LIMITS OF REAL FUNCTIONS

Consider the functions $f(x)=x^2$. The values of $f(x)$ when x approaches to two are tabulated below.

x	1.5	1.9	1.9999	2	2.01	2.5	2.9	2.99
$f(x)=x^2$	2.25	3.61	3.9996009	4	4.0401	6.25	8.41	8.9401

From the table it can be seen that when 'x' approaches or tends to '2' the value of $f(x)=x^2$ approaches to 4 and Mathematically we say the limit of $f(x)=x^2$ when x tends 2 is 4. In symbols we write.

$$\underset{x\to 2}{\text{Lt}}\ f(x)=\underset{x\to 2}{\text{Lt}}\ x^2=4$$

Right Hand Limit (RHL)

From the above table it can be seen that when 'x' tends to 2 from $x>2$ the value of $f(x)=x^2$ approaches to 4 and we write that in symbols $\underset{x\to 2^+}{\text{Lt}}\ f(x)=4$ and is called the RHL of $f(x)$ at $x=2$. When we write $x\to 2^+$ it means that 'x' is tending to 2 from $x>2$

Left Hand Limit (LHL)

From the table it can be seen that when 'x' approaches to 2 from values less than 2, then also graph of $f(x)=x^2\to 2$ and we write $\underset{x\to 2}{\text{Lt}}-f(x)=4$

Results

1) When $\text{RHL}=\text{LHL}=\text{K}$ we say $\underset{x\to a}{\text{Lt}}\ f(x)$ exists and $\underset{x\to a}{\text{Lt}}\ f(x)=k$

2) When $\text{RHL}\neq\text{LHL}$ we say $\underset{x\to a}{\text{Lt}}\ f(x)$ does not exist (DNE)

3) The RHL is the expected value of $f(x)$ when $x \to a$ from $x > a$ and LHL is the expected value of $f(x)$ when $x \to a$ from $x < a$

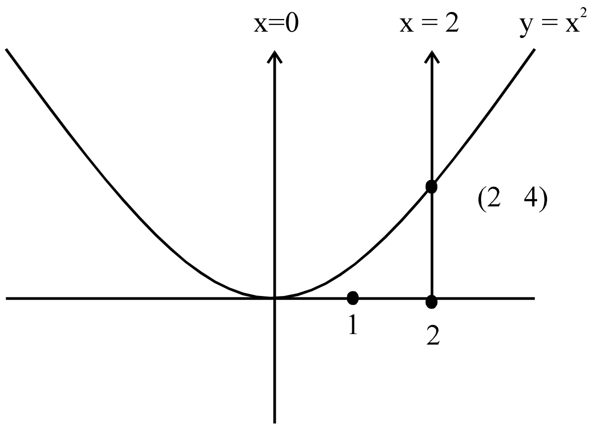
Geometrical Meaning of $\underset{x\to a}{\text{Lt}} f(x) = k$

In geometrical sense $\underset{x\to a}{\text{Lt}} f(x) = k$ means the graph of $f(x)$ meets the line $x = a$ at the point (a, k) where limiting point (a, k) need not be on the graph.

The limiting point (a k) is on the graph if the limit is obtained by direct substitution. If there is no limit on direct substitution and if the limit exists, then the limiting point (a k) is not on the graph it is a hole.

Examples

1) Let $f(x) = x^2$. $\underset{x\to 2}{\text{Lt}} x^2 = 4$. It means that the graph of $f(x) = x^2$ and the line $x = 2$ meets at the point $(2, 4)$. $f(x) = x^2$ $f(2) = 4$. Limit is obtained by direct substitution.
$\therefore$ Limiting (2, 4) is on the graph



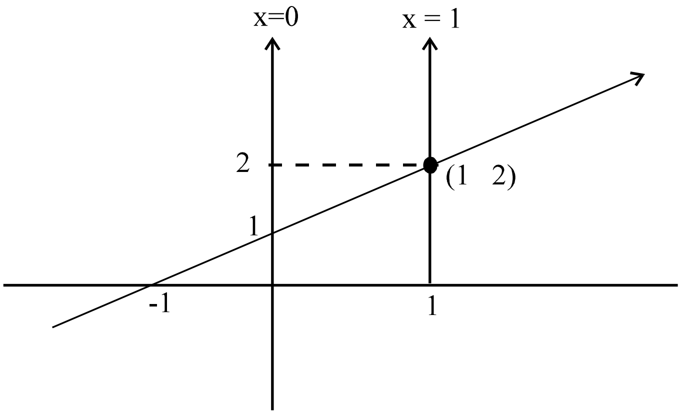
Example-2 Let $f(x) \left\{ \begin{array}{ll} = \dfrac{x^2-1}{x-1} & x \neq 1 \\ = 1 & x = 1 \end{array} \right.$

$$\underset{x\to 1}{\text{Lt}} f(x) = \underset{x\to 1}{\text{Lt}} \frac{x^2-1}{x-1} = \underset{x\to 1}{\text{Lt}} \frac{(x-1)(x+1)}{(x-1)}$$

$= \underset{x\to 1}{Lt}(x+1) = 1+1 = 2$

$\underset{x\to 1}{Lt} f(x)$ exist and is 2

$f(x) = \dfrac{x^2-1}{x-1}$ $\quad f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ undefined Thus limit can not be obtained by direct substitution. Hence limiting point (1, 2) is not on the graph. It is a hole as shown below.



From the above discussion it can be seen that the concept limit is defined to find the expected value (not the exact value) of a function at a point whose direct substitution is an undefined quantity.

Algebra of limits

1) $\underset{x\to a}{Lt} f(x) \pm g(x) = \underset{x\to a}{Lt} f(x) \pm \underset{x\to a}{Lt} g(x)$

2) $\underset{x\to a}{Lt} K f(x) = K \underset{x\to a}{Lt} f(x)$

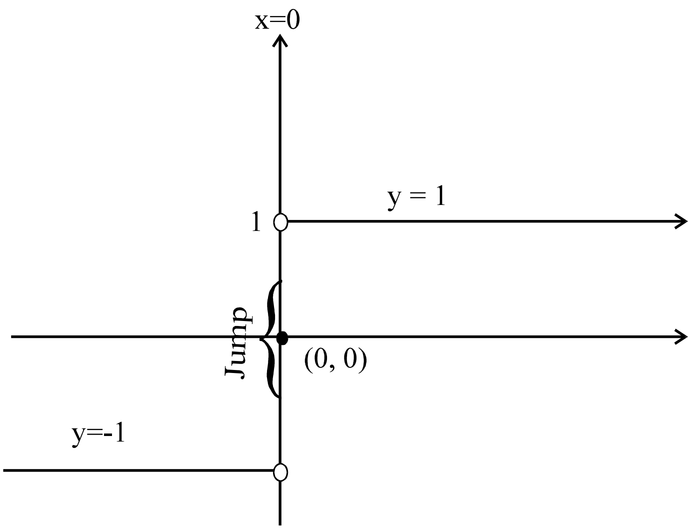
3) $\underset{x\to a}{Lt} f(x)\cdot g(x) = \underset{x\to a}{Lt} f(x)\cdot \underset{x\to a}{Lt} g(x)$

4) $\underset{x\to a}{Lt} \dfrac{f(x)}{g(x)} = \dfrac{\underset{x\to a}{Lt} f(x)}{\underset{x\to a}{Lt} g(x)}$ If $\underset{x\to a}{Lt} g(x) \neq 0$

Geometrical Meaning of $\underset{x\to a}{\mathrm{Lt}}\ f(x)$ Does not Exist (DNE)

In geometrical sense $\underset{x\to a}{\mathrm{Lt}}\ f(x)$ does not exist means there is a jump in the graph of $f(x)$ at the line x = a. For example consider the signum function

$$f(x)=\mathrm{sig}\ x \begin{cases} =1 & x>0 \\ =0 & x=0 \\ =-1 & x<0 \end{cases}$$



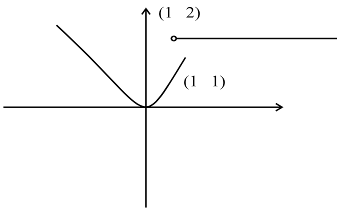
$\mathrm{RHL}=\underset{x\to 0^+}{\mathrm{Lt}}\ \mathrm{sig}\ x=1 \qquad \mathrm{LHL}=\underset{x\to 0^-}{\mathrm{Lt}}\ \mathrm{sig}\ x=-1$

$\therefore\ \underset{x\to 0}{\mathrm{Lt}}$ sig x does not exist and there is a jump in the graph at x = 0

Example

$$f(x)\begin{cases} =x^2 & x\le 1 \\ =2 & x>1 \end{cases}$$

(Piecewise function)



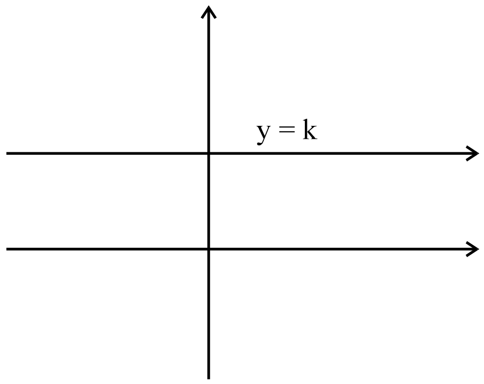
$\underset{x\to 0^+}{\mathrm{Lt}}\ f(x)=\underset{x\to 0^-}{\mathrm{Lt}}\ f(0)=0$

$\underset{x\to 1^+}{\mathrm{Lt}}\ f(x)=2 \qquad \underset{x\to 1^-}{\mathrm{Lt}}\ f(x)=1$

The limits of some Real valued functions

1) Constant function

$f(x) = K \quad \forall x$

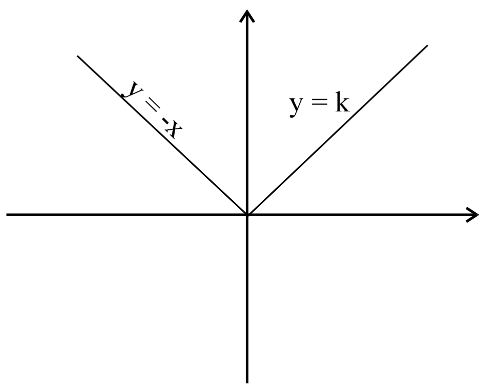


$\underset{x \to a}{\mathrm{Lt}} f(x) = \underset{x \to a}{\mathrm{Lt}} K = K$

ie limit of a constant is that constant itself.

2)Modulus function

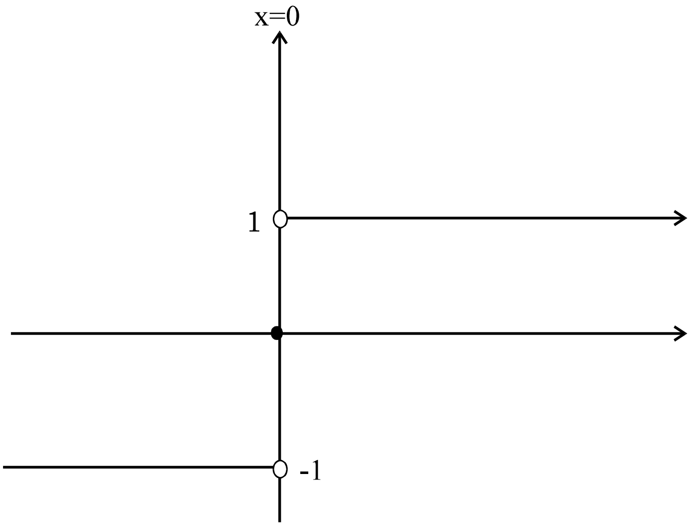
$$f(x) = |x| \begin{cases} = x & x > 0 \\ = 0 & x = 0 \\ = -x & x < 0 \end{cases}$$



There is no jump in the graph anywhere $\underset{x \to 0^+}{\mathrm{Lt}} |x| = 0, \quad \underset{x \to 0^-}{\mathrm{Lt}} |x| = 0 \Rightarrow \underset{x \to 0}{\mathrm{Lt}} |x| = 0$

3) Signum function

$$f(x)\begin{cases} =1 & x>0 \\ =0 & x=0 \\ =-1 & x<0 \end{cases}$$



There is jump at $x=0$ $\therefore \underset{x\to 0}{\text{Lt}}\ \text{sig}\ x$ DNE

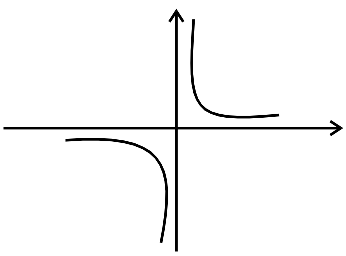
$\underset{x\to 0}{\text{Lt}}+\text{sig}\ x=1$ and $\underset{x\to 0^{-}}{\text{Lt}}\ \text{sig}\ x=-1$

4) Reciprocal function

$f(x)=\frac{1}{x}$

$\underset{x\to 0^{+}}{\text{Lt}}\frac{1}{x}=\infty \Rightarrow \frac{1}{0^{+}}=\infty$

$\underset{x\to 0^{-}}{\text{Lt}}\frac{1}{x}=-\infty \Rightarrow \frac{1}{0^{-}}=-\infty$



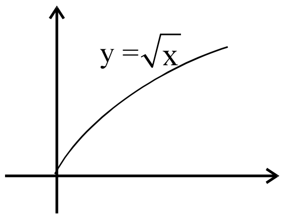
Also

$$\boxed{\begin{array}{l} \underset{x\to \infty}{\text{Lt}}\frac{1}{x}=\frac{1}{\infty}=0 \\ \underset{x\to -\infty}{\text{Lt}}\frac{1}{x}=\frac{1}{-\infty}=0 \end{array}}$$

5) Square Root Function

$Y=\sqrt{x}$ where $x>0$ and $y>0$ is the square root function.

$$\underset{x\to 0}{\text{Lt}}\sqrt{x}=\underset{x\to 0^+}{\text{Lt}}\sqrt{x}=0$$



An important Note: (Limits at the end points)

Let $y=f(x)$ be defined in $a\le x\le b$ ie domain is $[a\quad b]$

$\underset{x\to a}{\text{Lt}}\,f(x)$ is defined as $\underset{x\to a^+}{\text{Lt}}\,f(x)$

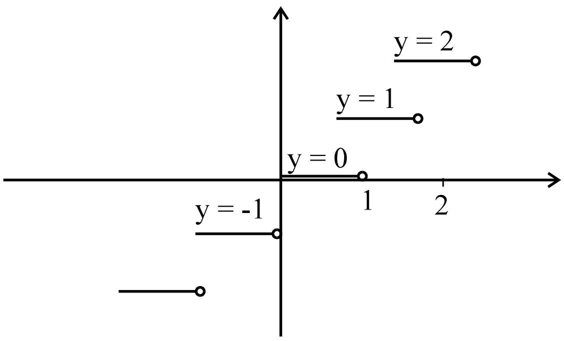
$\underset{x\to b}{\text{Lt}}\,f(x)$ is defined as $\underset{x\to b^-}{\text{Lt}}\,f(x)$

Thus limit as the first point of the domain is the RHL at that point. The limit at the last point of the domain is the LHL at that point.

Greatest Integer value function

$f(x)=[x]$ where $[x]$ is the greatest integer $\le x$

$[2.5]=2\ \ [2]=2\ \ [-2.5]=-3\ \ [0.5]=0$



Let $a=\text{Integer}$

$\underset{x\to a^+}{\text{Lt}}[x]=a$

$\underset{x\to a^-}{\text{Lt}}[x]=a-1$

Question : 09/01/2019(JEE (MAIN))

$$\underset{x\to 0^-}{\text{Lt}} \frac{x\left(\left[x\right]+\left|x\right|\right)\sin\left[x\right]}{\left|x\right|}$$

When $x \to 0^- \Rightarrow \left|x\right| = -x$ and $\left[x\right] = -1$

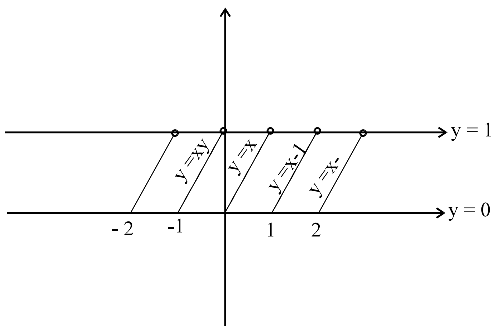
Rea. $\text{limit} = \underset{x\to 0}{\text{Lt}} \frac{x\left(-1-x\right)\sin\left(-1\right)}{-x}$

$$= \frac{\left(-1-0\right)\left(-\sin 1\right)}{-1} = \frac{-\sin 1}{-1} = \sin 1$$

Fractional value function

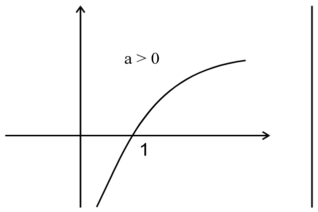
$f\left(x\right) = \left\{x\right\} = x - \left(x\right)$

Domain = R   Range $= 0 \le x < 1$

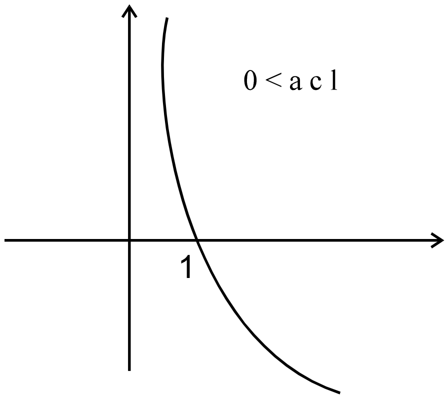


Logarithmic function

It is the inverse of exponential functions. It is defined as $f\left(x\right) = \log_a^x$ where $x > 0$ and $a > 0$ and $a \ne 1$



logo= $-\infty$
logo $\infty = \infty$

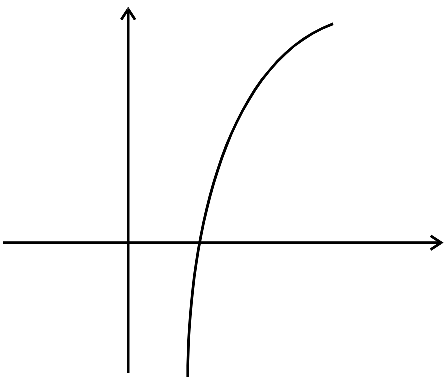


log 1 = 0

logo= ∞
logo $\infty = -\infty$

Natural logrithmic functions

Where base $a = e \Rightarrow f(x) = \log\ x$ is called natural logarithmic function. It is also denoted by $f(x) = \ell nx$

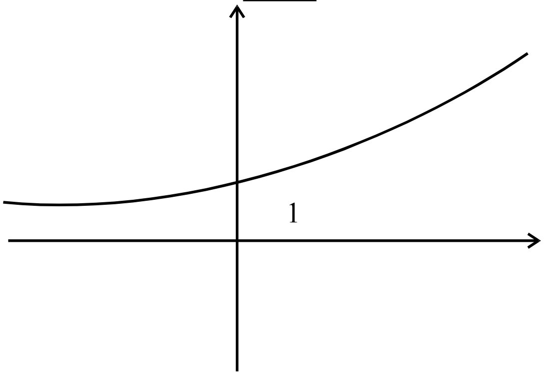


1) $\log o = -\infty,\ \log\ 1 = 0,\ \log o = 1$
   $\log 10 = 2.303,\ \log\ \infty = 0$
2) $\log\ a^x = \dfrac{\log\ x}{\log\ a}$
3) $-\log\ a = \log\left(\dfrac{1}{a}\right)$
4) $\log\ a < \log\ b \Rightarrow a < b$
5) $e^{\log f(x)} = f(x)$

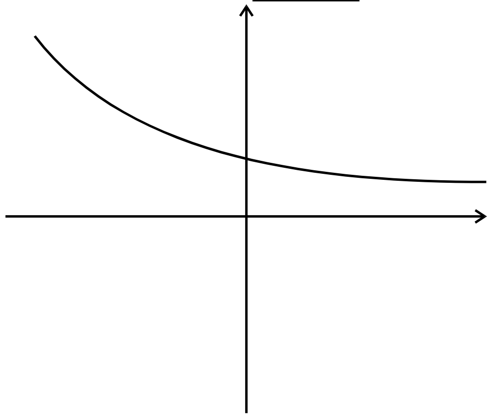
Exponential function

$f(x) = a^x \quad x \in R, \quad a > 0$ and $a \neq 1$

Case 1 : $\boxed{a > 1}$



Case 2 : $\boxed{0 < a < 1}$



$$\boxed{\begin{gathered}\operatorname*{Lt}_{x\to\infty} a^x = a^{\infty} = \infty \\ \operatorname*{Lt}_{x\to-\infty} a^x = a^{-\infty} = 0\end{gathered}}$$

$$\boxed{\begin{gathered}\operatorname*{Lt}_{x\to\infty} a^x = a^{\infty} = 0 \\ \operatorname*{Lt}_{x\to-\infty} a^x = a^{-\infty} = \infty\end{gathered}}$$

Question : Find $\operatorname*{Lt}_{x\to 0^-} \dfrac{1}{3-2^{\frac{1}{x}}} = \dfrac{1}{3}$

$$\boxed{\frac{3\times 2^{n+1} - 4\times 5^n}{5\times 2^n + 7\times 5^n} = \frac{-20}{7}}$$

$$\operatorname*{Lt}_{x\to 0^-} \frac{1}{3-2^{\frac{1}{x}}} = \frac{1}{3-2^{\frac{1}{0^-}}} = \frac{1}{3-2^{-\infty}} = \frac{1}{3-0} = \frac{1}{3}$$

Question: Find $\operatorname*{Lt}_{x\to\infty} \left(4^x + 5^x\right)^{\frac{1}{x}}$

$$\operatorname*{Lt}_{x\to\infty} \left(4^x + 5^x\right)^{1/x} = \operatorname*{Lt}_{x\to\infty} 5\left(\left(\frac{4}{5}\right)^x + 1\right)^{\frac{1}{x}}$$

$$= 5\left(\left(\frac{4}{5}\right)^{\infty} + 1\right)^{\frac{1}{\infty}} = 5(0+1)^o = 5$$

$$\boxed{\begin{array}{l} \underset{x\to\infty}{\mathrm{Lt}}\; x^{2n} \left.\begin{array}{ll} = 0 & -1 < x < 1 \\ = \infty & x < -1 \text{ a } x > 1 \\ = 1 & x = \pm 1 \end{array}\right. \end{array}}$$

$$\boxed{\begin{array}{l} \underset{x\to\infty}{\mathrm{Lt}}\; x^{n} \left.\begin{array}{ll} = 0 & -1 < x < 1 \\ = \infty & x > 1 \\ = 1 & x = 1 \\ \text{DNE} & x < -1 \end{array}\right. \\ (-2)^{10.5} = (-2)^{10} \times (-2)^{\frac{1}{2}} \\ = (-2)^{10} \times \sqrt{-2} \end{array}}$$

Evaluation of limits

Method 1 : Direct substitution

Directly put value of x and get limit. This is possible only if limiting point is on the graph.

Method 2: Direct substitution results in $\frac{0}{0}$ form

When the direct substitution results in $\frac{0}{0}$, Identify and eliminate the factor which makes denominator zero and evaluvate the limit.

Example $\boxed{\underset{x\to 0}{\mathrm{Lt}} \dfrac{(x+1)^5 - 1}{x} = \dfrac{0}{0}}$

$$\therefore \underset{x\to 0}{\mathrm{Lt}} \frac{(x+1)^5 - 1}{x} = \underset{x\to 0}{\mathrm{Lt}} \frac{\left(x^2 + 5x^4 + 10x^3 + 10x^2 + 5x + 1\right) - 1}{x}$$

$$= \underset{x\to 0}{\mathrm{Lt}}\; x \frac{\left(x^4 + 5x^3 + 10x^2 + 10x + 5\right)}{x} \quad \text{Factor which makes Dr} = 0 \text{ is eliminated}$$

$$= 0 + 5 = 5$$

Example $\underset{x\to 3}{\mathrm{Lt}} \dfrac{x^4 - 81}{2x^2 - 5x - 3} = \dfrac{3^4 - 81}{18 - 15 - 3} = \dfrac{0}{0}$

$$\underset{x\to 3}{\text{Lt}}\ \frac{x^4-81}{2x^2-5x-3}=\underset{x\to 3}{\text{Lt}}\ \frac{\left(x^2-9\right)\left(x^2+9\right)}{2x^2-6x+x-3}$$

$$\underset{x\to 3}{\text{Lt}}\ \frac{(x-3)(x+3)\left(x^2+9\right)}{2x(x-3)+(x-3)}$$

$$\underset{x\to 3}{\text{Lt}}\ \frac{(x+3)\left(x^2+9\right)}{2x+1}=\frac{108}{7}$$

Example $\boxed{\underset{x\to 0}{\text{Lt}}\ \frac{\sqrt{1+x-1}}{x}=\frac{0}{0}}$

$$\therefore\ \underset{x\to 0}{\text{Lt}}\ \frac{\left(\sqrt{1+x}-1\right)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}+1\right)}=\underset{x\to 0}{\text{Lt}}\ \frac{1+x-1}{x\left(\sqrt{1+x+1}\right)}$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1}{\sqrt{1+x}+1}=\frac{1}{2}$$

Method 3: Limits of piecewise Functions

In case of piecewise functions we find RHL and LHL at the points where the function is divided in to pieces. If $\text{RHL}=\text{LHL}$ the limit exists and if $\text{RHL}\neq\text{LHL}$ the limit does not exist (DNE).

Example 1:

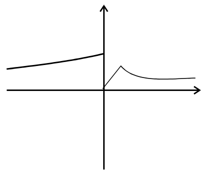
Jump at $x=1\Rightarrow$

$\underset{x\to 1}{\text{Lt}}\ f(x)$ DNE

Example 2:

$$f(x)\begin{cases} =ex & 0\le x\le 1\\ =2-e^{x-1} & 1<x\le 2\\ =x-e & 2<x\le 3\end{cases}$$

$$f(x)\begin{cases} =e^x & x\le 0\\ =x^2 & 0<x\le 1\\ =\dfrac{1}{x} & x>1\end{cases}$$



i) $\underset{x\to 0}{\text{Lt}}\ f(x) = \underset{x\to 0^+}{\text{Lt}}\ f(x) = \underset{x\to 0}{\text{Lt}}\ e^x = e^0 = 1$

ii) $\left.\begin{array}{l} \underset{x\to 1^-}{\text{Lt}}\ f(x) = \underset{x\to 1}{\text{Lt}}\ e^x = e \\ \underset{x\to 1^+}{\text{Lt}}\ f(x) = \underset{x\to 1}{\text{Lt}}\ 2 - e^{x-1} = 2-1 = 1 \end{array}\right\} \begin{array}{l} \text{RHL} \neq \text{LHL} \\ \underset{x\to 1}{\text{Lt}}\ f(x)\ \text{DNE} \end{array}$

iii) $\left.\begin{array}{l} \underset{x\to 2^-}{\text{Lt}}\ f(x) = \underset{x\to 2}{\text{Lt}}\ 2 - e^{x-1} = 2-e \\ \underset{x\to 2^+}{\text{Lt}}\ f(x) = \underset{x\to 2}{\text{Lt}}\ x - e = 2-e \end{array}\right\} \begin{array}{l} \text{RHL} = \text{LHL} = 2-e \\ \underset{x\to 2}{\text{Lt}}\ f(x) = 2-e \end{array}$

iv) $\underset{x\to 3}{\text{Lt}}\ f(x) = \underset{x\to 3^-}{\text{Lt}}\ f(x) = \underset{x\to 3}{\text{Lt}}\ x - e = 3-e$

$$f(x)\left\{\begin{array}{ll} = 5 & x \le 1 \\ = a+bx & 1 < x < 3 \\ = b+5x & 3 \le x < 5 \\ = 30 & x \ge 5 \end{array}\right.$$

Find a and b if $f(x)$ is has limits at $x = 3,\ x = 5$ and $x = 1$

Example $f(x)\left\{\begin{array}{ll} = a+bx, & x<1 \\ = 4, & x=1 \\ = b-ax, & x>1 \end{array}\right.$

Given $\underset{x\to 1}{\text{Lt}}\ f(x) = f(1)$. Find 'a' and 'b'

Given $\underset{x\to 1}{\text{Lt}}\ f(x) = f(1) = 4$ Since given f(1) = 4

$\therefore\ \underset{x\to 1^+}{\text{Lt}}\ f(x) = \underset{x\to 1^-}{\text{Lt}}\ f(x) = 4$

$\underset{x\to 1}{\text{Lt}}\ b - ax = \underset{x\to 1}{\text{Lt}}\ a + bx = 4$

$b - a = a + b = 4$

$\left.\begin{array}{l} b-a=4 \\ a+b=4 \end{array}\right\} \begin{array}{l} a=0 \\ b=4 \end{array}$

Method 4: Limits based on the trigonometric

$$\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}=1$$

Based on $\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}=1$ we can proove the following limits

i) $\underset{x\to 0}{\text{Lt}}\frac{x}{\sin x}=1$ ii) $\underset{x\to 0}{\text{Lt}}\frac{\tan x}{x}=1$ iii) $\underset{x\to 0}{\text{Lt}}\frac{x}{\tan x}=1$

When x is replaced by $f(x)$ the limit = 1 when $f(x)\to 0$. For example

$$\boxed{\underset{f(x)\to 0}{\text{Lt}}\frac{\sin f(x)}{f(x)}=1 \text{ Also } \underset{f(x)\to 0}{\text{Lt}}\frac{\tan f(x)}{f(x)}=1}$$

Example

i) Find $\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{x}$

$$\underset{x\to 0}{\text{Lt}}\frac{\sin 5x}{x}=\underset{x\to 0}{\text{Lt}}\,5\frac{\sin 5x}{5x}=5\times 1=5$$

ii) Find $\boxed{\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{\sin bx}}$

$$\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{\sin bx}=\underset{x\to 0}{\text{Lt}}\frac{ax\left(\frac{\sin ax}{ax}\right)}{bx\left(\frac{\sin bx}{bx}\right)}=\frac{a\times 1}{b\times 1}=\frac{a}{b}$$

iii) Find $\boxed{\underset{x\to \pi}{\text{Lt}}\frac{\sin(\pi-x)}{\pi(\pi-x)}}$

$$\underset{x\to \pi}{\text{Lt}}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\underset{(\pi-x)\to 0}{\text{Lt}}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

$$= \frac{1}{\pi} \underset{(\pi - x) \to 0}{\text{Lt}} \frac{\sin(\pi - x)}{(\pi - x)} = \frac{1}{\pi} \times 1 = \frac{1}{\pi}$$

iv) find $\boxed{\underset{x \to 0}{\text{Lt}} \frac{\cos 2x - 1}{\cos x - 1}}$

$$\underset{x \to 0}{\text{Lt}} \frac{\cos 2x - 1}{\cos x - 1} = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos 2x}{1 - \cos x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin^2 x}{2 \sin^2 \frac{x}{2}} = \underset{x \to 0}{\text{Lt}} \frac{\left(\frac{\sin^2 x}{x^2}\right)\left(x^2\right)}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\left(\frac{x^2}{4}\right)} = 4$$

v) $\underset{x \to 0}{\text{Lt}} (\operatorname{cosec} x - \cot x)$

$$\underset{x \to 0}{\text{Lt}} (\operatorname{cosec} x - \cot x) = \underset{x \to 0}{\text{Lt}} \frac{1}{\sin x} - \frac{\cos x}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{\sin x} = \underset{x \to 0}{\text{Lt}} \frac{2 \sin^2 \frac{x}{2}}{\sin x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \underset{x \to 0}{\text{Lt}} \tan \frac{x}{2} = \tan 0 = 0$$

Method 5: Limits based on series expansions of functions

The following series expansions of functions are used to evaluate limits.

1) $e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+.....$

2) $$\boxed{(1+x)^{n}=1+nx+\frac{n(n-1)}{1\times 2}n^{2}+\frac{n(n-1)(n-2)}{1\times 2\times 3}x^{3}+....|x|<1}$$

3) $e^{-x}=1-\frac{x}{1!}+\frac{x^{2}}{2!}-\frac{x^{3}}{3!}+.......$

4) $a^{x}=1+x\log a+\frac{(x\log a)^{2}}{2!}+\frac{(x\log a)^{3}}{3!}+.......a>0$

5) $\log(1+x)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+..........-1<x\le 1$

6) $\log(1-x)=-x-\frac{x^{2}}{2}-\frac{x^{3}}{3}-\frac{x^{4}}{4}-.....$

7) $\tan x=x+\frac{x^{3}}{3}+\frac{2x^{5}}{15}+....$

8) $\sin x=x-\frac{x^{3}}{3!}+\frac{x5}{5!}-....\left(\text{without factorial it the expansion of } \tan^{-1}x\right)$

9) $\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-.......$

10) $(1+x)^{1/x}=\left(1-\frac{x}{2}+\frac{11}{24}x^{2}-........\right)$

11) $\sin^{-1}x=x+\frac{1^{2}}{3!}x^{3}+\frac{1^{2}\times 3^{2}}{5!}x^{5}+\frac{1^{2}\cdot 3^{2}\cdot 5^{2}}{7!}x^{7}+.......$

12) $\sec^{-1} x$

$$\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$$

Examples

i) $$\underset{x\to 0}{\text{Lt}} \frac{e^x - 1}{x} = \underset{x\to 0}{\text{Lt}} \frac{\left(1 + x + \frac{x^2}{2} + ...\right) - 1}{x}$$

$$= \underset{x\to 0}{\text{Lt}}\ 1 + \frac{x}{2} + ... = 1$$

ii) $$\underset{x\to 0}{\text{Lt}} \frac{a^x - 1}{x} = \underset{x\to 0}{\text{Lt}} \frac{\left(1 + x \log a + \frac{x^2 (\log a)^2}{2} + ...\right) - 1}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\left(\log a + x(\log a)^2 + ...\right)}{x} = \log a$$

iii) $$\underset{x\to 0}{\text{Lt}} \frac{\log(1+x)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - .....}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\left(1 - \frac{x}{2} + \frac{x^2}{3} - ....\right)}{x} = 1$$

Thus we have 3 more important limits

i) $\underset{x\to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1$

ii) $\underset{x\to 0}{\text{Lt}} \frac{a^x - 1}{x} = \log a$ iii) $\underset{x\to 0}{\text{Lt}} \frac{\log(1+x)}{x} = 1$

When x is replaced by f(x) the limits remain unchanged if $f(x) \to 0$. For example

$$\underset{f(x)\to 0}{\text{Lt}} \frac{e^{f(x)}-1}{f(x)}=1, \quad \underset{f(x)\to 0}{\text{Lt}} \frac{a^{f(x)}-1}{f(x)}=\log a \quad \text{and} \quad \underset{f(x)\to 0}{\text{Lt}} \frac{\log\left(1+f(x)\right)}{f(x)}=1$$

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$$\boxed{\underset{x\to 0}{\text{Lt}} \frac{1}{x}\log\left(\frac{1+3x}{1-2x}\right)}$$

$$\boxed{\begin{array}{l}\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+3x)}{3x}\times 3-\dfrac{\log(1-2x)}{-2x}(-2) \\ \qquad =3-(-2)=5\end{array}}$$

Questions

i) find $\boxed{\underset{x\to 0}{\text{Lt}} \dfrac{\tan x-\sin x}{x^3}}$

$$\underset{x\to 0}{\text{Lt}} \frac{\tan x-\sin x}{x^3}=\underset{x\to 0}{\text{Lt}} \frac{x+\frac{x^3}{3}+\frac{2x^5}{15}+.....-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+....\right)}{x^3}$$

$$=\underset{x\to 0}{\text{Lt}} \frac{x^3\left[\left(\frac{1}{3}+\frac{2x^2}{15}+.....\right)-\left(-\frac{1}{6}+\frac{x^2}{5!}+.....\right)\right]}{x^3}$$

$$=\frac{1}{3}-\left(-\frac{1}{6}\right)=\frac{1}{2}$$

ii) Find $\boxed{\underset{x\to 0}{\text{Lt}} \dfrac{\log(1+x)-\sin x+\frac{x^2}{2}}{x\tan x\sin x}}$

Given limit $= \underset{x\to 0}{Lt} \dfrac{\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}\right)-\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-.....+\dfrac{x}{2}\right)}{x^3}$

$$= \underset{x\to 0}{Lt} \frac{x^3\left[\left(\frac{1}{3}-\frac{x}{4}+.....\right)-\left(-\frac{1}{6}+\frac{x^2}{5!}-....\right)\right]}{x^3}$$

$$=\frac{1}{3}-\left(-\frac{1}{6}\right)=\frac{1}{2}$$

iii) $\boxed{\underset{x\to 0}{Lt} \dfrac{\left(e^x-1\right)^2}{\sin\left(\dfrac{x}{a}\right)\log\left(1+\dfrac{x}{4}\right)}=12}$ Find a

$$\underset{x\to 0}{Lt} \frac{\left(\frac{e^x-1}{x}\right)^2\left(x^2\right)}{\left(\frac{\sin\frac{x}{a}}{\left(\frac{x}{a}\right)}\right)\left(\frac{x}{a}\right)\left(\frac{\log\left(1+\frac{x}{4}\right)}{\frac{x}{4}}\right)\left(\frac{x}{4}\right)}=12$$

$$\underset{x\to 0}{Lt} \frac{x^2}{\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)}=12 \Rightarrow 4a=12 \Rightarrow a=3$$

iv) $\boxed{\text{Find } \underset{x\to 0}{Lt} \dfrac{e-(1+x)^{1/x}}{\tan x}}$

Given Limit $= \underset{x\to 0}{Lt} \dfrac{e-e\left(1-\dfrac{x}{2}+\dfrac{11}{24}x^2-......\right)}{x}$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{e - e + \frac{ex}{2} - \frac{11}{24}ex^2 + ...}{x}$$

$$= \underset{x\to 0}{\mathrm{Lt}} \frac{x\left(\frac{e}{2} - \frac{11}{24}ex + ...\right)}{x}$$

$$= \frac{e}{2}$$

Method 6 : Limits Based on $\boxed{\underset{x\to a}{\mathrm{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}}$

i) $\underset{x\to 2}{\mathrm{Lt}} \frac{x^3 - 8}{x - 2} = \underset{x\to 2}{\mathrm{Lt}} \frac{x^3 - 2^3}{x - 2} = 3(2)^{3-1} = 12$

ii) $\underset{x\to 1}{\mathrm{Lt}} \frac{x^{15} - 1}{x^{10} - 1} = \underset{x\to 1}{\mathrm{Lt}} \frac{\left(\frac{x^{15} - 1}{x - 1}\right)}{\left(\frac{x^{10} - 1}{x - 1}\right)} = \frac{15(1)^{14}}{10(1)^9} = \frac{3}{2}$

iii) Find $\underset{x\to 0}{\mathrm{Lt}} \frac{(x+1)^5 - 1}{x}$

Put $u = x + 1$ when $x \to 0 \Rightarrow u \to 1$

Given limit $= \underset{u\to 1}{\mathrm{Lt}} \frac{u^5 - 1}{u - 1} = 5(1)^4 = 5$

iv) Find $\underset{u\to 0}{\mathrm{Lt}} \frac{\sqrt{1+x} - 1}{x}$

Put $u = 1 + x \Rightarrow x = u - 1 \quad x \to 0 \Rightarrow u \to 1$

Given limit $= \underset{u\to 1}{\mathrm{Lt}} \frac{u^{1/2} - 1}{u - 1} = \frac{1}{2}$

Method 7: Limits Based on Sandwich Theorem (squeese law)

$\underset{x\to a}{\text{Lt}}\ f_1(x)=K$ and $\underset{x\to a}{\text{Lt}}\ f_3(x)=K$

If $f_1(x)\le f_2(x)\le f_3(x)$ then $\underset{x\to a}{\text{Lt}}\ f_2(x)=K$

Example

Let $f(x)=x\ \sin\dfrac{1}{x}$

$-1\le \sin\dfrac{1}{x}\le 1\Rightarrow -x\le x\sin\dfrac{1}{x}\le x$

$\underset{x\to 0}{\text{Lt}}\ -x=0$ and $\underset{x\to 0}{\text{Lt}}\ x=0$ $\therefore$ By sandwich thereom

$\boxed{\underset{x\to 0}{\text{Lt}}\ x\ \sin\dfrac{1}{x}=0}$

Question

Find $\boxed{\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \dfrac{\cos^2\ \tan\ x}{\tan\ x}}$

We have $0\le \cos^2\ \tan\ x\le 1$

$0\le\dfrac{\cos^2\ \tan\ x}{\tan\ x}\le\dfrac{1}{\tan\ x}$ We have $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ 0=0$

Also $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \dfrac{1}{\tan\ x}=\dfrac{1}{\tan\ \frac{\pi}{2}}=\dfrac{1}{\infty}=0$

By sandwich Theorem $\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \dfrac{\cos^2\ \tan x}{\tan\ x}=0$

Method 8: Limits of Rational functions when $x \to \infty$

$$\underset{x\to\infty}{\text{Lt}} \frac{a_o x^n + a_1 x^{n-1} + \dots + an}{b_o x^m + b_1 x^{m-1} + \dots + bm} \left.\begin{matrix} = \dfrac{a_o}{b_o} & \text{If } m = n \\ = o & \text{If } m > n \\ = \infty & \text{If } n > m,\ a_o b_o > o \\ = -\infty & \text{If } n > m,\ a_o b_o < o \end{matrix}\right\}$$

Example

i) $\underset{x\to\infty}{\text{Lt}} \dfrac{5x^3 + 2x^2 + 1}{4x^3 - 3x + 1} = \dfrac{a_o}{b_o} = \dfrac{5}{4}$

ii) $\underset{x\to\infty}{\text{Lt}} \dfrac{5x^2 + 3x - 4}{2x^3 + 2x^2 + 1} = 0$

iii) $\underset{x\to\infty}{\text{Lt}} \dfrac{8x^5 - 3x^4 + 7x + 6}{9x^3 - 4x^4 + 11\,x^2 - 2} = -\infty$

Questions

i) $\boxed{\underset{n\to\infty}{\text{Lt}} \dfrac{1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2}{n^3} = \underset{x\to\infty}{\text{Lt}} \dfrac{n(n+1)\ (2n+1)}{6n^3} = \dfrac{2}{6} = \dfrac{1}{3}}$

ii) $\boxed{\underset{n\to\infty}{\text{Lt}} \dfrac{n\,Pr}{n^r} = \underset{x\to\infty}{\text{Lt}} \dfrac{n!}{(n-r)!nr}}$

$= \underset{n\to\infty}{\text{Lt}} \dfrac{n(n-1)(n-2)\dots.3\times2\times1}{(n-r)(n-r-1)\dots..3\times2\times1\ \ n^r}$

$= \underset{n\to\infty}{\text{Lt}} \dfrac{n(n-1)(n-2)\dots.(n-r+1)}{n^r} = 1$

Result

$$\underset{n\to\infty}{\text{Lt}} \frac{n\,Pr}{n^r} = 1 \text{ and } \underset{n\to\infty}{\text{Lt}} \frac{nCr}{n^r} = \frac{1}{r!}$$

Question:

$$\boxed{\underset{x\to\infty}{\text{Lt}} \left(\frac{x^2+1}{x-1} - ax - b\right) = 0 \text{ find 'a' and 'b'}}$$

$$\underset{x\to\infty}{\text{Lt}} \frac{x^2+1-ax(x-1)-b(x-1)}{(x-1)} = 0$$

$$\underset{x\to\infty}{\text{Lt}} \frac{x^2+1-ax^2+ax-bx+b}{x-1} = 0$$

$$\underset{x\to\infty}{\text{Lt}} \frac{(1-a)x^2+(a-b)x+(1+b)}{x-1} = 0$$

$\text{Limit} = 0 \Rightarrow \text{Deg of Dr} > \text{Deg of Nr}$

$\therefore\ 1-a=0 \Rightarrow a=1$ also $a-b=0 \Rightarrow a=b$

Home work

$$\boxed{\underset{x\to\infty}{\text{Lt}} \left[\frac{x^3+1}{x^2+1} - ax - b\right] = 2 \text{ find 'a' and 'b'}}$$

Method 9: Limits of undefined fom 1^∞

Let $\underset{x\to a}{\text{Lt}}\ f(x) = 0$ and $\underset{x\to a}{\text{Lt}}\ g(x) = 0$

Then $\underset{x\to 0}{\text{Lt}} (1+f(x))^{\frac{1}{g(x)}} = e^{\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)}}$

Examples

i) $\underset{x\to 0}{\text{Lt}} (1+\tan x)^{\frac{1}{x}} = e^{\underset{x\to 0}{\text{Lt}} \frac{\tan x}{x}} = e' = e$

ii) $\underset{x\to 0}{\text{Lt}}\,(1+x)^{\frac{1}{x}} = e^{\underset{x\to 0}{\text{Lt}}\frac{x}{x}} = e$

iii) $\underset{x\to\infty}{\text{Lt}}\left(1+\frac{1}{x}\right)^{x} = e^{\underset{x\to\infty}{\text{Lt}}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}} = e$

iv) $\underset{x\to 0}{\text{Lt}}\,(1-\tan x)^{\frac{1}{x}} = e^{\underset{x\to 0}{\text{Lt}} -\frac{\tan x}{x}} = e^{-1} = \frac{1}{e}$

Questions

i) $\boxed{\underset{x\to 0}{\text{Lt}}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}} = \underset{x\to 0}{\text{Lt}}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}}$

$$= \frac{\underset{x\to 0}{\text{Lt}}\,(1+\tan x)^{\frac{1}{x}}}{\underset{x\to 0}{\text{Lt}}\,(1-\tan x)^{\frac{1}{x}}} = \frac{e}{\left(\frac{1}{e}\right)} = e^2$$

ii) $\boxed{\underset{x\to\infty}{\text{Lt}}\left(\frac{x+6}{x+1}\right)^{x+4} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{x+6}{x+1}-1\right)^{x+4}}$

$$\underset{x\to\infty}{\text{Lt}}\left(1+\frac{x+6-(x+1)}{x+1}\right)^{x+4}$$

$$= \underset{x\to\infty}{\text{Lt}}\left(1+\frac{5}{x+1}\right)^{\frac{1}{\left(\frac{1}{x+4}\right)}} = e^{\underset{x\to\infty}{\text{Lt}}\frac{\left(\frac{5}{x+1}\right)}{\frac{1}{x+4}}}$$

$$= e^{\underset{x\to\infty}{\text{Lt}} \frac{5(x+4)}{(x+1)}} = e^5$$

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$\underset{x\to 0}{\text{Lt}} \left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$

Add and substract 1 and convert in to standard form.

iii) $\underset{x\to 0}{\text{Lt}} \left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}} = \underset{x\to 0}{\text{Lt}} \left(1+\frac{a^x+b^x}{2}-1\right)^{\frac{2}{x}}$

$$= \underset{x\to 0}{\text{Lt}} \left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{1}{\left(\frac{x}{2}\right)}} = e^{\underset{x\to 0}{\text{Lt}} \frac{\frac{\left(a^x+b^x-2\right)}{2}}{\left(\frac{x}{2}\right)}}$$

$$= e^{\underset{x\to 0}{\text{Lt}} \frac{a^x+b^x-2}{x}}$$

$$= e^{\underset{x\to 0}{\text{Lt}} \left(\frac{a^x-1}{x}+\frac{b^x-1}{x}\right)}$$

$$= e^{\log a+\log b} = e^{\log(ab)} = ab$$

Method 10: L - Hospital’s Rule (LHR)

LHR is a very powerful method of evaluating limits if direct substitution results in $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form. Let $\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ Then by LHR

$$\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Lt}} \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}$$

Example

$$\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = \frac{\sin o}{o} = \frac{o}{o} \text{ By LHR}$$

$$\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = \underset{x\to o}{\text{Lt}} \frac{\frac{d}{dx}\sin x}{\frac{d}{dx} x}$$

$$\boxed{\begin{array}{l} \frac{1-\cos x}{x^2} = \frac{1}{2} \\ \text{when } x \to 0 \end{array}}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\cos x}{1} = \cos o = 1$$

Questions

i) $$\underset{x\to o}{\text{Lt}} \frac{2^x - 1}{(1+x)^{1/2} - 1} = \frac{o}{o} \quad \therefore \text{ By LHR}$$

$$\boxed{\begin{array}{l} 07/01/2020 \\ \underset{x\to 2}{\text{Lt}} \frac{3^x + 3^{3-x} - 12}{3^{-\frac{x}{2}} - 3^{1-x}} \end{array}}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{d}{dx}\left(2^x - 1\right)}{\frac{d}{dx}(1+x)^{1/2} - 1} = \underset{x\to 0}{\text{Lt}} \frac{2^x \log 2}{\frac{1}{2}(1+x)^{-\frac{1}{2}}}$$

$$= \frac{\log 2}{\left(\frac{1}{2}\right)} = 2 \log 2 = \log 4$$

$$\boxed{\begin{array}{l} 219 \quad \text{JEEMAIN} \\ f(3) + f(2) = 0 \\ \underset{x\to 0}{\text{Lt}} \left[\frac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)}\right]^{\frac{1}{x}} \end{array}}$$

ii) $\underset{x\to\frac{\pi}{4}}{\text{Lt}} \dfrac{\cot^3 x - \tan x}{\cos\left(x+\frac{\pi}{4}\right)} = \dfrac{o}{o}$ By LHR

$$= \underset{x\to\frac{\pi}{4}}{\text{Lt}} \frac{3\text{cto}^2 x\left(-\text{cos ec}^2 x\right) - \sec^2 x}{-\sin\left(x+\frac{\pi}{4}\right)} = \frac{-3\left[3\times 2+2\right]}{-1} = 8$$

Method 11: Limits when direct substitution results in $o\times\infty$ or $\infty\times o$

When direct substitution results in $o\times\infty$ or $\infty\times o$ convert it in to $\dfrac{o}{o}$ or $\dfrac{\infty}{\infty}$ and use LHR

Example

$$\underset{x\to1}{\text{Lt}} (1-x)\tan\left(\frac{\pi x}{2}\right) = (1-0)\tan\frac{\pi}{2} = o\times\infty$$

$$\therefore \text{Limit} = \underset{x\to1}{\text{Lt}} \frac{1-x}{\cot\frac{\pi x}{2}} = \frac{1-1}{\cot\frac{\pi}{2}} = \frac{o}{o} \quad \therefore \text{ By LHR}$$

$$= \underset{x\to1}{\text{Lt}} \frac{-1}{-\text{cos ec}^2\frac{\pi x}{2}\times\frac{\pi}{2}} = \frac{2}{\pi}\underset{x\to1}{\text{Lt}} \sin^2\frac{\pi x}{2} = \frac{2}{\pi}$$

Method 12: Limits when direct substitution results in 0° form

When direct substitution is 0° take logaritham and evaluate the limits

Example

$$\underset{x\to0}{\text{Lt}} x^x = 0^\circ$$

$$\therefore \text{ Let } A = \underset{x\to0}{\text{Lt}} x^x \to \log A = \underset{x\to0}{\text{Lt}} x\log x$$

$$\log A = 0\times^{-}\infty \to \log A = \underset{x\to0}{\text{Lt}} \frac{\log x}{\left(\frac{1}{x}\right)} = \frac{-\infty}{\infty}$$

$$\therefore \text{ By LHR} \Rightarrow \log A = \underset{x\to 0}{\text{Lt}} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \underset{x\to 0}{\text{Lt}} -x = 0$$

$$\log A = 0 \Rightarrow A = 1$$

$$\therefore \underset{x\to 0}{\text{Lt}}\ x^x = 1$$

Find $\underset{x\to 0}{\text{Lt}}\ x^{x^x} - x^x$

Limit $= \underset{x\to 0}{\text{Lt}}\ x^{x^x} - \underset{x\to 0}{\text{Lt}}\ x^x$

we know $\underset{x\to 0}{\text{Lt}}\ x^x = 1$

Let $A = \underset{x\to 0}{\text{Lt}}\ x^{x^x} \Rightarrow \log A = \underset{x\to 0}{\text{Lt}}\ x^x \log x$

$$\log A = \underset{x\to 0}{\text{Lt}}\ x^x \times \underset{x\to 0}{\text{Lt}}\ \log x$$

$$\log A = -\infty \Rightarrow A = 0$$

Given limit $= 0 - 1 = -1$

13) **<u>Limit of an infinite sum as definite integral</u>**

a) $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

Example

$$\underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{n}{n^2 + r^2} = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \frac{n}{n^2\left[1 + \left(\frac{r}{n}\right)^2\right]}$$

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$f\left(\frac{r}{x}\right) = \frac{1}{1+\left(\frac{r}{x}\right)^2}$$

$$\text{pat} \frac{r}{n} x$$

$$f(x) = \frac{1}{1+x^2}$$

$$= \int_0^1 \frac{1}{1+x^2} dx$$

$$= \left[\tan^{-1} x\right]_0^1 = \tan_1^{-1} - \tan_0^{-1}$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$12(b)$ $$\boxed{\begin{array}{l} \underset{x\to\infty}{\mathrm{Lt}} \frac{1}{n} \sum_{r=\phi}^{g} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx \\ \text{where } a = \underset{n\to\infty}{\mathrm{Lt}} \left(\frac{\phi}{n}\right) \text{ and } b = \underset{n\to\infty}{\mathrm{Lt}} \left(\frac{g}{n}\right) \end{array}}$$

Example

i) Find $\underset{n\to\infty}{\mathrm{Lt}} \frac{n}{n^2+1} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots\dots\dots + \frac{1}{5n}$

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r^{th} term is $\frac{n}{n^2+r^2}$ $r = 1, 2, 3\dots\dots\dots\dots 2n$

Limit $= \underset{n\to\infty}{\mathrm{Lt}} \sum_{r=1}^{2n} \frac{n}{n^2+r^2} = \underset{n\to\infty}{\mathrm{Lt}} \sum_{r=1}^{2n} \frac{n}{n^2\left[1+\left(\frac{r}{n}\right)^2\right]}$

$$= \underset{n\to\infty}{Lt}\ \frac{1}{n}\sum_{r=1}^{2n} \frac{1}{1+\left(\frac{r}{n}\right)^2} = \int_a^b f(x)dx$$

$$a = \underset{n\to\infty}{Lt}\ \frac{1}{n} = 0 \quad b = \underset{n\to\infty}{Lt}\ \frac{(2n)}{n} = 2$$

$$f\left(\frac{r}{n}\right) = \frac{1}{1+\left(\frac{r}{n}\right)^2} \Rightarrow f(x) = \frac{1}{1+x^2}$$

$$\boxed{\begin{array}{l} a = \frac{1}{\infty} = 0 \\ b = \frac{2n}{n} = 2 \end{array}}$$

$$\therefore \int_a^b f(x)dx = \int_0^2 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^2$$

$$= \tan^{-1}2 - \tan^{-1}0 = \tan^{-1}2$$

Questions

2) $$\underset{n\to\infty}{Lt}\ \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}} = \underset{n\to\infty}{Lt}\ \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{n\sqrt{1+\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}dx = \left(\sqrt{1+x^2}\right)_0^2$$

$$= \sqrt{5} - 1 = \sqrt{5} - 1$$

$$\boxed{\begin{array}{l} \frac{d}{dx}\sqrt{1+x^2} = \frac{12x}{2\sqrt{1+x^2}} \\ \therefore \int \frac{x}{\sqrt{1+x^2}} = \sqrt{1+x^2} \end{array}}$$

Question :

1) $$\underset{n\to\infty}{Lt}\ \sum_{r=0}^{n-1} \left(\frac{r}{n^2+r^2}\right)$$

$$\underset{n\to\infty}{\mathrm{Lt}} \sum_{r=0}^{n-1} \frac{r}{n^2\left(1+\left(\frac{r}{n}\right)^2\right)} = \underset{n\to\infty}{\mathrm{Lt}} \frac{1}{n} \sum_{r=0}^{n-1}\left(\frac{r}{n}\right) \frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{1}{u}\frac{dx}{2}$$

$$\boxed{\begin{array}{l} \text{Put } u = 1+x^2 \\ x = 0 \Rightarrow u = 1 \\ x = 1 \Rightarrow u = 2 \\ \frac{du}{dx} = 2n \\ x\ dx = \frac{du}{2} \end{array}}$$

$$= \frac{1}{2}[\log\ u]_1^2 = \frac{1}{2}\log\ 2$$

2) $\underset{n\to\infty}{\mathrm{Lt}} \frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + ... + \frac{1}{n}\sec^2 1 =$

$$r^{th} \text{ term} = \frac{r}{n^2}\sec^2\left(\frac{r}{n}\right)^2 \quad r = 1, 2, 3, ...n$$

Given limit $\underset{x\to\infty}{\mathrm{Lt}} \sum_{r=1}^{n} \frac{r}{n^2}\sec^2\left(\frac{r}{n}\right)^2 = \frac{1}{2}\tan 1$

3) $\underset{x\to\infty}{\mathrm{Lt}} \frac{1}{n}\left[1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+.........+\sqrt{\frac{n}{n+3(n-1)}}\right]$

$$r^{th} \text{ term} = \sqrt{\frac{n}{n+3r}} \quad r = 0, 1, 2, ...(n-1)$$

Given Limit $= \underset{n\to\infty}{\mathrm{Lt}} \frac{1}{n}\sum_{r=0}^{n-1}\sqrt{\frac{n}{n+3r}}$

$$= \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{r=0}^{n-1} \frac{1}{\sqrt{\left(1+3\frac{r}{n}\right)}} 0$$

$$\boxed{\begin{array}{l} u = 1+3x \\ x = 0 \quad u = 1 \\ x = 1 \quad u = 4 \end{array}}$$

$$= \int_0^1 \frac{1}{\sqrt{1+3x}}dx = \int_1^4 \frac{1}{\sqrt{4}}\frac{24}{3} \qquad du = 3dx$$

$$\boxed{\begin{array}{l} \frac{d}{dx}\sqrt{x} = \frac{1}{2}\sqrt{x} \\ \int \frac{1}{\sqrt{x}}dx = 2\sqrt{x} \end{array}}$$

$$= \frac{1}{3}\times 2\left(\sqrt{1}\right)_1^4$$

$$= \frac{2}{3}\times(2-1) = \frac{2}{3}$$

4) $\underset{n\to\infty}{Lt} \dfrac{1^{2020}+2^{2020}+3^{2020}+........+n^{2020}}{n^{2021}}$

$$r^{th} \text{ term} = \frac{r^{2020}}{n^{2021}} \; r = 1,\ 2,\ 3, ...n$$

Given Limit $= \underset{n\to\infty}{Lt}\displaystyle\sum_{r=1}^{n}\frac{r^{2020}}{n^{2021}}$

$$= \underset{n\to\infty}{Lt}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{2020} \qquad f(x) = x^{2020}$$

$$= \int_0^1 x^{2020}dx = \frac{\left[x^{2021}\right]_0^1}{2021}$$

$$= \frac{1}{2021}$$

Results

1) $$\underset{n\to\infty}{\text{Lt}} \frac{[1f(x)]+[2f(x)]+...+[n\,f(x)]}{n^2} = \frac{f(x)}{2}$$

2) $$\underset{n\to\infty}{\text{Lt}} \frac{[1^2 f(x)]+[2^2\, f(x)]+....+[n^2\, f(x)]}{n^3} = \frac{f(x)}{3}$$

3) $$\underset{n\to\infty}{\text{Lt}} \frac{[1^3 f(x)]+[2^3\, f(x)]+....+[n^3\, f(x)]}{n^4} = \frac{f(x)}{4}$$

Additional Questions

1) $f(x) \begin{cases} = \dfrac{1}{1-e^{-\frac{1}{x}}} & n \neq 0 \\ = 0 & n = 0 \end{cases}$ Find $\underset{x\to 0}{\text{Lt}}\, f(x) =$

2) $f(x) \begin{cases} = \dfrac{e^{\frac{1}{x}}-1}{} & n \neq 0 \\ = 1 & n = 0 \end{cases}$ Find $\underset{x\to 0}{\text{Lt}}\, f(x) =$

2) (a) $\boxed{\text{Find the values of A, B, C so that } \underset{x\to 0}{\text{Lt}} \dfrac{Ae^x - B\cos x + Ce^{-x}}{x \sin x} = 2}$

3) $f(x)$ is a polynomial of degree 4 s.t

$f'(1) = f(2) = 0$ Given $\underset{x\to 0}{\text{Lt}} \left[1+\dfrac{f(x)}{x^2}\right] = 2$ find $f(2)$

Find $f(2) =$

4) $$\underset{x\to 0}{\text{Lt}} \frac{\tan[e^2]x^2 - \tan[-e^2]x^2}{\sin^2 x}$$ $\boxed{\begin{array}{l} e = 2.72 \\ e^2 \simeq 7.39 \end{array}}$

5) $\underset{x\to 0}{\mathrm{Lt}}\left[\left(\mathrm{Mini}\left(t^2+4t+6\right)\dfrac{\sin\ x}{x}\right)\right]$

Important Note

$$\frac{\sin\ x}{x}<1\Rightarrow \underset{x\to 0}{\mathrm{Lt}}\left[\frac{\sin\ n}{x}\right]=\left[\text{number}<1\right]=0$$

$$\frac{\tan\ x}{x}<1\Rightarrow \underset{x\to 0}{\mathrm{Lt}}\left[\frac{\tan\ x}{x}\right]=\left[\text{number}<1\right]=1$$

$$\underset{n\to\frac{\pi}{2}}{\mathrm{Lt}}\ \frac{\cot\ x-\cos x}{\left(\pi-2x\right)^3}=\frac{1}{16}$$

$$u=\pi-2x\Rightarrow x=\frac{\pi}{2}-\frac{u}{2}\Rightarrow x\to\frac{\pi}{2}\Rightarrow u\to 0$$

$$\underset{u\to 0}{\mathrm{Lt}}\ \frac{\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)-\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)}{u^3}$$

$$=\underset{u\to 0}{\mathrm{Lt}}\ \frac{\tan\frac{u}{2}-\sin\ \frac{u}{2}}{u^3}$$

$$=\underset{u\to 0}{\mathrm{Lt}}\ \frac{\tan\frac{u}{2}\left[1-\cos\ \frac{u}{2}\right]}{u^3}$$

$$=\underset{u\to 0}{\mathrm{Lt}}\ \frac{\dfrac{\tan\ \frac{u}{2}}{\left(\frac{u}{2}\right)}\left(\frac{u}{2}\right)\dfrac{1-\cos\frac{u}{2}}{\left(\frac{u}{2}\right)^2}\left(\frac{u}{2}\right)^2}{u^3}$$

$$=\underset{u\to 0}{\mathrm{Lt}}\ \frac{\frac{u}{2}\times\frac{1}{2}\ \frac{u^2}{4}}{43}=\frac{1}{16}$$

Method 14 : Leibnitz Theorem

$$\boxed{\frac{d}{dx}\int_{d(x)}^{g(x)} f(t)\,dt = g'(x) f\left[g(x)\right] - \phi'(x)\ f\left(\phi(x)\right)\phi(x)}$$

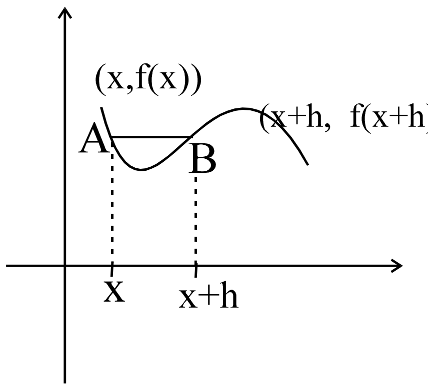
$$\frac{d}{dx}\int_{x^2}^{x^3} \log\ t\ dt = 3x^2\ \log\ x^3 - 2n\ \log\ x^2$$

$$= 9x^2\ \log\ x - 4x\ \log\ x$$

$$\frac{d}{dx}\int_{\sin\ x}^{e^x} \left(\frac{t+1}{t-1}\right)dt = e^x\left(\frac{e^x+1}{e^x-1}\right) - \cos\ x\left(\frac{\sin\ x+1}{\sin\ x-1}\right)$$

Derivative or differential coefficient

Let $y = f(x)$ be a differentiable function. Let $A\left(x, f(x)\right)$ and $B\left(x+h,\ A\left(x, f(x)\right)\ f(x+h)\right)$ be two points ont he grpah of f(x)



Explain

chord, secant
tangent

Slope of secant $AB = \dfrac{f(x+h) - f(x)}{x+h-x}$

Slope of tangent at $A = \underset{h\to 0}{\mathrm{Lt}}\ \dfrac{f(x+h) - f(x)}{h}$

This limit, if it exists, is called the derivative or differential coefficient of y=f(x) w.r.t x. It is denoted by $\frac{dy}{dx}$ or $f'(x)$

Geometrically $\frac{dy}{dx}$ is the slope of tangent and in physical sense $\frac{dy}{dx}$ is the rate of change of y w.r.t x

If we find the derivative using this method it is called differentiation from first principles or the ab-Initio method. The process of finding the derivative is called differentiation .

1) Find the derivative of f(x) = a^x using ab-initio method

$$f(x)=a^x \quad f(x+h)=a^{x+h}=a^x a^h$$

$$f'(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{Lt}\frac{a^x.a^h-a^x}{h}$$

$$=\underset{h\to 0}{Lt}\, a^x\frac{a^h-1}{h}=a^x\log a$$

$$y=a^x \Rightarrow \frac{dy}{dx}=a^x\log a$$

2) Find the derivative of $f(x)=\log x$ using first principle method

$$f(x)=\log x \quad f(x+h)=\log(x+h)$$

$$f'(x)=\underset{h\to 0}{Lt}\frac{\log(x+h)-\log x}{h}=\underset{h\to 0}{Lt}\frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$=\underset{h\to 0}{Lt}\frac{\log\left(1+\frac{h}{x}\right)}{h};=\underset{h\to 0}{Lt}\frac{\log\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)\times x}=\frac{1}{x}\underset{h\to 0}{Lt}\frac{\log\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)};\ \frac{1}{x}\times 1=\frac{1}{x}$$

$$\therefore y=\log x\Rightarrow\frac{d}{dx}\log x=\frac{1}{x}$$

Results

1) $\frac{d}{dx}k = 0$

2) $\frac{d}{dx}x^n = nx^{n-1}$

3) $\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$

4) $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$

5) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

6) $\frac{d}{dx}e^x = e^x$

7) $\frac{d}{dx}a^x = a^x \log a$

8) $\frac{d}{dx}\sin x = \cos x$

9) $\frac{d}{dx}\cos x = \sin x$

10) $\frac{d}{dx}\tan x = \sec^2 x$

11) $\frac{d}{dx}\sec x = \sec x \tan x$

12) $\frac{d}{dx}\operatorname{cos\,ec} x = -\operatorname{cose} x \cot x$

13) $\frac{d}{dx}\cot x = -\cos e^2\ x$

14) $\frac{d}{dx}\log x = \frac{1}{x}$

Chain Rule

$$\frac{d}{dx}f\left[g(x)\right] = f'\left[g(x)\right]\frac{d}{dx}g(x)$$

$$\frac{d}{dx}g\left[f(x)\right] = g'\left[f(x)\right]\frac{d}{dx}f(x)$$

Ex: $\frac{d}{dx}\log\ \sin x = \frac{1}{\sin x}\cos x = \cot x$

$$\frac{d}{dx}\sin\ \log x = \cos \log x \times \frac{1}{x}$$

Power Rule

$$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1}\frac{d}{dx}f(x)$$

Ex: $\frac{d}{dx}\sin^s x = 5\sin^4 x \cos x$

$$\frac{d}{dx}\log^3 x = 3\log^2 x \times \frac{1}{x}$$

Reciprocal Rule

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=\frac{-1}{\left[f(x)\right]^2}\frac{d}{dx}f(x)$$

Ex: $\frac{d}{dx}\frac{1}{\log x}=\frac{-1}{\log^2 x}\times\frac{1}{x}$

$$\frac{d}{dx}\frac{1}{2x^3+3}=\frac{-1}{\left(2x^2+3\right)^2}\times 6x^2$$