Binomial theorem for positive integral index

If a and b are real/ complex numbers/ expressions, then for all $n \in N$

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots\cdots + {}^nC_r a^{n-r}b^r + \cdots\cdots + {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n$$

$$= \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r ;$$

where ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$; ${}^nC_0 = {}^nC_n = 1$.

${}^nC_0,\ {}^nC_1, \cdots\cdots, {}^nC_n$ are called binomial coefficients.

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

and so on.

General term in the expansion of $(a+b)^n$

The $(r+1)^{th}$ term, denoted by T_{r+1}, is $T_{r+1} = {}^nC_r a^{n-r} b^r$, $r = 0, 1, 2, \cdots\cdots, n$.

Properties of Binomial coefficients

i. ${}^nC_0 = {}^nC_n = 1$

ii. ${}^nC_r = {}^nC_{n-r}$

iii. ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

iv. Greatest Binomial coefficient is ${}^nC_{\frac{n}{2}}$ if n is even; ${}^nC_{\frac{n-1}{2}}$ & ${}^nC_{\frac{n+1}{2}}$ if n is odd.

- $C_0 + C_1 + C_2 + \text{----} + C_n = 2^n$
- $C_0 - C_1 + C_2 - C_3 + \text{---} = 0$
- $C_0 + C_2 + C_4 + C_6 + \text{---} = C_1 + C_3 + C_5 + \text{---} = 2^{n-1}$

Characteristics of Binomial theorem

i. Total number of terms in the binomial expansion is $n+1$.

ii. Sum of the indices of a and b in each term is n.

iii. Since ${}^nC_r = {}^nC_{n-r}$, the coefficient of terms equidistant from the beginning and the end are equal.

iv. The coefficient of $(r+1)^{th}$ term in the expansion is ${}^nC_r$.

Middle term

Since the binomial expansion of $(a+b)^n$ has $(n+1)$ terms, so $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term if n is even; $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms is the middle term if n is odd.

Binomial expansion of $(1+x)^n$

i. $(1+x)^n = {}^nC_0 a^n + {}^nC_1 x + {}^nC_2 x^2 + \cdots\cdots + {}^nC_r x^r + \cdots\cdots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$

$$= \sum_{r=0}^{n} {}^nC_r x^r$$

ii. General term $T_{r+1} = {}^nC_r x^r$, $r = 0, 1, 2, \cdots\cdots, n$.

Some important expansions

i. $(a+b)^n + (a-b)^n = 2\left({}^nC_0 a^n + {}^nC_2 a^{n-2} b^2 + \cdots\cdots\right)$

ii. $(a+b)^n - (a-b)^n = 2\left({}^nC_1 a^{n-1} b + {}^nC_3 a^{n-3} b^3 + \cdots\cdots\right)$

iii. $(1+x)^n + (1-x)^n = 2\left({}^nC_0 + {}^nC_2 x^2 + \cdots\cdots\right)$

iv. $(1+x)^n - (1-x)^n = 2\left({}^nC_1 x + {}^nC_3 x^3 + \cdots\cdots\right)$

- **Multinomial theorem for positive integral index**

  If $x_1, x_2, \cdots\cdots, x_k$ are real/ complex numbers/ expressions, then for all $n \in N$

  $(x_1 + x_2 + x_3 + \cdots\cdots + x_k)^n = \sum \frac{n!}{r_1! r_2! r_3! \cdots\cdots r_k!} x_1^{r_1} x_2^{r_2} \cdots\cdots x_k^{r_k}$, where $r_1, r_2, \cdots\cdots, r_k$ are non-negative

  No. of terms in the expansion of $(a + b)^n$ is $(n + 1)$ and number of terms in the expansion of $(a + b + c)^n$ is $\frac{(n+1)(n+2)}{2}$

- For all rational values of n

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 +$$

$$- - - - + \frac{n(n-1)(n-2) - - - - (n-r+1)x^r}{r!} + - - -$$

Note: If n is not a positive integer; the expansion is an infinite series and valid only if $|x| < 1$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots\dots\dots \infty$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots\dots \infty$
- $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots\dots\dots \infty$

- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + --\infty$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + ---\infty$
- $(1 - x)^{-3} = 1 + \frac{2.3}{2}x + \frac{3.4}{2}x^2 + \frac{4.5}{2}x^3 + - - - -$