THEORY OF PROBABILITY PART -II

Random experiment: An experiment whose outcomes can not be predicted in advance and it satisfies the following conditions (i) It has more than one possible outcomes (ii) It is not possible to predict the outcomes in advance (iii) The outcomes of the experiment should vary irregularly (iv) when we repeat the experiment it should result in one of its different possibilities

Sample space: The set of all possible outcomes of the random experiment. For example when a coin is tossed the sample space $S = \{H, T\}$. When two coins are tossed the sample space

$S = [HH, HT, TH, TT]$

Event: Any finite sub set of the sample space is called an event

Let $S = \{1, 2, 3, 4, 5, 6\}$

A = Even face = $\{2, 4, 6\}$ and $B = \{1, 3, 5\}$ etc are events

$$\boxed{\begin{array}{l} \text{Let } n(s) = n \\ \text{number of events} = 2^n \\ S = \text{sure event} \\ \phi = \text{Impossible event} \end{array}}$$

Algebra of events

i) Complement of event A $\left(A' \text{ or } \overline{A} \text{ or } A^c\right)$

The set of all outcomes in S, but not in event A is called complement of A

ie $A' = S - A$ and $\boxed{A \cup A' = S,\ A \cap A' = \phi}$

ii) Union of two events: The union of two even A and B are the set of outcomes either in A or B

ie $\boxed{A \cup B = A \text{ or } B}$

III) Intersection of two events : The intersection of two events A and B is the set of outcomes in both A and B ie $\boxed{A \cap B = A \text{ and } B}$

iv) $'A \text{ but not } B'$:It is the event A–B

$\boxed{\text{A but not B=A-B=A} \cap B'}$

v) $'B \text{ but not } A'$: It is B–A

$\boxed{\text{B but not A=B-A=B} \cap A'}$

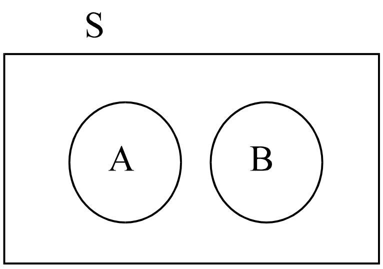
Type of Events: There are 3 different types of events namely equally likely events, mutually exclusive events and exhaustive events.

i) Equally likely events: Events are equally likely if they have the same chance to occur. For example when a fair coin is tossed $\{H\}$ and $\{T\}$ are equally likely events

ii) Mutually exclusive events

Events are mutually exclusive if they can not occur at the same time. For example when a fair coin is tossed $\{H\}$ and $\{T\}$ are mutually exclusive

$\boxed{\text{A and B are mutually exclusive} \Rightarrow A \cap B = \phi}$



$A \cap B = \phi \Rightarrow A$ and B are mutually exclusive

iii) Exhaustive events : Events are exhaustive if their union is the sample space

$$\boxed{\begin{array}{l} \text{A and B are exhaustive} \Rightarrow A \cup B = S \\ \text{A, B and C are exhaustive} \Rightarrow A \cup B \cup C = S \end{array}}$$

Probability: Probability is defined as a numerical measure of chance of future events. There are 3 different schools of thought on the concept of probability. They are classical or mathematical probability, statistical or empirical probability and the axiomatic probability

1) Mathematical or classical probability

In classical definition probability of an event 'A' is defined as

$$\boxed{P(n) = \frac{m}{n}}$$

Where n is the total numbers of outcomes in sample space and 'm' is the number of favourable outcomes in event A

The classici definition can be used only when 'n' is finite and the various outcomes in the sample space are equally likely

Examples

1) A fair die is thrown. The sample space is $S = \{1, 2, 3, 4, 5, 6\} \therefore n = 6$

Let A = Prime faces = $\{2, 3, 5\} \therefore m = 3$

$$\therefore P(A) = P(\text{Prime face}) = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$$

2) Three coins are tossed. What is the probability of gelting i) Exactly two heads ii) At least two heads and iii) At most two heads

$$S = \{HHH, HHT, THH, HTH, TTH, HTT, THT, TTT]$$

$$n = n(s) = 8$$

i) $P(A) = P[\text{Exactly two heads})$

$$= P(HHT, THH, HTH) = \frac{3}{8}$$

ii) P(A) = P(At least two heads)

$$= P(HHT, THH, HTH, HHH) = \frac{4}{8} = \frac{1}{2}$$

iii) P(A) = P(At most two heads)

$$= P(TTT, TTH, HTT, THT, HHT, THH, HTH) = \frac{7}{8}$$

3) $A = \{1, 2, 3, 4, 5\}$ $B = \{x, y\}$. A relation is selected from set A to B. What is the probability that it is a function.

$n(A) = m = 5 \quad n(B) = n = 2$

Total number of relations = $2^{mn} = 2^{10}$

Number of functions from A to B = n^m

$= n^m = 2^5$

$$P[\text{Relation is a function}] = \frac{2^5}{2^{10}} = \frac{1}{32}$$

4) 'a' and 'b' are obtained by throwing a pair of dice. What is the probability that $\underset{x \to 0}{Lt}\left(\frac{a^x + b^x}{2}\right)^{\frac{2}{x}} = 6$

Answer

Throwing a pair of dice means total number of outcomes = 36

$$\underset{x\to 0}{\mathrm{Lt}}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}} = \underset{x\to 0}{\mathrm{Lt}}\left(1+\frac{a^x+b^x}{2}-1\right)^{\frac{1}{x}}$$

$$= \underset{x\to 0}{\mathrm{Lt}}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{1}{\left(\frac{x}{2}\right)}}$$

$$= e^{\underset{x\to 0}{\mathrm{Lt}}\frac{\left(\frac{a^x+b^x-2}{2}\right)}{\left(\frac{x}{2}\right)}} = e^{\underset{x\to 0}{\mathrm{Lt}}\frac{a^x+b^x-2}{x}}$$

$$= e^{\underset{x\to 0}{\mathrm{Lt}}\frac{a^x-1}{x}+\frac{b^x-1}{x}} = e^{\log a+\log b}$$

$$= e^{\log ab} = ab$$

Given limit = 6

$\therefore P(\text{Limit}=6) = P(ab=6)$

'a' and 'b' are obtained by throwing a pair of dice

$\therefore P(ab=6) = P\{(1,6)(6,1)(2,3)(3,2)\}$

$$=\frac{4}{36}=\frac{1}{9}$$

5) The letters of the word 'SLEEPLESSNESS' are arranged at random. What is the probability that all the Ss come together

Answer : SLEEPLESSNESS

Total number of letters = 13

Number of S = 5 number of E = 4 number of L = 2

Total number of arrangements = $\frac{13!}{5!\times 2!\times 4!}$

When Ss are together

$$\overset{1}{\boxed{SSSSS}}\ \overset{2}{E}\ \overset{3}{E}\ \overset{4}{E}\ \overset{5}{E}\ \overset{6}{L}\ \overset{7}{L}\ \overset{8}{N}\ \overset{9}{P}$$

Number of favourable arrangements = $\frac{9!}{4!\times 2!}$

$$P(\text{Ss together}) = \frac{\left(\frac{9!}{4!\times 2!}\right)}{\left(\frac{(13!)}{5!\times 2!4!}\right)}$$

$$=\frac{9!\times 5!}{13!}=\frac{1\times 2\times 3\times 4\times 8}{10\times 11\times 12\times 13}=\frac{1}{143}$$

Statistical or Empirical probability

Let a random experiment be repeated 'n' times and let an event 'A' occurs 'r' times. Then $\left(\frac{r}{n}\right)$ is called the frequency Ratio. In statistical or empirical definition probability of event is defined as $P(A)=\underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)$ where the limit exists and is finite

Axiomatic probability

In Axiomatic theory probability is a real valued set function from the power set of sample space to the set of real nos in [0 1] satisfying the following anions

1) $P(A)\geq 0$

2) $P(A)\leq 1$

3) $P(S)=1\Rightarrow P(\text{sure event})=1$

4) $P(\phi)=0\Rightarrow P(\text{Impossible event})=0$

5) $P(A\cup B)=P(A)+P(B)$ if $A\cap B=\phi$ ie if A and B are mutually exclusive

As a function

Domain of probability = Power set of S

Range of probability = [0 1]

Show that $P(A')=1-P(A)$ where A' is the complement of A

Answer : $A\cup A'=S$ and $A\cap A'=\phi\ \therefore$ A and A' are mutually exclusive and exhaustive

$$P(A\cup A')=P(S)$$

$$P(A\cup A')=1\quad(\text{By axiom } 3)$$

$$\boxed{\begin{array}{l}P(A')=1-P(A)\\ P(A)=1-P(A')\end{array}}$$

Addition theorem on Probability

If A and B are any two events the addition theorem states that

$$\boxed{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$

But $A\cup B=A\text{ or }B\quad A\cap B=A\text{ and }B$

Hence the addition theorem can also be written as

$$\boxed{P\left(A \text{ or } B\right)=P\left(A\right)+P\left(B\right)-P\left(A \text{ and } B\right)}$$

Now suppose that A,B are C are any three event. The addition theorem states that

$$P\left(A\cup B\right)=P(A)+P(B)+P(C)-P(A\cap B)$$

$$-P\left(B\cap C\right)-P\left(A\cap C\right)+P\left(A\cap B\cap C\right)$$

Questions

1) The probability that a contractor may get an electric contract is $\frac{1}{2}$ and that he may get a plumbing contract is $\frac{1}{3}$. The probability that he will get both the contracts is $\frac{1}{4}$. What is the probability that he will get at least one contract.

A = Electric contract B = Plumbing contract

$$P(A)=\frac{1}{2}\ \ P(B)=\frac{1}{3}\ P(A \text{ and } B)=P\left(A\cap B\right)=\frac{1}{4}$$

$$P\left[\text{At least once}\right]=P\left[A \text{ or } B\right]=P\left(A\cup B\right)$$

$$P\left(A\cup B\right)=P\left(A\right)+P\left(B\right)-P\left(A\cap B\right)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}$$

2) An integer is taken at random from the first 200 natural numbers. What is the probability that it is divisible by 6 or 8.

A = Divisible by 6

B = Divisible by 8

$$n\left(A\right)=\left[\frac{200}{6}\right]=33\ \ \text{where}\left[.\right]=\text{GIV}$$

$$n\left(B\right)=\left[\frac{200}{8}\right]=25$$

$$n\left(A\cap B\right)=n\left(\text{Divisible by } 6 \text{ and } 8\right)=\left[\frac{200}{\text{LCM of } 6 \text{ and } 8}\right]$$

$$=\left[\frac{200}{24}\right]=8$$

$$P(A)=\frac{33}{200}\,P(B)=\frac{25}{200}\,P(A\cap B)=\frac{8}{200}$$

$$P(\text{Disible by 6 or 8})=P(\text{A or B})$$

$$=P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$=\frac{33}{200}+\frac{25}{200}-\frac{8}{200}=\frac{50}{200}=\frac{1}{4}$$

Odds in favour and against of an event A

Let n(A) = number of outcomes in favour of an event A and

$n(A')$ = number of outcomes against an event A

$$\boxed{\begin{array}{l}\text{Odds in favour of A}=\dfrac{n(A)}{n(A')}\\ \text{Odds against A}=\dfrac{n(A')}{n(A)}\end{array}}$$

Question

A party of n persons sit around a table. What are the odds against two persons sitting next to each other

Total number of persons = n

Total number of arrangements = $(n-1)!$

n(A) = number of arrangements in favour of 2 persons together = $(n-2)!\times 2!$

$$\therefore n(A')=\text{Total}-n(A)$$

$$=(n-1)!-2(n-2)!$$

$$=(n-2)!(n-1-2)=(n-2)!(n-3)$$

Odds against $A = \frac{n(A')}{n(A)} = \frac{(n-2)!(n-3)}{2(n-2)!} = \frac{n-3}{2}$

Problems based on packet of playing cards

The details regarding the packet of playing cards are given below

Sl. No.	Spade (Black)	Club (Black)	Hearts (Red)	Diamond (Red)	Total
1	KING	KING	KING	KING	4
2	QUEEN	QUEEN	QUEEN	QUEEN	4
3	JACK	JACK	JACK	JACK	4
4	ACE	ACE	ACE	ACE	4
5	2	2	2	2	4
6	3	3	3	3	4
7	4	4	4	4	4
8	5	5	5	5	4
9	6	6	6	6	4
10	7	7	7	7	4
11	8	8	8	8	4
12	9	9	9	9	4
13	10	10	10	10	4
TOTAL	**13**	**13**	**13**	**13**	**52**

Total number of cards = 13×4= 52
Number of Red cards = 26
Number of Black cards = 26
Number of Kings / Queens/ Jack / Ace cards = 4
Number of Spade / Clubs / Hearts / Diamonds = 13

Face cards | court cards: King + Queen + Jack = 12 cards

1) A card is taken from a packet of cards. What is the probability that it is a spade or Ace

$$P(\text{spade}) = \frac{13}{52} \quad P(\text{Ace}) = \frac{4}{52}$$

$$P(\text{Spade and Ace}) = \frac{1}{52}$$

$$P(\text{Spade or Ace}) = P(\text{spade}) + P(\text{Ace}) - P[\text{ Spade and Ace}]\text{-(Addition Theorem)}$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

2) Two cards are taken from a packet of cards. What is the probability that both are queens if cards are taken

i) at a time ii) one by one with replacement iii) one by one without replacement

i) At a time

$$\text{P (Both Queens)} = \frac{4C_2}{52C_2} = \frac{4 \times 3}{52 \times 51} = \frac{1}{221}$$

ii) with Replacement

$$\text{P (Both Queens)} = \frac{4C_1}{52C_1} \times \frac{4C_1}{52C_1} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

iii) with out replacement

$$\text{P (Both Queens)} \ \text{P (Both} = \frac{4C_1}{52C_1} \times \frac{3C_1}{51C_1} = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

3) A card is taken from packet of cards and gambler bets that it is a spade or Ace. What are the odds against his winning the bet.

Answer: By Addition Theorem

$$P(\text{spade or Ace}) = P(\text{spade}) + P(\text{Ace}) - P(\text{spade and Ace})$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

$$P(\text{wins bet}) = \frac{4}{13}$$

$$\therefore \text{Total} = 13 \quad n(A) = 4 \quad n(A') = 13 - 4 = 9$$

$$\text{Odds against} = \frac{n(A')}{n(A)} = \frac{9}{4}$$

4) What is the probability that in a hand of 7 cards drawn from a packet of 52 cards will contain

i) All kings

ii) 3 kings

iii) At least 3 kings

i) $P(\text{All kings}) = \dfrac{4C_4 \times 48C_3}{52C_7} = \dfrac{1}{7735}$

ii) $P(3\text{ kings}) = \dfrac{4C_3 \times 48C_4}{52C_7} = \dfrac{45}{7735}$

iii) $P[\text{At least 3 kings}] = P(\text{All}) + P(3)$

$= \dfrac{1}{7735} + \dfrac{45}{7735} = \dfrac{46}{7735}$

Conditional probability

Let A and B be two events having non zero probabilities. The conditional probability of the event A given that the event B has already occurred is denoted by P(A/B) (we read it as probability of A given B) and is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Similarly $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

Questions

1) The probability that student A passes in physics is 65% and he will pass in chemistry is 70%. The probability that he will pass in both the subjects is 40%. If he passes in physics what is the probability that he will pass in chemistry also

2) One ticket is taken at randon from ticket with serial nos 00,01,02,......,49. If the sum of digits is 8 what is the probability that product of digits is zero

3) An integer is choosen at random from the first 200 natural nos. If the integer is divisible by 8 what is the probability that it is divisible by 6

4) A parent has two children. If one of them is a boy what is the probability that other is also a Boy

S = [BB BG GB GG]

B= One Boy = [BB BG GG]

A = other Boy = [BB]

$$P[\text{other Boy} \mid \text{One is Boy}] = P[A \mid B] = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$$

5) A question bank consists of 300 easy T|F Questions, 200 difficult true or false questions, 500 easy MCQS and 400 difficult MCQS. If a question is selected at random what is the probability that it is an easy question given that it is an MCQ.

	Easy	Diff	Total
T\F	300	200	500
MCQ	500	400	900
Total	800	600	1400

$P[\text{Easy} \mid \text{MCQ}] =$

$$\frac{P[\text{Easy and MCQ}]}{P[\text{MCQ}]}$$

$$= \frac{\left(\frac{500}{1400}\right)}{\left(\frac{900}{1400}\right)} = \frac{5}{9}$$

Multiplication theorem

Let A and B be two events having non zero probabilities. Then

$$P(A \cap B)\left.\begin{array}{l} = P(A)PCB \mid A \\ \text{or} \\ = P(B)P(A \mid B) \end{array}\right\}$$

$$\boxed{\begin{array}{l} \text{Let there be 3 events A,B and C} \\ D(A \cap B \cap C)\left.\begin{array}{l} = P(A)P(B \mid A)P(C \mid A \cap B) \\ = P(B)P(A \mid B)P(C \mid A \cap B) \\ = P(C)P(B \mid C)P(A \mid B \cap C) \end{array}\right\} \end{array}}$$

1) A card is drawn sucessively without replacement. From a packet of 52. What is the probability that the first 2 cards are king and the 3rd is ace

$A = I^{st}\text{king} \quad B = 2^{nd}\text{king} \ C = 3^{rd}\text{Ace}$

P(A and B and C) = $P(A \cap B \cap C)$

$= P(A)P(B \mid A)P(C \mid A \cap B)$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$$

Question

X and Y are events such that $P(X) = \frac{1}{3}$ $P(X \mid Y) = \frac{1}{2}$ $P(Y \mid X) = \frac{2}{5}$. Which of the following are true

A) $P(y) = \frac{4}{15}$ B) $P(x \mid y) = \frac{1}{2}$

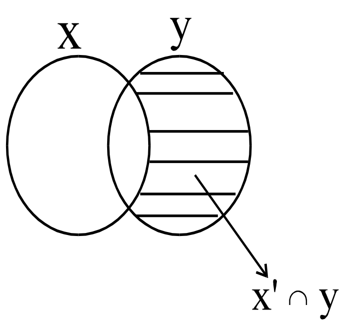
C) $P(x \cup y) = \frac{2}{5}$ D) $(x \cap y) = \frac{1}{5}$

BY M.T $P(x \cap y) = P(x)P(y \mid x) = P(y)P(x \mid y)$

$$P(x \cap y) = \frac{1}{3}\frac{2}{5} = P(y)\frac{1}{2}$$

$$P(x \cap y) = \frac{2}{15} \quad p(y) = \frac{4}{15}$$

$$P(x \cup y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{7}{15}$$

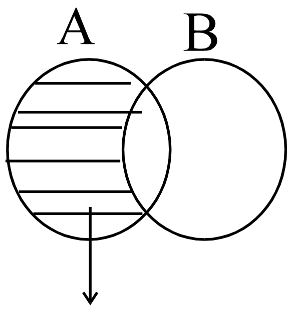


$$P(x'y) = \frac{P(x' \cap y)}{p(y)} = \frac{P(y) - P(x \cap y)}{P(y)}$$

$$= \frac{\frac{4}{15} - \frac{2}{15}}{\left(\frac{4}{15}\right)} = \frac{\left(\frac{2}{15}\right)}{\left(\frac{4}{15}\right)} = \frac{1}{2}$$

Result I

When A and B are independent show that A and B are independent. A and B are independent $\Rightarrow$
$P(A \cap B) = P(A)P(B)$



$A \cap B'$

$A \cap B' = A - (A \cap B)$

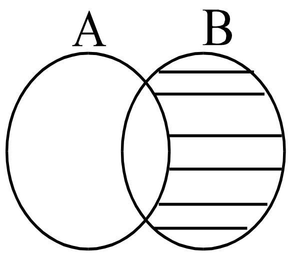
$P(A)(1 - P(B'))$

$= P(A)P(B')$

$\therefore P(A \cap B') = P(A)P(B') \Rightarrow A \text{ and } B'$ are independent

Result 2

When A and B are independent show that A’ and B are independent



$B \cap A'$

$P(B \cap A')P(B) - P(A \cap B)$

$P(B \cap A') = P(B) - P(A)P(B)$

$= P(B)(1 - P(A))$

$= P(B)P(A') \Rightarrow B \text{ and } a' \text{ are independent}$

Result 3

When A and B are independent show that A' and B' are independent

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - \left[P(A) + P(B) - P(A)P(A)\right]$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= P(A') - P(B)(1 - P(A'))$$

$$= P(A') - P(B)P(A')$$

$$= P(A')(1 - P(B))$$

$$= P(A')P(B') \Rightarrow A' \text{and } B' \text{ are independent}$$

Part III

Independent Events

When A and B are independent

$$\boxed{P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B)}$$

Results

When A and B are independent

$$\boxed{\begin{array}{l} \text{i) } A' \text{ and } B' \text{are independent} \\ \text{ii)} A \text{ and } B' \text{are independent} \\ \text{iii)} A' \text{ and } B \text{ are independent} \end{array}}$$

Multiplication theorem for independent events

When A and B are independent

$$\boxed{P(A \cap B) = P(A)P(B)}$$

Addition theorem for independent events

$$\boxed{\begin{array}{l} P(A \cup B) = 1 - P(A')P(B') \\ P(A \cup B \cup C) = 1 - P(A')P(B')P(C') \end{array}}$$

$$\text{Where } \left\| \begin{array}{l} P(A') = 1 - P(A) \\ P(B') = 1 - P(B) \\ P(C') = 1 - P(C) \end{array} \right\|$$

Question

A and B are independent events such that probability both A and B is $\frac{1}{12}$ and probability neither A nor B is $\frac{1}{2}$ find P(A) and P(B)

$$P(A \cap B) = \frac{1}{12} \quad P(A' \cap B') = \frac{1}{2}$$

$$P(A)P(B) = \frac{1}{12} \quad P(A')P(B') = \frac{1}{2}$$

$$(1 - P(A))(1 - P(B)) = \frac{1}{2} \Rightarrow 1 - [P(A) + P(B)] + P(A)P(B) = \frac{1}{2}$$

$$1 + \frac{1}{12} - \frac{1}{2} = P(A)P(B)$$

$$P(A)P(B) = \frac{1}{12}$$

$$P(A) + P(B) = \frac{7}{12}$$

$$P(A) = \frac{1}{3} \; P(B) = \frac{1}{4}$$

$$P(A) = \frac{1}{4} \; P(B) = \frac{1}{3}$$

Question

A fair coin and an unbiased die are tossed. Let A =Head on coin and B = 3 on die. Check whether A and B are independent. $A = [H_1 \; H_2 \; H_3 \; H_4 \; H_5 \; H_6]$

$$B = \text{face } 3 = [H_3 \; T_3]$$

$$S = [H_1, H_2, H_3, H_4, H_5, H_6, T_1, T_2, T_3, T_4, T_5, T_6]$$

$$P(A) = \frac{6}{12} = \frac{1}{2} \; P(B) = [H_3, T_3] = \frac{2}{12}$$

$$A \cap B = [H_3] \Rightarrow P(A \cap B) = \frac{1}{12}$$

$$P(A) \times P(B) = \frac{6}{12} \times \frac{2}{12} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

$P(A)P(B) = P(A \cap B)$

$\therefore$ A and B are independent

Mutual and pairwise independent events

3 events A,B and C are said to be pairwise independent if

i) $P(A \cap B) = P(A)P(B)$

ii) $P(B \cap C) = P(B)P(C)$

iii) $P(A \cap C) = P(A)P(C)$

In addition to these 3 properties if $P(A \cap B \cap C) = P(A)P(B)P(C)$ then the events A,B and C are mutually independent.

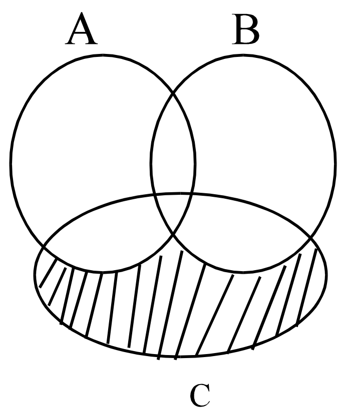
Question

A, B are C are pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$

Then $P(A' \cap B' | C) =$

A) $P(B') - P(A)$ B) $P(B') - P(A')$

C) $P(A') - P(B')$ D) $P(B') - P(A)$



$$P(A' n B' | C) = \frac{P(A' \cap B' n C)}{P(C)} = \frac{P[\text{shaded part}]}{P(C)}$$

$$= \frac{P(C) - P(B \cap C) - P(A \cap C)}{P(C)}$$

$$= \frac{P(C) - P(B)P(C) - P(A)P(C)}{P(C)}$$

$$= 1 - P(B) - P(A) \left.\begin{matrix} = P(B') - P(A) \\ = P(A') - P(B) \end{matrix}\right.$$

Random Variabiles and Expected value of A Random Variable

Definition

A variable taking values based on the outcomes of a sample space is called a random variable (RV). It is denoted by X. For example consider the tossing of two coins

$S = [HH \;\; HT \;\; TH \;\; TT]$

Let 'X' be the number of Heads

X = 0 when TT occurs

X=1 when HT or TH occur

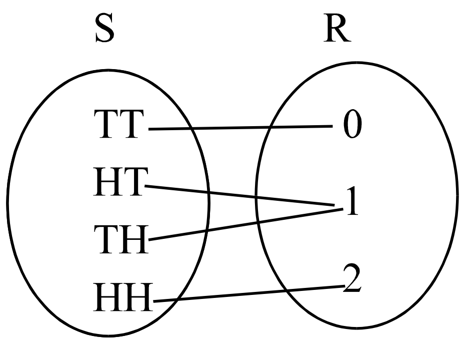
X = 2 when HH occurs

Thus 'X' takes values '0' , '1' and '2' based on the outcomes of the sample based on the outcomes of the sample space and hence 'X' is a R.V.

Random variable as a function

Consider the tossing of two coins $S = [HH \;\; HT \;\; TH \;\; TT]$

Let X = number of Heads



Thus in Mathematical sense a R.V is a function from sample space tot he set of real number

$\boxed{X : fn : S \to R}$

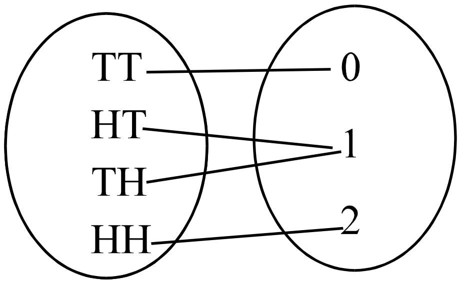
Discrete R.V. : Taking either finite and countably infinite number of values. Continuous R.V: Taking any value in an interval.

Probability Mass function (pmf)

The probability that a discrete R.V. 'X' takes a particular value x_i is denoted by $P(x_i) = P[X = x_i]$ and is called the pmf

Ex: Consider the tossing of two coins

Let X = no of Heads



$$P(0) = P(X = 0) = P(TT) = \frac{1}{4}$$

$$P(1) = P(X = 1) = \frac{3}{4}$$

$$P(2) = P(X = 2) = \frac{1}{4}$$

$$\boxed{\begin{array}{l} P(0) = \frac{1}{4}\ P(1) = \frac{2}{4} \text{ and} \\ P(2) = \frac{1}{4} \text{ are called pmf} \end{array}}$$

Probability Distribution of a Random variable

A table presenting the various values of Random variable and the corresponding pmf is called the probability distribution of the R.V. For example consider the tossing of 2 coins. The probability distribution of the number of heads is given below

X no.of HS	O TT	1 HT or TH	2 HH
pmf P(x)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Properties of pmf

i) $P(x_i) \geq 0$

ii) $P(x_i) \leq 1$

iii) $\sum P(x_i) = 1$

Expected value of a Random variable

The expected value of the R.V is the A.M of the Random variable and it is given by

$$\boxed{E(x) = \text{A.M of } X = \sum x_i P(x_i)}$$

The variance of the Random variable is given by

$$\boxed{V(x) = E(x^2) - [E(x)]^2}$$

The S.D of the Random variable is given by

$$\boxed{S.D(x) = \sqrt{V(x)}}$$

Qn. 3 coins are tossed. Find the mean variance and S.D of the number of Heads

$S = [HHH, HHT, THHH, HTH, TTH, HTT, THT, TTT]$

X = no. of Heads

X = 0 when TTT occurs $P(0) = \frac{1}{8}$

$X = 1$ when HTT, TTH, GTH $\Rightarrow$ P(1)=2/8

X=2 when HHT, THH, HTH $\Rightarrow$ P(2)=3/8

X= 3 when HHH $\Rightarrow$ P(3) = 1/8

$\therefore$ Probability distribution of X is

x	0	1	2	3
$P(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean = $E(x) = \sum x_i P(x_i) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8} = 1.5$

$E(x^2) = \sum x_i^2 P(x_i) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2$

$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$

$V(x) = E(x^2) - (E(x))^2 = 3 - (1.5)^2 = 0.75$

$$S.D = \sqrt{V(x)} = \sqrt{0+5} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Bernoulli Trials and Binomial Distribution

Consider the tossing of a coin. When the coin is tossed there are only two possible outcomes head (H) and Tail (T). Such experiments where there are only two possible outcomes are called Binomial experiments or Bernoulli trails

The Binomial outcomes are named as “Success” (S) and “Failure” (F). When a coin is tossed if getting a head is a success, then getting a Tail is failure. If the coin is tossed ‘n’ times we say there are ‘n’ trials. In each trial there are only two possible outcomes success (S) or failure (F). The outcomes in any trial are independent. Such independent trials which have only two possible outcomes namely success and failure are called Binomial or Bernocelli trials.

Definition

Trails of a random experiment are called Bernoulli trials if they have the following conditions

i) There should be a finite number of trails

ii) The trials should be independent

iii) Each areal has only two outcomes which are named as success and failure

iv) The probability of success remains the same in each trail

For example if die is thrown 50 times we say there are 50 trials. Let us define success as even face and failure as odd face. Let

$$p = P[\text{success}] = P[\text{Even face}] = \frac{2}{6} = \frac{1}{2}$$

$$P = P[\text{success}] = P[\text{Even face}] = \frac{2}{6} = \frac{1}{2}$$

Obviously $\boxed{p + q = 1 \text{ so that } q = 1 - p}$

Binomial Distribution

Let a Binomial experiment be repeated independently a total of ‘n’ times. ie we consider ‘n’ independent Bernoulli trials. In each trial there are only two possible outcomes namely success (s) and failure (F) Let p=P(success) and q=p(failure)

In a single trial of the experiment. Here p+q = 1 so that q= 1-p. The Binomial distribution gives the probability of getting ‘x’ success out of ‘n’ independent repetitions or trials of the Binomial experiment. This probability is given by

$$\boxed{\begin{array}{l} P(x) = nC_n p^x q^{n-x} \\ \text{where } x = 0,1,2,3....n \end{array}}$$

The Binomial distribution is discreate. ‘n’ and ‘p’ are called the parameters of the Binomial distribution. If a Random variable ‘X’ follows a Binomial distribution with parameters ‘n’ and ‘p’ it is denoted by $X \sim B(n, p)$

The mean, SD and variance of the Binomial distribution are given by

$$\boxed{\begin{aligned} &\text{Mean} = np \\ &SD = \sqrt{npq} \\ &\text{Variance} = npq \end{aligned}}$$

Partition of a sample space

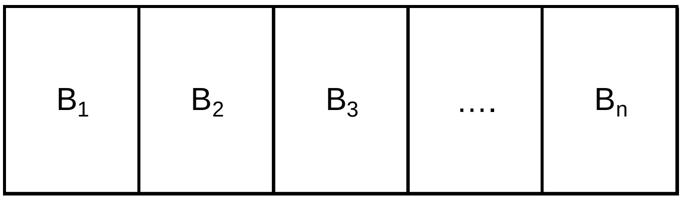
A set of events $B_1, B_2, B_3 \ldots B_n$ is said to be a partition of the sample space if

i) $B_i \cap B_j = \phi \quad i \neq j \quad i, j = 1, 2, 3 \ldots\ldots n$

ii) $B_1 \cup B_2 \cup B_3 \ldots\ldots \cup B_n = S$

iii) $P(B_i) > 0$

Or in otherwords a set of events $B_1, B_2, B_3 \ldots B_n$ is a partition of the sample space if they are pairwise M.E and exhaustive having non-zero probabilities.



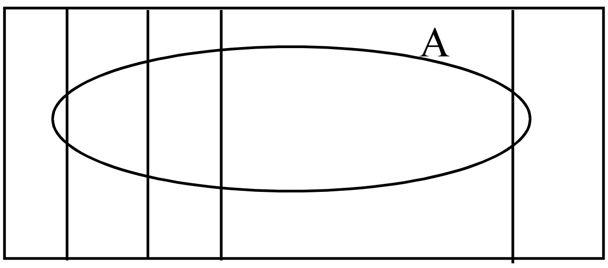
$P(B_i \cap B_j) = \phi \quad B_1 \cup B_2 \ldots \cup B_n = S$

$P(B_i) > 0$

$B_1, B_2 \ldots B_n$ is a partition of the sample space

Theorem on total probability

Let $B_1, B_2, B_3 \ldots B_n$ be a set mutually exclusive and exhaustive events with $P(B_i) > 0$. Let A be an event which can occur with any one $B_1, B_2 \ldots B_n$



Then the theorem of total probability states that

$A=(A\cap B_1)\cup(A\cap B_2)\cup(A\cap B_3)\cup....\cup(A\cap Bn)$ states that

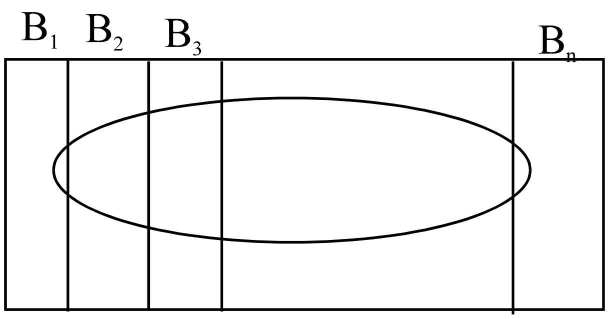
$P(A)=P(A\cap B_1)+P(A\cap B_2)+P(A\cap B_3)+....+P(A\cap B_n)$

$P(A)=P(B_1)P(A|B_1)+P(B_2)P(A|B_3)+....+....+P(B_n)P(A|B_n)$

$\boxed{\text{In symbol } P(A)=\sum P(B_i)P(A|B_i)}$

Baye's Theorem

Let B_1, B_2, B_3 ....B_n be a set of mutually exclusive and exhaustive events with $P(B_i)>0$. ie $B_1, B_2, B_3....B_n$ is a partition of the sample space. Let 'A' be an event which can occur with any one of B_1,B_2,B_3....B_n



we are given $P(B_1)P(B_2)....P(B_n)$ which are positive. The conditional probability $P(A|B_1),P(A|B_i),P(A|B_3).....P(A|B_n)$ are also given. From diagram

$A=(A\cap B_1)\cup(A\cap B_2)\cup(A\cap B_3)\cup....\cup(A\cap B_n)$

$P(A)=P(A\cap B_1)+P(A\cap B_2)+....+P(A\cap B_n)$ By axion

$P(A)=P(B_1)P(A|B_1)+P(B_2)P(A|B_3)+....+P(B_n)P(A|B_n)$ by multiplication there

In symbols $\boxed{P(A)=\sum P(B_i)P(A|B_i)}$

This is known as the total probability in a Bayessian situation

The Bayes theorem state that in a Bayessian situation

$$\boxed{\begin{array}{l} P(B_i|A)=\dfrac{P(B_i)P(A|B_i)}{P(A)} \\ \text{where} P(A)=\displaystyle\sum_{i=1}^{n}P(B_i)P(A|B_i) \end{array}}$$

$$P(B_1 | A) = \frac{P(B_1)P(A | B_1)}{P(A)}$$

$$P(B_2 | A) = \frac{P(B_2)P(A | B_2)}{P(A)}$$

For example let there be 3 boxes s.t. Box B_1 has 4 w and 1 black balls, box B_2 contain 3w 2 black balls and box B_3 contains 2w 3 black balls. One box is is taken at random and a ball is taken if it is white what is the probability that the 2nd box was taken

$$P(B_1) = \frac{1}{3} \; P(B_2) = \frac{1}{3} \;\; P(B_3) = \frac{1}{3}$$

$$A = \text{white Ball}$$

$$P(A | B_1) = \frac{4}{5} \; P(A | B_2) = \frac{3}{5} \;\; P(A | B_3) = \frac{2}{5}$$

$$P(A) = \frac{1}{3} \times \frac{4}{5} + \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{2}{5} = \frac{9}{15}$$

$$P(B_2 | \text{white ball}) = P(B_2 | A) = \frac{P(B_2)P(A | B_2)}{P(A)}$$

$$= \frac{\frac{1}{3} \times \frac{3}{5}}{\left(\frac{9}{15}\right)} = \frac{1}{3}$$

Question

A factory has 3 machines which respectively produces 30%, 50% and 20% of the total output. It is known that 3%, 5% and 2% of the items produced by the machines taken in order are defective. An item is taken at random from the total output of a day. What is the probability that it is defective. If the item taken is not defective what is the probability that it was produced by 3rd machine

$$P(B_1) = 0.3 \;\; P(B_2) = 0.5 \quad P(B_3) = 0.2$$

$$A = \text{Defective item}$$

$$P(A | B_1) = 0.03 \;\; P(A | B_2) = 0.05 \quad P(A | B_3) = 0.02$$

$$P(A) = P(\text{Defective}) = 0.3 \times 0.03 + 0.5 \times 0.05 + 0.2 \times .02$$

$$= .009 + .025 + 0.004$$

$= 0.0.8 = 3.8\%$

$P(A') = P(\text{Not defective}) = 1 - 0.038 = 0.962 = 96.2$

$P[\text{3rd Machine} \mid \text{Net defective}] = P[B_3 \mid A'] = \dfrac{P(B_3)P(A' \mid B_3)}{P(A')}$

$= \dfrac{0.2 \times 0.98}{0.962} = 0.2037$

Continuous Random Variable

If a Random variable takes any value is an interval, it is called a continuous random variable.

If 'X' is a continuous Random variable with probability function f(x) then the probability that the variable takes values in the interval $[a \ \ b]$ is given by

$$P(x) = \int_a^b f(x)\,dx$$

The A.M. ie the expected value of a continuous R.V is given by

$E(x) = \int_a^b x\, f(x)\,dx$. The variance is given by

$$V(x) = E(x^2) - [E(x)]^2$$

If $f(x) = ke^{-\frac{x}{2}}$ is the probability function of a continuous R.V $0 \le x < \infty$. Find K

Total probability = $\int_0^\infty f(x)\,dx = 1$

$$\int_0^\infty Ke^{-\frac{x}{2}}dx = 1 \Rightarrow K\left[e^{-\frac{x}{2}}\right]_0^\infty \times (-2) = 1$$

$$\therefore K[0 \quad -1] \times -2 = 1 \Rightarrow K = \frac{1}{2}$$