CHAPTER - 08
CIRCLES

JEE MAIN - SECTION I

1. A Centre (5,5) and radius = 5

2. D Let cent $(h, 2)$ $r = h$

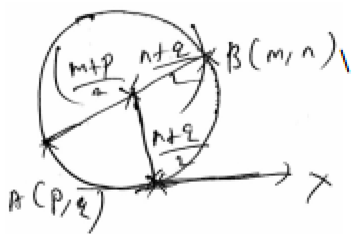
$$(h+1)^2 + (0-2)^2 = h^2$$

$$h^2 + 24 + 1 + 4 = h^2$$

$$h = \frac{-5}{2}$$

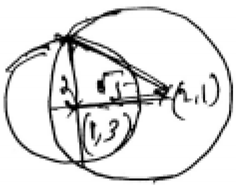
Eq of the circle is $\left(x+\frac{5}{2}\right)^2 + \left(y-2^2\right) = \frac{25}{4}$

3. D



$$(m-p)^2 + (n-q)^2 = (n+q)^2$$

4. A



$$r - \sqrt{(5)^2 + 2^2} = 3$$

5. B The circum circle passes through the centre of the given circle

$\therefore (1,8)$ and $(3,2)$ are the ends point of the diameter

6. B $x^2+y^2+3x-6y-9=0$; $c=\left(\frac{-3}{2},3\right)$ $r=\sqrt{\frac{9}{4}+9+9}=\frac{9}{2}$

locus is circle with centre $\left(\frac{-3}{2},2\right)$ $r=\frac{9}{2}+2=\frac{13}{2}$

7. 4 Let the centre be (h, k), then radius = h

Also $CC_1 = R_1 + R_2$ or $\sqrt{(h-3)^2+(k-3)^2} = h+\sqrt{9+9-14}$

$\Rightarrow (h-3)^2+(k-3)^2 = h^2+4+4h$

$\Rightarrow k^2-10h-6k+14=0$ or $y^2-10x-6y+14=0$.

8. 2 Suppose (x_1, y_1) be any point on first circle from which tangent is to be drawn, then

$x_1^2+y_1^2+2gx_1+2fy_1+c_1=0$ ....(i)

and also length of tangent

$=\sqrt{S_2}=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$ ....(ii)

From (i), we get (ii) as $\sqrt{c-c_1}$.

9. 3

Here, $g_1=\frac{k}{2}, f_1=2, c_1=2$

$g_2=-1, f_2=\frac{-3}{4}, c_2=\frac{k}{2}$

Condition for orthogonal intersection,

$\Rightarrow 2(g_1g_2+f_1f_2)=c_1+c_2$

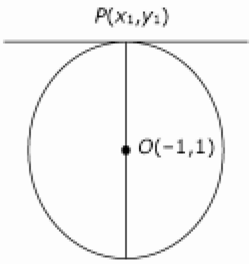
$\Rightarrow 2\left[\frac{-k}{2}+\left(\frac{-3}{2}\right)\right]=2+\frac{k}{2}$

$\Rightarrow -k-3=2+\frac{k}{2} \Rightarrow \frac{3k}{2}=-5;\ k=\frac{-10}{3}$.

10. 2

Let point of contact be $P(x_1, y_1)$.
This point lies on line
$x_1 + 2y_1 = -12$ ....(i)

Gradient of OP $= m_1 = \dfrac{y_1 - 1}{x_1 + 1}$



Gradient of $x + 2y + 12 = m_2 = -\dfrac{1}{2}$

The two lines are perpendicular,
$\therefore m_1 m_2 = -1$

$\Rightarrow \left(\dfrac{y_1 - 1}{x_1 + 1}\right)\left(\dfrac{-1}{2}\right) = -1 \Rightarrow y_1 - 1 = 2x_1 + 2$

$\Rightarrow 2x_1 - y_1 = -3$ ....(ii)

On solving equation (i) and (ii), we get

$(x_1, y_1) = \left(\dfrac{-18}{5}, \dfrac{-21}{5}\right)$.

11. 3

Let any point on the circle $x^2 + y^2 = a^2$ be $(a\cos t, a\sin t)$ and $\angle OPQ = \theta$

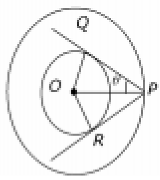
Now; PQ = length of tangent from P on the circle $x^2 + y^2 = a^2 \sin^2\alpha$

$\therefore PQ = \sqrt{a^2\cos^2 t + a^2\sin^2 t - a^2\sin^2\alpha} = a\cos\alpha$

OQ = Radius of the circle $x^2 + y^2 = a^2\sin^2\alpha$

$OQ = a\sin\alpha$, $\therefore \tan\theta = \dfrac{OQ}{PQ} = \tan\alpha \Rightarrow \theta = \alpha$

$\therefore$ Angle between tangents $= \angle QPR = 2\alpha$.



12. 2

The equation of required circle is $S_1 + \lambda S_2 = 0$.

$\Rightarrow x^2(1+\lambda) + y^2(1+\lambda) + x(2+13\lambda) - y\left(\dfrac{7}{2} + 3\lambda\right) - \dfrac{25}{2} = 0$

Centre $= \left(\dfrac{-(2+13\lambda)}{2}, \dfrac{\frac{7}{2} + 3\lambda}{2}\right)$

$\because$ Centre lies on $13x + 30y = 0$

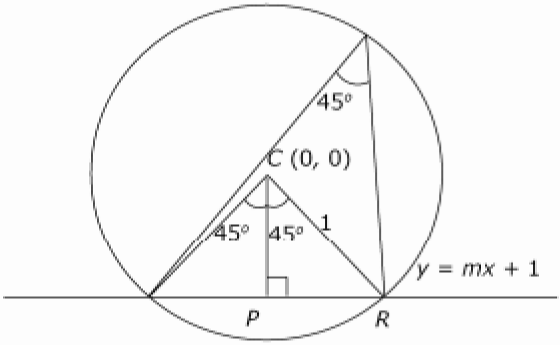
$\Rightarrow -13\left(\dfrac{2+13\lambda}{2}\right) + 30\left(\dfrac{\frac{7}{2} + 3\lambda}{2}\right) = 0 \Rightarrow \lambda = 1.$

Hence the equation of required circle is
$4x^2 + 4y^2 + 30x - 13y - 25 = 0.$

13. 3

Given circle is $x^2 + y^2 = 1$

$C(0,0)$ and radius = 1 and chord is $y = mx + 1$, $\cos 45^o = \frac{CP}{CR}$



CP = Perpendicular distance from (0,0) to chord $y = mx + 1$

$CP = \frac{1}{\sqrt{m^2+1}}$ (CR = radius = 1)

$\cos 45° = \frac{1/\sqrt{m^2+1}}{1} \Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{m^2+1}}$

$m^2 + 1 = 2 \Rightarrow m = \pm 1.$

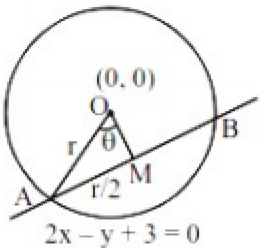
14. 2

Let chord AB = r

$\because$ ΔAOM is right angled triangle

$\therefore$ OM = $\frac{r\sqrt{3}}{2}$ = perpendicular distance of line AB from (0, 0)

$\frac{r\sqrt{3}}{2} = \left|\frac{3}{\sqrt{5}}\right|$, $r^2 = \frac{12}{5}$.



15. 2

Slope of tangent to $x^2+y^2=1$ at $P\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

$2x+2yy'=0 \Rightarrow m_T|_P=-1$

$y=mx+c$ is tangent to $(x-3)^2+y^2=1$

$y=x+c$ is tangent to $(x-3)^2+y^2=1$

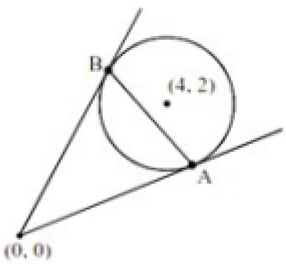
$\left|\frac{c+3}{\sqrt{2}}\right|=1 \Rightarrow c^2+6c+7=0$

16. 4

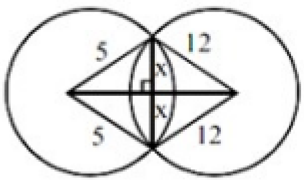
$R=\sqrt{16+4-16}=2,\ L=\sqrt{S_1}=4$

AB(Chord of contact) = $\frac{2LR}{\sqrt{L^2+R^2}}=\frac{8}{\sqrt{5}}$

$(AB)^2=\frac{64}{5}$



17. 2



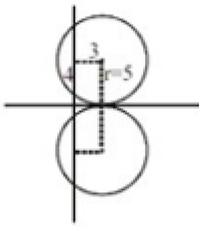
Let length of common chord = $2x$

$\sqrt{25-x^2}+\sqrt{144-x^2}=13$ after solving

$x=\frac{12\times 5}{13},\ 2x=\frac{120}{13}$

18. 1

Equation of circles are $\begin{cases}(x-3)^2+(y-5)^2=25\\(x-3)^2+(y+5)^2=25\end{cases}$



$\Rightarrow \begin{cases}x^2+y^2-6x-10y+9=0\\x^2+y^2-6x+10y+9=0\end{cases}$

19. 1

Centre of circles are opposite side of line $(3+4-\lambda)(27+4-\lambda)<0$

$(\lambda-7)(\lambda-31)<0$

$\lambda\in(7,31)$ distance from $S_1 = \left|\frac{3+4-\lambda}{5}\right|\geq 1$

$\Rightarrow\ \lambda\in(-\infty,2]\cup(12,\infty]$

Distance from $S_2=\left|\frac{27+4-\lambda}{5}\right|\geq 2$

$\Rightarrow\ \lambda\in(-\infty,21]\cup[41,\infty)$

So, $\lambda\in[12,21]$

20. 1

Clearly, $P(\sqrt{2},\sqrt{6})$ lies on $x^2+y^2=8$, which is director circle of $x^2+y^2=4$

Therefore, tangents PA and PB are perpendicular to each other

So, OAPB is a square,

Hence, area of OAPB= $(\sqrt{S_1})^2=S_1$

$=(\sqrt{2})^2+(\sqrt{6})^2-4=4$

$\therefore$ Both statements are true and statement II is correct explanation of statement-I

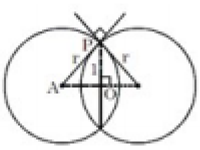
SECTION II (NUMERICAL)

21. 2

In ΔAPO, $\left(\frac{\sqrt{2}r}{2}\right)^2+1^2=r^2$

$\Rightarrow\ r=\sqrt{2}$

So, distance between centres = $\sqrt{2}r=2$



22. 3

23. 7

Let $P(3\cos\theta, 3\sin\theta)$, $Q(-3\cos\theta, -3\sin\theta)$

$$\Rightarrow \alpha\beta = \frac{|(3\cos\theta+3\sin\theta)^2-4|}{2}$$

$$\Rightarrow \alpha\beta = \frac{5+9\sin 2\theta}{2} \le 7.$$

24. 9

Circle $x^2+y^2-2x-4y+4=0$

$\Rightarrow (x-1)^2+(y-2)^2=1$

Centre: (1, 2), Radius = 1

Line $3x+4y-k=0$ intersects the circle at two distinct points.

$\Rightarrow$ distance of centre from the line < radius

$\Rightarrow \left|\frac{3\times 1+4\times 2-k}{\sqrt{3^2+4^2}}\right| < 1 \Rightarrow |11-k| < 5$

$\Rightarrow 6 < k < 16 \Rightarrow k \in \{7, 8, 9, ..., 15\}$ since $k \in I$

Number of k is 9.

25. 23

$2g_2(g_1-g_2)+2f_2(f_1-f_2)=c_1-c_2$

$2(1)(3-1)+2(-3)(-1+3)=k+15$

$4-12=k+15$ or $-8=k+15 \Rightarrow |k|=23$.

JEE ADVANCED LEVEL

SECTION III

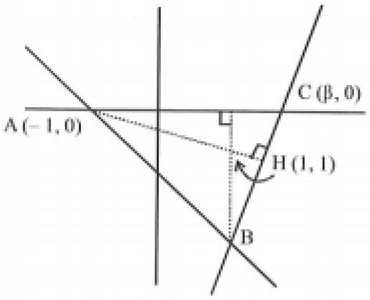
26. D

The given expression can be written as

$$6\left(l^2+m^2\right)=9l^2+6l+1 \text{ i.e. } \frac{3l+1}{\sqrt{l^2+m^2}}=\sqrt{6}.$$

From this expression we can infer that the perpendicular distance of the point (3, 0) from the line $lx+my+1=0$ is $\sqrt{6}$. This proves that the given line is a tangent to the circle $(x-3)^2+y^2=6$.

27. B



$(1, -2) = (\alpha, -\alpha - 1)$
$\Rightarrow \alpha = 1$

It is clear from question that one of the vertex of triangle is intersection of x-axis and
$x + y + 1 = 0 \Rightarrow A(-1, 0)$
Let vertex B be $(\alpha, -\alpha - 1)$
Line AC $\perp$ BH so, $m_{AC} \cdot m_{BH} = -1$

$\Rightarrow \ O = -\dfrac{(1-\alpha)}{\alpha+2} \Rightarrow \alpha = 1 \Rightarrow B(1, -2)$

Let vertex C be $(\beta, 0)$
Line AH $\perp$ BC
$\therefore \ m_{AH} \cdot m_{BC} = -1$

$\Rightarrow \ \dfrac{1}{2} \cdot \dfrac{2}{\beta - 1} = -1 \Rightarrow \ \beta = 0$

Centroid of ΔABC is $\left(0, -\dfrac{2}{3}\right)$

We know that G (centroid) divides line joining circumcentre (O) and orthocentre (H) in the ratio 1 : 2.

$$\Rightarrow \ \underset{O}{(h, k)} \xrightarrow{\ 1\ } \underset{G}{\left(0, -\tfrac{2}{3}\right)} \xrightarrow{\ 2\ } \underset{H}{(1, 1)}$$

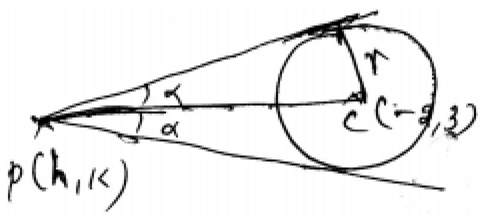
$2h + 1 = 0 \Rightarrow \dfrac{2k+1}{3} = -\dfrac{2}{3}$

$\Rightarrow \ h = -\dfrac{1}{2} \Rightarrow k = -\dfrac{3}{2}$

$\Rightarrow$ Circumcentre is $\left(-\dfrac{1}{2}, -\dfrac{3}{2}\right)$.

Equation of circum circle is (passing through C (0, 0)) is $x^2 + y^2 + x + 3y = 0$

28. B



$$r = \sqrt{4 + 9 - 9\sin\alpha - 13\cos^2\alpha} = 2\sin\alpha$$

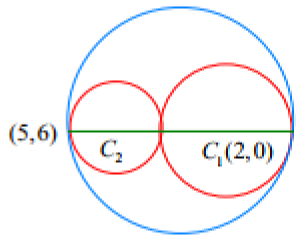
$\sin\alpha = \dfrac{2\sin\alpha}{pc}$; $pc = 2$; $(pc)^2 = 4$

29. C Length of the transversal common tangent $= \sqrt{d^2 - (r_1 + r_2)^2}$

30. B Given circle is $(x-2)^2 + y^2 = 4$

Centre is (2, 0) and radius = 2

Therefore, distance between (2,0) and (5,6) is

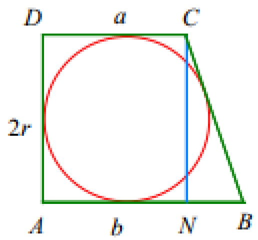


$\sqrt{9+36} = 3\sqrt{5} \Rightarrow r_1 = \dfrac{3\sqrt{5}-2}{2}$

and $r_2 = \dfrac{3\sqrt{5}+2}{2} = r_1 r_2 = \dfrac{41}{4}$

SECTION IV (More than one correct)

31. B,C $DC + AB = AD + CB \Rightarrow CB = a + b - 2r$



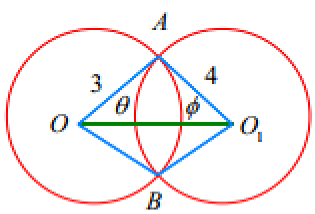
The triangle CNB gives

$(2r)^2 + (b-a)^2 = (a+b-2r)^2$

32. B,D The two circles are orthogonal.

$\therefore \angle OAO_1 = \dfrac{\pi}{2}, O_1O = 5$

The common area = Area of sector $AOB +$ Area of sector $AO_1B -$ Area of the kite OAO_1B.



Let $\angle AOO_1 = \theta, \angle AO_1O = \phi = \dfrac{\pi}{2} - \theta$

Area $= \dfrac{9}{2}.2\theta + \dfrac{16}{2}.2\phi - 3\times 4$

$= 9\theta + 16\phi - 12 = 9\theta + 16\left(\dfrac{\pi}{2} - \theta\right) - 12$

$$= 8\pi - 7\theta - 12 = 8\pi - 12 - 7\tan^{-1}\frac{4}{3}$$

$$= 8\pi - 12 - 7\left(\frac{\pi}{2} - \tan^{-1}\frac{3}{4}\right) = \frac{9\pi}{2} - 12 + 7\tan^{-1}\frac{3}{4}.$$

33. A,C

$a > 2 \quad b > 2$

$\frac{1}{a} > \frac{1}{2} \quad \frac{1}{b} > \frac{1}{2}$

$\frac{1}{a} + \frac{1}{b} < 1$

Equation of AB is $\frac{x}{a} + \frac{y}{b} = 1$

$\left|\frac{\frac{1}{a} + \frac{1}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = 1$; $\frac{1}{a} + \frac{1}{b} - 1 < 0$

$\frac{1}{a} + \frac{1}{b} - 1 = -\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

34. A,B

When two circles intersect, the common chord AB of maximum length will be the diameter of smaller circle.

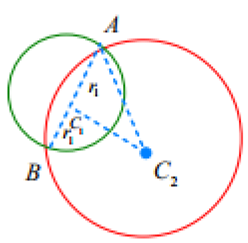
$S_1 : x^2 + y^2 = 16, C_1(0,0), r_1 = 4$

$S_2 : (x-h)^2 + (y-k)^2 = 5^2, C_2(h,k), r_2 = 5$, then S_1 is smaller circle.

$r_2^2 = r_1^2 + (C_1C_2)^2 \Rightarrow S^2 = 4^2 + (h^2 + k^2)$

$h^2 + k^2 = 3^2$

Slope of $AB = m_1 = \frac{3}{4}$ (gien);



Slope of $C_1C_2 = m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{k-0}{h-0} = \frac{k}{h}$

$C_1C_2 \perp AB; m_1m_2 = -1; \frac{k}{h}\left(\frac{3}{4}\right) = -1; 3k = -4h$

$\frac{h}{-3} = \frac{k}{4} = t \quad h = -3t, \quad k = 4t(2)$

Put (2) in (1), $t^2(3^2 + 4^2) = 3^2; t = \pm\frac{3}{5}$

If $t = \frac{3}{5}$, then (2); $h = -\frac{9}{5}, k = \frac{12}{5}$;

$C_2(h,k)=\left(-\frac{9}{5},\frac{12}{5}\right)$ If $t=\frac{3}{5}$, then (2) $h=\frac{9}{5},k=-\frac{12}{5}$

$C_2\left(\frac{9}{5},-\frac{12}{5}\right)$

35. A,D Note that if side of a square is x then radius of the inscribed circle must be $\frac{x}{2}$ and if the radius of the circle is R then side of the square inscribed is $R\sqrt{2}$.

Now $a_n=\pi\left(r_1^2+r_2^2+r_3^2+.....\right)$

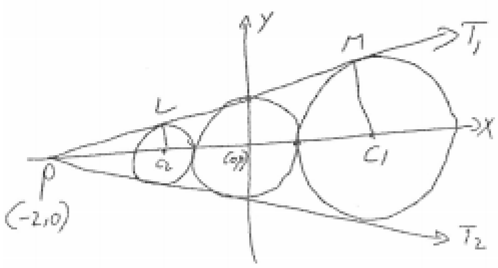
$=\pi\left[\left(\frac{1}{2}\right)^2+\left(\frac{1}{4}\sqrt{2}\right)^2+\left(\frac{1}{8}(\sqrt{2})(\sqrt{2})\right)^2+.....\right]$

36. A,B,C,D

The given circle is $x^2+y^2=1\,(1)$

Centre O (0,0) & radius =1

Let T_1 & T_2 be the targents drawn from (-2,0) to the circle (1)



Let m be the slope of targent then equations of targents are $y-0=m(m+2)$

$r_1=5;\ r_2=\sqrt{15};C_1C_2=\sqrt{40}$

$\Rightarrow\quad r_1+r_2>C_1C_2>r_1-r_2$

Hence,circles intersect in two distinct points

There are two common tangents

Also $2g_1g_2+2f_1f_2=2(1)(3)+2(2)(-4)=-10$

and $\quad c_1+c_2=-20+10=-10$

Thus, two circle are orthogonal

Length of common chord is $\dfrac{2r_1r_2}{\sqrt{r_1^2+r_2^2}}=5\sqrt{\dfrac{3}{2}}$

Length of common tangent is $\sqrt{C_1C_2^3-(r_1-r_2)}=5\left(\dfrac{12}{5}\right)^{\frac{1}{4}}$

Now any given .... is such that its centre lies on x-axis

Let (h,o) be the centre of such circle, then from fig.

$OC_1=OA+AC_1 \Rightarrow |h|=1+AC_1$

But AC_1 - ...... dist of (h,o) to tgt

$|h|=1+\left|\dfrac{h+2}{2}\right| \Rightarrow |h|-1=\left|\dfrac{h+2}{2}\right|$

squaring, $h^2-2|h|+1=\dfrac{h^2+4h+4}{4}$; $h=4$ or $\dfrac{-4}{3}$

Thus, centres of circles are $(4,0),\left(\dfrac{-4}{3},0\right)$

$\therefore$ radius of circle with centre

$\left(\dfrac{-4}{3},0\right)=\dfrac{4}{3}-1=\dfrac{1}{3}$

$\therefore$ Two possible circles are

$(x-4)^2+y^2=3^2$ & $\left(x+\dfrac{4}{3}\right)^2+y^2=\left(\dfrac{1}{2}\right)^2$

37. B,C Distance of line $x+y-1=0$ from centre

$\left(\dfrac{1}{2},\dfrac{-1}{2}\right)$ is $\dfrac{\left|\dfrac{1}{2}-\dfrac{3}{2}-1\right|}{\sqrt{2}}=\sqrt{2}$

Let req.d line by y - mx =0

Dist of $\left(\frac{1}{2},\frac{-3}{2}\right)$ from $y-mx=0$ should also be $\sqrt{2}$

$$\frac{\left|\frac{-3}{2}-\frac{m}{2}\right|}{\sqrt{1+m^2}}=\sqrt{2}\Rightarrow m=1,\frac{-1}{7}\Rightarrow x-y=0 \quad \text{or } x+7y=0$$

38. A,B,C,D

$r_1=5; r_2=\sqrt{15}; c_1c_2=\sqrt{40}$

$r_1+r_2>c_1c_2>r_1-r_2$

Hence, circles intersect in 2 dist points

There are two common targets

Also $2g_1g_2+2f_1f_2=2(1)(3)+2(2)(-4)=-10$

& $c_1+c_2=-20+10=-10$

Thus, two circles are orthogonal

Length of common chord $\frac{2r_1r_2}{\sqrt{r_1^2+r_2^2}}=5\sqrt{\frac{3}{2}}$

Length of common tgt $=\sqrt{C_1C_2^2-(r_1-r_2)^2}\ ;=5\left(\frac{15}{5}\right)^{\frac{1}{4}}$

39. A,C

Equation of tgt of slope 'm' to $x^2+y^2=1$ is

$y=mx\pm(1)\sqrt{1+m^2}$ (1)

Sine it passes $(0,5)\Rightarrow m=\pm\sqrt{24}$

$\therefore\ y=\pm\sqrt{24}x+5$

$\pm\sqrt{24}\,x-y+5=0$ (2)

Let the target intersect $x^2+y^2=4$ at $P\ \&\ Q$

If tgt at P & Q intersect at (h,k), then chord of certact of (h,k) to

$x^2+y^2=4$ is $\pm\sqrt{24}x-y+5=0$

Also chord of certact of (h,k) w.r.t $x^2+y^2=4$ is

$hx+ky=4\Rightarrow\ hx+ky-4=0$

(2) & (3) are considered

$$\frac{h}{\pm\sqrt{24}}=\frac{k}{-1}=\frac{-4}{5}\Rightarrow(h,k)=\left(\frac{8\sqrt{6}}{5},\frac{4}{5}\right)(\text{or})\left(\frac{-8\sqrt{6}}{5},\frac{4}{5}\right)$$

SECTION VI - (Matrix match type)

40. A-S; B-R; C-Q; D-P

a) $\sqrt{2}(b-a)=(b+a)$

b) $a^2+b^2-4ab=0\Rightarrow\dfrac{b}{a}=2+\sqrt{3}$

c) Radicual axis passes through the centre of small circle

d) conceptual