-module, we let $C^0(G, A) = A$ and $C^n(G, A)$ be the group of A-valued maps on $G^n = G \times \cdots \times G$ for $n \geq 1$. These C^n are called the group of n**-cochains** of G with values in A. We also define the **differential operators** $d_n : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \cdots, g_{n+1}) := {} & g_1 \cdot f(g_2, \cdots, g_{n+1}) \\ & + \Sigma_{i=1}^n (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \cdots, g_n). \end{aligned}$$

The cases of most interest are $n = 0, 1, 2$:

- when $n = 0$, we have $f = a \in C^0(G, A) = A$, and then we get

$$d_0(f)(g) = g \cdot a - a.$$

- when $n = 1$, then f is a function with one input and we have

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

- when $n = 2$, then f is a function with two inputs and we have

$$d_2(f)(g,h,k) = g \cdot f(h,k) - f(gh,k) + f(g,hk) - f(g,h).$$

  Amazingly, people wrote down this formula correctly before the general theory came out.

It can be shown that $d_n \circ d_{n+1} = 0$. Thus, we can define the group of n-**cocycles** as $Z^n(G,A) := \ker(d_n)$ for $n \geq 0$ and the group of n-**coboundaries** as $B^n(G,A) := \mathrm{im}(d_{n-1})$ for $n \geq 1$ (or $B^0(G,A) = 1$ when $n = 0$). Then we define the n**-th cohomology group** as

$$H^n(G,A) := Z^n(G,A)/B^n(G,A).$$

Note that $H^0(G,A) = A^G$.

2.13.4 Example. Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathcal{F}$ a $\mathcal{O}_X$-module. Let $\Gamma(-,X)$ be the "global sections" functor, $\mathcal{F} \mapsto \mathcal{F}(X)$, from $\mathbb{O}_X$-modules to R-modules for $R = \mathcal{O}_X(X)$. This functor is left-exact (to make sense of this we need to make sure we know what exact sequences of sheaves are - defining "kernels" is easy but to define "images" we need to sheafify).

Here there is the Grothedieck (right) derived cohomology functor, $R^i\Gamma(-,X)$, which we denote $H^i(X,-)$ (we also call $H^i(X,\mathcal{F})$ the *sheaf cohomology of X with values in $\mathcal{F}$*); in particular, we have $H^0(X,-) = \Gamma(-,X)$. This is given abstractly by the theory earlier in this section; but like group cohomology, we will really need to work with sheaf cohomology, so we need to a way to compute them explicitly.

Here, we have a more concrete one, which is called the *Čech cohomology*, $\check{H}^i(X,\mathcal{F})$. In general, the two cohomologies are not equal! Fortunately they are equal in the settings we will be interested in.

The Čech cohomology is more explicitly computable, and always gives us a map $\phi : H^i(X,\mathcal{F}) \to \check{H}^i(X,\mathcal{F})$. We have:

1. For $i = 0, 1$, ϕ is an isomorphism.
2. (Grothendieck): If X is a Noetherian, separated scheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then ϕ is an isomorphism.
3. (Godement): If X is a paracompact and Hausdorff topological space, then ϕ is an isomorphism.

However, ϕ is not an isomorphism in general! Here are some counterexamples:

- In his Tôhoku paper (p.177), Grothendieck provides a counterexample where $H^2 \neq \check{H}^2$, for $X = \mathbb{A}^2$ with the Zariski topology and $\mathcal{F}$ comes from taking $\underline{\mathbb{Z}}$ and modifying it based on a space Y that is a union of two circles. This example is explicit but the proof is somehow deep!
- A recent paper of Schröer (arxiv post 1309.2524) gives a Hausdorff (not not paracompact) topological space constructed from 2-dimensional discs that is a counterexample.

2.13.5 Čech cohomology

So how do we define Čech cohomology? The idea is that if $\mathcal{U}$ is an open cover on X, the "nerve" of $\mathcal{U}$ approximates X. We define a q-simplex σ of $\mathcal{U}$ as an ordered collection of $q+1$ elements in $\mathcal{U}$ with nonempty intersection that we call $|\sigma|$. Suppose $\sigma = (U_i)$ (for $0 \leq i \leq q$), we define $\partial_j \sigma := (U_i)_{i \neq j}$ and then $\partial\sigma = \Sigma_{j=0}^{q}(-1)^{j+i}\partial_j\sigma$. Note that $|\sigma| = \cap U_i$.

We then define q-**cochains** of $\mathcal{U}$ with coefficients in $\mathcal{F}$ to be the set $C^q(\mathcal{U}, \mathcal{F})$ of functions $\sigma \mapsto f_\sigma \in \mathcal{F}(|\sigma|)$. Therefore, we get the **boundary maps** $C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by

$$(\delta_q\omega)(\sigma) = \Sigma_{j=0}^{q+1}(-1)^j \mathrm{res}_{|\partial_j\sigma|,|\sigma|}\omega(\partial_j\sigma).$$

One can check that $\delta_{q+1} \circ \delta_q = 0$ and hence can define **cocycles** $Z^q(\mathcal{U}, \mathcal{F}) := \ker(\delta_q)$ and **coboundaries** $B^q(\mathcal{U}, \mathcal{F}) := Im(\delta_{q-1})$, and then the **Čech cohomology** of $\mathcal{U}$ is given by

$$\check{H}^q(\mathcal{U}, \mathcal{F}) := Z^q(\mathcal{U}, \mathcal{F})/B^q(\mathcal{U}, \mathcal{F}).$$

So this gives us the cohomology $\check{H}^i(\mathcal{U}, \mathcal{F})$ of an open cover $\mathcal{U}$; However, we want a cohomology $\check{H}^i(X, \mathcal{F})$ associated to the whole space X! There are two ways to solve this:

1. If X has a "good" cover $\mathcal{U}$ (with all finite intersections of U_i to be contractible), then $\check{H}^i(\mathcal{U}, \mathcal{F})$ is canonical.

2. In general, we can define $\check{H}^i(\mathcal{X}, \mathcal{F})$ as $\varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$. But then we need to make sense of this direct limit. (It might be over an index set that is a proper class?) [fix farbod]

Remark: Let G be a topological group. Let $BG = K(G, 1)$ be the *Eilenberg-MacLane space*. (For example, $B\mathbb{Z} = S^1$.) Note that $\pi_1 = G$ and $\pi_n = 0$ for all $n > 1$. Then if A is a G-module, the sheaf cohomology $H^n(BG, \underline{A})$ (here $\underline{A}$ is the constant sheaf, which is the sheafification of the constant presheaf) is isomorphic to the usual CW complex cohomology $H^n(BG, A)$ and to the group cohomology $H^n(G, A)$, if A has a trivial G-action. (If A has a nontrivial G-action we can still make sense of this but the $H^n(BG, \underline{A})$ needs to be reinterpreted in terms of "local coefficient systems".)

2.14 Back to complex tori (sort of)

We want to understand line bundles on a locally ringed space $(X, \mathcal{O}_X)$. For now, we can think of it as a complex manifold or a variety. Let $\mathcal{F}$ be a sheaf. We call $\mathcal{F}$ (globally) **free** if $\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_X$ is the direct sum of copies of the structure sheaf; r is called the **rank** of $\mathcal{F}$. $\mathcal{F}$ is called **locally free** if there exists an open cover $\{U_i\}$ such that each $\mathcal{F}|_{U_i}$ is free. (There is a correspondence between locally free sheaves $\mathcal{F}$ of rank n and vector bundles of rank n. If $\pi : E \to X$ is a vector bundle, we get a locally free sheaf with $\mathcal{F}(U)$ being the sections of π over U; conversely, if $\mathcal{F}$ is locally free, we can construct an associated line bundle as $\coprod U_i \times \mathbb{C}^n$ modulo gluing data.)

2.14.1 Line bundles

Line bundles are vector bundles of rank 1 (equivalently, locally free sheaves of rank 1). Our goal is to give a cohomological interpretation on the set of line bundles. Let $\pi : L \to X$ be a line bundle. Let $\{U_\alpha\}$ be an open cover with trivializations given by (holomorphic) $\phi_\alpha : L|_\alpha \xrightarrow{\sim} U_\alpha \times \mathbb{C}$ where $L|_\alpha := \pi^{-1}[U_\alpha]$. Define the **transition functions** for L with respect to $\{\phi_\alpha\}$ as $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ which are given by

$$g_{\alpha\beta}(z) := \phi_\alpha \circ \phi_\beta^{-1}|_{L_z}$$

where $L_z := \{z\} \times \mathbb{C}$. By itself this is a linear (and hence holomorphic) map on $\{z\} \times \mathbb{C}$, which is determined by the complex number we are calling $g_{\alpha\beta}(z)$. We check that $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$ (so it is nonzero) and also $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = 1$. Rewriting this latter condition gives a cocycle condition

$$g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1.$$

To summarize, a line bundle (trivialized by an open cover $\mathcal{U}$) determines a collection of

- holomorphic functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$; which satisfy
- $g_{\alpha\beta} \circ g_{\beta\alpha} = 1$, and
- $g_{\alpha\beta} g_{\gamma\beta}^{-1} g_{\gamma\alpha} = 1$.

Conversely, if we are given $\{g_{\alpha\beta}\}$ satisfying these properties, then we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ as a quotient of $\coprod U_\alpha \times \mathbb{C}$ by the appropriate gluing relations.

2.14.2 Choices

If $f_\alpha \in \mathcal{O}^\times(U_\alpha)$ is a nonvanishing holomorphic function on U_α, and we construct new trivializations $\phi'_\alpha = f_\alpha \circ \phi_\alpha$, then the new transition functions $g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$ give the same bundle L.

Now, the collection of $\{g_{\alpha\beta} \in \mathcal{O}^\times(U_\alpha \cap U_\beta)\}$ is a Čech 1-cochain. The conditions we wrote down that it satisfies implies that it is actually a 1-cocycles. Moreover, the ambiguity mentioned above is exactly the 1-coboundaries. Therefore, we can then conclude the set of (isomorphism classes of) line bundles $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$. Moreover, this is a group homomorphism; for the group structure on line bundles coming from tensor product and the group structure naturally showing up on $H^1(X, \mathcal{O}_X^\times)$.

2.14.3 Line bundles vs "factors of automorphy"

Let X be a complex torus, and $\tilde{X} = \mathbb{C}^g$ its universal cover, with $\pi : \tilde{X} \to X$ the covering map.

2.14.4 Theorem. *There exists a canonical exact sequence*

$$0 \to H^1(\pi(X), H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)) \xrightarrow{\phi} H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times).$$

The first term in the above exact sequence is the group of *"factors of automorphy"* and the latter two are Picard groups as above. (Note that in general $\mathcal{O}_X^\times$ is the sheaf of invertible elements in our rings with respect to multiplication. In our holomorphic setting, this is equivalent to the sheaf of nonvanishing functions, because you can prove that if f is holomorphic and nonzero, then $1/f$ is holomorphic.)

2.14.5 Corollary. *If $\pi^*(L)$ is trivial, then it is completely described by a "factor of automorphy".*

2.14.6 Corollary. *If $\mathrm{Pic}(\tilde{X}) = 0$ (eg when $\tilde{X} = \mathbb{C}^g$), then ϕ is an isomorphism.*

Now, we prove the theorem above.

Proof. The 1-cochains $C^1(G, M)$ where $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)$. Note that M contains elements like holomorphic $g : \tilde{X} \to \mathbb{C}^\times$. Therefore, for $(h : G \to M) \in C^1(G, M)$, we can naturally get holomorphic $f : G \times \tilde{X} \to \mathbb{C}^\times$. Note that G acts on $\tilde{X}$ by $g \cdot \tilde{x}$ which gives $f(\tilde{x}) \cdot g = f(g \cdot \tilde{x})$.

Also, note that the 1-cocycles satisfy

$$(h(\mu) \cdot \lambda)(h(\lambda\mu)^{-1})(h(\lambda)) = 1.$$

Therefore, we have the following cocycle condition: For $\lambda, \mu \in \pi_1(X), \tilde{x} \in \tilde{X}$,

$$f(\lambda\mu, \tilde{x}) = f(\mu, \lambda\tilde{x})f(\lambda, \tilde{x})$$

Also the 1-coboundary condition: For $h_1 \in M = C^0(G, M)$,

$$f(\lambda, \tilde{x}) = \frac{h_1(\lambda \cdot \tilde{x})}{h_1(\tilde{x})}$$

and we define $H^1(G, M) = Z^1/B^1$ like before.

Now, each $f \in Z^1(G, M)$ gives a Line Bundle on X. Let $L = \tilde{X} \times \mathbb{C} \to \tilde{X}$ be the trivial line bundle map. Then $G = \pi_1(X)$ acts on $\tilde{X}$, and so acts on L: $\lambda \cdot (\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that this action is

- free (If $g \cdot * = *$ for some g and $*$, then $g = id$.)
- properly discontinuous (For all K_1, K_2 compact subsets, $\{g \in G : gK_1 \cap K_2 \neq \emptyset\}$ is finite.)

There is a theorem saying that complex manifolds came from complex manifolds moding out free properly discontinuous actions. Therefore, the induced $L' \to X$ is a holomorphic line bundle. And this gives a map $Z^1(G, M) \to H^1(X, \mathcal{O}_X^\times)$.

Proof of Theorem: First we will show that there is a map ϕ_1 given by

$$0 \to H^1(G, M) \xrightarrow{\phi_1} \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^\times)).$$

and then we will try to prove that ϕ_1 is an isomorphism.

First, for existence of ϕ_1, we can just take the above map. Or a more explicit solution for this is given by the following: Let $\pi : \tilde{X} \to X$ be the covering map. Then fix a cover $\{U_i\}_I$ such that for all $i \in I$, there exists $W_i \subset \pi_i^{-1}(U_i)$, with biholomorphic $\pi_i := \pi|_{W_i} : W_i \to U_i$. For all i, j, there exists unique $\lambda_{i,j} \in \pi_1(X)$

such that for $x \in U_i \cap U_j$, $\pi_j^{-1}(x) = \lambda_{ij} \cdot \pi_i^{-1}(x)$. It is clear that $\lambda_{ij} \cdot \lambda_{jk} = \lambda_{ik}$. Therefore, now we have a map $Z^1(G, M) \to Z^1(X, \mathcal{O}_X^\times)$ (note that $Z^1(X, \mathcal{O}_X^\times)$ maps onto $H^1(X, \mathcal{O}_X^\times)$) given by

$$f \mapsto \{g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)\}$$

where $g_{ij}(x) = f(\lambda_{ij}, \pi_i^{-1}(x))$. It is easy to check the 1-cocycle condition

$$g_{ij} g_{jk} = g_{ik}.$$

(Explicitly, $g_{ij}(x) g_{jk}(x) = f(\lambda_{ij}, \pi_i^{-1}(x)) f(\lambda_{jk}, \pi_j^{-1}(x)) = f(\lambda_{ij}\lambda_{jk}, \pi_i^{-1}(x)) = f(\lambda_{ik}, \pi_i^{-1}(x)) = g_{ik}(x)$.)

So we have a well defined map

$$Z^1(G, M) \to \check{Z}^1(X, \mathcal{O}_X^\times) \twoheadrightarrow \check{H}^1(X, \mathcal{O}_X^\times).$$

It is easy to check that the kernel of this map contains $B^1(G, M)$, so this descends to a homomorphism $H^1(G, M) \to \check{H}^1(X, \mathcal{O}_X^\times)$. (To check this: if we have a 1-coboundary $f(\lambda, x) = h(\lambda \tilde{x})/h(x)$, this will go to a Čech 1-cocycle

$$g_{ij}(x) = \frac{h(\lambda_{ij}\pi_i^{-1}(x))}{h(\pi_i^1(x)} = \frac{h(\pi_j^{-1}(x))}{h(\pi_i^{-1}(x))}.$$

This tells us $g_{ij}(x)$ is a Čech 1-coboundary so is trivial in $\check{H}^1(X, \mathcal{O}_X^\times)$.) It is also easy to check that $Im(\phi)$ lies in $\ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Now, we want to show that ϕ_1 is an isomorphism. To do this, we want to find an inverse map. That is, given $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$, we want to find $f \in H^1(G, M)$ such that $L = \phi_1(f)$. (We want an explicit expression for this inverse map, we will use it very often when computing things.)

We know that $\pi^* L$ is trivial. So we fix $\alpha : \pi^* L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$. (Note that G acts on $\pi^* L$, and hence via α, G also acts on $\tilde{X} \times \mathbb{C}$.) For all $\lambda \in G$, there exists automorphisms ϕ_λ of $\tilde{X} \times \mathbb{C}$ such that $\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x})t)$. It is easy to check that

- $f(\lambda, \tilde{x}) \in Z^1(G, M)$
- For another map $\alpha' : \pi^* L \to \tilde{X} \times \mathbb{C}$, we have

  $$\phi'_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, h(\lambda \tilde{x}) f(\lambda \tilde{x}) h(\tilde{x})^{-1} t)$$

  where $\alpha' \alpha^{-1}(\tilde{x}, t) = (\tilde{x}, h(\tilde{x})t)$.

□

In fact, via a similar proof, one can also prove the following:

2.14.7 Theorem. *Let $\mathcal{F}$ be a sheaf of abelian groups on X. Let $G = \pi_1(X)$ and $M = H^0(\tilde{X}, \pi^* \mathcal{F})$. Then for each $n \geq 0$, there exist canonical homomorphisms $\phi_n : H^n(G, M) \to H^n(X, \mathcal{F})$. Moreover,*

- *when $n = 1$, for each $f \in Z^1(G, M)$, we have $(\phi_1 f)_{ij} = f(\lambda_{ij}, \pi_i^{-1})$.*
- *when $n = 2$, for each $f \in Z^2(G, M)$, we have $(\phi_2 f)_{ijk} = f(\lambda_{ij}, \lambda_{jk}, \pi_i^{-1})$.*

2.15 Global sections of $H^0(X, L)$

Our next goal is to describe global sections of $H^0(X, L) = \Gamma(L, X)$ in terms of $\phi_1(f) = L$, where $L \in \ker(H^1(X, \mathcal{O}_X^\times) \to H^1(\tilde{X}, \mathcal{O}_X^\times))$.

Since $\phi^* L$ is trivial, if we fix a trivialization $\alpha : \pi^* L \xrightarrow{\sim} \tilde{X} \times \mathbb{C}$, this gives us an explicit cocycle $f \in Z^1(G, M)$ (giving the cohomology class $\phi_1^{-1}(L)$). Observe that there exists a canonical isomorphism between $H^0(X, L) \xrightarrow{\sim} H^0(\tilde{X}, \pi^* L)^G$. And then we have (via α) an isomorphism $H^0(\tilde{X}, \pi^* L)^G \simeq H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$, which is explicitly given by

$$\phi_\lambda(\tilde{x}, t) = (\lambda \cdot \tilde{x}, f(\lambda, \tilde{x}) t).$$

$H^0(\tilde{X}, \tilde{X} \times \mathbb{C})^G$ then corresponds to the set $\{\vartheta : \tilde{X} \xrightarrow{hol} \mathbb{C} : \vartheta(\lambda, \tilde{x}) = f(\tilde{\lambda}, \tilde{x}) \vartheta(x)\}$. So $H^0(X, L)$ is in bijection with the set of theta functions ϑ with f as its factor of automorphy. (Remark: This does depend on the trivialization and thus the specific form of f; but changing α changes f in a predictable way and thus changes the set of ϑ.) Also, one can check functoriality of everything we've done.

Now, let's get back to the case where $X = V/\Lambda$ is complex torus.

2.15.1 Proposition. *Every holomorphic line bundle on X pulls back to a trivial bundle on V.*

Sketch of proof. We need

- the *exponential sequence* of sheaves:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_V \xrightarrow{e^{2\pi i}} \mathcal{O}_V^\times \to 1$$

  ($\underline{\mathbb{Z}}$ is the constant sheaf of $\mathbb{Z}$.)

- the $\overline{\partial}$-Poincaré lemma; and
- $H^2(V, \mathbb{Z}) = 0$.

This induces the exact sequence:

$$\cdots \to H^1(V, \mathcal{O}_V) \to H^1(V, \mathcal{O}_V^\times) \to H^2(V, \mathbb{Z}) \to \cdots$$

First term is trivial by the $\overline{\partial}$-Poincare lemma, and last term is trivial by $H^2(V, \mathbb{Z}) = 0$. Therefore, $H^1(V, \mathcal{O}_V^\times)$ is also trivial. So, all line bundles on V are trivial. □

2.15.2 Corollary. *The above exact sequence gives an isomorphism* $\mathrm{Pic}(X) := H^1(X, \mathcal{O}_X^\times) \simeq H^1(\Omega, H^0(V, \mathcal{O}_V^\times))$.

The proof can be found in Griffiths-Harris. [farbod add reference]

2.16 Next time

We will talk about the exponential exact sequence, and generalize to X. We will then get a similar long exact sequence, but here $H^2(X, \mathbb{Z})$ is nontrivial, and the map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is extremely important! Our next goal is the Appell-Humbert theorem, which is actually due to Lefschetz.

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