CHAPTER - 13

CONTINUITY, DIFFERENTIABILITY AND DERIVATIVES

JEE MAIN - SECTION I

1. 2 $\underset{x \to 0}{Lt} (1+x)^{\frac{1}{x}} = e = f(0)$

2. 2 $f(0) = e^2$

$\underset{x \to 0}{Lt} (1+2x)^{\frac{1}{x}} = e^{\underset{x \to 0}{Lt} \frac{2x}{x}} = e^2$

$\therefore$ Continuous at $x = 0$

3. 4 Continuous in $R \setminus \{1,2\}$.

4. 3 $x^y = e^{x-y} \Rightarrow y \log x = x - y$

$y(1+\log x) = x \Rightarrow y = \frac{x}{1+\log x}$

$\frac{dy}{dx} = \frac{(1+\log x) - x\frac{1}{x}}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$

5. 1 $y = 2^{ax}$ $\frac{dy}{dx} = 2^{ax} \log 2 \times a$

$\left.\frac{dy}{dx}\right]_{x=1} = a 2^a \log 2 = 8 \log 2$

$a 2^a = 8 \Rightarrow a = 2$

6. 1 $y = \cot^{-1} \tan \frac{x}{2} = \cot^{-1} \cot\left(\frac{\pi}{2} - \frac{x}{2}\right)$

$y = \frac{\pi}{2} - \frac{x}{2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

7. 3 $y = e^{\log \sec x} = \sec x$

$\frac{dy}{dx} = \sec x \tan x$

8. 2 $1 + \frac{3x^2}{3} + \frac{5x^4}{5} + \cdots$

$= 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$

9. 1 $\operatorname{sig}(x+1) = 1$ in $1 < x \le 2$

$f(1^-) = f(1^+) \Rightarrow 1 = a + b$

10. 4 $f'(x) = f(x) \Rightarrow f(x) = e^c e^x$

$f(1) = e^c e = 2 \Rightarrow e^c = \frac{2}{e}$

$\therefore f(x) = 2e^{x-1}$ $h'(x) = f'(f(x))\, f'(x)$

$h'(1) = f'(f(1))\, f'(1) = 4e$

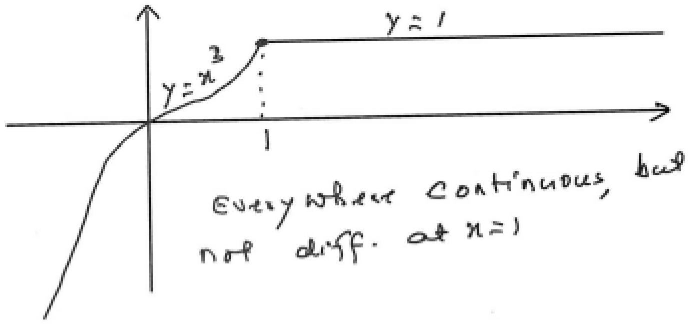
11. 2 $|f'(x)| = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x-y|} \le \lim_{y \to x} 2|x-y|$

$|f'(x)| \le 0 \Rightarrow |f'(x)| = 0 \Rightarrow f'(x) = 0$

$f(x) = k \Rightarrow f(0) = 1 \Rightarrow f(x) = 1$

$\int_0^1 f^2(x)\,dx = \int_0^1 1\,dx = [x]_0^1 = 1 - 0 = 1$

12. 3



Everywhere continuous, but not diff. at $x = 1$

13. 3

$f(x) = \frac{\tan(n\pi)}{1+[x]^2}$ when $n = [x-\pi]$ = integer

$f(x) = 0 \ \forall x$ ∴ Continuous at all points

14. 3

$f[g(x)] = x \qquad f'[g(x)]\, g'(x) = 1$

$\frac{1}{1+(g(x))^n}\, g'(x) = 1 \Rightarrow g'(x) = 1 + [g(x)]^n$

15. 2

Multiply Nr. and Dr. by $(x-a)$

16. 1

$\frac{dx}{dt} = 3\sec^2 t \qquad \frac{dy}{dt} = 3\sec t \tan t$

$\frac{dy}{dx} = \frac{3\sec t \tan t}{3\sec^2 t} = \sin t$

$\frac{d^2y}{dx^2} = \cos t \frac{dt}{dx} = \frac{\cos t}{3\sec^2 t} = \frac{\cos^3 t}{3}$

$\left.\frac{d^2y}{dx^2}\right]_{\frac{\pi}{4}} = \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{6\sqrt{2}}$ Ans 1

17. 4

$x=0 \quad y=0 \quad f(0)=(f(0))^2 \Rightarrow f(0)=1$

$x=x \quad y=0 \Rightarrow f(0)=f(x)f(0) \Rightarrow f(x)=1$

$\therefore \frac{dy}{dx}=1 \Rightarrow y=x \quad \therefore y(\frac{1}{4})+y(\frac{3}{4})=\frac{1}{4}+\frac{3}{4}=1$

18. 4

$y=\log^n x=\log\log\log\ldots\log x$

Put $n=1, 2, 3$ and get $\frac{dy}{dx}$.

19. 1

$f'(x)=f'(x)=2x\,f(x) \Rightarrow f(x)=e^{x^2}$

$f(\sqrt{t})=e^t \quad F(x)=\int_0^{x^2} e^t\,dt=\left[e^t\right]_0^{x^2}$

$F(x)=e^{x^2}-1 \Rightarrow F(2)=e^4-1$

20. 1

Use $\underset{x\to 0}{Lt}\ \frac{1-\cos x}{x^2}=\frac{1}{2}$ and

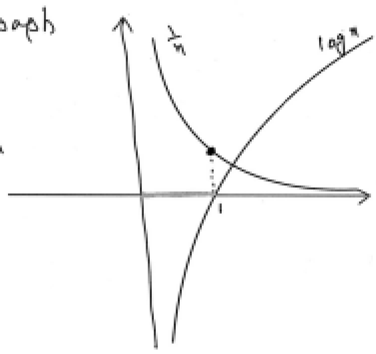
$\log\left(\frac{a}{b}\right)=\log a-\log b$ Ans 1

21. 1

Use graph

$0<x<1$

$\frac{1}{x}>\log x$



SECTION II (NUMERICAL)

22. 40

$\log(x+y) = 4xy$ when $x=0$ $y=1$

$x+y = e^{4xy}$

$1 + \frac{dy}{dx} = e^{4xy}\left(4x\frac{dy}{dx} + y4\right)$

$1 + \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = 4-1 = 3$

$1 + y_1 = e^{4xy}(4xy_1 + 4y)$

$y_2 = e^{4xy}(4xy_2 + y_1 4 + 4y_1)$
$\quad + (4xy_1 + 4y)\, e^{4xy}(4xy_1 + y4)$

Put $x=0$ $y=1$ $\frac{dy}{dx} = y_1 = 3$

$y_2 = 1(0 + 12 + 12) + (0+4)\,1\,(0+4)$
$= 24 + 4 \times 4 = 24 + 16 = \underline{\underline{40}}$

23. 17

$$\Rightarrow \left(y^{\frac{1}{4}}\right)^2 - 2xy^{\left(\frac{1}{4}\right)} + 1 = 0$$

$$\Rightarrow y^{\frac{1}{4}} = x + \sqrt{x^2-1} \text{ or } x - \sqrt{x^2-1}$$

$$\text{So, } \frac{1}{4}\frac{1}{y^{\frac{3}{4}}}\frac{dy}{dx} = 1 + \frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{1}{4}\frac{1}{y^{3/4}}\frac{dy}{dx} = \frac{y^{\frac{1}{4}}}{\sqrt{x^2-1}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y}{\sqrt{x^2-1}} \quad \ldots(1)$$

$$\text{Hence, } \frac{d^2y}{dx^2} = 4\,\frac{\left(\sqrt{x^2-1}\right)y' - \frac{yx}{\sqrt{x^2-1}}}{x^2-1}$$

$$\Rightarrow (x^2-1)y'' = 4\,\frac{(x^2-1)y' - xy}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)y'' = 4\left(\sqrt{x^2-1}\,y' - \frac{xy}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow (x^2-1)y'' = 4\left(4y - \frac{xy'}{4}\right) \text{ (from I)}$$

$$\Rightarrow (x^2-1)y'' + xy' - 16y = 0$$

So, $|\alpha - \beta| = 17$

24. 0.5

Put $x = \cos 2\theta$

$f'(1/2) = -k = \frac{-1}{2} \Rightarrow k = \frac{1}{2} = 0.5$

25. 3

When $x \to 0$

$e^x - 1 \approx x$, $\sin\left(\frac{x}{a}\right) \approx \frac{x}{a}$, $\log\left(1+\frac{x}{4}\right) \approx \frac{x}{4}$

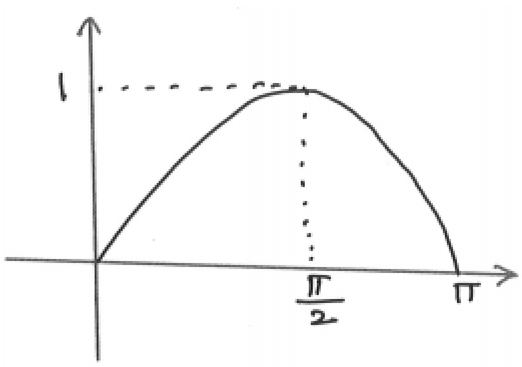
$\therefore a = 3$

JEE ADVANCED LEVEL

SECTION III

26. B

Let $g(t) = \sin t$



Graph of $g(t)\, 0 \le t \le x \;\; 0 \le x \le 1$

From graph Maximum

$$g(t) \begin{cases} = \sin x & 0 \le x \le \frac{\pi}{2} \\ = 1 & \frac{\pi}{2} < x \le 1 \end{cases}$$

Hence $$f(x) \begin{cases} = \sin x & 0 \le x \le \frac{\pi}{2} \\ = 1 & \frac{\pi}{2} < x \le \pi \\ = 2 + \sin x & x > \pi \end{cases}$$

Now check continuity and differentiability at $x = \frac{\pi}{2}$ and $x = \pi$

27. A $|x| \leq 1 \Rightarrow -1 \leq x \leq 1$

$|x|>| \Rightarrow x < -1$ or $x > 1$. Hence f(x) is

$\frac{1}{2}(\lvert x\rvert-1) = -\frac{1}{2}(x+1)$	$\frac{\pi}{4}+\tan^{-1}x$	$\frac{1}{2}(\lvert x\rvert-1) = \frac{1}{2}(x-1)$
	-1	1

$f(-1^+) = f(-1^-) \Rightarrow$ continuous at $x=-1$

$Rf'(-1) \neq Lf'(-1) \Rightarrow$ Not diff. at $x=-1$

$f(1^+) \neq f'(1^-) \Rightarrow$ Not continuous and not diff. at $x = 1$

28. 11 Use the result $\underset{f(x)\to 0}{\text{Lt}} \frac{1-\cos f(x)}{(f(x))^2} = \frac{1}{2}$

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{x^4}{x^n} = 2^{m+7} \Rightarrow n = 4$$

$$n = 4 \Rightarrow 2^{m+7} = 1 \Rightarrow m+7 = 0 \Rightarrow m = -7$$

$$n - m = 4-(-7) = 11$$

29. 1 $g(1) = \underset{x\to 1}{\text{Lt}} (1+\log x)^{\frac{1}{\frac{\log x}{2}}} = e^2$

$$g(1) = e^2$$

Continuous at $x = 1$

$$\underset{x\to 1}{\text{Lt}}\ g(x) = g(1) = e^2$$

$$e^2 = \text{RHL} = \underset{x\to 1^+}{\text{Lt}}\ g(x) = \underset{m\to\infty}{\text{Lt}} \frac{x^m f(1)+h(x)+1}{2x^m+3x+3}$$

$$= \underset{x\to 1^+}{\text{Lt}}\ \underset{m\to\infty}{\text{Lt}} \frac{x^m\left[f(1)+\frac{h(x)}{x^m}+\frac{1}{x^m}\right]}{x^m\left[2+\frac{3}{x^{m-1}}+\frac{3}{x^m}\right]}$$

$$x \Rightarrow 1^{+} \Rightarrow x > 1 \Rightarrow x^{\infty} = \infty \Rightarrow \frac{1}{x^{m}} = 0$$

$$\text{RHL} = \frac{f(1)}{2+0+0} = \frac{f(1)}{2} = e^{2} \Rightarrow f(1) = 2e^{2}$$

$$e^{2} = \text{LHL} = \underset{x \to 1^{-}}{\text{Lt}}\ \underset{m \to \infty}{\text{Lt}} \frac{x^{m} f(1) + h(x) + 1}{2x^{m} + 3x + 3}$$

$$e^{2} = \frac{0 + h(1) + 1}{0+3+3} \text{ since } 1^{-} = x < 1 \Rightarrow x^{\infty} = 0$$

$$6e^{2} - 1 = h(1)$$

$$2g(1) + 2\,f(1) - h(1) = 2e^{2} + 4e^{2} - \left(6e^{2} - 1\right) = 1$$

30. D Use Leibnitz theorem

31. A $y^{t} = \log(x+t) = t$

$$y^{t} = t \text{ and } x + t = e^{t}$$

$$y = t^{\frac{1}{t}} \text{and } x = e^{t} - t\ ;\ \frac{dy}{dt} = \frac{t^{\frac{1}{t}}}{t^{2}}(1 - \log t) \text{ and } \frac{dx}{dt} = e^{t} - 1$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} ; \text{ when } x = e^{2} - 2 \text{ and } y = \sqrt{2} \text{ then } t = 2$$

SECTION IV (More than one correct)

32. B,D

$$\underset{x \to \infty}{\text{Lt}} f_1(x) = \underset{x \to \infty}{\text{Lt}} \frac{|x|}{x} \quad \text{DNE}$$

$$-1 \le x < e^{\frac{\pi}{2}} - 2$$

$$1 \le x + 2 < e^{\frac{\pi}{2}}$$

$$0 \le \log(x+2) < \frac{\pi}{2}$$

$$0 \le \sin\log(x+2) < 1$$

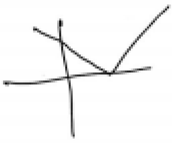
$$f_2(x) = [\sin\log(x+2)] = 0$$

$$\underset{x \to 0}{\text{Lt}} f_3(x) = 0$$

33. C, D

Put $x-1=u$ | Also $P=1$

$x \to 1 \Rightarrow u \to 0$



$$\underset{u\to 0}{Lt} \frac{u^n}{m \log \cos u} = -1$$

$$\underset{u\to 0}{Lt} \frac{n u^{n-1}}{m(-\tan u)} = -1$$

$$\underset{u\to 0}{Lt} -\frac{n}{m} u^{n-2} = -1$$

$$\underset{u\to 0}{Lt} u^{n-2} = \frac{m}{n} \Rightarrow \underline{\underline{m=n=2}}$$

34. A, B

Ans A, B. split in to piecewise functions.

SECTION V - (Numerical type)

35. 0.5

$f(x) = e^{\frac{x}{2}} \quad \therefore \log f(x) = \frac{x}{2}$

$\log f(4) = \frac{4}{2} = 2$

36. 0.25

$F'(x) = f'(x) = 2x\, f(x)\, dx$

$\therefore f(x) = e^{x^2} \qquad f(\sqrt{t}) = e^t$

$$F(x) = \int_0^{x^2} e^t\, dt = e^{x^2} - 1$$

$F(2) = e^4 - 1 \qquad F(0) = 0$

$$\frac{e^4 - F(2) + F(0)}{4} = \frac{e^4 - (e^4-1) - 0}{4}$$

$$= \frac{1}{4} = 0.25$$

37. 6.4

$$x = \sec\theta - \cos\theta \qquad y = \sec^8\theta - \cos^8\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$

$$x = \sec\theta - \cos\theta \qquad y = \sec^8\theta - \cos^8\theta$$

$$\frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$$
$$= \tan\theta(\sec\theta + \cos\theta)$$

$$\frac{dy}{d\theta} = 8\sec^7\theta\sec\theta\tan\theta + 8\cos^7\theta\sin\theta$$
$$= 8\tan\theta\left[\sec^8\theta + \cos^8\theta\right]$$

$$\frac{dy}{dx} = \frac{8\tan\theta\left[\sec^8 + \cos^8\theta\right]}{\tan\theta\left[\sec\theta + \cos\theta\right]}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{64(\sec^8\theta + \cos^8\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$\left(\frac{dy}{dx}\right)^2 = 64\left\{\frac{(\sec^8\theta - \cos^8\theta)^2 + 4}{(\sec\theta - \cos\theta)^2 + 4}\right\}$$

$$= 64\frac{(y^2 - 4)}{(x^2 - 4)}$$

$$\frac{1}{10}\left[\frac{x^2 - 4}{y^2 - 4}\right]\left(\frac{dy}{dx}\right)^2 = \frac{64}{10} = 6.4$$

38. 0.5

$$\underset{x\to 0}{Lt} \frac{\left(\frac{e^x-1}{x}\right)x\left[\sqrt{1+x^2}+\sqrt{1-x^2}\right]}{\left[\sqrt{1+x^2}-\sqrt{1-x^2}\right]\left[\sqrt{1+x^2}+\sqrt{1-x^2}\right]} = \frac{x\left[\sqrt{1+x^2}+\sqrt{1-x^2}\right]}{\sqrt{(1+x^2)^2-(1-x^4)}}$$

$$= \frac{x[\quad]}{\sqrt{1+x^4+2x^2-1+x^2}} = \frac{x[\quad]}{\sqrt{x^4+3x^2}} = \frac{x[\quad]}{\sqrt{x^2(x^2+3)}}$$

$$= \underset{x\to 0}{Lt} \frac{x}{|x|}\frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\sqrt{x^2+3}} = \pm\frac{\sqrt{2}}{\sqrt{3}}$$

$$RHL = \frac{\sqrt{2}}{3} \qquad LHL = -\frac{\sqrt{2}}{3}$$

39. 1.4

$$f(0-) = \underset{x\to 0}{Lt}\ a\sin\frac{\pi}{2}(x-1)$$

$$= a\sin(-\pi/2) = -a$$

$$f(0+) = \underset{x\to 0}{Lt}\ \frac{\tan 2x - \sin 2x}{bx^3} = -a$$

$$= \frac{2x+\frac{8x^3}{3}-\left(2x-\frac{8x^3}{6}\right)}{bx^3} = -a$$

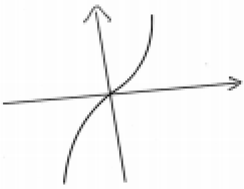
$$\frac{8}{3b}+\frac{8}{6b} = -a \qquad\qquad 4 = -ab$$

$$\frac{8}{3}+\frac{8}{6} = -ab \qquad\qquad 4+10 = -ab+10$$

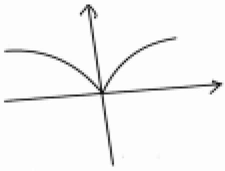
$$\frac{24}{6} = -ab \qquad\qquad 10-ab = \underline{\underline{14}}$$

40. A

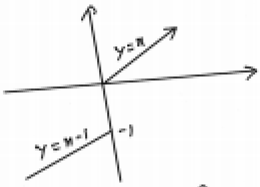
$$|x|x \begin{cases} = x^2 & x > 0 \\ = 0 & x = 0 \\ = -x^2 & x < 0 \end{cases}$$



$f(x) = \sqrt{|x|}$



$f(x) = x + |x|$



$f(x) = |x-1| + |x+1|$