CHAPTER - 10
INVERSE TRIGONOMETRIC FUNCTIONS

JEE MAIN - SECTION I

1. 3 $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

2. 1 $\cot \sum_{n=1}^{50} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$

$$= \cot \sum_{n=1}^{50} \tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$$

$$= \cot \sum_{n=1}^{50} \left(\tan^{-1}(n+1) - \tan^{-1} n\right)$$

$$= \cot\left(\tan^{-1}51 - \tan^{-1}1\right)$$

$$= \cot \cot^{-1}\left(\frac{52}{50}\right)$$

$$= \frac{26}{25}$$

3. B $\sin\cos^{-1}\left(\cos\left(\tan^{-1}x\right)\right) = p$ for x ∈ R, $\tan^{-1}x \in (-\pi/2, \pi/2)$

$\cos^{-1}\left(\cos\left(\tan^{-1}x\right)\right) \in [0, \pi/2)$

$\sin\left(\cos^{-1}\left(\cos\left(\tan^{-1}x\right)\right)\right) \in [0,1)$

4. 1 $$2\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{24}{25}=\sin^{-1}2\times\frac{3}{5}\sqrt{1-\frac{9}{25}}+\cos^{-1}\frac{24}{25}=\sin^{-1}\frac{24}{25}+\cos^{-1}\frac{24}{25}=\frac{\pi}{2}.$$

5. 2
$$\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right]+\cos[\tan^{-1}(2\sqrt{2})]$$
$$=\sin\left[\tan^{-1}\frac{2/3}{1-1/9}\right]+\cos[\tan^{-1}(2\sqrt{2})]$$
$$=\sin[\tan^{-1}3/4]+\cos[\tan^{-1}2\sqrt{2}]$$
$$=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}.$$

6. 3
$$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$$
$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$$
$$\frac{\pi}{2}-\sin^{-1}x>\sin^{-1}x$$
$$\frac{\pi}{2}>2\sin^{-1}x$$
$$\Rightarrow \frac{\pi}{4}>\sin^{-1}x \quad -(1)$$
$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2} \quad -(2)$$
(1) and (2) $$\Rightarrow -\frac{\pi}{2}\le\sin^{-1}x<\frac{\pi}{4}$$
$$\Rightarrow -1\le x<\frac{1}{\sqrt{2}}$$

7. 2 Given that $\tan\{\cos^{-1}(x)\} = \sin\left(\cot^{-1}\frac{1}{2}\right)$

Let $\cot^{-1}\frac{1}{2} = \phi \Rightarrow \frac{1}{2} = \cot\phi \Rightarrow \sin\phi = \frac{1}{\sqrt{1+\cot^2\phi}} = \frac{2}{\sqrt{5}}$

Let $\cos^{-1}x = \theta \Rightarrow \sec\theta = \frac{1}{x} \Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1}$

$\Rightarrow \tan\theta = \sqrt{\frac{1}{x^2} - 1} \Rightarrow \tan\theta = \frac{\sqrt{1-x^2}}{x}$

So, $\tan\{\cos^{-1}(x)\} = \sin\left(\cot^{-1}\frac{1}{2}\right) \Rightarrow \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$

$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow \sqrt{(1-x^2)5} = 2x$

Squaring both sides, we get $x = \pm\frac{\sqrt{5}}{3}$.

8. 2

$\cos^{-1}p + \cos^{-1}q = \pi - \cos^{-1}r$

$\cos^{-1}\left(pq - \sqrt{1-p^2}\sqrt{1-q^2}\right) = \cos^{-1}(-r)$

$pq - \sqrt{1-p^2}\sqrt{1-q^2} = -r$

Squaring and simplifying

$p^2 + q^2 + r^2 + 2pqr = 1$

9. 1

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

(Putting $x^2=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2$)

$$=\tan^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right]=\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]=\tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right]$$

$$=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2.$$

10. 1

We have $\sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)=\sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{1+(m^2+m+1)(m^2-m+1)}\right)$

$$=\sum_{m=1}^{n}\tan^{-1}\left(\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)}\right)=\sum_{m=1}^{n}[\tan^{-1}(m^2+m+1)-\tan^{-1}(m^2-m+1)]$$

$$=(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}7-\tan^{-1}3)+(\tan^{-1}13-\tan^{-1}7)+\ldots\ldots+[\tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1)]$$

$$=\tan^{-1}(n^2+n+1)-\tan^{-1}1=\tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right).$$

11. 3

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{4}$$

$\tan^{-1}\sqrt{x(x+1)}$ is defined when

$x(x+1)\ge 0$ .....(i)

$\sin^{-1}\sqrt{x^2+x+1}$ is defined when

$0\le x(x+1)+1\le 1$ or $0\le x(x+1)\le 0$ .....(ii)

From (i) and (ii), $x(x+1)=0$ or $x=0$ and -1.

Hence number of solution is 0.

12. 1

$$2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]=\cos^{-1}\left[\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}\right]\left(\because 2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}\right)$$

$$=\cos^{-1}\left[\frac{(a+b)-(a-b)\tan^2\frac{\theta}{2}}{(a+b)+(a-b)\tan^2\frac{\theta}{2}}\right]=\cos^{-1}\left[\frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}\right]$$

$$=\cos^{-1}\left[\frac{\frac{a\left(1-\tan^2\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right]=\cos^{-1}\left[\frac{a\cos\theta+b}{a+b\cos\theta}\right].$$

13. 4

$$2\tan^{-1}(\cos x)=\tan^{-1}(\operatorname{cosec}^2x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}\left(\frac{1}{\sin^2 x}\right)$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x}=\frac{1}{\sin^2 x}\Rightarrow 2\cos x=1$$

$$\Rightarrow x=\frac{\pi}{3}.$$

14. D

$$\cos^{-1}\frac{1}{5\sqrt{2}}=\tan^{-1}\frac{\sqrt{(5\sqrt{2})^2-1}}{1}=\tan^{-1}7$$

$$\sin^{-1}\frac{4}{\sqrt{17}}=\tan^{-1}\left(\frac{4}{\sqrt{17-16}}\right)=\tan^{-1}4$$

So the expression is $\tan(\tan^{-1}7-\tan^{-1}4)=\tan\left\{\tan^{-1}\frac{7-4}{1+7\times4}\right\}$

$$=\tan\tan^{-1}\frac{3}{29}=\frac{3}{29}$$

15. 1 We have $\cos^{-1}\left[\frac{x}{a}.\frac{y}{b}-\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}\right]=\alpha$

$$\Rightarrow \frac{xy}{ab}-\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}=\cos\alpha$$

$$\therefore \left(\frac{xy}{ab}-\cos\alpha\right)^2=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2}+\cos^2\alpha-\frac{2xy}{ab}\cos\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2}{y^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha=\sin^2\alpha.$$

16. 2 We have $\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$

$$=\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\ldots\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)$$

$$=(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+\ldots\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})$$

$$=\tan^{-1}a_n-\tan^{-1}a_1=\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)=\tan^{-1}\left(\frac{(n-1)d}{1+a_1a_n}\right).$$

17. C *Let* $\cos^{-1}(1-x)=\alpha\Rightarrow\cos\alpha=1-x,$

$$\cos^{-1}x=\beta\Rightarrow\cos\beta=x$$

$$\therefore \alpha-2\beta=\frac{\pi}{2}\Rightarrow\alpha-\frac{\pi}{2}=2\beta$$

introduce sin on both sides and proceed

18. 3

$$\cot\left(\sum_{n=1}^{19}\cot^{-1}(1+n(n+1))\right)$$

$$\Rightarrow \cot\left(\sum_{n=1}^{19}\cot^{-1}(n^2+n+1)\right) = \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{1}{1+n(n+1)}\right)$$

$$\Rightarrow \sum_{n=1}^{19}(\tan^{-1}(n+1)-\tan^{-1}n)$$

$$\cot(\tan^{-1}20-\tan^{-1}1)=\frac{\cot A\cot\beta+1}{\cot\beta-\cot A}$$ (where $\tan A=20$, $\tan B=1$)

$$=\frac{1\left(\frac{1}{20}\right)+1}{1-\frac{1}{20}}=\frac{21}{19}$$

19. 2

We have, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}=\frac{\pi}{4}$

Let x = tan A, y = tan B and z = tan C

Then $A+B+C=\frac{\pi}{4}$.

Now, $\tan(A+B+C)=\frac{x+y+z-xyz}{1-(xy+yz+zx)}$

$\Rightarrow 1=\frac{x+y+z-xyz}{1-(xy+yz+zx)}$

$\Rightarrow 1-(xy+yz+zx)=x+y+z-xyz$.

$\Rightarrow (x-1)(y-1)(z-1)=0$

$\Rightarrow$ One of x, y, z is equal to 1

If z = 1, x + y = 0

$\therefore (x)^{odd}+(-x)^{odd}+1^{odd}=1$

Thus, AM of odd powers of x, y, z is equal to 1/3.

20. 1

$$\tan^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y-x}{y+x}\right)$$

$$=\tan^{-1}\frac{x}{y}+\tan^{-1}\left(\frac{1-\frac{x}{y}}{1+\frac{x}{y}}\right)$$

$$=\tan^{-1}\frac{x}{y}+\tan^{-1}1-\tan^{-1}\frac{x}{y}$$

$$=\frac{\pi}{4}$$

$$\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{3}{4}+\tan^{-1}\left(\frac{4-3}{4+3}\right)$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\left(\frac{1-\frac{3}{4}}{1+\frac{3}{4}}\right)$$

$$=\tan^{-1}(1)$$

SECTION II (NUMERICAL)

21. 15

Let $\tan^{-1}2=\alpha\Rightarrow\tan\alpha=2$ and $\cot^{-1}3=\beta\Rightarrow\cot\beta=3$

$\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$

$=\sec^2\alpha+\operatorname{cosec}^2\alpha=1+\tan^2\alpha+1+\cot^2\alpha$

$=2+(2)^2+(3)^2=15$.

22. 3 $\quad \dfrac{n^2-10n+26}{2\sqrt{3}}<\sqrt{3}$ minimum value of $n=3$

23. 3

$$\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=\cos\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$=\cos\cos^{-1}\sqrt{1-\frac{3}{4}}=\frac{1}{2}.$$

24. 23

$$\tan\left(\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$=\tan\left[\tan^{-1}\left(\frac{\sqrt{5^2-4^2}}{4}\right)+\tan^{-1}\frac{2}{3}\right]$$

$$=\tan\left[\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right]$$

$$=\tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)=\frac{17}{6}$$

$$\therefore m+n=17+6=23$$

25. 4

$$\sin^{-1}(\sin x)=x;\frac{-\pi}{2}\le x\le\frac{\pi}{2}\quad=\pi-x;\frac{\pi}{2}\le x\le\frac{3\pi}{2}$$

$$=-2\pi+x;\frac{3\pi}{2}\le x\le\frac{5\pi}{2}=3\pi-x;\frac{5\pi}{2}\le x\le\frac{7\pi}{2}$$

$$\cos^{-1}(\cos x)=x;0\le x<\pi\quad=2\pi-x;\pi\le x\le 2\pi$$

JEE ADVANCED LEVEL
SECTION III

26. B

$$T_n=\cot^{-1}\left(n^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{n^2+\frac{3}{4}}\right)=\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

$$s_n=\sum_{n=1}^{n}t_n=\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\frac{1}{2}$$

$$s_\infty=\tan^{-1}(\infty)-\tan^{-1}\frac{1}{2}=\frac{\pi}{2}-\tan^{-1}\frac{1}{2}=\cot^{-1}\frac{1}{2}=\tan^{-1}2$$

27. D

We have $b\sin^{-1}x+b\cos^{-1}x=\frac{b\pi}{2}$...... (1)

and $a\sin^{-1}x-b\cos^{-1}x=c$...... (2) (given)

$\therefore$ On adding (1) and (2),

we get $(a+b)\sin^{-1}x=\frac{b\pi}{2}+c\Rightarrow\sin^{-1}x=\frac{\frac{b\pi}{2}+c}{a+b}$.

Similarly $\cos^{-1} x = \dfrac{\dfrac{a\pi}{2} - c}{a + b}$

Hence $(a \sin^{-1} x + b \cos^{-1} x) = \dfrac{\pi ab + c(a - b)}{a + b}$

28. B

Given equation

$\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$

Now, $\sin^{-1}\left[x^2 + \frac{1}{3}\right]$ is defined if

$-1 \le x^2 + \frac{1}{3} < 2 \Rightarrow \frac{-4}{3} \le x^2 < \frac{5}{3}$

$\Rightarrow \boxed{0 \le x^2 < \frac{5}{3}}$ ....(1)

and $\cos^{-1}\left[x^2 - \frac{2}{3}\right]$ is defined if

$-1 \le x^2 - \frac{2}{3} < 2 \Rightarrow \frac{-1}{3} \le x^2 < \frac{8}{3}$

$\Rightarrow \boxed{0 \le x^2 < \frac{8}{3}}$ ....(2)

So, form (1) and (2) we can conclude

$\boxed{0 \le x^2 < \frac{5}{3}}$

Case - I if $0 \le x^2 < \frac{2}{3}$

$\sin^{-1}(0) + \cos^{-1}(-1) = x^2$

$\Rightarrow x + \pi = x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[0, \frac{2}{3}\right)$

$\Rightarrow$ No value of 'x'

Case - II if $\frac{2}{3} \le x^2 < \frac{5}{3}$

$\sin^{-1}(1) + \cos^{-1}(0) = x^2$

$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} = x^2$

$\Rightarrow x^2 = \pi$

but $\pi \notin \left[\frac{2}{3}, \frac{5}{3}\right)$

$\Rightarrow$ No value of 'x'

So, number of solutions of the equation is zero.

29. D $\sin^{-1}x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; $\cos^{-1}y\in[0,\pi]$; $\sec^{-1}z\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

$\Rightarrow \sin^{-1}x+\cos^{-1}y+\sec^{-1}z\le\frac{\pi}{2}+\pi+\pi=\frac{5\pi}{2}$; Also $t^2-\sqrt{2\pi}\;t+3\pi$

$=t^2-2\sqrt{\frac{\pi}{2}}\;t+\frac{\pi}{2}-\frac{\pi}{2}+3\pi=\left(t-\sqrt{\frac{\pi}{2}}\right)+\frac{5\pi}{2}\ge\frac{5\pi}{2}$

The given inequality holds $\Leftrightarrow x=1, y=-1, z=-1$

LHS=RHS $=\frac{5\pi}{2}\Rightarrow x=1, y=-1, z=-1$ and

$t=\sqrt{\frac{\pi}{2}}\Rightarrow \cos^{-1}\left(\cos 5t^2\right)=\cos^{-1}\left(\cos\left(\frac{5\pi}{2}\right)\right)=\frac{\pi}{2}$

$\cos^{-1}\left(\min\{x,y,z\}\right)=\cos^{-1}(-1)=\pi$

30. B

31. C

SECTION IV (More than one correct)

32. D $-1\le\dfrac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\le 1$

$-\pi/2\le\sin^{-1}\frac{1}{4x^2-1}\le\pi/2$

Always $-1\le\frac{1}{4x^2-1}\le 1$

$x\in\left(\infty,\frac{1}{\sqrt{2}}\right)\cup\left[\frac{1}{\sqrt{2}},\infty\right)$

33. A,B $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x \Rightarrow \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}(1-x)=\cos^{-1}x$

$\Rightarrow 2\cos^{-1}x=\pi-\cos^{-1}(1-x) \Rightarrow \cos^{-1}(2x^2-1)=\cos^{-1}(x-1)$

$\Rightarrow 2x^2-x=0\Rightarrow x(2x-1)=0$

$\therefore x=0,\ \frac{1}{2}.$

34. A,B,C $\sin\left(2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right)=0$

$\Rightarrow 2\cos^{-1}\{\cot(2\tan^{-1}x)\} = n\pi, n \in I$

$\Rightarrow \cos^{-1}(\cot\{2\tan^{-1}x)\} = \frac{n\pi}{2} = 0, \frac{\pi}{2}, \pi$ $\quad (\because 0 \le \cos^{-1}x \le \pi)$

35. A,B $S_n(x) = \sum_{k=1}^{n}\tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$

$= \sum_{k=1}^{n}\tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$

$S_n(x) = \tan^{-1}(nx+x) - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

(A) $S_{10}(x) = \tan^{-1}\frac{10x}{1+11x^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$ $(x>0)$

(B) $\lim_{n\to\infty}\cot(S_n(x)) = \lim_{n\to\infty}\frac{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}{x} = x \ (x>0)$

(C) $S_3(x) = \tan^{-1}\frac{3x}{1+4x^2} = \frac{\pi}{4} \Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$

(D) $\tan(S_n(x)) = \frac{nx}{1+(n+1)x^2}$; $\forall\, n \ge 1$; $x > 0$

We need to check the validity of $\frac{nx}{1+(n+1)x^2} \le \frac{1}{2}$ $\forall\, n \ge 1$; $x > 0$; $n \in \mathbb{N}$

$\Rightarrow 2nx \le (n+1)x^2 + 1$

$\Rightarrow (n+1)x^2 - 2nx + 1 \ge 0\ \forall\, n \ge 1$; $x > 0$; $n \in \mathbb{N}$

Discriminant of $y = (n+1)x^2 - 2nx + 1$ is

$D = 4n^2 - 4(n+1)$ and $n \in \mathbb{N}$

$D < 0$ for $n = 1$; true for $x > 0$

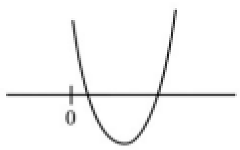
$D > 0$ for $n \ge 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of

$y = 0$ will be positive.

$y = (n+1)x^2 - 2nx + 1$, $n \ge 2$

So, $y \ge 0\ \forall\, n \ge 1$; $\forall\, x > 0$; $n \in \mathbb{N}$ is false.



SECTION V - (Numerical type)

36. 5 $\left(\cot^{-1}x\right)\left(\tan^{-1}x\right)+\left(2-\frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2-\frac{\pi}{2}\right) > 0$

$\Rightarrow \left(\cot^{-1}x - 3\right)\left(2-\cot^{-1}x\right) > 0$

$\Rightarrow \left(\cot^{-1}x - 3\right)\left(\cot^{-1}x - 2\right) < 0$

$\Rightarrow 2 < \cot^{-1}x < 3$

$\Rightarrow \cot 3 < x < \cot 2$

(as $\cot^{-1}x$ is a decreasing function)

$$\Rightarrow Hence, x \in (\cot 3, \cot 2)$$

$$\Rightarrow \cot^{-1} a + \cot^{-1} b = \cot^{-1}(\cot 3) + \cot^{-1}(\cot 2) = 5$$

37. 9 $\quad 1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + ...\infty = 2$

$$\Rightarrow \frac{1}{1 - \sin(\cos^{-1} x)} = 2 \Rightarrow \frac{1}{2} = 1 - \sin(\cos^{-1} x)$$

$$\Rightarrow \sin(\cos^{-1} x) = \frac{1}{2} \Rightarrow \cos^{-1} x = \frac{\pi}{6} \Rightarrow x = \frac{\sqrt{3}}{2} \Rightarrow 12x^2 = 9$$

38. 1 $\quad \tan^{-1}\sqrt{\frac{xr}{yz}} + \tan^{-1}\sqrt{\frac{yr}{zx}} + \tan^{-1}\sqrt{\frac{zr}{xy}}$

$$= \tan^{-1}\left[\frac{\sqrt{\frac{xr}{yz}} + \sqrt{\frac{yr}{zx}}}{1 - \sqrt{\frac{xr}{yz}} \cdot \sqrt{\frac{yr}{zx}}}\right] + \tan^{-1}\sqrt{\frac{zr}{xy}}$$

$$= \tan^{-1}\left[\frac{\sqrt{z}\, r\,(x+y)}{\sqrt{xy}\,(z-r)}\right] + \tan^{-1}\frac{\sqrt{zr}}{\sqrt{xy}}$$

$$= \tan^{-1}\left[\frac{\frac{\sqrt{z}\, r\,(x+y)}{\sqrt{xy}\,(z-r)} + \sqrt{\frac{zr}{xy}}}{1 - \frac{zr}{xy}\left(\frac{x+y}{z-r}\right)}\right]$$

$$= \tan^{-1}\frac{\sqrt{\frac{zr}{xy}}\left[-\frac{(z-r)}{z-r} + 1\right]}{1 + \frac{zr}{xy}\,\frac{z-r}{z-r}}$$

$$= \tan^{-1} 0 = \pi, \quad k = 1$$

39. 4 $\quad$ We have $f(x) = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$

$= (\tan^{-1}x + \cot^{-1}x)$

$$\left((\tan^{-1}x)^2-(\tan^{-1}x)(\cot^{-1}x)+(\cot^{-1}x)^2\right)$$

$$=\frac{\pi}{2}\left((\tan^{-1}x)^2-(\tan^{-1}x)\left(\frac{\pi}{2}-\tan^{-1}x\right)+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2\right) \qquad \left(\text{Using } \cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x\right)$$

$$=\frac{3\pi}{2}\left(\left(\tan^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{48}\right)$$

Clearly, $f(x)$ will be minimum when $\left(\tan^{-1}x-\frac{\pi}{4}\right)^2=0$

and $f(x)$ will be maximum when $\left(\tan^{-1}x-\frac{\pi}{4}\right)^2=\left(-\frac{\pi}{2}-\frac{\pi}{4}\right)^2$

$$\therefore a=f(x)_{min}=\frac{3\pi}{2}\left(0+\frac{\pi^2}{48}\right)=\frac{\pi^3}{32} \text{ and } b=f(x)_{max}=\frac{3\pi}{2}\left(\left(\frac{-3\pi}{4}\right)^2+\frac{\pi^2}{48}\right)=\frac{7\pi^3}{8}$$

Hence $\frac{b}{7a}=\frac{\frac{\pi^3}{8}}{\frac{\pi^3}{32}}=4$

SECTION VI - (Matrix match type)

40. A

a) $\left(\sin^{-1}x\right)^2+\left(\sin^{-1}y\right)^2=\frac{\pi^2}{2} \qquad \Rightarrow\left(\sin^{-1}x\right)^2=\left(\sin^{-1}y\right)^2=\frac{\pi^2}{4}$

$\Rightarrow \sin^{-1}x=\pm\frac{\pi}{2}, \sin^{-1}y=\pm\frac{\pi}{2} \Rightarrow x=\pm1 \text{ and } y=\pm1$

$\Rightarrow x^3+y^3=-2,0,2$

b) $\left(\cos^{-1}x\right)^2+\left(\cos^{-1}y\right)^2=2\pi^2 \Rightarrow \left(\cos^{-1}x\right)^2=\left(\cos^{-1}y\right)^2=\pi$

$\Rightarrow x=y=-1 \Rightarrow x^5+y^5=-2$

c) $\left(\sin^{-1}x\right)^2\left(\cos^{-1}y\right)^2=\frac{\pi^4}{4} \Rightarrow \left(\sin^{-1}x\right)^2=\frac{\pi^2}{4} \text{ and } \left(\cos^{-1}y\right)^2=\pi^2$

$\Rightarrow \left(\sin^{-1}x\right)=\pm\frac{\pi}{2} \text{ and } \left(\cos^{-1}y\right)=\pi \Rightarrow x=\pm1 \text{ and } y=-1$

$\Rightarrow |x-y|=0,2$