Portfolio backtesting in practice

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Agenda

1. Returns: theory and facts
2. Portfolio backtesting
3. Portfolio performance metrics
4. Wrap-up

Returns

Two ways to proceed

Arithmetic returns

$r_{t-\Delta t,t} = \frac{P_t}{P_{t-\Delta t}} - 1$

$P_t = P_0 \prod_{s=1}^{t} (1 + r_{s-1,s})$

Good for cross-section aggregation

Often used in discrete time

ex: portfolio allocation

Logarithmic returns

$r_{t-\Delta t,t} = \log\left(\frac{P_t}{P_{t-\Delta t}}\right)$

$P_t = P_0 \exp\left(\sum_{s=1}^{t} r_{s-1,s}\right)$

Good for time aggregation

Often used in continuous time

ex: option pricing

When Δt is sufficiently small so that returns are small too, both definitions are close because $\log(1+x) = x + o(x)$.

Example of difference: aggregating AR in time: $(1+r)(1-r) \neq 1$!

Note: despite their practical importance, we do not include dividends in this lecture.

CS aggregation with arithm. returns

A simple proof

Consider 2 assets, A and B and a portfolio consisting of n^A and n^B units of each. The corresponding weights at time t are $w^A = \frac{n^A P_t^A}{n^A P_t^A + n^B P_t^B}$ and $w^B = \frac{n^B P_t^B}{n^A P_t^A + n^B P_t^B}$.
The total wealth is equal to $W_t = n^A P_t^A + n^B P_t^B$.
Its return is

$$\begin{aligned}
\frac{W_{t+1}}{W_t} - 1 &= \frac{n^A P_{t+1}^A + n^B P_{t+1}^B}{n^A P_t^A + n^B P_t^B} - 1 \\
&= w^A \frac{P_{t+1}^A}{P_t^A} + w^B \frac{P_{t+1}^B}{P_t^B} - 1 \\
&= w^A \frac{P_{t+1}^A}{P_t^A} + w^B \frac{P_{t+1}^B}{P_t^B} - w^A - w^B \\
&= w^A r_{t,t+1}^A + w^B r_{t,t+1}^B
\end{aligned}$$

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TS aggregation with log-returns

A simple proof

Take any three dates $s < t < u$,

$$r_{s,u} = \log\left(\frac{P_u}{P_s}\right) = \log\left(\frac{P_u}{P_t}\frac{P_t}{P_s}\right) = \log\left(\frac{P_u}{P_t}\right) + \log\left(\frac{P_t}{P_s}\right)$$
$$= r_{s,t} + r_{t,u}$$

This is why, in continuous time, prices are often exponentials of processes with independent increments (e.g., Brownian motion, Lévy processes)

TS aggregation: link with CLT

Squeezing the tails

If log-returns are iid, then for a uniform sequence $0 = t_1 < t_2 < \cdots < t_n = t$, $r_t = \sum_{i=1}^n r_{t_i}$ and

$$\frac{r_t - n\bar{r}}{\sigma_r \sqrt{n}} \to N(0,1),$$

so that when n is large,

$$r_t \overset{d}{\approx} \sigma_r \sqrt{n} \times N(0,1) + n\bar{r},$$

i.e., Gaussian models become **good approximations**.

This is why long term returns (e.g., annual returns) are *close* to normally distributed.

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Returns in theory

Very often, iid Gaussian

IN THEORY, non-overlapping returns are often assumed independent and their distribution depends only on Δt, μ, σ.

$$r_{t-\Delta t,t} \overset{d}{=} \mathcal{N}\left(\mu\Delta t, \sigma^2\Delta t\right),$$

which (e.g.) matches the **Black-Scholes** model when $\mu = r - \frac{\sigma^2}{2}$.

$\rightarrow$ very tractable in computations!

$\rightarrow$ can be extended to Lévy processes to account for extreme events.

BUT!

Stylized Facts

Returns in **practice**! See Cont 2001[1]

Empirically, the following properties have been identified

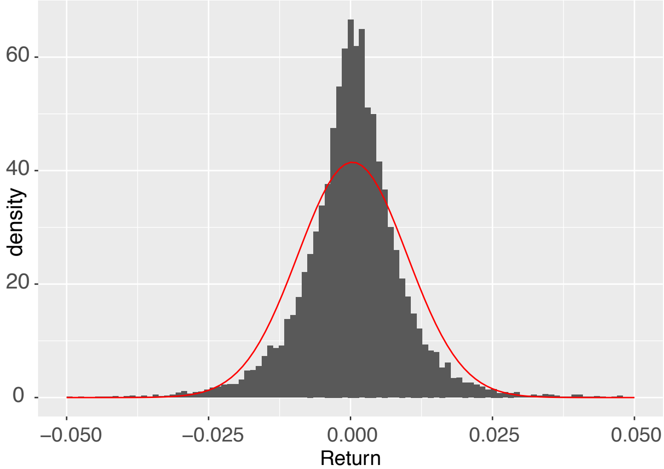
- **distribution**: returns are heavy-tailed when Δt is small but converge to Gaussian laws when the horizon increases
- **lack of auto-correlation**: at daily frequencies (and above), returns are only mildly serially correlated
- **stochastic volatility**: returns exhibit clusters during which their dispersion increases
- **persisting absolute returns**: relatedly, the autocorrelogram of *absolute* returns is slowly decaying

$\rightarrow$ complex stochastic processes reflect these empirical properties (e.g., combining stochastic volatility to jump processes).

[1] *Empirical properties of asset returns: stylized facts and statistical issues* - **Quant. Finance**

S&P 500 returns (1/3)

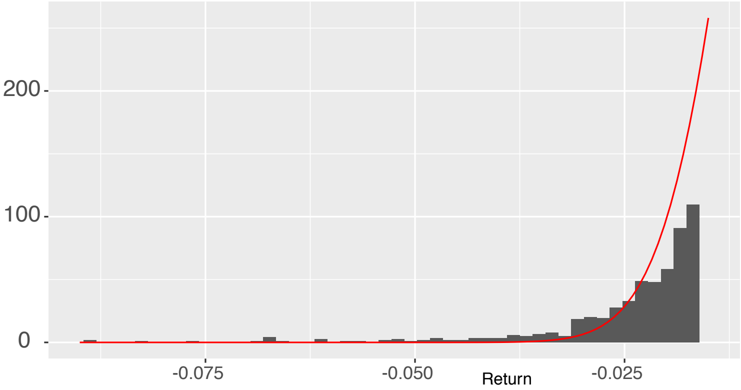
Distribution of daily returns Jan 1950 - May 2018:



The red curve is the fitted **Gaussian density**.

S&P 500 returns (2/3) - Tails!

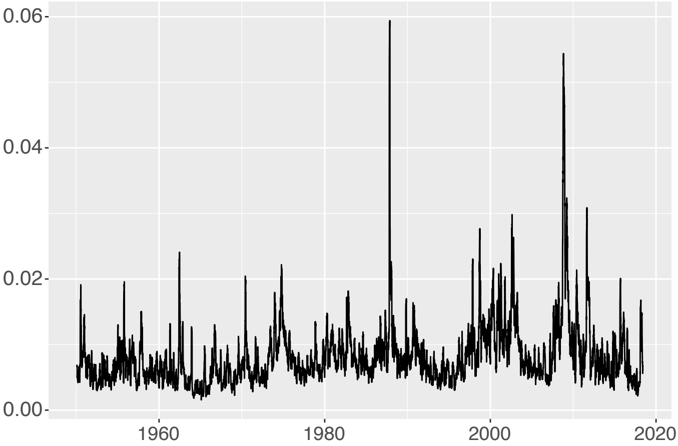
The devil is in the ‘de-tails’.



The tails of empirical returns are much heavier than those of the corresponding Gaussian distribution. **Extreme events are not that rare**!

S&P 500 returns (3/3) - Volatility

Standard deviation over 21 daily returns ($\sim$ monthly).



$\rightarrow$ Empirically, returns **do not seem stationary** over the long term.

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Protocol

User specified rules

- Investment universe
- Portfolio strategy/strategies:
  - **strategy type**: agnostic, price-based (ex: chartist), characteristics-based, etc.
  - **implementation constraints**: leverage, liquidity, factor/region/industry exposure, etc.
  - possibly, a benchmark for comparison purposes (EW-CW)
- Backtesting period (usually driven by data and investment universe) & rebalancing frequency (usually driven by policy and transaction costs)
- Performance metrics (covered below)

In practice (1/4)

Pseudo code form

```
initiate portfolio weights
initiate portfolio return

for t in (rebalancing_dates){
    1. filter data appropriately (retain PAST data)
    2. determine portfolio weights
    3. compute FUTURE portfolio returns (with present values)
}
```

Then, given portfolio weights and portfolio returns, compute performance metrics

It's as simple as that. To keep things clear: **compartmentalise**!

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In practice (2/4)

Four crucial steps

We must:

1. prepare data (session 4)
2. build ML engine (sessions 3, 5, 6, 7, 8)
3. convert signal (ML output) into investment decision
4. be attentive to perf metrics for both signal (ML) and portfolios (session 7)

The **meta-framing task** is even more important: does what you do make sense economically and technically/statistically?
It does require a bit of expertise (knowledge + experience) to figure this out.

In practice (3/4)

This stage is incredibly important.

A word on signal processing

Usually, the ML engine will yield forecasts/signals: future returns, profitability probability, confidence score, etc. This must be translated into **portfolio weights**. Several choices are possible:

1. immediate optimisation (Markowitz-like, the output can, e.g., serve as expected return vector)
2. heuristic two-stage construction:
   - start by choosing the assets to invest in (those with positive score, those with score above a threshold (median), etc.)
   - then weight: EW, CW, Inverse vol, Min vol, etc.

+ **constraints** (see next slide)

In practice (4/4)

Classical constraints

- (hard): $b_- < w_i < b^+$;
- turnover, liquidity: $(\boldsymbol{w}_t - \boldsymbol{w}_{t-1})'\boldsymbol{\Lambda}(\boldsymbol{w}_t - \boldsymbol{w}_{t-1}) < \delta$ with $\boldsymbol{\Lambda}$ diag;
- diversification, $\boldsymbol{w}_t'\boldsymbol{w}_t < \kappa$ (Herfindhal index);
- factor exposure (proxy), $v_k^- < [\boldsymbol{F}\boldsymbol{w}_t]_k < v_k^+$
- etc.

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Notations

Weights and returns

- asset (stock) returns: r_t^n, t for time, n for asset index
- portfolio weights: w_t^n, proportion invested at time t in asset n
- **portfolio returns**: $r_t^P = \sum_{n=1}^N w_{t-1}^n r_t^n$ (to state the obvious)

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Raw performance

Realised returns

We start from the uniform time grid: $t = 1, \dots, T$ with 1 being the first date of the backtest and T the last one.
We assume that ν periods make one year (e.g. $\nu = 12$ for monthly rebalancing and $\nu = 4$ for quarterly rebalancing)
The full portfolio trajectory is given by $P_t = \prod_{s=1}^{t}(1 + r_t^P)$. Initial value equals one (usually).

- Annualised arithmetic returns: $\bar{r}^P = \frac{\nu}{T}\sum_{t=1}^{T} r_t^P$
- Annualised geometric returns: $\left(\prod_{s=1}^{T}(1 + r_s^P)\right)^{\nu/T} - 1$

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Risk measures & relative perf.

A large palette of metrics

- Annualised volatility: $\sigma^P = \sqrt{\frac{\nu}{T}\sum_{t=1}^T(r_t^P - \bar{r}^P/\nu)^2}$
- VaR at q (often, $q = 0.05$): q quantile of r_t^P: $P[r_t^P \le \text{VaR}_q^P] = q$
- Maximum Drawdown: maximum decline from a historical peak:

$$MDD^P = \max_{t\in[0,T]}\left(\left(\max_{\tau\in[0,t]} P_\tau\right) - P_t\right)$$

- See also: downwards vol, expected shortfall (CVaR).

Risk adjusted performance:

- **Sharpe ratio**: $\text{SR}^P = \bar{r}^P/\sigma^P$ (originally in excess of riskfree rate version)
- **MAR ratio**: $\text{MAR}^P = \bar{r}^P/MDD^P$ (originally used with geometric returns)

Relative performance: **Information ratio**: IR=$(\bar{r}^P - \bar{r}^B)/TE(P,B)$, where P=portfolio, B=benchmark and TE is the tracking error: $TE(P,B) = \mathbb{V}[r^P - r^B]$.

Turnover

Asset rotation is costly

Turnover assesses the changes in the portfolio when rebalancing takes place. A simple proxy is the following:

$$\text{Turnover} \approx \frac{1}{T-1}\sum_{t=2}^{T}\sum_{n=1}^{N}|w_t^n - w_{t-1}^n|,$$

as it requires the portfolio weights only. The *true* turnover is

$$\text{Turnover} = \frac{1}{T-1}\sum_{t=2}^{T}\sum_{n=1}^{N}|w_t^n - w_{t-}^n|,$$

where $t-$ denotes the time **just before** rebalancing: the trajectory of the weight must be computed using the asset returns.

Cap-weighted strategies have very low turnover.

TC-adjusted performance: $SR_{TC} = \frac{\bar{r}-\delta\text{Turnover}}{\sigma}$, where δ (30-50 bps) is a TC constant. **NOTE**: TC could also be evaluated at the asset level: liquid assets have lower TC (that's preferable, but that's harder to do).

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Portfolios in practice

It's all about returns

- empirically: heavy-tailed, volatility clusters, etc.
- backtesting requires many choices, i.e., many degrees of freedom, though some are dictated by data or policy contingencies.
- Each step counts and being careful at each stage can make a difference.

Thank you for your attention

Any questions?