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**ESSEC BUSINESS SCHOOL
Ph.D. Program**

**EXOTIC OPTIONS, INFINITELY DIVISIBLE DISTRIBUTIONS
AND LÉVY PROCESSES
THEORETICAL AND APPLIED PERSPECTIVES**

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RÉSUMÉ

Cette thèse comporte trois parties indépendantes.

La première traite des formes fermées de la factorisation de Wiener-Hopf pour les processus de Lévy. Nous recensons la demie-douzaine de cas pour lesquels la factorisation peut être écrite explicitement, et mettons l'accent sur les fonctions méromorphes ayant des pôles d'ordre deux.

La deuxième partie se focalise sur l'inversion de la transformée de Laplace. Son but est de présenter une nouvelle méthode approximative, dans un contexte probabiliste. Si la transformée de Laplace a un comportement facilement identifiable en zéro et si la densité associée est bornée, alors cette méthode permet d'obtenir une borne uniforme pour l'erreur commise sur la fonction de répartition. L'efficacité de cette méthode est testée sur deux exemples non triviaux.

Enfin, la troisième et dernière partie est dédiée au pricing d'options exotiques dans le modèle log-stable aux moments finis de Carr et Wu. Dans certains cas, il est possible d'obtenir des formules fermées sous forme de séries convergentes pour les prix d'options lookback et barrières. Pour tous les autres cas, nous étudions divers techniques de simulation pour les trajectoires du processus sous-jacent, dans le but d'une évaluation par méthode de Monte-Carlo.

MOTS-CLÉS

- Processus de Lévy
- Factorisation de Wiener-Hopf
- Fonctions méromorphes
- Inversion de transformée de Laplace
- Approximation
- Evaluation d'options exotiques
- Modèle Log-Stable aux Moments Finis
- Processus stable spectrallement négatif

Exotic options, infinitely divisible distributions and Lévy processes: Theoretical and applied perspectives

SUMMARY

This thesis consists of three independent chapters.

The first one deals with closed forms of the Wiener-hopf factorization for Lévy processes. We list the known cases for which this factorization can be explicitly written and provide a detailed account when the underlying functions are meromorphic of order two.

The second chapter focuses on the inversion of the Laplace transform. We present an approximative method in a probabilistic setting. If the behavior of the Laplace transform near zero is known and if the underlying density is bounded, then this method yields a uniform bound for the error on the cumulative distribution function. We test this technique on two non-trivial examples.

The final chapter of the thesis is dedicated to the pricing of exotic options in the Finite Moment Log-Stable model of Carr and Wu. In some cases, it is possible to obtain closed forms (converging series) for the prices of lookback and barrier options. In all other cases, we study several simulation techniques for the trajectories of the underlying for the purpose of Monte-Carlo valuation.

KEY WORDS

- Lévy processes
- Wiener-Hopf factorization
- Meromorphic functions
- Laplace transform inversion
- Approximation
- Pricing of exotic options
- Finite Moment Log Stable model
- Spectrally negative stable process

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Chapter 1

Notations

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$: the sets of natural, integer, rational, real and complex numbers, respectively
- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$: real line without zero
- \mathbb{R}_+ (*resp.* \mathbb{R}_-): the positive (*resp.* negative) real line (including 0)
- $\mathcal{B}(\mathbb{R})$: the Borel set of all open intervals within \mathbb{R}
- $\Re(z)$ (*resp.* $\Im(z)$): the real (*resp.* imaginary) part of the complex number z
- $\delta_x(\cdot)$: the Dirac measure, $\delta_x(B) = 1$ if $x \in B$ and 0 if not, see also
- $\mathbb{1}_A$: the indicator function, which is worth 1 on the set A and 0 elsewhere
- $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$
- $X \stackrel{d}{=} Y$: X and Y have the same distribution
- a.s.: almost surely: an event or a set A is a.s (or P -a.s.) if $P[A] = 1$
- $o(\cdot)$, $O(\cdot)$: the classical Landau notation
- $f^{(n)}$: n^{th} derivative of f
- $\Gamma(\cdot)$: the gamma function: $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$
- $\Gamma(\cdot, \cdot)$: the upper incomplete gamma function: $\Gamma(z, x) = \int_x^\infty t^{z-1} e^{-t} dt$
- $\psi(\cdot)$: the characteristic exponent of a Lévy process: $\psi(z) := \log(\mathbb{E}[e^{izX_t}])/t$
- $\phi(\cdot)$: the Laplace exponent of a Lévy process, i.e. $\phi(z) = \psi(-iz)$

Chapter 2

Introduction

Bien qu'elles aient les processus de Lévy pour dénominateur commun, les trois parties de cette thèse sont largement indépendantes.

L'objectif de la première partie est double. En premier lieu, il s'agit d'introduire rapidement les outils de base dont nous aurons besoin par la suite, ainsi que les concepts que nous étudierons. La grande majorité du premier chapitre est ensuite consacrée à l'étude des processus de Lévy et plus spécifiquement à la factorisation de Wiener-Hopf, qui est un des résultats centraux dans la théorie des fluctuations de ces processus (étude de leur minimum et de leur maximum notamment). Cette factorisation est très populaire en finance de marché, dans le cadre des modèles dits d'exponentielles de Lévy.

Il y a deux manières d'aborder (et de prouver) cette formule. La première est probabiliste et repose sur l'étude des temps locaux et des excursions du processus (voir par exemple les chapitres VI et 6 de [11] et [68]).

La seconde approche de la factorisation de Wiener-Hopf utilise des méthodes purement analytiques (comme dans [63] ou chez [94], section 45). Afin d'obtenir des formules fermées, la seconde approche est logiquement la plus appropriée. En effet, si l'on spécifie une mesure de saut, une partie gaussienne et un drift (et donc l'exposant de Lévy-Khintchine ψ), une étude approfondie de la fonction $q(q - \psi(z))^{-1}$ peut déboucher sur la factorisation recherchée.

Le cas où la mesure de saut est une somme finie de densités de lois gamma est traité dans [77]. Dans ce cas, ψ est une fonction méromorphe et ses singularités peuvent avoir des ordres quelconques. Si la mesure de Lévy est une somme infinie de densités de lois exponentielles, alors ψ est également méromorphe, et tous ses pôles sont d'ordre 1. Ce cas est détaillé dans [61] où l'on découvre que toutes les solutions de l'équation $\psi(z) = q$ sont réelles. Quand les sauts sont bornés par le haut, la factorisation est donnée dans [67] et les zéros de $\psi(z) - q$ sont dans le plan complexe.

Dans tous les cas, la localisation de ces zéros est cruciale et la preuve de la factorisation requiert certaines informations concernant leur répartition asymptotique dans \mathbb{C} . L'enjeu principal du premier chapitre est l'étude de cette répartition dans

un cas particulier. Si l'on impose à tous les pôles d'être d'ordre deux (la mesure de Lévy est une somme infinie de densités $\text{gamma}(2, \cdot)$), alors la partie imaginaire de ces zéros est par exemple forcément plus petite que leur partie réelle. Sous une condition peu contraignante, il est possible d'être encore plus précis et d'en déduire la factorisation recherchée.

Le second chapitre est consacré à l'inversion de la transformée de Laplace (ITL) dans un cadre probabiliste. La transformée de Laplace a de multiples applications, que ce soit en physique, en théorie des probabilités ou bien en finance quantitative. La littérature dédiée à l'ITL est immense et plusieurs centaines d'articles lui ont déjà été consacrés.

Nous commençons donc par un inventaire succinct des différentes techniques d'approximation existantes. La plupart d'entre elles souffrent d'un défaut de taille: il est impossible de connaître l'erreur induite par l'inversion. Certaines approches permettent de borner l'erreur commise, mais la borne est rarement uniforme d'une part, et d'autre part, elle peut devenir assez conséquente lorsque l'on s'éloigne de l'origine.

Nous présentons ici une méthode qui permet de borner différents types d'erreurs. La procédure est itérative: la $(n + 1)$ -ième fonction de répartition approchante est prise en sandwich (sur \mathbb{R}_+) entre la n -ième fonction de répartition approchante et la fonction de répartition inconnue.

Dans tous les cas, il est possible d'obtenir la distance de Kantorovich associée à l'approximation. Quand la densité sous-jacente est bornée, alors on obtient une borne supérieure de la distance de Kolmogorov-Smirnoff: l'erreur commise sur les fonctions de répartition est majorée de manière uniforme sur \mathbb{R}_+ . Notre approche permet de plus d'obtenir toute la courbe de la fonction approchée d'un seul coup, alors que beaucoup de méthodes procèdent point par point.

Enfin, le troisième et dernier chapitre s'intéresse au modèle log-stable aux moments finis (FMLS), introduit dans [24]. Ce modèle est appliqué à l'évaluation de produits dérivés classiques, les options. Il est très utile car, comme le montrent Carr et Wu dans leur article [24], les erreurs sur les prix d'options induites par ce modèle sont comparables à celles d'autres modèles dépendants de 3 à 6 paramètres, alors que le calibrage du FMLS n'en nécessite que 2 (contre un seul - la volatilité -, pour le modèle de Black-Scholes).

Les options "vanille", qui dépendent de la valeur finale du sous-jacent, sont la plupart du temps liquides sur les marchés et leurs prix obéissent à la loi de l'offre et de la demande. Les formules théoriques d'évaluation servent donc à calibrer les modèles pour ensuite déterminer le prix de produits plus complexes, typiquement des options dites "exotiques", lesquelles dépendent de toute la trajectoire du sous-jacent.

Etonnamment, aucun résultat n'est disponible concernant l'évaluation d'options

exotiques dans le modèle FMLS. Dans certains cas, et notamment pour les options dites "lookback" et "barrières", il est pourtant possible d'obtenir des formules fermées.

Pour tous les autres cas, nous étudions exhaustivement les techniques numériques d'évaluation dont, plus particulièrement, les méthodes de Monte-Carlo. L'erreur induite par la discrétisation des trajectoires peut parfois être explicitée, comme dans le cas des options lookback.

Afin d'être complet, nous présentons finalement une procédure qui permet de calculer numériquement les prix des options vanille ainsi que leurs sensibilités à différents paramètres.

Chapter 3

Closed forms of the Wiener-Hopf factorization for Lévy processes

3.1 Random processes

Stochastic (or random) processes are families of random variables defined on a probability space (Ω, \mathcal{F}, P) - see definition below. These families are indexed by some set \mathcal{I} which is most of the time used to represent time; it can be discrete (for instance \mathbb{N} or $\{1, 2, \dots, n\}$) or continuous (\mathbb{R}_+ or $[0, t]$). Stochastic processes may have a dimension greater than one (random vectors), but we will stick in this thesis to the one-dimensional *continuous* case.

These processes are a relevant tool to model the evolution of phenomena which are, from the perspective of mankind, random. These phenomena can relate to Physics (behavior of particles during an experiment), to Meteorology (weather forecasting), to Biology (genetics), to Finance (stock markets, interest rates), to Economics (GDP growth), to Demography ...

Of course, random processes have been introduced in these fields with a purpose towards prediction: one observes the past behavior of a phenomenon and one wants to be able to predict its future (or sometimes one wants to reproduce *virtually* sample paths of this phenomenon). Obviously, if X_t is the process chosen to model this phenomenon, it is impossible to forecast the future value X_T with certainty. Usually, the focus is set on confidence intervals $P[X_T \in (a, b)]$ or average values of the type $\mathbb{E}[f(X_T)]$, $\mathbb{E}\left[f\left(\int_0^T |X_s| ds\right)\right]$ or $\mathbb{E}\left[f\left(\sup_{0 \leq s \leq T} X_s\right)\right]$ for some positive function f such that these expectations make sense.

3.1.1 Characterization of stochastic processes

There are many ways to characterize random processes. One very intuitive way to do so would be to specify the law to the random variable X_t for any fixed time t . However, this is not enough to uniquely determine this process. Roughly speaking, this gives an information on where the process is at time t , but not *how* it got there.

The trajectory (or sample path) of a process is crucial. For example, if one considers that X_t has a Gaussian law with mean zero and variance t , than one must specify the covariance function $\mathbb{E}[X_s X_t]$ (for $s, t \geq 0$) as well. This function will determine the smoothness of the sample paths of X . For instance, a process known as the fractional Brownian motion is Gaussian and satisfies $\mathbb{E}[X_s X_t] = (s^{2H} + t^{2H} - |t - s|^{2H})/2$ for $H \in (0, 1)$. Its paths are all the more regular than H is close to one, in the sense that there is $c > 0$ such that $\forall \epsilon \in (0, H)$,

$$|X_t - X_s| \leq c|t - s|^{H-\epsilon}, \quad a.s., \quad \forall s, t > 0;$$

this property being known as the Hölder continuity of the process. Moreover, if $H > 1/2$, the increments of X are positively correlated while if $H < 1/2$, they are negatively correlated. If $H = 1/2$, they are independent and X is called a Brownian motion.

Before we describe two ways to characterize random process we recall basic definitions related to probability spaces and random variables.

Basic definitions

Given the set Ω , a σ -algebra \mathcal{F} on Ω is a subset of the power set of Ω which is closed under complementation and countable unions. A function will be called \mathcal{F} -measurable (i.e. measurable for the σ -algebra \mathcal{F}) if $f^{-1}(A) \in \mathcal{F}$ for any Borel set A and where $f^{-1}(A) = \{\omega \in \Omega | f(\omega) \in A\}$. A Borel function is a $\mathcal{B}(\mathbb{R})$ -measurable function.

When \mathcal{F} is a σ -algebra on Ω , the couple (Ω, \mathcal{F}) is called measurable space.

Definition 3.1. *Given a measurable space (Ω, \mathcal{F}) , a real-valued random variable X is a measurable function of (Ω, \mathcal{F}) onto $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

The σ -algebra generated by this random variable is the set

$$\sigma(X) := \{X^{-1}(A), A \in \mathcal{B}(\mathbb{R})\}.$$

Definition 3.2. *Given a measurable space (Ω, \mathcal{F}) , a probability measure P on (Ω, \mathcal{F}) is an application from \mathcal{F} onto $[0, 1]$ such that $P[\Omega] = 1$ and*

$$P\left[\bigcup_{n=0}^{\infty} A_n\right] = \sum_{n=1}^{\infty} P[A_n],$$

where the A_i are mutually disjoint subsets of \mathcal{F} .

A measurable space (Ω, \mathcal{F}) equipped with a probability measure P is called a probability space.

Definition 3.3. *If X is a real-valued random variable on (Ω, \mathcal{F}, P) , then its law is defined by*

$$P[X \in A] = P[\{\omega \in \Omega | X(\omega) \in A\}], \quad \forall A \in \mathcal{B}(\mathbb{R}).$$

The function $F : x \mapsto P[X \leq x]$ is called the cumulative distribution function (c.d.f.) of X . If F is continuous with continuous derivative, then $f := F'$ is called the density of X . In which case,

$$P[X \in [a, b]] = \int_a^b f(x)dx = \int_a^b P[X \in dx]$$

The average value (mean or expectation) of X is given by

$$\mathbb{E}[X] = \int_{\mathbb{R}} xP[X \in dx],$$

and, more generally

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} f(x)P[X \in dx],$$

where these integrals may not converge, in which case, the expected values are infinite.

We now turn to filtrations. The natural filtration of a process can be thought of as the information given by the knowledge of the past of this process.

Definition 3.4. *Given a probability space (Ω, \mathcal{F}, P) , a family $(\mathcal{F}_t)_{\{t \geq 0\}}$ of subsets of \mathcal{F} is called a filtration if it is increasing, that is to say if $\mathcal{F}_s \subset \mathcal{F}_t$ when $s < t$. Any given stochastic process engenders its own natural filtration $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$.*

Note that \mathcal{F}_t^X encompasses all the events of the type $\{\omega, X_{t_1}(\omega) = x_1, \dots, X_{t_n}(\omega) = x_n\}$ for $0 \leq t_1 \leq \dots \leq t_n = t$, $x_1, \dots, x_n \in \mathbb{R}$ and $n = 1, 2, \dots$.

We will also need the following tool:

Definition 3.5. *Given a probability space (Ω, \mathcal{F}, P) , $\mathcal{G} \subset \mathcal{F}$, and a random variable X (with finite mean), there exists a unique random variable which is \mathcal{G} -measurable and such that*

$$\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Z\mathbb{1}_A],$$

for any $A \in \mathcal{G}$. Z is called the conditional expectation of X given \mathcal{G} and is usually written $Z = \mathbb{E}[X|\mathcal{G}]$.

Lastly, we define the Markov property.

Definition 3.6. *A random process $X = (X_t)_{t \geq 0}$ has the Markov property if, for any bounded Borel function f ,*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s^X] = \mathbb{E}[f(X_t)|X_s], \quad t > s.$$

A process having the Markov property is quite logically called a Markov process: all the information of the past of such a process is contained in its present value. Moreover, its future behavior does not depend on the past but only on its present position.

Transition functions and Markov processes

A natural way to characterize the way X behaves through time is to give the probability that X_t belongs to a Borel set B , knowing that its value was x at time $s < t$. This probability, which we denote $P_{s,t}(x, B)$ is called a transition function must satisfy, for an \mathbb{R} -valued process and $u \geq t \geq s \geq 0$,

- $P_{s,t}(x, \cdot)$ is a probability measure,
- $P_{s,t}(\cdot, B)$ is a $\mathcal{B}(\mathbb{R})$ -measurable function,
- $P_{s,s}(x, B) = \delta_x(B)$,
- $\int_{\mathbb{R}} P_{s,t}(x, dy) P_{t,u}(y, B) = P_{s,u}(x, B)$,

the last condition being known as the Chapman-Kolmogorov identity.

A very important result is that any Markov process is entirely characterized by its transition function plus its distribution at time 0. A transition function (and thus Markov process) is called temporally homogeneous if

$$P_t(x, B) = P_{s,s+t}(x, B), \quad \forall s \geq 0.$$

Moreover, if for any Borel set B ,

$$P_{s,t}(x, B) = P_{s,t}(0, B - x),$$

where $B - x = \{y - x : y \in B\}$, then the process enjoys the spatial homogeneity property.

Semimartingales

We now turn to another possible characterization of stochastic processes and begin with a definition.

Definition 3.7. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space (that is to say a probability space equipped with a filtration). A random process $X = (X_t)_{t \geq 0}$ is a martingale if $\forall t \geq 0$, $\mathbb{E}[|X_t|] < \infty$ and

$$\forall s \leq t, \quad \mathbb{E}[X_t | \mathcal{F}_s] = X_s.$$

One obvious property of martingales is that their mean is constant through time.

The process X is said to be \mathcal{F}_t -adapted if X_t is \mathcal{F}_t -mesurable for all $t \geq 0$. We call \mathcal{A}^+ the set of all increasing (\mathcal{F}_t) -adapted processes with path that are right-continuous with left limits and \mathcal{A} the set of all processes which are differences of two processes of \mathcal{A}^+ .

Definition 3.8. A process $X = (X_t)_{t \geq 0}$ which can be written $X_t = X_0 + M_t + A_t$ where X_0 is \mathcal{F}_0 -adapted, M is a martingale and $A \in \mathcal{A}$, is called a semimartingale.

Remark 3.1.1. In fact, in the general definition of a semimartingale, M can be a local martingale, a concept which will not be defined here.

3.1.2 Lévy processes

Definition 3.9. A process $X = (X_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}_t, P)$ is a Lévy process if

- $P[X_0 = 0] = 1$,
- $\forall u \geq t \geq s \geq 0$, $X_u - X_t$ and $X_t - X_s$ are independent,
- $\forall t \geq s \geq 0$, $X_t - X_s$ has the same distribution as X_{t-s} ,
- the sample paths of X are P -almost surely right continuous with left limits,

First of all, by their second and third defining properties, Lévy processes are time and space homogeneous. This double homogeneity makes it possible to further characterize their law at any fixed time t : pick $n \geq 1$ any strictly positive integer; by the defining properties of the Lévy process we see that

$$X_t = (X_t - X_{t(1-1/n)}) + (X_{t(1-1/n)} - X_{t(1-2/n)}) + \cdots + (X_{t/n} - X_0),$$

where all the random variables within the brackets have the same distribution and are independent from one another. This property of the law of X_t is called infinite divisibility. The most important result on infinitely divisible distribution is the so-called Lévy-Khintchine formula (here in a Lévy process setting):

Theorem 3.1. For any Lévy process $X = (X_t)_{t \geq 0}$, there are real numbers m, σ and a measure ν concentrated on \mathbb{R}^* and satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$ such that

$$\mathbb{E}[e^{izX_t}] = \exp \left(t \left(imz - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}} (e^{izx} - 1 - izx \mathbf{1}_{|x| < 1}) \nu(dx) \right) \right) = e^{t\psi(z)}. \quad (3.1)$$

Note that measure satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$ is very often called a "Lévy measure". This theorem can be further interpreted in the following way. The characteristic function of X_t can be factorized into three terms. Hence, it seems possible that X_t be the sum of three independent processes. The formal proof and statement of this fact, which is called the Lévy Ito decomposition, will not be provided here. It suffices to point out that X has three independent layers: the first one is a deterministic (non-random) linear drift; the second one is a scaled Brownian motion B_t and the third one is a pure-jump process:

$$X_t = mt + \sigma B_t + J_t.$$

The pure jump process J can itself be decomposed into a martingale plus a process which belongs to \mathcal{A} . Since B is itself a martingale and the drift is also in \mathcal{A} , it is plain that any Lévy process is a semimartingale. If $\mathbb{E}[X_t] = t\mathbb{E}[X_1] = 0$, then it is a martingale.

Moreover, by the independence of their increments, all Lévy processes are also Markov processes and are in fact completely characterized by their law at unit time.

3.1.3 Nomenclature for Lévy processes

Depending on the characteristics of the Lévy triplet (m, σ^2, ν) , some terms have been coined to describe Lévy processes.

- if $m \geq 0$, $\nu((-\infty, 0)) = 0$ and $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$, then X is a.s. increasing and it is called a subordinator.
- if $\nu((-\infty, 0)) = 0$ (resp. $\nu((0, +\infty)) = 0$), then X is said to be spectrally positive (resp. spectrally negative): it jumps only upwards (resp. downwards).
- if $\sigma^2 = 0$, then X is a purely non-gaussian process and if, in addition, $m = 0$, it is a pure jump process.
- if $\nu(\mathbb{R}) = \infty$, then X is said to have an infinite activity: it jumps infinitely many times in any finite time interval. In this case, X_t has a continuous density (this is also true whenever $\sigma^2 > 0$).
- if we define the variation of the function f over the interval $(s, t]$, by $V((s, t], f) = \sup_{\Delta} \sum_{j=1}^n |f(s_j) - f(s_{j-1})|$ where Δ is a partition with $n + 1$ points of the interval $(s, t]$, then the sample paths of X have a.s. finite variation if and only if $\sigma^2 = 0$ and $\int_{-1}^1 |x|\nu(dx) < \infty$.
- a Lévy process with finite variation and such that $m = 0$ is called a compound Poisson process.
- if X is a compound Poisson process, then $Y_t = X_t + mt + \sigma B_t$ is called a jump-diffusion process.

3.1.4 Financial applications of Lévy processes

Lévy processes have numerous fields of application (population models for instance), but we focus here on their financial purposes. We underline that the following classification is not meant to be exhaustive.

- **Financial applications:** they can be divided into two categories:
 - the pricing and hedging of derivatives: the price of an underlying (a stock, an interest rate, an FX rate, a commodity, a future, etc.) is modeled by the exponential of a Lévy process. It is then possible to compute (or approximate) the price of a derivative (a vanilla option, an exotic option, or a swap for instance) written on this underlying using various techniques (see for instance, chapter 5 of this thesis for a specific example). References for this category are [30] and [69].
 - the study of credit risk: the aim is to model the probability of default of one (or several) asset(s) and the implications in terms of (aggregate) losses for the holder of this (those) asset(s). Derivatives are also priced and hedged in this setting: Credit Default Swaps, Collateralized Debt Obligations, Asset Backed Securities. Two references for this topic are [13] and [21].

- **Applications in insurance:** they too can be divided into two categories:
 - ruin theory: the reserves of an insurance company are modeled by a Lévy process plus a positive constant and the focus is set on whether these reserves will become negative or not in some finite or infinite time horizon. A central reference is [8].
 - insurance portfolio: the price of a portfolio can be modeled using a Lévy process. This allows for various risk analyses. This field is the less developed in the literature, but one reference is [31].

3.2 The Wiener-Hopf factorization

3.2.1 Statement of the result

One of the most interesting (and challenging) facets of random processes is the study of their running supremum and infimum. More precisely, if $(X_t)_{t \geq 0}$ is a stochastic process, we are interested in $S_t = \sup_{0 \leq s \leq t} X_s$ and $I_t = \inf_{0 \leq s \leq t} X_s$. Explicitly formulating the law of S_t or I_t is practically impossible, except in a very limited number of cases (the Brownian motion for instance). Usually, Lévy processes are defined by their Lévy measure or their characteristic function (via the Lévy-Khintchine formula). However, neither the density or the characteristic function of S_t or I_t can be easily inferred from them. There is, however, a way to characterize, in some sense, the behavior of the supremum and infimum of a Lévy process.

We state the result and comment on it. We refer to [68], chapter 6, for technical details on the subject.

Theorem 3.2. *Let e_q be an exponential random variable with parameter q and X an independent Lévy process such that either $\nu(\mathbb{R}) = \infty$, $\sigma \neq 0$ or $m \neq 0$. Then, the following unique factorization holds*

$$\mathbb{E}[e^{izX_{e_q}}] = \frac{q}{q - \psi(z)} = \psi_q^-(z) \psi_q^+(z) = \mathbb{E}[e^{izI_{e_q}}] \mathbb{E}[e^{izS_{e_q}}], \quad z \in \mathbb{R}. \quad (3.2)$$

where ψ is the characteristic exponent defined in (3.1).

We underline the fact that the Wiener-Hopf factorization is part of a greater (and much richer) whole which is usually referred to as the *fluctuation identities* and involve tools that would take too long to define here, such as, local times, reflected processes, ladder height processes and excursions.

First note that for any process X , $\mathbb{E}[e^{izX_{e_q}}]$ is the Laplace transform of the characteristic function of X with respect to the time parameter: $\int_0^\infty qe^{-qt} \mathbb{E}[e^{izX_t}] dt$. Next, we recall the Frullani integral, which holds for $a, b > 0$ and $\Re(z) \leq 0$:

$$\frac{1}{(1 - z/a)^b} = \exp \left(\int_0^\infty (e^{zx} - 1) b x^{-1} e^{-ax} dx \right).$$

Using the fact that $\Re(\psi(z)) = -\frac{\sigma^2 z^2}{2} - \int_{\mathbb{R}} (1 - \cos(zx))\nu(dx) \leq 0$, this leads to

$$\frac{q}{q - \psi(z)} = \exp \left(\int_0^\infty \int_{\mathbb{R}} (e^{izx} - 1)t^{-1} e^{-qt} P[X_t \in dx] dt \right),$$

and in fact it is true (though it is not a direct implication from the above equation) that

$$\psi_q^\pm(z) = \exp \left(\int_0^\infty \int_{\mathbb{R}_\pm} (e^{izx} - 1)t^{-1} e^{-qt} P[X_t \in dx] dt \right), \quad \pm \Im(z) \geq 0.$$

The formal proofs of these results are quite lengthy and will not be provided here. They make use of two families of techniques. The first family is purely analytical, as in [94] and [63]. The other family of proofs relies on probabilistic arguments (see [11], chapter VI and [68], chapter 6 for instance).

We will also make use of the following notation

$$\phi_q^+(z) := \psi_q^+(iz) = \mathbb{E}[e^{-zS_{e_q}}], \quad \phi_q^-(z) := \psi_q^-(-iz) = \mathbb{E}[e^{zI_{e_q}}], \quad \Re(z) > 0,$$

which are the Laplace transforms of the positive random variables S_{e_q} and $-I_{e_q}$.

3.2.2 Hitting times and the term "Wiener Hopf"

In section 6.6 of his monograph [68], Kyprianou explains why the term "Wiener-Hopf factorization" was originally coined. We want to show another connection between fluctuations of compound Poisson processes and integral equations which are usually solved using the so-called Wiener-Hopf method.

Let $(X_t)_{t \geq 0}$ be a compound Poisson process with Laplace exponent (at least defined on the imaginary axis) given by

$$\phi(z) = \lambda \int_{\mathbb{R}} (e^{zx} - 1)\nu(dx),$$

and infinitesimal generator

$$\mathcal{L}[f](x) = \lambda \int_{\mathbb{R}} (f(x+y) - f(x))\nu(dy),$$

defined on the space of real bounded continuous functions. Let $T_a = \inf\{t \geq 0, X_t \geq a\}$ be the first passage time of X over the level a and $T_{(0,\infty)} = \inf\{t \geq 0, X_t > 0\}$ be the first time X takes a strictly positive value. Denote

$$v_q = \mathbb{E}[e^{-qT_{(0,\infty)}} \mathbf{1}_{T_{(0,\infty)} < \infty}], \quad u_q(x) = \mathbb{E}[e^{-qT_x} \mathbf{1}_{T_x < \infty}],$$

then, u_q solves Dynkin's system (see [37]):

$$(*) \quad \begin{cases} \mathcal{L}[u_q](x) &= q u_q(x), \text{ for } x < 0 \\ u_q(x) &= v_q \text{ for } x \geq 0 \end{cases}$$

Given the definition of \mathcal{L} in the compound Poisson case, $(*)$ is in fact a non-homogeneous Wiener-Hopf equation of the second kind (see [90], section 13.11-4). The connection with the Wiener-Hopf factorization of Lévy processes stems from the fact that

$$u_q(x) = P[S_{e_q} \geq x],$$

where, as usual, e_q is an independent exponential variable with parameter q . When the Wiener-Hopf factors are low order meromorphic functions as in [60], [61], [66] and (3.8), then $P[S_{e_q} \geq x]$ is easily computed using residues.

Given the supremum S and first passage time T , we end this chapter by providing other identities, some of which hold for processes other than Lévy processes:

$$P[T_{e_q} \geq x] = \int_0^1 P[e^{-qS_x} \geq y] dy = \mathbb{E}[e^{-qS_x}], \quad q, x > 0.$$

If e_q and $e_{q'}$ are independent, then

$$\mathbb{E}[e^{-qT_{e_{q'}}}] = \int_0^\infty q' e^{-q'u} \int_0^1 P[e^{-qT_u} \geq y] dy du = 1 - \mathbb{E}[e^{-q'S_{e_q}}].$$

If, for $q, x > 0$, $\hat{T}_x = \inf\{t > 0, X_t \leq -x\}$ and $\mathbb{E}[e^{-xI_{e_q}}] < \infty$, we have another factorization:

$$\mathbb{E}[e^{-xX_{e_q}}] = P[T_{e_x} \geq e_q] P[\hat{T}_{e_x} \geq e_q],$$

where e_x is independent of e_q and x -exponentially distributed.

It is also possible to introduce other positive random variables. If p_m has a Pareto density of the form $f_{p_m}(x) = m(1+x)^{-m-1} \mathbf{1}_{\{x \geq 0\}}$ for $m > 0$, then

$$P[S_{p_m} \geq x] = \mathbb{E}[(1+T_x)^{-m}], \quad P[T_{p_m} \geq x] = \mathbb{E}[(1+S_x)^{-m}], \quad x > 0,$$

Lastly, if g_σ has a density $f_{g_\sigma}(x) = \sqrt{\frac{2}{\pi\sigma}} e^{-\frac{x^2}{2\sigma}} \mathbf{1}_{\{x \geq 0\}}$, then,

$$P[S_{g_\sigma} \leq x] = \mathbb{E}[\text{erf}(T_x/\sqrt{2\sigma})], \quad P[T_{g_\sigma} \leq x] = \mathbb{E}[\text{erf}(S_x/\sqrt{2\sigma})], \quad x > 0.$$

In fact, these results can be extended to any positive random variable V_θ with density f_θ where θ is a set of parameters. Then, if there is h_θ , a positive strictly monotonous function such that $h'_\theta(x) = g(\theta)f_\theta(x)$, then,

$$P[\pm S_{V_\theta} \geq \pm x] = \mathbb{E}[h_\theta(T_x)]/g(\theta),$$

where the \pm depends on whether h_θ is increasing or decreasing. h_θ should satisfy $h_\theta(0) = 1$ if h_θ is decreasing and $h_\theta(0) = 0$ if it is increasing.

3.3 A brief history of closed-forms of WH factorization

We collect in this section the various closed forms of ψ_q^\pm or ϕ_q^\pm , in their chronological order of publication. Like most results with Lévy processes, the Wiener-Hopf

factorization was first expressed for the Brownian motion. It can be stated in the following way

$$\mathbb{E}[e^{izB_{e_q}}] = \int_0^\infty qe^{-qx}e^{-xz^2/2}dx = \frac{q}{q+z^2/2} = \frac{\sqrt{2q}}{\sqrt{2q}-iz} \frac{\sqrt{2q}}{\sqrt{2q}+iz} = \mathbb{E}[e^{izI_{e_q}}]\mathbb{E}[e^{izS_{e_q}}].$$

This purely analytical factorization can be also obtained using fluctuation arguments. If L is the Laplace transform operator (see next chapter): $L[f](t) = \int_0^\infty e^{-xt}f(x)dx$, then if F is the antiderivative of f , $L[F](t) = L[f](t)/t$. Now, recall the well-known Laplace transform of the first passage time for the Brownian motion $T_a = \inf\{t \geq 0, B_t \geq a\}$, for $a > 0$: $\mathbb{E}[e^{-zT_a}] = e^{-a\sqrt{2z}}$. Then, noticing that in this case,

$$\mathbb{E}[e^{-qT_{ez}}] = \int_0^\infty ze^{-zx}e^{-x\sqrt{2q}}dx = \frac{z}{z+\sqrt{2q}},$$

and more generally that

$$\begin{aligned} \mathbb{E}[e^{-qT_{ez}}] &= \int_0^\infty ze^{-zx}\mathbb{E}[e^{-qT_x}]dx = \int_0^\infty \int_0^\infty ze^{-zx}e^{-qy}d_yP[T_x \leq y]dx \\ &= \int_0^\infty \int_0^\infty zqe^{-zx}e^{-qy}P[T_x \leq y]dydx \\ &= \int_0^\infty \int_0^\infty zqe^{-zx}e^{-qy}(1 - P[S_y \leq x])dydx \\ &= 1 - \mathbb{E}[e^{-zS_{e_q}}], \end{aligned}$$

we get that $\mathbb{E}[e^{-zS_{e_q}}] = \frac{\sqrt{2q}}{\sqrt{2q}+z}$. The symmetry of the Brownian motion also yields $\mathbb{E}[e^{zI_{e_q}}] = \frac{\sqrt{2q}}{\sqrt{2q}+z}$.

For the drifted Brownian motion $(X_t = B_t + mt)$, the factorization reads

$$\begin{aligned} \mathbb{E}[e^{izX_{e_q}}] &= \frac{q}{q+z^2/2-imz} \\ &= \frac{m+\sqrt{2q+m^2}}{iz+m+\sqrt{2q+m^2}} \times \frac{m-\sqrt{2q+m^2}}{iz+m-\sqrt{2q+m^2}}, \end{aligned}$$

and in fact, S_{e_q} is an exponential random variable with parameter $\sqrt{2q+m^2}-m$, which is a well known fact (see Corollary 2, chapter VII in [11]).

3.3.1 Stable processes

The most general formulations of the Wiener-Hopf factors and fluctuation identities for stable processes were recently given in [46] and [62], but we focus here on the first results provided by Doney in [36]. We call $X^{(\alpha)}$ a (strictly) stable process if its Lévy-Khintchine exponent is given by

$$\psi(z) = \log \left[\mathbb{E} \left[e^{izX_1^{(\alpha)}} \right] \right] = -c|z|^\alpha \left(1 - i\beta \operatorname{sgn}(z) \tan \left(\frac{\pi\alpha}{2} \right) \right), \quad (3.3)$$

where $\alpha \in (1, 2)$ and $\beta \in [-1, 1]$ or $\alpha \in (0, 1)$ and $\beta \in (-1, 1)$. Without loss of generality, we fix $c = (1 + \beta^2 \tan^2(\pi\alpha/2))^{-1/2}$. The cases when $X^{(\alpha)}$ or $-X^{(\alpha)}$ is a subordinator are trivial and are henceforth excluded.

A stable process enjoys a scaling (or self-similarity) property: for $c > 0$, the process $Y_t := (c^{-1/\alpha} X_{ct}^{(\alpha)})_{t \geq 0}$ has the same finite dimensional margins as $X^{(\alpha)}$. This property makes it possible to study the Wiener-Hopf factors for $q = 1$ without any loss of generality. Moreover, it implies that the probability $P[X_t^{(\alpha)} > 0]$ does not depend on t and is thus constant through time. It was shown by Zolotarev (see [102] for instance) that

$$\rho = P[X_1^{(\alpha)} > 0] = \frac{1}{2} + \frac{1}{\pi\alpha} \tan^{-1} \left(\beta \tan \left(\frac{\pi\alpha}{2} \right) \right).$$

When ρ and α satisfy a particular relationship, then it is possible to explicitly write the Wiener-Hopf factors. More precisely, if, for some integers $k \geq 1$ and $l \geq 1$, $\rho + k = l/\alpha$, then,

$$\begin{aligned} \phi_q^+(-z) = \mathbb{E}[e^{zS_{e_1}}] &= \frac{\prod_{j=0}^{k-1} (-1)^l (-z)^\alpha + e^{i\pi(k-1-2j)\alpha}}{l-1} , \quad \Re(z) < 0, \\ \phi_q^-(z) = \mathbb{E}[e^{zI_{e_1}}] &= \frac{\prod_{j=0}^{l-1} (-1)^{k-1} z + e^{i\pi(l-1-2j)/\alpha}}{k} , \quad \Re(z) > 0. \end{aligned}$$

3.3.2 The result of Lewis and Mordecki

We summarize in this subsection the results of [77]. If the Lévy measure of a Lévy process has the form

$$\nu(dx) = \nu^+(dx) \mathbb{1}_{\{x>0\}} + \mathbb{1}_{\{x<0\}} \sum_{j=1}^J c_j \frac{(-x)^{m_j-1}}{(m_j-1)!} e^{\rho_j x},$$

where ν^+ is any Lévy measure on \mathbb{R}_+ , then the characteristic exponent is given by

$$\psi(z) = -\frac{\sigma^2 z^2}{2} + imz + \int_0^\infty (e^{izx} - 1 - izx \mathbb{1}_{\{|x|<1\}}) \nu^+(dx) + \sum_{j=1}^J \frac{c_j}{(\rho_j + iz)^{m_j}} - \lambda,$$

where $\lambda = \sum_{j=1}^J c_j \rho_j^{-m_j}$.

In this case, if $\int_0^1 x \nu^+(dx) < \infty$ and $m > 0$, $\psi(z) = q$ has exactly $N = \sum_{j=1}^J m_j$ zeros ζ_k (counted according to their multiplicity n_k : $N = \sum_k n_k$) in the upper complex half-plane (for $q \geq 0$).

In other case, there are $N = 1 + \sum_{j=1}^J m_j = \sum_k n_k$ zeros ζ_k in the upper complex half-plane and in any case, for $\Re(z) > 0$,

$$\begin{aligned}\mathbb{E}[e^{zI_{e_q}}] &= \prod_k \left(\frac{\zeta_k}{\zeta_k + z} \right)^{n_k} \times \prod_{j=1}^J \left(\frac{\rho_j + z}{\rho_j} \right)^{m_j}, \\ \mathbb{E}[e^{-zS_{e_q}}] &= \frac{q}{q - \phi(-z)} \prod_k \left(\frac{\zeta_k}{\zeta_k - z} \right)^{-n_k} \times \prod_{j=1}^J \left(\frac{\rho_j - z}{\rho_j} \right)^{-m_j}.\end{aligned}\tag{3.4}$$

If $N = \sum_{j=1}^J m_j$, this implies $P[I_{e_q} = 0] = \prod_{n=1}^N \zeta_n / \prod_{j=1}^J \rho_j^{m_j}$; if not, then the law of I_{e_q} has no atom at 0.

Of course, (3.4) can be transposed to S_{e_q} whenever the positive part of the jump measure of the underlying Lévy process has a Fourier transform of rational type.

3.3.3 Meromorphic Lévy processes

These processes were introduced in [66], inspired by [60] and [61]. In this case, the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure and its density is defined by

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n=1}^{\infty} a_n \rho_n e^{-\rho_n x} + \mathbb{1}_{\{x<0\}} \sum_{n=1}^{\infty} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x},$$

where $a_n, \hat{a}_n, \rho_n, \hat{\rho}_n$ are positive and ρ_n and $\hat{\rho}_n$ are strictly increasing and going to infinity. This is of course a generalization of the so-called hypergeometric class, for which ν has a density of the form

$$\pi_{J,\hat{J}}(x) = \mathbb{1}_{\{x>0\}} \sum_{n=1}^J a_n \rho_n e^{-\rho_n x} + \mathbb{1}_{\{x<0\}} \sum_{n=1}^{\hat{J}} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}, \quad J, \hat{J} < \infty,$$

and for which financial applications have been derived (see [7], [53] and [58] for instance).

Now, we assume that $\sum_{n=1}^{\infty} a_n \rho_n^{-2}$ and $\sum_{n=1}^{\infty} \hat{a}_n \hat{\rho}_n^{-2}$ are finite so that ν is indeed a Lévy measure. The Laplace exponent is then given by

$$\phi(z) := \psi(-iz) = mz + z^2 \left[\frac{\sigma^2}{2} + \sum_{n=1}^{\infty} \frac{a_n}{\rho_n(\rho_n - z)} + \sum_{n=1}^{\infty} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} \right].$$

It is obvious that ϕ is analytic on the whole complex plane, except for the isolated points $z = \rho_n$ and $z = -\hat{\rho}_n$, hence the term "meromorphic Lévy processes".

The Wiener-Hopf factorization can in this case quite naturally be stated as follows

$$\frac{q}{q - \phi(z)} = \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\hat{\zeta}_n}} = \phi_q^+(-z) \phi_q^-(z), \quad |\Re(z)| < \min(\zeta_1, \hat{\zeta}_1),$$

where the $(\zeta_n, -\hat{\zeta}_n)$ are the zeros of $\phi(z) - q$ and satisfy the following interlacing property:

$$\begin{aligned} 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \dots \\ 0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 < \dots \end{aligned}$$

The factorization can be Laplace-inverted into the density of S_{e_q} :

$$f_{S_{e_q}}(x) = \sum_{n=1}^{\infty} \left(\left(1 - \frac{\zeta_n}{\rho_n}\right) \prod_{k \neq n} \frac{\zeta_k(\rho_k - \zeta_n)}{\rho_k(\zeta_k - \zeta_n)} \right) \zeta_n e^{-\zeta_n x}, \quad x > 0,$$

and $P[S_{e_q} = 0] = \prod_{n=1}^{\infty} \frac{\zeta_n}{\rho_n}$. Replacing ρ_n and ζ_n by $\hat{\rho}_n$ and $\hat{\zeta}_n$ gives the density of $-I_{e_q}$.

3.3.4 Lévy processes with bounded positive jumps

One of the latest papers on the subject, [67], provides the Wiener-Hopf factors for Lévy processes with Laplace exponent of the form

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \int_{-\infty}^k (e^{zx} - 1 - zx \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx),$$

where $k > 0$. In this case, for $q > 0$, equation $\phi(z) = q$ has a unique positive solution ζ_0 and infinitely many solutions in the quadrant $\mathcal{Q} := \{z \in \mathbb{C}, \Re(z) > 0, \Im(z) > 0\}$, which are denoted by ζ_n and ordered according to their increasing absolute value. The following holds,

- i) ζ_0 has multiplicity one and $\Re(\zeta_n) \geq \zeta_0$ for all $n \geq 1$,
- ii) the series $\sum_{n \geq 1} \Re(\zeta_n^{-1})$ converges,
- iii) all of the numbers $\{\zeta_n\}_{n \geq 1}$ except possibly those of a set of zero density, lie inside an arbitrary small angle $\pi/2 - \epsilon < \arg(z) < \pi/2$,
- iv) the Wiener-Hopf factors are, for $\Re(z) \geq 0$

$$\begin{cases} \phi_q^+(z) &= e^{kz/2} \left(1 + \frac{z}{\zeta_0}\right)^{-1} \prod_{n \geq 1} \left(1 + \frac{z}{\zeta_n}\right)^{-1} \left(1 + \frac{z}{\bar{\zeta}_n}\right)^{-1}, \\ \phi_q^-(z) &= \frac{q}{q - \phi(z)} \frac{1}{\phi_q^+(-z)}. \end{cases}$$

Using partial fraction decomposition, this leads to the density of S_{e_q} , when the law of the latter does not have an atom at zero:

$$\frac{d}{dx}P[S_{e_q} \leq x] = a_0 e^{-\zeta_0 x} + 2 \sum_{n \geq 1} \Re[a_n e^{-\zeta_n x}],$$

where

$$a_0 = \zeta_0 e^{-k\zeta_0/2} \prod_{m \geq 1} \left| 1 - \frac{\zeta_0}{\zeta_m} \right|^{-2},$$

and

$$a_n = \frac{\zeta_0 |\zeta_n|^2 e^{-\zeta_n k/2}}{2\Im(\zeta_n)(\zeta_n - \zeta_0)} \prod_{\substack{m \geq 1 \\ m \neq n}} \left(1 - \frac{\zeta_n}{\zeta_m} \right)^{-1} \left(1 - \frac{\zeta_n}{\bar{\zeta}_m} \right)^{-1}.$$

For example, if X is the standard Poisson process, then $\phi(z) = e^z - 1$ and $\zeta_0 = \log(1+q)$, $\zeta_n = \log(1+q) + 2\pi ni$. We then have

$$\begin{aligned} \mathbb{E}[e^{-zS_{e_q}}] &= \mathbb{E}[e^{-zX_{e_q}}] = \frac{q}{q+1-e^{-x}} \\ &= e^{z/2} \left(1 + \frac{z}{\log(1+q)} \right)^{-1} \prod_{n=1}^{\infty} \frac{\log(1+q)^2 + (2\pi n)^2}{(\log(1+q) + z)^2 + (2\pi n)^2} \\ &= e^{z/2} \frac{\sinh(\log(1+q)/2)}{\sinh((z + \log(1+q))/2)}, \end{aligned}$$

where we have used 1.431-2 from [47] for the last equality.

3.4 The case when ϕ has poles of order two

This case will be studied in detail, as it is one of the contributions of this thesis.

3.4.1 Notations and main result

Let $(X_t)_{t \geq 0}$ be a real-valued Lévy process starting at 0, with Lévy-Khintchine representation given by (3.1). The Lévy measure ν we are interested in is absolutely continuous and its density is given by

$$\pi(x) = \mathbb{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \rho_n^2 x e^{-\rho_n x} + \mathbb{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \hat{\rho}_n^2 (-x) e^{\hat{\rho}_n x},$$

where the $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ are positive, real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{a_n}{\rho_n} < \infty, \quad \sum_{n=1}^{\infty} \frac{\hat{a}_n}{\hat{\rho}_n} < \infty. \quad (3.5)$$

Moreover, the sequences $\rho_n, \hat{\rho}_n$ are increasing and satisfy

$$\forall n \geq 1, \quad \min(\rho_n, \hat{\rho}_n) \geq cn^{1+\varepsilon}, \quad \max\left(\frac{\rho_{n+1}}{\rho_n}, \frac{\hat{\rho}_{n+1}}{\hat{\rho}_n}\right) \leq C, \quad (3.6)$$

for some strictly positive constants C, c, ε . It is easy to check that under (3.5), ν is indeed a Lévy measure since $\int_{\mathbb{R}^*} |x| \pi(x) dx < \infty$. The first condition in (3.6) will ensure that the functions we are interested in are of order less than one, which is crucial in our proofs.

In this case, the truncation function is not necessary and we set $h := 0$, which gives

$$\psi(z) = imz - \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\rho_n^2}{(\rho_n - iz)^2} - 1 \right) + \sum_{n=1}^{\infty} \hat{a}_n \left(\frac{\hat{\rho}_n^2}{(\hat{\rho}_n + iz)^2} - 1 \right).$$

In fact, it will be more convenient to work with the Laplace exponent:

$$\phi(z) = \log(\mathbb{E}[e^{zX_1}]) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \frac{2\rho_n z - z^2}{(\rho_n - z)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2\hat{\rho}_n z + z^2}{(\hat{\rho}_n + z)^2}. \quad (3.7)$$

This Laplace exponent is well defined on an open set containing zero because $c > 0$.

Before stating our main result, we introduce a condition which will be discussed later on, see Section 3.4.3 thereafter.

$$(*) \quad \begin{cases} \forall j \geq 1, \quad \forall b^2 > 0, \\ \frac{\sigma^2}{2} + \sum_{n=1}^{\infty} a_n \frac{\rho_j^2 - 4\rho_j \rho_n + 3\rho_n^2 + b^2}{((\rho_n - \rho_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\rho_j^2 + 4\rho_j \hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \rho_j)^2 + b^2)^2} > 0 \\ \frac{\sigma^2}{2} + \sum_{n=1}^{\infty} a_n \frac{\hat{\rho}_j^2 - 4\hat{\rho}_j \rho_n + 3\rho_n^2 + b^2}{((\rho_n - \hat{\rho}_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_j^2 + 4\hat{\rho}_j \hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \hat{\rho}_j)^2 + b^2)^2} > 0 \end{cases}$$

We are now ready to proceed with our main result, which states that the Wiener-Hopf factorization in this setting has the same form as in [60] and the related literature. We formulate it in a probabilistic fashion using the Laplace transform of S_{e_q} and I_{e_q} .

Theorem 3.3. *For $q, z > 0$, under (3.5), (3.6) and (*),*

$$\mathbb{E}[e^{-zS_{e_q}}] = \frac{1}{1 + \frac{z}{\zeta_0^+}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^-}}, \quad \mathbb{E}[e^{zI_{e_q}}] = \frac{1}{1 - \frac{z}{\zeta_0^-}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^-}}, \quad (3.8)$$

where ζ_0^+ , ζ_n^+ and ζ_n^- (resp. ζ_0^- , $\hat{\zeta}_n^+$ and $\hat{\zeta}_n^-$) are the zeros of $\phi(z) - q$ with positive (resp. negative) real part.

The notation used for the zeros stems from the fact that they go by pairs since, as we shall see, each pole of order two ρ_n and $\hat{\rho}_n$ will engender two roots for $\phi(z) - q$. The proof of the theorem relies heavily on the location of the zeros of $\phi(z) - q$, a topic discussed in the next section.

3.4.2 The location of the zeros of $\phi(z) - q$

In [60], Kuznetsov addresses the problem of root localization in two steps: first he finds obvious locations and then proves that they are sufficient (there are no other zeros) using an asymptotic result. Here, we proceed otherwise: we show that the function has exactly two zeros in an infinite number of well-defined zones, and none outside these zones.

More specifically, we define $R_0 = \{z \in \mathbb{C}, -\hat{\rho}_1 < \Re(z) < \rho_1, |\Im(z)| \leq \max(\rho_1, \hat{\rho}_1)\}$ and the two series of rectangles: $\forall n \geq 1$,

$$R_n = \{z \in \mathbb{C}, \rho_n < \Re(z) < \rho_{n+1}, |\Im(z)| \leq \rho_{n+1}\},$$

$$\hat{R}_n = \{z \in \mathbb{C}, -\hat{\rho}_n > \Re(z) > -\hat{\rho}_{n+1}, |\Im(z)| \leq |\hat{\rho}_{n+1}|\}.$$

The main result of the section follows.

Proposition 3.1. *For any $q > 0$, under (*), all the zeros of $\phi(z) - q$ are located in $R_0 \cup \bigcup_{n \geq 1} (R_n \cup \hat{R}_n)$. Moreover, for any $n \geq 1$, R_n and \hat{R}_n both contain exactly 2 zeros or one double zero and so does R_0 .*

The proof of the proposition will require a few intermediate results. We first introduce two test functions, defined for $q_n, \hat{q}_n > 0$,

$$\Phi_n(z) = \frac{1}{(\rho_n - z)^2} + \frac{1}{(\rho_{n+1} - z)^2} - q_n, \quad \hat{\Phi}_n(z) = \frac{1}{(\hat{\rho}_n + z)^2} + \frac{1}{(\hat{\rho}_{n+1} + z)^2} - \hat{q}_n.$$

Our objective is to apply an extension of Rouché's theorem (see below) to the functions ϕ and Φ_n . This technique was already used in a similar context for the proof of the main theorem of [77]. Rouché's theorem will be applied on some contours K_n which will be closely related to the R_n . The condition (*) will ensure the strict inequality required in the theorem on the vertical parts of the K_n . A geometric condition will then imply that the zeros cannot lie outside of $R_0 \cup \bigcup_{n \geq 1} (R_n \cup \hat{R}_n)$.

We start by proving that the functions $\Phi_n(z)$ and $\hat{\Phi}_n(z)$ have the following property.

Lemma 3.1. *For all $n \geq 1$, Φ_n (resp $\hat{\Phi}_n$) has only two zeros in R_n (resp \hat{R}_n). Furthermore, the real part of Φ_n (resp $\hat{\Phi}_n$) is strictly negative on ∂R_n (resp $\partial \hat{R}_n$).*

Proof. We will often use the usual complex notation $z = a + ib$. We will prove the lemma for Φ_n since the transposition to $\hat{\Phi}_n$ will be straightforward. First notice that Φ_n has exactly four zeros

$$z_q^{1,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n \left(4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2} \right)}}{2q_n},$$

$$z_q^{2,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n \left(4 + q_n(\rho_n - \rho_{n+1})^2 + 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2} \right)}}{2q_n}.$$

The zeros $z_q^{2,\pm}$ are both real and outside R_n and since $|4 + x - 4\sqrt{1 + x}| \leq x$ for $x \geq 0$, it is obvious that

$$|4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2}| \leq q_n(\rho_n - \rho_{n+1})^2,$$

hence, the $z_q^{1,\pm}$ belong to R_n and are either both real or both complex with $|\Im(z_q^{1,\pm})|$ smaller than $(\rho_{n+1} - \rho_n)/2$.

Moreover,

$$\Re(\Phi_n(a + ib)) = -q_n + \frac{(\rho_n - a)^2 - b^2}{((\rho_n - a)^2 + b^2)^2} + \frac{(\rho_{n+1} - a)^2 - b^2}{((\rho_{n+1} - a)^2 + b^2)^2},$$

so that for $a = \rho_n$ and denoting $\rho = (\rho_n - \rho_{n+1})^2$,

$$\Re(\Phi_n(\rho_n + ib)) = -q_n + \frac{-1}{b^2} + \frac{\rho - b^2}{(\rho + b^2)^2} = -q_n + \frac{-\rho^2 - b^2\rho - 2b^4}{b^2(\rho + b^2)^2} < 0, \quad \forall b \in \mathbb{R}.$$

The proof is the same for $a = \rho_{n+1}$. Lastly, for $b = \pm\rho_{n+1}$,

$$\Re(\Phi_n(a \pm i\rho_{n+1})) = -q_n + \frac{(\rho_n - a)^2 - \rho_{n+1}^2}{((\rho_n - a)^2 + \rho_{n+1}^2)^2} + \frac{(\rho_{n+1} - a)^2 - \rho_{n+1}^2}{((\rho_{n+1} - a)^2 + \rho_{n+1}^2)^2},$$

and both numerators are strictly negative for $a \in (\rho_n, \rho_{n+1})$. \square

The proof of Proposition 3.1 will rely on the following reinforcement, due to Estermann (see [38] p. 156), of Rouché's theorem.

Theorem 3.4 (Estermann-Rouché's Theorem). *Let f and g be two holomorphic functions inside and on some simple contour ∂K . If $|f(z) - g(z)| < |f(z)| + |g(z)|$ on ∂K , then f and g have the same number of zeros (counting multiplicities) inside K .*

We must choose proper contours K_n to proceed with the application of this theorem. To this purpose, we introduce two series of disks: given their radiuses $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$, they are defined for all $n \geq 1$ by

$$D_{n,q} = \{z \in \mathbb{C}, |z - \rho_n| \leq \varepsilon_{n,q}\}, \quad \hat{D}_{n,q} = \{z \in \mathbb{C}, |z - \hat{\rho}_n| \leq \hat{\varepsilon}_{n,q}\}.$$

We will need the following result related to these disks.

Lemma 3.2. *For any $n \geq 1$ and $q > 0$, there exist $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$ such that*

- i) $\forall z \in D_{n,q}, |\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|,$*
- i') $\forall z \in \hat{D}_{n,q}, |\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|,$*
- ii) there are no zeros of $\phi(z) - q$ inside $D_{n,q},$*
- ii') there are no zeros of $\phi(z) - q$ inside $\hat{D}_{n,q}.$*

Proof. The proof of both *i)* and *ii)* relies on the fact that as $\varepsilon_{n,q}$ decreases, both ϕ and Φ_n behave like $t(z) = (\rho_n - z)^{-2}$ inside $D_{n,q}$. Indeed, $\phi(z) - a_n \rho_n^2 t(z)$ is bounded inside $D_{n,q}$ and so is $\Phi(z) - t(z)$.

We divide $D_{n,q}$ into eight radial subsets of equal size (and angle), as shown in Figure 3.4.1. The function t has the following properties

- in areas Im+, $\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough,
- in areas Im-, $-\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough,
- in areas Re+, $\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough,
- in areas Re-, $-\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough.

First, this means that for $\varepsilon_{n,q}$ small enough, $|\phi(z) - q| > 1$, yielding *ii)*. Moreover, for $\varepsilon_{n,q}$ small enough, there is either $\Im(\phi(z))\Im(\Phi_n(z)) > 0$ or $\Re(\phi(z) - q)\Re(\Phi_n(z)) > 0$ for any $z \in D_{n,q}$. This implies *i)* since $|x - y| = |x| + |y|$ if and only if 0 belongs to the segment $[x, y]$ in the complex plane (both imaginary and real parts must have opposite signs).

We prove *i')* and *ii')* likewise. □

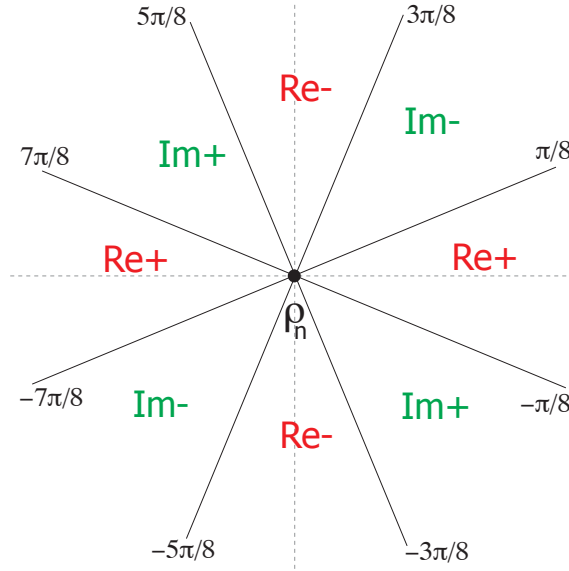


Figure 3.4.1: The subdivisions of $D_{n,q}$

Lastly, to cover the whole complex plane, we need to show the following lemma.

Lemma 3.3. *For $q > 0$, let $z^* = a + ib$ be a non-real zero of $\phi(z) - q$, then $|a| \geq \sqrt{3}|b|$.*

Proof. First, we have the following identities

$$\begin{aligned} \frac{2\rho(a+bi) - (a+bi)^2}{(\rho-a-bi)^2} &= \frac{-a^4 + 4a^3\rho - 5a^2\rho^2 - 2a^2b^2 + 2a\rho^3 - 3b^2\rho^2 + 4ab^2\rho - b^4}{((\rho-a)^2 + b^2)^2} \\ &+ i \frac{2b\rho^2(\rho-a)}{((\rho-a)^2 + b^2)^2}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \frac{2\rho(a+bi) + (a+bi)^2}{(\rho+a+bi)^2} &= \frac{a^4 + 4a^3\rho + 5a^2\rho^2 + 2a^2b^2 + 2a\rho^3 + 3b^2\rho^2 + 4ab^2\rho + b^4}{((\rho-a)^2 + b^2)^2} \\ &+ i \frac{2b\rho^2(\rho+a)}{((\rho+a)^2 + b^2)^2}, \end{aligned} \quad (3.10)$$

which leads to

$$\begin{aligned} \Re(\phi(z^*) - q) &= -q + \sum_{n=1}^{\infty} a_n \frac{-a^4 + 4a^3\rho_n - 5a^2\rho_n^2 - 2a^2b^2 + 2a\rho_n^3 - 3b^2\rho_n^2 + 4ab^2\rho_n - b^4}{((\rho_n-a)^2 + b^2)^2} \\ &+ \frac{\sigma^2}{2}(a^2 - b^2) + am \\ &- \sum_{n=1}^{\infty} \hat{a}_n \frac{a^4 + 4a^3\hat{\rho}_n + 5a^2\hat{\rho}_n^2 + 2a^2b^2 + 2a\hat{\rho}_n^3 + 3b^2\hat{\rho}_n^2 + 4ab^2\hat{\rho}_n + b^4}{((\hat{\rho}_n-a)^2 + b^2)^2}, \end{aligned} \quad (3.11)$$

and

$$\Im(\phi(z^*)) = \sigma^2 ab + bm + \sum_{n=1}^{\infty} a_n \frac{2b\rho_n^2(\rho_n-a)}{((\rho_n-a)^2 + b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2b\hat{\rho}_n^2(\hat{\rho}_n+a)}{((\hat{\rho}_n+a)^2 + b^2)^2}, \quad (3.12)$$

so that as $b \neq 0$ and $\Im(\phi(z^*)) = \Re((\phi(z^*) - q)) = 0$, after simplifications,

$$\begin{aligned} 0 &= \Re(\phi(z^*) - q) - \frac{a}{b} \Im(\phi(z^*)) \\ &= -q - (a^2 + b^2) \left(\frac{\sigma^2}{2} + \sum_{n=1}^{\infty} a_n \frac{a^2 - 4a\rho_n + 3\rho_n^2 + b^2}{((\rho_n-a)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{a^2 + 4a\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n+a)^2 + b^2)^2} \right). \end{aligned} \quad (3.13)$$

If $a > 0$ and $|a| < \sqrt{3}|b|$, then $\forall n \geq 1$,

$$a^2 - 4a\rho_n + 3\rho_n^2 + b^2 > \frac{4}{3}(a - 3\rho_n/2)^2 \geq 0,$$

and $a^2 + 4a\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2 > 0$ so that z^* cannot be a zero of $\phi(z) - q$. If $a < 0$ and $|a| < \sqrt{3}|b|$, then the contradiction is the same. Hence, for $a \neq 0$, the zero $z^* = a + ib$ must satisfy $|a| \geq \sqrt{3}|b|$.

Lastly, for $a = 0$,

$$\Re(\phi(z^*) - q) = -q - \frac{\sigma^2 b^2}{2} - \sum_{n=1}^{\infty} a_n \frac{3b^2\rho_n^2 + b^4}{(\rho_n^2 + b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{3b^2\hat{\rho}_n^2 + b^4}{(\hat{\rho}_n^2 + b^2)^2},$$

which is strictly negative for any real b , hence there are no purely imaginary zeros of $\phi(z) - q$. \square

The lemma tells us that there are no zeros of $\phi(z) - q$ in the angles $(\pi/6, 5\pi/6)$ and $(-5\pi/6, -\pi/6)$ of the complex plane. We are now ready to prove the proposition.

Proof of Proposition 3.1. We begin with R_n and \hat{R}_n . The aim of the proof is to show that there are $\varepsilon_{n,q}, \varepsilon_{n+1,q} > 0$ (resp $\hat{\varepsilon}_{n,q}, \hat{\varepsilon}_{n+1,q} > 0$) such that on the boundary of $K_n = R_n \setminus \{D_{n,q} \cup D_{n+1,q}\}$ (see Figure 3.4.2) (resp $\hat{K}_n = \hat{R}_n \setminus \{\hat{D}_{n,q} \cup \hat{D}_{n+1,q}\}$), the condition of Rouché's theorem applies, that is,

$$(**) \quad |\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|$$

(resp $|\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|$). To prove this, we will rely on the following equivalence, for $x, y \in \mathbb{C}$

$$|x - y| = |x| + |y| \iff \begin{cases} \Re(x)\Im(y) = \Im(x)\Re(y) \\ \Im(x)\Im(y) \leq 0 \\ \Re(x)\Re(y) \leq 0 \end{cases} \quad (3.14)$$

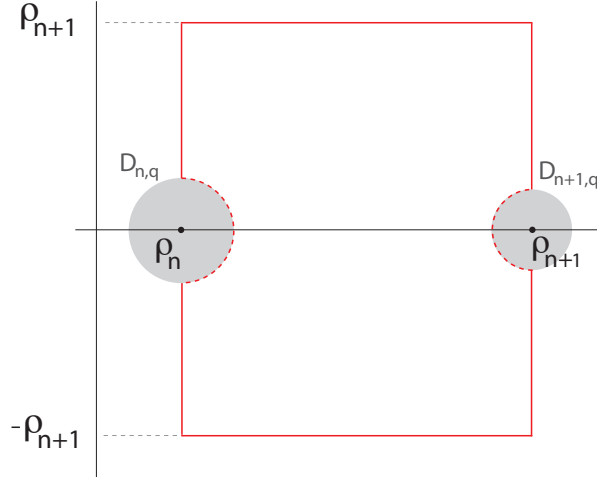


Figure 3.4.2: the contour K_n with S_n in hard (red) line

We are first interested in the horizontal and vertical segments of ∂K_n , a set which we denote by S_n (see Figure 2). Recall the expression of $\Re(\phi(a+ib)) - \frac{a}{b}\Im(\phi(a+ib))$ given by (3.13). Because of (*) (vertical segments), and Lemma 3.3 (horizontal segments), we have, for $z = a+ib \in S_n$, $\Re(\phi(a+ib)) - \frac{a}{b}\Im(\phi(a+ib)) < 0$. Hence, near the zeros of $\Im(\phi)$, $\Re(\phi)$ is negative. More precisely, there are two cases involving $\epsilon_n := \inf\{|\Im(\phi(z))|, z \in S_n\}$:

- either $\epsilon_n > 0$ and $\Im(\phi)$ has no zero on S_n ,
- or $\epsilon_n = 0$ and there exists $\epsilon > 0$ such that

$$V_{n,q,\epsilon} = \{z \in S_n, |\Im(\phi(z))| < \epsilon, \Re(\phi(z) - q) < 0\} \neq \emptyset.$$

In either case, on $S_n \setminus V_{n,q,\epsilon}$, $|\Im(\phi)|$ is bounded from below by some $\epsilon > 0$.

We want to prove (**) on the following three sets: $V_{n,q,\epsilon}$, $S_n \setminus V_{n,q,\epsilon}$ and $\partial K_n \setminus S_n$ (this last set consisting in the two semi-circles).

- by Lemma 3.1, $\Re(\Phi_n) < 0$ on S_n thus (3.14) ensures that (**) holds on $V_{n,q,\epsilon}$ if it is not empty.
- on $S_n \setminus V_{n,q,\epsilon}$, $|\Im(\phi)| \geq \epsilon > 0$ and both $\Im(\Phi_n)$ and $\Re(\phi)$ are bounded, it is therefore possible to find a q_n (in the definition of Φ_n) such that

$$|\Re(\phi(z) - q)\Im(\Phi_n(z))| < |\Im(\phi(z))\Re(\Phi_n(z))|, \quad z \in S_n \setminus V_{n,q},$$

hence, by (3.14), (**) holds on $S_n \setminus V_{n,q}$.

- for the two semi-circles of ∂K_n , we invoke Lemma 3.2. The radii must be chosen small enough for the zeros of Φ_n to be in K_n .

Therefore, by Rouché's theorem, for any $q > 0$, under (*), the function $\phi(z) - q$ has exactly 2 zeros (or a double zero) in R_n .

The proof is identical for the sets \hat{R}_n . For R_0 , it is easy to see that the properties of Φ_n and $\hat{\Phi}_n$ described in Lemma 3.1 also hold for

$$\Phi_0 = \frac{1}{(\hat{\rho}_1 + z)^2} + \frac{1}{(\rho_1 - z)^2} - q_0,$$

on ∂R_0 and hence the same reasoning applies (with the proper K_0, S_0 and $V_{0,q,\epsilon}$).

Lastly, the 3 sets were constructed so that with Lemma 3.3, the whole complex plane is covered. This is shown in the following representation of the complex plane (figure 3.4.3). □

We denote by ζ_n^+ and ζ_n^- the two roots in R_n . If they are complex then $\Im(\zeta_n^+) > \Im(\zeta_n^-)$, if not, then $\Re(\zeta_n^+) \geq \Re(\zeta_n^-)$. The equivalent notations hold for R_0 and \hat{R}_n .

3.4.3 Proof and discussion of the theorem

This section is divided into three parts. First, we prove Theorem 3.3, using Proposition 3.1 and Kuznetsov's paper [63]. Then we discuss a possible generalization when the poles of ϕ are allowed to have any finite order. Lastly, we introduce a simple condition which implies (*).

Proof of Theorem 3.3

The proof will rely on the following lemma. We denote by \log the principal branch of the complex logarithm defined on $\mathbb{C} \setminus \mathbb{R}_-$.

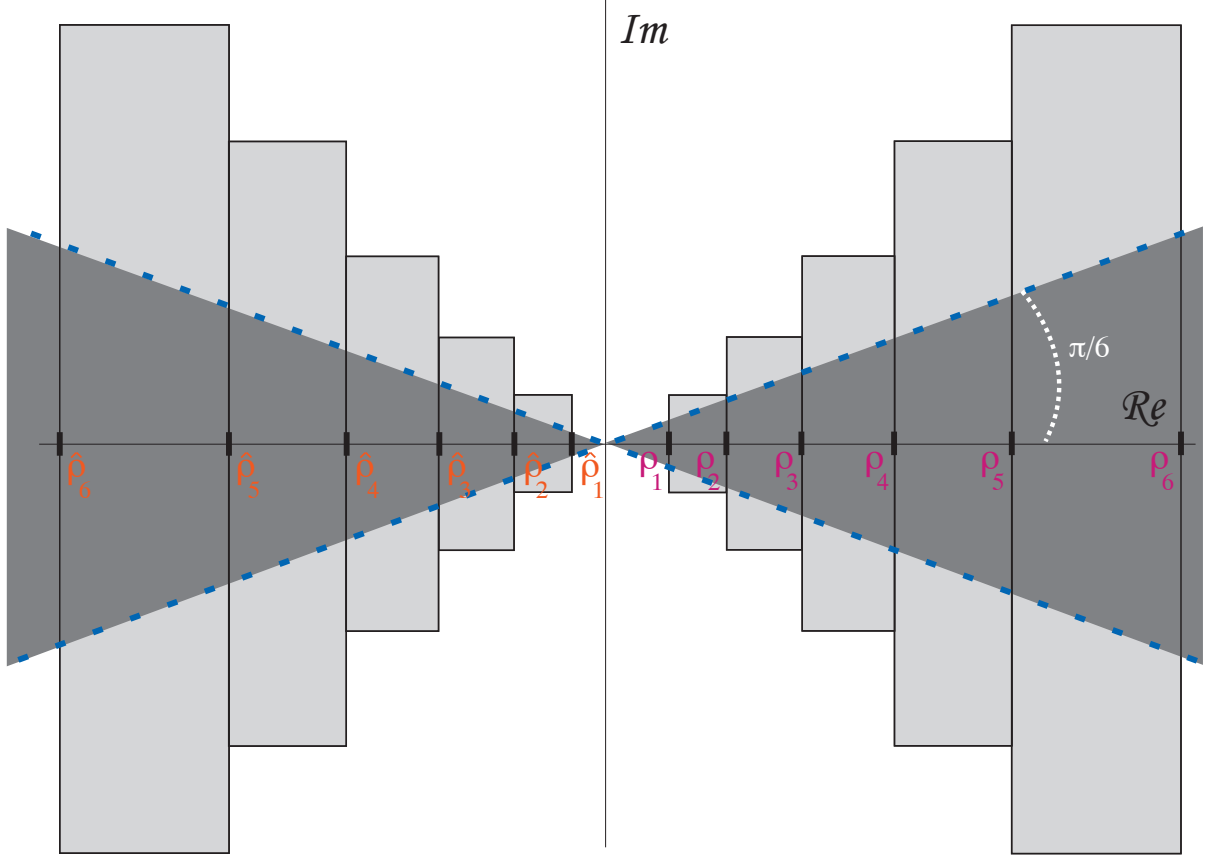


Figure 3.4.3: The zeros lie within the dark grey area

Lemma 3.4. *Define*

$$A^\pm(z) = \left| \frac{1}{z} \log \left(\prod_{n=1}^{\infty} \frac{\zeta_n^\pm(\rho_n - z)}{\rho_n(\zeta_n^\pm - z)} \right) \right|, \quad \hat{A}^\pm(z) = \left| \frac{1}{z} \log \left(\prod_{n=1}^{\infty} \frac{\hat{\zeta}_n^\pm(\hat{\rho}_n + z)}{\hat{\rho}_n(\hat{\zeta}_n^\pm - z)} \right) \right|,$$

Then,

$$A^\pm(z) \text{ (resp } \hat{A}^\pm(z)) \rightarrow 0, \quad \text{as } |z| \rightarrow \infty, \quad \Re(z) \leq 0 \text{ (resp } \Re(z) \geq 0).$$

Proof. First note that by Proposition 3.1 and (3.6),

$$\left| \frac{\zeta_n^\pm(\rho_n - z)}{\rho_n(\zeta_n^\pm - z)} - 1 \right| = \left| z \frac{\rho_n - \zeta_n^+}{\rho_n(\zeta_n^+ - z)} \right| = O(n^{-1-\varepsilon}), \quad n \rightarrow \infty,$$

so that the infinite product is well defined. Again by Proposition 3.1 and (3.6),

$|\zeta_n^+/\rho_n| > 1$. Moreover, $\left| \frac{\rho_n - z}{\zeta_n^+ - z} \right| \rightarrow 1$ as $|z| \rightarrow \infty$, $\Re(z) \leq 0$. The function

$z \mapsto \left| \frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} \right|$ is thus bounded from below for $\Re(z) \leq 0$ and there is therefore

$C > 0$ such that for $\Re(z) \leq 0$ and $n \geq 1$,

$$\left| \log \left(\frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} \right) \right| \leq C \left| \frac{\zeta_n^+(\rho_n - z)}{\rho_n(\zeta_n^+ - z)} - 1 \right|.$$

Hence,

$$A^+(z) \leq \left| \sum_{n=1}^{\infty} \frac{\rho_n - \zeta_n^+}{\rho_n(\zeta_n^+ - z)} \right| = O \left(\sum_{n=1}^{\infty} \frac{1}{|\zeta_n^+ - z|} \right).$$

Proposition 3.1 and (3.6) ensure that the infinite sum is finite for any z with negative real part and by dominated convergence, $A^+(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

The proof for A^- , \hat{A}^\pm is achieved in a similar fashion. \square

Proof of Theorem 3.3. First note that due to condition (3.6) and Proposition 3.1,

$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n^\pm|}$ is finite, so that the order of the entire function

$$z \mapsto (q - \phi(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n} \right)^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\hat{\rho}_n} \right)^2$$

is less than one (see [76], lecture 5, for instance). Hence, the only possible Hadamard/Weierstrass representation ([76] page 26) for $\frac{q}{q - \phi(z)}$ is

$$\frac{q}{q - \phi(z)} = e^{kz} \frac{1}{1 - \frac{z}{\zeta_0^+}} \frac{1}{1 - \frac{z}{\zeta_0^-}} \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^-}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\zeta_n^-}}, \quad (3.15)$$

for some $k \in \mathbb{C}$ after arrangement.

We rely on Lemma 3.4 to prove that $k = 0$ (in the same way as in the end of the proof of Theorem 5 in [60], using the fact that $\phi(iz) = O(z^2)$ for $z \rightarrow \infty$, $z \in \mathbb{R}$). Lastly, all the conditions of Theorem 1 (f) from [63] are fulfilled, which completes the proof. \square

Towards a generalization

It is natural to ask what might happen if the density of the Lévy measure had the more general form

$$\pi(x) = \mathbb{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \frac{\rho_n^{m_n}}{(m_n - 1)!} x^{m_n-1} e^{-\rho_n x} + \mathbb{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{m}_n - 1)!} (-x)^{\hat{m}_n-1} e^{\hat{\rho}_n x},$$

where $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ satisfy the usual conditions and $(m_n, \hat{m}_n) \in \{1, \dots, M\}$, ($M < \infty$), which would give

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\rho_n^{m_n}}{(\rho_n - z)^{m_n}} - 1 - \frac{zm_n}{\rho_n} \right) + \sum_{n=1}^{\infty} \hat{a}_n \left(\frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{\rho}_n + z)^{\hat{m}_n}} - 1 + \frac{z\hat{m}_n}{\hat{\rho}_n} \right).$$

The case $M = 2$ can be treated using the ideas of the present paper (with a slightly different condition $(*)$ and four pairs of test functions). The central problem remains to get precise results on the location of the zeros of $\phi(z) - q$.

In the general case, finding a multiple zero is not easy and we cannot apply Theorem 3.3. However, heuristically, if we consider $q \rightarrow \infty$, we see that, asymptotically, the $(\rho_n, -\hat{\rho}_n)$ become zeros of order m_n and \hat{m}_n . Hence, when q decreases, the zeros should be located in the vicinity, in some sense, of the ρ_n and $-\hat{\rho}_n$. Empirically, many computations show that, in fact, each ρ_n engenders m_n zeros, all of which are located inside circles with center ρ_n , radius ρ_n and all the more close to ρ_n than q is large. However, this fact (which requires formal proof) does not suffice to show the convergence $A(z) \rightarrow 0$ in the proof of the theorem.

In this spirit, we would like expose a set of conditions under which the theorem remains valid. Namely,

Theorem 3.5. *If there is an ordering of the zeros and poles, repeated according to multiplicity, such that*

i) $\phi(z) - q$ is meromorphic with real poles $\rho_n, -\hat{\rho}_n$ and zeros $\zeta_n, \hat{\zeta}_n$ with $\Re(\zeta_n) > 0$, $\Re(\hat{\zeta}_n) < 0$,

ii) the series with terms $\frac{1}{\rho_n}, \frac{1}{\hat{\rho}_n}, \frac{1}{\zeta_n}, \frac{1}{\hat{\zeta}_n}$ are absolutely convergent,

iii) $\forall n > 0$, $|\zeta_n - \rho_n| < C|\rho_n|$, $|\hat{\zeta}_n - \hat{\rho}_n| < \hat{C}|\hat{\rho}_n|$ for some $C, \hat{C} > 0$,

then (3.8) holds.

Proof. Condition *ii)* ensures that $q/(\phi(z) - q)$ is the ratio of two entire functions of order less than one and has thus a similar form as (3.15), namely

$$\frac{q}{q - \phi(z)} = e^{cz} \frac{h_+(z)}{g_+(z)} \frac{h_-(z)}{g_-(z)},$$

where h_{\pm}, g_{\pm} are holomorphic in \mathbb{C} , with zeros in $\mathbb{C}^{\pm} = \{z \in \mathbb{C}, \pm \Re(z) > 0\}$ normalized so that $h_{\pm}(0) = g_{\pm}(0) = 1$. Then, as in the proof of Theorem 3.3, condition *iii)* will ensure that $z^{-1} \log \left(\frac{h_{\pm}(z)}{g_{\pm}(z)} \right) \rightarrow 0$ as $z \rightarrow \mp \infty$, so that the factorization is indeed of Wiener-Hopf type (and $c = 0$). \square

Remark 3.4.1. We wish to show the connexion between the papers on meromorphic Lévy-Khintchine exponents and the ideas in [40]. For simplicity, we will consider $q \in \mathcal{Q} := \{q > 0, \phi(z) - q \text{ has only simple zeros}\}$. Once the zeros are located, Theorem 1 in [61] is a special case of Theorem 4.1 in [40]. It is however possible to consider complex zeros, as long as the proof of Lemma 3.1 in [40] remains valid. This can only happen if the arguments of the zeros stay away from $\pi/2$ and $-\pi/2$. Condition *iii)* in Theorem 3.5 implies this and it is verified in [60], [61] and [66], as long as the $(\rho_n, \hat{\rho}_n)$ in those papers have at most an exponential growth. It is thus possible to very briefly prove Theorem 3.5 using the fact that under *i)*, *ii)* and *iii)*,

the series

$$\sum_{n=1}^{\infty} \frac{h_+(\zeta_n)}{g'_+(\zeta_n)} e^{-x\zeta_n}, \quad \sum_{n=1}^{\infty} \frac{h_-(\hat{\zeta}_n)}{g'_-(\hat{\zeta}_n)} e^{-x\hat{\zeta}_n}$$

are convergent for $x > 0$ and can then be proven to be the densities of S_{e_q} and I_{e_q} using dominated convergence, as in [40], Theorem 4.1 (using Lemma 3.2 of the same article).

Conditions *ii*) and *iii*) in the theorem depend directly on the location of the $\zeta_n, \hat{\zeta}_n$. In the next section, we provide an example for which their position is asymptotically determined. But before we can proceed, we must find conditions which are easily verified and under which (*) holds.

Some remarks on (*)

It is easy to find examples for which (*) fails. However, for many cases when a_n, \hat{a}_n and $\rho_n, \hat{\rho}_n$ are expressed using basic functions (exponential, power), (*) will in fact hold. We provide an example below.

We split (*) into its two inequalities: the upper $(*)_1$ and the lower $(*)_2$, and we will only comment on $(*)_1$ because the transposition to $(*)_2$ will be immediate.

We are in fact only interested in the term $T(\rho_j, b) = \sum_{n=1}^{\infty} a_n \frac{(\rho_j - 3\rho_n)(\rho_j - \rho_n) + b^2}{((\rho_n - \rho_j)^2 + b^2)^2}$,

since it is the only one which can be negative. Furthermore, if we consider $l(j) = \sup\{k \geq 1, 3\rho_{j-k} \geq \rho_j\}$ (with $l(j) = 0$ if $3\rho_{j-1} \leq \rho_j$), we have

$$\begin{aligned} T(\rho_j, b) &> \sum_{k=1}^{l(j)} \left[\frac{a_j}{b^2 l(j)} + a_{j+k} \frac{(\rho_{j+k} - \rho_j)(3\rho_{j+k} - \rho_j) + b^2}{((\rho_{j+k} - \rho_j)^2 + b^2)^2} \right. \\ &\quad \left. + a_{j-k} \frac{(\rho_{j-k} - \rho_j)(3\rho_{j-k} - \rho_j) + b^2}{((\rho_{j-k} - \rho_j)^2 + b^2)^2} \right] \\ &:= \sum_{k=1}^{l(j)} t_k(\rho_j, b). \end{aligned} \tag{3.16}$$

Now, $t_k(\rho_j, b)$ can only be negative if $3\rho_{j-k} - \rho_j > 0$. This seldom happens if ρ_n increases quickly. For instance, if $\rho_n = c^n$ for $c > 1$, then $3\rho_{j-k} - \rho_j > 0 \iff k < \frac{\ln(3)}{\ln(c)}$. In this case, if $(a_n)_{n \geq 1}$ is smooth enough (increasing, for instance), it is not hard to show that $(*)_1$ holds since for any $j \geq 1$, there are only a fixed number of negative terms. However, if ρ_n increases at a slower rate (typically of power type), then the number of negative terms increases with j . To show that $(*)_1$ holds then requires some additional conditions, a set of which is detailed below.

Proposition 3.2. *If the following conditions are fulfilled,*

- i)* $(a_n)_{n \geq 1}$ is increasing,
- ii)* $\forall n \geq 2, \rho_{n+1} - \rho_n \geq (\rho_n - \rho_{n-1}),$

$$iii) \forall j > k \geq 1,$$

$$3\rho_{j-k} - \rho_j \geq 0 \implies 2(\rho_j - \rho_{j-k})^3 \geq (3\rho_{j-k} - \rho_j)(\rho_{j+k} - \rho_j)(\rho_{j+k} - 2\rho_j + \rho_{j-k}),$$

then $(*)_1$ holds.

Proof. We want to study $t_k(\rho_j, b)$ as a function of b . The idea is to show that any possible negative term indexed by $j - k$ in (3.16) is absolutely smaller than its $j + k$ counterpart. It is obvious that, by *i*), it is sufficient to prove this for a constant sequence $(a_n)_{n \geq 1}$; hence we set $a_j := 1$ for all $j \geq 1$. For notational convenience, we denote

$$A_1 = \rho_{j+k} - \rho_j, \quad A_2 = 3\rho_{j+k} - \rho_j, \quad B_1 = \rho_j - \rho_{j-k}, \quad B_2 = 3\rho_{j-k} - \rho_j,$$

which are all positive (the case $B_2 < 0$ is irrelevant) and satisfy $A_2 = 3A_1 + 3B_1 + B_2$. Omitting the constant term in (3.16),

$$\begin{aligned} t_k(\rho_j, b) &\geq \frac{A_1 A_2 + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} = \frac{A_1(3A_1 + 3B_1 + B_2) + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} \\ &\geq \frac{c_0 + c_2 b^2 + c_4 b^4 + 2b^6}{(A_1^2 + b^2)^2 (B_1^2 + b^2)^2}, \end{aligned}$$

where

$$\begin{aligned} c_0 &= -A_1^4 B_1 B_2 + A_1 B_1^4 B_2 + 3A_1^2 B_1^4 + 3A_1 B_1^5, \\ c_2 &= -2A_1^2 B_1 B_2 + 2A_1 B_1^2 B_2 + 6A_1^2 B_1^2 + 6A_1 B_1^3 + A_1^4 + B_1^4, \\ c_4 &= -B_1 B_2 + A_1 B_2 + 5A_1^2 + 3A_1 B_1 + 2B_1^2, \end{aligned}$$

Note that by *ii*), $A_1 \geq B_1$, hence $c_4 > 0$. Condition *iii*) : $2B_1^3 \geq A_1 B_2 (A_1 - B_1)$ implies

$$c_0 \geq -2B_1^6 + A_1^2 B_1^4 + A_1 B_1^5 \quad c_2 \geq -4B_1^4 + 6A_1^2 B_1^2 + 6A_1 B_1^3 + A_1^4 + B_1^4,$$

which are both positive, by *ii*). This leads to $t_k(\rho_j, b) \geq 0$; the sum in (3.16) is therefore positive and $(*)_1$ holds. \square

This result calls for a few comments. First of all, many positive terms have been left out in the proof, thus $(*)_1$ holds under much weaker conditions. Furthermore, for any explicit formulation of a_n and ρ_n , *i*) is usually easily verified, and so is the convexity condition *ii*) which is in fact not too restrictive, given (3.6). However, *iii*) is much harder to prove. If we consider $\rho_n = n^\alpha$ for $\alpha > 1$, then we are interested in

$$2(j^\alpha - (j - k)^\alpha)^3 - (3(j - k)^\alpha - j^\alpha)((j + k)^\alpha - j^\alpha)((j + k)^\alpha - 2j^\alpha + (j - k)^\alpha),$$

and denoting k as a proportion of j : $k = cj$, this becomes

$$[2(1 - (1 - c)^\alpha)^3 - (3(1 - c)^\alpha - 1)((1 + c)^\alpha - 1)((1 + c)^\alpha - 2 + (1 - c)^\alpha)]j^{3\alpha},$$

Using numerical softwares, it is possible to show that this function of the variable c is increasing and positive on $(0, 1)$ for $1 < \alpha < 15$ (and in fact positive for $1 < \alpha \leq 15.87$). Note that $3\rho_{j-k} - \rho_j \geq 0 \iff k \leq \frac{3^{1/\alpha}-1}{3^{1/\alpha}}j$, hence the positivity criterion should only be checked for $c \in \left(0, \frac{3^{1/\alpha}-1}{3^{1/\alpha}}\right)$.

Other techniques can be used to show that $(*)_1$ also holds for $\alpha > 15.87$ when a_n is increasing. They rely on the fact that, as in the exponential case, there are, proportionally, very few negative terms in $T(\rho_j, b)$.

3.4.4 An example

Our goal in this section is to be able to locate precisely the zeros of ϕ for one exponent involving explicit functions. We will show that even in this very simple case, this is not so easy to achieve. We set $\rho_n = \hat{\rho}_n = n^2$ and unit a_n and \hat{a}_n , which leads to the the following Lévy measure,

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n=1}^{\infty} x n^4 e^{-n^2 x} + \mathbb{1}_{\{x<0\}} \sum_{n=1}^{\infty} (-x) n^4 e^{n^2 x}.$$

First note that in this case, given the remarks of the preceding section, $(*)$ holds. Moreover, it is easy to show (see Proposition 4 in [61]) that as $x \rightarrow 0^\pm$, $\pi(x) = O(|x|^{-3/2})$, which confirms that we can take a zero cut-off function ($h := 0$) and the following simplified representation (see (3.7)) for ϕ :

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} \frac{2zn^2 - z^2}{(n^2 - z)^2} - \sum_{n=1}^{\infty} \frac{2zn^2 + z^2}{(n^2 + z)^2}.$$

This particular choice of ϕ was made to exhibit known functions. Using the fact that

$$\frac{2zn^2 - z^2}{(n^2 - z)^2} = \frac{1}{4} \left(\frac{-6z}{z - n^2} + 2 \frac{z^2 + zn^2}{(z - n^2)^2} \right),$$

and relations 1.421-3, 1.422-4 (second equality) in [47], we get

$$\sum_{n=1}^{\infty} \frac{2n^2 z - z^2}{(n^2 - z)^2} = \frac{1}{4} \left(2 - 3\pi\sqrt{z} \cot(\pi\sqrt{z}) + \pi^2 z \csc(\pi\sqrt{z})^2 \right),$$

and substituting $i\sqrt{z}$ for \sqrt{z} ,

$$- \sum_{n=1}^{\infty} \frac{2n^2 z + z^2}{(n^2 + z)^2} = \frac{1}{4} \left(2 - i3\pi\sqrt{z} \cot(i\pi\sqrt{z}) - \pi^2 z \csc(i\pi\sqrt{z})^2 \right),$$

where \cot and \csc are the usual cotangent and cosecant functions. Recalling

$$\csc(z)^2 = \sin(z)^{-2} = \frac{\cos(z)^2 + \sin(z)^2}{\sin(z)^2} = 1 + \cot(z)^2, \quad (3.17)$$

ϕ can in this case be expressed solely in terms of the cotangent function

$$\begin{aligned}\phi(z) &= mz + \frac{\sigma^2}{2}z^2 + 1 \\ &+ \frac{1}{4} \left[\pi\sqrt{z} \cot(\pi\sqrt{z})(-3 + \pi\sqrt{z} \cot(\pi\sqrt{z})) + i\pi\sqrt{z} \cot(i\pi\sqrt{z})(-3 + i\pi\sqrt{z} \cot(i\pi\sqrt{z})) \right] \\ &:= mz + \frac{\sigma^2 z^2}{2} + 1 + \frac{1}{4}(c_+(z) + c_-(z)).\end{aligned}$$

We refer to section 4.3 of [2] for the behavior of the cotangent function in the complex plane. One of its useful properties is that if z has a large imaginary part, then both the real and imaginary parts of $\cot(z)$ can be accurately estimated. More precisely, by Euler's formula and equation 4.3.58 from [2], for any $a_n > 0$ and for any $b_n > 0$ large enough,

$$\begin{aligned}\cot(a_n \pm ib_n) &= \cot(-a_n \pm ib_n) \\ &= 2 \sin(2a) e^{-2b_n} + i(\mp 1 + 2 \cos(2a) e^{-2b_n}) + o(e^{-2b_n})(1 + i). \quad (3.18)\end{aligned}$$

Lastly, recall that the zeros $\zeta_n^\pm = a_n \pm ib_n$ of $\phi(z) - q$ are ordered so that the series $(a_n)_{\{n \geq 1\}}$ is increasing.

Lemma 3.5. *There are, asymptotically, no real zeros of $\phi(z) - q$, except if $\sigma = 0$ and $m \geq \pi^2/4$ (resp $m \leq -\pi^2/4$), in which case, $\hat{\zeta}_n^\pm$ (resp ζ_n^\pm) are real.*

Proof. The proof lies in the fact that, by (3.18), for z real large enough, $c_-(z) = c_+(-z) \sim (-3\pi\sqrt{z} + \pi^2 z)/4$. Moreover, both c_+ and c_- are U -shaped between their poles with a negative local minimum which is close to -0.5 . The local minima of ϕ thus go to $+\infty$ (yielding only complex zeros) or to $-\infty$ (yielding only real zeros). \square

We will now focus on ζ_n^\pm , as the transposition to $\hat{\zeta}_n^\pm$ is immediate. Note that ζ_n^\pm verifies

$$\begin{aligned}\frac{\sigma^2}{2}(\zeta_n^\pm)^2 + m\zeta_n^\pm + 1 - q + \frac{1}{4} \left[-3i\pi\sqrt{\zeta_n^\pm} \cot\left(i\pi\sqrt{\zeta_n^\pm}\right) - \pi^2\zeta_n^\pm \cot\left(i\pi\sqrt{\zeta_n^\pm}\right)^2 \right] \\ = -\frac{1}{4} \left[-3\pi\sqrt{\zeta_n^\pm} \cot\left(\pi\sqrt{\zeta_n^\pm}\right) + \pi^2\zeta_n^\pm \cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2 \right]. \quad (3.19)\end{aligned}$$

For ζ_n^\pm away from the zeros of $\Re(\cot(\pi\sqrt{z}))$, dividing (3.19) by ζ_n^\pm yields

$$\Im \left(\frac{3\pi \cot(\pi\sqrt{\zeta_n^\pm})}{4\sqrt{\zeta_n^\pm}} - \frac{\pi^2}{4} \cot(\pi\sqrt{\zeta_n^\pm})^2 \right) = \pm \frac{\sigma^2}{2} b_n + O(\Im((\zeta_n^\pm)^{-1})), \quad n \rightarrow +\infty, \quad (3.20)$$

and

$$\Re \left(-\frac{\pi^2}{4} \cot(\pi\sqrt{\zeta_n^\pm})^2 \right) = \frac{\sigma^2}{2} a_n + m + \frac{\pi^2}{4} + R(n) + o(R(n)), \quad n \rightarrow +\infty, \quad (3.21)$$

where $R(n) = -\frac{3\pi}{4} \Re((\zeta_n^\pm)^{-1/2} [\cot(\pi\sqrt{\zeta_n^\pm}) + i \cot(i\pi\sqrt{\zeta_n^\pm})])$.

We are now able to accurately locate the ζ_n^\pm , asymptotically, in all possible cases.

Proposition 3.3. *As $n \rightarrow +\infty$,*

- *if $\sigma > 0$, $\zeta_n^\pm = n^2 \pm i\sqrt{2/\sigma^2} + O(n^{-2})(1+i)$,*
- *if $\sigma = 0$ and $m > 0$, then $\zeta_n^\pm = n^2 + c + O(n^{-1}) \pm i\frac{n}{\pi} \left(\cosh^{-1} \left(1 + \frac{\pi^2}{2m} \right) + O(n^{-1}) \right)$,*
- *if $\sigma = 0$ and $m \in (-\pi^2/4, 0)$, then*

$$\zeta_n^\pm = n^2 + n + c + O(n^{-1}) + \pm i\frac{n}{\pi} \left(\cosh^{-1} \left(-1 - \frac{\pi^2}{2m} \right) + O(n^{-1}) \right),$$
- *if $\sigma = 0$ and $m = 0$, $\Re(\sqrt{\zeta_n^\pm}) = n + 3/8 + O(\log(n)n^{-2})$ and $\frac{2\pi\Im(\sqrt{\zeta_n^+})}{\log(\frac{4\pi}{3\sqrt{2}}n)} \rightarrow 1$,*
- *if $\sigma = 0$ and $m \leq -\pi^2/4$, then $\zeta_n^- = n^2 + \frac{n}{\pi} \left(\cos^{-1} \left(1 + \frac{\pi^2}{2m} \right) + cn^{-1} + o(n^{-1}) \right)$
and $\zeta_n^+ = (n+1)^2 - \frac{n}{\pi} \left(\cos^{-1} \left(1 + \frac{\pi^2}{2m} \right) + cn^{-1} + o(n^{-1}) \right)$,*

for some irrelevant constants c , and where \cos^{-1} and \cosh^{-1} are defined on $[-1, 1]$ and $[1, +\infty)$ respectively.

Proof. The proof being quite lengthy, some minor steps and details will be omitted. Since the zeros will be located in the vicinity, in some sense, of the poles of ϕ , we will keep the following notation throughout the proof $\sqrt{\zeta_n^\pm} = n + d_n \pm i\frac{b_n}{2(n+d_n)}$. The periodicity of the cotangent function in the real variable yields

$$\begin{aligned} \cot \left(\pi \sqrt{\zeta_n^\pm} \right) &= \cot \left(\pi d_n \pm i \frac{\pi b_n}{2(n+d_n)} \right) \\ &= \cot \left(\pi (2d_n^2 + 2nd_n \pm ib_n) \frac{1}{2(n+d_n)} \right). \end{aligned} \quad (3.22)$$

If $\sigma > 0$, then by (3.21), $b_n/n \rightarrow 0$ and $d_n \rightarrow 0$ (if not the real part on the l.h.s. would be bounded).

Using the following series expansions for $z \rightarrow 0$, (4.3.70 in [2])

$$\cot(\pi(a+ib)z)^2 = (\pi z(a+ib))^{-2} - 2/3 + O(((a+ib)z)^2),$$

we have the following asymptotics as $n \rightarrow +\infty$,

$$\begin{aligned}
-\frac{\pi^2}{4} \cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2 &= -\frac{(n+d_n)^2}{(2d_n^2+2nd_n \pm ib_n)^2} + \pi^2/6 + o(1)(1+i) \\
&= -(n+d_n)^2 \left(\frac{4(d_n^2+nd_n)^2 - b_n^2}{(4(nd_n+d_n^2)^2 + b_n^2)^2} \mp i \frac{4(nd_n+d_n^2)b_n}{(4(nd_n+d_n^2)^2 + b_n^2)^2} \right) \\
&\quad + \pi^2/6 + o(1)(1+i) \\
&= n^2 \frac{b_n^2 - 4n^2 d_n^2}{(4n^2 d_n^2 + b_n^2)^2} + \frac{\pi^2}{6} \pm in^2 \frac{4nd_n b_n}{(4n^2 d_n^2 + b_n^2)^2} + o(1)(1+i). \tag{3.23}
\end{aligned}$$

From (3.21) and (3.20), it follows that

$$n^2 \frac{b_n^2 - 4n^2 d_n^2}{(4n^2 d_n^2 + b_n^2)^2} = \frac{\sigma^2}{2} n^2 + o(n^2), \quad n \rightarrow +\infty, \tag{3.24}$$

$$\pm n^2 \frac{4nd_n b_n}{(4(nd_n)^2 + b_n^2)^2} = \pm \frac{\sigma^2}{2} b_n + o(1), \quad n \rightarrow +\infty. \tag{3.25}$$

Because $\sigma > 0$, (3.24) imposes that both b_n and nd_n do not diverge. Therefore, the l.h.s. of (3.25) implies that either b_n or nd_n goes to 0 and because the r.h.s. in (3.24) is positive, then it must be $nd_n \rightarrow 0$, which gives $b_n \rightarrow \sqrt{2/\sigma^2}$. With (3.25), this yields $d_n = O(n^{-3})$. All these facts imply that the $o(n^2)$ in (3.24) (which stems from (3.21)) is in fact a $O(1)$, which completes the proof.

If $\sigma = 0$ and $m \neq 0$, then using the eulerian representation of the cotangent function (see 4.3.58 in [2] for instance), (3.20) and (3.21) are rewritten into

$$\frac{\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 - \sin(2\pi d_n)^2}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)^2} = 1 + \frac{4m}{\pi^2} + \frac{4}{\pi^2} R(n) + o(R(n)), \quad n \rightarrow \infty, \tag{3.26}$$

$$\begin{aligned}
\pm \frac{2\pi \sinh\left(\frac{\pi b_n}{n+d_n}\right) \sin(2\pi d_n)}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)^2} &- \frac{3 \left(\frac{b_n}{2(n+d_n)} \sin(2\pi d_n) \pm (n+d_n) \sinh\left(\frac{\pi b_n}{n+d_n}\right) \right)}{\left((n+d_n)^2 + \frac{b_n^2}{4(n+d_n)^2} \right) \left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n) \right)} \\
&= O(\Im((\zeta_n^\pm)^{-1})), \quad n \rightarrow \infty, \tag{3.27}
\end{aligned}$$

where both $R(n)$ and $O(\Im((\zeta_n^\pm)^{-1}))$ converge to 0.

Notice that for $b_n/n \rightarrow +\infty$, the l.h.s. of (3.26) converges to 1, thus b_n/n must be bounded (since $m \neq 0$). Hence, because 0 is a pole for $\Im(\cot(z))$, (3.27) imposes that either $\sin(2\pi d_n)$ or b_n/n goes to 0. In fact, because of the positivity of (3.26), $\sin(2\pi d_n) \rightarrow 0$ while $b_n = O(n)$. Note that this gives $\Im((\zeta_n^\pm)^{-1}) = O(n^{-3})$ and $R(n) = O(n^{-1})$. Equation (3.26) then imposes $d_n \rightarrow 0$ if $m > 0$ and $d_n \rightarrow 1/2$ if $m < 0$. For $m > 0$, it can then be rewritten into

$$\frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right)^2 - 1}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1\right)^2} = \frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right) + 1}{\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1} = 1 + \frac{4m}{\pi^2} + O(n^{-1}), \quad n \rightarrow \infty,$$

thereby yielding the constant in the imaginary part. Using the Taylor expansion of the sinus function at 0, (3.27) implies $d_n = (d + \delta_n)/n$ and is simplified into

$$n^{-1} \left(\pm \frac{4\pi^2(d + \delta_n)}{\cosh\left(\frac{\pi b_n}{n+d_n}\right) - 1} \mp 3 + O(n^{-2}) \right) = O(n^{-3}), \quad n \rightarrow \infty,$$

from which d and $\delta_n = O(n^{-2})$ can be inferred. Writing $b_n = (b + \beta_n)n$ for $\beta_n \rightarrow 0$, and recalling the expansion

$$\sinh(\pi(b_n + \beta_n)) = \sinh(\pi b_n) + \pi\beta_n \cosh(\pi b_n) + o(\beta_n),$$

implies that $\beta_n = O(n^{-1})$, by (3.26). Note that in this case, the constant c in the proposition depends on d and b . The case $m \in (-\pi^2/4, 0)$ is treated similarly.

The case $\sigma = 0$ and $m = 0$ is very special as, by (3.26), it is in fact necessary that $b_n/n \rightarrow +\infty$. More precisely, (3.27) yields $\frac{\pi b_n}{n \log(\delta n)} \rightarrow 1$ for some constant $\delta > 0$ which verifies

$$\frac{4\pi \sin(2\pi d_n)}{\delta n} - 3 \left(\frac{\log(\delta n) \sin(2\pi d_n)}{\delta n^3} + \frac{1}{n} \right) = O(\log(n)n^{-3}), \quad n \rightarrow \infty,$$

From the Taylor expansion of the sinus function at 0 and the fact that $\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$, we can deduce that $d_n = d + O(\log(n)n^{-2})$ with $4\pi \sin(2\pi d) = 3\delta$. Because $b_n/n \rightarrow +\infty$, $|\Im((\zeta_n^\pm)^{1/2})| \rightarrow +\infty$ and hence, by (3.18), $R(n) = -3\pi/4n^{-1} + o(n^{-1})$. As $\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \cosh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \delta n/2$, $n \rightarrow +\infty$, it can be deduced from (3.26) that

$$\frac{4 \cos(2\pi d_n)}{\delta n} + O(n^{-2}) = -\frac{3}{\pi n} + o(n^{-1}), \quad n \rightarrow \infty,$$

from which we infer $d = 3/8$ and $\delta = \frac{4\pi}{3} \sin(3\pi/4)$.

Lastly, **if $\sigma = 0$ and $m < \pi^2/4$** , writing $\sqrt{\zeta_n^-} = n + d_n$ and again using the eulerian form of the cotangent function and (3.21),

$$-\frac{\sin(2\pi d_n)^2}{(\cos(2\pi d_n) - 1)^2} = \frac{\cos(2\pi d_n) + 1}{\cos(2\pi d_n) - 1} = 1 + \frac{4m}{\pi^2} + \frac{3}{\pi} \left[1 + \frac{\sin(2\pi d_n)}{\cos(2\pi d_n) - 1} \right] n^{-1} + o(n^{-1}),$$

which gives $d_n \rightarrow d = \cos^{-1}\left(1 + \frac{\pi^2}{2m}\right)/2\pi$. Recalling $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$ yields $d_n = d + cn^{-1} + o(n^{-1})$. The same method holds for $\sqrt{\zeta_{n-1}^+} = n - d_n$. \square

This result enables us to compute interesting fluctuations quantities, such as the Laplace transform of the first passage time of X : $T_x = \inf\{t \geq 0, X_t \geq x\}$. Indeed, since $P[S_t \geq x] = P[T_x \leq t]$ it is easy to show that

$$P[S_{e_q} \geq x] = \int_0^1 P[e^{-qT_x} \geq y] dy = \mathbb{E}[e^{-qT_x}] := h_q(x).$$

We aim at providing a graph of h_q for some fixed values of q , m and σ^2 . Using residues, it is possible to perform a Laplace transform inversion on the Wiener-Hopf factors to get a series representation of the law of S_{e_q} (see Theorem 1 in [61] for instance):

$$\frac{d}{dx} P[S_{e_q} \leq x] = \sum_{n=1}^{\infty} \left[c_n^+ \zeta_n^+ e^{-\zeta_n^+ x} + c_n^- \zeta_n^- e^{-\zeta_n^- x} \right] + c_0 \zeta_0^+ e^{-\zeta_0^+ x}, \quad x > 0 \quad (3.28)$$

$$\text{where } c_n^{\pm} = \frac{1 - \frac{\zeta_n^{\pm}}{n^2}}{1 - \frac{\zeta_n^{\pm}}{\zeta_0^+}} \prod_{\substack{k \geq 1 \\ k \neq n}} \frac{1 - \frac{\zeta_n^{\pm}}{k^2}}{1 - \frac{\zeta_n^{\pm}}{\zeta_k^{\pm}}}, \quad c_0 = \prod_{k \geq 1} \frac{1 - \frac{\zeta_0^+}{k^2}}{1 - \frac{\zeta_0^+}{\zeta_k^+}} \frac{1 - \frac{\zeta_0^+}{k^2}}{1 - \frac{\zeta_0^+}{\zeta_k^-}},$$

$$\text{and } P[S_{e_q} = 0] = \zeta_0^+ \prod_{k \geq 1} \frac{\zeta_k^+}{k^2} \frac{\zeta_k^-}{(k+1)^2} = \mathbb{E}[e^{-qT_{0+}}],$$

where $T_{0+} = \inf\{t \geq 0, X_t > 0\}$. The Laplace transform of this latter random variable is not equal to 1 whenever 0 is not regular for $(0, \infty)$ (see chapter 6 in [68] for further details). Note that the cumulative distribution functions of S_{e_q} and I_{e_q} can be used for simulation purposes (see subsection 5.4.3).

We start by providing a sample of the the locations of the roots with positive real and imaginary parts for the following parametrization: $\sigma^2/2 \in \{0, 1\}$, $m = 0$ and $q = 1$:

	Case $\sigma^2/2 = 1$	Case $\sigma^2 = 0$
n	ζ_n^+	
0	0.4431	0.4596
1	1.5284+0.4173i	1.800+0.273i
2	4.2785+0.9257i	5.4705+1.2401i
3	9.1176+0.9813i	11.1627+2.2600i
4	16.0638+0.9884i	18.866+3.3642i
5	25.0403+0.9914i	28.5772+4.5378i
6	36.0278+0.9933i	40.2934+5.7690i
7	49.0204+0.9947i	54.0134+7.0492i
8	64.0156+0.9957i	69.7364+8.3720i
9	81.0123+0.9964i	87.4620+9.7321i
10	100.0100+0.9971i	107.1894+11.1263i
\vdots	\vdots	\vdots
50	2500.0004+0.9998i	2536.771+80.085i

The last line of the table is coherent with the asymptotic results of Proposition 3.3. Integrating (3.28) and taking $x \downarrow 0$ yields

$$c_0^+ + \sum_{n \geq 1} (c_n^+ + c_n^-) = 1 - P[S_{e_q} = 0] \leq 1,$$

from which we infer that the c_n are bounded (in fact, their moduli decrease very rapidly). Hence, it is possible to compute h_q using a finite number of terms, even for x close to zero. We provide below the graphs for the cases $\sigma^2/2 \in \{0, 1\}$ and $q \in \{1, 2, 3\}$, where we have truncated the series above $n = 50$.

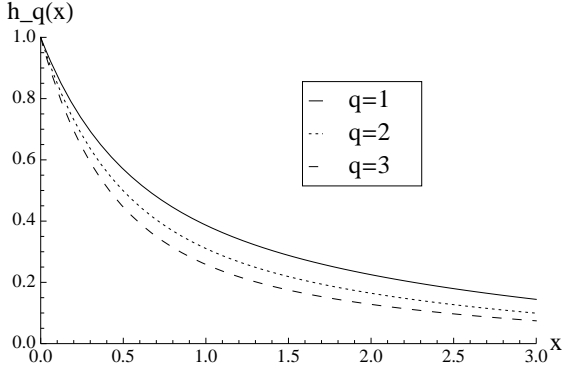


Figure 3.4.4: Plot of the function h_q for $\sigma^2 = 2$ and $m = 0$

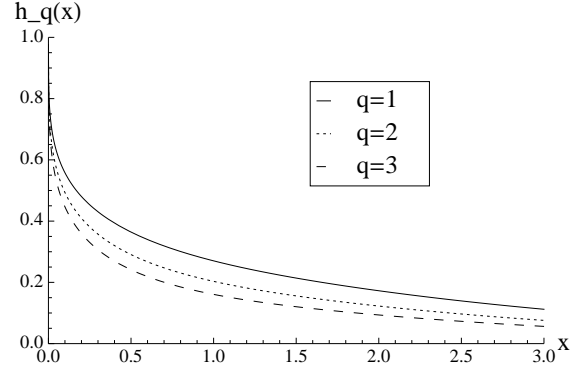


Figure 3.4.5: Plot of the function h_q for $\sigma^2 = 0$ and $m = 0$

Chapter 4

Approximations of probabilistic Laplace transforms and their inverses

4.1 Introduction

4.1.1 Laplace transforms

The Laplace transform is a linear operator which transforms a positive function f into another positive function L , namely,

$$\mathbb{L}[f(x)](t) = L(t) := \int_0^\infty e^{-tx} f(x) dx.$$

Of course, depending on f , this integral may diverge, and we will therefore only consider the functions f which are integrable near the origin and bounded at infinity.

A striking property of the Laplace transform is that it converts derivatives into polynomial/power functions and vice-versa. More precisely, if $f^{(n)}$ denotes the n^{th} derivative of f , then

$$\mathbb{L}[f^{(n)}(x)](t) = t^n L(t) - t^{n-1} f(0) - t^{n-2} f'(0) - \dots - f^{(n-1)}(0),$$

and

$$\mathbb{L}[x^n f(x)](t) = (-1)^n L^{(n)}(t).$$

In fact, similar relationships hold for integrals:

$$\mathbb{L}\left[\int_0^x f(y) dy\right](t) = L(t)/t,$$

and

$$\mathbb{L}[f(x)/x](t) = \int_t^\infty L(s) ds.$$

Because of these properties, Laplace transforms are very useful to handle differential equations: applying the Laplace transform on a simple ordinary differential equation

gives a polynomial expression which usually easier to solve. In many disciplines, some results can only be obtained via Laplace transforms; in the financial field, for instance, it is a widespread approach, see: [27], [53] and [70]. However, in order to get the sought result, one must resort to Laplace transform inversion.

The topic of Laplace transform inversion is the purpose of this chapter and will be addressed in the subsequent sections. We continue, in this introductory paragraph, a presentation of the various applications of the Laplace transform. Even though it is massively used in Physics and various mathematical fields, we will only focus here on its links with Probability Theory.

Given a positive random variable X , the function $L(t) = \mathbb{E}[e^{-tX}]$ will be called the Laplace transform of X . When X has a density f , this is quite straightforward, as L is the Laplace transform of f . It is known that the characteristic function of a distribution uniquely determines this distribution; however, as they are complex-valued (for non-symmetric distributions), they are not as easy to manipulate as real-valued function. This is why, for positive random variables, the Laplace transform is used instead. Indeed,

Theorem 4.1. *Two distinct positive probability distributions have distinct Laplace transforms*

Two proofs of this result are given in [39], section XIII.1. In fact, another result exists, which state that if f is continuous, then no other function has Laplace transform L . For details on this, see section 2.1 in [29].

A nice property of Laplace transforms is that if X and Y are independent, then

$$\mathbb{E}[e^{-t(X+Y)}] = \mathbb{E}[e^{-tX}]\mathbb{E}[e^{-tY}].$$

Moreover, the Laplace transform of X can be used to compute the moments of X :

$$\mathbb{E}[X^n] = (-1)^n L^{(n)}(0),$$

with obvious conventions in case of divergence.

Laplace transforms in probability have many other properties (see for instance chapter XIII in [39]), but we would like to emphasize their connection with Lévy processes.

An important class of Lévy processes are those with almost surely increasing sample paths, also known as subordinators. These processes are obviously always positive (since $X_0 = 0$ a.s.) and are thus defined by their Laplace transform, which reads:

$$\mathbb{E}[e^{-zX_t}] = \exp\left(t\left(-zd + \int_0^\infty (e^{-zx} - 1)\nu(dx)\right)\right) := \exp(t\phi(z)),$$

where d is a positive number and ν a measure on \mathbb{R}_+ satisfying $\int_{\mathbb{R}_+} (1 \wedge x)\nu(dx) < \infty$.

Subordinators (and their Laplace transforms) are an essential tool in the fluctuation theory of Lévy processes (local times, ladder heights ... see for instance, chapters IV and VI in [11]).

Another class of Lévy processes of interest are those with no positive jumps. In this case again, it is possible to compute their Laplace transform (formally, their moment generating function):

$$\mathbb{E}[e^{zX_t}] = \exp \left(t \left(dz + \frac{\sigma^2 z^2}{2} + \int_{-\infty}^0 (e^{zx} - 1 - zx \mathbf{1}_{|x| \leq 1}) \nu(dx) \right) \right) := \exp(t\phi(z)).$$

In this case, ϕ is strictly convex and goes to infinity when $z \rightarrow \infty$, hence ϕ is strictly increasing and positive on some half-line. On this half-line, it is thus possible to define the inverse ϕ^{-1} on this half-line. In this case, the process

$$T_x = \inf\{t \geq 0, X_t \geq x\}, \quad x \geq 0,$$

is a subordinator and its Laplace exponent is ϕ^{-1} .

In fact, a similar result is available for one-dimensional diffusion processes: the Laplace transform of their first passage time is characterized by a differential equation related to the infinitesimal generator of the process. We refer to [51], chapters 3 and 4 for more details.

Lastly, we would like to recall that the Wiener-Hopf factors (studied in the previous chapter) are simply a double Laplace transform of the law of the supremum and infimum of a Lévy process with respect to the time and space variables:

$$\phi_q^+(z) = q \int_0^\infty \int_0^\infty e^{-qt-zx} P[S_t \in dx] dt, \quad \phi_q^-(z) = q \int_0^\infty \int_0^\infty e^{-qt+zx} P[I_t \in dx] dt.$$

4.1.2 Laplace transform inversion

The topic of Laplace transform inversion is an old problem which relates to physics, probability theory, analysis and numerical methods. The number of publications dedicated to it is so large that it is possible to write surveys of surveys on the subject (chapter 9 of [29]). Roughly speaking, there are two major approaches to numerical inversion of Laplace transforms. If f is a function with sub-exponential growth, and

$$\mathbb{L}[f(x)](t) = L(t) = \int_0^\infty e^{-tx} f(x) ds$$

its Laplace transform, the first family of methods uses results with closed forms to compute the original function f :

$$\mathbb{L}^{-1}[L(t)](x) = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xt} L(t) dt \quad (\text{Bromwich integral, see 2.2 in [29]}), \quad (4.1)$$

$$f(x) = \lim_{n \rightarrow +\infty} \frac{(-1)^n}{n!} \left(\frac{n}{x}\right)^{n+1} L^{(n)}(n/x) \quad (\text{Post-Widder formula, see [39], VII.6}), \quad (4.2)$$

for some c chosen so that the path of integration makes sense for L .

For instance, the papers [26] and [4] make use of these formulae to obtain numerical approximations. Other examples of such methods can be found in chapters 6 and 7 of [29]. A recent survey on the efficiency of some of these procedures was carried out by Masol and Teugels in [81].

Simply put, the target function (L) in these methods is exact, but the inversion is approximative: discretization of the integral in (4.1) and choice of a large, but finite, n in (4.2).

The second family of numerical inversions proceeds otherwise: the inversion is exact, but it is applied to an approximation of the initial Laplace transform. Both the original function and the Laplace transform are usually approximated by sums (truncated series for instance) of functions. The case when these functions are of rational type was studied in [75] in the scope of least square optimization. Orthogonal polynomials (Legendre, Chebyshev and Laguerre polynomials) have also been investigated (see section 3.2 in [29]). In any case, the approximations take the form

$$f(x) \approx \sum_{k=1}^N f_k(x) \quad \Leftrightarrow \quad L(t) \approx \sum_{k=1}^N L_k(t). \quad (4.3)$$

The core idea of our method is to take advantage of the specificities of Laplace transforms in probability to choose the L_k (and thus f_k) wisely, depending on some properties of f . More precisely, the technique is adapted to fit the asymptotics of f near zero and infinity. In many cases, our technique will provide satisfying results for $N = 2$ or $N = 3$ in (4.3).

Such a procedure can be used, for instance, to approximate the law of a subordinator (an increasing Lévy process), which is usually characterized by its Laplace transform.

We proceed in three steps. First, we make a quick review of the different families of methods of Laplace transform inversion. Second, we propose a naive method to construct an approximate in (4.3) which will have a pedagogical role. While any approximation method has its flaws, our naive method seems to cumulate most of the drawbacks of the methods in the literature, which we also review. Our second method was precisely designed to avoid these flaws and will be presented afterwards.

4.2 A short review of classical methods

We summarize the various families of procedures which can be found in the literature. This subsection is inspired by [29].

4.2.1 Series expansion

If L possesses a series expansion of the form

$$L(t) = \sum_{n=1}^{\infty} \frac{a_n}{t^{\nu_n}},$$

then, since $\mathbb{L}^{-1}[t^{-\nu}](x) = x^{\nu-1}/\Gamma(\nu)$,

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n x^{\nu_n-1}}{\Gamma(\nu_n)}.$$

In most cases, one must truncate this series. Depending on the a_n and the value x at which we choose to approximate f , the accuracy of this method varies. Of course, any series expansion

$$L(t) = \sum_{n=1}^{\infty} a_n L_n(t),$$

where the L_n have a known Laplace inverse works as well. The following functions can be approximated and inverted this way (see [29], section 3.1): $1/(1+t)$, $1/(t^2+1)$, $e^{-a/t}/\sqrt{t}$, $\log(t)/(1+t)$, $1/(t^{1/2} + t^{1/3})$.

This method can be further extended. If L can be decomposed into

$$L(t) = \sum_{n=1}^{\infty} a_n P_n(p(t)),$$

where the P_n are orthogonal polynomials and p is a simple function such that $p(t)^n$ is easily Laplace-inverted, then

$$f(x) = \sum_{n=1}^{\infty} a_n \mathbb{L}^{-1}[P_n(p(t))](x),$$

and again, a truncation of the series provides an approximation.

4.2.2 Around the Bromwich integral

The methods which aim at approximating (4.1) often rely on the computation of Fourier integrals. We present here only two of them (see [1] and [97]) and refer to [29], chapters 4 and 6, for an exhaustive account of these methods.

We start by considering the original function f , defined on \mathbb{R}_+ . We set, for some $c > 0$, such that there are no singularities of L to the right of c (so that (4.1) holds),

$$h(x) = \begin{cases} e^{-cx} f(x) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

and, for $T > 0$ and n even,

$$g_n(x) = \begin{cases} h(nT+x) & \text{if } x \in [0, T] \\ h(nT-x) & \text{if } x \in [-T, 0] \\ 0 & \text{otherwise} \end{cases},$$

while if n is odd,

$$g_n(x) = \begin{cases} h((n+1)T - x) & \text{if } x \in [0, T] \\ h((n+1)T + x) & \text{if } x \in [-T, 0] \\ 0 & \text{otherwise} \end{cases}.$$

The g_n were built so that their Fourier cosine representation is

$$g_n(x) = \frac{1}{2}A_{n,0} + \sum_{k=1}^{\infty} A_{n,k} \cos\left(\frac{k\pi x}{T}\right),$$

where

$$A_{n,k} = \frac{2}{T} \int_{nT}^{(n+1)T} h(y) \cos\left(\frac{k\pi y}{T}\right) dy.$$

This leads to

$$\sum_{n=0}^{\infty} g_n(x) = \frac{2}{T} \left[\frac{B_0}{2} + \sum_{k=1}^{\infty} B_k \cos\left(\frac{k\pi x}{T}\right) \right],$$

where

$$B_k = \int_0^{\infty} h(y) \cos\left(\frac{k\pi y}{T}\right) dy.$$

From this it is possible to recognize the real parts of the complex-valued Laplace transform of f with $t = c + i(k\pi)/T$:

$$\sum_{n=0}^{\infty} e^{cx} g_n(x) = \frac{2e^{cx}}{T} \left[\frac{L(c)}{2} + \sum_{k=1}^{\infty} \Re\left(L\left(c + i\frac{k\pi}{T}\right)\right) \cos\left(\frac{k\pi x}{T}\right) \right], \quad (4.4)$$

but, moreover, by the definition of the g_n and for $x \in (0, T)$,

$$\begin{aligned} \sum_{n=0}^{\infty} e^{cx} g_n(x) &= f(x) + \sum_{k=1}^{\infty} e^{cx} h(2nT + x) + \sum_{n=1}^{\infty} e^{cx} h(2nT - x) \\ &= f(x) + \sum_{k=1}^{\infty} e^{-2cTn} [f(2nT + x) + e^{2cx} f(2nT - x)]. \end{aligned}$$

Abate and Dubner ([1]) show that for $x \in (0, T/2)$, the second term can be made arbitrarily small and hence that $L(t)$ can be approximated by (4.4). However, given the exponential term in this equation, the performance of this method decreases when x increases. Some enhancements of this procedure can be found in section 4.4 in [29].

The second method we present is based on another type of approximation. We denote by γ the smallest real part of $z \in \mathbb{C}$ for which $L(z)$ is convergent. The aim is to replace the line of integration $(c - i\infty, c + i\infty)$ by a contour \mathcal{C} starting and ending in the left half-plane, with infinite negative real part. If

- i) \mathcal{C} encloses all singularities of L ,
- ii) $|L(z)| \rightarrow 0$ uniformly in $\Re(z) \leq \gamma$ as $|z| \rightarrow \infty$,

then (4.1) can be rewritten into

$$f(x) = \frac{\lambda e^{\sigma x}}{2\pi i} \int_{\mathcal{C}} e^{\lambda x t} L(\lambda t + \sigma) dt, \quad x > 0,$$

where λ and σ are chosen so that i) holds. It is possible to further specify the contour: if M is the interval $(-\pi, \pi)$ and S is an analytic function such that

- a) it has simple poles at $\pm\pi$ and residues there with imaginary parts respectively positive and negative,
- b) it has no singularities in the strip $|\Re(z)| < 2\pi$,
- c) it maps M into a valid contour \mathcal{C} ,
- d) it maps the half-strip $\{z \in \mathbb{C}, \Im(z) > 0, |\Re(z)| < 2\pi\}$ into the exterior of \mathcal{C} ,

then

$$f(x) = \frac{\lambda e^{\sigma x}}{2\pi i} \int_M e^{\lambda S(z)x} L(\lambda S(z) + \sigma) S'(z) dz.$$

Possible parametrizations for S include

$$S(z) = z \cot(z) + i\nu z, \quad S(z) = 1 + \frac{2z^2}{z^2 - \pi^2} + i\nu z,$$

where the second stems from the first (the original Talbot contour, see figure 4.2.1) via truncated partial fraction expansion.

Then, f can be approximated using the trapezoidal rule:

$$\begin{aligned} f(x) \approx f_N(x) &= \frac{1}{2N i} \sum_{k=-N}^{N-1} e^{x(\sigma + \lambda S(z_k))} L(\sigma + \lambda S(z_k)) S'(z_k) \\ &= \frac{1}{N} \Im \left(\sum_{k=0}^{N-1} e^{x(\sigma + \lambda S(z_k))} L(\sigma + \lambda S(z_k)) S'(z_k) \right), \end{aligned}$$

where $z_k = \pi(2k + 1)/2N$.

One pleasant feature of this method is that, as is shown in [29], section 6.2, it is possible to obtain bounds on the error $|f(x) - f_N(x)|$, which is, however, exponentially increasing in x . An interesting subject is also the optimization of the contour of integration, an issue discussed in [100] in a multidimensional setting.

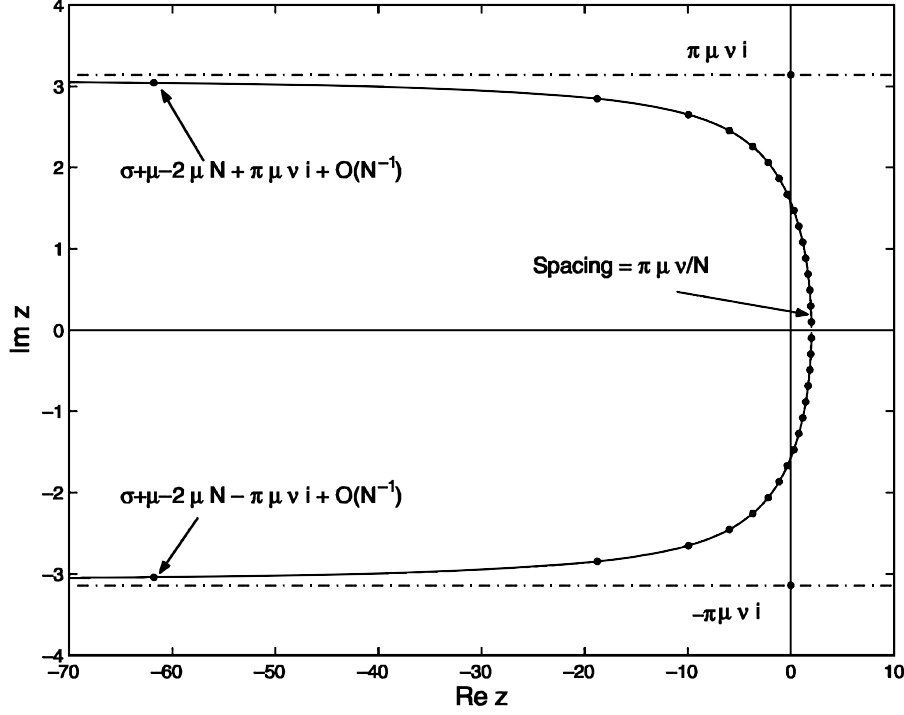


Figure 4.2.1: The original Talbot contour with $\sigma = 0$, $\lambda = 2$, $\nu = 0.5$ ([100])

4.2.3 Rational approximation

When the Laplace transform is a rational function, then, by partial fraction expansion, it has the form

$$L(t) = \sum_{k=1}^n \frac{c_k}{(t - r_k)^{\nu_k}},$$

and thus,

$$f(x) = \sum_{k=1}^n \frac{c_k x^{\nu_k} e^{r_k x}}{\Gamma(\nu_k)}.$$

Of course, this approach is quite restrictive. If f can be approximated by a function g depending on a finite number of parameters, for instance, for $n \in \mathbb{N}^*$ and parameters $(A_k, \alpha_k)_{k \in \{1, \dots, n\}}$,

$$g(x) = \sum_{k=1}^n A_k e^{-\alpha_k x}, \quad x > 0,$$

then Longman, in [75] proposed the following procedure. The idea is to minimize the following quantity

$$S := S(w, (A_k, \alpha_k)_{k \in \{1, \dots, n\}}) = \int_0^\infty e^{-wx} [f(x) - g(x)]^2 dx, \quad (4.5)$$

where e^{-wx} is some weight function required to ensure convergence. A necessary condition for S to be a minimum is

$$\frac{\partial S}{\partial A_k} = 0, \quad \frac{\partial S}{\partial \alpha_k} = 0, \quad \forall k \in [1, n],$$

which yields, after calculations,

$$(\diamond) \quad \forall k \in [1, n] \left\{ \begin{array}{l} \sum_{i=1}^n \frac{A_i}{w + \alpha_k + \alpha_i} = L(w + \alpha_k) \\ \sum_{i=1}^n \frac{A_i}{(w + \alpha_k + \alpha_i)^2} = -L(w + \alpha_k) \end{array} \right.,$$

and further numerical methods are required to solve this system.

However, because of the rapidly decreasing weight function e^{-wx} , the approximation is often only good near the origin, hence Sidi, in [96], introduced the weight function $x^n e^{-wx}$.

Other approaches in the field of rational approximations are possible using Padé approximants and their extensions (see section 5.3 in [29]).

4.2.4 Around the Post-Widder formula

Inversion techniques based on (4.2) have a complicated implementation because they require the computation of high order derivatives. We do not provide any explicit method here, but we refer to chapter 7 in [29], as well as [4], [42], [98] and [99] (for convergence results).

4.3 A first naive approximation method

4.3.1 The procedure

Our goal is to approximate the unknown function f via an approximation of L . We are seeking proxies of L of the form

$$L(t) \approx \sum_{k=1}^N c_k L_k(t) := L(N, t), \quad \forall t \geq 0. \quad (4.6)$$

One obvious choice for the determination of the c_k would be the minimizing of (4.5) with any possible relevant weight function. However, this is not possible, since Longman's method was tailored for exponential functions, which are the only functions that can lead to a system of equations of the type (\diamond) . When other functions are considered, this method is not tractable.

Once the choice of the L_k is made, a naive way to proceed is simply to make sure that both L and $L(N, \cdot)$ behave alike in some critical zones.

The choice of the L_k will be discussed in the next section and in the examples. When $f(0+) > 0$, for a given set of points t_i with $i \in \{1, \dots, N-1\}$, the c_k are then chosen such that

$$\left\{ \begin{array}{l} L(N, t_i) = \sum_{k=1}^N c_k L_k(t_i) = L(t_i), \quad \forall i \in \{1, \dots, N-1\} \\ \lim_{t \rightarrow \infty} t \sum_{k=1}^N c_k L_k(t) = f(0+) \end{array} \right., \quad (4.7)$$

which amounts to solve a simple $N \times N$ linear system. When the evaluation points t_i are well chosen (a topic discussed in the next subsection), this system is easily solved by any quantitative software.

If $f(0+) = 0$, then the L_k can be chosen such that $L_k(t) = o(t^{-1})$ ($t \rightarrow \infty$) for any $k \geq 1$. In this case, the limit part of (4.7) vanishes and the calibration can be performed on N points instead of $N-1$.

Of course, the L_k will be chosen so that they are easily inverted into f_k , in order to yield a simple density approximation

$$f(x) \approx \sum_{k=1}^N c_k f_k(x) := f(N, x). \quad (4.8)$$

4.3.2 Choosing the points and the parameters

We start by discussing the choice of the t_i . There are two major conditions that must be fulfilled to ensure a good performance of the procedure. First, as the Tauberian theorems show (as we shall see in the next section), $L(N, \cdot)$ must behave like L both near the origin and far from it, hence it should hold that $t_1 = 0$, $t_2, t_3 < 1$ and $t_{N-1}, t_N > 10$. The second condition is that there should be a minimal distance between the t_i (especially for small i) in order to avoid quasi colinear vectors in the resolution of the linear system.

One a priori fairly logic way to proceed is to consider L as a survival function and to choose the t_i according to quantiles. However, when $-L'(0) = \infty$, this does not work well because the t_i are too close to each other and to 0. A rule that works in all the cases presented below is to keep at least one third of the t_i in $(0, 1)$ and one third in $(5, +\infty)$.

It is also very important that the c_k in (4.6) do not increase too much, when $N > 10$.

4.3.3 Numerical examples

We test the approximations against known densities. We provide two examples: one with heavy tail (Lévy distribution) and one with a light tail (one-sided Gaussian).

For the Gaussian law, we chose $L_k(t) = e^{-a_k t}$ (and hence Longman's method would have worked, but here the algorithm is faster) and for the Lévy distribution,

$$L_k(t) = \frac{3}{4a_k t} (1 - e^{a_k t} (a_k t)^b (a_k t + 3/2) \Gamma(-3/2, a_k t)) \Leftrightarrow f_k(x) = \frac{3\sqrt{a_k} x}{4(a_k + x)^{5/2}}.$$

This latter choice was made because it enables a density approximation which is 0 near the origin and behaves like $x^{-3/2}$ near infinity.

The approximation points belonged to the interval $(0, 15)$ in the Gaussian case and $(0, 7)$ in the Lévy case. In both cases, the $a_k \in (0, 11/2)$. The errors of the approximations are plotted below (Figures 4.3.1 and 4.3.2).

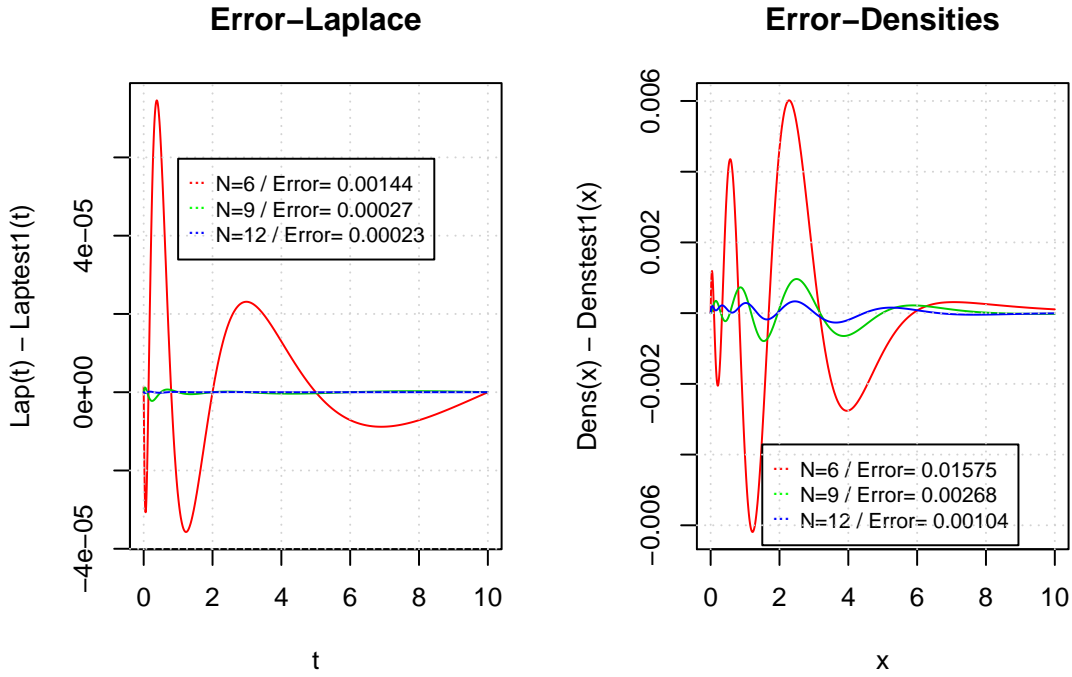


Figure 4.3.1: Error on the Laplace transform and density for the approximation of a Half-Gaussian(1) distribution. The caption indicates the corresponding L^1 errors

4.3.4 The drawbacks

This simplistic inversion procedure has many flaws, which we now list.

1. $f(N, \cdot)$ in (4.8) is rarely a density: it can have negative values and most of the time $\int_0^\infty f(N, x) dx \neq 1$, hence $f(N, \cdot)$ requires a normalizing factor;
2. it is impossible to know what the error with respect to f is and there is no obvious way to reduce it. Intuitively, increasing N seems a good idea, but,

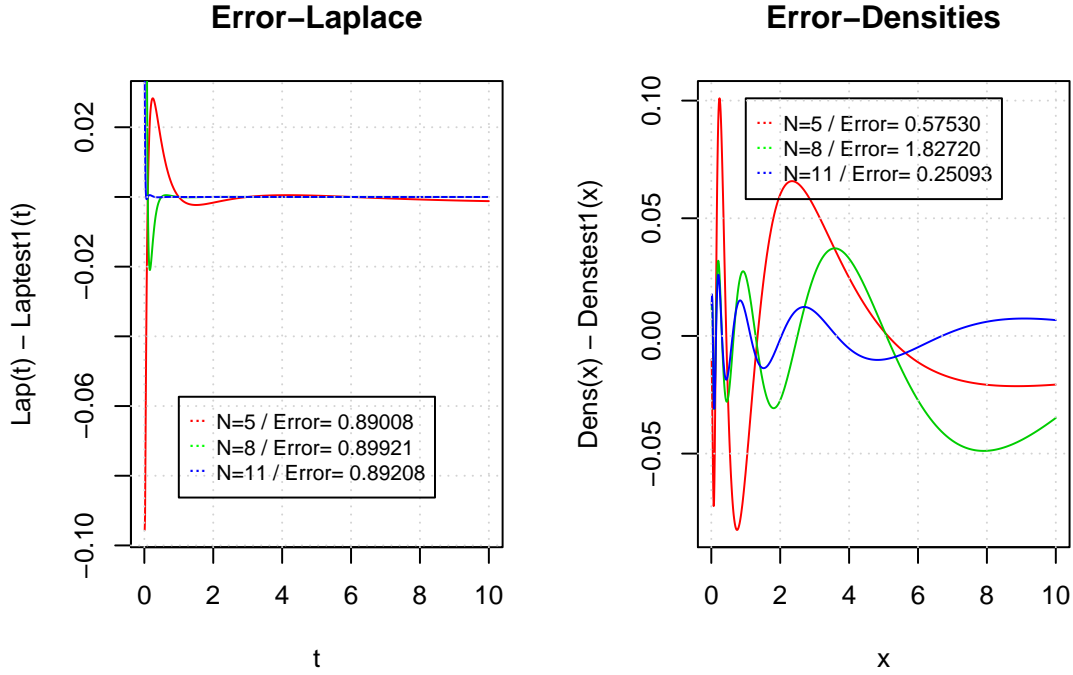


Figure 4.3.2: Error on the Laplace transform and density for the approximation of a Lévy(1) distribution. The caption indicates the corresponding L^1 errors

3. as N increases, the c_k in (4.6) increase at a power rate, which makes it possible for $f(N, \cdot)$ to oscillate from large positive values to large negative values for x small;
4. the computation of the c_k is highly sensitive to the parameters of the L_k and to the calibration points t_i . A small variation in any of those can lead to completely different proxies.

The methods described in subsections 4.2.1 to 4.2.4 also have their shortcomings.

The Bromwich integral and Talbot methods for instance are pointwise formulae. They provide an approximation at point x only, and to get $f(y)$ for $y \neq x$, one must perform the whole procedure again. This is usually time consuming, especially when computing integrals.

The rational approximation has at least two flaws: the error on the density is unknown and it can occur that the target Laplace transform has a positive pole (after solving the system of equations), in which case the density proxy goes to infinity as $x \rightarrow +\infty$.

Lastly, it is well-known that the Post-Widder formula converges very slowly. Some enhancements of the formula (see subsection 4.2.4) perform better (under some conditions, in $\log(n)/n$ in [99]), but they still remain pointwise methods.

We are now ready to present a method which was designed to get rid of all of the weaknesses mentioned above.

4.4 Laplace transform inversion with completely monotone functions

4.4.1 Some properties of the density

The aim of this subsection is to recall a few classical results which show that many properties of f can be derived from a thorough study of L .

We begin with some notations. Until the end of the chapter, we will consider two positive and absolutely continuous random variables X and Y with densities f and g , cumulative distribution functions (c.d.f.s) F and G and Laplace transforms L and M respectively. We also denote by $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ their survival functions. L will be the original Laplace transform and M its approximation.

For a function $f = f^{(0)}$, $f^{(n)}$ will denote its n -th derivative while $f^{(-n)}$ will denote its n -th antiderivative:

$$f^{(-n)}(x) = \int_x^\infty f^{(-n+1)}(y)dy,$$

whenever this integral makes sense.

In some asymptotic settings, we will also write $f(x) \sim g(x)$ for $f(x)/g(x) \rightarrow 1$.

Support

The first basic information that is required to characterize a distribution is its support.

Theorem 4.2. *Let A denote the left point of the support of the positive random variable X . Then if B is the set of real numbers b such that $e^{bt}L(t) = O(1)$ as $t \rightarrow \infty$, we have*

$$A = \sup_{b \in B} b.$$

Proof. If $A = 0$, then, for any $x < 0$, $e^{xt}L(t) \rightarrow 0$ ($t \rightarrow +\infty$). For $x > 0$,

$$e^{xt}L(t) \geq \int_0^x e^{st}f(x-s)ds \geq \eta e^{\delta t} \text{Leb}\{s \in [\delta, x], f(x-s) \geq \eta\} \rightarrow \infty, \quad t \rightarrow +\infty,$$

where Leb is the Lebesgue measure and $\delta, \eta > 0$ were chosen such that $\text{Leb}\{s \in [\delta, x], f(x-s) \geq \eta\} > 0$, which is possible, since $A = 0$. The case $A > 0$ follows by direct translation. \square

Another way to compute the lower bound of the support of X is in fact to compute the limit of $-L'(t)/L(t)$ when $t \rightarrow \infty$. Indeed, by Hölder's inequality, $\text{Log}L$ is convex, hence L'/L is increasing. Since it is bounded by zero, it converges to some negative limit. A simple analysis shows that this limit at infinity is in fact $-A$.

In order to find the upper bound of the support of X , we propose a test, based on the following proposition. Note that it is easy to compute $\mathbb{E}[X]$ with the sole knowledge of L , since $\mathbb{E}[X] = L'(0)$.

Proposition 4.1. *If the positive random variable X is almost surely bounded by C , then for any $A > 0$ and $\gamma \geq 1$,*

$$L(t) \leq 1 - \mathbb{E}[X^\gamma] \frac{1 - e^{-A}}{A^\gamma} t^\gamma, \quad \forall t \in [0, A/C].$$

Proof. The proof relies on the inequality

$$y^\gamma - x^\gamma \geq e^{-x} y^\gamma - e^{-y} x^\gamma, \quad \gamma \geq 1, \quad 0 < x < y.$$

Then setting $x = tX$, $y = A$ and applying the expectation operator yields the result. \square

Hence, if in the vicinity of 0, $L(t) \geq 1 - t\mathbb{E}[X](1 - e^{-A})/A$, then X is unbounded. The test usually performs better for $A \ll 1$.

In the same spirit, note that Theorem 7(b) in [48] makes it possible to build another test based on $\mathbb{E}[X^\gamma]$ for $\gamma < 1$. Since they depend on the interval $[0, A/C]$, these results make it even possible to derive bounds for C .

We will henceforth consider distributions which are **supported over the whole positive half-line**.

Tail behaviours

This subsection recalls classical Tauberian theorems in probability (see for instance section XIII.5 in [39]). These results show the strong link that exists between the behaviour of f near zero and that of L near infinity and vice-versa. The general form of the de Bruijn exponential Tauberian theorem can be found in [14], Theorem 4.12.9, but we recall below a more peculiar form, derived from Corollary 4.12.6 of the same monograph.

Theorem 4.3. *Let $0 < \gamma < 1$, $\delta \in \mathbb{R}$, $C > 0$ and X a positive random variable. Then,*

$$\log \mathbb{E}[e^{-tX}] \sim -Ct^\gamma (\log(t))^\delta, \quad t \rightarrow \infty,$$

if and only if

$$\log P[X \leq x] \sim -[C\gamma^\gamma(1 - \gamma)^{1-\gamma-\delta}x^{-\gamma}(-\log x)^\delta]^{1/(1-\gamma)}, \quad x \downarrow 0.$$

In a series of papers, Nakagawa provides conditions on L to determine whether a distribution has a heavy or a light tail. We state one of his results below (see [86] and the references therein). For a complex number $z = a + ib$, if $L(z)$ converges for $a > a_0$ and diverges for $a < a_0$, then a_0 is said to be the abscissa of convergence of $L(z)$.

Theorem 4.4. *If a_0 is the abscissa of L such that $-\infty < a_0 < 0$ and a_0 is a pole of L , then*

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log P[X > x] = a_0.$$

When the asymptotic behaviors are not exponential but of power form, we must resort to Corollary 8.1.7 in [14] and Theorem 2, section XIII.5 in [39], which we recall below (with $l(x) = C \log(x)^\beta$).

Proposition 4.2. *For $0 \leq \alpha < 1$, $\beta \geq 0$ and $C > 0$, the following are equivalent*

$$\begin{aligned} 1 - L(t) &\sim -Ct^\alpha \log(t)^\beta, \quad t \downarrow 0, \\ 1 - F(x) &\sim C \frac{\log(x)^\beta}{x^\alpha \Gamma(1 - \alpha)}, \quad x \rightarrow \infty. \end{aligned}$$

Proposition 4.3. *For $\alpha, \beta \geq 0$ and $C > 0$, the following are equivalent*

$$\begin{aligned} L(t) &\sim C \frac{\log(t)^\beta}{t^\alpha}, \quad t \rightarrow \infty, \\ F(x) &\sim -C \frac{\log(x)x^\alpha}{\Gamma(1 + \alpha)}, \quad x \downarrow 0. \end{aligned}$$

Proposition 4.2 allows to accurately determine the tail of a distribution when it is very heavy. For other power tail behaviours (when $\alpha > 1$), we refer to Theorem 8.1.6 in [14].

Lastly, we recall the initial value theorem:

$$f(0+) = \lim_{t \rightarrow \infty} tL(t).$$

Boundedness

It can be very convenient to know whether a distribution has a bounded density. In order to do so, it is possible to build tests based on the following corollary of the Post-Widder formula.

Lemma 4.1. *A function L , defined on \mathbb{R}_+ is the Laplace transform of a probability density bounded by c if and only if $L(0) = 1$ and*

$$0 \leq (-1)^n L^{(n)}(t) \leq \frac{cn!}{t^{n+1}},$$

for all $n = 0, 1, \dots$ and $t > 0$.

Unimodality

For the sake of completeness, we recall a characterization of unimodal densities, due to Khinchine [57]. We recall that a positive random variable is said to be unimodal at point x if its cumulative distribution function is convex on $(0, x)$ and concave on $(x, +\infty)$.

Theorem 4.5. *A positive random variable is unimodal at point 0 if and only if its Laplace transform is representable in the form*

$$L(t) = \frac{1}{t} \int_0^t v(s) ds,$$

where v is the Laplace transform of a positive random variable.

4.4.2 The approximation method

Introductory remarks

We shall henceforth consider a given positive function L defined on \mathbb{R}_+ , satisfying $L(0) = 1$ and

$$(-1)^n L^{(n)}(t) \geq 0, \quad \forall t > 0, \quad \forall n \geq 0. \quad (4.9)$$

Any function for which (4.9) holds is called a complete monotone function. Such functions have the following well-known property (Theorem 7.11 in [101] for instance)

Theorem 4.6. *A function h is completely monotone on \mathbb{R}_+ if and only if it is the Laplace transform of a nonnegative finite Borel measure ν , i.e., if and only if*

$$h(x) = \int_0^\infty e^{-xt} \nu(dt).$$

Therefore, (4.9) and $L(0) = 1$ are necessary and sufficient conditions for L to be a probabilistic Laplace transform.

Our approach is essentially error driven: a classical problem which arises after an approximation is that of the sign of the error induced by the approximation. In many cases, this sign is not constant, which makes some computations complicated, if not impossible. Indeed, if one is interested in the L^1 error for instance, then $\int_0^\infty |F(x) - G(x)| dx$ cannot be retrieved from $\left| \int_0^\infty (F(x) - G(x)) dx \right|$, unless the sign of $F - G$ does not change. Notice that the choice of cumulative distribution functions is critical since it can occur that G is dominated by F on \mathbb{R}_+ while this is impossible for two densities.

The aim of our method is thus to find G as close to F as possible, satisfying $G(x) \geq F(x)$ (or $G(x) \leq F(x)$) for all $x \geq 0$.

This property is connected to a notion called stochastic ordering. We will say that the positive random variable X is less than Y i.u.s.o. (in the usual stochastic order) if

$$1 - F(x) = P[X \geq x] \leq P[Y \geq x] = 1 - G(x), \quad \forall x \geq 0.$$

Note that if X is less than Y i.u.s.o., then an integration by parts yields $L(t) \geq M(t)$ for all $t \geq 0$. Sadly, the converse is not always true. A counter-example is given by the densities $f(x) = (\mathbf{1}_{(0,1)}(x) + \mathbf{1}_{(2,3)}(x))/2$ and $g(x) = \mathbf{1}_{(1,2)}(x)$. In this case, the c.d.f.s are not ordered, while

$$L(t) = \frac{1 - e^{-t} + e^{-2t} - e^{-3t}}{2t} \geq \frac{e^{-t} - e^{-2t}}{t} = M(t).$$

In order to make sure that $G(x) \geq F(x)$ for all $x \geq 0$, we will require an additional tool: completely monotone functions.

Given L , our aim is thus to find (or build) another Laplace transform M , as close as possible (in some sense) to L and such that $t \mapsto \mathbb{L}[G(x) - F(x)](t) = \frac{M(t) - L(t)}{t}$ is a completely monotone function. Under these conditions, the error made on the cumulative distribution functions will have a constant sign.

Practical implementation

We proceed in two steps.

Step 1. The first step is to find M , a *rough* proxy of L . Inspired by the results of subsection 4.4.1, we propose families of approximants depending on tail behaviours.

If X is light-tailed, then a relevant tool to work with is the gamma distribution. Indeed, its tail is light and it allows for any power behaviour near the origin, including $f(0+) > 0$. M and G then have the form

$$M(t) = \frac{a^b}{(a+t)^b}, \quad G(x) = \gamma(b, ax)/\Gamma(b), \quad g(x) = \frac{a^b x^{b-1} e^{-ax}}{\Gamma(b)}, \quad a, b > 0,$$

where $\gamma(\cdot, \cdot)$ is the lower gamma function.

If X has heavy tails, then the choice of the Pareto distribution seems quite straightforward when $f(0+) > 0$. That is,

$$M(t) = ba^b e^{at} t^b \Gamma(-b, at), \quad G(x) = 1 - \frac{a^b}{(a+x)^b}, \quad g(x) = \frac{ba^b}{(a+x)^{b+1}}, \quad a, b > 0,$$

where $\Gamma(\cdot, \cdot)$ is the upper gamma function. In this case, $M(t) \sim ba^{-1}/t$, $t \rightarrow \infty$.

If $f(0+) = 0$ (and X is heavy-tailed), we propose the following two choices

◦ if f goes slowly to 0 ($x \downarrow 0$),

$$M(t) = \frac{b(b-1)}{at} (1 - e^{at} (at)^b (at+b) \Gamma(-b, at)), \quad G(x) = 1 - a^{b-1} \frac{a+bx}{(a+x)^b}, \quad a > 0, b > 1,$$

with corresponding density $g(x) = \frac{b(b-1)a^{b-1}x}{(a+x)^{b+1}}$, and

◦ if f goes rapidly to 0,

$$M(t) = \frac{2}{\Gamma(b)} (at)^{b/2} K_b(2\sqrt{at}), \quad G(x) = \frac{\Gamma(b, a/x)}{\Gamma(b)}, \quad g(x) = \frac{a^b e^{-a/x}}{\Gamma(b)x^{b+1}}, \quad a, b > 0,$$

where $K_\nu(x)$ is the modified Bessel function of the second kind with index ν (see 3.471-9 in [47] for the computation of the Laplace transform). This is a generalization of both the Lévy and the inverse Chi-square laws, often referred to as the inverse gamma distribution.

These choices for M were made because they require well-known special functions and are thus easily computed by any software. Other choices are of course possible.

The purpose of the rough proxy is to mimic as well as possible the behaviour of L at 0 and/or infinity while satisfying $\mp(M-L)$ is a completely monotonic function.

Step 2. The error made with the rough proxy $N(t) = M(t) - L(t)$ is usually not satisfactory and requires improvement. The trick is to find a close minorant μ of N such that $(N(t) - \mu(t))/t$ is again a completely monotone function. The implication is that $M - \mu$ is a better approximation of L than M taken alone. The aim of step 2 is to reduce the error of a prior approximation, hence it can be carried out several times. However, in our examples, we will show that only one iteration of step 2 may be sufficient to obtain a reasonably small error.

Good candidates for μ are differences of Laplace transforms of stochastically ordered distributions. Taking, for instance, gamma or 1/2-stable laws yields the following forms

$$\mu(t) = c \left(\frac{a^\nu}{(a+t)^\nu} - \frac{b^\nu}{(b+t)^\nu} \right), \quad c, \nu > 0, \quad a > b > 0, \quad (4.10)$$

or

$$\mu(t) = c \left(e^{-\sqrt{at}} - e^{-\sqrt{bt}} \right), \quad c > 0, \quad b > a > 0. \quad (4.11)$$

We underline that the choice of a proper μ is crucial as it will enhance the approximation in a very peculiar way. For instance, choosing (4.10) will have a considerable impact on the tail of the Laplace approximation and thus on the behavior of the

new c.d.f. near 0; however, on the contrary, μ defined in (4.11) is negligible near infinity, but it will induce a significant modification of the tail of the approximating distribution.

Once μ is chosen (this task usually requires a fitting tool from a quantitative software), the critical point is to check that $h(t)/t := (N(t) - \mu(t))/t$ is indeed a completely monotone function. We recall that for any C^n function h ,

$$\left(\frac{h(\cdot)}{\cdot}\right)^{(n)}(t) = \frac{1}{t^{n+1}} \sum_{i=0}^n \frac{(-1)^i n!}{(n-i)!} t^{n-i} h^{(n-i)}(t) = \frac{h^{(n)}(t) - n \left(\frac{h(\cdot)}{\cdot}\right)^{(n-1)}(t)}{t}, \quad (4.12)$$

which can be proven iteratively.

The function $h(t) - th'(t)$ requires a particular focus since it is associated with the first derivative (and also with the leading term $n!$ in (4.12)). If the functions $t^n h^{(n)}(t)$ are smooth then some patterns can be identified for n small. If $h(\cdot)/\cdot$ is indeed completely monotone, then there is a good chance that, in (4.12), the relative weight of $h^{(n)}$ compared to that of $n(h(\cdot)/\cdot)^{(n-1)}$ will decrease as n increases. The idea, based on empirical results, is to test whether

$$d_n(t) := -t \frac{(h(\cdot)/\cdot)^{(n)}(t)}{(h(\cdot)/\cdot)^{(n-1)}(t)} = n - \frac{h^{(n)}(t)}{\left(\frac{h(\cdot)}{\cdot}\right)^{(n-1)}(t)} \approx n, \quad \forall t \geq 0. \quad (4.13)$$

We provide examples below to illustrate this matter (Figures 4.4.1 and 4.4.2).

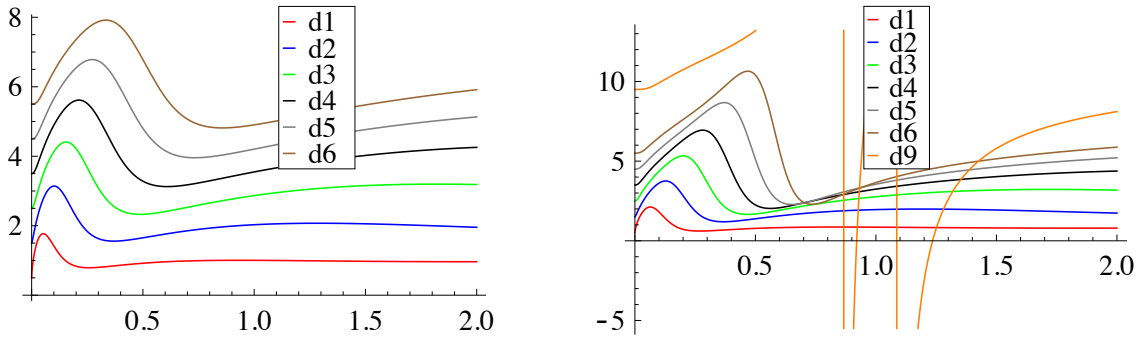


Figure 4.4.1: Graph of d_n for various n in two cases (related to the first example below)

Among these four graphs, the two on the right fail the test (not only do the d_n drift away from n , but they exhibit a sign change at some point). The wave shape in three of the graphs is due to the fact that the function $h(t) - th'(t)$ has a local minimum away from zero. In this case, the successive derivatives may progressively (as n increases) hit zero in the vicinity of this local minimum. When there is no

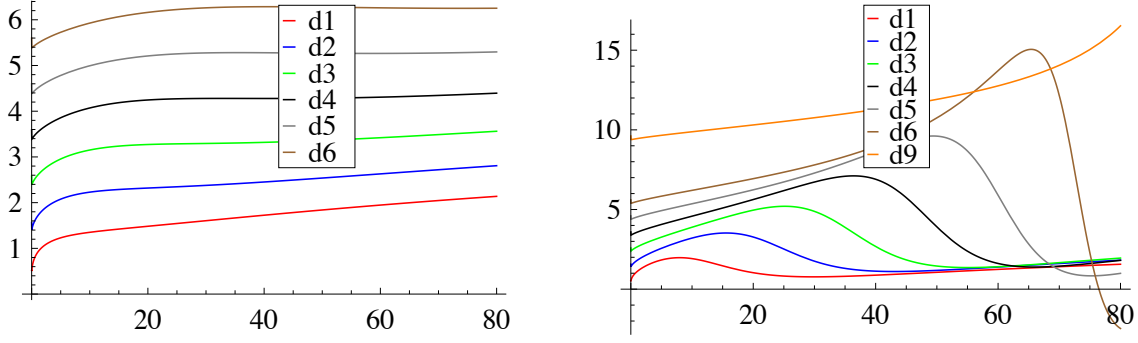


Figure 4.4.2: Graph of d_n for various n in two cases (related to the second example below)

local minimum, our tests have shown that the d_n are close to a constant or a slightly increasing affine function (like on the left graph of Figure 4.4.2).

Error results

We define $N = M - L$, $H = G - F$ and recall that $\mathbb{L}[H(x)](t) = N(t)/t$ is the Laplace transform of the error on the c.d.f.s. The following proposition provides the Mellin transform of H , given N , and the Kantorovich distance between X and Y , which we define in the following way

$$K(X, Y) = \sup \left\{ \int_0^\infty f(x)(F(dx) - G(dx)); f \in \text{Lip} \right\},$$

where Lip is the set of 1-Lipschitz functions. Dall'Aglio proved in [32] that in fact,

$$K(X, Y) = \int_0^1 |F^{-1}(x) - G^{-1}(x)| dx = \int_0^\infty |F(x) - G(x)| dx,$$

because the support of X and Y is \mathbb{R}_+ .

We recall that in our setting, the functions N and H are either nonnegative or nonpositive. For simplicity, and without any loss of generality, we consider henceforth assume that they are nonnegative.

Proposition 4.4. *For $0 < b < 1$, whenever these integrals make sense,*

$$\int_0^\infty \frac{N(t)}{t^{1+b}} dt = \Gamma(1-b) \int_0^\infty x^{b-1} H(x) dx = \frac{\Gamma(1-b)}{b} \int_0^\infty H(x^{1/b}) dx.$$

Moreover,

$$\lim_{t \downarrow 0} N(t)/t = \int_0^\infty H(x) dx = K(X, Y). \quad (4.14)$$

Proof. The first equality is simply Fubini's theorem combined with the identity $\int_0^\infty e^{-xt} t^{-b} dt = \Gamma(1-b)x^{b-1}$ and a standard change of variable; the second equality is obvious. \square

In some cases, it is possible to obtain an upper bound for the L^p quasi-norm of H for $p \in (0, 1)$, using Jensen's (reversed) inequality.

Lastly, we would like to recall the link between the Kantorovich distance and the Kolmogorov (uniform) distance $\sup_{x \geq 0} |F(x) - G(x)|$. Intercalating the Lévy and Prohorov metrics, (using the results from [49] pp. 35-36 and [88] p43), we get

$$\sup_{x \geq 0} |F(x) - G(x)| \leq \left((1+c) \int_0^\infty |F(x) - G(x)| dx \right)^{1/2} = \sqrt{(1+c)K(X, Y)},$$

where c is the maximum value (over \mathbb{R}_+) of $f = F'$, the density of X , if it exists.

4.4.3 Examples

We test our method on two distributions for which a rather simple closed form for f or F is available. The driving criterion for our approximations will be to get a finite Kantorovich distance.

A generalized Mittag-Leffler distribution

We follow the notations of [54]. Generalized Mittag-Leffler distributions are a two parameter family of laws with Laplace transforms

$$L(t) = (1 + t^\alpha)^{-\beta}, \quad \beta > 0, \quad 0 < \alpha \leq 1,$$

and cumulative distribution function

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta + k) x^{\alpha(\beta+k)}}{k! \Gamma(\beta) \Gamma(1 + \alpha(\beta + k))}.$$

We will focus on the simple case $\alpha = 1/2$, and $\beta = 2$. First notice that since $\frac{\Gamma(2k+2)}{\Gamma(k+2)(2k)!} = \frac{1}{k!} (2 - 1/(k+1))$,

$$\sum_{k=0}^{\infty} \frac{\Gamma(2+2k) x^{(2+2k)/2}}{(2k)! \Gamma(1 + (2+2k)/2)} = e^x (2x - 1) + 1.$$

Next, the odd integers are dealt with using the infinite series representation of the error function (8.253-1 in [47])

$$e^x \operatorname{erf}(\sqrt{x}) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2^k x^{k+1/2}}{(2k+1)!!},$$

where $(2k+1)!! = 1 \cdot 3 \cdot 5 \dots 2k+1$, and the identity

$$\frac{\Gamma(2k+3)}{(2k+1)!\Gamma(k+5/2)} = (2k+2) \frac{2^{k+2}}{\sqrt{\pi}(2k+3)!!},$$

which yields in the end

$$F(x) = e^x(2x-1)\operatorname{erfc}(\sqrt{x}) - 2\sqrt{x/\pi} + 1.$$

From $L(t) = (1 + \sqrt{t})^{-2}$, we know that $f(0+) = 1$ and that f has a heavy tail. We thus choose the Pareto family with $a = n$ in order to have the proper asymptotic behaviour for L ($t \rightarrow \infty$). In fact, the domination condition imposes $a, n \geq 1/2$ and a few tests show that $a = n = 1/2$ is a relevant choice, yielding

$$M(t) = \frac{\sqrt{t}e^{t/2}\Gamma(-1/2, t/2)}{2\sqrt{2}}, \quad t \geq 0.$$

As expected, the approximation is not satisfactory and we must resort to a proper μ .

We wish to stress the importance of the choice of μ and we will test the performance of two functions, namely μ_1 and μ_2 . The first naive choice was to take μ of the form $\mu_1(t) = c \left(\frac{a^{3/2}}{(a+t)^{3/2}} - \frac{b^{3/2}}{(b+t)^{3/2}} \right)$ and an admissible set of parameters was $a = 3$, $b = 0.05$ and $c = 0.135$. This triple was the result of a fitting algorithm from a quantitative software.

Nevertheless, this approximation does not allow to compute the Kantorovich error because it is not good enough near 0. In order to be able to compute (4.14), we recall the expansion of L at zero (derived from that of $(1+t)^{-2}$):

$$(1 + \sqrt{t})^{-2} = 1 - 2\sqrt{t} + 3t - 4t^{3/2} + O(t^2), \quad t \downarrow 0.$$

Therefore, a strong improvement of the approximation should satisfy $M(t) - \mu_2(t) \sim 1 - 2\sqrt{t} + O(t)$, $t \downarrow 0$. By 45:5:2 in [87] combined with 6.5.17 and 7.1.5 in [2],

$$\frac{\sqrt{t}e^{t/2}\Gamma(-1/2, t/2)}{2\sqrt{2}} = 1 - \sqrt{\frac{\pi t e^t}{2}} \left(1 - \operatorname{erf}\left(\sqrt{t/2}\right) \right) = 1 - \sqrt{\pi t/2} + t + O(t^{3/2}), \quad t \downarrow 0.$$

Moreover,

$$e^{-\sqrt{at}} = 1 - \sqrt{at} + \frac{at}{2} + O(t^{3/2}), \quad t \downarrow 0,$$

hence we propose $\mu_2(t) = c(e^{-\sqrt{at}} - e^{-\sqrt{bt}})$ with a, b, c satisfying $c(\sqrt{a} - \sqrt{b}) = -2 + \sqrt{\pi/2}$. The triple $a = 0.777$, $b = 20$ and $c = 0.206$ yields promising results with a Kantorovich distance of approximately 0.02. This figure was computed using two techniques:

- formula (4.14) as well as the series representation of the involved functions near $t \downarrow 0$ (the expansion of $N(t)$ has one term in t remaining as well as terms with higher orders). For instance, in our case, the limit in (4.14) is computed using the term in t in the expansions of L , M and $e^{-\sqrt{a \cdot}}$ at zero; the figure is thus $3 - 1 - 0.206 * (20/2 - 0.777/2) \approx 0.02$.

- numerical integration of the difference of the cumulative distribution functions on the interval $(0, 100)$.

Of course, in both cases, we have checked (using our d_n -based test (4.13)) that the error h was such that $h(t)/t$ was a completely monotone function.

We provide the graphical results below (Figures 4.4.3 and 4.4.4). M is the Laplace transform of the Pareto distribution with $a = n = 1/2$, $M_1(t) = M(t) - \mu_1(t)$ and $M_2(t) = M(t) - \mu_2(t)$. Their c.d.f. counterparts are G , G_1 and G_2 . It is plain on the graphs that M_1 and M_2 are quite close, except near zero; this explains why only G_2 is a good fit for F for x large (as expected).

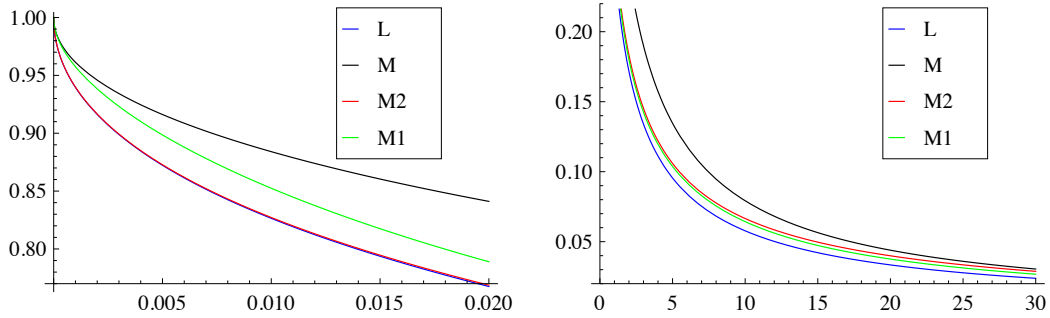


Figure 4.4.3: Graph of L and its proxies for $t \in (0, 0.02)$ and $t \in [0.02, 30]$

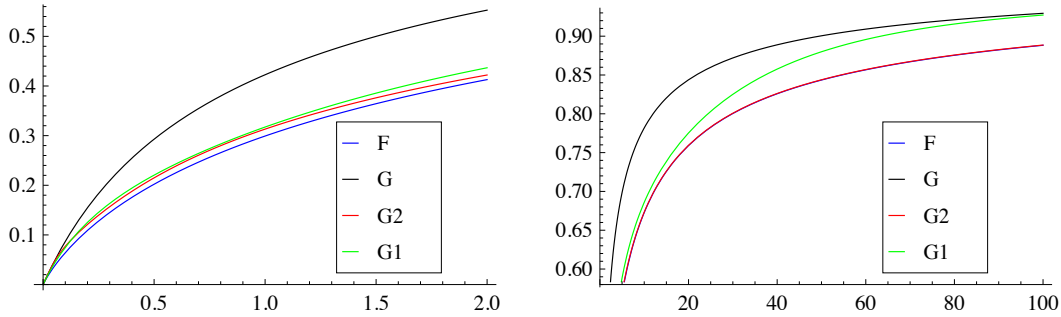


Figure 4.4.4: Graph of F and its proxies for $x \in (0, 2)$ and $x \in (2, 100)$

A positive stable distribution

Our second example is the one parameter one-sided stable laws with Laplace transform

$$L(t) = e^{-t^\alpha}, \quad \alpha \in (0, 1).$$

It is the Laplace transform of an α -stable subordinator taken at time $t = 1$. The case $\alpha = 1/2$ is sometimes referred to as the Lévy distribution, which is connected with the first passage time of the Brownian motion over fixed levels. The case

$\alpha = 1/3$ also has a closed-form density, which is given by (see B.25 in ([79]) for instance):

$$f(x) = \frac{K_{1/3}\left(\sqrt{4/(27x)}\right)}{3\pi x^{3/2}}, \quad x \geq 0.$$

We will thus aim at approximating $L(t) = e^{-t^{1/3}}$. In this case, $f(0+) = 0$ and f has a fat tail. Moreover,

$$L(t) = 1 - t^{1/3} + \frac{t^{2/3}}{2} - t/6 + O(t^{4/3}), \quad t \downarrow 0. \quad (4.15)$$

The choice of the inverse gamma family with $n = 1/3$ seems relevant, as it satisfies (see 51:6:1 in [87])

$$M(t) = \frac{2}{\Gamma(1/3)}(at)^{1/6}K_{1/3}(2\sqrt{at}) = 1 + \frac{a^{1/3}\Gamma(-1/3)}{\Gamma(1/3)}t^{1/3} + 3at/2 + O(t^{4/3}), \quad t \downarrow 0.$$

Hence, for $a = -\Gamma(1/3)^3/\Gamma(-1/3)^3$, the $t^{1/3}$ term in the error will vanish (by (4.15)), but the $t^{2/3}$ term will remain. This leads to the following choice of μ :

$$\begin{aligned} \mu(t) &= 2c \left(\frac{(bt)^{1/3}K_{2/3}(2\sqrt{bt})}{\Gamma(2/3)} - \frac{(dt)^{1/3}K_{2/3}(2\sqrt{dt})}{\Gamma(2/3)} \right) \\ &= c \frac{\Gamma(-2/3)}{\Gamma(2/3)} (b^{2/3} - d^{2/3}) t^{2/3} + 3c(b - d)t + O(t^{5/3}), \quad t \downarrow 0. \end{aligned}$$

Notice that this time, the ordering is in the opposite way: $L(t) \geq M(t)$ for all $t \geq 0$. An admissible set of parameters is $c = 6$, $b = 0.4$ and $d = 0.43$ which yields a Kantorovich distance of less than 0.07 (see Figures 4.4.5 and 4.4.6).

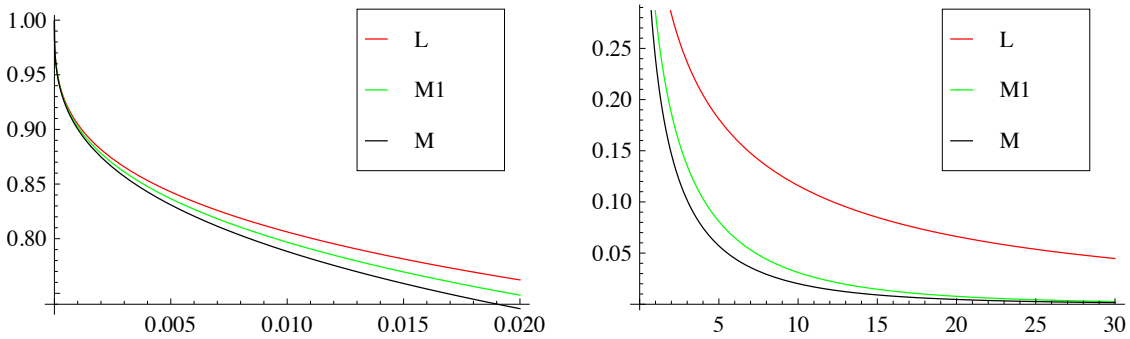


Figure 4.4.5: Graph of L and its proxies for $t \in (0, 0.02)$ and $t \in [0.02, 30]$

Of course, in both examples, it is possible to further reduce the error by repeating step 2 at least one time (using a minorant μ^* of $L - M - \mu$ for instance).

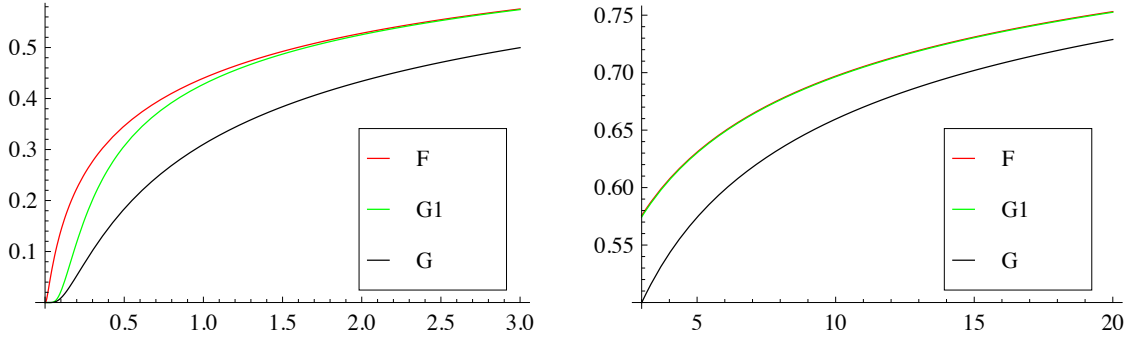


Figure 4.4.6: Graph of F and its proxies for $x \in (0, 2)$ and $x \in (2, 100)$

Remark 4.4.1. We did not study distributions with lighter tails in the examples because when $1 - F(x) \leq cx^{-\alpha}$, with $c > 0$ and $\alpha > 1$ for any large x , then it is much easier to obtain a finite Kantorovich error, as the original survival function is already integrable.

Using the exact same procedure as in the second example, it would thus take $n - 2$ iterations of step 2 to obtain an approximation with finite Kantorovich error for the stable law with Laplace transform equal to $e^{-t^{1/n}}$. The same holds for generalized Mittag-Leffler distributions defined by $L(t) = (1 + t^{1/n})^{-p}$, for any real $p \geq n$. These assertions are a consequence of the Taylor expansion of L at 0. In the stable case, when $\alpha \in (1/2, 1)$, it is possible to obtain a finite Kantorovich measure by taking $M(t) = e^{-\sqrt{t}}$ and μ such that $\mu(t) \sim t^\alpha - \sqrt{t}$ when $t \downarrow 0$.

Furthermore, we would like to underline that even though we have assumed (for simplicity) that the law of X was absolutely continuous, our method remains valid for most positive laws. It is indeed possible to make do without densities throughout the whole process. However, it is not clear whether this method can perform well for some rather unusual distributions, such as those which possess an infinite number of atoms.

Lastly, the results we provided in subsection 4.4.1 were sufficient to treat both of our examples; however, many Laplace transforms require more complicated functions and in such cases, more elaborate proxies should be used, along with the most general formulations of the Tauberian theorems in [14], section 4.12.

Chapter 5

Pricing exotic options in the Finite Moment Log-Stable model

5.1 Introduction

During the XXth century, the financial markets have invented many products which were very often used for two very different purposes: speculating and hedging risks. One very large family of such products are called derivatives. They were named this way because their value depends essentially on that of an underlying asset or financial rate, e.g. a stock, a currency rate, or an interest rate. Popular families of derivatives are options. A European Call option enables its holder to buy a specific stock¹ at a future (maturity) date T at some pre-determined price K , the strike of the option. The (European²) Put option enables its holder to sell the stock at time T at a pre-determined price of K . Hence, at maturity, because the security gives the right, but not the *obligation* to buy or sell, the payoffs of these two options are

$$p^{Call} = (S_T - K)^+, \quad p^{Put} = (K - S_T)^+, \quad (5.1)$$

where S_T is the value of the stock at time T and $(x)^+ = \max(0, x)$. When the random part of the payoff of an option depends solely on the terminal value of S , then this option is said to be "vanilla". If it depends on the whole trajectory of S up to T (maximum value or average value for instance), the option is said to be "exotic".

Now, 1€ today is not equal to 1€ at time T . This is because some institutions (countries, banks, people ...) lend money to other institutions with very limited default risk. Because of the interests, lending 1€ today will generate almost surely a little more than 1€ in the future. We therefore introduce a risk-free rate, which we will assume constant and equal to r and such that e^{-rT} € today will be worth 1€ at time T .

¹Even though the results of this chapter will apply to any type of underlying, we will, for simplicity, consider the case of derivatives written on stocks

²We will deal only with European options in this chapter and we will henceforth omit this precision in our notations

Therefore, when a Call or a Put is issued, its present value is equal to

$$Call_0 = Call = e^{-rT} \mathbb{E}_P[(S_T - K)^+] \quad \text{or} \quad Put_0 = Put = e^{-rT} \mathbb{E}_P[(K - S_T)^+], \quad (5.2)$$

where P is a relevant probability measure. Any option can be bought or sold on a secondary market and we write $Call_t$ and Put_t for their value at time t . Buying at time $t \in [0, T)$ a Call and selling the corresponding Put with strike K and maturity T yields an almost sure payoff of $S_T - K$ at time T . Therefore,

$$Call_t - Put_t = S_t - e^{-rT} K, \quad \forall t \in [0, T],$$

a formula which is known as the Put-Call parity.

In order to provide a closed-form for the values of the Call and the Put, we need to provide a particular dynamic for S under P .

The first person who tried to do this was Louis Bachelier, in his 1900 thesis, *Théorie de la spéculation*. He assumed independent Gaussian increments for S_t , but this could lead to negative prices. A tractable solution was provided more than seven decades afterwards by Black and Scholes [15] and Merton [83] (the BS-M model), following the earlier work of Samuelson [93], among others. In order to introduce this model, we focus on one of the most important concept in Finance: returns. If S_t is the value of any underlying at time t , then the relative increase (or decrease) of S between t and $t + \Delta t$ is $(S_{t+\Delta t} - S_t)/S_t$. This value is called the return of S between t and $t + \Delta t$. When working with time-continuous model, it is preferable to consider log-returns, which are defined by

$$r_{t,t+\Delta t} = \log(S_{t+\Delta t}/S_t),$$

and which are very close to $(S_{t+\Delta t} - S_t)/S_t$ when Δt is small, because of the Taylor expansion of $\log(1 + x)$ at $x = 0$.

In the BS-M model, the log-returns are assumed to be i.i.d. Gaussian variables. Consequently, we can write

$$S_t = S_0 e^{\mu t + \sigma B_t}, \quad (5.3)$$

where B is a standard Brownian motion. The parameter σ , which is the standard error of the log-returns of S is called the volatility. The estimation of μ , the average of the log-returns is a very complicated task. Fortunately, in the option pricing setting, an elegant and abstract solution consists in the introduction of the risk-neutral probability P . We do not provide details here, but under mild technical assumptions (including the absence of arbitrage opportunities), then, under P , $\mu = r - \sigma^2/2$ for all stocks³. The process $L_t = e^{-rt} S_t$ is an \mathcal{F}_t -martingale, where \mathcal{F}_t is the natural filtration of the process $\log(S_t/S_0)$. The prices of the standard options

³Prices are observed under the historical probability, but, in order to satisfy the condition of no arbitrage opportunities, an absolutely continuous change of measure is required, which leads to $\mu = r - \sigma^2/2$. See [52], section 2.3.1 for details on the subject

are then easily computed:

$$Call_t = S_t N(d_1) - K e^{-r(T-t)} N(d_2), \quad (5.4)$$

$$Put_t = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1), \quad (5.5)$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and N is the Gaussian cumulative distribution function.

For the sake of completeness, we provide another hint for the proof of this result. The dynamics of the underlying can be rewritten in a differential fashion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad t > 0.$$

Using Ito's lemma, we can show that it is possible to construct a risk-free portfolio consisting of -1 $Call_t$ and $\partial Call_t / \partial S_t$ shares of the underlying. If there are no arbitrage opportunity, then the return of this portfolio must be r . From this, we can infer the following partial differential equation, due to Black, Scholes and Merton:

$$\frac{\partial Call_t}{\partial t} + r S_t \frac{\partial Call_t}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 Call_t}{\partial S_t^2} = r Call_t.$$

This equation, with the boundary condition

$$Call_T = (S_T - K)^+$$

yield the price of the call. We refer to [28], sections 2.2 and 2.3 for the formal proofs.

One feature of the formulae (5.4) and (5.5) is that they have only one free parameter: σ . All of the other parameters are either given in the definition of the option (S_t , K , $T - t$), or by the market (r). It can further be shown (see subsection 5.5) that

$$\frac{\partial Call}{\partial \sigma} = \frac{\partial Put}{\partial \sigma} > 0,$$

for any choice of positive S_0 , K , T and r . If we denote by $Call^M(K, T)$ the observed price, on the market, for a call with strike K and maturity T , then there is a unique value σ^* of σ such that the price in the BS-M model coincides with the market price. The value σ^* is called the implied volatility. According to the BS-M model, σ^* should not depend on K . However, it has empirically been shown that σ^* taken as a function of K (i.e. observed for several values of $Call^M(K, T)$) exhibits various shapes, often referred to as "smiles", "smirks" or "skews". More generally, the mapping $(K, T) \mapsto \sigma^*(K, T)$ is called the volatility surface. The fact that it is, in practice, not completely flat points out one of the limitations of the BS-M model.

In fact, many empirical observations have shown that this model is too restrictive (see section 1.1 in [18] for instance) and alternatives have blossomed even before the seventies. For instance, in the early sixties, Mandelbrot ([80]) had already raised the question of whether or not stock returns could be modeled by stable distributions. Scholars and practitioners have published contradicting results on this topic for over four decades (a non-exhaustive sample is: [41], [3], [72], [55]).

If returns can indeed be modeled by stable laws, then the next logical step is the pricing of options written on stocks driven by such distributions (i.e. providing closed-forms for (5.2)). In order to do so, the classical framework is to use exponentials of Lévy processes (hence postulating that returns are i.i.d., regardless of their time scale). This is quite problematic since the heavy tails of the stable laws imply infinite prices for standard Call options under these assumptions⁴. Empirically, one way to circumvent this inconvenience is to consider options with very short maturity (see [82]). Unfortunately, this is not satisfactory because in practice, many options are long-lived (warrants, for instance). A tractable solution was provided by the Finite Moment Log-Stable (FMLS) model, due to Carr and Lu in [24] (even though a hint towards this direction was already in [82]).

Their idea is to resort to completely asymmetric stable distributions. In this case, the left tail remains heavy, but the right tail becomes sub-exponential, thereby yielding finite option prices. Another way to ensure finite prices is to force the damping of the tails of the distribution. The result is a wide class of distribution, known as the tempered-stable (also referred to as CGMY or KoBoL) laws, which has been extensively studied in Finance ([12], [17], [22], [69] (chapter 12), [89], [92]).

The remainder of chapter 5 is structured as follows. In section 2, we introduce the model while in section 3 we present our results on exact prices. Section 4 is devoted to approximative methods (Monte-Carlo mainly), and section 5 is dedicated to the pricing of vanilla options and to the computation of their sensitivities with respect to some parameters (greeks).

5.2 Presentation of the model

We start by postulating that the stock value under consideration can be modeled, under the risk-neutral measure P , as follows

$$S_t = S_0 e^{(r-d-\sigma^\alpha)t + \sigma X_t^{(\alpha)}}, \quad t \geq 0, \quad (5.6)$$

where r and d are the continuous risk-free rate and dividend rate respectively, σ a strictly positive constant and $X^{(\alpha)}$ a spectrally negative α -stable Lévy process with

⁴The prices of Put options are finite in any model, since their payoffs are bounded by K . Obviously, when Call prices are infinite, the Put-Call parity does not hold.

$\alpha \in (1, 2)$ and such that

$$\mathbb{E}_P \left[e^{\sigma X_t^{(\alpha)}} \right] = e^{t\sigma^\alpha}, \quad t \geq 0, \quad (5.7)$$

that is to say, $X^{(\alpha)}$ verifies (3.3) with $\beta = -1$ and $c = \cos(\pi(\alpha - 2)/2)$. The model therefore has two free parameters, α and σ , while the BS-M only has one (the volatility). For notational convenience, we will henceforth omit the dependence in P in our notations, since all of our results will hold under the risk-neutral measure. Compared to [24], we have introduced a scaling factor which further simplifies the calculations. It is easy to see with (5.7) that $\mathbb{E}[S_t^n]$ is finite for any $n > 0$, which justifies the denomination of the model.

We denote by \mathcal{F}_t the natural filtration of the process $X^{(\alpha)}$. Equation (5.7) ensures that the process $L_t = e^{-(r-d)t}S_t$ is an \mathcal{F}_t -martingale under P . An important feature of $X^{(\alpha)}$ is that it has only negative jumps, hence the distribution of the asset's log-returns $r_{t,T} = \log(S_T/S_t)$ (for $T > t$) is strongly negatively skewed: its density has a power decaying tail on the left and an exponentially decaying tail on the right. Empirically, this can be justified by the fact that stocks usually increase slowly, with daily positive returns rarely above 5% or 10%, while they can experience massive daily losses due either to macro-economic shocks or to the publication of unfavorable stock-specific news or reports.

In their paper [24], Carr and Wu show that the representation (5.6) for the stock value is able to generate any type of slope for the implied volatility skews observed in the S&P 500 option market. In order to compute vanilla price options in their model, they use the method developed by Carr and Madan in [23]. With the help of this technique, they show that the post-calibration pricing error implied by (5.6) is never worse (in fact often better) than that of other popular models with 3 to 6 free parameters (for instance, the Merton Jump Diffusion process with stochastic volatility).

Nowadays, the option market for most stocks with large market capitalization is very liquid. Hence, the model calibration can be used to price more complicated options. We will focus on two types of such products. We begin with the lookback options which have the following payoffs at maturity $t = T$:

$$p^{LC} = S_T - I_T \text{ for a Lookback Call, } p^{LP} = M_T - S_T \text{ for a Lookback Put, } \quad (5.8)$$

where

$$I_T = \inf_{0 \leq t \leq T} S_t, \quad M_T = \sup_{0 \leq t \leq T} S_t,$$

which guarantees that the payoffs are almost surely positive. The prices at $t = 0$ for these derivatives under the risk-neutral measure are

$$LC = LC_0 = e^{-rT} \mathbb{E}[S_T - I_T] \quad \text{and} \quad LP = LP_0 = e^{-rT} \mathbb{E}[M_T - S_T]. \quad (5.9)$$

The second family of options is much wider. Barrier options are classical puts and calls which are activated or killed if a barrier has been hit before the maturity of the

option. The option is called "In" (*resp.* "Out") if it is activated (*resp.* *killed*) upon hitting the barrier. If the barrier is to be reached from below (*resp.* *above*), then the option is "Up" (*resp.* "Down"). For instance, we define the payoff at maturity of these options in two cases

$$p^{UIP} = (K - S_T)^+ \mathbf{1}_{\{M_T > B\}} \text{ for an Up and In Put,}$$

$$p^{DOC} = (S_T - K)^+ \mathbf{1}_{\{I_T > B\}} \text{ for a Down and Out Call,}$$

where K is the strike of the option and B its barrier.

The corresponding prices are

$$UIP = e^{-rT} \mathbb{E}[(K - S_T)^+ \mathbf{1}_{\{M_T > B\}}], \quad DOC = e^{-rT} \mathbb{E}[(S_T - K)^+ \mathbf{1}_{\{I_T > B\}}]. \quad (5.10)$$

It is possible to provide exact formulae for (5.9) and (5.10) in some cases. It is the purpose of the next section.

5.3 Exact valuation

5.3.1 Pricing lookback options at $t = 0$ - case $r - d = \sigma^\alpha$

We first deal with the lookback options and start by a simplified case, i.e. when S is modeled by the exponential of a Lévy process without drift. In this case, some results are available, both for the running maximum and the running minimum of the driving Lévy process. The first valuation of Lookback options, in the Black-Scholes setting, is due to Goldman, Sosin and Gatto in [45]; another reference is [43]. In the FMLS model, the following holds.

Proposition 5.1. *If $r - d = \sigma^\alpha$, then, for $T=1$,*

$$LP = S_0 e^{-r} (E_{1/\alpha}(\sigma) - e^{\sigma^\alpha}), \quad (5.11)$$

and

$$LC = S_0 e^{-r} \left(e^{\sigma^\alpha} - \frac{\alpha}{\Gamma(1/\alpha)} e^{\sigma^\alpha} \int_\sigma^\infty e^{-z^\alpha} dz \right), \quad (5.12)$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \quad \alpha > 0. \quad (5.13)$$

Remark 5.3.1. For $\alpha = 2$, $E_{1/2}(z) = e^{z^2}(\text{erfc}(-z))$ (see section 45:14 in [87]). By the Désiré-André identity, the density of the supremum M^B of the Brownian motion B_t is given by

$$f_{M_t^B} = \frac{2}{\sqrt{2\pi t}} e^{-x^2/(2t)} \mathbf{1}_{\{x \geq 0\}}, \quad t > 0,$$

and the change of variable $z = \sqrt{(x - \sigma t)^2 / (2t)}$ then leads to

$$\int_0^\infty \frac{2e^{\sigma x}}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx = e^{\sigma^2 t/2} \int_0^\infty \frac{2}{\sqrt{2\pi t}} e^{-(x - \sigma t)^2/(2t)} = e^{\sigma^2 t/2} \operatorname{erfc}(-\sqrt{t\sigma^2/2}),$$

hence the results are consistent with the Brownian case, for $S_t = S_0 e^{\sigma\sqrt{2}B_t}$.

We provide graphs of both prices as functions of α and σ (with $S_0 = 1$, $r = \sigma^\alpha$ and $d = 0$). The values were computed using Mathematica (NIntegrate function for the integral and a truncated (above $k = 400$) version of (5.13).

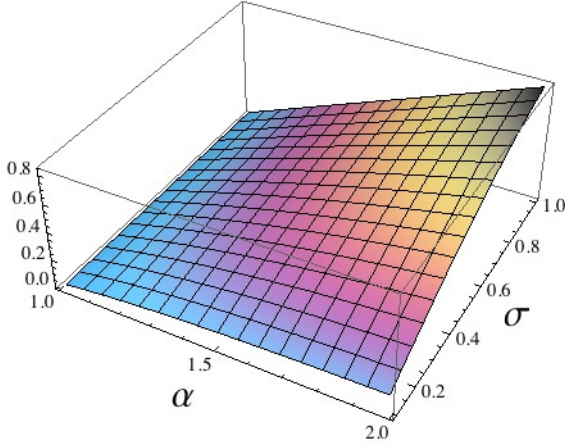


Figure 5.3.1: Graph of LP for $\alpha \in (1, 2)$ and $\sigma \in (0.1, 1)$

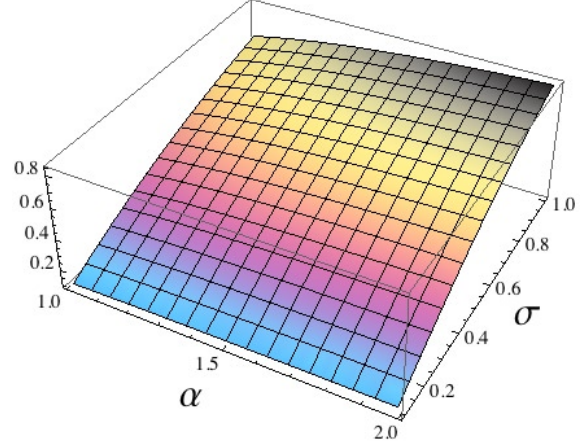


Figure 5.3.2: Graph of LC for $\alpha \in (1, 2)$ and $\sigma \in (0.1, 1)$

Notice that for the Lookback Put, the price is increasing, both in α and σ , but for the Lookback Call, it is almost constant in α . Qualitatively, this difference stems from the negative jumps which imply that a variation in α has more impact on the running minimum of S than on its running maximum.

Before proving Proposition 5.1, we recall some basic facts. $X^{(\alpha)}$ is a Lévy process with absolutely continuous jump measure

$$\nu(dx) = \frac{\mathbf{1}_{\{x < 0\}}}{\Gamma(-\alpha)(-x)^{1+\alpha}} dx, \quad (5.14)$$

so that its Lévy-Khintchine representation is indeed given by

$$\log \left(\mathbb{E} \left[e^{\sigma X_1^{(\alpha)}} \right] \right) = \int_{-\infty}^0 (e^{\sigma x} - 1 - \sigma x) \nu(dx) = \sigma^\alpha.$$

Because its jumps are fully compensated, $X^{(\alpha)}$ is a martingale. The prices (5.9) require the knowledge of the law of the supremum and infimum of X_t^α , or more generally of $X_t^{\alpha, \mu} = X_t^{(\alpha)} + \mu t$. Some results in this direction are given in [85], but they are not exactly what we seek here. We further recall the positivity parameter of $X^{(\alpha)}$:

$$\rho = P \left[X_t^{(\alpha)} \geq 0 \right] = (1 + (2 - \alpha)/\alpha)/2 = \frac{1}{\alpha}.$$

Lastly, we recall some notations: I_t, M_t are the infimum and supremum of the price process S_t while I_t^X and M_t^X are the infimum and supremum of the underlying Lévy process. X can be $X^{(\alpha)}$, $X^{\alpha,\mu}$, $\tilde{X}^{(\alpha)}$ or $\tilde{X}^{\alpha,\mu}$. The two latter processes are independent copies of $X^{(\alpha)}$, $X^{\alpha,\mu}$ which will be used later on. In our proofs, we will use the series representation of stable densities extensively. Our main source is [94], and proofs can be found in [102].

Proof of Proposition 5.1. Because $r-d = \sigma^\alpha$, we only need to consider $X^{(\alpha)}$ (without any drift). Note that by the self-similarity property of the process, it is sufficient to work with $T = 1$. Moreover, the same property yields

$$P \left[M_1^{X^{(\alpha)}} \leq x \right] = P[T_x \geq 1] = P[(T_1)^{-1/\alpha} \leq x],$$

hence $M_1^{X^{(\alpha)}}$ and $(T_1)^{-1/\alpha}$ have the same distribution, where $T_a = \inf \left\{ t > 0, X_t^{(\alpha)} > a \right\}$ is the first passage time of $X^{(\alpha)}$ over a fixed level $a > 0$.

Because $X^{(\alpha)}$ has no positive jumps, then we can apply Theorem 1, section VII from [11] to get that T_a is a subordinator with characteristic exponent $\log(\mathbb{E}[e^{-zT_a}]) = -az^{1/\alpha}$, thus is an $1/\alpha$ -subordinator. Equation (5.11) then stems from exercise 29.18 in [94] (see also exercise 6.6 in [68]).

Equation (5.12) is a much deeper result, which is the combination of (2.55) in [10] and the fact that for any process X , $\inf_{0 \leq t \leq T} X_t = - \sup_{0 \leq t \leq T} -X_t$. \square

5.3.2 Pricing lookback options for $t \in (0, T)$ - case $r - d = \sigma^\alpha$

The valuation of these products on the secondary market for $t \in (0, T)$ is a more complicated task. We need to consider an updated model:

$$S_{t+s} = S_t e^{(r-d-\sigma^\alpha)s + \sigma \tilde{X}_s^{(\alpha)}}, \quad s \geq 0, \quad (5.15)$$

where $\tilde{X}^{(\alpha)}$ is an independent α -stable spectrally negative Lévy process (that is, an independent copy of $X^{(\alpha)}$). The payoff of the lookback options are given by

$$p_t^{LC} = S_T - \min(I_t, I_{t,T}), \quad p_t^{LP} = \max(M_t, M_{t,T}) - S_T,$$

where

$$I_{t,T} = \inf_{t \leq s \leq T} S_s, \quad M_{t,T} = \sup_{t \leq s \leq T} S_s.$$

Therefore, under (5.15) and $r - d = \sigma^\alpha$, the prices, at time t , of lookback options issued at time zero are detailed in the following formulae.

Theorem 5.1. At time $t \in (0, T)$, when $S_{t+s} = S_t e^{\sigma \tilde{X}_s^{(\alpha)}}$, the lookback prices are given by,

$$\begin{aligned}
LC_t &= S_t e^{(T-t)(\sigma^\alpha - r)} - e^{-r(T-t)} \sum_{n=1}^{\infty} \frac{I_t (\log(S_t/I_t)/\sigma)^{\alpha n - 1}}{\Gamma(\alpha n - 1) \Gamma(1 - n + 1/\alpha) (\alpha n - 1) (T-t)^{n-1/\alpha}} \\
&\quad + S_t e^{(T-t)(\sigma^\alpha - r)} \left[\sum_{n=1}^{\infty} \frac{\int_0^{\log(S_t/I_t)/\sigma} e^{-\sigma x} x^{\alpha n - 2} dx}{\Gamma(\alpha n - 1) \Gamma(1 - n + 1/\alpha) (T-t)^{n-1/\alpha}} - \frac{\alpha}{\Gamma(1/\alpha)} \int_{\sigma(T-t)^{1/\alpha}}^{\infty} e^{-z^\alpha} dz \right], \\
LP_t &= e^{-r(T-t)} \frac{\alpha M_t}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1) \sin(\pi n/\alpha) (\log(M_t/S_t)/\sigma)^n}{n! n (T-t)^{-n/\alpha}} - S_t e^{(T-t)(\sigma^\alpha - r)} \\
&\quad + e^{-r(T-t)} S_t \left[E_{1/\alpha}[\sigma(T-t)^{1/\alpha}] - \frac{\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(n/\alpha + 1) \sin(\pi n/\alpha) \int_0^{\log(M_t/S_t)/\sigma} e^{\sigma x} x^{n-1} dx}{n! (T-t)^{-n/\alpha}} \right].
\end{aligned}$$

Proof. We start by recalling the densities of $-I^{X^{(\alpha)}}$ and $M^{X^{(\alpha)}}$: for $t > 0$, $x > 0$,

$$\frac{P[-I_t^{X^{(\alpha)}} \in dx]}{dx} = \sum_{n=1}^{\infty} \frac{1}{\Gamma(\alpha n - 1) \Gamma(1 - n + 1/\alpha)} \frac{x^{\alpha n - 2}}{t^{n-1/\alpha}}, \quad (5.16)$$

$$\frac{P[M_t^{X^{(\alpha)}} \in dx]}{dx} = \frac{\alpha}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) x^{n-1} t^{n/\alpha}. \quad (5.17)$$

The first identity is simply (2.54) in [10]. For $t = 1$, the second identity stems from the first term of (5.11) and equation (2.10.9) from [102]. The self-similarity property gives the formula for any $t > 0$.

Then, for the Lookback Call, when $(s > t)$,

$$\begin{aligned}
\mathbb{E} \left[\min \left(I_t, e^{\sigma I_s^{X^{(\alpha)}}} \right) \right] &= \int_0^{\infty} \min(I_t, S_t e^{-\sigma x}) f_{-I_s^{X^{(\alpha)}}}(x) dx \\
&= I_t \int_0^{-\log(I_t/S_t)/\sigma} f_{-I_s^{X^{(\alpha)}}}(x) dx + S_t \int_{-\log(I_t/S_t)/\sigma}^{\infty} e^{-\sigma x} f_{-I_s^{X^{(\alpha)}}}(x) dx,
\end{aligned}$$

the first term is computed by integrating term by term (5.16). For the second one, we use the decomposition

$$\int_{-\log(I_t/S_t)/\sigma}^{\infty} e^{-\sigma x} f_{-I_s^{X^{(\alpha)}}}(x) dx = \int_0^{\infty} e^{-\sigma x} f_{-I_s^{X^{(\alpha)}}}(x) dx - \int_0^{-\log(I_t/S_t)/\sigma} e^{-\sigma x} f_{-I_s^{X^{(\alpha)}}}(x) dx,$$

and, as in Proposition 5.1,

$$\int_0^\infty e^{-\sigma x} f_{-I_s^{\bar{X}^\alpha}}(x) dx = \frac{\alpha e^{s\sigma^\alpha}}{\Gamma(1/\alpha)} \int_{\sigma s^{1/\alpha}}^\infty e^{-z^\alpha} dz,$$

the second integral can be expressed in terms of the upper incomplete gamma function, but we have chosen to leave it unchanged in the formula.

The formula for the Lookback Put can be obtained using the exact same steps and the density (5.17). \square

We underline that when $S_t = I_t$ or $S_t = M_t$, these results are coherent with the pricing formulae for $t = 0$.

5.3.3 Pricing of an Up and In Put - case $r = d$

Because the computation of the expectations (5.10) requires the knowledge of the distribution of the couples (S_T, M_T) and (S_T, I_T) , we are able to provide an exact result only in a particular case.

We follow Bowie and Carr [16]. They show that under the assumption $r = d$, barrier options can be hedged using linear combinations of vanilla options and barrier Bonds, that is, options with terminal payoffs $\mathbf{1}_{\{M_T > B\}}$, $\mathbf{1}_{\{M_T < B\}}$, $\mathbf{1}_{\{I_T > B\}}$ or $\mathbf{1}_{\{I_T < B\}}$. Unfortunately, one of their results requires a symmetry formula which is not available for stable process with $\alpha < 2$. Moreover, their second family of results depends on the Put-Call parity upon touching the barrier, therefore, the negative jumps in the FMLS model make it impossible to value "Down" type options using their methods. In fact, the only case we can consider is the Up and In Put with $K > B$. If $r = d$, it is shown in [16] that the replicating portfolio consisting of one standard call with strike K and $K - B$ Up and In Bonds is exactly equivalent to the UIP. If the barrier is never hit, both their values are zero but if the barrier is hit (continuously, i.e. without jumps), then, by the classical (vanilla) Put-Call parity, the portfolio is exactly worth the price of the vanilla put. The payoff of the digital barrier option is $\mathbf{1}_{\{M_T > B\}}$, which leads to the following valuation.

Proposition 5.2. *If $r = d$ and $K > B$, the price, at $t = 0$, of an Up and In Put is given by*

$$\begin{aligned} UIP &= e^{-rT}(K - B)P[M_T > B] + C(K, T) \\ &= e^{-rT}(K - B)P[M_T^{X^{\alpha, \mu}} > \log(B/S_0)/\sigma] + C(K, T), \end{aligned} \tag{5.18}$$

where $\mu = -\sigma^\alpha$ is the drift of $\log(S_T/S_0)$ and $M_T^{X^{\alpha, \mu}} = \sup_{0 \leq t \leq T} \{X_t^\alpha + \mu t\}$. $C(K, T)$ is the price (at $t = 0$) of a vanilla call of strike K and maturity T .

Of course, under the event $(M_t < B)$, the price at time $t < T$ is easily derived:

$$\begin{aligned} UI P_t &= e^{-r(T-t)}(K - B) \int_B^\infty f_{\tilde{M}_{T-t}}(x) dx + C_t(K, T - t) \\ &= e^{-r(T-t)}(K - B) P \left[M_{T-t}^{\tilde{X}^{\alpha, \mu}} > \log(B/S_t)/\sigma \right] + C_t(K, T - t), \end{aligned} \quad (5.19)$$

where $C_t(K, T - t)$ is the price, at time t , of a vanilla call with strike K and maturity $T - t$. The following Proposition provides a closed form for the probability in the above formulae. We define $T_x^{\alpha, \mu} = \inf\{t > 0, X_t^\alpha + \mu t \geq x\}$.

Theorem 5.2. *If α is not a rational number, then for any finite $t > 0$, $x > 0$, and $\mu \neq 0$,*

$$P[M_t^{X^{\alpha, \mu}} \leq x] = \frac{\alpha}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(k/\alpha + 1)}{k! k} \sin(\pi k/\alpha) {}_2F_1 \left(1 - k, -\frac{k}{\alpha}; 1 - \frac{k}{\alpha}; \frac{\mu t}{x} \right) t^{-k/\alpha} x^k,$$

where ${}_2F_1$ is the hypergeometric function:

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad c \notin \mathbb{Z} \setminus \mathbb{N} \cup \{0\},$$

with the Pochhammer symbol $(x)_k$ defined by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1) \dots (x+n-1) & \text{if } k > 0 \end{cases}.$$

Proof. We keep the same notations as above, with $X^{(\alpha)}$ replaced by $X_t^{\alpha, \mu} = X_t^{(\alpha)} + \mu t$. Because $P[M_t^{X^{\alpha, \mu}} \leq x] = P[T_x \geq t]$, we will work with the first hitting time T_x of $X^{\alpha, \mu}$. Since $X^{\alpha, \mu}$ has no positive jumps, we can use Corollary 3, section VII from [11]:

$$\frac{P[T_x \in dt]}{dt} = \frac{x}{t} \frac{P[X_t^{\alpha, \mu} \in dx]}{dx},$$

and the series representation of the density of $X^{\alpha, \mu}$ which is given by 14.28 and 14.30 in [94], yielding

$$\frac{P[T_x \in dt]}{dt} = \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) t^{-n/\alpha-1} (x - \mu t)^{n-1}.$$

The integration of this function on a finite interval should be handled carefully. The classical argument, which invokes the normal convergence of the series, does not hold here (the series is not normally convergent). Instead, we consider the partial sum

$$S_k(t) = \sum_{n=1}^k (-1)^{n-1} \frac{x}{\pi} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) t^{-n/\alpha-1} (x - \mu t)^{n-1}, \quad t > 0,$$

which makes sense since the Stirling formula implies that the term $\Gamma(n/\alpha + 1)/\Gamma(n + 1)$ mitigates any power term c^n at infinity (the term of the series is in fact $o(n^{-\gamma})$ for any $\gamma > 1$, as $n \rightarrow \infty$). For $0 < a < b < \infty$, not only does $S_k(t) \rightarrow S_\infty(t) = P[T_x \in dt]/dt$ for any $t \in [a, b]$ but $|S_k(t)|$ is also bounded for any $k \geq 1$ and $t \in [a, b]$. It is thus possible to apply the Arzela-Osgood theorem (see [78] and the references therein) in order to integrate $S_k(t)$ term by term and let $k \rightarrow \infty$.

It can then be shown (using 3.194 in [47], or the properties from subsection 2.1.2 in [9]) that the application

$$F_n := F_{n,\alpha,x,\mu} : t \mapsto \alpha x^{n-1} t^{-n/\alpha} {}_2F_1 \left(1 - n, -\frac{n}{\alpha}; 1 - \frac{n}{\alpha}; \frac{\mu t}{x} \right), \quad t > 0,$$

is one anti-derivative of $t \mapsto t^{-n/\alpha-1}(x - \mu t)^{n-1}$, which yields

$$P[T_x \in [t, u]] = \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) (F_n(t) - F_n(u)) \quad (5.20)$$

$$= G_x(t) - G_x(u), \quad (5.21)$$

with

$$G_x : u \mapsto \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) F_n(u), \quad x, u > 0,$$

where the series is absolutely convergent for any $x, u > 0$ because of the asymptotics of the Hypergeometric function (see equation (9) from subsection 2.3.2 in [9]). Equation (5.21) implies that $G_x(\cdot)$ is both bounded (in $[0, 1]$) and decreasing. We thus have

$$\lim_{u \rightarrow \infty} G_x(u) = P[T_x = +\infty] \in [0, 1],$$

which is strictly positive if $\mu < 0$ ($X_t \rightarrow -\infty$ a.s. when $t \rightarrow \infty$) and equal to zero if $\mu \geq 0$. This can easily be shown when $\mu > 0$ using Wald's identity on X_{T_x} (implying $\mathbb{E}[T_x] < \infty$ in this case). If $\mu = 0$, then X oscillates (see Th.12, section VI in [11]) and thus touches x at some point in time. Therefore, $P[T_x \geq t] = G_x(t)$. \square

Remark 5.3.2. Numerically, the maturity will always be normalized so that $T = 1$. The series in the theorem will be truncated beyond $k = 200$ in which case they remain valid for $x \in (0, 6 + \alpha\mu)$ when $\alpha = \sqrt{\pi}$ and for $x \in (0, 1.5 + \alpha\mu)$ when $\alpha = \sqrt{5}/2$.

Lastly, for the sake of completeness, we wish to point out that a result exists for the supremum of a drifted spectrally positive stable process (see [84]). It can be used to compute $P[I_t^{\alpha,\mu} \leq x]$ for $x < 0$. However, because of the negative jumps, $X_{\hat{T}_{-x}}^{\alpha,\mu} \neq -x$ ($\hat{T}_{-x} = \inf\{t > 0, X_t \leq -x\}$) and the above reasoning does not apply for Down and In Calls. Nevertheless, these formulae can be used to (numerically) compute the prices of the lookback options in the general case.

5.4 Approximative pricing

5.4.1 Classical Monte-Carlo, first method

Whenever the jump measure of a Lévy process is known, it is possible to simulate approximative sample paths of this process and hence to generate sample payoffs of exotic options. Repeating this procedure many times and invoking the law of large numbers gives a close proxy to the average value of the payoff. This technique is usually called Monte-Carlo pricing. If the payoffs have a finite variance (which is the case for all classical exotic options in the FMLS model), the speed of convergence of this method is $\hat{\sigma}_N/\sqrt{N}$ where N is the number of simulations and $\hat{\sigma}_N$ is the empirical standard error of the N generated random payoffs. The central limit theorem provides confidence intervals for the average value of the proxy.

The critical issue in the FMLS model is that the underlying Lévy process has infinite activity: it has an infinite number of very small jumps within any time interval. For stable processes, the best simulation technique to date was developed by Asmussen and Rosinski in [6]. If we are aiming at simulating $X_t^{\alpha,\mu} = X_t^{(\alpha)} + \mu t$, then we should consider the following jump-diffusion process

$$X_t^\epsilon = \mu_\epsilon t + v_\epsilon B_t + \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| > \epsilon}, \quad t > 0, \epsilon \in (0, 1), \quad (5.22)$$

where B_t is a standard Brownian motion. In this representation, the small jumps have been omitted and the values μ_ϵ and v_ϵ account for the mean and standard error of this truncation. In our case, because the jumps are fully compensated $\mu_\epsilon = \mu$, and

$$v_\epsilon = \left(\int_{-\epsilon}^0 x^2 \nu(dx) \right)^{1/2} = \left(\int_{-\epsilon}^0 \frac{x^2}{\Gamma(-\alpha)(-x)^{1+\alpha}} dx \right)^{1/2} = \frac{\epsilon^{1-\alpha/2}}{\sqrt{\Gamma(-\alpha)(2-\alpha)}}.$$

Rosinski and Asmussen have shown in the general case that whenever

$$\lim_{\epsilon \downarrow 0} \frac{v_\epsilon}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{\left(\int_{-\epsilon}^\epsilon x^2 \nu(dx) \right)^{1/2}}{\epsilon} = \infty,$$

then

$$v_\epsilon \left(\sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \leq \epsilon} \right)_{t \in [0,1]} \xrightarrow{D} (W_t)_{t \in [0,1]},$$

where \xrightarrow{D} is the convergence in probability in the space of càdlàg functions on $[0, 1]$ - see [6], section 2 for more details. Note that the latter condition is verified in the case of stable processes. Hence the truncated jumps can indeed be replaced by a Brownian component when ϵ is sufficiently small.

The process defined in (5.22) is decomposed into a drift, a Brownian component and a compound Poisson process. We measure the maximum of the process at

the simulation points of the (drifted) Brownian motion and after the jumps of the compound Poisson process.

In his PhD thesis, Dia, [33], proved the following error bounds, which we have adapted to our setting. The prices LC , LP and DOP are given by (5.9) and (5.10) while LC^ϵ , LP^ϵ , DOP^ϵ are their approximated counterparts, using the process defined by (5.22).

Proposition 5.3. *For $\epsilon \in (0, 1/2)$ and $T = 1$, the pricing errors satisfy the following bounds*

$$\begin{aligned} |LP - LP^\epsilon| &\leq S_0 C \max \left(\epsilon^{2-\alpha}, \epsilon^{1-\alpha/3} \sqrt{\log(\epsilon^{-\alpha/3})} \right), \\ |LC - LC^\epsilon| &\leq S_0 C \max \left(\epsilon^{2-\alpha}, \epsilon^{1-3\alpha/8} \sqrt{\log(\epsilon^{-\alpha/4})} \right), \\ |DOP - DOP^\epsilon| &\leq S_0 C \epsilon^{1/2-\alpha/6} \sqrt{\log(\epsilon^{-\alpha/6})}, \end{aligned}$$

where C is a generic constant. The last rate, remains valid for any barrier put option.

We added S_0 in the formulae to recall that the error increases linearly with this variable.

Proof. In the FMLS model, the functions defined by Dia [33] are equal to $\sigma_0(\epsilon) = \epsilon^{1-\alpha/2}$, $\beta(\epsilon) = C\epsilon^\alpha$ where C is a generic constant which does not depend on ϵ . The error on $\mathbb{E}[S_1]$ is given by Proposition 4.18 (and the remark subsequent to Proposition 4.4) and it is equal to $C\epsilon^{2-\alpha}$. It must be compared with the error on $\mathbb{E}[I_1]$ which is smaller than $C\epsilon^{1-\alpha/3} \sqrt{\log(\epsilon^{-\alpha/3})}$ (Theorem 4.22 with $f(x) = e^{-x}$). Lastly, the error on $\mathbb{E}[M_1]$ is bounded by $C\epsilon^{1-3\alpha/8} \sqrt{\log(\epsilon^{-\alpha/4})}$, see Proposition 4.28 (with $p = 2$ and $\theta = 1/2$).

The result for barrier options stems from Proposition 5.50 in [33], with $\rho = \theta = 1/2$ and $\tilde{\beta}$ defined in Proposition 4.29. \square

These error bounds are quite problematic when α is not close to 1. For instance, if $\alpha = 1.5$, and we want to price a Lookback Put, then in order to obtain a 10^{-2} precision for $S_0 = C = 1$, we must choose $\epsilon = 2.10^{-5}$, which is very small. This corresponds to a jump intensity of 3.10^6 , which means that we have to simulate, on average three million jumps per unit interval.

Thanks to the exact results (5.11) and (5.12), it is possible to test these bounds. We have performed Monte-Carlo simulations for various values of ϵ and α , with $\sigma = 0.5$ and $S_0 = 1$. In order to be precise, we chose $N = 10^6$ so that $\hat{\sigma}_N/\sqrt{N} \leq 10^{-3}$. The absolute errors on the prices are provided in the table below. The NA="Not Available" cells were not computed, since, for $\alpha = 1.5$ and $\epsilon = 10^{-4}$, the simulations lasted twenty hours. The case $\alpha = 1.9$ and $\epsilon = 10^{-4}$ would have required several days.

There are two main conclusions to be drawn from this table. First, the error bounds in Proposition 5.3 are not optimal. The convergence is in fact faster. The

	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	exact value
Lookback Call ($\alpha = 1.1$)	<0.001	-0.002	-0.002	-0.002	0.419
($\alpha = 1.5$)	+0.001	-0.007	<0.001	+0.02	0.483
($\alpha = 1.9$)	+0.008	-0.002	+0.002	NA	0.515
Lookback Put ($\alpha = 1.1$)	+0.002	<0.001	+0.001	+0.001	0.066
($\alpha = 1.5$)	+0.012	+0.010	+0.001	+0.002	0.296
($\alpha = 1.9$)	+0.033	+0.023	+0.011	NA	0.481

Table 5.1: Absolute errors on lookback prices for various values of ϵ and α ($\Delta t=0.005$, $\sigma = 0.5$)

second conclusion is more technical. In order to be able to simulate the sample paths of the Brownian motion, we chose a time discretization of $\Delta t=0.005$ for the piecewise constant Euler scheme, that is to say, for $i \in [1, (\Delta t)^{-1}]$ and $t_i = i\Delta t$,

$$B_t \approx \tilde{B}_{t_i} = \sqrt{\Delta t} \sum_{k=1}^i N_k, \quad \forall t \in [t_i, t_{i+1}),$$

where the N_k are independent normal laws. $\Delta t = 0.005$ represents 200 points per unit interval. In comparison, the case $\alpha = 1.1$ and $\epsilon = 0.1$ implies on average 1.4 jumps per unit interval while the case $\alpha = 1.5$ and $\epsilon = 0.0001$ requires on average more than 280,000 jumps per unit time. Hence, the choice of the time discretization Δt introduces a bias and should be made in accordance with ϵ . This is what may explain why, as ϵ decreases, the prices of the Lookback Call do not gain accuracy, except when $\alpha = 1.9$.

It seems appealing to think that taking $v_\epsilon = 0$ would probably have given better results in some cases. We have thus run the same computations, but without the Brownian part and the outcome is summarized in the table below.

	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
Lookback Call ($\alpha = 1.1$)	-0.006	-0.017	-0.004	-0.001
($\alpha = 1.5$)	+0.028	+0.003	+0.005	+0.015
($\alpha = 1.9$)	+0.287	+0.216	+0.173	NA
Lookback Put ($\alpha = 1.1$)	+0.007	+0.006	+0.001	<0.001
($\alpha = 1.5$)	+0.075	+0.029	+0.005	+0.003
($\alpha = 1.9$)	+0.321	+0.238	+0.179	NA

Table 5.2: Absolute error on lookback prices for various values of ϵ and α ($\sigma = 0.5$, no Brownian component)

For $\alpha = 1.1$, the errors are comparable to the simulations embedding a Brownian component. However, in the other cases, the error is quite sizable, especially for $\alpha = 1.9$. This can be explained by the fact that $v_\epsilon = 0.015$ is negligible when $\epsilon = 0.001$ and $\alpha = 1.1$, while $v_\epsilon = 0.95$ when $\alpha = 1.9$. Consequently, it seems reasonable to keep the Brownian part whenever $\alpha \geq 3/2$ and $\epsilon \in (10^{-4}, 10^{-1})$.

5.4.2 Classical Monte-Carlo, second method

The second method of Monte Carlo simulation relies on the closeness of stable distribution under convolution. Indeed, if X has an α -stable distribution and X_1, \dots, X_n are n independent copies of it, then

$$\frac{X_1 + \dots + X_n}{n^{1/\alpha}} \stackrel{d}{=} X + d_n, \quad (5.23)$$

for some real number d_n .

If we consider the process $X^{(\alpha)}$, then $\forall t \geq 0$, $\mathbb{E}[X_t^{(\alpha)}] = 0$ and $d_n = 0$ (the random variable is in fact strictly stable, see [94], Definition 13.1 and Theorem 14.7 (vi)) Because of the independence of its increments, is therefore possible to simulate $X^{\alpha, \mu}$ using a discrete uniform skeleton $t_i = i\Delta t = i/n$ for $i = 0, \dots, n$ with $t_0 = 0$ and $t_n = T$. The approximation $\tilde{X}^{\alpha, \mu}$ is a piecewise constant process such that $\tilde{X}_0^{\alpha, \mu} = Y_0 = 0$ and

$$\begin{cases} Y_i &= Y_{i-1} + Z_i^\alpha / n^{1/\alpha}, \quad i \in [1, n] \\ \tilde{X}_t^{\alpha, \mu} &= Y_i + i\mu T/n, \quad \forall t \in [t_i, t_{i+1}), i \in [0, n] \end{cases}, \quad (5.24)$$

where the Z_i are independent and have a completely asymmetric α -distribution (that is, they have the same law as X_1^α).

We use the method of Chambers, Mallows and Stuck [25] to simulate the stable random variables, that is

$$Z^\alpha = \frac{\sin\left(\alpha\left(U + \frac{\pi(2-\alpha)}{2\alpha}\right)\right)}{\cos(U)^{1/\alpha}} \left(\frac{\cos\left(U - \alpha\left(U + \frac{\pi(2-\alpha)}{2\alpha}\right)\right)}{E} \right)^{(1-\alpha)/\alpha}, \quad (5.25)$$

is α -asymmetrically distributed if U is a uniform random variable on $[-\pi/2, \pi/2]$ and E is an independent 1-exponentially distributed variable.

This simulation method has two advantages over the previous one: first, there is no interference between the simulation of the pure-jump part and that of the Brownian part; second, the number of simulation points is deterministic.

We provide below some results on the valuation of two barrier options. The first one is an Up and In Put with $S_0 = 40$, $B = 45$, $K = 50$ and $T = 1$, while the second one is a Down and Out Call with $S_0 = 50$ and $K = 40$. In order to compare our results to those of subsection 5.3.3, we consider the case $r = d$. The parameters are $\sigma = 1/2$, $\alpha = 3/2$, and we set $r = d = 0$. We first put the stress on the effect of n on the convergence of the price of the barrier from a discretely monitored process to a quasi-continuously observed process. The number of simulations is $N=200,000$ in all of the cases.

Up and In Put ($\alpha = 1.5$)	$n = 50$	$n = 150$	$n = 500$	$n = 2000$
Average	8.389	8.851	9.145	9.298
Std Error ($\hat{\sigma}_N$)	12.795	13.047	13.223	13.154
CPU time (sec.)	5	14	44	174
Down and Out Call ($\alpha = 1.5$)	$n = 50$	$n = 150$	$n = 500$	$n = 2000$
Average	10.909	10.535	10.345	9.986
Std Error ($\hat{\sigma}_N$)	21.953	21.799	21.864	21.453
CPU time (sec.)	5	14	44	174

Table 5.3: MC results on barrier option prices for various values of n

Let us focus on the UIP. Because the simulation is stepwise constant, the actual behavior of $X^{\alpha,\mu}$ inside the interval (t_i, t_{i+1}) is unknown and the supremum of the process inside this time interval is very likely to be greater than $X_{t_i}^{\alpha,\mu}$ or $X_{t_{i+1}}^{\alpha,\mu}$. This is why, once $i = n$ and the discretization is over, both S_T and $-I_T$ are underestimated by such a procedure. As n increases, the range of this underestimation decreases and the event $\{\tilde{S}_T > B\}$ (embedded in the payoff) becomes slightly more likely. This explains why the price of the UIP increases as n increases. The opposite effect is of course logical in the case of the DOC.

We wish to underline that formula (5.18) combined to Theorem 5.2 yields a value of 9.38 for the UIP (with $\alpha = \sqrt{2} + 0.086 \approx 3/2$). This is 0.08 above our mean value for $n = 2000$ (we recall that $\hat{\sigma}_N/\sqrt{N} \approx 0.03$), hence the convergence to a continuous observation is quite slow.

For $\alpha = \sqrt{15/4} \approx 1.93$, the approximative value for the UIP is 11.286 (with the same parameters and $n = 2000$), for a theoretical value of 11.892. For $\alpha = \sqrt{5/4} \approx 1.118$, the values are 3.304 versus 3.283. The convergence (as n increases) is thus faster as $\alpha \downarrow 1$.

In the Lookback put case, a theoretical result is available to characterize this error induced by the discrete scheme. If we define, for $t > 0$ and $n > 1$,

$$M_t^{X^{(\alpha)},n} = \sup_{k \in \{1, \dots, n\}} X_{kt/n}^{(\alpha)},$$

then we have the following proposition.

Proposition 5.4. *For any $\epsilon > 0$, as $n \rightarrow \infty$,*

$$\mathbb{E} \left[e^{M_t^{X^{(\alpha)}}} \right] = \mathbb{E} \left[e^{M_t^{X^{(\alpha)},n}} \right] + O(n^{-1/\alpha+\epsilon}).$$

Proof. The proof is based on the following result which can be found in the proof of Proposition 3.9 in [33]:

$$\mathbb{E} \left[M_t^{X^{(\alpha)}} - M_t^{X^{(\alpha)},n} \right] = -\frac{t^{1/\alpha} \zeta(1 - 1/\alpha) \mathbb{E}[(X_1)^+]}{n^{1/\alpha}} + o(n^{-1/\alpha}),$$

where ζ is the usual Riemann Zeta function.

The remainder of the proof is simply an application of Lemma 6.3 in [34] with a large enough β . \square

We provide empirical evidence of this result. We have computed the absolute errors $\mathbb{E} \left[e^{M_t^{X^{(\alpha)}}} \right] - \mathbb{E} \left[e^{M_t^{X^{(\alpha)},n}} \right]$ in the following table for $\sigma = 1$ and $N = 2.10^6$ so that the error related to the Monte-Carlo simulations is close to 10^{-3} :

n	$\alpha = 1.2$	$\alpha = 1.8$
100	0.024	0.259
200	0.016	0.176
500	0.008	0.105
1000	0.005	0.078
2000	0.002	0.060

Table 5.4: Absolute errors induced by the discretization scheme

A similar result holds for the Lookback call. However, there are no available results for barrier options. When the underlying is a drifted Brownian motion, the continuity correction was obtained in [20] for the drifted Brownian motion and it was recently found for general jump-diffusion processes in [35]. Note that with the Euler scheme in the FMLS model, we are able to exactly simulate the increments of $X^{(\alpha)}$ and hence there is no error due to the discretization on the term $\mathbb{E}[e^{\sigma X_T^{(\alpha)}}]$. This is not often the case for general Lévy processes.

Lastly, for the sake of completeness, we recall that the weak error rate for vanilla options (i.e. which depend only on the terminal value S_T) is given in [91]. The results in this article in fact encompass stochastic differential equations driven by Lévy processes, which generalize simple Lévy processes. They aim at providing an upper bound for

$$|\mathbb{E}[g(X_T)] - \mathbb{E}[g(X_T^n)]|,$$

where g is some function such that these integrals make sense and X^n is a discrete version of X obtained through a Euler scheme.

5.4.3 Wiener-Hopf Monte-Carlo

A recent method, described in [65], also makes it possible to avoid the technical arbitrage between Δt and ϵ of the first method. The idea is to concatenate small samples of independent trajectories of $X^{\alpha,\mu}$ stopped at an independent exponential time and to use the Wiener-Hopf factorization (see Chapter 3 for details). More precisely, if we consider I_{e_q} and S_{e_q} the running infimum and supremum of $X^{\alpha,\mu}$ stopped at time e_q , an independent q -exponentially distributed time, then $X_{e_q}^{\alpha,\mu} \stackrel{d}{=} I_{e_q}^{X^{\alpha,\mu}} + M_{e_q}^{X^{\alpha,\mu}}$. If we define $t(q, n) = \sum_{k=1}^n e_q^{(i)}$ where the $e_q^{(i)}$ are independent copies

of e_q , then it is formally proved in [65] that

$$\left(X_{t(q,n)}^{\alpha,\mu}, S_{t(q,n)}^{\alpha,\mu} \right) \stackrel{d}{=} (V(q,n), J(q,n)), \quad (5.26)$$

where V and J are defined iteratively for $n \geq 1$ by

$$\begin{aligned} V(q,n) &= V(q,n-1) + I_{e_q}^{(n)} + M_{e_q}^{(n)}, \\ J(q,n) &= \max \left(J(q,n-1), V(q,n-1) + M_{e_q}^{(n)} \right), \end{aligned}$$

where $V(q,0) = J(q,0) = 0$ and $M_{e_q}^{(n)}$ (resp. $I_{e_q}^{(n)}$) are independent copies of $M_{e_q}^{X^{\alpha,\mu}}$ (resp. $I_{e_q}^{X^{\alpha,\mu}}$). This construction is in fact fairly natural as it corresponds to a simple concatenation of independent trajectories of X .

Invoking the law of large numbers, we then have, for k large enough

$$\mathbb{E} [F(X_t^{\alpha,\mu}, M_t^{X^{\alpha,\mu}})] \approx \frac{1}{k} \sum_{m=1}^k F(V^{(m)}(q,n), J^{(m)}(q,n)),$$

where $V^{(m)}(q,n)$ and $J^{(m)}(q,n)$ are independent copies of $V(q,n)$ and $J(q,n)$ under the obvious condition $\mathbb{E}[t(q,n)] = n/q = t$.

Theoretically, this technique seems very appealing. In practice, however, things are more complicated. This algorithm is only efficient when it is possible to simulate $I_{e_q}^{(n)}$ and $M_{e_q}^{(n)}$ very quickly, which is not the case for stable processes. $M_{e_q}^{(n)}$ is of course not the problem, since, by Corollary 2, Chapter VII in [11], it is an exponential variable with a parameter which is easy to compute. There is, nonetheless, no simple way to simulate $I_{e_q}^{(n)}$. One can proceed

- either with the acceptance-rejection method; but it requires a companion distribution with a density that behaves like that of $I_{e_q}^{(n)}$. This is problematic since not only does $f_{I_t^{X^{\alpha,\mu}}}$ go to infinity when $x \downarrow 0$ - this could have been handled with a gamma distribution -, but the tail of $I_t^{X^{\alpha,\mu}}$ is also of polynomial type (see (5.16) in the Appendix and (2.62 in [10]). There is, to our knowledge, no easy-to-simulate random variable with such characteristics.
- or with the c.d.f. inversion technique, once e_q has been drawn. In this case, a truncation of the series is required and even with an enhanced Newton-Raphson algorithm, this method is quite lengthy (at least one hundred loops to compute $P[I_t^{X^{\alpha,\mu}} \leq x]$ for a single value $x \dots$).
- in both cases, only the density and c.d.f. of $I_t^{X^{\alpha,\mu}}$ is known and only in the driftless case ($\mu = 0$). The random variable e_q must be drawn first. The major issue is that since we can only work with truncated series, the computation of (5.16) may explode if just one sample value of $t \stackrel{d}{=} e_q$ is too small.

Even though the Wiener-Hopf Monte Carlo method can prove to be very efficient when the stopped supremum and infimum are easily simulated, it seems that it is

in fact less tractable than classical Monte-Carlo techniques in the case of stable processes.

5.4.4 A word on Quasi Monte-Carlo

In order to increase the speed of convergence of the simulation methods described above, a popular solution is Quasi-Monte Carlo (QMC) and the use of sequences with low discrepancy. We refer to chapter 5 in [44] for technical details.

The pricing of exotic options using QMC was investigated in [71] (section 5) and [50]. Before we discuss our numerical results, we wish to comment on these references. Once the parameters of the process are fixed, two choices remain: the number of simulations N and the number of points in the simulation grid (n in (5.24), τ in [71] and d in [50]). The maximum value for N is 80,000 in [71] and around 120,000 in [50]. An intuitive property, which is observed in both articles (see tables 2 in [71] and 5.3 in [50]), is that the competitiveness of QMC methods decrease as n increases.

In practice, exotic options are discretely monitored. The monitoring can be monthly, weekly, daily, etc. Therefore, the less frequent the monitoring is, the more relevant the QMC methods become. We underline that QMC methods require *a priori* the knowledge of the number of random variables to be simulated, this is why it is not suited to techniques relying on jump diffusions or on the rejection method.

Using (5.24) and (5.25), we have computed the price of an Up and In Put with $S_0 = 40$, $B = 45$ and $K = 50$ and of a Down and Out Call with $S_0 = 50$, $B = 45$ and $K = 40$. In both cases, we fixed $\alpha = 3/2$ and $\sigma = 1/2$ in order to compare with the results of the classical Monte-Carlo procedure in Table 5.3. The pseudo random numbers were generated by a $2n$ dimension Sobol sequence: the first n numbers being used for the uniform variable and the last n numbers for the exponential variable.

We compare MC and QMC methods in the graphs below (figures 5.4.1 and 5.4.2).

For vanilla payoffs, it is well known (see [44], chapter 5) that the convergence of QMC methods is $O(\log(N)^n/N)$ while it is $O(N^{-1/2})$ for MC methods. Figures 5 and 6 in [71] illustrate this feature. However, for path-dependent payoffs, the competitiveness of QMC versus MC is (much) less obvious. As a rule of thumb, it seems that the prices are close to stable for $N > 50,000$ in the Sobol case, and when $N > 100,000$ for the classical Monte-Carlo method. QMC thus appears slightly more effective than MC, but requires a few extra seconds of computation.

5.4.5 PIDE methods

Another family of methods for computing barrier option prices in exponential Lévy models consists in solving Partial Integro-Differential Equations (PIDE). A short review of these techniques is given in the introduction of [59]. When the small

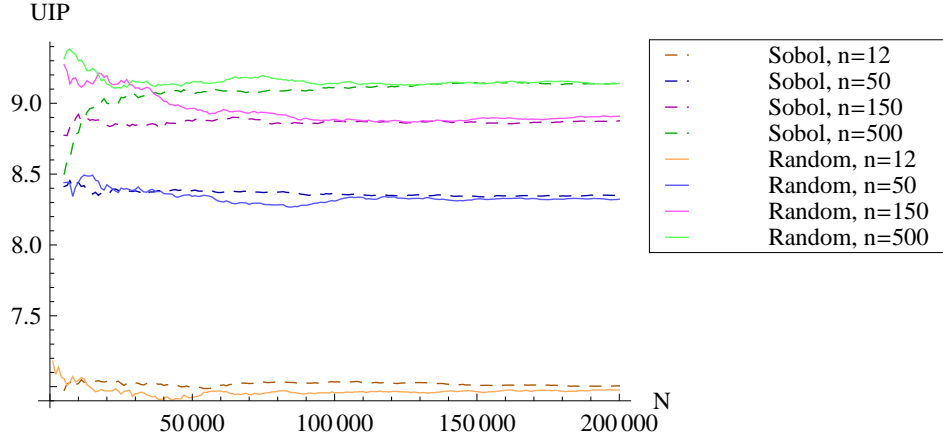


Figure 5.4.1: Graph of the UIP price for $n \in \{12, 50, 150, 500\}$ and $N \in (5\,000, 200\,000)$

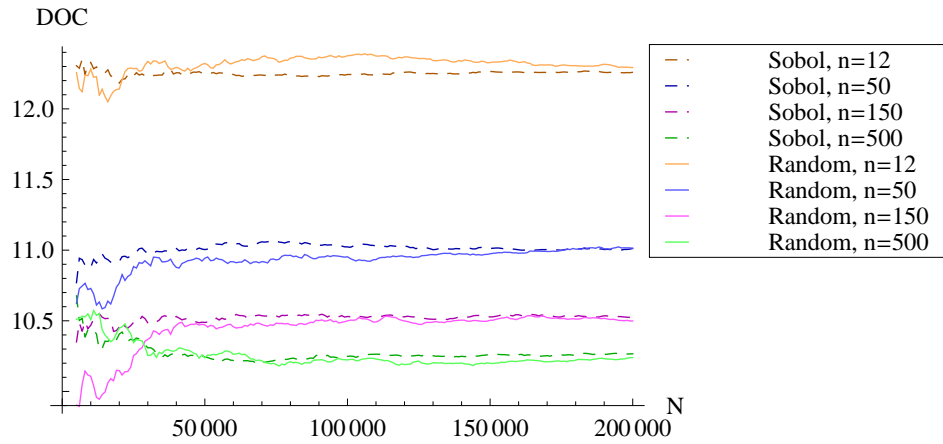


Figure 5.4.2: Graph of the DOC price for $n \in \{12, 50, 150, 500\}$ and $N \in (5\,000, 200\,000)$

jumps are replaced by the Brownian component, Kudryavstev and Levendorskii show that the error can be quite sizable. In the case of infinite activity, they strongly recommend not to truncate the small jumps but rather to resort to the Wiener-Hopf factorization of the underlying Lévy process.

However, these methods do not apply for stable processes, for two reasons:

- the Wiener-Hopf factorization for stable processes is very complicated in the general case (see [62] for instance) and only available in the driftless case. In the spectrally negative case, the Wiener-Hopf factors are given by Theorem 4, section VII in [11], but the inverse Lévy-Khintchine exponent Φ is only known in a completely closed-form for a very limited number of α .
- stable processes do not belong to the class of "regular Lévy processes of ex-

ponential type" (see section 2.3 in [59]) and are therefore not suited to these techniques.

5.5 Numerical computation of greeks and vanilla prices

We end this chapter with numerical recipes for the computation of the basic tools of a market operator: the vanilla prices and the greeks.

In [24], Carr and Wu use the method described in [23] to compute vanilla option prices. It is however possible to make do without the Fast Fourier transform and simply to use the series representation of the densities instead. Of course, in practice, one needs to truncate the series and hence, the formulae are only valid for an interval around zero. Without any loss of generality (by (14.28) in [94], or simply by self-similarity), we set $T = 1$. Since the discounting factor and the dividend rate are not an issue in our model, we further assume $r = d = 0$. If we define, according to (14.34), (14.30) and (14.35) from [94],

$$f(x) = \begin{cases} \sum_{n=1}^{n^*} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{\pi n!} \frac{\sin(\pi n(\alpha - 1))}{(-x)^{\alpha n + 1}} & \text{if } x \leq x_* < -1 \\ \sum_{n=1}^{N^*} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{\pi n!} \sin(\pi n/\alpha) x^{n-1} & \text{if } x \in (x_*, x^*) \\ C_1 \left(\frac{x}{\alpha}\right)^{\frac{2-\alpha}{2(\alpha-1)}} e^{-(\alpha-1)(\frac{x}{\alpha})^{\alpha/(\alpha-1)}} \left[1 + C_2 \left(\frac{\alpha}{x}\right)^{\alpha/(\alpha-1)}\right] & \text{if } x \geq x^* > 1 \end{cases}, \quad (5.27)$$

where

$$C_1 = \sqrt{\frac{\alpha - 1}{2\pi\alpha}}, \quad C_2 = \frac{2\alpha + 1 + 2/\alpha}{12(\alpha - 1)},$$

then the error on the target stable density is $O((-x_*)^{-\alpha(n^*+1)-1})$ at point $x = x_*$ (and any point to the left of x_*) while it is $O\left((x^*)^{\frac{6-5\alpha}{2(\alpha-1)}} e^{-(\alpha-1)(x^*/\alpha)^{\alpha/(\alpha-1)}}\right)$ at point $x = x^*$ (and any point to the right of x^*). For instance, if $N^* = 400$ and $n^* = 8$, then appropriate choices for the couple (x_*, x^*) are $(-7.5, 6.5)$ if $\alpha = 1.8$, $(-5, 4)$ if $\alpha = 1.5$ and $(-2.5, 2.5)$ if $\alpha = 1.2$. These values ensure that the discontinuity of f at points x_* and x^* is negligible. If $T \neq 1$, then one can use (14.28) in [94] and the couples (x_*, x^*) are completely different.

The price of vanilla options readily follows. For a Call issued at time $t = 0$, for instance,

$$\begin{aligned} \mathbb{E}[(S_1 - K)^+] &= \int_{\mathbb{R}} (S_0 e^{\sigma x - \sigma^\alpha} - K)^+ f(x) dx \\ &= S_0 \int_{(\log(K/S_0) + \sigma^\alpha)/\sigma}^{\infty} e^{\sigma x - \sigma^\alpha} f(x) dx - K \int_{(\log(K/S_0) + \sigma^\alpha)/\sigma}^{\infty} f(x) dx, \end{aligned}$$

which can be computed very accurately using (5.27). The delta can be derived upon differentiating this formula:

$$\frac{\partial}{\partial S_0} \mathbb{E}[(S_1 - K)^+] = \int_{(\log(K/S_0) + \sigma^\alpha)/\sigma}^{\infty} e^{\sigma x - \sigma^\alpha} f(x) dx. \quad (5.28)$$

The gamma follows immediately

$$\frac{\partial^2}{\partial S_0^2} \mathbb{E}[(S_1 - K)^+] = \frac{K}{\sigma S_0^2} f\left(\frac{\log(K/S_0) + \sigma^\alpha}{\sigma}\right) = \frac{\partial^2}{\partial S_0^2} \mathbb{E}[(K - S_1)^+]. \quad (5.29)$$

The vega is usually the sensitivity with respect to the volatility. In our case, σ is not a volatility proxy per se, but it is a scaling factor for the stock returns, thus it makes sense to provide the sensitivity with respect to this parameter. We use the classical rule for interchanging differentiation and expectation (see chapter 8 in [19] for instance):

$$\frac{\partial}{\partial \sigma} \mathbb{E}[(S_1 - K)^+] = S_0 \int_{(\log(K/S_0) + \sigma^\alpha)/\sigma}^{\infty} e^{\sigma x - \sigma^\alpha} (x - \alpha \sigma^{\alpha-1}) f(x) dx.$$

We provide graphs for the delta, gamma and vega of a vanilla Call option, as a function of S_0 for various values of α , namely: $\alpha = 1.4$, $\alpha = 1.7$ and $\alpha = 2$. The integrals were computed using Mathematica. Under the risk-neutral measure, the underlying has the following representation

$$\begin{cases} S_t = e^{\sigma X_t^{(\alpha)} - \sigma^\alpha t} & \text{if } \alpha \in (1, 2) \\ S_t = e^{\sqrt{2}\sigma B_t - \sigma^2 t} & \text{if } \alpha = 2 \end{cases}.$$

The parameters are set to $\sigma = 1/2$, $K = 50$, $T = 1$ and $r = 0$ (the processes are martingales). We recall that in the BS-M model ($\alpha = 2$), the sensitivities (derived from (5.4)) are

$$\frac{\partial}{\partial S_0} \mathbb{E}[(S_T - K)^+] = N\left(\frac{\log(S_0/K) + (r + \sigma^2)T}{\sigma\sqrt{2T}}\right) = 1 + \frac{\partial}{\partial S_0} \mathbb{E}[(K - S_T)^+],$$

$$\frac{\partial^2}{\partial S_0^2} \mathbb{E}[(S_T - K)^+] = \frac{\partial^2}{\partial S_0^2} \mathbb{E}[(K - S_T)^+] = \frac{\exp\left(-\frac{1}{2}\left(\frac{\log(S_0/K) + (r + \sigma^2)T}{\sigma\sqrt{2T}}\right)^2\right)}{\sigma S_0 \sqrt{4\pi T}},$$

$$\frac{\partial}{\partial \sigma} \mathbb{E}[(S_T - K)^+] = \frac{\partial}{\partial \sigma} \mathbb{E}[(K - S_T)^+] = S_0 \sqrt{T/2\pi} \exp\left(-\frac{1}{2}\left(\frac{\log(S_0/K) + (r + \sigma^2)T}{\sigma\sqrt{2T}}\right)^2\right),$$

where N is the cumulative distribution function of the Gaussian distribution.

The outcome is hard to comment for the vega. For the gamma, a maximum is reached close to the value K/α and this maximum increases as α decreases. When

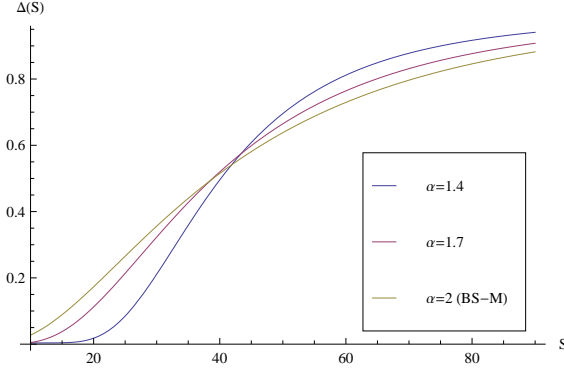


Figure 5.5.1: Graph of the Delta of a vanilla Call option

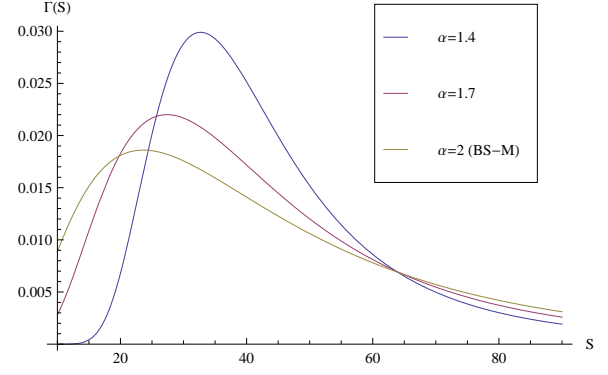


Figure 5.5.2: Graph of the Gamma of a vanilla Call (or Put) option

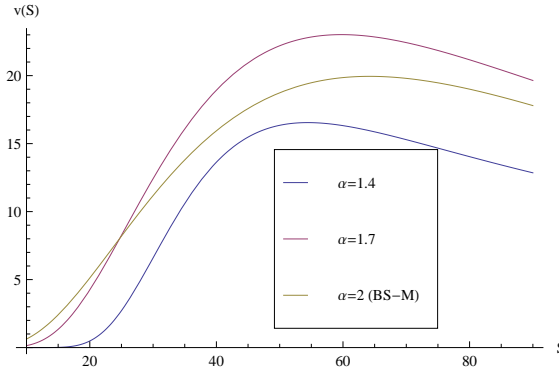


Figure 5.5.3: Graph of the Vega of a vanilla Call option

the underlying is within the interval $(20, 60)$, the delta hedging requires frequent adjustments when α is small. For $S_0 \geq 60$, the index α has little impact on both the delta and the gamma.

Lastly, we wish to underline that another reference for this topic is [56], where the setting is slightly more general, but the results less explicit.

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