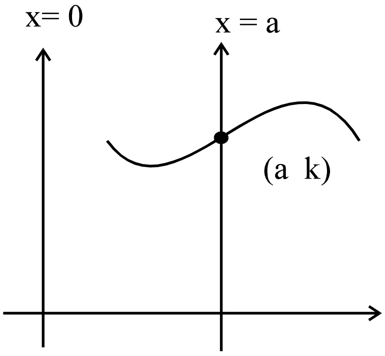
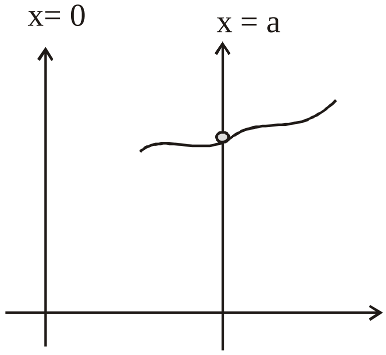
CHAPTER - 00
CONTINUITY, DIFFERENTIABILITY AND DERIVATIVES

Revision

$\underset{x\to a}{\mathrm{Lt}}\ f(x)=k \Rightarrow$ Graph of f(x) meets line x = a at the point (a, k) where limiting point (a k) need not be on the graph.





$f(x)\begin{cases} =x^2 & x\neq 0 \\ =1 & x=0 \end{cases}$ $\qquad$ $f(x)=x^2 \quad \forall x$

$\frac{1}{0+}=\infty$, $\frac{1}{0-}=-\infty$, $\frac{1}{\infty}=0$, $\frac{1}{-\infty}=0$

$\left.\begin{array}{l} a^{\infty} \\ a^{-\infty} \end{array}\right\} \begin{array}{l} =\infty, a>1 \\ 0, a>1 \end{array}$

$\left.\begin{array}{l} a^{\infty} \\ a^{-\infty} \end{array}\right\} \left.\begin{array}{l} =0 \\ =\infty \end{array}\right\} 0<a<1$

$e^{\infty}=\infty$, $e^{-\infty}=0$, $\log 0=-\infty$, $\log 1=0$

$\log e=1$, $\log 10=2.303$, $\log\infty=\infty$

Revise Important Methods of evaluating limits.

Continuity at a point

The function y = f(x) is continuous at x = a if

i) f(x) is defined at x = a

ii) $\underset{x\to a^+}{Lt} f(x) = \underset{x\to a^-}{Lt} f(x) = f(a)$

ie $\overset{Lt}{n \to a}$ f(x) exists and is finite

If these two conditions are satisfied at every point in an intervel [a b], then f(x) is continuous in the interval [a b]

eg : 1) f(x) = x^2

f(x) is defined at x=0 and f(0) = 0 is finit $\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^-}{Lt} f(x) = f(0) = 0 \Rightarrow f(x) = x^2$ is continuous at x = 0.

2) $f(x) = \frac{1}{x}$

$f(0) = \frac{1}{0}$ is not finite (not defined)

$f(x) = \frac{1}{x}$ is not continuous at n= 0

$$\boxed{f(x)\begin{cases} = e^x & x \le 0 \\ = x^2 & 0 < x \le 1 \\ = \frac{1}{x} & x > 1 \end{cases}}$$

i) $f(0) = e^o = 1(\text{finite})$

$\underset{x\to 0^+}{Lt} f(x) \ne \underset{x\to 0^-}{Lt} f(x) \Rightarrow$ Not continuous at x = 0

ii) $\underset{x\to 1^+}{Lt} f(x) = \underset{x\to 1^-}{Lt} f(x) = 1 \Rightarrow$ continuous at x = 1

Note 1: If f(x) is continuous at x = a then graph can be drawn through (around) ‘a’ without lifting the pen from the plane of the paper.

Note 2: If f(x) is discontinuous at x = a there is a break at x = a so that the graph can not be drawn around ‘a’ without lifting the pen from the plane of the paper.

Questions

1. $$f(x)\begin{cases} =\dfrac{x4^x - x}{2x^2} & x \neq 0 \\ =\dfrac{k}{2} & x = 0 \end{cases}$$

is continuous at x = 0. Find k

Types of Discontinuities

Types of Discontinuities

- Removable Discontinuity (RD)
  - Isolated point RD
  - Missing point RD
- Non Removable Discontinuity
  - Type - 1 Jump Discontinuity
  - Type II Infinite Discontinuity

i) Removable Discontinuity (RD)

y = f(x) has a removable discontinuity at x = a if

$$\underset{x\to a^+}{\text{Lt}} f(x) = \underset{x\to a^-}{\text{Lt}} f(x) \neq f(x)$$

ie; $\underset{x\to a}{\text{Lt}} f(x)\,\text{exists} \neq f(x)$

Removable discontinuity

$$\boxed{fx\begin{cases} =\dfrac{x^{2-1}}{x-1} & x \neq 1 \\ =1 & x = 1 \end{cases}}$$

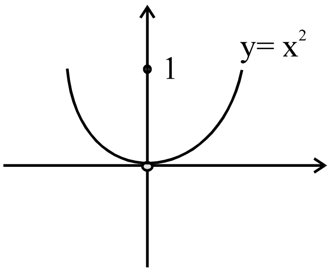
1) Ex: $$\boxed{fx\begin{cases} x\sin\dfrac{1}{x} & x \neq 0 \\ =1 & x = 0 \end{cases}}$$

Given f(0) = 1

$$\underset{x\to 0}{\text{Lt}}\, f(x) = \underset{x\to 0}{\text{Lt}}\, x\sin\frac{1}{x} = 0 \quad (\text{sand wich Theorem})$$

$$x \xrightarrow{\text{Lt}} 0 \;\; f(x) \text{exists} \neq f(0) \Rightarrow \text{RD at } x = 0$$

2) $f(x) \begin{cases} = x^2 & x \neq 0 \\ = 1 & x = 0 \end{cases}$



$$n \xrightarrow{\text{Lt}} 0 f(x) = 0 \neq f(0)$$

At RD the limiting point is not on the graph. It is hole.

Dirchlet function

$$f(x) \begin{cases} = 1, & x = \text{rational} \\ = 0, & x = \text{irrational} \end{cases}$$

Defined at every real number and discontinuous at every real number.

1) $f(x) \left.\begin{cases} x & x = \text{rational} \\ = 0 & x = \text{Irrational} \end{cases}\right\} \text{single point continuous function}$

2) $\left.\begin{matrix} \text{Single point function} \\ f(x) = \sqrt{1-x} + \sqrt{x-1} \end{matrix}\right\} \text{single point function}$

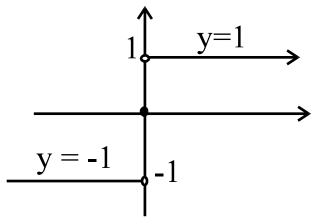
3) $f(x) = \dfrac{1}{\{x\} + \{-x\} - 1} \Rightarrow \text{Point function}$

Non Removable Discontinuity (NRD)

y = f(x) has a Non- Removable Discontinuity at x = a. If $\underset{x\to a^+}{\text{Lt}}\, f(x) \neq \underset{x\to a^-}{\text{Lt}}\, f(x)$

ie ; $\underset{x\to a}{\text{Lt}}\, f(x)$ does not exist. If both $\underset{x\to a^+}{\text{Lt}}\, f(x)$ and $\underset{x\to a^-}{\text{Lt}}\, f(x)$ are finite, but unequal, then the NRD is called Jump Discontinuity (JD).

Ex : $f(x) = \text{sigx} \begin{cases} = 1 & x > 0 \\ = 0 & x = 0 \\ = -1 & x < 0 \end{cases}$



RHL at x = 0 is 1

LHL at x = 0 is -1

RHL and LHL are finite, but not equal

$\therefore f(x)$ = sigx has jump discontinuity at x = 0

Special causes

1)DIRICHLET Function : $f(x) \begin{cases} = 1, & x = \text{rational} \\ = 0, & x = \text{Irrational} \end{cases}$
2)Single point function : $\sqrt{1-x} + \sqrt{x-1} \Rightarrow$ Continuous
3)Single point continuous fx : $f(x) \begin{cases} x, & x = \text{rational} \\ = 0, & x = \text{Irrational} \end{cases}$
(continuous at $x = 0$ only)
4) $f(x) = \dfrac{1}{\{x\} + \{-x\} - 1}$ Defined only at integers

1) $f(x) \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & x \neq 0 \\ = 1 & x = 0 \end{cases}$

check whether continuous at x = 0

2) $f(x) = (x+1)^{\cot x}$ when $x \neq 0$ is continuous at x = 0. Find f(0)

3) $f(x) = \dfrac{(e^x - 1)^2}{\sin\left(\dfrac{x}{a}\right)\log\left(1 + \dfrac{x}{a}\right)}$ $x \neq 0$ $f(0) = 12.$ Find ‘a’ if(x) is continuous at x = 0

4) $f(x)\begin{cases} = e^x & 0 \le x \le 1 \\ = 2 - e^{x-1} & 1 < x \le 2 \\ = x - e & 2 < x \le 3 \end{cases}$ check continuity at $x = 1$ and $x = 2$

5) $f(x)\begin{cases} = 5 & x \le 1 \\ = a + bx & 1 < x < 3 \\ = b + 5x & 3 \le x < 5 \\ = 30 & x \ge 5 \end{cases}$

For what value of 'a' and 'b' f(x) is continuous

6) $f(x) = |[x]x| -1 < x \le 2$

Find the points of discontinuities of f(x)

$\boxed{\text{Algebra of Continuous function}}$

Let f(x) and g(x) be continuous at x=a

i) k f(x) is continouus at x = a

ii) $f(x) \pm g(x)$ is continuous at x = a

iii) $f(x) \times g(x)$ is continuous at x =a

iv) $\dfrac{f(x)}{g(x)}$ is continuous at x = a

v) f(x) and g(x) s.t. f [g(x)] is defined at x = a

Let g(x) is continuous at x = a and f(x) is continuous at g(a) then f[g(x)] is continuous at x =a.

Right hand derivatives at x =a (RHD)

$$Rf'(a) = \underset{x \to 0}{Lt} \frac{f(a+h) - f(a)}{h} \text{ where } h > 0$$

Left hand derivative (LHD) at x = a

$$Lf'(a) = \underset{x \to 0}{Lt} \frac{f(a-h) - f(a)}{-h} \text{ where } h > 0$$

Ex: i) $f(x) = |x|$

$f(0) = |0| = 0,\ f(h) = |h| = h,\ \ f(-h) = |-h| = h$

$$Rf'(0) = \underset{x\to 0}{Lt} \frac{f(h) - f(0)}{h} = \underset{x\to 0}{Lt} \frac{h-0}{h} = 1$$

$$Lf'(0) = \underset{x\to 0}{Lt} \frac{f(-h) - f(0)}{-h} = \underset{x\to 0}{Lt} \frac{h-0}{-h} = -1$$

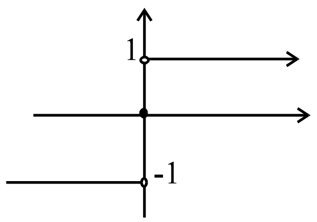
$$\boxed{f(x) = |x| \Rightarrow Rf'(0) = 1, \quad Lf^1(0) = -1}$$

Ex 2: Let $f(x) = \text{sign}\left.\begin{matrix} =1 & x>0 \\ =0 & x>0 \\ -1 & x<0 \end{matrix}\right\}$

$$Rf'(0) = \underset{h\to 0}{Lt} \frac{f(h) - f(0)}{h} = \underset{h\to 0}{Lt} \frac{1-0}{h} = \underset{h\to 0}{Lt} = \infty$$

$$Lf'(0) = \underset{h\to 0}{Lt} \frac{f(-h) - f(0)}{-h} = \underset{h\to 0}{Lt} \frac{-1-0}{-h} = \underset{h\to 0}{Lt} \frac{1}{h} = \infty$$

$$\boxed{f(x) = \text{sigx} \quad Rf'(0) = \infty \text{ and } Lf'(0) = -\infty}$$

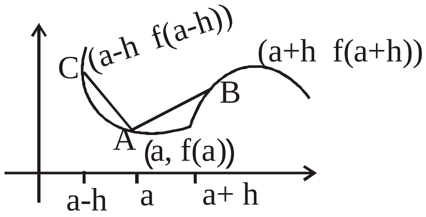


$$Rf'(0) = \infty$$
$$Lf'(0) = -\infty$$

Result: If f(x) is continuous at x = a then $Rf'(a)$ and $Lf'(a)$ are respectively the derivatives of the Right and Left branches of f(x) at x = a

Ex: $f(x)\left.\begin{matrix} x^2 & x \le 0 \\ = \sin x & x > 0 \end{matrix}\right\}$ $\boxed{\begin{matrix} Rf'(0) = 0 \\ Lf'(0) = 1 \end{matrix}}$

Geometrical Meaning of $Rf'(a)$ and $Lf'(a)$



Slope of secant AB = $\dfrac{f(a+h)-f(a))}{a+h-a} = \dfrac{f(a+h)-f(a)}{h}$

$$\left.\begin{matrix}\text{Slope of tangent at } x=a \\ \text{to the right of } x=a\end{matrix}\right\} = \underset{h\to 0}{Lt}\ \frac{f(a+h)-f(a)}{h} = Rf'(a)$$

Slope of secant AC = $\dfrac{f(a-h)-f(a)}{a-h-a} = \dfrac{f(a-h)-f(a)}{-h}$

$$\left.\begin{matrix}\text{Slope of tangent at } x=a \\ \text{to the left of } x=a\end{matrix}\right\} = \underset{h\to 0}{Lt}\ \frac{f(a-h)-f(a)}{-h} = Lf'(a)$$

Differentiability at x = a

The functions y = f(x) is differentiable at x =a if the following conditions are satisfied.

i) f(x) is continuous at x = a

ii) $Rf'(a) = Lf'(a)$

In geometrical sense if f(x) is differentiable at x = a then these exist a unique tangent at x = a.

Relation between continuity and Differentiability

All differentiable functions are continuous, but all continuous functions need not be differentiable

$$\boxed{\begin{matrix}\text{Differentiable} \Rightarrow \text{Continuous} \\ \text{Continuous} \not\Rightarrow \text{Differentiable}\end{matrix}}$$

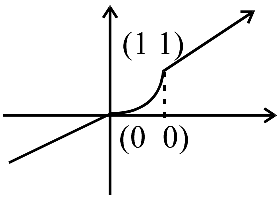
f(x) =|x| is continuous, but not differentiable at x = 0

SHARP POINT

A point of which a function is continuous, but not differentiable having finite RHD and LHD is called a sharp point. For ex : f(x) = |x| has a sharp point at x = 0.

Ex:(2) f(x) = Min: (x x^2)

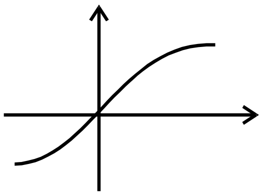
Sharp points at x = 0 and x = 1 $\Rightarrow$ Not differentiable at x = 0 and x = 1



$\boxed{f(x)=|x-1|+|x+1| \text{ .Find RHD and LHD at sharp points}}$

$$\left.\begin{array}{c} f(x) \text{ is continuous at } x=a \\ \text{and} \\ \text{unique tangent at } x=a \end{array}\right\} \left.\begin{array}{c} \text{Is the function} \\ \text{Different at } x=aq \end{array}\right\} \text{Need not be}$$

Ex : $f(x)=x^{1/3}$



$$\left.\begin{array}{l} Rf'(0)=\underset{h\to 0}{Lt}\dfrac{h^{1/3}-0}{h}=\underset{h\to 0}{Lt}\dfrac{1}{h^{2/3}}=\dfrac{1}{0}=\infty \\ Lf'(0)=\underset{h\to 0}{Lt}\dfrac{-h^{1/3}-0}{-h}=\underset{h\to 0}{Lt}\dfrac{1}{h^{2/3}}=\dfrac{1}{0}=\infty \end{array}\right\} \text{y axis is the unique tangent}$$

$f(x)=x^{1/3}$ is continuous at x = 0 and there exist a unique tangent at x = 0 without being differentiable at x = 0

Results

1) $Rf'(a)$ and $Lf'(a)$ are finite and equal $\Rightarrow$ f(x) is continuous and different at x = 0

Ex: $f(x)=x^2\sin\dfrac{1}{x} \quad x\neq 0$ and $f(x)=0,\ x=0$

$$Rf'(0)=\underset{h\to 0}{Lt}\frac{h^2\sin\frac{1}{x}-0}{h}=0.\text{Also}Lf'(0)=0$$

Continuous and different at x = 0

2) $Rf'(0)$ and $Lf'(0)$ are finite and unequal $\Rightarrow$ continuous, but not different at $x=0$

$$f(x) = \begin{cases} x e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3) If either $Rf'(a)$ or $Lf'(a)$ or both are infinite then the function may be continuous $\left(f(x)=x^{1/3}\right)$ or may not be continuous. (Ex: f(x) = sigx)

4) If the value of a derivative at a point is finite then function is continuous and differentiable at that point

Ex: $f(x)=x\left(\sqrt{x}-\sqrt{x+1}\right)$

$$f(x)=x\left(\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x+1}}\right)+\sqrt{x}-\sqrt{x+1}=\frac{\sqrt{x}}{2}-\frac{x}{2\sqrt{x+1}}+\sqrt{x}-\sqrt{x+1}$$

$$f'(0)=0-0+0-1=-1(\text{Finite})$$

$\therefore f(x)$ is continuous and differentiable at x = 0

Question:

$$f(x)=x-x^2 \quad 0\le x\le 1$$

$$g(x) \begin{cases} = \text{Maxi } f(t) & 0\le t\le x, 0\le x\le 1 \\ = \sin \pi x & x>1 \end{cases}$$

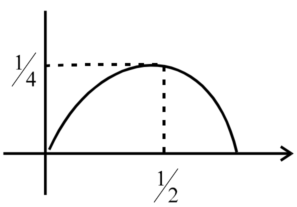
Find Non Differentiable points of f(x)

$f(t)=t-t^2 \Rightarrow$ downward parabola

$f'(t)=1-2t=0 \Rightarrow t = 1/2$ at maximum

Also $f\left(1/2\right)=1/4$

Graph of f(t)



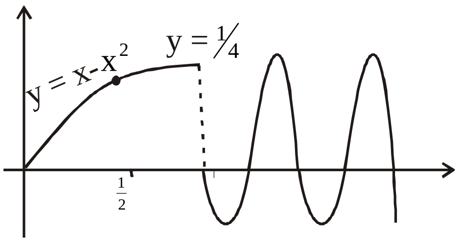
$0\le t\le \frac{1}{2} \Rightarrow$

Maxi $f(t)=f(x)$

$\frac{1}{2}<x\le 1 \Rightarrow$

Maxi $f(t)=1/4$

$$g(x)\begin{cases} = f(x) = x = x^2 & 0 \le x \le 1/2 \\ = 1/4, & \frac{1}{2} < x \le 1 \\ = \sin \pi x & x > 1 \end{cases}$$



$$y = x - x^2 \Rightarrow \frac{dy}{dx} = 1 - 2x$$

$$\left.\frac{dy}{dx}\right\}_{x=1/2} = \frac{1}{4} \Rightarrow \text{Differentiable at } x = 1/2$$

Methods of Differentiation

Derivative or Differential coefficient at x = a

Let y = f(x) be a differentiable function. The derivative or differential coefficient at x = a is defined as

$$f'(a) = \underset{h \to 0}{\text{Lt}} \frac{f(a+h) - f(a)}{h} \quad h > 0$$

In general the derivative of y = f(x)

w.r.t x is given by $f'(x) = \frac{dy}{dx} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} h > 0$

When $h = \Delta n$

$$f'(x) = \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x} --(1)$$

Let $y = f(x)$

$$y + \Delta y = f(x + \Delta x)$$

$$\Delta y = (y + \Delta y) - y = f(x + \Delta x) - f(x)$$

From (1) $\boxed{\frac{dy}{dx} = \Delta x \overset{\text{Lt}}{\to} 0 \frac{\Delta y}{\Delta x}}$

In Geometrical sense $\frac{dy}{dx}$ is the slope of tangent and inphysical sense $\frac{dy}{dx}$ is the rate of change of y w.r.t x. Process of finding the derivative is called **Differentiation.**

Have a quick revision of

$$\left.\begin{array}{l}\text{i) Product Rule}\\ \text{ii) Quotient Rule}\\ \text{iii) Power Rule}\\ \text{iv) Reciprocal Rule}\\ \text{v) Chain Rule}\end{array}\right\}\begin{array}{l}\text{Refer 1st year notes on}\\ \text{Limits and Derivatives}\end{array}$$

Extension of Chain Rule

$$\frac{d}{dx} f\left[g\left(h(x)\right)\right] = f'\left[g\left(h(x)\right)\right] g'\left(h(x)\right) h'(x)$$

Ex:

1) $\frac{d}{dx} \sin\log\sqrt{x} = \cos\ \log\ \sqrt{x} \times \frac{1}{\sqrt{x}} \times \frac{1}{2\sqrt{x}}$

2) y = sec tan e^x ; $\frac{dy}{dx}\ \sec e\ \tan e^x\ \left(\tan\ \tan e^x\right) \sec^2\ e^x\ e^x$

3) sin log cos x = y

$$\frac{dy}{dx} \cos\left(\log\ \cos x\right) \frac{1}{\cos x}\left(-\sin x\right)$$

$$\boxed{\left.\begin{array}{l}\text{Exercise 5.2}\\ \text{Page 166}\end{array}\right\}\text{NCERT Text}}$$

Implicit and Explict Functions

Functions of the form $y = f(x)$ or $x = \phi(y)$ are called explicit functions

Ex : $y = \frac{\sin x}{2x + y}$ or $x = \frac{2y + 1}{\cos y + e^y}$

Functions which are not explicit are implicit functons

Ex : $x^2 + y^2 + \sin xy = k$

Derivative of Explicit Functions

a) $y = f(x) \Rightarrow$ standard results can be used directly.

b) $x = \phi(y)$. The procedure is given below

Step 1 : Differentiable with respect to y and get $\frac{dx}{dy}$

Step 2 : $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

Ex: 1) $x = \sin y + e^y \quad \text{Different w.r.t } x$ $\qquad \frac{dx}{dy} = \cos y + e^y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y + e^y}$

2) $x = \frac{2\sin y}{y + \log y}$

$x = \frac{2\sin y}{y + \log y}$ Different w.r.t y

$$\frac{dx}{dy} = \frac{(y + \log y)2\cos y - 2\sin y\left(1 + \frac{1}{y}\right)}{(y + \log y)^2}$$

$$\frac{dx}{dy} = \frac{2y\cos y(y + \log y) - 2\sin y(1 + y)}{y(y + \log y)^2}$$

Now $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$

Derivative of Implicit functions

Method 1 : If possible convert the impicit function in to explicit and differentiate .

Ex : xy = x+ y - implicit

$$xy - y = x \Rightarrow y(x - 1) = x$$

$$\therefore y = \frac{x}{x - 1}; \qquad \therefore \frac{dy}{dx} = \frac{(x - 1) - x}{(x - 1)^2} = \frac{-1}{(x - 1)^2}$$

(Answer in x only)

2) $2x^2 - 3y = 7$ $-\text{Implicit}$

$2x^2 - 7 = 3y$; $y = \frac{1}{3}\left(2x^2 - 7\right)$

$\frac{dy}{dx} = \frac{1}{3}\left(4x\right) = \frac{4}{3}x$

(Answer in x only)

3) Find $\frac{dy}{dx}$ If $\sin y = x \sin\left(a+y\right)$

$\sin y = x \sin\left(a+y\right)$ $-\text{Implicit}$

$x = \frac{\sin y}{\sin\left(a+y\right)}$ $-\text{explicit}$

Different w.r.t x

$$\boxed{\begin{array}{l} x\sqrt{1+y}+y\sqrt{1+x}=0 \\ \text{s.t} \\ \frac{dy}{dx}=-\frac{1}{(1-x)^2} \end{array}}$$

$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cos\left(a+y\right)}{\sin^2\left(a+y\right)}$

$\frac{dx}{dy} = \frac{\sin\left(a+y-y\right)}{\sin^2\left(a+y\right)}$

$\therefore \frac{dx}{dy} = \frac{\sin a}{\sin^2\left(a+y\right)}$; $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2\left(a+y\right)}{\sin a}$

(Answer in y only)

Questions:

1)

$x>1$ If $(2x)^{2y}=4e^{2x-2y}$ then $\left(1+\log^{1x}\right)^2\frac{dy}{dx}=$

1) $\log^2 x$ 2) $\frac{x\ \log^2 x+\log^2}{x}$ 3) $x\ \log^2 x$

4) $\frac{x\ \log^2 x-\log^2}{x}$

[Option in x only convert in to $y=f(x)$]

2) $x\sqrt{1+y}+y\sqrt{1+x}=0$

show that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$

$x\sqrt{1+y}=-y\sqrt{1+x}\ ;\ x^2(1+y)=y^2(1+x)$

$x^2+x^2y=y^2+y^2x\ ;\ x^2-y^2=y^2x-(x^2y)$

$(x+y)(x-y)=xy(y-x);\ (x+y)=-xy;\ x=-xy-y$

$x=y(1+x);\ y=-\frac{x}{1+x}$ (explicit)

$$\frac{dy}{dx}=-\left[\frac{1+x-x}{(1+x)^2}\right]=-\frac{1}{(1+x)^2}$$

Derivative of Implicit Functions

In case of implicit function we differentiate term by term w.r.t x and arrange the terms of $\frac{dy}{dx}$

1) $x^2+y^2+\sin xy=k$ Different w.r.t x

$$2x+2y\frac{dy}{dx}+\cos xy\left(x\frac{dy}{dx}+y\right)=0\ ;\ \frac{dy}{dx}[2y+x\ \cos xy]=-[2x+y\ \cos xy]$$

$$\frac{dy}{dx}=-\left[\frac{2x+y\cos xy}{2y+x\cos xy}\right]$$

2) Find $\frac{dy}{dx}$ If $x^3+x^2y+xy^2+y^3=81$

$$3x^2+x^2\frac{dy}{dx}+y2x+x\,2y\frac{dy}{dx}+y^2+3y^2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[x^2+2xy+3y^2\right]=-\left[3x^2+2xy+y^2\right]$$

$$\frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Exercise 5.3
Page 169 NCERT TEXT

Derivative of inverse Trignometric functions

1) $f(x)=\sin^{-1}x$; $y=\sin^{-1}x\Rightarrow x=\sin y$

$$\frac{dx}{dy}=\cos y\Rightarrow\frac{dy}{dx}=\frac{1}{\cos y}\quad-(1)$$

$$x=\sin y\Rightarrow x^2=\sin^2 y\Rightarrow 1-x^2=1-\sin^2 y$$

$$1-x^2=\cos^2 y\Rightarrow\cos y=\sqrt{1-x^2}\ ;\ \frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}\text{ where }1-x^2>0$$

$$\therefore x^2<1\Rightarrow -1<x<1$$

2) $f(x)=\cos^{-1}x$

$$y=\cos^{-1}x\Rightarrow x=\cos y\Rightarrow\frac{dx}{dy}=-\sin y$$

$$\therefore\frac{dy}{dx}=-\frac{1}{\sin y};\ x^2=\cos^2 y\Rightarrow 1-x^2=1-\cos^2 y=\sin^2 y;\ \therefore\sin y=\sqrt{1-x^2}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = \frac{-1}{\sqrt{1-x^2}} \quad -1 < x < 1$$

3) Derivative of $f(x) = \tan^{-1} x$

$$y = \tan^{-1} x \Rightarrow x = \tan y; \ \frac{dx}{dy} = \sec^2 y \ \Rightarrow \ \frac{dy}{dx} = \frac{1}{\sec^2 y} \quad -(1)$$

$$x = \tan y \Rightarrow 1 + \tan^2 y = \sec^2 y$$

$$\therefore \boxed{1 + x^2 = \sec^2 x}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad \boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \quad x \in R}$$

4) Derivative of f(x) = sec x

$$y = \sec x \Rightarrow x = \sec y \Rightarrow \frac{dx}{dy} = \sec y \ \tan \ y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec y \ \tan y} \quad -(1) \quad \boxed{1 + \tan^2 x = \sec^2 x}$$

$$x = \sec y \Rightarrow 1 + \tan^2 y = \sec^2 y; \ \tan^2 y = \sec^2 y - 1 \Rightarrow 1$$

$$\tan^2 y = x^2 - 1 \Rightarrow \tan y = \sqrt{x^2-1}; \ \frac{dy}{dx} = \frac{1}{\sec y \ \tan y} = \frac{1}{x\sqrt{x^2-1}} \ |x| > 1$$

$$y = \sec x \text{ is a } S\uparrow \text{ fx} \Rightarrow \frac{dy}{dx} > 0 \quad \boxed{f(x) = \sec x \text{ is } s\uparrow}$$

$$\boxed{\frac{d}{dx} \sec x = \frac{1}{|x|\sqrt{x^2-1}}}$$

5) Derivative of f(x) = $\text{cosec}^{-1}x$

$$y = \text{cosec}^{-1} x \Rightarrow x = \text{cosec} \ y$$

$$\frac{dx}{dy} = -\text{cosec}\, y \ \cot y \Rightarrow \frac{dy}{dx} = \frac{-1}{\text{cosec}\, y \ \cot y} \ -(1)$$

$$x = \text{cosec} \ y \Rightarrow 1 + \cot^2 y = \text{cosec}^2 y; \ 1 + \cot^2 y = x^2 \Rightarrow \cot^2 y = x^2 - 1 \Rightarrow \cot y = \sqrt{x^2-1}$$

$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$ But $f(x) = \text{cosec}^{-1}x$ is $S\downarrow$

$\therefore \frac{dy}{dx} < 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

6) Derivative of f(x) = cot x

$y = \cot^{-1} x \Rightarrow x = \cot y \Rightarrow \frac{dx}{dy} = -\text{cosec}^2 y$

$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y}$

$1 + \cot^2 y = \text{cosec}^2 y \Rightarrow 2 + x^2 = \text{cosec}^2 y$

$\frac{dy}{dx} = \frac{-1}{1+x^2}$ $x \in R$; $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$ $x \in R$

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ $-1 < x < 1$

$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$ $-1 < x < 1$

$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ $x \in R$; $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$ $x < -1$ or $x > 1$

$\frac{d}{dx}\text{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$ $x < -1$ or $x > 1$; $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$ $x \in R$

1) $y = \sin^{-1}\frac{2x}{1+x^2}$ Find $\frac{dy}{dx}$; Put $x = \tan\theta$

$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2 a}\right) = \sin^{-1}\sin 2\theta = 2\theta$; $y \Rightarrow \theta = 2\tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

2) $f(x) = \sin^{-1} 2x\sqrt{1-x^2}$ Find $\frac{dy}{dx}$; Put $x = \sin\theta \sqrt{1-x^2} = \cos\theta$;

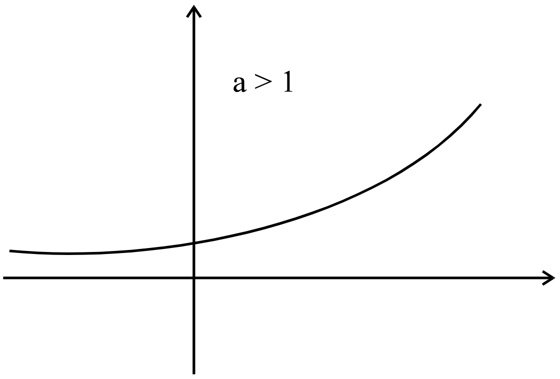
$f(x) = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$; $f(x) = 2\sin^{-1} x \Rightarrow f'(x) = \frac{2}{\sqrt{1-x^2}}$

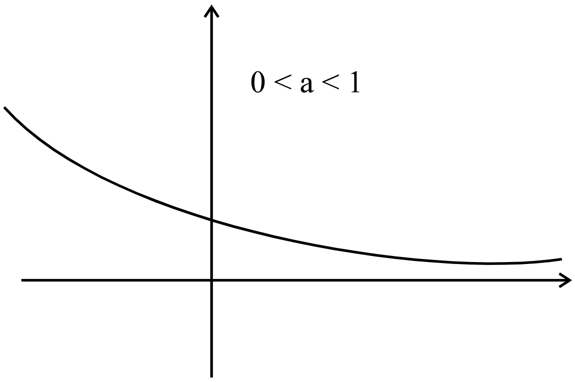
3) Find the Derivative of $\tan^{-1}\sin\sqrt{x}$

4) $\frac{d}{dx}\sin^{-1}\left(2+3\cos x\right) = \frac{1}{\sqrt{1-\left(2+3\cos x\right)^2}}\ (-3\sin x)$

Exponential and logarithmic functions

$f\left(x\right)=a^x, x \in R, a = \text{base} > 0,\ a \neq 1$

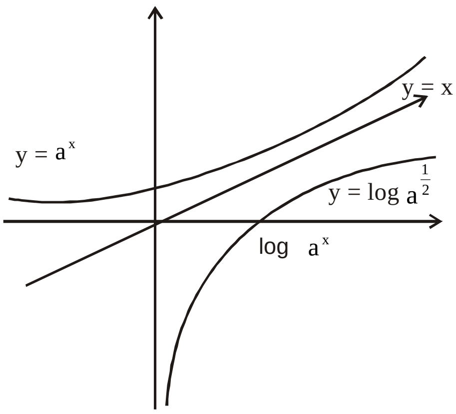




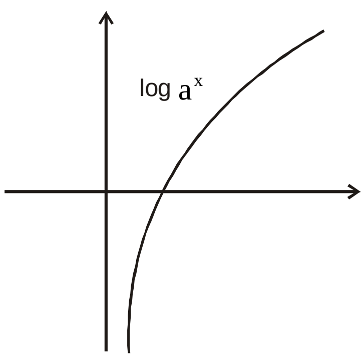
$y=10^x$: Common exponential function	$e^{\infty}=\infty$
$y=e^x$: Natural exponential function	$e^{-\infty}=0$

Logarithmic Function

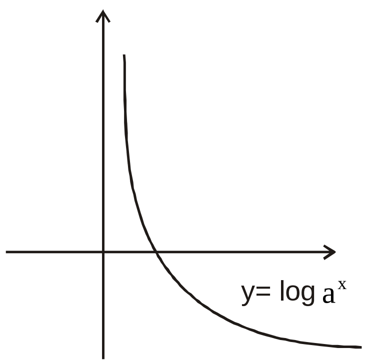
A function defined by $f\left(x\right)=\log a^x$ where $x>0$ and $a>0$ and $a \neq 1$. It is the inverse of exponential function $f(x) = a^x$.



a > 1



O < a < 1



$\boxed{a=10 \Rightarrow \text{common log}}$ and $\boxed{a=e \text{ Natural log}}$

Results ($\frac{d}{dx}$ of exponential and logarithmic function)

i) $\frac{d}{dx}a^x = a^x \log a$

ii) $\frac{d}{dx}e^x = e^x$

iii) $\frac{d}{dx}a^{mx} = ma^{mx}\log a$

iv) $\frac{d}{dx}a^{mx} = m\,e^{mx}$

v) $\frac{d}{dx}\log x = \frac{1}{x}$

v) $\frac{d}{dx}\log 10^x = \frac{1}{x\log 10}$

$$\boxed{\begin{array}{l} \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a\frac{d}{dx}f(x) \\ \frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x) \\ \frac{d}{dx}\log f(x) = \frac{1}{f(x)}f'(x) \end{array}}$$

1) Find derivative of $y = \sin^{-1}e^x$; $\frac{dy}{dx} = \frac{1}{\sqrt{1-e^{2x}}}xe^x$

Exercise 5.4
Page 174

2) $\frac{dy}{dx}2^{\sin x+ex} = 2^{\sin x+ex}\left(\cos x + e^x\right)$

3) Find $\frac{dy}{dx}$ i) $y = e^{x^2+\log x+2y}$

4) $\frac{d}{dx}\sqrt{e^{\sqrt{x}}} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}\frac{d}{dx}e^{\sqrt{x}} = \frac{1}{2\sqrt{e^{\sqrt{x}}}}e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$

Question

$e^{xy} + \log xy + \cos xy + 5 = 0$
$x > 0 \quad y > 0$
Find $\frac{dy}{dx}$

Logarithmic Differentiation

When the functions are in the form $a^{f(x)}, (f(x))^{g(x)}, f(x)g(x)$ and $\frac{f(x)}{g(x)}g(x) \neq 0,$ before finding the Derivative we take logarithms and it is called logarithmic differentiation.

Find the Derivative of

1) $y = a^x$ 2) $y = 3^{\sin x}$ 3) $y = x^x$ 4) $y = x^{-x}$

5) $y = x^{1/x}$ 6) $y = x^{-1/x}$ 7) $y = x^{x^x}$ 8) $y = (\log x)^x + x^{\log x}$

9) $y = (\log x)^x + x^{\log x}$ 10) $y = (\sin x)^x + x^{\sin x}$

11) $y = e^x \cos^3 x \sin^2 x$ 12) $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\ x \neq 1,2,3,4$

13) $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ find $f'(1) =$

Ans : 120

Find $\dfrac{dy}{dx}$ if $y^x + x^y + x^x = a^b$; $u = y^x \quad v = x^y \quad z = x^x \Rightarrow u+v+z = a^b$

$$\frac{du}{dx}+\frac{dv}{dx}+\frac{dz}{dx}=0\ ;\ \frac{du}{dx}=u\left[\frac{x}{y}\frac{dy}{dx}+\log y\right],\frac{dv}{dx}=v\left(\frac{y}{x}+\log x\frac{dy}{dx}\right),\ \frac{dz}{dx}=x^x(1+\log x)$$

$$\frac{dy}{dx}=\frac{-\left[y^x\log y+x^y\dfrac{y}{x}+x^x\log(1+x)\right]}{\left(y^x\left(\dfrac{x}{y}\right)\right)+x^y\log x}$$

Important Results

$$\boxed{\begin{aligned}&\frac{d}{dx}[f(x)]^{g(x)}=[f(x)]^{g(x)}\left[\frac{f'(x)}{f(x)}g(x)+g'(x)\log f(x)\right]\\&\frac{d}{dx}a^{(f(x))^{g(x)}}=a^{(f(x))^{g(x)}}\log a\ \frac{d}{dx}[f(x)]^{g(x)}\end{aligned}}$$

1) $y = (\sin x)^{ex}$ 2) $(\log x)^{\sin x}$ 3) $y = (2x^3+1)^{(x+e^x+1)}$

4) $y = (3x^2 + 2\log x)^{2x+1}$ find $\dfrac{dy}{dx}$ at $x = 1$

and

Find derivative of

1) $2^{(\sin x)^{x+1}}$ 2) $y=10^{\left(x^2+1\right)^{\sin x}}$ 3) $\boxed{y=e^{x^{e^x}}\ \frac{dy}{dx}\text{at } x=1}$

4) $\boxed{y=x^{x^{x^x}}}$ 5) $\boxed{y=10^{x^{10x}}\ \text{Find } \frac{dy}{dx} \text{ at } x=1}$

6) $y=x^{x^x} \Rightarrow \frac{dy}{dx}=x^{x^x}\left[\frac{1}{x}x^x+x^x\left(1+\log x\right)1\right]$

Derivative of Parametric functions

If the relation between two variables x and y is expressed via a third variable 't' then the function y = f(x) is called a parametric function.'t' is called the parameter.

ie; If $x=f\left(t\right)$ and $y=g\left(t\right)$ Then $y=f\left(x\right)$ is called a parametric function in parameter 't'

Ex : $x=a\cos\theta \quad y=a\sin\theta$

i) $x^2+y^2=a^2$. Hence $x=a\cos\theta \quad y=a\sin\theta$ is the parametric form (equation) of the circle $x^2+y^2=a^2$

2) $x=a\cos\theta \quad y=b\sin a$

$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$ is the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

3) $x=at^2 \quad y=2at$; $y^2=4a^2t^2=4a^2\frac{x}{a}=4ax$

$x=at^2 \quad y=2at$ is the parametric equation of parabola $y^2=4ax$

Parametric differentiation

Method 1 : In case of a parameteric function if possible eliminate the parameter and then differentiate

Ex :

1) $x=a\cos\theta \quad y=a\sin\theta$; $x^2+y^2=a^2 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$

2) Find $\frac{dy}{dx}$ if $x=\sqrt{a^{\sin^{-1}}x}$ and $y=\sqrt{a^{\cos^{-1}}x}$

$xy=\sqrt{a^{\sin^{-1}}x+\cos^{-1}x} \Rightarrow xy=\text{cons}\tan t$

$$\therefore \frac{dy}{dx} = \frac{-y}{x} \qquad \boxed{\begin{array}{l} x = \sqrt{\dfrac{1-t^2}{1+t^2}} \quad y = \dfrac{\sqrt{1+t^2}-\sqrt{1-t^2}}{\sqrt{1+t^2}+\sqrt{1-t^2}} \\ y = \dfrac{1-x}{1+x} \end{array}}$$

3) $x = \theta - \frac{1}{\theta} \qquad y = \theta + \frac{1}{\theta};\ y^2 - x^2 = \left(\theta + \frac{1}{\theta}\right)^2 - \left(\theta - \frac{1}{\theta}\right)^2 = 4$

$$y^2 - x^2 = 4;\ 2y\frac{dy}{dx} - 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

4) $x = at^2 \qquad y = 2at;\quad y^2 = 4a^2t^2 = 4a^2\frac{x}{a} = 4ax$

$$y^2 = 4ax \Rightarrow \frac{dy}{dx} \times 2y = 4a \Rightarrow \frac{dy}{dx} = \frac{4a}{2y}$$

Parametric Differentiation : Method 2

In case of parametric function the derivative can also be obtained by $\boxed{\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}}$

Ex : 1

$$\left.\begin{array}{l} x = at^2 \quad y = 2at \\ \dfrac{dx}{dt} = 2at \quad \dfrac{dy}{dt} = 2a \end{array}\right| \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{2a}{2at} = \frac{1}{t}$$

Ex : 2

$$x = \theta - \frac{1}{\theta} \qquad y = \theta + \frac{1}{\theta};\ \frac{dx}{d\theta} = 1 + \frac{1}{\theta^2} \qquad \frac{dy}{d\theta} = 1 - \frac{1}{\theta^2}$$

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{\theta^2 - 1}{\theta^2 + 1} = \frac{\theta\left(\theta - \dfrac{1}{\theta}\right)}{\theta\left(\theta + \dfrac{1}{\theta}\right)} = \frac{x}{y}$$

Ex: 3

$$x = a\left(\cos t + \log\ \tan\frac{t}{2}\right) \quad y = a\sin t \Rightarrow \text{s.t}\frac{dy}{dx} = \tan t$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}\ \sec^2\frac{t}{2}\times\frac{1}{2}\right) = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{\left(1-\sin^2 t\right)}{\sin t} = \frac{a\cos^2 t}{\sin t};\ \frac{dy}{dx} = \frac{a\cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)} = \frac{\sin t\cos t}{\cos^2 t} = \tan t$$

$$x = a\left(t+\sin t\right) \quad y = a\left(1-\cos t\right).\ \left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{dy}{dx} = a\left(1+\cos t\right) \quad \frac{dy}{dt} = a\sin t$$

$$\frac{dy}{dx} = \frac{a\sin t}{a\left(1+\cos t\right)} = \frac{\sin t}{1+\cos t}$$

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{1}{1} = 1$$

$$\boxed{\begin{array}{l} x = a\cos^2\theta \quad y = a\sin^3\theta \\ \text{Find } 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta - \sin a}\right)^2 \\ = 1+\left(-\tan\theta\right)^2 = 1+\tan^2\theta = \sec^2\theta \end{array}}$$

Derivative of special type Functions Containing an infinite expression

1) $y = \sqrt{x+\sqrt{x+\sqrt{x+....}}}$ Find $\frac{dy}{dx}$

$$y = \sqrt{x+y} \Rightarrow y^2 = x+y\ ;\ 2y\frac{dy}{dx} = 1+\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[2y-1\right] = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2y-1}}$$

2) $y = \sqrt{\sin x + \sqrt{\sin x + ...}}$

$$y=\sqrt{\sin x+y} \Rightarrow y^2=\sin x+y \Rightarrow 2y\frac{dy}{dx}=\cos x+\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{\cos x}{(2y-1)}$$

3) $$y=\sqrt{a^x+\sqrt{a^x+\sqrt{a^x+.....}}}$$

$$y=\sqrt{a^x+y} \Rightarrow y^2=a^x+y\ ;\ 2y\frac{dt}{dx}=a^x\log a+\frac{dy}{dx}\ ; \qquad \therefore \frac{dy}{dx}=\frac{a^x\log a}{2y-1}$$

Result

$$\boxed{\begin{array}{l} y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.....}}} \\ \dfrac{dy}{dx}=\dfrac{f'(x)}{2y-1} \end{array}}$$

$$y=\sqrt{(x^2+1)\sqrt{(x^2+1)+\sqrt{(x^2+1)+.....}}}\ ;\ \frac{dy}{dx}=\frac{2x}{2y-1}$$

$$y=(\sin x)^{(\sin x)^{(\sin x)}} \text{ Find } \frac{dy}{dx}$$

$$y=(\sin x)^y \Rightarrow \log y=y\log\sin x\ ;\ \frac{1}{y}\frac{dy}{dx}=\frac{y}{\sin x}\cos x+\log\sin x\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\frac{1}{y}-\log\ \sin x\right]=y\cot x\ ;\ \frac{dy}{dx}=\frac{y^2\cot x}{1-y\log\sin x}=\frac{y^2\cot x}{1-\log y}$$

$$y=(\cos x)^{(\cos x)^{\cdot^{\cdot}}} \text{ Find } \frac{dy}{dx} \Rightarrow y=(\cos x)^y$$

$$\log y=y\log\cos x \Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{y}{\cos x}-\sin x+\log\cos x\frac{dy}{dx}$$

$$\frac{dy}{dx}[t_1-\log\cos x]=-y\tan x\ ;\ \frac{dy}{dx}=\frac{-y>\tan x}{1-y\log\cos x}=\frac{y^2\tan x}{\log^{y-1}}$$

In General

$$y=\left(f(x)\right)^{\left(f(x)\right)^{f(x)^{\cdot^{\cdot^{\cdot}}}}}$$

$$\frac{dy}{dx}=\frac{y^2\frac{f'(x)}{f(x)}}{1-\log y}$$; where log y = y log f(x)

$$y=(ax)^{(ax)^{(ax)^{\cdot^{\cdot^{\cdot}}}}}\ ;\ \frac{dy}{dx}=\frac{y^2\frac{a x\log a}{ax}}{1-\log y}=\frac{y^2\log a}{1-y\log ax}=\frac{y^2\log a}{1-xy\log a}$$

$$y=\left(x^x\right)^{\left(x^x\right)^{\left(x^x\right)}}\ ;\ \frac{dy}{dx}=$$

1) $\boxed{\sqrt{y-\sqrt{y-\sqrt{y....}}}=\sqrt{x+\sqrt{x+\sqrt{x+....}}}\ \text{Find}\ \frac{dy}{dx}}$

$$\text{Let}\sqrt{y-\sqrt{y-\sqrt{y-.....}}}=\sqrt{x+\sqrt{x+\sqrt{x+....}}}=t$$

$$\sqrt{y-t}=\sqrt{x+t}=t$$

$y-t=t^2$ and $x+t=t^2$; $y=t^2+t$ and $x=t^2-t$

$$\frac{dy}{dt}=\frac{2t+1}{2t+1}=\frac{y-x+1}{y-x-1}$$

$$\boxed{\begin{array}{l} y=t^2+t \\ x=t^2-t \\ \hline y-x=2t \end{array}}$$

2) $y^{y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}}=\log\left(x+\log\left(x+....\right)\right)$

Find $\frac{dy}{dx}$ at $x=e^2-2\quad y=\sqrt{2}$

3) $\left(x^m\right)^{\left(x^m\right)^{\left(x^m\right)^{\cdot^{\cdot^{\cdot}}}}}=\left(y^n\right)^{\left(y^n\right)^{\left(y^n\right)^{\cdot^{\cdot^{\cdot}}}}}$ Find $\frac{dy}{dx}$

Put $x^m=u$ and $y^n=u\Rightarrow u^{u^{u^{u^{\cdot^{\cdot^{\cdot}}}}}}=v^{v^{v^{\cdot^{\cdot^{\cdot}}}}}=t$

$\therefore u^t = v^t = t \Rightarrow u = t^{1/t}$ and $v = t^{1/t} \Rightarrow u = v$

$\therefore n^m = y^n \Rightarrow mn^{m-1} = ny^{n-1}\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{my}{nx}$ since $x^m = y^n$

Differentiation by substitution

1) $\tan y = \dfrac{2t}{1+t^2}$ $\quad \sin x = \dfrac{2t}{1-t^2}$ put $t = \tan\theta$

Result

$$\boxed{\begin{array}{l} xy = K \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x} \\ \dfrac{x}{y} = K \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x} \end{array}}$$

2) $y = \sin^2 \cot^{-1}\sqrt{\dfrac{1+x}{1-x}}$ put $x = \cos_2\theta$

3) $y = \tan^{-1}\left(\dfrac{4x}{1+5x^2}\right) + \tan^{-1}\left(\dfrac{2+3x}{3-2x}\right)$

Hint: $4x = 5x - x$ and $5x^2 = 5x.x$

$$\frac{2+3x}{3-2x} = \frac{\frac{2}{3}+x}{1-\frac{2}{3}x}$$

Put $5x = \tan A$ $x = \tan B$ and $\dfrac{2}{3} = \tan C$

Result

$$\boxed{\begin{array}{l} \text{When ; } \sin f(x) = K \cos f(x) = K \text{ etc} \\ \text{Take inverse Diff.} \end{array}}$$

Ex : $\sec\dfrac{x^2-y^2}{x^2+y^2} = e^a$.Find $\dfrac{dy}{dx}$

4) $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

$x=\sin A \ \ y=\sin B$

$\boxed{\cos A+\cos B=2\ \cos\ \frac{A+B}{2}\cos\ \frac{A-B}{2}}$

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

5. $Y=\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right)=\tan^{-1}\left(\frac{5\frac{x}{a}}{1-6\frac{x^2}{ax}}\right)=\tan^{-1}\frac{\frac{3x}{a}+\frac{2x}{a}}{1-\frac{3x}{a}\left(\frac{2x}{a}\right)}$

6. $\boxed{\sec\left(\frac{x^2-y^2}{x^2+y^2}\right)=e^a\ \text{ find }\ \frac{dy}{dx}}$

7. $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}\ \text{ find }\ \frac{dy}{dx}$

$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$\sin^{-1}y=\cos^{-1}x\Rightarrow x=\cos\left(\sin^{-1}y\right),y=\sin\left(\cos^{-1}x\right)$

$1-y^2=1-\sin^2\cos^{-1}x=\cos^2\left(\cos^{-1}x\right)=x^2$. Now differentiate

8. $3\sin xy+4\cos xy=5$

$\boxed{\begin{array}{l}3,4,5\\3^2+4^2=5^2\end{array}}\Rightarrow(3,4,5)\text{ is a pythagorian triple}$

Put $\frac{3}{5}=\sin A$ and $\frac{4}{5}=\cos A$

9. $y=\tan^{-1}\left[\frac{6x-8x^3}{1-12x^2}\right]$

$\boxed{2x=\tan\theta}$

10. $y=\cos^{-1}\left(\frac{3x-4\sqrt{1-x^2}}{5}\right)$ find $\frac{dy}{dx}$

$x=\cos\theta \therefore y=\cos^{-1}\left(\frac{3}{5}\cos\theta-\frac{4}{5}\sin\theta\right)$

Put $\frac{3}{5}=\cos A$ and $\frac{4}{5}=\sin A$

$y=\cos^{-1}\cos\left(A+\theta\right)=A+\theta$

11. $y=\tan^{-1}\sqrt{\frac{e^x-1}{e^x+1}}=\tan^{-1}\sqrt{\frac{e^x\left(1-\frac{1}{e^x}\right)}{e^x\left(1+\frac{1}{e^x}\right)}}$

Put $e^x=\frac{1}{\cos\theta}\Rightarrow\frac{\frac{1}{\cos\theta}-1}{\frac{1}{\cos\theta}+1}=\frac{1-\cos\theta}{1+\cos\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\tan^2\frac{\theta}{2}$

$y=\tan^{-1}\sqrt{\tan^2\frac{\theta}{2}}=x\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2}$

$\cos\theta=e^{-x}\Rightarrow\theta=\cos^{-1}e^{-x}$

$y=\frac{1}{2}\cos^{-1}e^{-x}\Rightarrow\frac{dy}{dx}=\frac{1}{2}\frac{-1\ \ x-e^{-x}}{\sqrt{1-e^{-2x}}}=\frac{1}{2}\left(\frac{-1}{\sqrt{1-e^{-2x}}}\right)\times\left(-e^{-x}\right)$

$\frac{dy}{dx}=\frac{1}{2}\frac{e^{-x}}{\sqrt{\frac{e^{2x}-1}{e^{2x}}}}=\frac{1}{2}\frac{e^{-x}e^{x}}{\sqrt{e^{2x}-1}}=\frac{1}{2\sqrt{e^{2x}-1}}$

12. $\boxed{y=\left(x+a\right)\left(x^2+a^2\right)\left(x^4+a^4\right)\left(x^8+a^8\right),x\neq a \quad \text{find}\ \frac{dy}{dx}}$

Derivative at a particular point

Differentiate directly and substitute the point

1) $f(x)=\tan^{-1}\left(\sqrt{1+x^2}-x\right)$ find $f'(0)$

2) $\tan^{-1}\left(\dfrac{5ax}{a^2-6x^2}\right)=y$ then $\dfrac{dy}{dx}$ at $x=0$

3) $y=\cot^{-1}\sqrt{\cos^2 x}$ $\dfrac{dy}{dx}$ at $x=\dfrac{\pi}{6}$

4) $f(x)=\cot^{-1}\left(\dfrac{x^x-x^{-x}}{2}\right)$ find $f'(x)$

In $f'(x)$ direct differentiation is used. Otherwise

Put $x^x=t\Rightarrow x^{-x}=\dfrac{1}{t}\to\dfrac{t-\frac{1}{t}}{2}=\dfrac{-\left(1-t^2\right)}{2t}$ Put $t=\tan\theta\Rightarrow\dfrac{-\left(1-\tan^2\theta\right)}{2\tan\theta}$

$$f(x)=\cot^{-1}-\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)=\tan^{-1}-\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$=\tan^{-1}\left(-\tan 2\theta\right)=-2\theta\Rightarrow x^x=-2\tan^{-1}t=-2\tan^{-1}x^x$$

Successive Differentiation (Higher order Derivation)

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d^2y}{dx^2}$ or $f''(x)$ or y_2 is the second order derivative.

$\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)=\dfrac{d^3y}{dx^3}$ or $f'''(x)$ is the third order derivative

$$\boxed{\begin{array}{l}\dfrac{d^ny}{dx^n}=n^{th}\text{ order Derivative}\\ \left(\dfrac{dy}{dx}\right)^n=n^{th}\text{ Degree Derivative}\end{array}}$$

1) $f(x)=x^4$ $f'(x)=4x^3$

$f^{II}(x)=12x^2$ $f^{III}(x)=24x$ $f^{IV}(x)=24$

2) $y=(2x+3)^2$ Find $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx}=(2x+3)^2=4(2x+3)\,;\ \frac{d^2y}{dx^2}=4\times 2=8$$

n^th^ Derivatives of some Functions

1) $$\frac{d^n}{dx^n}\,x^m=\frac{m!}{(m-n)!}x^{m-n}$$

2) $$\frac{d^n}{dx^n}\,x^n=n!$$

3) $$\frac{d^n}{dx^n}\,(ax+b)^m=\frac{m!}{(m-n)!}a^n\,(ax+b)^{m-n}$$

4) $$\frac{d^n}{dx^n}(ax+b)^n=n!a^n$$

5) $$\frac{d}{dx}\,e^{mn}=m^n e^{mn}$$

6) $$\frac{d^n}{dx^n}\sin(ax+b)=a^n\sin\left(ax+b+n\frac{\pi}{2}\right)$$

7) $$\frac{d^n}{dx^n}\sin x=\sin\left(x+n\frac{\pi}{2}\right)$$

8) $$\frac{d^n}{dx^n}\cos(ax+b)=a^n\cos\left(ax+b+n\frac{\pi}{2}\right)$$

9) $$\frac{d^n}{dx^n}\cos x=\cos\left(x+n\frac{\pi}{2}\right)$$

10) $$\frac{d^n}{dx^n}\log(ax+b)=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

11) $$\frac{d^n}{dx^n}\log x=\frac{(-1)^{n-1}(n-1)!}{x^n}$$

12) $\frac{d^n}{dx^n}\frac{1}{ax+b}=\frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$

13) $\frac{d^n}{dx^n}\frac{1}{n}=\frac{(-1)^n n!}{x^{n+1}}$

14) $\frac{d^n}{dx^n}xe^x = e^x(x+n)$

1) $y = ae^{mx}+be^{-mx}$ Find y_{10}

Find $y_{10} = a(m)^{10}e^{mx}+b(-m)^{10}e^{-mx}$

$y_{10} = m^{10}\left[ae^{mx}+be^{-mx}\right]=m^{10}y$

2) $\frac{d^{20}}{dx^{20}}2\cos x\ \cos 3x = \frac{d^{20}}{dx^{2}}\cos 4x+\cos 2x$ $\boxed{2\cos A\cos B=\cos(A+B)+\cos(A-B)}$

$=\frac{d^{20}}{dx^{20}}\cos 4x+\frac{d^{20}}{dx^{20}}\cos 2x$

Now use n^{th} derivatives

3) $\frac{d5}{dx5}\log(2x+3)$ use $\frac{dx}{dx^n}\log(ax+b))$

4) $f(x)=\tan^{-1}x$ Find $\frac{d^5}{dx^5}$ at$x=0$

$f'(x)=\frac{1}{1+x^2}=\frac{1}{(1+ix)(1-ix)}=\frac{A}{1+ix}+\frac{B}{1-ix}$

$1=A(1-ix)+B(1+ix)$ $\boxed{\begin{matrix}A+B=1\\ -A+B=0\end{matrix}}$ $A=\frac{1}{2}$ $B=\frac{1}{2}$

$\frac{d5}{dx5}=\frac{d4}{dx4}\left[\frac{1}{2}\frac{1}{1+ix}+\frac{1}{2}\frac{1}{1-ix}\right]$

Now use $\frac{dn}{dx^n}\frac{1}{(ax+b)}$; $\frac{d5}{dx^5}=\frac{1}{2}\left[\frac{(-1)^4(i)^4 4!}{(1+ix)^5}+\frac{(-1)^4(-i)^4 4!}{(1-ix)^5}\right]=\frac{1}{2}(24+24)=24$

Relation between y, y_1 and y_2

1) Find y_1 2) Square and cross multiply y_1

3) Get back function y 4) Differentiate once again w.r.t x 5) Divide by $2y_1$

1) $y=\cos(m\sin^{-1}x)=\cos m(\sin^{-1}x)$

$s.t(1-x^2)^2-xy_1+x^2y=0$

1) $y_1=-\sin(m\sin^{-1}x)\left(\frac{m}{\sqrt{1-x^2}}\right)$

2) $y_1^2(1-x^2)=m^2\sin^2(m\sin^{-1}x)$

3) $y_1^2(1-x^2)=m^2(1-y^2)$

4) $y_1^2(-2x)+(1-x^2)2y_1y_2=-m^2 2y\,y_1$

5) $-xy_1+(1-x^2)y_2=-m^2y$; $(1-x^2)y_2-xy_1+m^2y=0$

2) $y=e^a\sin^{-1x}$

$y_1=e^a\sin^{-1x}\times\frac{a}{\sqrt{1-x^2}}$; $y_1^2(1-x^2)=a^2y^2$

$y_1^2(-2x)+(1-x^2)^2y_1y_2=a^2 2yy_1$

$-xy_1+(1-x^2)y_2-a^2y=0$

3) $y=\left[x+\sqrt{1+x^2}\right]^m$

$y=\left(x+\sqrt{1+x^2}\right)^n$ Find relation y, y_1 and y_2

$y_1=n\left[x+\sqrt{1+x^2}\right]^{n-1}\left[1+\frac{2x}{2\sqrt{1+x^2}}\right]$; $y_1=n\left[x+\sqrt{1+x^2}\right]^{n-1}\left[\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right]$

$y_1^2\left(1+x^2\right)=x^2y^2$; $y_1^2 2x_1+\left(1+x^2\right)2y_1y_2=x^2yy_1$

$xy_1+\left(1+x^2\right)y_2=x^2y$

4) $y=\sin\ \log x$ Find relation y y_1 and y_2

$y_1=\cos\log x\times\frac{1}{x}\Rightarrow y_1^2x^2=\left(\cos\log x\right)^2$

$y_1^2x^2=1-\sin^2\log x\Rightarrow y_1^2x^2=1-y^2$

$y_1^2 2x+x^2 2y_1y_2=-2yy_1$; $xy_1+x^2+y_2=-y\Rightarrow x^2y_2+xy_1+y=0$

$$\left\{\begin{array}{l} y(x)=f\boxed{\cos\left(3\cos^{-1}x\right)}.\ \text{Find}\ \frac{1}{y(x)}\left[\left(x^2-1\right)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right]= \\ \text{The Qn is}\ \frac{1}{y}\left[\left(x^2-1\right)y_2+xy_1\right]=\text{Ans.}9 \end{array}\right\}$$

$y_1=-\sin\left(3\cos^{-1}x\right)\ x\frac{-3}{\sqrt{1-x^2}}$; $y_1^2\left(1-x^2\right)=9\left(1-y^2\right)$

$y_1^2\left(1-x^2\right)=9\left(1-y^2\right)$; $y_1^2\left(-2x\right)+\left(1-x^2\right)2y_1y_2=9\left(-2yy_1\right)$

$\left(1-x^2\right)y_2-xy_1=-9y\Rightarrow\frac{1}{y}\left[\left(x^2-1\right)y_2+xy_1\right]=9$

Partial Differentiation

If a dependent variable u depends on two independent variable x and y, it is denoted by $u=f\left(xy\right)$ and is called a Bivariate function.

Let $u=f\left(x\,y\right))$ be a bivariate funtion. The derivative u w.r.t x when y remains a constant is called the partial derivative of u w.r.t x and is denoted by $\frac{\partial y}{\partial x}$. Thus

$$\frac{\partial u}{\partial x}=\underset{\Delta x\to 0}{\text{Lt}}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\quad f(x,y)$$

$\frac{\partial u}{\partial y}$ is the rate of change of u w.r.t. y when x remains constant.

$$u = 3x^2 + 3x^2y + 4xy^2 + 3y^3$$

$$\frac{\partial u}{\partial x} = 6x + 6xy + 4y^2$$

$$\frac{\partial u}{\partial y} = 3x^2 + 8xy + 9y^2$$

$$u = 4x^7 + 3x^5y^2 + 5y^7$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(28x^6 + 15x^4y^2\right) + y\left(6x^5y + 35y^6\right)$$

$$= 28x^7 + 15x^5y^2 + 6x5y^2 + 35y^7 \quad = 28x^7 + 21x^5y^2 + 35y^7 = 7\left(4x^7 + 3x^5y^2 + 5y^7\right) = 7u$$

Euler' Theorem

$u = f(x,y)$ is a bivariate homogenous function in degree n . Then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$\boxed{\text{Derivative of implicit function}}$

Consider the implicit function $u = f(x,y) = 0$

Then $$\boxed{\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial y}}}$$

Ex:1) Let $\boxed{x^2 + x^2y^3 + y^2 = 0}$

$$\frac{dy}{dx} = \frac{-\frac{\partial y}{\partial x}}{\frac{\partial u}{\partial y}} = -\left[\frac{2x + 2xy^3}{3x^2y^2 + 2y}\right]$$

2) $\mathrm{Sin}(x+y)=\log(x+y)$ Find $\frac{dy}{dx}$

$\sin(x+y)-\log(x+y)=0$

$$\frac{\partial y}{\partial}\frac{dy}{dx}=\frac{\frac{-\partial y}{\partial x}}{\left(\frac{\partial x}{\partial y}\right)}=-\left[\frac{\cos(x+y)-\frac{1}{x+y}}{\cos(x+y)-\frac{1}{x+y}}\right]=-1$$

3) $x^2+y^2=2-\sin xy$; $\therefore$ $x^2+y^2+\sin xy-2=0$

$$\frac{dy}{dx}=\frac{-\frac{\partial u}{\partial x}}{\frac{\partial x}{\partial y}}=-\left[\frac{2x+\cos xy\times y}{2y+\cos xy\times x}\right]$$

4) $e^{xy}+\log xy+\cos xy+5=0$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx}=-\left[\frac{ye^{xy}+y\frac{1}{xy}-\sin xy.y}{xe^{xy}+x\frac{1}{xy}-\sin xy\times x}\right]=-\frac{y}{x}\left[\frac{e^{xy}+\frac{1}{xy}-\sin xy}{e^{xy}+\frac{1}{xy}-\sin xy}\right]=-\frac{y}{x}$$

5) $ax^2+2hxy+by^2+2gx+2fy+c=0$

$$\frac{dy}{dx}=\frac{-\frac{\partial y}{\partial x}}{\left(\frac{\partial x}{\partial y}\right)}=-\left[\frac{2ax+2by+2g}{2bx+2by+2f}\right]$$

Leibnitz Theorem

Let $u=f(x)$ be a bivariable function.

$$\boxed{\frac{d}{dx}\int_{\phi(x)}^{g(x)}f(x\,t)\,dt=\int_{\phi(x)}^{g(x)}\frac{\partial}{\partial x}f(x,t)\,dt+g'(x)f(x\;g(x))-\phi'(x)f(x\;\phi(x))}$$

Ex: $\frac{d}{dx}\int_{x^2}^{e^x}(\sin x+\log t)\,dt=\int_{x^2}^{e^x}\cos x\,dt+e^x(\sin x+x)-e^x(\sin x+2\log x)$

$=\cos x\left(x^x-x^2\right)+e^x(x+\sin x)-e^x(\sin x+2\log x)$

Particular cases

1) $\frac{d}{dx}\int_{\phi(x)}^{g(x)}f(t)\,dt=g'(x)f[g(x)]-\phi'(x)f(\phi(x))$

2) $\frac{d}{dx}\int_{k}^{\phi(x)}f(t)\,dt=\phi'(x)f(\phi(x))$

3) $\frac{d}{dx}\int_{\phi(x)}^{k}f(t)\,dt=-\phi'(x)f(\phi(x))$

Questions

1) $g(x)=\frac{d}{dx}\int_{x^2}^{x^3}\log t\,dt$ Find $g'(e)$

$g(x)=3x^2(3\log x)-2x(2\log x)$

$g'(x)=9\left(\frac{x^2}{x}+2x\log x\right)-4\left(\frac{x}{x}+\log x\right)$

$g'(e)=9(e+2e)-4(2)=27e-8$

2) $x=\int_0^y\frac{1}{\sqrt{1+4t^2}}dt$ find $\frac{d^2y}{dx^2}$

3) $f(x)=\frac{1}{x^2}\int_4^x\left(4t^2-2f'(t)\right)dt$ then $f'(4)=$

4) $f(x)$ is a continuous different function such that $\int_0^x f(t)dt=f(x)$. Find log f(5)

5) $f(x)$ is a non- negative fx defined in [0 1] s.t

$\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)\,dt$. Given f(0) = 0 . Then

A) $f\left(1/2\right) < 1/2$ B) $f\left(1/2\right) > 1/2$ C) $f\left(1/3\right) < 1/3$ D) $f\left(\pi/2\right) = 1$

Inportant Result

$$\boxed{\begin{array}{l}
\text{When } f'(x) = f(x) \\
\dfrac{f'(x)}{f(x)} = 1 \\
\dfrac{f'(x)}{f(x)}dx = \int 1dx \\
\log f(x) = x + cf(x) = e^c e^x;\ f(0) = e^c \Rightarrow f(0) = 0 \Rightarrow f(x=0) \\
f(0) = 1 \Rightarrow f(x) = e^x \\
f(0) = 2 \Rightarrow f(x) = 2e^x \\
f(0) = k \Rightarrow f(x) = ke^x
\end{array}}$$

Derivative of a function w.r.t another function

$$\left.\begin{array}{l}\text{Derivative of } f(x) \\ \text{w.r.t } g(x)\end{array}\right\} = \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)} = \frac{f'(x)}{g'(x)}$$

1) $$\left.\begin{array}{l}\text{Derivative of } \sin x \\ \text{w.r.t } \cos x\end{array}\right\} = \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}\cos x} = -\cot x$$

2) $$\left.\begin{array}{l}\text{Derivative of } a^x \\ \text{w.r.t } x^a\end{array}\right\} = \frac{a^x \log a}{ax^{a-1}}$$

3) $\left.\begin{array}{l}\text{Derivative of } x^x \\ \text{w.r.t } \cos x^{-x}\end{array}\right\} = \dfrac{x^x(1+\log x)}{-x^{-x}(1+\log x)} = -x^{2x}$

4) Derivative of $\sin x^3$ w.r.t x^3

5) Derivative of $a^{\sin^{-1}x}$ w.r.t $\sin^{-1}x$

6) Derivative of $\sin^2 x$ w.r.t$(\log x)^2$

Derivative of a Determinant

$$\frac{d}{dx}\begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x)\end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x)\end{vmatrix} + \begin{vmatrix} f1(x) & f_2(x) \\ g_1^1(x) & g_2^2(x)\end{vmatrix}$$

1) $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x\end{vmatrix}$ $\Delta_2 = \begin{vmatrix} x & b \\ a & x\end{vmatrix}$ S.T $\dfrac{d}{dx}\Delta_1 = 3\Delta_2$

$$\frac{d}{dx}\Delta_1 \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x\end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x\end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1\end{vmatrix} = (x^2-ab)+(x^2-ab)+(x^2-ab) = 3\Delta_2$$

2) $f(x)\begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ si(\alpha-\beta) & \sin(\beta-\gamma) & \sin(\alpha-\gamma)\end{vmatrix}$

$f(x) = 5$ Find $\sum_{r=1}^{20} f(x)$