CHAPTER - 16

DIFFERENTIAL EQUATIONS

An equation involving an independent variable, dependent variable and differential coefficients is called a differential equation.

The order of a differential equation is the order of the highest order derivative in that equation (it is always a positive integer).

The degree of a differential equation is the degree of the highest order derivative, after the differential coefficients are made free from radicals and fractions.

Ex: $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{3}{4}\frac{d^2y}{dx^2}$$

order 2, degree 2

The solution of a differential equation is a relation between dependent and independent variables which satisfies the differential equation.

The general solution will contain as many arbitrary constants as the order of the differential equation. Solution obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution.

Remember, every differential equation represents a family of curves. Each member of the family can be obtained by giving particular values to the arbitrary constants.

Any differential equation of the form $a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+..........+a_{n-1}\frac{dy}{dx}+a_n y=b$, where a_0, a_1, a_2..........a_n , b are either constants or function of the independent variable is called linear differential equation. Note that the degree of every differential coefficient and that of the dependent variable is one. Also no product terms of differential coefficients and dependent variable are present.

Formation of a Differential Equation:-

If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using these (n+1) equations, we eliminate the n constants. The resulting equation is the differential equation of order n of the given equation of n arbitrary constants.

Note:-

1. If an equation contains 'n' arbitrary constants, the corresponding differential equation is of order 'n'.

   Conversely, the solution of a differential equation of order 'n' will contain 'n' arbitrary constants.

2. d (xy) = xdy + ydx; $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$; $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$ and

$$d\left[\log\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy} \text{ or } \frac{-(ydx-xdy)}{xy};\ d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{x^2+y^2} \text{ or } -\frac{(ydx-xdy)}{x^2+y^2}$$

Equation of the Ist order Ist degree:-

A differential equation of the Ist order and Ist degree involves x,y and $\frac{dy}{dx}$. The general form is Mdx + Ndy = 0, where M and N are functions of x and y.

1. Variable separable

   A differential equation of the form Mdx + Ndy = 0 is called a variable separable equation if M is purely a function of x and N is purely a function of y. The solution of such an equation is given by $\int M dx + \int N dy = c$, an arbitrary constant.

Equations reducible to variable separable

Some times the equation is not in the form of variable separable, but some proper substitution reduces the equation to the variable separable form. If the equation is of the form $\frac{dy}{dx} = f(ax + by)$, then substitute ax + by = z reduces the equation to the form in which the varibles are separable.

2. Linear Differential Equation:-

An equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x alone can be solved by finding the integrating factor (I.F.) $e^{\int \rho dx}$, and the solution in given by $y(\text{I.F.}) = \int Q.(\text{I.F})dx + C$

Special Case : Sometimes the given differential equation can be converted into a linear differential equation by rewriting in the form $\frac{dx}{dy} + Rx = S$, where R and S are functions of y alone. Now the integrating factor is $e^{\int R dy}$. The solution is $x(\text{I.F}) = \int S(\text{I.F})dy + c$

3. Homogenous Differential Equation:-

A function f(x,y) is said to be a homogenous function of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

Such a function can be reduced to the variable separable form by the substitution .

y = vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Note:-

1. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if $h^2 = ab$, an ellipse if $h^2 < ab$ and a hyperbola if $h^2 > ab$

2. Equation of the circle passing through the origin and having centres on the x axis or Equation of the circle touching the y axis at origin is $x^2 + y^2 + 2gx = 0$

3. Equation of the circle passing through the origin and having centres on the y axis or

Equation of the circle touching the x axis at origin is $x^2 + y^2 + 2fy = 0$

4. $y = mx \pm a\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = a^2$, $y = mx + a/m$ is a tangent to the parabola $y^2 = 4ax$. $y = mx \pm \sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $y = mx \pm \sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

5. General equation of all non horizontal lines in the xy plane is $ax + by = 1$ where $a \neq 0$

6. General equation of all non vertical lines in the xy plane is $ax + by = 1$ where $b \neq 0$

7. General equation of all parabola whose axis of symmetry is parallel to x axis is $x = ay^2 + by + c$

8. General equation of all parabolas having the axis of symmetry coinciding with x axis is $y^2 = 4a(x - h)$

PART I - (JEEMAIN)

PART 1

SECTION - I - Straight objective type questions

1. The degree and order of the differential equation of the family of all parabolas whose axis is x–axis, are respectively

(1) 2, 1 (2) 1, 2 (3) 3, 2 (4) 2, 3

2. The order and degree of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{\frac{1}{3}}+x^{\frac{1}{4}}=0$ are respectively

(1) 2, 3 (2) 3, 3 (3) 2, 6 (4) 2, 4

3. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant, is

(1) $(x^2 + y^2)y' = 2xy$ (2) $2(x^2 + y^2)y' = 2xy$

(3) $(x^2 - y^2)y' = 2xy$ (4) $2(x^2 - y^2)y' = xy$

4. The solution of the differential equation $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

(1) $\tan y + \cot x = c$ (2) $\tan y \cot x = c$

(3) $\tan y - \cot x = c$ (4) None of these

5. The solution of the differential equation $(x^2 + y^2)dx = 2xydy$ is

(1) $x = c(x^2 + y^2)$ (2) $x = c(x^2 - y^2)$

(3) $x + c(x^2 - y^2) = 0$ (4) None of these

6. The solution of the differential equation $xdy - ydx = (\sqrt{x^2 + y^2})\, dx$ is

(1) $y - \sqrt{x^2 + y^2} = cx^2$ (2) $y + \sqrt{x^2 + y^2} = cx^2$

(3) $y + \sqrt{x^2 + y^2} + cx^2 = 0$ (4) None of these

7. Integrating factor of differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$ is

(1) $\cos x$ (2) $\tan x$

(3) $\sec x$ (4) $\sin x$

8. The solution of the equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$ is

(1) $x\sqrt{1-y^2} - y\sqrt{1-x^2} = c$ (2) $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$

(3) $x\sqrt{1+y^2} + y\sqrt{1+x^2} = c$ (4) None of these

9. If $\frac{dy}{dx}=1+x+y+xy$ and $y(-1)=0$, then function y is

(1) $e^{(1-x)^2/2}$ (2) $e^{(1+x)^2/2}-1$ (3) $\log_e(1+x)-1$ (4) $1+x$

10. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt}-ty=1$ and $y(0)=-1$, then $y(1)$ is equal to

(1) $-\frac{1}{2}$ (2) $e+\left(\frac{1}{2}\right)$ (3) $e-\frac{1}{2}$ (4) $\frac{1}{2}$

11. The solution of the equation $x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$ is

(1) $x\sin\left(\frac{x}{y}\right)+c=0$ (2) $x\sin y+c=0$

(3) $x\sin\left(\frac{y}{x}\right)=c$ (4) None of these

12. The solution of the differential equation $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$, is

(1) $(x-2)=ke^{\tan^{-1}y}$ (2) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$

(3) $xe^{\tan^{-1}y}=\tan^{-1}y+k$ (4) $xe^{2\tan^{-1}y}=e^{\tan^{-1}y}+k$

13. The solution of the differential equation $x\frac{dy}{dx}=y(\log y-\log x+1)$ is

(1) $y=xe^{cx}$ (2) $y+xe^{cx}=0$

(3) $y+e^{x}=0$ (4) None of these

14. The solution of the equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ is

(1) $e^y = e^x + \frac{x^3}{3} + c$ (2) $e^y = e^x + 2x + c$

(3) $e^y = e^x + x^3 + c$ (4) $y = e^x + c$

15. Solution of $(x+y-1)dx + (2x+2y-3)dy = 0$ is

(1) $y + x + \log(x+y-2) = c$ (2) $y + 2x + \log(x+y-2) = c$

(3) $2y + x + \log(x+y-2) = c$ (4) $2y + 2x + \log(x+y-2) = c$

16. The equation of the curve passing through the origin and satisfying the equation $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$ is

(1) $3(1+x^2)y = 4x^3$ (2) $3(1-x^2)y = 4x^3$

(3) $3(1+x^2) = x^3$ (4) None of these

17. The curve amongst the family of curves, represented by the differential equations, $(x^2 - y^2)dx + 2xy\,dy = 0$ which passes through (1, 1) is:

(1) A circle with centre on the y-axis

(2) A circle with centre on the x-axis

(3) An ellipse with major axis along the y-axis

(4) A hyperbola with transverse axis along the x-axis.

18. The solution of the differential equation $\frac{dy}{dx} = (x-y)^2$, when $y(1) = 1$, is:

(1) $\log_e\left|\frac{2-y}{2-x}\right| = 2(y-1)$ (2) $\log_e\left|\frac{2-x}{2-y}\right| = x - y$

(3) $-\log_e\left|\frac{1+x-y}{1-x+y}\right| = x + y - 2$ (4) $-\log_e\left|\frac{1-x+y}{1+x-y}\right| = 2(x-1)$

19. If $y = y(x)$ is the solution of the differential equation $e^y\left(\frac{dy}{dx}-1\right)=e^x$ such that $y(0)=0$, then $y(1)$ is equal to

(1) $2+\log_e 2$ (2) $2e$ (3) $\log_e 2$ (4) $1+\log_e 2$

20. Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2)=\frac{2}{\sqrt{3}}$

Statement-I: $y(x)=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)$ and

Statement-II: $y(x)$ is given by $\frac{1}{y}=\frac{2\sqrt{3}}{x}-\sqrt{1-\frac{1}{x^2}}$.

(1) If both Statement-I and Statement-II are true and the reason is the correct explanation of the statement-I.

(2) If both Statement-I and Statement-II are true but reason is not the correct explanation of the statement-I.

(3 If Statement-I is true but Statement-II is false.

(4) If Statement-I is false but Statement-II is true.

SECTION - II

Numerical type Questions

21. Let $y = y(x)$ be the solution of the differential equation $\frac{2+\sin x}{y+1}\cdot\frac{dy}{dx}=-\cos x, y>0$, $y(0)=1$. If $y(\pi)=a$ and $\frac{dy}{dx}$ at $x=\pi$ is b, then the sum of pair $(a+b)$ is ___________

22. Let $y = y(x)$ be the solution of the differential equation, $(x^2+1)^2\frac{dy}{dx}+2x(x^2+1)y=1$ such that $y(0)=0$. If $\sqrt{\frac{1}{a}}y(1)=\frac{\pi}{32}$, then the value of '$a$' is __________

23. Let $y = y(x)$ be the solution of the differential equation, $x\frac{dy}{dx}+y=x\log_e x$, $(x>1)$.

If $2y(2)=\log_e 4-1$. If $y(e)=\frac{e}{k}$, then the 'k' value is________

24. Let $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y\tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$. If the maximum value of the function $y(x)$ over R is $\frac{1}{k}$, then the 'k' value is ________

25. Let $y = y(x)$ be the solution of the differential equation $x\tan\left(\frac{y}{x}\right)dy = \left(y\tan\left(\frac{y}{x}\right) - x\right)dx$, $-1 \le x \le 1$, $y\left(\frac{1}{2}\right) = \frac{\pi}{6}$. If the area of the region bounded by the curves $x = 0$, $x = \frac{1}{\sqrt{2}}$ and $y = y(x)$ in the upper half plane is $\frac{1}{k}(\pi - 1)$, then the 'k' value is __________

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. $y = c_1e^{2x} + c_2e^{x} + c_3c^{-x}$ satisfies the differential equation $\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, then $\frac{a^3+b^3+c^3}{abc}$ is equal to

(A) 1 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{1}{4}$

27. The differential equation of the system of circle touching the x-axis at origin is

(A) $\left(x^2 - y^2\right)\frac{dy}{dx} - 2xy = 0$ (B) $\left(x^2 - y^2\right)\frac{dy}{dx} + 2xy = 0$

(C) $\left(x^2 + y^2\right)\frac{dy}{dx} - 2xy = 0$ (D) $\left(x^2 + y^2\right)\frac{dy}{dx} + 2xy = 0$

28. Solution of $\left(xy^4 + y\right)dx - xdy = 0$ is

(A) $\frac{x^4}{4} + \left(\frac{x}{y}\right)^3 = c$ (B) $\frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = c$ (C) $\frac{x^4}{4} + 3\left(\frac{x}{y}\right)^2 = c$ (D) $\frac{x^4}{4} - \frac{1}{3}\left(\frac{x}{y}\right)^3 = c$

29. The equation of a curve passing through $\left(1, \frac{\pi}{4}\right)$ and having slope $\frac{\sin 2y}{x + \tan y}$ at (x, y) is

(A) $x = \tan y$ (B) $y = \tan x$ (C) $x = 2\tan y$ (D) $y = 2\tan x$

30. The solutions of differential equation $2y\sin x\left(\frac{dy}{dx}\right) = 2\sin x\cos x - y^2\cos x$ at $x = \frac{\pi}{2}, y = 1$ is

A) $y^2 = \sin x$ B) $y = \sin^2 x$ C) $y^2 = \cos x + 1$ D) none of these

31. Let f(x) be differentiable on the interval $(0,\infty)$ such that f(1)=1, and $\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each x>0. Then f(x) is

A) $\frac{1}{3x}+\frac{2x^2}{3}$ B) $-\frac{1}{3x}+\frac{4x^2}{3}$ C) $-\frac{1}{x}+\frac{2}{x^2}$ D) $\frac{1}{x}$

SECTION - IV (More than one correct answer)

32. Solution of the differential equation $\frac{dy}{dx}=e^{x-y}\left(e^x-e^y\right)$ is

A) $ye^{e^x}=e^{e^x}\left(e^x-1\right)+C$ B) $y=e^x-e^{x-e^x}+C$

C) $y=(e^e-1)+ce^{-e^x}+C$ D) $ye^{ex}=e^x\left(e^{e^x}-1\right)+C$

33. If $y(x)$ satisfies the differential equation $y'-y\tan x=2x\sec x$ and $y(0)=0$, then

A) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$ B) $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$ C) $y\left(\frac{\pi}{3}\right)=\frac{\pi^2}{19}$ D) $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

34. Let $y=(A+Bx)e^{3x}$ is a solution of the differentialequation $\frac{d^2y}{dx^2}+m\frac{dy}{dx}+ny=0, m,n\in I$, then

A) $m=-6$ B) $n=-6$ C) $m=9$ D) $n=9$

35. The differential equation representing the family of curves $y^2=2c\left(x+\sqrt{c}\right)$, where c is positive parameter, is of

A) order 1 B) order 2 C) degree 3 D) degree 4

SECTION - V (Numerical Type)

36. A curve passing through the poin (1,1) has the property that the perpendicular distance of the orign from normal at any pint P of the curve is equal to the distance of P from the x-axis is a circle with radius = __________

37. The curve represented by the differential equation xdy - ydx = ydy intersects the y axis at A(0, 1) and the line y = e at (a, b), then find a + b

38. Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1)=2$. If $6\int_1^x f(t)\,dt=3xf(x)-x^3$ for all $x\ge 1$, then the value of $f(2)$ is ________

39. Let y(x) be the general solution of $x(x-1)\frac{dy}{dx}-y=x^2(x-1)^2$. Then 4y(2)-y(-1) equals.

SECTION VI - (Matrix match type)

40. Match List I with List II and select the correct answer using the codes given below the lists:

Column - I (Differential Equation)	**Column - II (Solution)**
A) $\dfrac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\dfrac{y^4}{x^2}$	P) $x^2+2xy=c$
B) $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}$	Q) $y=\dfrac{1}{2}\log(1+x^2)+c$
C) $x\,dx+(x+y)\,dx=0$	R) $e^y=e^x+\dfrac{x^3}{3}+c$
D) $\dfrac{dy}{dx}=\dfrac{x}{1+x^2}$	S) $\dfrac{2y}{x}-\dfrac{1}{x^2+y^2}=c$

A) A-S; B-R; C-P; D-Q
B) A-R; B-S; C-P; D-Q
C) A-P; B-Q; C-R; D-S
D) A-S; B-Q; C-R; D-P