SEQUENCE AND SERIES

Sequence:- A sequence is a function from the set of natural numbers to the set of real numbers. Or we can say in a sequence, the number of term increasing according to a specified condition

eg: 1,3,5,7,........

2,4,8,16,........

1,2,4,7,11,16,......

2,5,10,17,26,37,50,.........

Progression: A sequence is called a progression if we can find any term using a specified n^{th} term

eg: $\{a_n\} = 3n - 1$

then the sequence is $\{a_1, a_2, a_3......\}$

$a_1 = 3\times1 - 1 = 2$

$a_2 = 3\times2 - 1 = 5$

$a_3 = 3\times3 - 1 = 8$

$\therefore \{a_n\} = \{2, 5, 8, ...\}$

Finite and infinite sequences

A sequence is said to be finite or infinite, if the number of terms are finite or infinite respectively

Series

The sum of terms of a sequence is called a series

eg: 2,5,7,9,......is a sequence

then 2+5+7+9+.....is a series

Finite and infinite series

The sum of terms of a finite sequence is known as finite series and the sum of terms of an infinite sequence is known as infinite series.

Arithmetic progression (A.P)

A sequence of numbers is called an A.P. if its term, after the first term is obtained by adding a fixed number to the immediate proceeding term. The fixed number is known as common difference (d) of the A.P.

eg: 1,3,5,7,......d=2

n^{th} term of an A.P.

If a_1, a_2, a_3........is an A.P with common difference d, then

$a_1 = a$

$a_2 = a+d$

$a_3 = a+2d$

....

....

$a_n = a+(n-1)d$

Sum of n time of an A.P.

Let $s_n = a + (a+d) + (a+2d) + ... + (a+(n-2)d) + (a+(n-1)d)$.........(1)

$s_n = (a+(n-1)d) + (a+(n-2)d) + ... + (a+d) + a$.....(2)

$(1)+(2) \Rightarrow 2S_n = (2a+(n-1)d) + (2a+(n-1)d) + ...n \text{ terms}$

$= n[2a+(n-1)d]$

$\therefore S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[a+a+(n-1)d]$

$= \frac{n}{2}[a+a_n]$

Condition for a,b,c are in A.P

Since a,b,c are in A.P.

b-a=c-b

$\boxed{\Rightarrow 2b = a+c}$

Number of terms between two terms

Consider the A.P. $a_1, a_2, ..., a_n$

$a_n = a + (n-1)d$

$\Rightarrow (n-1)d = a_n - a$

$\Rightarrow n-1 = \frac{a_n - a}{d} \Rightarrow \boxed{n = \frac{a_n - a}{d} + 1}$

Arithmetic means between two numbers

Let $a, x_1, x_2, \ldots\ldots x_n, b$ are in A.P. then $x_1, x_2, \ldots. x_n$ are the n arithmetic means between a and b

$$x_1 + x_2 + \ldots.. + x_n = \frac{n}{2}(x_1 + x_n)$$

$$= \frac{n}{2}(a + d + b - d) = \frac{n}{2}(a + b)$$

* Three consecutive terms of an A.P. can be taken as a-d, a, a+d
* Four consecutive terms $\rightarrow$ a-3d,a-d,a+d,a+3d
* Five consecutive terms $\rightarrow$ a-2d, a-d, a,a+d, a+2d

Geometric progression (G.P.)

A sequence of numbers, where each term after the first term, is obtained by multiplying a fixed number to the immediate proceeding term, is called a G.P. and the fixed number is called the common ratio (r) of the G.P.

eg: 2,6,18,54,..... r=3

1,–1,1,–1,.......r=–1

n^{th} term of a G.P.

let $a_1, a_2, \ldots\ldots. a_n$ is the G.P.

then $a_1 = a$

$a_2 = ar$

$a_3 = ar^2$

$$\therefore \; a_n = ar^{n-1}$$

$$\therefore \boxed{a_n = ar^{n-1}}$$

Sum of n terms of a G.P.

Let $a, ar, ar^2, \ldots.. ar^{n-1}$ is the GP

then let $s_n = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}$........(1)

(1) $r \Rightarrow r S_n = ar + ar^2 + ar^3 + \ldots\ldots.. + ar^{n-1} + ar^n \ldots..(2)$

(2)- $(1) \Rightarrow r \; S_n - S_n = ar^n - a$

$$\Rightarrow S_n(r-1) = a(r^n - 1)$$

$$\therefore \boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

a,b,c in G.P.

If a,b,c are three consecutive terms of a G.P. then $\frac{b}{a}=\frac{c}{b}\Rightarrow \boxed{b^2=ac}$

Geometric means

Let $a, x_1, x_2, x_3, \ldots\ldots, x_n, b$ are in G.P. then $x_1, x_2 \ldots\ldots x_n$ are the n geometric means between a and b

Since b is the $(n+2)^{th}$ term, $b=ar^{n+1}$

$$x_1 \times x_2 \times \ldots\ldots \times x_n = ar \times ar^2 \times \ldots\ldots \times ar^n$$

$$=a^n \times r^{1+2+\ldots\ldots+n}$$

$$=a^n \times r^{\frac{n(n+1)}{2}} = a^{n/2} \times a^{n/2} \times r^{\frac{n(n+1)}{2}}$$

$$=a^{n/2} \times \left(ar^{n+1}\right)^{n/2}; \ = a^{n/2} \times b^{n/2}$$

$$=(ab)^{n/2}$$

Infinite G.P.

We can find the sum of an infinite G.P., if its common ratio, r satisfies $|r|<1$

$$S_n = \frac{a\left(1-r^n\right)}{1-r} \text{ when } n \to \infty, r^n \to 0, \text{since } |r|<1$$

$$\therefore S_\infty = \frac{a}{1-r}$$

* Three consecutive terms of a G.P. can be lathen as $\frac{a}{r}, a, ar$

* Four consecutive $\Rightarrow \frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

* Five consecutive $\Rightarrow \frac{a}{r^2}, \frac{a}{r}, r, ar, ar^2$

Harmonic progression (H.P)

A sequence is called an H.P, if its terms are reciprocals of corresponding terms of an A.P.

eg: $2, 4, 6, 8, \ldots\ldots$ is an A.P.

then $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\ldots$ is an H.P.

a,b,c are in H.P.

Since a,b,c are in H.P.

$\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P.

$\therefore 2\times\frac{1}{b}=\frac{1}{c}+\frac{1}{a};\ \Rightarrow \frac{2}{b}=\frac{a+c}{ac}\ \Rightarrow \boxed{b=\frac{2ac}{a+c}}$

* a,b,c are in A.P. $\Rightarrow b=\frac{a+c}{2}$
* a,b,c are in G.P. $\Rightarrow b=\sqrt{ac}$
* a,b,c are in H.P. $\Rightarrow b=\frac{2ac}{a+c}$
* If A,G,H are the A.M., G.M and H.M. respectively of two numbers a and b, then

  $\rightarrow A\geq G\geq H$

  $\rightarrow A,G,H$ from a G.P. $\Rightarrow G^2=AH$
* Consider $a_1,a_2, a_3, ...a_n$

  $\rightarrow A.M=\frac{a_1+a_2+....+a_n}{n}$

  $\rightarrow G.M=\left(a_1a_2.....a_n\right)^{\frac{1}{n}}$

  $\rightarrow H.M=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+....+\frac{1}{a_n}}$

Arithmetico - Geometric progression (A.G.P)

A sequence of numbers, where each terms is the product of corresponding time of an A.P. and G.P. is called an A.G.P.

eg: 3,5,7,9,...... is an A.P.

1,2,4,8,.....is a G.P.

then 3×1, 5×2, 7×4, 9×8,.....is an A.G.P.

Sum to n terms of an A.G.P.

Let $a, a+d, a+2d, ...+a+(n-1)d$ is the A.P.

$1, r_1, r^2, ...r^{n-1}$ is the G.P.

then $a, (a+d)r, (a+2d)r^2, ...(a+(n-1)d)r^{n-1}$ is the A.G.P.

Let $S_n = a+(a+d)r+(a+2d)r^2+......+(a_n(n-2)d)r^{n-2}+(a+(n-1)d)r^{n-1}$ ....(1)

(1) $r \Rightarrow rS_n = ar+(a+d)r^2+(a+2d)r^3+....+(a+(n-2)d)r^{n-1}+(a+(n-1)d)r^n$ ......(2)

(1)-(2) $\Rightarrow S_n - rS_n = a+dr+dr^2+.....+dr^{n-1}-(a+(n-1)d)r^n$

$\Rightarrow S_n(1-r) = a+dr(1+r+r^2+.....+r^{n-2})-(a+(n-1)d)r^n$

$$= a+dr\frac{(1-r^{n-1})}{1-r}-(a+(n-1)d)r^n$$

$$\therefore S_n = \frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{(a+(n-1)d)r^n}{1-r} .....(1)$$

Sum to infinity

when $|r|<1, r^n \to 0, r^{n-1} \to 0$

$$\therefore (1) \Rightarrow S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

Note: The above two formulae's are applicable only when the first term of G.P. is 1

* Sum of 1st n natural numbers = $\frac{n(n+1)}{2}$
* Sum of squares of 1st n natural numbers = $\frac{n(n+1)(2n+1)}{6}$
* Sum of cubes of 1st n natural numbers = $\left[\frac{n(n+1)}{2}\right]^2$