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CHAPTER - 14

APPLICATION OF DIFFERENTIATION

Physical Meaning of Derivative

$\frac{dy}{dx}$ is the rate of change of Y w.r.t x

Geometrical Meaning of Derivative

$\frac{dy}{dx}$ at a point is the slope of the tangent drawn at that point

Equation of tangent at $(x_1\ y_1)$

Let $\left.\frac{dy}{dx}\right]_{(x_1 y_1)} = f'(x_1)$ = slope of tangent at $(x_1\ \ y_1)$

Then the equation of the tangent is $y - y_1 = f'(x_1)(x - x_1)$

Equation of Normal at $(x_1\ y_1)$

Normal is a line perpendicular to tangent

Slope of normal = $-\frac{1}{f'(x_1)}$, Hence the equation of Normal is $(x - x_1) + f'(x_1)(y - y_1) = 0$

Sub Tangent (ST) and Sub Normal (SN)

$ST = \frac{Y}{\left(\frac{dy}{dx}\right)}$ = Projection of Tangent on X-axis

$SN = Y\left(\frac{dy}{dx}\right)$ = Projection of Normal on X-axis

Notes

1. ST, ordinate and SN are in G.P
2. (ST) (SN) = Y^2

Length of Tangent (LT) and Length of Normal (LN)

$$LN = Y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \ ; \quad LT = \frac{LN}{\left(\frac{dy}{dx}\right)}$$

Increasing and Decreasing functions

1. Y = f(x) is increasing if
   $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
2. $f'(x) > 0 \Rightarrow$ f(x) increases
3. f(x) increases not necessarily mean $f'(x) > 0$
4. F(x) is decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
5. $f'(x) < 0 \Rightarrow$ f(x) decreases
6. f(x) decreases not necessarily mean $f'(x) < 0$

Local Maximum and Local Minimum Points

1. Local Maximum : Turning from increasing to decreasing. At a local Maximum
   $f'(x) = 0$ and $f''(x) < 0$
2. Local Minimum : Turning from decreasing to increasing . At a local minimum
   $f'(x) = 0$ and $f''(x) > 0$
3. Local maximum at x = a means f(x) - f(a) < 0 in a delected nbd of 'a'
4. Local minimum at x = a means f(x) - f(a) > 0 in a delected nbd of 'a'
5. Extreme point or turning point : All local maximum and minimum points are known as extreme points or turning points
6. f(x) has absolute maximum at x = a if $f(a) > f(x) \ \forall x$
7. f(x) has an absolute minimum at x = b if $f(b) < f(x) \ \forall x$
8. Absolute maximum and minimum need not be turning points
9. At an extreme point $f'(x) = 0$ only if $f'(x)$ exists. There can be an extreme point even when $f'(x)$ does not exist

Critical Points

A point at which $f'(x) = 0$ or $f'(x)$ does not exist is called a critical point or stationary point.

Note

i) All extreme points are critical points

ii) All critical points need not be extreme points

Rolle's theorem

Let Y = f(x) be continuous in [a b] and differentiable at least in (a b). Let f(a) = f(b). Then the Rolle's theorem states that there exist at least one 'c' e (a b) where $f'(c) = 0$

Geometrically Rolle's theorem states that b/w 'a' and 'b' there exist at least one 'c' where a tangent can be drawn parallel to x-axis

In algebriac sense Rolle's Theorem says that b/w any two roots 'a' and 'b' of f(x) = 0, there exist at least one root for $f'(x) = 0$

Langrange's Mean Value Theorem (LMVT)

Let Y = f(x) be continuous in [a b] and differentiable at least in (a b). Then LMVT states that there exist at least one 'c' e (a b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically LMVT states that b/w 'a' and 'b' there exist at least one 'c' where the tangent drawn is parallel to the chord joining point [a f(a)] and [b f(b)]

Angle of Intersection

Angle of intersection is the angle between the tangents drawn at the point of intersection. Let m_1 be the slope of the tangent at $(x_1\ y_1)$ to the curve y = f(x) and m_2 be the slope of the tangent at$(x_1\ y_1)$ to the cure Y = g(x). Let q be the angle b/w the curves

Then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Notes

1. Let $m_1 m_2 = -1$, then $\theta = \frac{\pi}{2}$ and the curve are said to intersect orthogonally.

2. When $m_1 = m_2 \Rightarrow \theta = 0$ then the two curves touch each other

PART I - (JEEMAIN)

QUESTIONS

SECTION - I - Straight objective type questions

1. A spherical ball is inflated so that its volume increases uniformly at the rate of 40 cc/s. What is the rate of increase of its surface area when the radius is 8 cm:

   1) 20 2) 10 3) 5 4) 16

2. The interval in which $f(x)=||x|-1|$ is increasing :

   1) $(-1,\ 0)\cup(1,\ \infty)$ 2) $(-\infty,\ -1)\cup(0,\ 1)$

   3) $(-1,\ 0)\cup(0,\ 1)$ 4) $(-\infty,\ -1)\cup(1,\ \infty)$

3. $h(x)=f(x)-[f(x)]^2+[f(x)]^3$. Then which of the following is true?

   1) h (x) increases when f(x) increases 2) h (x) increases when f(x) decreases

   3) h (x) decreases when f(x) increases 4) h (x) is independent of f(x)

4. The equation $a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots + a_1 x = 0$ where $a_1 \neq 0$ $n \geq 2$ has a positive root at x = α. Then the equation

   $n\, a_n x^{n-1} + (n-1)\, a_{n-1} x^{n-2} + \ldots\ldots\ldots + a_1 = 0$ has a root

   1) Positive and greater than α 2) Positive and less than α

   3) negative and < α 4) negative and > α

5. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). The nearest distance between the soldier and the jet is:

   1) $\sqrt{5}$ 2) 2 3) 5 4) 4

6. α and β are the roots of the equation $x^2 - (a-2)x - (a+1) = 0$. Then the minimum value of $\alpha^2+\beta^2$ is

   1) 5 2) 7 3) 4 4) 6

7. The volume V and depth x of water in a vessel are connected by the relation $V=5x-\frac{x^2}{6}$ and the volume of water is increasing at the rate of 5cm³/sec, when x= 2cm. The rate at which the depth of water is increasing, is

   1) $\frac{5}{18}$ cm / sec 2) $\frac{1}{4}$ cm / sec 3) $\frac{5}{16}$ cm / sec 4) $\frac{15}{13}$ cm / sec

8. A particle is moving in a straight line. Its displacement at time t is given by $s=-4t^2+2t$, then its velocity and acceleration at time $t=\frac{1}{2}$ second are

1) – 2, – 8 2) 2, 6 3) – 2, 8 4) 2, 8

9. The function $f(x)=x^3-3x^2-24x+5$ is an increasing function in the interval given below

1) $(-\infty,-2)\cup(4,\infty)$ 2) $(-2,\infty)$ 3) $(-2,4)$ 4) $(-\infty,4)$

10. For the function $f(x)=e^x, a=0, b=1$, the value of c in mean value theorem will be satisfied is

1) log x 2) $\log(e-1)$ 3) 0 4) 1

11. A curve in the plane is defined by the parametric equations $x=e^{2t}+2e^{-t}$ and $y=e^{2t}+e^{t}$. An equation for the line tangent to the curve at the point t = ln 2 is

1) $5x-6y=7$ 2) $5x-3y=7$ 3) $10x-7y=8$ 4) $3x-2y=3$

12. A point is moving along the curve $y^3=27x$. The interval in which the abscissa changes at slower rate than ordinate, is

1) $(-3,3)$ 2) $(-\infty,\infty)$ 3) $(-1,1)$ 4) $(-\infty,-3)\cup(3,\infty)$

13. Let f(x) be a increasing function defined on $(0,\infty)$. If $f(2a^2+a+1)>f(3a^2-4a+1)$ the range of a is

1) $(0,\infty)$ 2) $\left(0,\frac{1}{3}\right)$ 3) $\left(0,\frac{1}{3}\right)\cup(1,5)$ 4) $(0,5)$

14. Let S be the set of all x such that $x^4-10x^2+9\le 0$. The maximum value of $f(x)=x^3-3x$ on S, is

1) 16 2) 17 3) 18 4) 19

15. A police flying squad approaching a right angled intersection from the north is chasing a speeding car that has turned the corner and is moving straight east. When the police is 0.6 km north of intersection and the car is 0.8 km to the east, the police determined with radar that the distance between them and the car is increasing 20 km/hr. If the police is moving at 60 km/hr what is the speed of the moving car:

1) 60 2) 70 3) 80 4) 90

Assertion Reason Type

1) Both assertion and reason are true and reason is the correct explanation of assertion

2) Both assertion and reason are true but reason is not the correct explanation of assertion

3) Assertion is true but reason is false

4) Assertion is false but reason is true

16. Statement I : $f(x)=\begin{cases} e^x & 0\le x\le 1 \\ 2-e^{x-1}, & 1<x\le 2 \\ x-e, & 2<x\le 3 \end{cases}$

f(x) has local Maximum at x = 1 and local Minimum at x = 2

Statement II : Then function f(x) has Local Maximum at x = a if f(a) > f(a + h) and f(a) > f (a - h). It has a local Minimum at x = b if f(b) < f(b + h) and f (b) < f (b - h) where h > 0.

17. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness the melts at a rate of $50\,cm^3/min$. When the thickness of ice is 5 cm, then the rate of (in cm/min) at which of the thickness of ice decreases, is:

1) $\frac{1}{36\pi}$ 2) $\frac{5}{6\pi}$ 3) $\frac{1}{18\pi}$ 4) $\frac{1}{54\pi}$

18. Let $a_1, a_2, a_3, ...$ be an A.P., with $a_6 = 2$. Then the common difference of this A.P., which maximizes the produce $a_1a_4a_5$ is

1) $\frac{6}{5}$ 2) $\frac{8}{5}$ 3) $\frac{2}{3}$ 4) $\frac{3}{2}$

19. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function $f(x)=9x^4+12x^3-36x^2+25, x\in R$, then

1) $S_1=\{-2,1\}; S_2=\{0\}$ 2) $S_1=\{-2,0\}; S_2=\{1\}$

3) $S_1=\{-2\}; S_2=\{0,1\}$ 4) $S_1=\{-1\}; S_2=\{0,2\}$

20. The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3, is

1) $2\sqrt{3}$ 2) $\sqrt{3}$ 3) $\sqrt{6}$ 4) $\frac{2}{3}\sqrt{3}$

SECTION - II

Numerical Type Questions

21. The value of 'λ' for which $f(x)=x^3+3(\lambda-7)x^2+3(\lambda^2-9)x-1$ has a positive point of maximum lies in the interval $(\alpha,\beta)\cup(\gamma,\delta)$. Then the value of $\beta+11\gamma+70\delta$ is

22. If $M(x_0, y_0)$ is the point on the curve $3x^2-4y^2=72$, which is nearest to the line $3x+2y+1=0$, then the value of (x_0+y_0) is equal to

23. Let P(x) be a polynomial of degree 5 having extremum at $x=-1,1$ and $\lim_{x\to 0}\left(\frac{P(x)}{x^3}-2\right)=4$. If M and m are the maximum and minimum value of the function $y=P'(x)$ on the set $A=\{x|x^2+6\le 5x\}$ then $\frac{M}{m}$ is

24. Consider the function $f(x)=\begin{cases}(2+x)^3, -3<x\le -1\\ x^{2/3}, -1<x<2\end{cases}$

Let α be the number of extremum values, β be the number of non-differentiable points and γ be the number of inflection points of f(x) then $\frac{\alpha+\gamma}{\beta}$ is

25. Consider the cubic $f(x)=8x^3+4ax^2+2bx+a$ where $a,b\in R$

For b = 1, if y = f(x) is non monotonic then the sum of all the integral values of $a\in[1,100]$, is

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. A circle with centre at (15,–3) touches to $y=\frac{x^2}{3}$ at a point in the first quadrant, then the radius of the circle is

A) $5\sqrt{6}$ B) $8\sqrt{3}$ C) $9\sqrt{2}$ D) $6\sqrt{5}$

27. Tangents are drawn from the origin to the curve $y=\sin x$ and their points of contact lie on the curve, then the equaiton of the curve is

A) $\frac{1}{x^2}-\frac{1}{y^2}+1=0$ B) $\frac{1}{x^2}+\frac{1}{y^2}-1=0$ C) $x^2-y^2=1$ D) $x^2+y^2=1$

28. The lines $y=\frac{-3}{2}x$ and $y=\frac{-2}{5}x$ intersect the curve $3x^2+4xy+5y^2-4=0$ at the points A and B respectively, then the tangents drawn to the curve at A and B

A) Intersect each other at an angle of 45°

B) are parallel to each other

C) are perpendicular to each other

D) Intersect each other at an angle of 60°

29. On the curve $x^3 = 12y$, the abscissa changes at a faster rate than the ordinate. Then x belongs to the interval

A) $(-2,2)$ B) $(-1,1)$ C) $(0,2)$ D) (–2,0)

30. If $1^\circ = \alpha$ radians, then the approximate value of $\cos 60^\circ\, 2'$

A) $\frac{1}{2}+\frac{\alpha\sqrt{3}}{60}$ B) $\frac{1}{2}-\frac{\alpha}{60}$ C) $\frac{1}{2}-\frac{\alpha\sqrt{3}}{60}$ D) none of these

31. The real values of x satisfying $1-f(x)-f(x)^3 > f(1-5x)$ such that $f(x)=1-x-x^3$ is

A) $(1,3)\cup(2,\infty)$ B) $(-1,0)\cup(1,\infty)$ C) $(-2,0)\cup(2,\infty)$ D) $(-5,0)\cup(1,\infty)$

32. Statement 1 : The maximum value of $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x \mid x^2+20\le 9x\}$ is 7

Statement 2 : If f(x) is strictly increasing in $[a,b]$, then the maximum value of $f(x)$ is $f(b)$

A) S1 is true B) S2 is true

C) S1 and S2 are true D) S1 is true and S2 are false

SECTION - IV (More than one correct answer)

33. $f(x)=\begin{cases} e^x, 0\le x\le 1 \\ 2-e^{x-1}, 1<x\le 2 \\ x-e, 2<x\le 3 \end{cases}$ and

$g(x)=\int_0^x f(t)\,dt$ where $x\in[1,\ 3]$. Then which of the following are true

A) g(x) has local maximum at $x=1+\log 2$ and local minimum at x = e

B) f(x) has local maximum at x = 2 and local minimum at x = 1

C) f(x) has local maximum at x = 1 and local minimum at x = 2

D) f(x) is discontinuous at x = 1, continuous at x = 2 and not differentiable at x = 1 and x = 2

34. Let $f(x)=\int_{-2}^{x}(t^2-t+2)(t^2-t-2)(t^2-t-6)(t^2-t-12)\,dt$ for all $x\in R$, then which of the following statement(s) is / are correct?

A) The equation of normal to f(x) at $x=-2$ is $x+2=0$

B) f(x) increases in $(-\infty,-3)\cup(-2,-1)\cup(2,3)\cup(4,\infty)$

C) f(x) decreases in $(-3,-2)\cup(-1,2)\cup(3,4)$

D) the sum of values of x at which f(x) has local maximum equals –1

35. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function in the set of real numbers R. Then a and b satisfy the condition

A) $a^2 - 3b - 15 \leq 0$
B) $a^2 - 3b + 15 \geq 0$
C) $a^2 - 3b + 15 \leq 0$
D) $a > 0$ and $b > 0$

SECTION - V (Numerical Type)

36. If the equation $x^3 - 3x + 1 = 0$ has 3 real roots α, β, γ where $\alpha < \beta < \gamma$, then the value of $\{\alpha\} + \{\beta\} + \{\gamma\}$ equals, where $\{x\}$ denots functional part of x

37. Let $f : R \to R$ be a funciton defined by $f(x) = 2x^3 - 21x^2 + 78x + 24$, then the number of integers satisfying the inequality $f\left(f\left(f(x) - 2x^3\right)\right) \geq f\left(f\left(2x^3 - f(x)\right)\right)$ is

38. Let f(x) be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(6-x)$ and $f'(0) = 0 = f'(2) = f'(5)$. Then the minimum number of zeroes of $g(x) = (f''(x))^2 + f'(x)f'''(x)$ in the interval $[0,6]$ is

39. The minimum value of $\left((x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2\right)^{\frac{1}{2}}$ where $x_1 \in (0, \sqrt{2})$ and $x_2 \in R^+$

SECTION VI - (Matrix match type)

40. Let $f(x)$ be a real valued function defined by $f(x) = x^2 - 2|x|$ and

$$g(x) = \begin{cases} \min imum\{f(t); -2 \leq t \leq x\}, & x \in [-2, 0) \\ \max imum\{f(t); 0 \leq t \leq x\}, & x \in [0, 3] \end{cases}$$

Column-I	Column-II
A) $f(x)$ is not continous at x equal to	P) -2
B) g(x) is not derivable at x equal to	Q) 0
C) The points of local extremum of g(x) is/are	R) 1
D) Absolute maximum value of g(x) is equal to	S) 2
	T) 3

A) A-(Q); B(QS); C-(QRS); D-(T)
B) A-(Q); B(RS); C-(QRS); D-(S)
C) A-(R); B(QS); C-(QRS); D-(T)
D) A-(Q); B(RT); C-(QRS); D-(T)