# CIRCLES

**General second degree equation**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \rightarrow (1)$$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \text{discriminant}$$

1) If $\Delta = 0$ then (1) represent equation of a pair of straight lines

2) If $\Delta \neq 0, a = b \neq 0$, and h =0 then (1) represent equation of a circle

3) If $\Delta \neq 0, h^2 = ab$ then (1) represent equation of a parabola

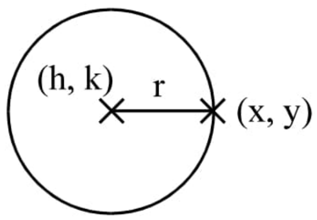
4) If $\Delta \neq 0, h^2 < ab$ then (1) represent equation of an ellipse

5) If $\Delta \neq 0, h^2 > ab$ then (1) represent equation of a hyperbola

**Standard Equation of a Circle**

$$(x-h)^2 + (y-k)^2 = r^2$$

Centre = (h, k)
Radius = r



## General Equation of a Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Centre = $(-g, -f), r = \sqrt{g^2 + f^2 - c}$

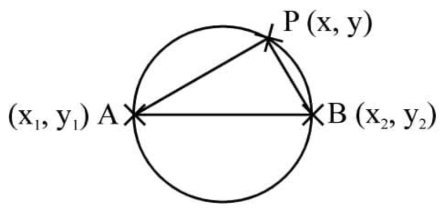
**Note :**

1) If $g^2 + f^2 - c > 0$ then the circle is real

2) If $g^2 + f^2 - c = 0$ then circle is a point circle (degenerate circle)

3) If $g^2 + f^2 - c < 0$ then circle is imaginary

## Diameter form of a circle




$\angle APB = 90^0$

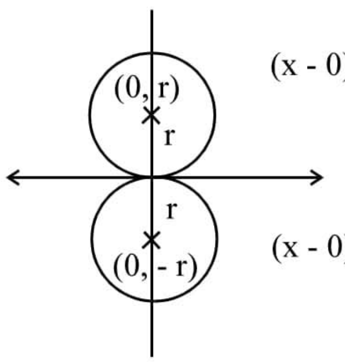
Slope of AP × slope of BP = - 1

$$\frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

## Equation of circle touching x - axis at origin




$(x - 0)^2 + (y - r)^2 = r^2$

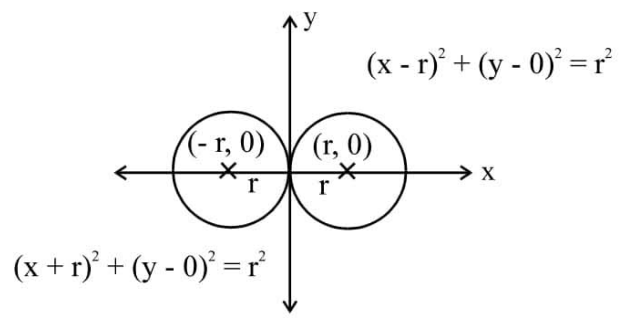
$(x - 0)^2 + (y + r)^2 = r^2$

Combined equation is

$$x^2+(y\pm r)^2=r^2$$

$$x^2+y^2\pm 2ry=0$$

**Equation of circle touching y - axis at origin**

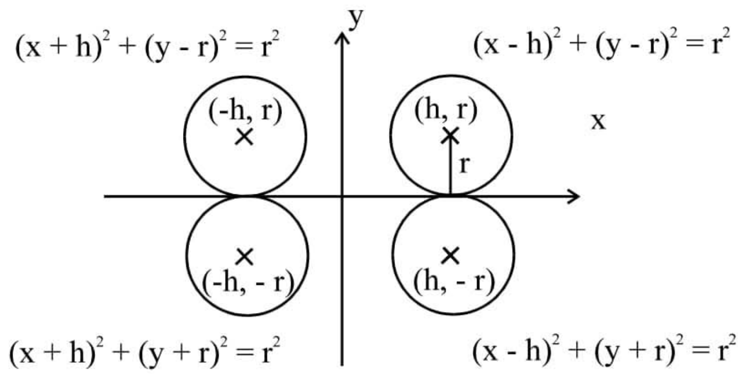



Combined equation is

$$(x\pm r)^2+y^2=r^2$$

$$x^2+y^2\pm 2rx=0$$

**Equation of circle touching x -axis**

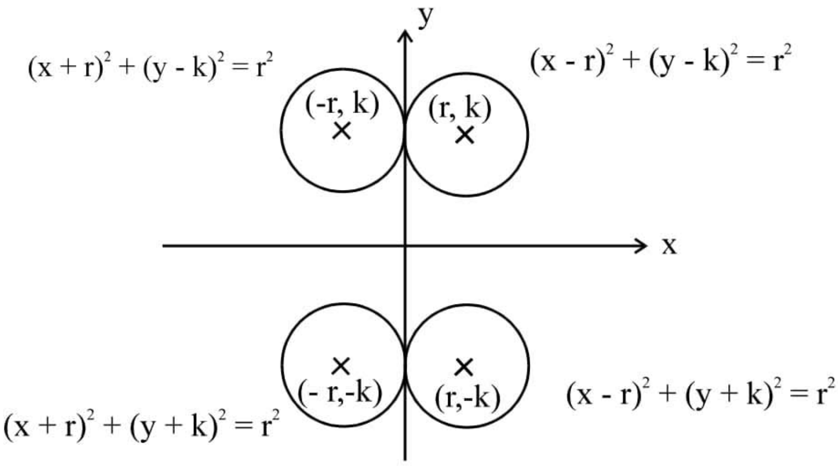



Combined equation is

$$(x\pm h)^2+(y\pm r)^2=r^2$$

$$x^2+y^2\pm 2hx\pm 2ry+h^2=0$$

## Equation of circle touching y - axis

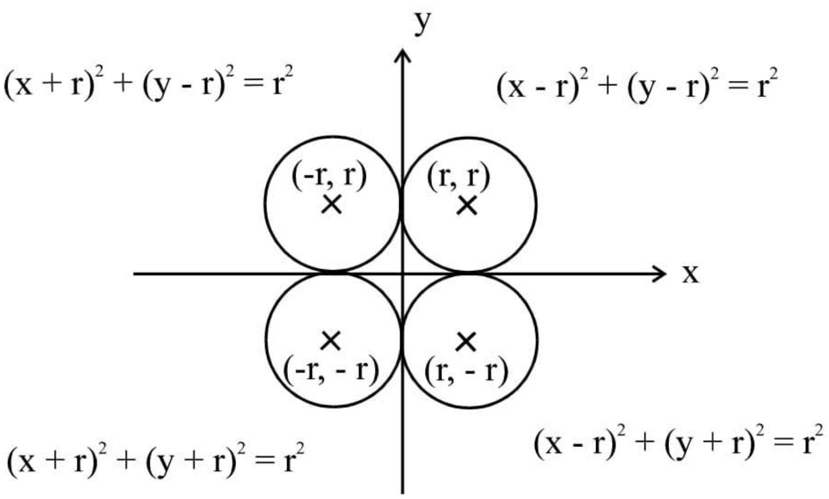



Combined equation is

$$(x \pm r)^2 + (y \pm k)^2 = r^2$$

$$x^2 + y^2 \pm 2rx \pm 2ky + k^2 = 0$$

## Equation of circle touching both axes

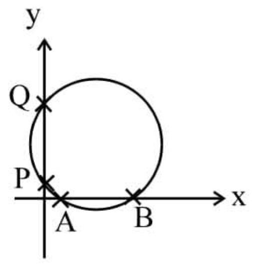



Combined equation is

$$(x \pm r)^2 + (y \pm r)^2 = r^2$$

## Intercept made by a circle on the axes

Consider the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

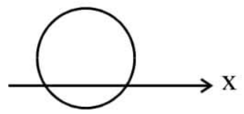



AB = x intercept = $2\sqrt{g^2 - c}$

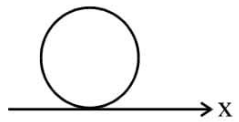
PQ = y intercept = $2\sqrt{f^2 - c}$

**Note :**

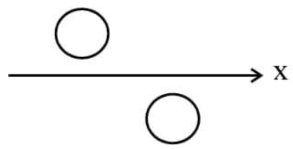
If $g^2 - c > 0$ then the circle cut x-axis at two points




If $g^2 - c = 0$ then the circle touch x- axis at one point




If $g^2 - c < 0$ then there is no contact between the circle and x-axis




## Similar relation between the circle and y-axis

### Equation of a circle passing through three points $(x_1y_1)(x_2y_2)(x_3y_3)$

First write the general equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \rightarrow (1)$$

Put $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ in (1)

we get three equations

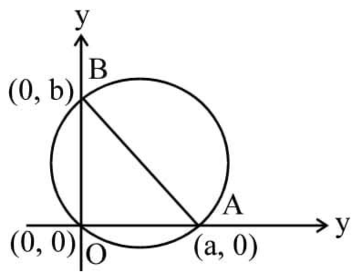
$$x_1^2 + y_1^2 + 2gx_1 + 2f\,y_1 + c = 0 \quad \rightarrow (2)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2f\,y_2 + c = 0 \quad \rightarrow (3)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2f\,y_3 + c = 0 \quad \rightarrow (4)$$

Solving (2), (3) and (4) we get g, f, c

**Circle passing through three points (0, 0), (a, 0) and (0, b)**




$\angle AOB$ is $90^0$ $\therefore$ AB is a diameter

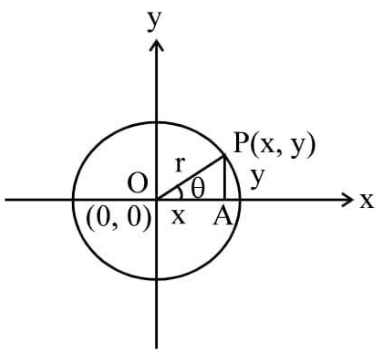
$E_2$ of the circle is

(x - a) (x - 0) + (y - 0) (y - b) = 0

$$x^2 + y^2 - ax - by = 0$$

## Parametric equation of a circle

Consider the circle $x^2 + y^2 = r^2$




OA = x, PA = y

$$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$$

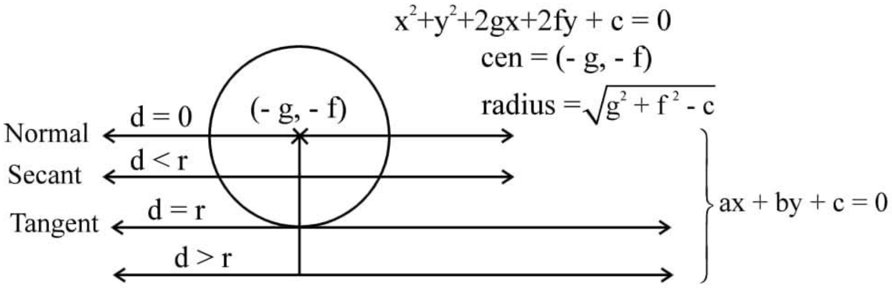
$$x = r\cos\theta,\ y = r\sin\theta$$

$\theta \in [0, 2\pi]$ is called parametric equation of a circle

Consider the circle $(x-h)^2 + (y-k)^2 = r^2$

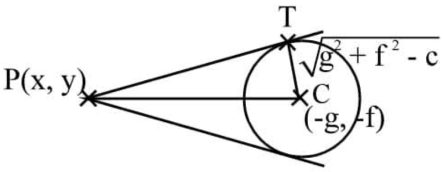
Its parametric equation is $x = h + r\cos\theta, y = k + r\sin\theta,\ \theta \in [0, 2\pi]$

**Relation between a circle and a line**




$$d = \left|\frac{a(-g) + b(-f) + c}{\sqrt{a^2 + b^2}}\right|$$

**Length of the tangents**




$$CT = \sqrt{g^2 + f^2 - c} = r$$

$$PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

**Power of the point**

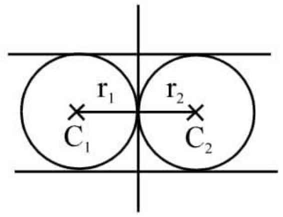
$PT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is called the power of the point $(x_1y_1)$ w.r.t the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. It is denoted by $S_1$.

ie, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore$ Length of the tangent = PT = $\sqrt{S_1}$

**Relation between two circles and number of common tangents**

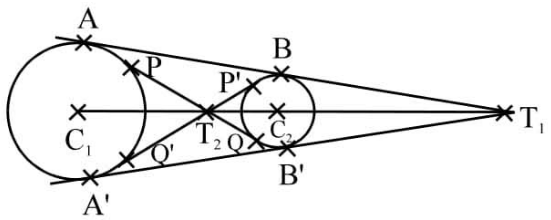
**1) The circles touch externally**




$|C_1C_2| = |r_1 + r_2|$

Number of common tangent is 3

**2) The circles do not touch externally**




$|C_1C_2| > |r_1 + r_2|$

The number of common tangents = 4

$T_1$ = point of intersection of external common tangent.

$T_1$ divides $C_1C_2$ in the ratio $r_1 : r_2$ externally

$T_2$ = point of intersection of internal common tangent

$T_2$ divides $C_1C_2$ in the ratio $r_1 : r_2$ internally

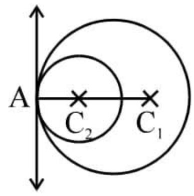
Length of internal common tangent, $PQ = \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2} = P'Q'$

Length of external common tangent, $AB = \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2} = A'B'$

Shortest distance between two circles = $C_1C_2 - (r_1 + r_2)$

Maximum distance between two circles = $C_1C_2 + (r_1 + r_2)$

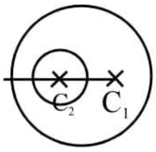
**The circles touch internally**




$|C_1C_2| = |r_1 - r_2|$

Number of common tangent = 1

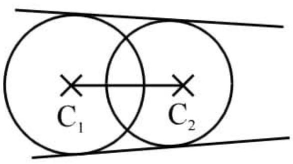
**The circles donot touch internally**




$|C_1C_2| < |r_1 - r_2|$

The number of common tangent = 0

**The circles are intersecting**

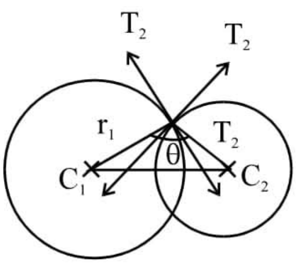



$|r_1 - r_2| < |C_1C_2| < |r_1 + r_2|$

The number of common tangent = 2

**Angle between two circles**

Angle between two circles is the angle between their tangents at their point of intersection and angle between two tangents is same as angle between their radii.



$$\cos\theta = \left|\frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2}\right|$$

**Orthogonal circles**

When $\theta = 90^0$, the circles are orthogonal condition for orthogonality is

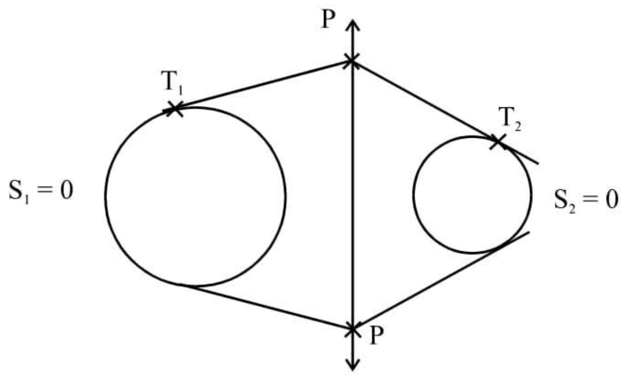
$$(C_1C_2)^2 = r_1^2 + r_2^2$$

If the circles are $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

orthogonal then $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

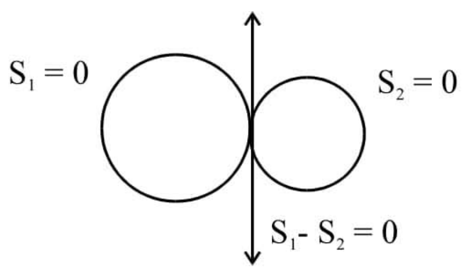
**Radical axis of two circles**

Radical axis the locus of a point P such that $PT_1 = PT_2$.



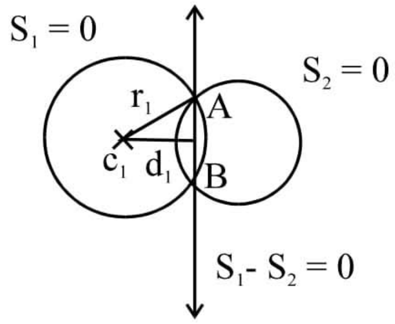

The line passing through P is called radical axis its equation is $S_1 - S_2 = 0$.

If the circles are touch externally then radical axis become common tangent.



**Common chord**

If two circles are intersecting then the radical axis become the common chord.

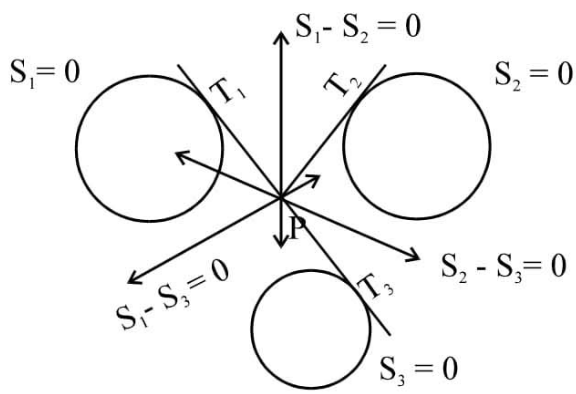



Length of the common chord = $2\sqrt{r_1^2 - d_1^2}$

**Radical centre**

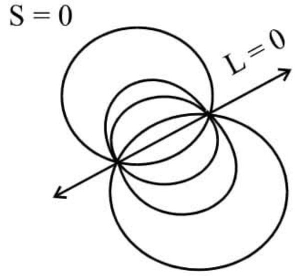
Radical centre is the point of concurrence of three radical axis of three circles taken two at a time.

If P is the radical centre then $PT_1 = PT_2 = PT_3$




**Family of circles**

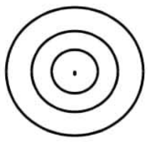
1) Equation of the family of circles passing through the intersection of the circle $S_1 = 0$ and the line $L = 0$ are $S + \lambda L = 0, \lambda \in R$



2) Equation of the circles passing through the intersection of the circle $S_1 = 0$ and $S_2 = 0$ are $S_1 + \lambda S_2 = 0$ or $S_1 + \lambda(S_1 - S_2) = 0$

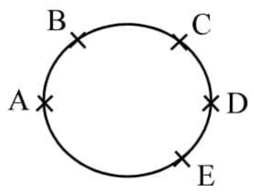
**Concentric circles**

Circles with **same centre** but different radii are called **concentric circles.**



**Concyclic points**

If four or more points lie on the same circle is known as concyclic points.




A, B, C, D, E are called concyclic points.

**Equation of tangents in different form**

**1) Equation of the tangent when slope of the tangent (m) is given and the circle is given**

Let the equation of the circle is $x^2 + y^2 = r^2$ and the equation of the tangents is $y = mx + c$

Condition for tangency is $c^2 = r^2(1 + m^2)$

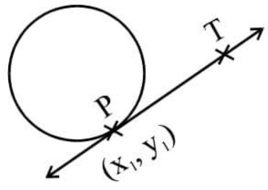
Equation of the tangents are $y = mx \pm r\sqrt{1 + m^2}$

If the circle is $(x - h)^2 + (y - k)^2 = r^2$ then equation of the tangents are

$$y - k = m(x - h) \pm r\sqrt{1 + m^2}$$

2) **Equation of the tangent at a point on the given circle**

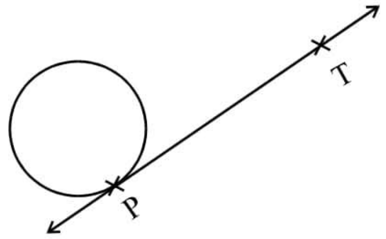
Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 = r^2$



Equation of PT is $T_1 = 0$

$$xx_1 + yy_1 - r^2 = 0$$

Let $P(x_1, y_1)$ be a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$



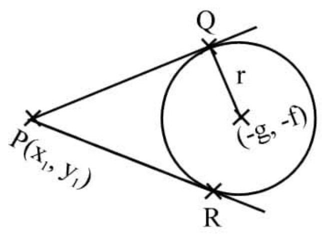

$$x^2 \to xx_1$$

$$y^2 \to yy_1$$

$$x \to \frac{x + x_1}{2}, \quad y \to \frac{y + y_1}{2}$$

Equation of PT is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

**3)** **<u>Equation of the tangent from a point outside of the circle</u>**



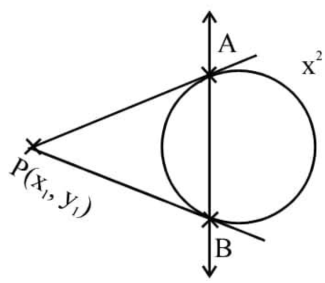

$x^2 + y^2 + 2gx + 2fy + c = 0$

Let the equation of PQ or PR is

$y - y_1 = m(x - x_1)$ convert into the general form $ax + by + c = 0$

Using distance from centre to tangent = r, we can find the value of m

**<u>Chord of contact</u>**

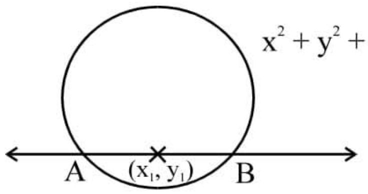



$x^2 + y^2 + 2gx + 2fy + c = 0$

The line AB is called chord of contact of the circle w.r.t the point $P(x_1, y_1)$ its equation is $T_1 = 0$.

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

**Equation of the chord whose midpoint is given**



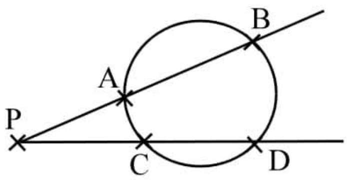

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Equation of the chord AB is $T_1 = S_1$

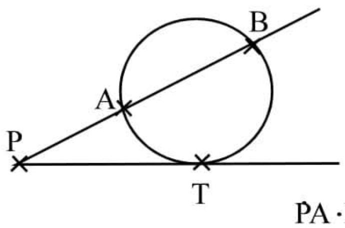
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Simplify in the form $ax + by + c = 0$

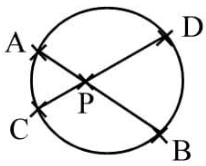
**Intersecting chord theorm**




$$PA \cdot PB = PC \cdot PD$$




$$PA \cdot PB = PT^2$$




$$PA \cdot PB = PC \cdot PD$$