1. Suppose that $f(n) = \Theta(g(n))$. Assume that both functions increase without limit.
   (a) Must it be true that $\log f(n) = \Theta(\log g(n))$? Prove or disprove.
   (b) Must it be true that $2^{f(n)} = \Theta(2^{g(n)})$? Prove or disprove.

Answer:

(a)

Since $f(n) = \Theta(g(n))$, there exists constant c_1, c_2 and n_0 such that

$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n), \forall n \geq n_0$,

then $log(c_1 \cdot g(n)) \leq log(f(n)) \leq log(c_2 \cdot g(n))$,

$log(c_1) + log(g(n)) \leq log(f(n)) \leq log(c_2) + log(g(n))$,

since $log(c_1)$ and $log(c_2)$ are constants,

$\log f(n) = \Theta(\log g(n))$.

(b)

as $f(n)$ and $g(n)$ goes larger, the value of $2^{f(n)}$ and $2^{g(n)}$ has significant different exponentials.

As the exponential goes too big or too small, it goes against the defination of big Theta, where there's

$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n), \forall n \geq n_0$,

as the constant c_1 and c_2 will hardly make a difference in the exponential function.

Thus, this is not always true.

2. (a) Suppose you have a function of two variables, n and k; for instance $h(n, k)$.
If you are told that $h(n, k) = O(n + k)$, what should that mean mathematically?

(b) Let $f(n) = O(n)$ and $g(n) = O(n)$. Let c be a positive constant.
Prove or disprove that $f(n) + c \cdot g(k) = O(n + k)$.

Answer:

(a)

Having $h(n, k) = O(n + k)$ means that function $h(n, k)$ grows depends on variables n and k, and there should exists constant c such that:

$c \cdot (n + k) \geq h(n, k), \forall n \geq n_0, \forall k \geq k_0$.

(b)

$\because g(n) = O(n)$,

$\therefore g(k) = O(k)$

$\because f(n) = O(n)$,

$\therefore$ there exists constant c_1 and c_2, n_0, k_0 such that:

$c_1 \cdot n \geq f(n), c_2 \cdot k \geq g(k), \forall n \geq n_0, \forall k \geq k_0$

$\therefore c_1 \cdot n + c_2 \cdot k \geq f(n) + g(k)$

make $h(n, k) = f(n) + g(k)$,

$\therefore c_1 \cdot n + c_2 \cdot k \geq h(n, k)$.

when $c_1 \leq c_2$,

$c_2 \cdot n + c_2 \cdot k \geq h(n, k)$.

$\therefore c_2 \cdot (n + k) \geq h(n, k)$, which fits the defination we defined previously.

same logic applies when $c_1 \geq c_2$.

$\therefore f(n) + c \cdot g(k) = O(n + k)$ is always true.

3. Let $f(n) = \sum_{y=1}^{n}(n^6 \cdot y^{23})$.

   Find a simple $g(n)$ such that $f(n) = \Theta(g(n))$, by proving that $f(n) = O(g(n))$, and that $f(n) = \Omega(g(n))$.

   Don't use induction / substitution, or calculus, or any fancy formulas.
   Just exaggerate and simplify for big-O, then underestimate and simplify for Ω.

   Answer:

   $f(n) = \sum_{y=1}^{n}(n^6 \cdot y^{23}) \leq n \cdot (n^6 \cdot n^{23})$ $f(n) \leq n^{30}$

   $\therefore$ there exists constants c, n_0 such that,

   $c \cdot n^{30} \geq f(n), \forall n \geq n_0$.

   $\therefore f(n) = O(n^{30})$.

   $f(n) = \sum_{y=1}^{n}(n^6 \cdot y^{23}) \geq \sum_{y=n/2}^{n} n^6 \cdot y^{23}$

   $f(n) \geq \sum_{y=n/2}^{n} n^6 \cdot (\frac{n}{2})^{23}$

   $f(n) \geq (\frac{n}{2}) \cdot n^6 \cdot (\frac{n}{2})^{23}$

   $= (\frac{1}{2})^{24} \cdot n^6 \cdot n^{24}$

   $\therefore$ there exists constants c, n_0 such that,

   $c \cdot n^{30} \leq f(n), \forall n \geq n_0$.

   $\therefore f(n) = \Omega(n^{30})$.

   $\therefore f(n) = \Theta(n^{30})$.