CHAPTER - 06

RIGID BODY DYNAMICS

SYNOPSIS

Rigid body

- Ideally a rigid body is a body with a perfectly definite and unchanging shape
- The distance between different points of such a body do not change
- No real body is truly rigid

Centre of Mass

- The centre of mass of a body is a point where the whole mass of the body is supposed to be concentrated
- Position vector of the centre of mass

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$$

- Co-ordinates of the centre of mass

$$X_{cm} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}$$

$$Y_{cm} = \frac{m_1y_1 + m_2y_2 + ... + m_ny_n}{m_1 + m_2 + ... + m_n}$$

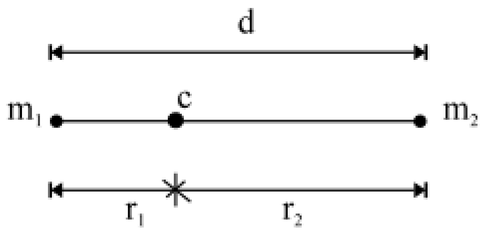
$$Z_{cm} = \frac{m_1z_1 + m_2z_2 + ... + m_nz_n}{m_1 + m_2 + ... + m_n}$$

- The sum of the moments of masses of all the particles of the body about the centre of mass is zero

$$\sum_{i=1}^{n} m_i\vec{r}_i = 0$$

- Centre of mass of a continuous mass distribution $X_{cm} = \frac{\int X dm}{\int dm}$, $Y_{cm} = \frac{\int Y dm}{\int dm}$, $Z_{cm} = \frac{\int Z dm}{\int dm}$

Centre of mass of two particle system



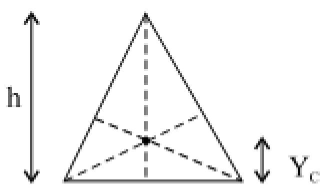
$$r_1 = \left[\frac{m_2}{m_1 + m_2}\right] d$$

$$r_2 = \left[\frac{m_1}{m_1 + m_2}\right] d$$

- The position of centre of mass of a body depends on the shape, size and distribution of mass within the body
- The centre of mass does not lie necessarily within the object
- In symmetrical bodies with homogeneous distribution of mass centre of mass coincides with the geometrical centre
- The centre of mass changes its position in translatory motion but remains unchanged in rotatory motion about an axis through the centre of mass
- The centre of gravity has no relevance where there is no force of gravity, where as the centre of mass is independent of gravitational forces

Centre of mass of some homogeneous bodies

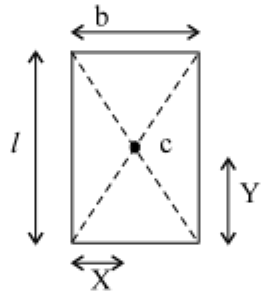
A triangular plate



at the centroid

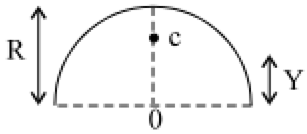
A rectangular plate

A semi-circular ring



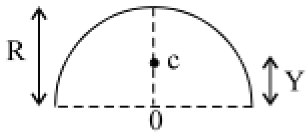
$$X = \frac{b}{2}$$

$$Y = \frac{l}{2}$$

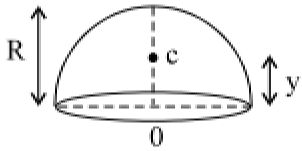


$$Y = \frac{2R}{\pi}$$

A semi-circular disc A hemispherical shell



$$Y = \frac{4R}{3\pi}$$

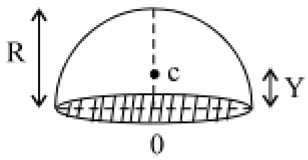


$$Y = \frac{R}{2}$$

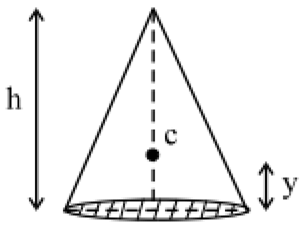
A solid hemisphere

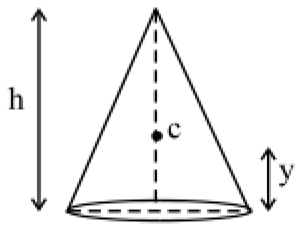
A circular Cone

(Solid) (Hollow)



$$Y = \frac{3R}{8}$$





$$Y = \frac{h}{4}$$

$$Y = \frac{h}{3}$$

Motion of Centre of Mass

- Velocity of centre of mass

$$\vec{V}_{cm} = \frac{m_1\overline{v_1} + m_2\overline{v_2} + ... + m_n\overline{v_n}}{m_1 + m_2 + ... + m_n}$$

- Acceleration of centre of mass

$$\vec{a}_{cm} = \frac{m_1\overline{a_1} + m_2\overline{a_2} + ... + m_n\overline{a_n}}{m_1 + m_2 + ... + m_n}$$

- Total momentum of the system of particles

  $\vec{P} = M\vec{V}_{cm}$

- The centre of mass of the system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.

  $M\vec{a}_{cm} = \vec{F}_{ext}$

- When no external force acts on a body its centre of mass will remain either at rest or move with a constant velocity.

Moment of Inertia

- Moment of inertia gives a measurement of the resistance of a body to a change in its rotational motion.
- Moment of inertia of a body about an axis depends on the mass as well as its distribution about that axis.
- For a single particle of mass m rotating about an axis at a distance r from the axis

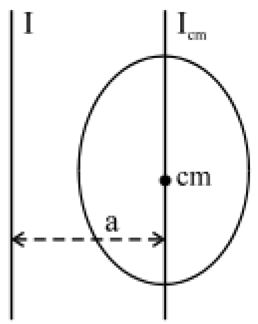
  $I = mr^2$

- For a system of particles about an axis $I = \Sigma mr^2$
- For a rigid body rotating about an axis $I = \int r^2 dm$
- Radius of gyration of a body about an axis may be defined as the distance from the axis to a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

Parallel Axes Theorem

The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between two parallel axes.

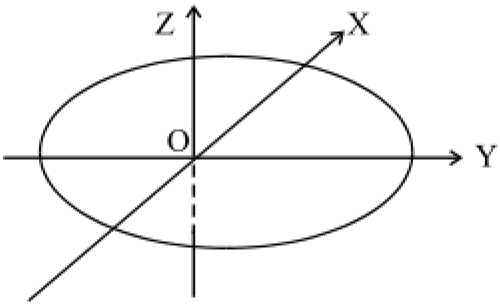
$\boxed{I = I_{cm} + Ma^2}$



Perpendicular Axes Theorem

Moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes cuncurrent with the perpendicular axis and lying in the plane of the lamina.

$$\boxed{I_Z = I_X + I_Y}$$



Moment of inertia of some regular shaped bodies about specific axes

Body	Axis	Moment of Inertia
Thin circular ring of radius R	a) Passing through centre and perpendicular to the plane b) Diameter	MR^2 $\frac{MR^2}{2}$
Circular disc of radius R	a) Passing through the centre and perpendicular to the plane b) Diameter	$\frac{MR^2}{2}$ $\frac{MR^2}{4}$
Uniform thin rod of length L	a) Passing through the mid point and perpendicular to the length b) Passing through one end and perpendicular to the length	$\frac{ML^2}{12}$ $\frac{ML^2}{3}$
Solid sphere of radius R	a) Diameter	$\frac{2}{5}MR^2$
Hollow sphere of radius R	a) Diameter	$\frac{2}{3}MR^2$
Hollow cylinder of radius R and length L	a) Axis of cylinder b) Passing through centre and perpendicular to length	MR^2 $M\left[\frac{L^2}{12}+\frac{R^2}{2}\right]$
Solid cylinder of radius R and length L	a) Axis of the cylinder b) Passing through centre and perpendicular to length	$\frac{MR^2}{2}$ $M\left[\frac{L^2}{12}+\frac{R^2}{4}\right]$

Thin rectangular plate of length l & bredth b	a) Axis passing through centre and perpendicular to its plane	$\frac{M}{12}\left[l^2+b^2\right]$
Thin square plate of side a	a) Axis passing through centre and perpendicular to plane	$\frac{Ma^2}{6}$
Cone of radius R	a) Axis joining vertex to the centre of the base	$\frac{3}{10}MR^2$

Torque

- Torque or moment of force is the rotational analogue of force
- Torque or the moment of force about a point is measured as the product of force and the perpendicular distance from the point to the line of action of the force

  $\tau = rF\sin\theta$; $\vec{\tau} = \vec{r} \times \vec{F}$

- Torque is related to angular acceleration as $\tau = I \propto$

Angular Momentum

- Angular momentum is the rotational analogue of linear momentum
- Angular momentum L = rpsin θ

  $\vec{L} = \vec{r} \times \vec{p}$

- Geometrical meaning of angular momentum $L = 2m\frac{dA}{dt}$
- Angular momentum of a rotating rigid body L = I ω
- Relation between torque and angular momentum $\tau = \frac{dL}{dt}$

Kinematic Equations of Rotational Motion

$$\omega_t = \omega_0 + \propto t \qquad \theta = \omega_0 t + \frac{1}{2} \propto t^2$$

$$\omega_t^2 = \omega_0^2 + 2 \propto \theta$$

Conservation of angular momentum

In the absence of an external torque, the angular momentum of the rotating system is conserved.

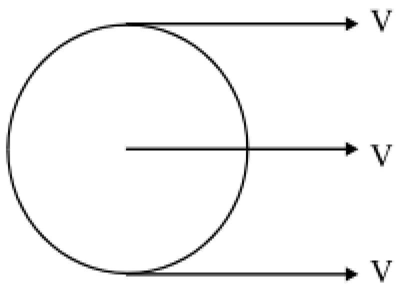
When I = 0 L = I ω = constant

$I_1\omega_1 = I_2\omega_2$

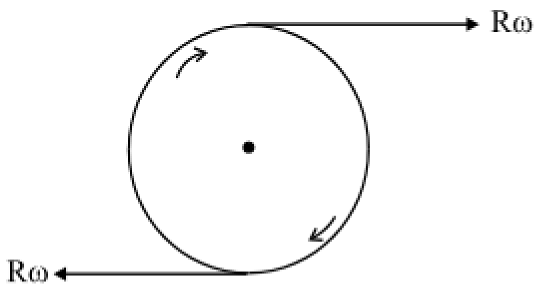
- Kinetic energy of a rotating rigid body KE = $\frac{1}{2}I\omega^2$

Rolling Motion

Rolling motion is the combination of translatory and rotatory motions

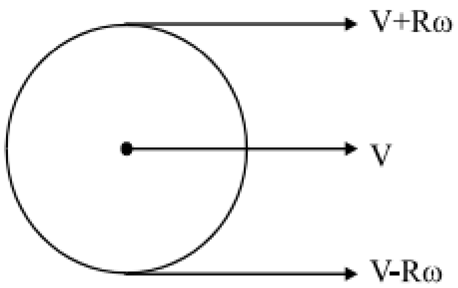


Pure translational motion



Pure rotational motion

$$KE = \frac{1}{2} Iw^2$$



Rolling motion

For pure rolling without slipping V = Rw

Kinetic Energy of Rolling Motion

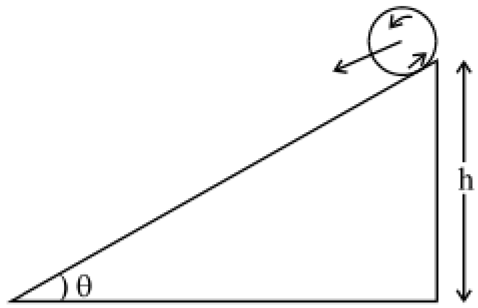
- $KE_{Total} = KE_{Translation} + KE_{Rotation}$

$$KE_{Total} = \frac{1}{2} MV^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} Mv^2 \left[1 + \frac{K^2}{R^2}\right]$$

Motion of a body rolling down without slipping along an inclined plane

- Velocity on reaching the bottom

$$V = \sqrt{\frac{2gh}{\beta}}$$



- Acceleration down the plane a = $\frac{g\sin\theta}{\beta}$

- Time taken to reach the bottom t = $\frac{1}{\sin\theta}\sqrt{\frac{2h\beta}{g}}$

  Where $\beta=1+\frac{I}{MR^2}$ or $\beta=1+\frac{K^2}{R^2}$

- Minimum friction required for pure rolling

  $F_{min}=\frac{MgSin\theta}{1+\frac{MR^2}{I}}$; $\mu_{min}=\frac{\tan\theta}{1+\frac{MR^2}{I}}$

Analogy between translational motion and rotational motion

Translational Motion	Rotational Motion
1. Linear displacement $\vec{r}$	1. Angular displacement $\vec{\theta}$
2. Linear velocity $\vec{V}=\frac{d\vec{r}}{dt}$	2. Angular velocity $\vec{\omega}=\frac{d\vec{\theta}}{dt}$
3. Linear Acceleration $\vec{a}=\frac{d\vec{v}}{dt}$	3. Angular acceleration $\vec{\propto}=\frac{d\vec{\omega}}{dt}$
4. Time t	4. Time t
5. Mass m	5. Moment of inertia I
6. Linear momentum $\vec{P}=m\vec{v}$	6. Angular momentum $\vec{L}=I\vec{\omega}$
7. Linear impulse $\vec{F}\times\Delta t=\overline{\Delta P}$	7. Angular impulse = $=\vec{\tau}\times\Delta t=\Delta\vec{L}$
8. Force $\vec{F}=m\vec{a}$	8. Torque $\vec{\tau}=I\vec{\propto}$
9. $W=\vec{F}.\vec{s}$	9. $W=\vec{\tau}.\vec{\theta}$
10. KE = $\frac{1}{2}mv^2$	10. KE = $\frac{1}{2}I\omega^2$

11. Power $P = \vec{F}.\vec{v}$

12. Kinematic eq.

$V = u + at$

$S = ut + 1/2\, at^2$

$V^2 - u^2 = 2\, as$

$S_n = u + \frac{a}{2}(2n-1)$

11. Power $P = \vec{\tau}.\vec{\omega}$

12. Kinematic equations.

$w_t = w_0 + \propto t$

$\theta = \omega_0 t + 1/2 \propto t^2$

$\omega_t^2 - \omega_0^2 = 2 \propto \theta$

$\theta_n = \omega_0 + \frac{\propto}{2}(2n-1)$

Part 1 - Jee main

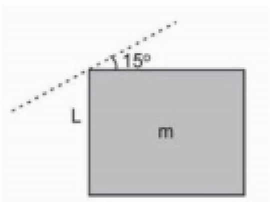
Section 1- Straight objective type questions

1. Mass per unit area of a circular disc of radius a depends on radial distance r from centre as $\sigma(r) = A + Br$ where A and B are constants. The moment of inertia of the disc about the axis perpendicular to plane and passing through its centre is

A) $2\pi a^4\left[\frac{A}{4}+\frac{aB}{5}\right]$ B) $2\pi a^4\left[\frac{A}{4}+\frac{aB}{6}\right]$

C) $2\pi a^4\left[\frac{2A}{4}+\frac{B}{5}\right]$ D) $2\pi a^4\left[\frac{A}{4}+\frac{B}{5}\right]$

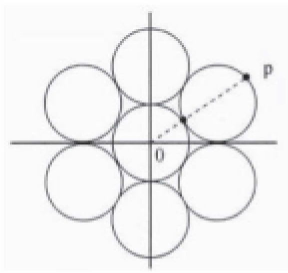
2. A square plate of mass m and edge length L is shown in figure. The M.I. of the plane about the axis in the plane of the plate and passing through one of its vertex making an angle 15° with horizontal



A) $\frac{mL^2}{12}$ B) $\frac{11}{24}mL^2$

C) $\frac{7}{12}mL^2$ D) $\frac{3}{2}mL^2$

3. Seven identical planar discs each of mass M and radius R are welded symmetrically as shown. The M.I. of the system about an axis passing through P and perpendicular to its plane is



A) $\frac{23}{2}MR^2$

B) $\frac{181}{2}MR^2$

C) $\frac{19}{2}MR^2$

D) $\frac{55}{4}MR^2$

4. A particle of mass 2 kg is dropped from rest from a point O. Which is 500m above ground, as shown, at t = 0

The torque on it about point P and also about O at t=3s respectively is

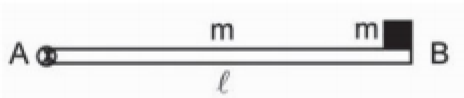
A) 100 Nm, zero

B) Zero, 100Nm

C) Zero, Zero

D) 100Nm, 100Nm

5. The rod AB of length ℓ and mass m can rotate freely about A in vertical plane. A point mass m is attached to the other end of the rod and the system is released from its horizontal position. The initial angular acceleration is



A) $\frac{2g}{3\ell}$ B) $\frac{4g}{\ell}$

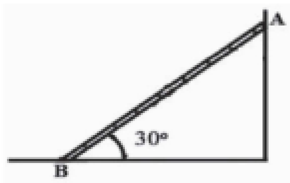
C) $\frac{9g}{8\ell}$ D) $\frac{5g}{3\ell}$

6. Two men are carrying a uniform bar of length L on their head, the bar is held horizontally such that the younger man get $(1/4)^{th}$ load. Suppose the younger man is at the end of the bar, what is the distance of other man from that end

A) $\frac{2L}{3}$ B) $\frac{L}{3}$

C) $\frac{2L}{5}$ D) $\frac{L}{6}$

7. A uniform rod of mass 15 kg leans against a smooth vertical wall making an angle 37° with horizontal. The other end rests on a rough horizontal floor. The reaction forces of he wall and floor are



A) 100N and 150N

B) 100N and 180N

C) 150N and 100N

D) 120N and 80N

8. A particle of mass 1 kg moving along the line y = x + 2 (x and y in m) with speed 2m/s. The magnitude of angular momentum of particle about origin

A) 4 kg m^2s^{-1}

B) $2\sqrt{2}\,kg\,m^2\,s^{-1}$

C) $4\sqrt{2}\,kg\,m^2s^{-1}$

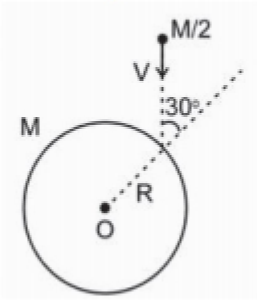
D) 2 kg m^2s^{-1}

9. A thin circular ring of mass M and radius r is rotating about its axis with a constant angular velocity ω. Two particles each of mass m are attached gently to the opposite ends of the diameter of the ring. The wheel now rotates with an angular velocity

A) $\frac{m\omega}{M+m}$ B) $\left(\frac{M-2m}{M+2m}\right)\omega$

C) $\frac{M\omega}{M+2m}$ D) $\left(\frac{M+2m}{M}\right)\omega$

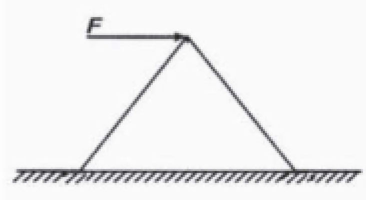
10. A circular disc of mass M and radius M can rotate freely about an axis passing through its centre and perpendicular to its plane. A bullet of mass M/2 travelling with speed v hits the disc as shown and gets stuck to it. The angular velocity of the system if the disc was initially at rest



A) $\frac{V}{4R}$ B) $\frac{V}{6R}$

C) $\frac{V}{R}$ D) $\frac{3V}{2R}$

11. A wedge in the form of equilateral triangle is placed on a rough horizontal surface as shown, for what value of coefficient of friction, can the block topple without slipping



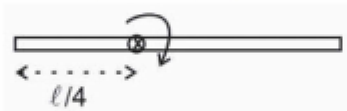
A) $\mu \geq \frac{1}{\sqrt{7}}$

B) $\mu \geq \frac{1}{\sqrt{5}}$

C) $\mu \geq \frac{1}{\sqrt{3}}$

D) $\sqrt{\frac{3g}{2\ell}}$

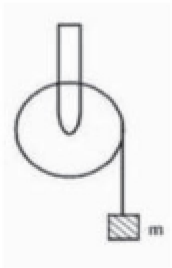
12. A uniform rod is hinged as shown and is released from a horizontal position. The angular velocity of the rod as it passes the vertical position (neglect friction)



A) $\sqrt{\frac{24g}{7\ell}}$ B) $\sqrt{\frac{48g}{8\ell}}$

C) $\sqrt{\frac{12g}{17\ell}}$ D) $\sqrt{\frac{3g}{2\ell}}$

13. A mass m is supported by massless string wound round a uniform cylinder of mass m and radius R. On releasing the mass from rest, it will fall with an acceleration



A) $\frac{2g}{3}$ B) $\frac{3g}{4}$

C) $\frac{g}{2}$ D) $\frac{g}{4}$

14. A meter stick is held vertically with one end of it on the floor and is then allowed to fall. Find the speed of ther end when it hits the floor (assuming that the end of stick does not slip)

A) 3.2 m/s B) 5.4 m/s

C) 7.5 m/s D) 9.2 m/s

15. A solid sphere rolls without slipping on a horizontal surface. The ratio of translational KE total KE

A) 5:7 B) 3:7

C) 2:7 D) 4:7

16. A small object of uniform density rolls up a rough curved surface with an initial velocity V. It reaches upto a maximum height of $\frac{3V^2}{4g}$ with respect to the initial position. The object is

A) ring B) solid sphere

C) hollow sphere D) disc

17. A ring rolls down from the top of an inclined plane of vertical height h. It will reach the ground after a time

A) $\frac{1}{\sin\theta}\sqrt{\frac{4h}{g}}$ B) $\frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$

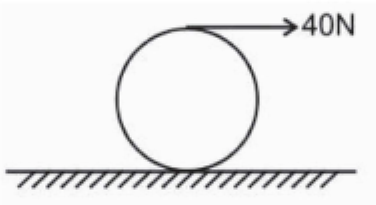
C) $\frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}$ D) $\frac{1}{\sin\theta}\sqrt{\frac{h}{g}}$

Section 2- Integer type questions

18. When a ceiling fan is switched off its angular velocity falls to half while it makes 36 rotations. How many more rotations will it make before coming to rest if angular retardation is constant

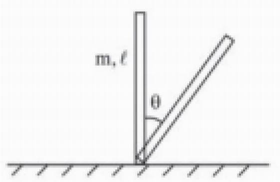
19. When a person throws a meter stick, it is found that the centre of the stick is moving with with speed 10m/s and left end of the stick with speed 20m/s. Both points move vertixally upwards at that moment. The angular speed of the stick is

20. A string is wound around a thin hollow cylinder of mass 5 kg and radius 0.5m. If the string is now pulled with a horizontal force of 40N and the cylinder is rolling with out slipping on a horizontal surface, the angular acceleration of the cylinder will be (neglect mass and thickness of string)



Section 3 - Jee Advanced Level

21. A rod of mass m and length l is kept on a rough floor in vertical position. The co-efficient of friction is sufficient so that the lower end of the rod does not slip over the floor. The rod is disturbed from the vertical position. The force of friction between the foot of the rod and the floor when rod makes an angle θ with vertical is



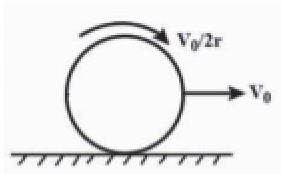
A) $\frac{3mg}{2}\sin 2\theta$

B) $\frac{mg}{2}\left[\sin 2\theta - \frac{\sin\theta}{4}\right]$

C) $\frac{3mg}{2}\left[\frac{4}{3}\sin 2\theta - 1\right]$

D) $\frac{3mg}{2}\left[\frac{3\sin 2\theta}{4} - \sin\theta\right]$

22. A solid sphere of mass m and radius r slips on a rough horizontal plane. At some instant, it has translational velocity v_0 and rotational velocity about the the centre $\frac{v_0}{2r}$. The translational velocity when sphere starts pure rolling is



A) $\frac{v_0}{3}$ B) $\frac{6v_0}{7}$

C) $\frac{v_0}{5}$ D) $\frac{2v_0}{7}$

23. Two boys support by the ends a uniform rod or mass M and length 2L. The rod is horizontal the two boys decided to change the ends of the rod by throwing the rod into air and catching it. The boys do not move from their position and the rod remained horizontal throughout its flight. Find the minimum impulse applied by each boy on the rod when it was thrown

A) $\frac{M}{2}\sqrt{\frac{\pi Lg}{3}}$

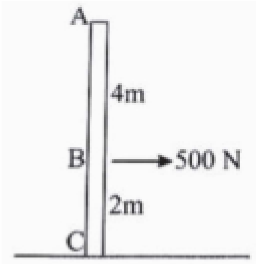
B) $\frac{M}{3}\sqrt{\frac{\pi Lg}{2}}$

C) $M\sqrt{\frac{\pi Lg}{2}}$

D) $\frac{2M}{3}\sqrt{\pi Lg}$

Section 4 - More than one correct answer type

24. A uniform 50 kg pole ABC of length 6m is balanced in the vertical position. A 500N horizontal force is suddenly applied at B. If the coefficient of friction between the pole and the ground is 0.3. The initial acceleration of point A is



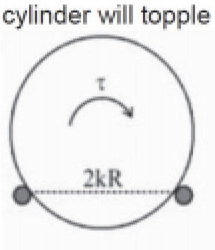
A) 4 m/s² towards right

B) 4 m/s² towards left

C) 6 m/s² towards right

D) 6 m/s² towards left

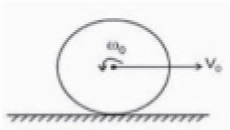
25. A cylinder of diameter 2R and mass M rests on two rough pegs (coefficient of static friction μ), the distance 2kR apart. A gradually increasing torque τ is applied as shown. The cylinder will topple before it slips if



A) $\mu < \frac{k}{\sqrt{1-k^2}}$ B) $\mu > \frac{k}{\sqrt{1-k^2}}$

C) $\mu > \sqrt{1-k^2}$ D) $\mu < \sqrt{1-k^2}$

26. A solid sphere of radius R is set in to motion on a rough horizontal surface with linear speed V_0 in forward direction and angular velocity $\omega_0 = \frac{V_0}{2R}$ in counterclock wise direction shown in figure (coefficient of kinetic friction μ)



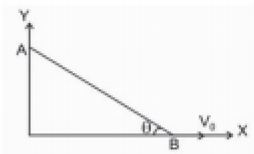
A) Time after which the sphere starts pure rolling is $t = \frac{3V_0}{7\mu g}$

B) The speed of C.M when it starts pure rolling is $\frac{4V_0}{7}$

C) Work done by friction is zero

D) Work done by friction is $-\frac{9}{28}mV_0^2$

27. The end B of a uniform rod AB of length l which makes an angle θ with the floor is pulled with a velocity V_0 as shown. At the instant when θ=37°



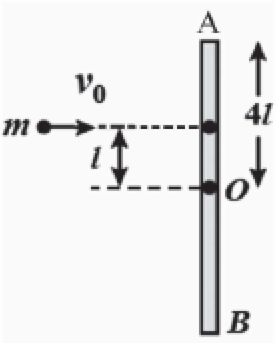
A) Velocity of end A is $\frac{4}{3}V_0$

B) Angular velocity of the rod is $\frac{5V_0}{3\ell}$

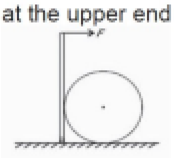
C) Velocity of C.M of rod is $\frac{5V_0}{6}$

D) Kinetic energy of the rod is $\frac{25}{54}mV_0^2$

28. A rod of mass M and length $8l$ lies on a smooth horizontal surface. A particle of mass m and velocity V_0 strikes the rod perpendicular to its length as shown. As a result of collision, the centre of mass of rod attains a speed of $\frac{V_0}{8}$ and the particle rebounds back with a speed of $\frac{V_0}{4}$



29. A uniform rod of mass M and length L is hinged at its lower and on table. The rod can rotate freely in vertical plane and there is no friction at the hinge. A ball of mass M and radius $R = \frac{L}{3}$ is placed in contact with the vertical rod and a horizontal force F is applied at the upper end of the rod



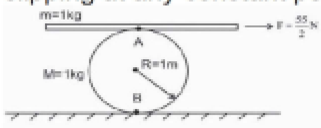
A) Acceleration of the ball immediately after the force starts acting is $\frac{3F}{4m}$

B) The normal contact force between the rod and ball is $\frac{3F}{4}$

C) Acceleration of the centre of mass of the rod is $\frac{9F}{8M}$

D) The horizontal component of hinge force acting on the rod immdediately after force F starts acting is $\frac{7F}{8}$

30. A solid cylinder of mass M = 1kg and radius R=1m lies on a rough horizontal surface. A plank of mass m=1 kg lies on its top. A force $F = \frac{55}{2}N$ applied on the plank as shown in figure causes the cylinder to roll. The plank always remains horizontal and there is no slipping at any constant point.



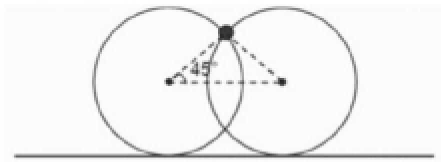
A) The acceleration of the cylinder is 10 m/s^2

B) friction force at A is 7.5N to right at cylinder top

C) The friction force at B is 2.5N to left at cylinder bottom

D) The acceleration of the plank is 20 m/s^2

31. Two identical uniform large rings each of mass m are connected through a bead of same mass, which can move freely. When bead is released it starts sliding down. The large rings roll apart over a sufficiently rough horizontal surface. Whole system is released from rest as shown in figure. Choose the correct option(s). (Neglect friction between bead and large rings)



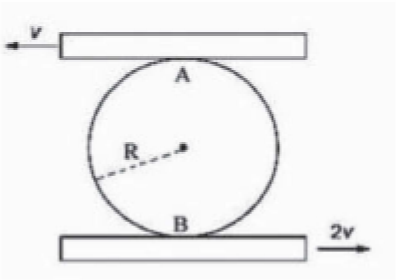
A) Acceleration of the bead at the initial moment is g/9

B) Acceleration of the bead at the initial moment is g/5

C) At the initial moment normal contact between bead and any of the ring is $\frac{4\sqrt{2}mg}{9}$

D) At the initial moment normal contact between bead and any of the ring is $\frac{2\sqrt{2}mg}{9}$

32. The disc of mass m and radius R is confirmed to roll without slipping at A and B as shown



A) The distance of instantaneous centre from point A is $\frac{2R}{3}$

B) The angular velocity of the disc is $\frac{3V}{2R}$

C) The velocity of the centre of mass is V/2

D) The total energy of the disc is 11/16 mv^2

33. A uniform disc of radius 2m lies on a smooth horizontal plane. A similar disc spinning with an angular velocity 5 rad/sec is carefully lowered on to the first disc. After what time both discs spin with the same angular velocity if the coefficient of kinetic friction between them is 1/8 [$g=10\ m/s^2$]

34. As shown in the figure rod of mass m and length l is hinged at O at one end and free to rotate in vertical plane. A force F of magnitude mg is applied on another end of rod which always act perpendicular to the rod. Force is start acting at time t = 0 (as shown). Find vertical component of normal reaction acting on rod when rod become horizontal for 3rd time: (given m = 2kg, l= 1m, g = $10m/s^2$]