PROJECTILE MOTION

An object that is in fight after being thrown or projected is called a projectile. The motion of a projectile may be thought as the result of two separate, simultaneously occurring perpendicular components of motions. One component is along horizontal direction without any acceleration and other is along vertical direction with constant acceleration due to gravity. It was Galileo who fist stated this independency of the horizontal and the vertical components of projectile motion.
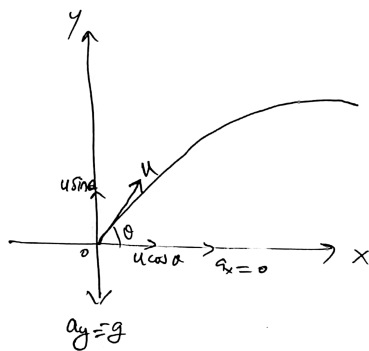
A particle is projected with a velocity u (velocity of projection) making using a angle θ with the horizontal. θ is known as angle of projection. Only force that control the projectile is gravity. We will neglect air resistance. Projectile is subjected to acceleration due to gravity $\vec{a} = -g\hat{j}$ ($a_x = 0$, $a_y = -g$).

$u\cos\theta$ is the horizontal component of velocity which remains constant. $u\sin\theta$ is initial vertical component of velocity. O is the point of projection which is taken as origin.

The velocity of the projectile after t second
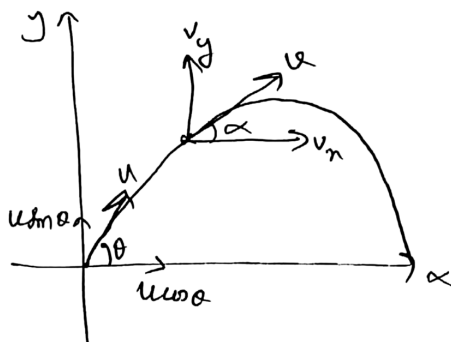
$V_y = u_y + a_y t$

$V_y = u\sin\theta - gt$

$V = \sqrt{V_x^2 + V_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$

$V = \sqrt{u^2 - 2u\sin\theta gt + g^2 t^2}$

Velocity v make an angle α with horizontal such that, $\tan\alpha = \dfrac{\text{opposite side}}{\text{adj.side}} = \dfrac{V_y}{V_x} = \dfrac{u\sin\theta - gt}{u\cos\theta}$

In vector form $\vec{a} = -g\hat{j}$

$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

$\vec{v} = \vec{u} + \vec{a}\,t$

$\vec{v} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} - gt\hat{j}$

$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$

$|\vec{v}| = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$, $\tan\alpha = \dfrac{u\sin\theta - gt}{u\cos\theta}$

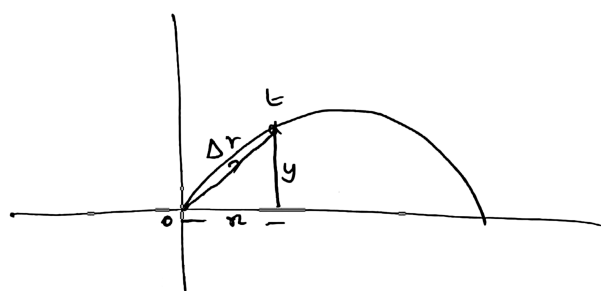**To find the displacement of the projectile after t seconds**

along x axis

disp = velocity × time

$$\boxed{x = (u\cos\theta)t}$$

This is the equation for x-coordinate of the projectile at any time t.

along y axis $y = u_y t + \dfrac{a_y t^2}{2}$

$\boxed{y = u\sin\theta\, t - \dfrac{1}{2}gt^2}$ Equation for y coordinate or height of projectile at any time t.

displacement = $r = \sqrt{x^2 + y^2}$

To find it in vector form we can use the equation $\vec{r} = x\hat{i} + y\hat{j}$ where $\vec{u} = u\sin\theta\hat{j},\ \vec{a} = -g\hat{j}$

Equation for path of a projectile

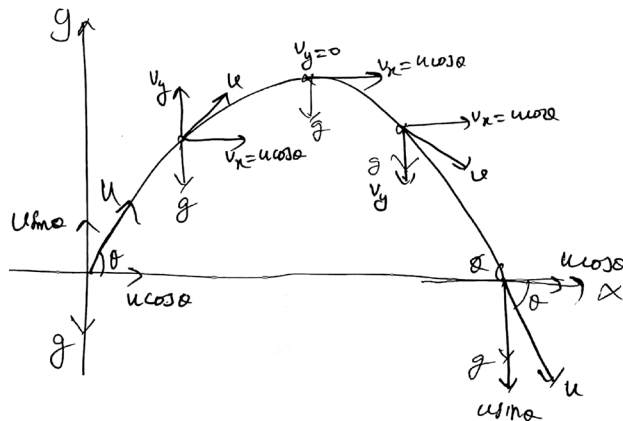$$x = u\cos\theta\, t,\ t = \frac{x}{u\cos\theta}$$

$$y = u\sin\theta\, t - \frac{1}{2}gt^2$$

$$y = u\sin\theta\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\boxed{y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}}$$

This is the equation of a parabola. Thus the path of a projectile is parabola.

At the highest point of the projectile vertical component of velocity is zero. Horizontal component is $u\cos\theta$ because it remains constant. At the highest point speed of the projectile is minimum and purely horizontal and is equal to $u\cos\theta$. Angle between acceleration and instantaneous velocity decreases from $(90 + \theta)$ to $(90 - \theta)$

Time of Flight of the projectile (T)
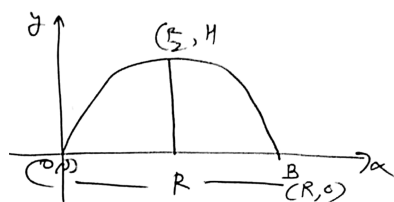
Consider the motion of the projectile along y-axis
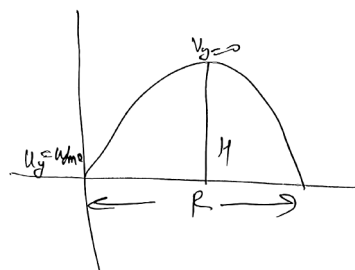
$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$t = T$$

$$0 = u\sin\theta T - \frac{1}{2}gT^2$$

$$\boxed{T = \frac{2u\sin\theta}{g}} \quad \text{or} \quad \boxed{T = \frac{2u_y}{g}}$$

Time of ascend = Time of descend = $\dfrac{u\sin\theta}{g}$

Maximum height of a projectile H

Consider the motion along y-axis

$$V_y^2 = u_y^2 + 2a_y g_y$$

$$0 = (u\sin\theta)^2 + 2(-gH)$$

$$2gH = (u\sin\theta)^2$$

$$\boxed{H = \frac{u^2 \sin^2\theta}{2g}} \quad \text{or} \quad \boxed{H = \frac{u_y^2}{2g}}$$

Horizontal range (R) of the projectile

Horizontal range = Horizontal velocity × time of flight

$$R = u\cos\theta\, T, \quad T = \frac{2u\sin\theta}{g}$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}} \quad \text{or} \quad \boxed{R = \frac{2u_x u_y}{g}}$$

Relation connecting R, H, T and angle of projection θ

$$\frac{H}{R} = \frac{u^2 \sin^2\theta}{2g} \cdot \frac{1}{\frac{u^2 2\sin\theta\cos\theta}{g}}$$

$$4H = R\tan\theta, \quad \boxed{H = \frac{gT^2}{8}}$$

$$\therefore 4\,\frac{gT^2}{8} = R\tan\theta \; ; \quad \boxed{R = \frac{gT^2}{2\tan\theta}}$$

Angle of projection for maximum range for a given speed of projection

$$R = \frac{u^2 \sin 2\theta}{g}, \quad [\sin 2\theta]_{max} = 1$$

$$R_{max} = \frac{u^2}{g} \qquad \sin 90^0 = 1$$
$$2\theta = 90^0$$
$$\theta = 45^0$$

To get maximum height we should throw vertically up $\theta = 90^0$

$$H_{max} = \frac{u^2}{2g}\,(1 - \text{dimensional motion})$$

$$H_{max} = \left(\frac{u^2}{g}\right) = \frac{R_{max}}{2}$$

Galileo in his book two new sciences stated that for elevations which exceed or fall short of 45^0 by equal amount, the ranges are equal.

i.e., there are two different angles of projection for same range. If one angle is θ other angle is $90 - \theta$ for same sped of projection.

Let θ_1 and θ_2 to be two different angles of projection for same range.
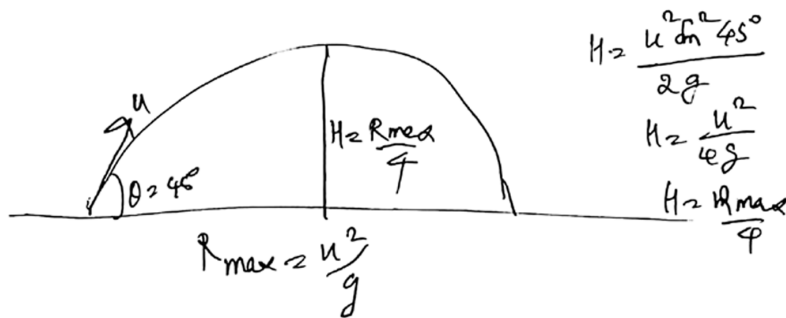
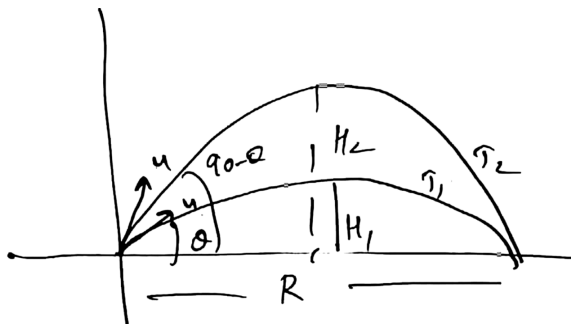
$$R = \frac{u^2 \sin 2\theta}{g}, \text{ since } R_1 = R_2$$

$$\sin 2\theta_1 = \sin 2\theta_2 \quad \sin(180 - A) = \sin A$$

$$\sin 2\theta_1 = \sin(180 - 2\theta_2)$$

$$2\theta_1 = 180 - 2\theta_2 \; ; \; \theta_1 = 90 - \theta_2$$

$$\theta_1 + \theta_2 = 90^0$$

e.g. for same speed at angles of projection 30^0 and 60^0 range is same. Also at 15^0 and 75^0 range is same.
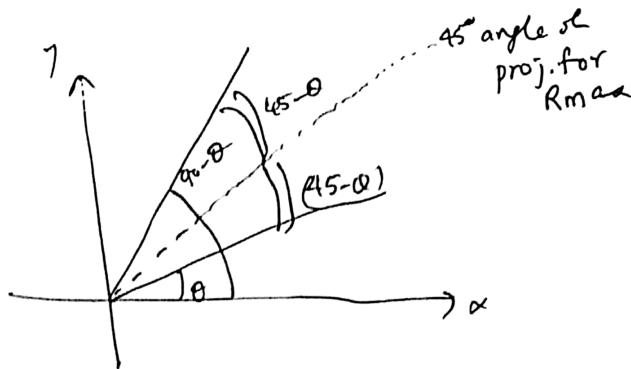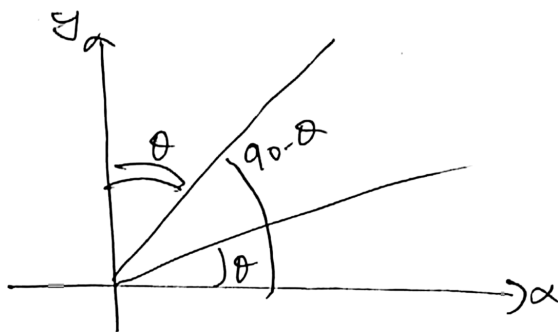

$\Rightarrow$ In the above situation when $R_1 = R_2$

$$T_1 = \frac{2u \sin \theta}{g} \; ; \; T_2 = \frac{2u \sin(90 - \theta)}{g}$$

$$\frac{T_1}{T_2} = \frac{\sin \theta}{\cos \theta} = \tan \theta \qquad [\sin(90 - \theta) = \cos \theta)$$

$$H_1 = \frac{u^2 \sin^2 \theta}{2g}; H_2 = \frac{u^2 \sin^2(90 - \theta)}{2g} \; ; \; \frac{H_1}{H_2} = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$$

Two different angles of projection for same range are equally inclined to the angle of projection for maximum range i.e. 45^0.

The different angles of projection for same range are equally inclined to the vertical and horizontal
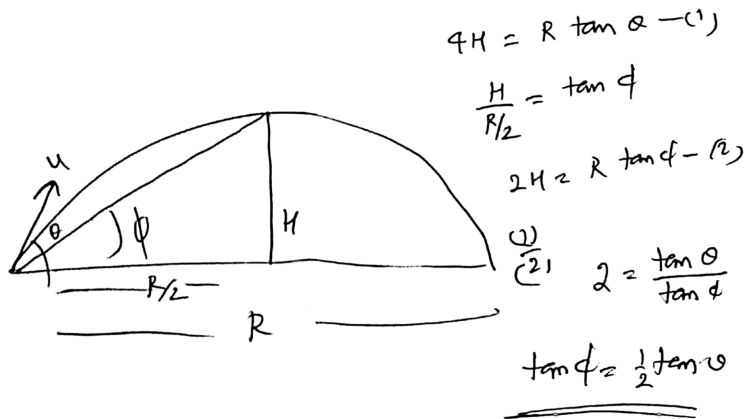
Equation for path of a projectile

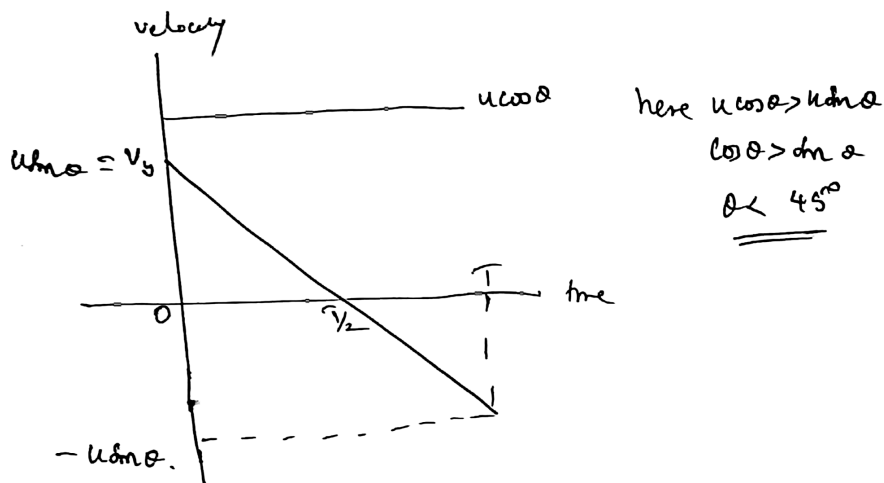$$y = x \tan\theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2\theta}$$

$$y = x \tan\theta \left[1 - \frac{x}{\dfrac{(2u^2 \sin\theta \cos\theta}{g}} \right]$$

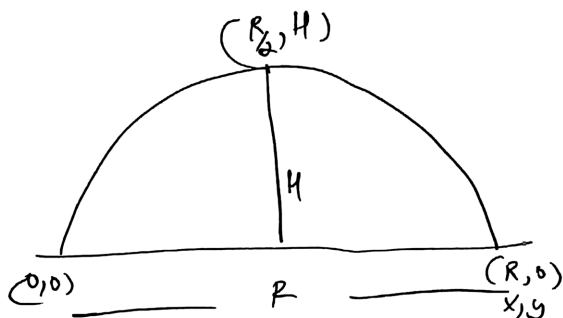$$y = x \tan\theta \left[1 - \frac{x}{R} \right]$$

Relation between angle of projection 'θ' and angle of elevation ϕ, at maximum height

$$4H = R \tan\theta \quad —(1)$$

$$\frac{H}{R/2} = \tan\phi$$

$$2H = R \tan\phi \quad —(2)$$

$$\frac{(1)}{(2)} \quad 2 = \frac{\tan\theta}{\tan\phi}$$

$$\underline{\tan\phi = \frac{1}{2}\tan\theta}$$

Velocity time graph

$u\cos\theta$

here $u\cos\theta > u\sin\theta$

$\cos\theta > \sin\theta$

$\theta < 45°$

$u\sin\theta = V_y$

$-u\sin\theta$

If the path of a projectile is given by the equation $y = ax - bx^2$ find R, H, T

$(\frac{R}{2}, H)$

H

$(0,0)$

$(R, 0)$

R

x, y

y = ax – bx², if y = 0, x = R

0 = ax – bx²

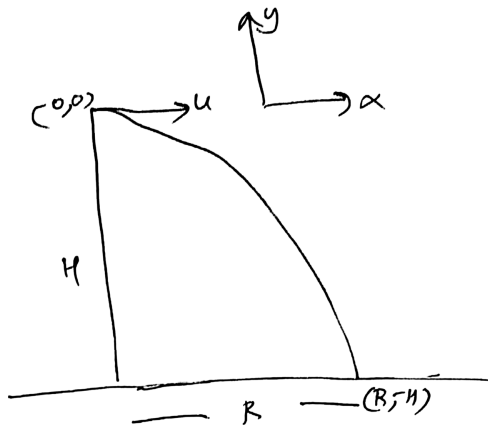ax = bx²

x = R = a/b

Compare $y = x\tan\theta[1 - x/R]$

$\tan\theta = a$

$4H = R\tan\theta$

$4H = \dfrac{a}{b}a, \qquad H = \dfrac{a^2}{4b}$

$H = \dfrac{T^2 g}{8} = \dfrac{a^2}{4b}$

Horizontal projection

A particle is projected horizontally with a velocity u from a height H. It follows a parabolic path and stroke the ground, horizontal component of velocity u remains constant vertical component is subjected to acceleration due to gravity.

$u_x = u, \ u_y = 0 \quad \vec{u} = u\hat{i}$

$a_x = 0, a_y = -g \quad \vec{a} = -g\hat{i}$

To find time of flight we consider the motion along y-axis

$S_y = u_y t + \dfrac{1}{2}a_y t^2$

$-H = 0 \times t + \dfrac{1}{2}(-g)t^2$

$$H = \frac{1}{2}gt^2$$
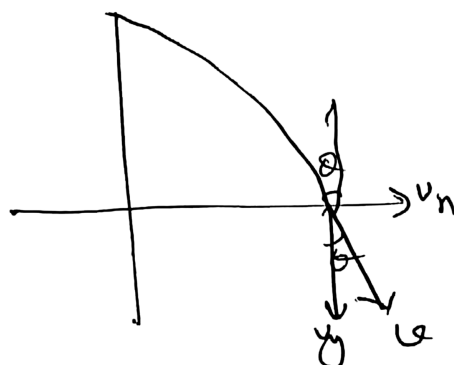
$$t = \sqrt{\frac{2}{g}H}$$

This is same as the time take by a dropped body to reach the ground dropped from rest.

To find range R, consider the horizontal motion,

$$S_x = u_x t$$

$$R = ut \quad R = u\sqrt{\frac{2H}{g}}$$

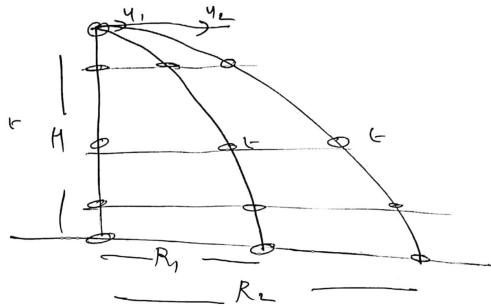**Velocity with which it hits the ground**

$$V_x = u$$

$$V_y^2 = u_y^2 + 2a_y y$$

$$Vy^2 = 0 + 2(-g)(-H)$$

$$V_y = \sqrt{2gH} \qquad V = \sqrt{V_x^2 + V_y^2} \quad ; \quad \tan\theta = \frac{V_x}{V_y}$$

θ is the angle made by the velocity with vertical.

Three projectiles one is dropped, other two are thrown with some velocities are shown below. Position are drawn at different intervals.


t is same for all $\qquad$ $R_1 = u_1 t$ $\qquad$ $R_2 = u_2 t$

$$\frac{R_1}{R_2} = \frac{u_1}{u_2}$$

All of them reach the ground at the same time. Their vertical motion are identical because they have same initial vertical velocity (zero in this case) and same acceleration.

Path of a projectile with respect to another projectile is a straight line. Their relative acceleration is zero.

Projectile Projected from the top of a building (Projected upwards)
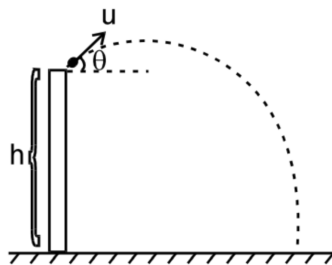
Horizontal motion $\qquad$ Vertical motion

$u_x = u \cos \theta$ $\qquad\qquad$ $u_y = u \sin \theta$

$a_x = 0$ $\qquad\qquad\qquad$ $a_y = -g$

Time of flight (T)

at $t = T$, $s_y = -h$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$-h = -u \sin \theta T + \frac{1}{2}(-g)T^2$$

Solving this equation 'T' will be obtained

Range (R)

at t = T, s_x = R

$$s_x = u_x t + \frac{1}{2} a_x t^2$$

$$R = u\cos\theta \times T + 0$$

$$\boxed{R = u\cos\theta \times T}$$

Projectile Projected from the top of a building (Projected downwards)

Horizontal motion	Vertical motion
$u_x = u\cos\theta$	$u_y = u\sin\theta$
$a_x = 0$	$a_y = -g$

Time of flight (T)

at t = T, s_y = –h

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$-h = -u\sin\theta T + \frac{1}{2}(-g)T^2$$
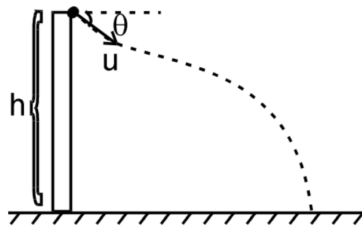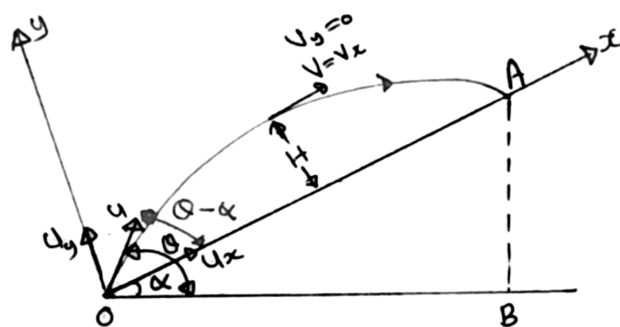
Solving this equation 'T' will be obtained

Range $R = u\cos\theta \times T$

Projection From Inclined Plane

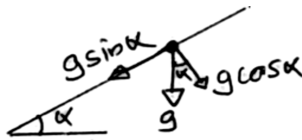$u_x = u\cos(\theta - \alpha)$

$u_y = u\sin(\theta - \alpha)$

$a_x = -g\sin\alpha$

$a_y = -g\cos\alpha$

For motion from 'O' to 'A' the displacement along the y-direction is zero.

$\therefore \ y = u_y t + \dfrac{a_y t^2}{2}$

$O = u\sin(\theta - \alpha)T - \dfrac{g\cos\alpha\, T^2}{2}$

$$\boxed{T = \dfrac{2u\sin(\theta - \alpha)}{g\cos\alpha}} \quad \text{or} \quad \boxed{T = \dfrac{2U_y}{|a_y|}}$$

Maximum Height from inclined surface (H)

$V_y^2 - U_y^2 = 2a_y y$

$\therefore O - U_y^2 = -2a_y H$

$$\boxed{\begin{array}{c} H = \dfrac{U_y^2}{2a_y} \\[2mm] H = \dfrac{U^2 \sin^2(\theta - \alpha)}{2g\cos\alpha} \end{array}}$$

Horizontal displacement $OB = (u\cos\theta)T$

$OB = u\cos\theta \times \dfrac{2u\sin(\theta - \alpha)}{g\cos\alpha}$

Range Along the inclined surface

$$R = OA = \frac{OB}{\cos\alpha}$$

$$\boxed{R = \frac{2u^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2\, 2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$$

$$\boxed{R = \frac{u^2[\sin(2\theta-\alpha)-\sin\alpha]}{g\cos^2\alpha}} \qquad 2\cos A\sin B = \sin(A+B)-\sin(A-B)$$

Range R is maximum, when $\sin(2\theta-\alpha)=1$
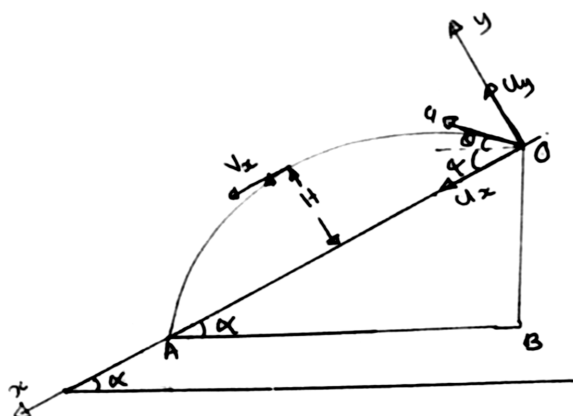
$$2\theta-\alpha = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4}+\frac{\alpha}{2}} \qquad \theta-\alpha = \frac{\pi}{4}+\frac{\alpha}{2}-\alpha = \frac{\pi-2\alpha}{4}$$

$$R_{max} = \frac{u^2[1-\sin\alpha]}{g\cos^2\alpha} = \frac{u^2(1-\sin\alpha)}{g(1-\sin^2\alpha)} \; ; \quad R_{max} = \frac{u^2[1-\sin\alpha]}{g[1+\sin\alpha]\,[1-\sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g[1+\sin\alpha]}}$$

Projectile Motion from an Inclined Plane

$$u_x = u\cos(\theta+\alpha)$$

$$u_y = u\sin(\theta+\alpha)$$

$$a_x = g\sin\alpha, \; a_y = -g\cos\alpha$$

When the object hits on the inclined plane

$$y = 0, \quad \therefore y = u_y t + \frac{a_y t^2}{2}$$

$$O = u\sin(\theta + \alpha)T - \frac{g\cos\alpha}{2}T^2 \qquad \boxed{T = \frac{2u\sin(\theta + \alpha)}{g\cos\alpha} = \frac{2U_y}{a_y}}$$

$$BA = (U_x \cos\theta)T = u\cos\theta \times \frac{2u\sin(\theta + \alpha)}{g\cos\alpha}$$

Range along the inclined surface

$$R = OA = \frac{AB}{\cos\alpha} = \frac{2u^2\cos\theta\sin(\theta + \alpha)}{g\cos^2\alpha} \qquad \boxed{R = \frac{2u^2\cos\theta\sin(\theta + \alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2}{g\cos^2\alpha}[\sin(2\theta + \alpha) - \sin(-\alpha)]$$

$$\boxed{R = \frac{u^2[\sin(2\theta + \alpha) + \sin\alpha]}{g\cos^2\alpha}}$$

For maximum range $\sin(2\theta + \alpha) = 1$, $\quad 2\theta + \alpha = \frac{\pi}{2}$

$$\boxed{\begin{aligned} \theta + \alpha &= \frac{\pi - 2\alpha}{4} + \alpha \\ \theta + \alpha &= \frac{\pi + 2\alpha}{4} \end{aligned}} \qquad \theta = \frac{\pi - 2\alpha}{4} \quad \boxed{R_{max} = \frac{u^2[1 + \sin\alpha]}{g\cos^2\alpha}}$$

$$R_{max} = \frac{u^2(1 + \sin\alpha)}{g(1 - \sin^2\alpha)} = \frac{u^2(1 + \sin\alpha)}{g[1 + \sin\alpha][1 - \sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g[1 - \sin\alpha]}}$$

Maximum height (H) from the inclined surface.

At maximum height $V_y = 0$

$$\therefore V_y^2 - u_y^2 = 2a_y y \text{ becomes } 0 - u_y^2 = 2a_y H$$

$$H = \frac{U_y^2}{2a_y} = \frac{u^2\sin^2(\theta + \alpha)}{2g\cos\alpha}$$

Note : For a given speed, the direction which gives the maximum range of the projectile on an inclined plane, bisects the angle between the incline and the vertical, for upward or downward projection.

Standard results for projectile motion on an incline plane

	Up the incline	Down the incline
Range	$\dfrac{2u^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$	$\dfrac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}$
Time of flight	$\dfrac{2u\sin(\theta-\alpha)}{g\cos\alpha}$	$\dfrac{2u\sin(\theta+\alpha)}{g\cos\alpha}=\dfrac{2u_y}{a_y}$
Maximum Range	$\dfrac{u^2}{g\,[1+\sin\alpha]}$	$\dfrac{u^2}{g\,[1-\sin\alpha]}$
Angle of projection for maximum range (from inclined surface)	$\dfrac{\pi-2\alpha}{4}$	$\dfrac{\pi+2\alpha}{4}$