WORK ENERGY POWER

In everyday language, we often use the terms work, energy and power. A teacher teaching a class, a student preparing for examination, a farmer ploughing the field, all are said to be working. A person who can put in long hours of work is said to have large stamina or more energy. In karate or boxing we talk of powerful punches, that are delivered at greater speed for small time durations. But all these terms have specific scientific meanings, which are to be learnt in detail in this chapter.

WORK

Any agent which can deliver a force can perform a work also. In science, work done by a force may be zero even if the force is acting. Suppose a man is pushing a wall and in that case the person is applying a force. But scientifically the work done by that man is zero. Also, when a man with a load on his head is walking on a straight horizontal road, then also work done by that man in displacing that load is zero. Scientifically a force is said to perform a work only when the following two conditions are satisfied.

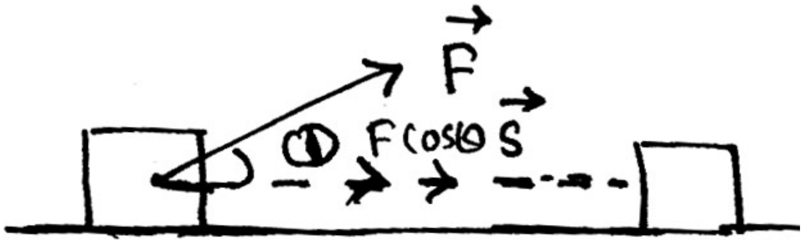
(i) The body on which the force is acting must undergo a displacement.

(ii) The direction of this displacement must not be perpendicular to the direction of that force.

So work is said to be by a force acting on a body, only if that body undergo a displacement under the influence of that force such that the direction of displacement is not perpendicular to the direction of that force.

It is to be noted that, even if the force is not directly responsible for the motion of the body, it can perform work on the body.

Consider a body placed on a horizontal surface. Let a force F is applied on the body as shown. Let the body undergo a displacement s along the surface such that the direction of $\vec{s}$ makes an angle θ with the direction of $\vec{F}$



$F\cos\theta$ is the component of force $\vec{F}$ in the direction of $\vec{s}$

Work done by the force is

$W = (F\cos\theta)\, s$ $\boxed{w = FS\cos\theta}$

OR

$\boxed{w = \vec{F}\cdot\vec{S}}$

So the work done by a force is defined as the product of component of the force in the direction of displacement and the magnitude of the displacement. So in general work done by a force can be defined as the dot product of force vector and displacement vector.

Calculation of work done by a force can be carried out in three different ways.

(i) $w = Fs\cos\theta$

(ii) $w = s \times (f\cos\theta)$

w = value of displacement × component of force in the direction of displacement

(iii) $w = F \times (s\cos\theta)$ = value of force x component of displacement in the direction of force

In terms of rectangular components, if

$\vec{F} = Fx\,\hat{i} + Fy\,\hat{j} + Fz\,\hat{k}$ and

$\vec{s} = x\,\hat{i} + y\,\hat{j} + z\hat{k}$

Then work done by the force

$$w = \vec{F}\cdot\vec{s} = (Fx\hat{i} + Fy\hat{j} + Fz\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$
$$w = xFx + yFy + zFz$$

Dimensions of work = ML^2T^{-2}; Unit of work done

In SI system ; unit of work is Joule (J)

In cgs system ; unit of work is erg

1 Joule = 10^7 ergs

Practical unit of work is Kilogram - metre (kg-m)

1 Kgm = 9.8 J

NATURE OF WORK DONE

Depending on the angle (θ) between the force vector and displacement vector, work done by a force can be positive, negative or zero.

a. **Positive work**

If θ is acute $(0 \le \theta < \pi/2)$, then $\cos\theta$ is positive and work done will be also positive.

eg :

(i) When a body falls freely under the action of gravity, weight mg acts downwards and body also displaces down wards. so $\theta = 0^0$ and hence work done by weight will be positive.

(ii) When a lawn roller is pulled by applying a force along the handle at an acute angle, work done by applied force is positive.

(iii) When a gas filled in a cylinder fitted with a movable piston is allowed to expand, work done by the gas is positive.

(iv) When a spring is stretched, work done by the stretching force is positive.

b. <u>Negative work</u>

If θ is obtuse $\left(\frac{\pi}{2} < \theta \leq \pi\right)$, $\cos\theta$ will be negative and hence work done will be negative.

eg: (i) When a body is thrown vertically upwards, weight of body is acting vertically downwards but the displacement of body is vertically upwards. So $\theta = 180^0$ and hence work done by weight will be negative.

(ii) When a body is moved over a rough horizontal surface, the force of friction opposes the motion. So angle between friction and displacement is 180^0 so work done by friction is negative.

(iii) When breaks are applied on a moving vehicle, work done by breaking force is negative.

c. <u>Zero work</u>

Work done by a force will be zero in two cases.

(i) If body is not moving even after the application of a force, then displacement of the body will be zero and hence work done will be zero.

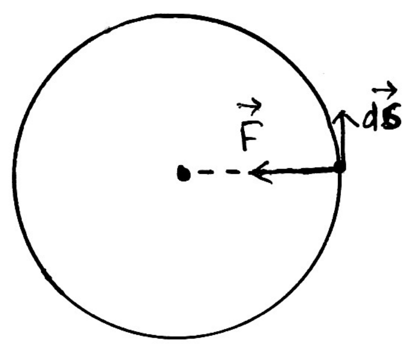
(ii) If the direction of displacement is perpendicular to the direction of force, then $\theta = 90^0$ and $\cos\theta = 0$. then work done by the force will be zero.

eg :

(i) when we push hard against a wall, the force we exert on the wall does no work because displacement is zero. However in this process our muscles are contracting and relaxing alternately and internal energy is being used up. That is why we get tired.

(ii) When a person carrying some load on his head moves on a horizontal platform. Then $\theta = 90^0$ and hence work done by him is zero.

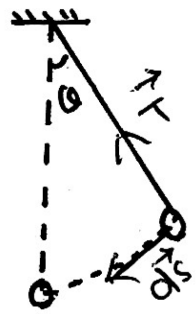
(iii) When a particle moves in a circular orbit, centripetal force always acts towards the centre of circle along the radius and the displacement is tangential to the circular orbit.



So angle between $\vec{F}$ and $\vec{ds}$ is 90^0. So work done by centripetal force is always zero.

If a satellite is revolving around the earth in a circular orbit, gravitational force on it from earth acts as the centripetal force. So work done by the gravitational force is zero on the satellite.

(iv) Tension in the string of a simple pendulum is always perpendicular to the displacement of bob.



So $\theta = 90^0$ hence work done by tension is zero.

WORK DONE BY A CONSTANT FORCE

If the value and direction of a force remains same throughout it is a constant force. Then work done by it to make a body to displace through s is;

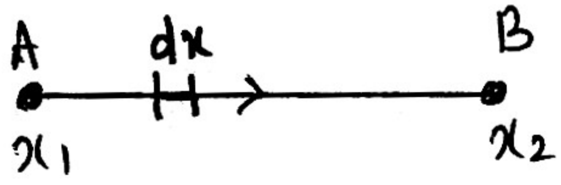
$w = \vec{F} \cdot \vec{S} = FS\cos\theta$

WORK DONE BY A VARIABLE FORCE

If the value or direction or both of a force varies either with position or time, it is a variable force.

eg: Restoring force developed in a spring.

Consider a variable force $\vec{F}(x)$ acting in the x direction. Under the influence of this force, let a body displaces in the x direction from an initial position x_1 to a final position x_2 as shown.



To find the work done by this force, let us consider an elementary displacement dx in the path. Then small work done to displace the body through the element.

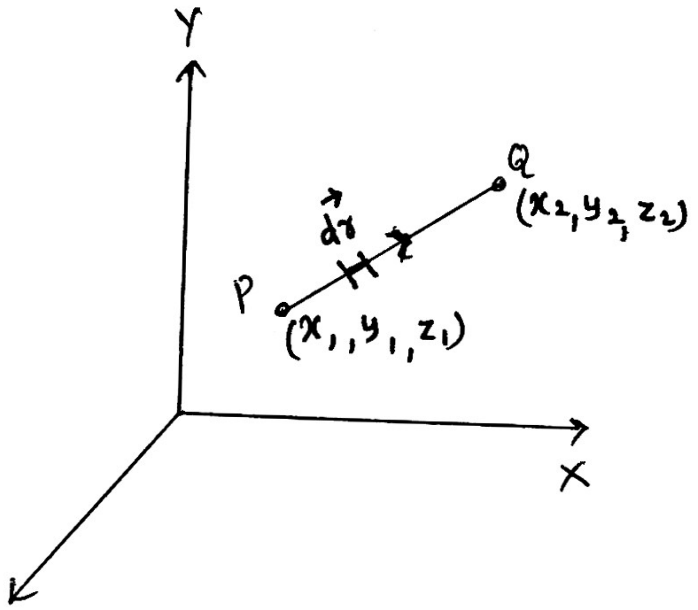
$dw = \vec{F} \cdot \vec{dx}$

To find the total work done to displace the body from A to B, the path length AB is split in to a number of such elements, then find dw for each element and take their sum. Instead, we can integrate dw within limits x_1 to x_2

$$\boxed{w = \int_{x_1}^{x_2} \vec{F} \cdot \overrightarrow{dx}}$$

The above situation is used when force vector and displacement vector are one dimensional.

Now let us consider a variable force in three dimensions. $\vec{F} = Fx\hat{i} + Fy\hat{j} + Fz\,\hat{k}$



Let this force displaces a body from a position p(x_1, y_1, z_1) to Q (x_2, y_2, z_2)

Consider a small element $\overrightarrow{dr}$ in the path, given by

$\overrightarrow{dr}$ = dx $\hat{i}$ + dy $\hat{j}$ + dz $\hat{k}$

Then small work done to displace the body through this element.

dw = $\vec{F} \cdot \overrightarrow{dr}$ = $(Fx\hat{i} + Fy\hat{j} + Fz\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$ = Fx dx + Fy dy + Fz dz

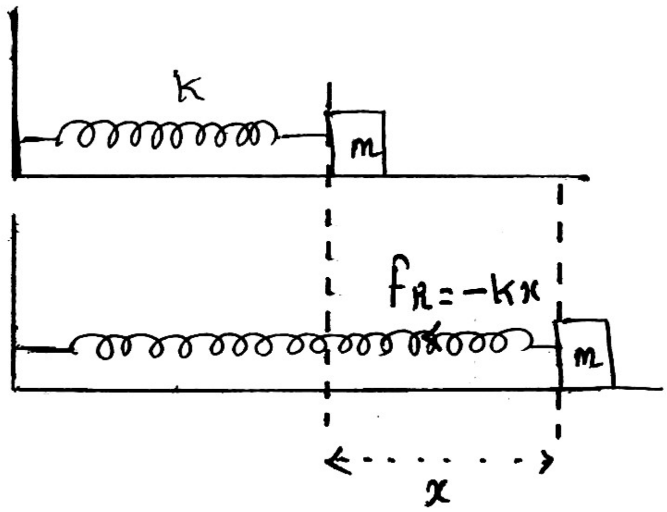
Then total workdone to displace the body from P to Q;

$$w = \int_{x_1, y_1, z_1}^{x_2, y_2, z_2} Fx\,dx + Fy\,dy + Fzdz$$

$$\boxed{w = \int_{x_1}^{x_2} Fx\,dx + \int_{y_1}^{y_2} Fy\,dy + \int_{z_1}^{z_2} Fz\,dz}$$

WORK DONE BY A SPRING FORCE

Consider a spring of force constant k with one end fixed and the other end connected to a block of mass M. Let the spring is in its natural length position. Let the spring is elongated or compressed through a distance x as shown.



If the spring is released, the system will return. It means that a restoring force acts on the spring given by

F = -kx

This is a variable force work done by spring force is given by

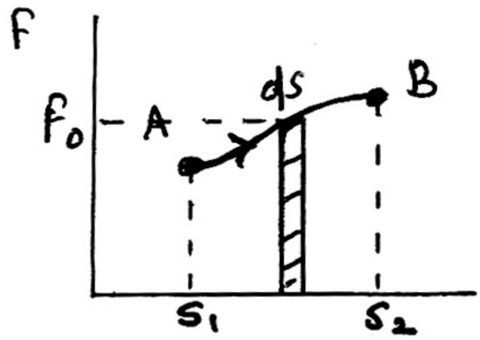
$$\boxed{w = \frac{1}{2}k\left[x_{initial}^{2} - x_{final}^{2}\right]}$$

$x_{initial} \rightarrow$ initial elongation or compression on the spring

$x_{final} \rightarrow$ final elongation or compression on the spring

WORK DONE FROM FORCE - DISPLACEMENT GRAPH

Consider a graph draw between a force F acting on a body and displacement s the body has undergone from position A to position B.



To find the workdone by the force consider a small element as shown over which the displacement is ds and force is almost constant at F_0. Then, element is like a rectangular strip.

Area of element

dA = F_0 x ds = work done by force for a small displacement ds

Then to find total workdone, split the area of region below the graph into such small elements, find area of each element and then take their sum.

$w = \int_{s_1}^{s_2} F ds =$ area of the region between graph AB and displacement axis.

Hence workdone can be calculated as the area under force displacement graph.

$\boxed{\text{Work done} = \text{area of the region between the force displacement graph and displacement axis}}$

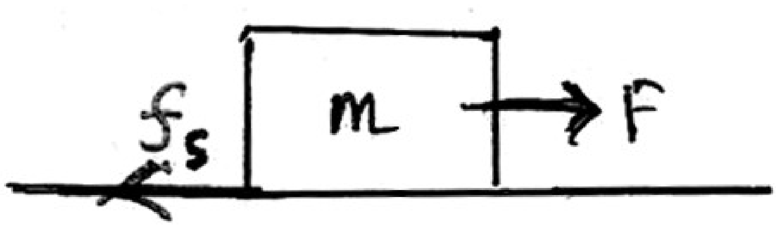
While taking the area, sign of force and displacement must also be considered. If Fs product is positive work done must be taken as positive and if the product is negative, workdone must be taken as negative.

WORK DONE BY STATIC FRICTION

Static friction is the frictional force developed at the contact surface between two bodies when there is no relative motion between them at their contact surface.

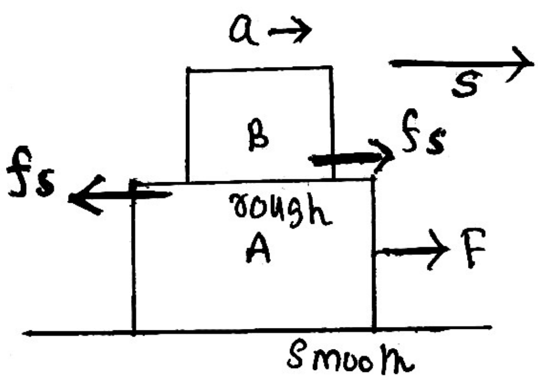
eg

(1) : Consider a block placed on a rough horizontal surface acted upon by an external force F as shown.



Let the body continues to be at rest even if F is acting. This is because F is nullified by friction f_s and is static. Since displacement of body is zero, work done by static friction in this case is zero.

eg (2): Consider a block A placed over a smooth horizontal floor. Another block B is placed over A and the cotact surface between A and B is rough. Let an external force F acts on the lower body as shown so that both the blocks move together with a common acceleration.



Since both move with a common acceleration, there is no relative motion or slipping between A and B. So friction at their contact surface is static (f_s). Since F on A is towards right, f_s will act towards left on A. So the direction of fs on B is towards right. So fx behaves like an applied force on B and makes it move along with A with same acceleration a. Let s be the displacement of the system after time t. Then static friction perform work on both A and B. Work done by static friction on A

$$w_A = f_s \, s \cos 180 = -f_s \, s$$

Work done by static friction on B;

$$w_B = f_s \, s \cos 0 = + f_s \, s$$

$\therefore$ Total work done by static friction on the system = $w_A + w_B = 0$

Points

(i) Work done by static friction can be positive, negative or zero.

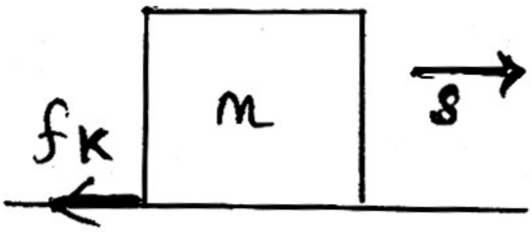
(ii) Static friction perform work on both bodies forming the contact surface

(iii) Total work done by the static friction on the system forming the contact surface is zero.

WORK DONE BY KINETIC FRICTION

Kinetic friction is the frictional force developed at the contact surface between two bodies when there is a relative motion between them at their contact surface.

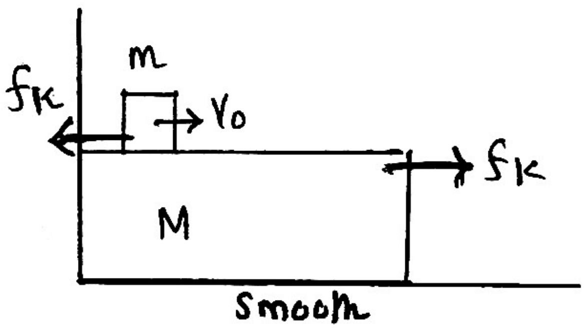
eg : Consider a block sliding on a fixed rough horizontal surface as shown. then friction is kinetic, opposite to its direction of motion.



Let s be the displacement . Then work done by kinetic friction = $f_k S \cos 180 = -f_k S$

So here workdone by kinetic friction is negative.

eg : A block of mass m is projected with a velocity Vo on a plank of mass M such that the block slides through a displacement over the plank.



So friction is kinetic.

(f_k) between them

It acts towards left on M as it slides towards right. So f_k acts towards right on m due to which it also starts moving towards right. Let S_1 and S_2 be the displacements of m and M until the slipping between them stops. So $S_1>S_2$.

Work done by kinetic friction on M ; $W_1 = f_k S_1 \cos 180 = -f_k S_1$

Work done by kinetic friction on M ; $W_2 - f_k S_2 \cos 0 = f_k S_2$

Total work done by Kinetic friction = $W_1 + W_2$

[Since $S_1 > S_2$; $|w_1|>|W_2|$] = (-)

Points

(i) Work done by kinetic friction can be positive or negative.

(ii) Total workdone by kinetic friction on a system forming the contact surface is always negative.

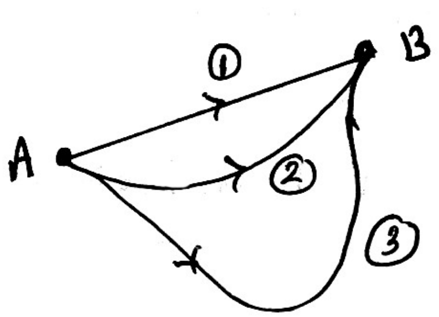
CONSERVATIVE FORCES

Depending on the work done, forces are generally classified into two types

(i) Conservative forces

(ii) Non conservative forces

Let a body is displaced from an initial point A to final point B under the influence of a force F, through different paths as shown.



Let W_1, W_2 and W_3 respectively represents the workdone by F along paths (1), (2) and (3).

If $W_1 = W_2 = W_3$, then the force F is said to be conservative.

A force is said to be conservative if work done by or against the force in moving a body depends only the initial and final positions of the body and is completely independent of the path through which the body is displaced.

eg : Gravitational force, spring force, electric force between charges, magnetic force between magnetic poles etc.

It is to be noted that work done by an external agent against a conservative force is also independent of the path and depends only on initial and final points.

If we consider mountain roads, they do not go straight up but wind up gradually in hair pin mode. Since engine of vehicles do work against gravitational force (conservative), work done will be same along both the paths. Since w = Fs cos θ, if s is large F will be small for same amount of work. So in hair pin mode, S will be very large and hence engine force required will be small.

Now let us check the conservative nature of gravitational force. Let a body of mas m is taken from the ground to a platform at a height h, through different paths as shown

Properties

(i) Work done by a conservative force or work done by an external agent against a conservative force to displace a body from one point to another depends only on the initial and final points and is completely independent of the path through which the body is displaced.

(ii) Work done by a conservative force to move a body around a closed loop is always zero.

(iii) If only conservative forces are performing work on a body, then total mechanical energy (KE + PE) of that body remains a constant. That is why the name conservative is used.

(iv) If an external agent performs a work against a conservative force to displace a body, then that energy does not lost but is stored as potential energy on that body.

(v) A conservative force on a body is always directed towards a fixed point.

NON CONSERVATIVE FORCES

Properties of a non conservative force is exactly opposite to those of conservative forces.

A force is said to be non conservative, if work done or against the force in moving a body from one position to another, depends on the path followed between these two positions.

eg : frictional force, viscous force etc.

Properties

(i) Work done by a non conservative force or by an external agent against a non conservative force depends on the path through which body is moved.

(ii) Work done by a non conservative force to move a body around a closed loop is non zero.

(iii) If non conservative forces are performing work on a moving body, its total mechanical energy (KE + PE) will not be conserved.

(iv) If an external agent performs a work against a non conservative force. it is lost as heat or other energy forms. That is why the non conservative forces are known as dissipative forces or damping forces.

(v) Non conservative forces on a body are not always directed to a fixed point.

It is to be noted that all the forces which are not conservative are non conservative. There are many forces which are neither purely conservative or purely nonconservative.

ENERGY

Energy of a body is defined as the capacity or ability of the body to do the work. When a body is capable of doing more work, it is said to pusess more energy. Like work, energy is also a scalar quantity.

There are many forms of energy like mechanical energy, heat energy, chemical energy, light energy, sound energy, electrical energy, nuclear energy etc.

Unit of energy is also joule (SI)or ergs (cgs). There are many practical units of energy. Calorie for heat energy; I calorie = 4.18 J electron volt for nuclear energy; 1 eV = 1.6×10^{-19} J

Kilowatt hour for electrical energy; 1 KWh = 3.6×10^{6} J

Mechanical energy of a body is defined as the ability of the body to do mechanical work. It is equal to the sum of kinetic energy and potential energy. ME = KE + PE

KINETIC ENERGY

Kinetic energy of a body is defined as the energy posessed by the body by virtue of its motion.

eg :

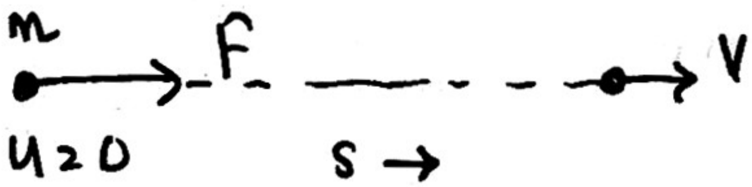
(i) A bullet fired from a gun can pierce through a target on account of Kinetic energy of the bullet

(ii) Wind mills work on the Kinetic energy of air.

(iii) A nail is driven into a wooden block on account of KE of the hammer striking the nail.

EQUATION FOR KE

Consider a body of mass m placed at rest. Let a constant force F is applied on it so that it starts moving with a constant acceleration a. Let it gains a velocity v when it is moved through a displacement s.



$a = \frac{F}{m}$ using $V^2 = U^2 + 2as$

$V^2 = U + 2\frac{F}{m}s \Rightarrow S = \frac{mv^2}{2F}$

Work done by the force $W = Fs\cos\theta = F \times \frac{mv^2}{2F} \times \cos 0$

$= \frac{1}{2}mv^2$

This work done appears as translational or linear KE

$$\boxed{KE = \frac{1}{2}mv^2}$$

(i) The expression $\frac{1}{2}mv^2$ holds even when the force applied varies. That is expression is valid irrespective of how the body acquires velocity v.

(ii) Kinetic energy of a body is always positive.

(iii) Kinetic energy of a body depends upon the frame of reference.

RELATION BETWEEN KE AND MOMENTUM

$$KE;\ \ E = \frac{1}{2}mv^2 \times \frac{m}{m}$$

$$= \frac{m^2v^2}{2m} \qquad\qquad mv = p\text{, linear momentum}$$

$$\boxed{E = \frac{p^2}{2m}}$$

KE α momentum^2

POTENTIAL ENERGY

The potential energy of a body is defined as the energy possessed by the body by virtue of its position or configuration in some field.

Two important types of potential energy are

(i) Gravitational Potential energy

(ii) Elastic Potential energy

Potential energy of a body refers to an energy stored in a system which gives that system an ability to do mechanical work. For that this PE is converted into KE.

For example, when we wind the spring of our watch, PE is stored in the spring on account of configuration of the turns of the spring. As the spring unwinds, it works to move the hands of the watch. Thus wound spring has the potential to do the work.

Two important points are need to be considered while considering PE

Point 1 : Potential energy is associated only with conservative forces
Point 2 : If an external agent performs work against a conservative force to displace a body, without an acceleration, the entire workdone by the agent appearas change in potential energy of that system or body

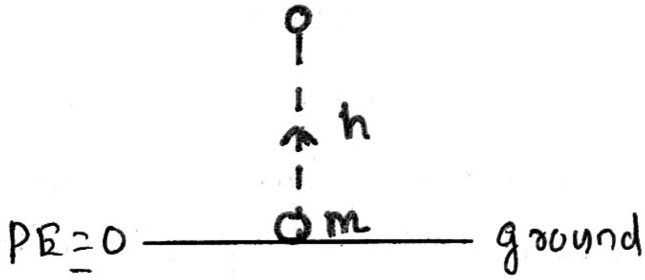
What is presented as point 2 above is the method by which we can provide PE to a body. The process is done without an acceleration because, in such a case, speed will not change and hence KE will also not change. So entire work done will appear as PE change only.

eg: Let a body of mass m is bring under the influence of gravity, which is conservative. Let the body is moved to a height h by an external agent very slowly without an acceleration. then the agent is performing work against gravitational force which is conservative and this work has a value mgh. This mgh is stored as PE on that body. If released, this mgh will convert back to KE.

GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy of a body is the energy posessed by the body by virtue of its position with respect to earth. This is kind of interaction PE developed due to the gravitational ineraction between the body and earth. It means that, if a body is placed at the surface of earth, its gravitational PE is not zero, because gravitational force exist between that body and earth.

But for practical calculations of PE, we will usually assume that PE of a body placed at the surface of earth is zero. Consider a body of mas m placed on the ground, where PE is assumed to be zero.



Let an external agent moves this body to a height h without an acceleration by doing work against gravitational force, which is conservative. So this workdone will store as gravitational PE on the body.

Work done by external agent

wext = F_{ext} s cos θ

[F_{ext} is the force given by external agent to overcome mg.

= mgh cos 0 = mgh

so F_{ext} = mg (upwards)]

This work done appears as change in PE, given as

Δ u = mgh

Here $U_{initial}$ = 0, U_{final} = mgh

$U_{final} - U_{initial}$ = mgh

Let U_{final} = U

U - 0 = mgh

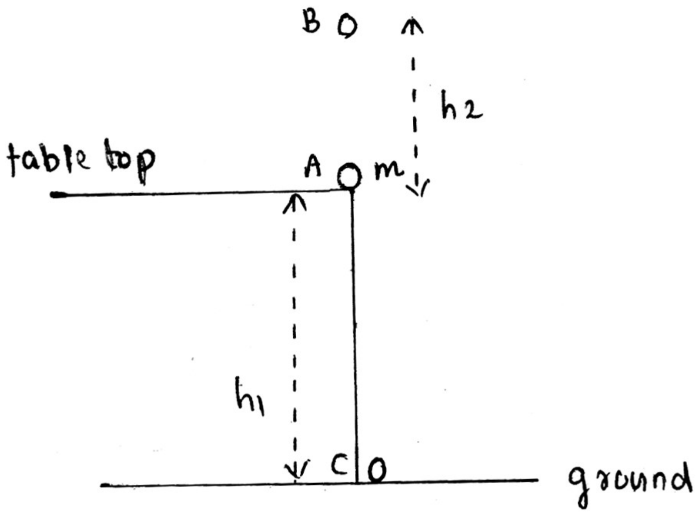
U = mgh

This equation is obtained on the assumption that PE is zero at the ground level. Such an assumption taken is known as zero reference. Any position or level can be taken as zero reference, according to o u r

Point : To define absolute PE or to express the value of PE of a body, we have to first assume that the value of PE is zero at a particular level called zero reference. then h must be taken from that reference level. For positions above that level, h is given a positive sign and for positions below that level, h is given a negative sign.

Consider the following example

Gravitational PE of a body can be positive, negative or zero.



If ground is taken as zero reference

when body is at A	when body is at B	when body is at C
$U = mgh_1$	$U = mg(h_1 + h_2)$	$U = 0$

If table top is taken as zero reference

When body is at A	when body is at B	when body is at c	
u = 0		$u = mgh_2$	$u = mg(-h_1)$
			$= -mgh_1$

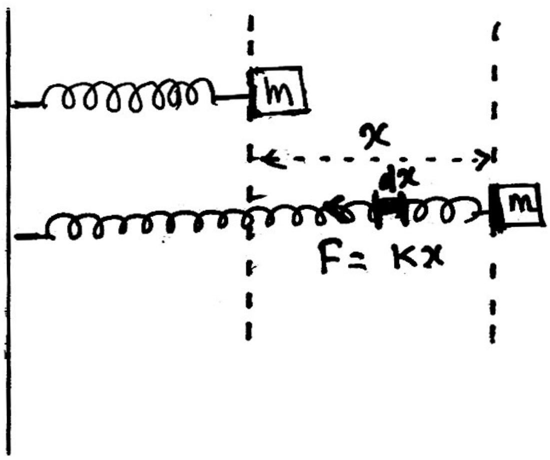
ELASTIC POTENTIAL ENERGY

Elastic potential energy of a body is the potential energy gained by that body due to a deformation happening on it.

For example, if we stretch or compress a spring, an external agent performs a work against the spring force, which is conservative. So this work done is stored as PE in the spring.

POTENTIAL ENERGY OF A SPRING

Consider a spring of constant k fixed at one end. its free end is connected to a block of mass m. Let the spring is at its natural length.



Now the block is displaced towards right through a distance x by an external agent so that the spring stretches through the same amount x. Then a spring force F= Kx is developed in it in a direction opposite to the direction of stretching. This spring force is a conservative force and is a variable force. Here external agent is performing a work against this conservative force without an acceleration. So this work done will be stored as elastic PE in the spring. Now let us find that workdone. For that consider a small element of length dx in the path. Then workdone by the external agent to displace the mass through the element.

dw = Fext dx cos θ = Kx dx cos 0

= kx dx

Total work done by external agent to displace the body from x = 0 (natural length) to x = x

Fext is the force given by external agent to overcome spring force kx Fext = kx, in the direction of dx so $\theta = 0^0$

$$w = \int_0^x kx\,dx = K\left(\frac{x^2}{2}\right)_0^x = \frac{1}{2}Kx^2$$

$$\boxed{U = \frac{1}{2}Kx^2}$$

This work done is stored as PE in the spring where x is the elongation or compression in the spring. Potential energy stored in a spring is always positive.

RELATION BETWEEN PE AND CONSERVATIVE FORCE

Potential energy is always associated with conservative forces only

(i) Gravitational PE

U = mgh; differentiating with respect to h and giving a (-) sign

$$\frac{^-dU}{dh} = \frac{^-d}{dh}(mgh) = -mg \times 1 = -mg$$

$$\therefore \frac{-dU}{dh} = F, \text{gravitational force } F = -mg$$

= mg, downwards

$$\boxed{\therefore F = \frac{-dU}{dh}}$$

(ii) Elastic PE

PE stored in a spring;

U = $\frac{1}{2}Kx^2$

differentiating with respect to x and giving a (-) sign.

$$\frac{^-dU}{dx} = \frac{^-d}{dx}\left(\frac{1}{2}Kx^2\right) = -\frac{1}{2}K\,2x$$

$$= -kx$$

= F, restoring force on spring

$\therefore \boxed{F = \frac{-dv}{dx}}$

So if we diffrentiate the equation for PE with respect to distance and give negative sign, we get the corresponding conservative force.

conservative force - Differential of PE function with respect to distance

differential of a quantity with respect to distance is called gradient of that quantity

$\therefore \boxed{\text{conservative force} = -\text{gradiant of PE}}$

$$\boxed{F = \frac{-dv}{dr}}$$

Note:

The equation F = $F = \frac{-dU}{dr}$ is used when PE is given as a function of a single position co-ordinate x or y or z. But in many cases, U may be given as a function of more than one position co-ordinates. In such a case we need to do an another type of mathematical calculation.

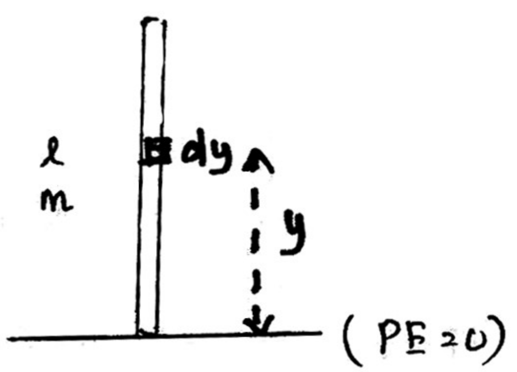
Negative sign always exist in the relations between potential energy and conservative force. this negative sign indicates that, the direction of a conservative force is same as the direction in which PE is reducing. So if we move in the direction of a conservative force, PE is found to be reducing.

Taking the reverse process ; change in PE is given by

dU = $-\int_{r_1}^{r_2} \vec{F} \cdot \overrightarrow{dr}$ (from position r_1 to r_2) and is known as the negative line integral of conservative force.

PE OF A UNIFORM ROD PLACED VERTICALLY

Consider a uniform rod of mass m and length ℓ placed vertically on the ground. Let PE is assumed to be zero at ground level (zero reference)



Here all the elements of the rod are not at the same height from the ground. So we have to use integration method to find its PE.

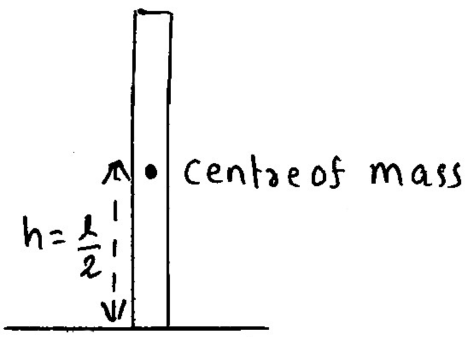
Consider a small element of length dy at a height y from the surface mass of the element; dm = $\frac{m}{\ell}$

dy PE of the element, du = (dm) g y = $\frac{m}{\ell}$ g dy

$\therefore$ Total PE of the rod; $U = \int dU = \frac{m}{\ell} g \int_0^{\ell} y\,dy = \frac{m}{\ell} g \left(\frac{y^2}{2}\right)_0^{\ell}$

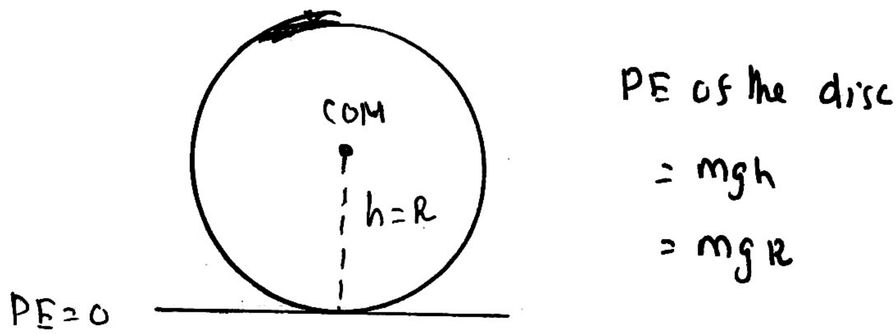
$\frac{m}{\ell} g \left(\frac{\ell^2}{2}\right) = \frac{mg\ell}{2}$; $U = \frac{1}{2} mg\ell$

That is ; $U = mg\left(\frac{\ell}{2}\right)$



So in these type of situations, where the size of the body is large, to calculate the gravitational PE using the equation mgh, h must be taken as the height to the centre of mass of that body from the zero reference level.

eg : Let a uniform disc of mass m and radius R is placed on the ground. Let ground be the zero reference level.



TRANSLATIONAL EQUILIBRIUM

If the net external force acting on a body is zero, it is in a state of translational equilibrium.

If Fnet = 0, using Newton's law

ma = 0 or a - 0

So acceleration is zero for a body in translational equilibrium; This can happen in two situations

(i) When body is at a permanent rest

(ii) When body is in uniform motion

Equilibrium status are classified into three types.

1. Stable equilibrium

2. Unstable equilibrium

3. Neutral equlibrium

This classification is made according to the potential energy values posessed by the bodies at these equilibrium states. We know that

Conservative force ; $F = \frac{-dU}{dr}$

When body is in translational equilibrium

F = 0

$\therefore \frac{dU}{dr} = 0$

This can happen in three situations

(i) When U is a constant (Neutral equilibrium)

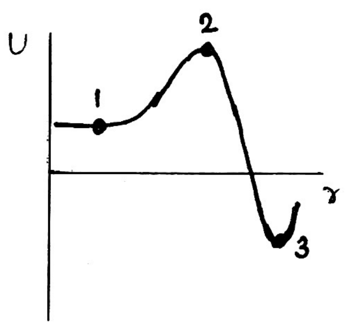
(ii) When U is maximum (unstable equlibrium)

(iii) When U is minimum (stable equlibrium)

if we consider a graph connecting PE (U) and distance (r), then its slope ; $\frac{dy}{dx} = \frac{dU}{dr}$

So at positions of equilibrium; slope = 0 so at those positions, graph will be parallel to x axis or r axis.

In this graph, slopes are found to do zero at positions 1, 2 and 3.



The equlibrium status at which net force on the body is zero and PE is minimum is called stable equilibrium. The equlibrium status at which net force is zero and PE is maximum is called unstable equlibrium. The equlibrium status at which net force is zero and PE remains constant is called neutral equilibrium.

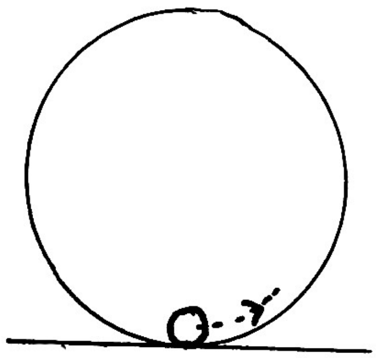
STABLE EQULIBRIUM

Following properties are followed by a body in stable equilibrium.

(i) net force acting on the body is zero

(ii) PE of the body will be a minimum

$\therefore \frac{du}{dr} = 0$ and slope of u - r graph = 0



(iii) A ball is placed inside a smooth spherical shell as shown. This ball is in equilibrium at that position. Now the ball is slightly displaced in the direction as shown and released. Then the ball will return to the earlier equilibrium position. It means that, the earlier equilibrium position was stable and the ball does not want to displace from that position. So when a body is displaced from its stable equilibrium position, a net force starts acting on the body in a direction opposite to the direction of displacement or towards the earlier equilibrium position. This force tends the body to bring back it to the earlier equilibrium position. So not force developed here is said to be negative.

(iv) At stable equilibrium

U is minimum

$\therefore \frac{du}{dr} = 0$ and $\frac{d^2u}{dr^2} = (+)$

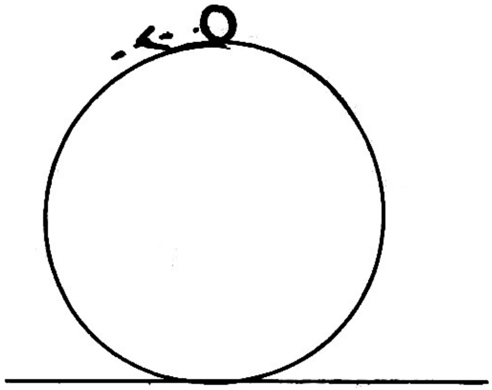
UNSTABLE EQUILIBRIUM

Following properties are followed by a body in unstable equilibrium

(i) Net force on the body is zero

(ii) PE of the body will be maximum

$\therefore \frac{du}{dr} = 0$ and slope of U - r graph is zero



(iii) A ball is placed at the top of a smooth spherical shell as shown. This ball is in equilibrium at that position . Now the ball is slrghtly displaced in the direction as shown and then released. Then the ball will further move away. So the earlier position of equilibrium was unstable and the ball does not want to stay there. So when a body is slightly displaced from unstable equilibrium position a net force is developed in it in a direction same as the direction of displacement, away from equilibrium position. That is why body further move away, when released. So net force developed here is said to be positive.

(iv) At unstable equilibrium,

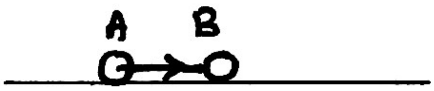
u is maximum $\therefore \frac{du}{dr}=0$ and $\frac{d^2u}{dr^2}=(-)$

NEUTRAL EQUILIBRIUM

(i) Net force acting on the body is zero

(ii) PE of the body will be a constant $\therefore \frac{du}{dr}=0$ and slope of u-r graph = 0

(iii) When a body is in neutral equilibrium, it will be in equilibrium over a region at a number of points.



Let a ball is placed on a

horizontal surface at point A. From there, it is displaced to point B as shown. At point A, ball was in equilibrium. Now at point B also, the ball is in equlibrium. So it will not show any tendency to return or move away, as no net force is developed on displacing.

(iv) At neutral equilibrium

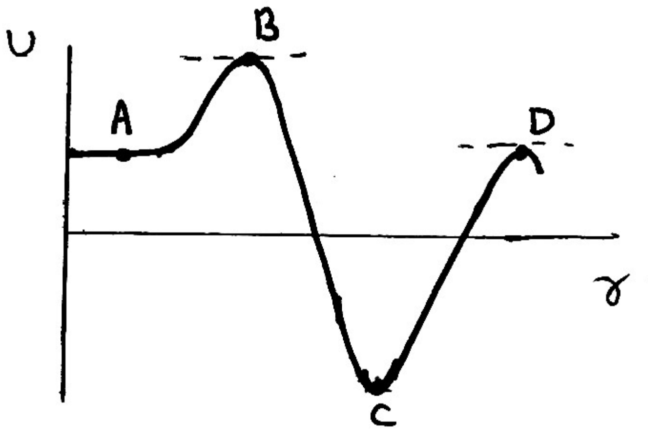
U is a constant

$\therefore \frac{du}{dr}=0$ and $\frac{d^2u}{dr^2}=0$

So to check whether the given equilibrium is stable unstable or neutral, displace the body slightly from the equilibrium position in the specified direction and then release it. Then a net force will be developed on it and find the direction of that net force. If it is towards the direction of earlier equilibrium position the body will show a tendency to return. Then the equlibrium was stable, in that direction. But if the direction of net force is away from the equilibrium position, the body will further move away. Then the equilibrium was unstable in that direction. But if no net force develops, body will continue in its displaced position. Then the equilibrium was neutral.

PE- DISTANCE AND FORCE - DISTANCE GRAPHS

Consider a graph connecting PE and distance of a system as shown.



All the four points A_1B_1 C and D corresponds to equilibrium. Let us find the nature of equilibrium.

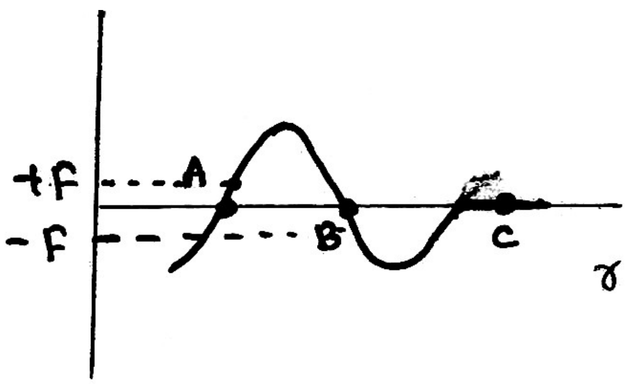
Slope of U - r graph = $\frac{du}{dr}$ at points A, B, C and D, graph is parallel to axis. So slope $\frac{du}{dr}$ = 0. So at all these positions, body is in equilibrium.

Point A : Near point A, PE is a constant over a region. So equilibrium is neutral at A

Point B : At point B, PE is maximum. So equlibrium is unstable

Point C : At point C, PE is minimum. So equlibrium is stable.

point D : At point D, PE is maximum. So equlibrium is unstable.



Consider the force -distance graph of a body as shown. In such a graph, the points at which graph meets the x axis or distance axis, the force value will be zero. So those are the points of equilibrium. So here points A,B,C are corresponding to equilibrium.

Point A : Take the next instant after point A. Then a force develops which is positive. so at A, equilibrium was unstable

Point B : Take the next instant after point B. Then a force develops which is negative. So at A, equilibrium was stable.

Point C : Take the next instant after point c. Then no force develops so equilibrium was neutral at C

WORK ENERGY THEOREM

Newton's second law is a basic law that can be used to solve force related problems in mechanics. Newton's second law has a vector form. It does not consider internal forces during solving mechanical systems. This demerit is overcome by suggesting work-energy theorem. It has a scalar form as it suggest scalar quantities like work and energy.

Work - energy theorem is a scalar form of Newton�s second law. This theorem consider work done by internal forces also

Work - energy theorem states that; total work done by all the forces acting on a body or system is equal to change in Kinetic energy of that body or system. So here we consider all the forces acting on the system, which includes conservative nonconservative, external, internal, etc. forces. If we apply this theorem from a non inertial frame, we consider the work done by pseudo forces also while applying this theorem.

$\therefore \boxed{\text{work done by all force} = \text{Change in KE}}$

This is the original form of work - energy theorem

SUPPLEMENTARY FORMS

Work - energy theorem has two supplementary forms.

(i) Let only conservative forces are performing work on the body.

For conservative force

$$F_c = \frac{-dU}{dr} \Rightarrow dU = -F_C\, dr$$

This equation suggests that, when a body is displaced through a small distance dr under the influence of a conservative force Fc, then change in its PE is du

$$\therefore \boxed{\begin{array}{l}\text{Change in PE} = -\text{Work done by conservative forces} \\ \quad \Delta PE = -W_{\text{conservative force}}\end{array}}$$

(ii) Original form of work - energy theorem is $W_{all} = \Delta KE$

$\therefore W_{\text{Conservative forces}} + W_{\text{Forces other than conservative}} = \Delta KE$

but W conservative forces = - Δ PE

$\therefore\ -\Delta PE + W_{\text{forces other than conservative}} = \Delta KE$

$\therefore\ W_{\text{forces other than conservative}} = \Delta KE + \Delta PE$ but $\Delta KE + \Delta PE = \Delta ME$, change in mechanical energy

$\therefore$ $\boxed{\text{Work done by all forces other than conservative} = \text{change in mechanical energy of system}}$

So to change the total mechanical energy of a body, forces other than conservative forces has to perform work

So consider the following three forms of energy change and work done.

1

$$\boxed{\begin{array}{l}\Delta KE = \text{Work done by all forces}\\ \Delta PE = -\text{work done by conservative forces only}\\ \Delta ME = \text{Work done by all forces other than conservative}\end{array}}$$

LAW OF CONSERVATION OF MECHANICAL ENERGY

The mechanical energy of a body is the sum of Kinetic energy and potential energy of the body

ME = KE + PE

We know that, to change the ME of a system, forces other than conservative force has to do work on the system. So

If only conservative forces are performing work on a system, then the total mechanical energy of that system remains a constant

This statement is known as law of conservation of mechanical energy.

Proof

Using work - energy theorem;

Work done by all forces = change in KE

Let us take all the forces either as conservative or forces other than conservative.

$$\therefore \left.\begin{array}{c}\text{work done by conservative forces}\\ +\\ \text{work done by forces other than conservative}\end{array}\right\} = \Delta KE \rightarrow (1)$$

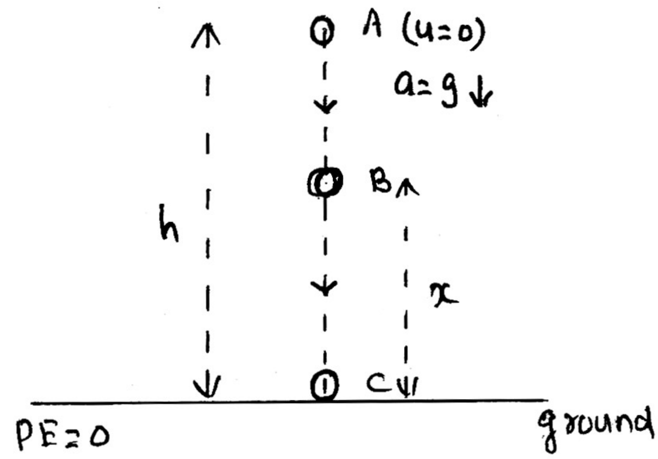
Let work done by forces other than conservative is zero. Also

Work done by conservative forces = - change in PE substituting in (1)

$$-\Delta PE + O + \Delta KE$$
$$\therefore \Delta KE + \Delta PE = O$$
$$\therefore \Delta ME = O \Rightarrow ME = \text{constant}$$

It means that, individually, Kinetic energy and Potential energy may very from point to point, but their sum is constant throughout.

eg : Consider a small ball of mass m is falling under gravity from a height, from point A.



Consider two points B and C also as shown. Let only gravitational forces (conservative) is performing work on the ball. So total ME of ball must be conserved.

Now let us calculate total ME of the body at points A, B and C

Point A	Point B	Point C
KE = 0	PE = mgx	PE = 0
PE = mgh	$V_B^2 = u^2 + 2as = 2g(h-x)$	$V_c^2 = u^2 + 2as = 2gh$
$ME_A = KE + PE = mgh$	$KE = \frac{1}{2}m\,V_B^2 = \frac{1}{2}m\;2g(h-x)$	$KE = \frac{1}{2}m\,V_c^2 = \frac{1}{2}m\;2gh$
	$= mg(h-x) = mgh - mgx$	$= mgh$
	$ME_B = mgh - mgx + mgx = mgh$	$ME_c = 0 + mgh = mgh$

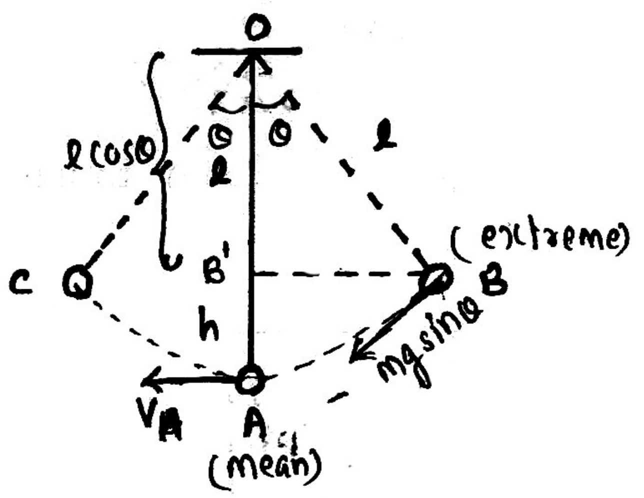
$\therefore ME_A = ME_B = ME_c$

This is because gravitational force is conservative from this, it is clear that, at point A, entire ME is PE. But when it reaches point C, the entire ME appears as KE only. So from A to c, entire PE is converting into KE. So we can conclude that, in cases where ME is conserved;

Loss in PE = gain in KE also Loss in KE = gain in PE

eg2: Oscillations of a simple pendulum

OA is the normal position of rest called mean position or equilibrium position. When the bob of the pendulum is displaced to B, through a height, it is given PE = mgh, where m is the mass of bob. On releasing the bob at B, it moves towards A. PE of the bob is being converted into KE. On reaching A, the entire PE has been converted in to KE. The bob, therefore cannot stop at A. On account of inertia, it overshoots the position A and reaches c at the same height h above A. The entire KE of the bob at A is converted into PE at c. The whole process is repeated and the pendulum vibrates about the equilibrium position A. At extreme position B and C, the bob is momentarily at rest. Hence KE = 0. So the entire energy at B and C is PE. At A entire energy is KE.



Point A

PE = 0

$KE = \frac{1}{2} m V_A{}^2$

$ME_A = kE_A + PE_A = mg\ell\ (1 - \cos\ \theta)$

$V_A{}^2 = u^2 + 2gh = 2g\ell(1 - \cos\theta)$

= $KE = \frac{1}{2} m V_A{}^2 = \frac{1}{2} m 2g\ell(1 - \cos\theta) = mg\ell(1 - \cos\theta)$

point B

KE = 0

PE = mgh = mg $AB^1 = mg(\ell - \ell\cos\theta)$

= $= mg\ell(\ell - \cos\theta)$

$\therefore ME_B = mg\ell(1 - \cos\theta)$

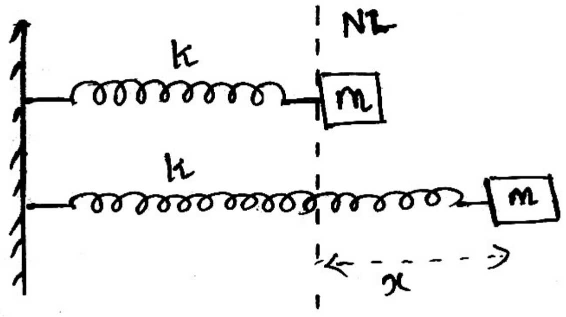
$ME_A = mg\ell(1 - \cos\theta)$

$\therefore ME_A = ME_B$

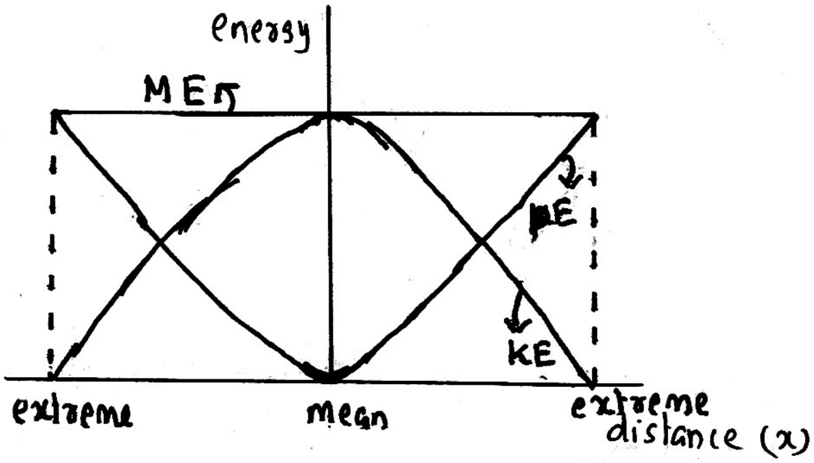
When a pendulum oscillates, work is done only by component of gravitational force mg sin θ. That is why ME is conserved.

eg3: Oscillations of a spring - block system

Consider a spring of constant K whose one end is fixed and other end is connected to a block of mass m, as shown.



Let the block is displaced towards right through a distance x and released. Then system oscillates. The only one force acting on the system is the spring force, which is conservative. So total ME of the system will remain conserved. The variation of KE and PE with distance x given below.



Here also KE + PE = ME, a constant at every position of the oscillating system.

POWER

When a mechanical or any other work is performed effectively in a very short duration, it is called a powerful performance. In science.

Power of an external agent or machine can be defined as the rate at which work is done by that agent or machine.

Power = Rate of doing work

$$= \frac{\text{work done}}{\text{time taken}}$$

usually this is taken as the average power. Thus power of a body measures how fast it can do the work. When a body takes lesser time to do a particular amount of work, its power is said to be greater and vice-versa.

The power at a particular instant of time or instantaneous power is the ratio of small work done (dw) to the small time interval (d t) around.

$$\boxed{P = \frac{dw}{dt}}$$

Let a force $\vec{F}$ is acting on a body to displace is through a small value $\overrightarrow{ds}$ in a small time interval dt. Then power delivered by the force.

$$P = \frac{dw}{dt} = \vec{F} \cdot \frac{\overrightarrow{ds}}{dt} \quad \left(\frac{\overrightarrow{ds}}{dt} = \vec{v}, \text{instantaneous velocity}\right)$$

$$\boxed{P = \vec{F} \cdot \vec{V}}$$

or $P = FV \cos\theta$, θ is the angle b/w $\vec{F}$ and $\vec{v}$.

If $\vec{F}$ is perpendicular to $\vec{v}$, ($\theta = 90^0$), then power delivered will be zero. for example, power delivered by a centripetal force is always zero. for doing work, energy has to be spend. So power can also be defined as the rate at which energy is spend.

Unit of power

$$\frac{\text{Joul}}{\text{second}} \text{ or Watt(W)}$$

Another popular unit of power is horse power (hp)

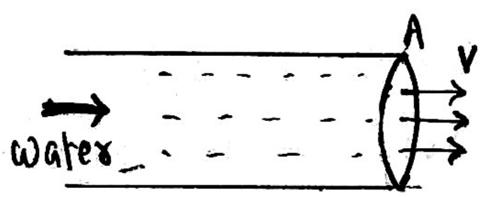
$$\boxed{1 \text{hp} = 746 W}$$

Dimensions of power = ML^2T^{-3}

Efficiency of an engine is usually defined in terms of power, given as

$$\eta = \frac{\text{output power}}{\text{input power}}$$

eg: Let a motor is pumping water, through a pipe of cross sectional area A, with a velocity V.



Then force due to water jet $F = AV^2P$

Where v is the relative velocity between water and the target on which it is hitting.

Then, power of water jet $P = FV = AV^2P\ V$; $P = AV^3P$ $\quad P \alpha V^3$

COLLISIONS

In common language, a collision is said to occur when objects crash into each other.

eg : collision between two automobiles, hammer striking a nail etc.

Often collisions are too brief in visible, though they involve significant distortion of the colliding bodies. Generally, a collision can be defined as an isolated event happening between two or more bodies in which relatively strong forces are exerted on each other for a relatively short time. Actual physical contact between two bodies is not necessary for a collision. For example, an alpha particle speeding towards a nucleus of a atom gets deflected by the electric force of repulsion. then scientifically, we will say that, a collision is occured between them.

When a collision happen, there will be an exchange or transfer of amount of motion or momentum between the bodies involved, but total momentum remain constant.

So scientifically, a collision is an isolated event happening between two or more bodies so that a transfer or exchange of momentum happens between the bodies involved in that process such that the total momentum of the systm remain conserved during that process.

During a collision process, forces are involved between the colliding bodies. For the entire system, these forces are internal. But for individual bodies involved, these forces are external. So for the entire system, total linear momentum will be surely conserved. But total kinetic energy of the system may or may not be conserved. So

Two important physical quantities which determines the nature of a collision are momentum and Kinetic energy.

Depending on the momentum and Kinetic energy, classification are made.

TYPE OF COLLISIONS

Collisions are classified into three types

(i) Elastic collision

(ii) Inelastic or partially elastic collision

(iii) Perfectly inelastic collision

(i) Elastic Collision

If both linear momentum and kinetic energy of the colliding system remain conserved during a collision, it is called an elastic collision. Elastic collisions are also known as perfectly elastic collision.

The basic characterestics of an elastic collission are :

(i) Linear momentum of system is conserved

(ii) KE of system is conserved

(iii) Bodies regain their original size and shape after an elastic collision.

(iv) The forces involved during elastic collisions must be conservative.

(ii) Inelastic collisions

A collision in which, only linear momentum of the system remain conserved but kinetic energy does not conserved is called an inelastic collision. So during such a collision, some loss of KE of the system will occur as heat, sound etc. As there is always some loss of KE in most of the collisions, therefore, collisions we come across in daily life are generally inelastic. the basic characterestics of an inelastic collision are ;

(i) Total linear momentum of system is conserved

(ii) Total KE of the system is not conserved

(iii) Bodies do not regain their size and shape perfectly after this collision.

(iv) Some or all the forces involved in an inelastic collision may not be conservative.

(iii) Perfectly inelastic collisions

If two bodies stick together after colliding, that collision is said to be perfectly inelastics.

Its characterestics are

(i) Total linear momentum is conserved

(ii) Total KE of system is not conserved

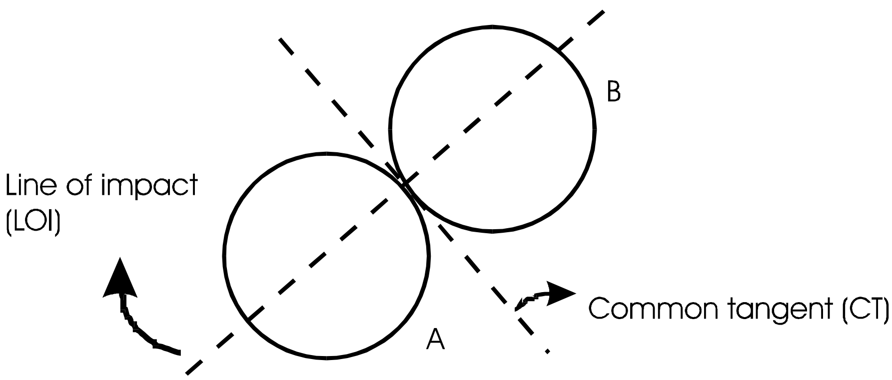
(iii) The bodies are well deformed and they do not show any tendency to regain their original size and shape.

(iv) All colliding forces are not conservative.

eg: Let a bullet hit a wooden block and stick to it. Collision is perfectly inelastic. If a meteorite collides earth and buried in earth, collision is perfectly inelastic.

Reference Directions In a Collision

In a collision, ther are two basic reference directions. Consider two spherical bodies A and B under going a collision, as shown.



There is a contact surface between the bodies

(i) Common tangent : It is a tangent drawn along the contact surface. Along common tangent,

the collision has no effect on the bodies. So the components of velocity of both the bodies do not change along common tangent during collision.

(ii) Common normal or line of impact (LOI))

This direction is perpendicular to common tangent. The collision has maximum effect along LOI. So the components of velocity of both the bodies suffer maximum change along LOI. So it is the most important direction, we consider, to solve a collision.

COEFFICIENT OF RESTITUTION (e)

This term gives a measure of amount of reformation happening to the bodies after collision. It is defined as the ratio of impulse of reformation to impulse of deformation.

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

Here v_2 - v_1 is the relative velocity of seperation between B and A, along LOI, after collision. Also u_1 - u_2 is the relative velocity of approach between B and A, along LOI, before collision.

$$\boxed{\therefore e = \frac{\text{relative velocity of separation after collision along LOI}}{\text{relative velocity of approach before collision along LOI}}}$$

for elastic collision $\qquad$ for inelastic collision

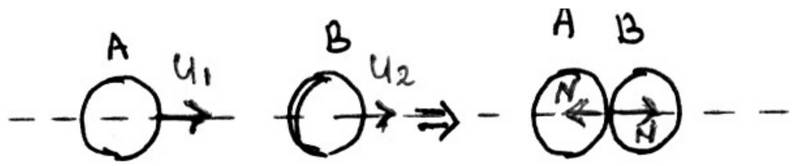
$\therefore \boxed{e = 1}$ $\qquad$ $\therefore \boxed{0 < e < 1}$

For perfectly inelastic collision [e depends on the nature of the surfaces coming in contact during collision]

$\therefore \boxed{e = 0}$

Head on collision

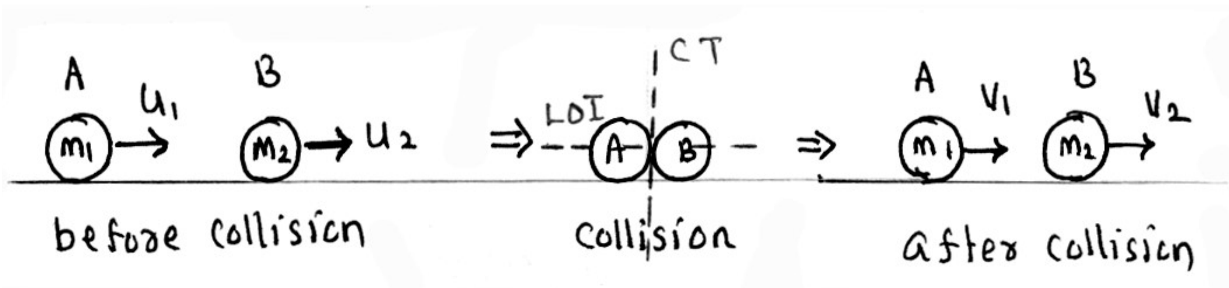
A collision is said to be head on or direct if the directions of the velocity of colliding objects are along the line of action of impulses, acting at the instant of collision.



Here the directions of $\vec{U_1}$ and $\vec{U_2}$ are same as the direction in which impulsive force $\vec{N}$ is acting. if just before collision, at least one of the colliding objects were moving in a direction different from the line of action of the impulses, the collision is called oblique or indirect.

HEAD ON ELASTIC COLLISION IN ONE DIMENSION

Consider two bodies of masses m_1 and m_2, initially moving along the same straight line with velocities u_1 and u_2 as shown.



They undergo a head on elastic collision. Let V_1 and V_2 are their respective velocities after collision. Since collision is elastic, both linear momentum and KE are conserved.

Using conservation of linear momentum along LOI

$m_1 u_1 + m_2u_2 = m_1v_1 + m_2 v_2$

$m_2u_2 + m_2v_2 = m_1v_1 - m_1u_1$

$m_2 (u_2 - V_2) = m_1 (v_1 = u_1) \rightarrow (1)$

Using conservation of KE

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2v_2^2$$

$$m_2u_2^2 - m_2 v_2^2 = m_1v_1^2 - m_1u_1^2$$

$$m_2 (u_2^2 - v_2^2) = m_1 (v_1^2 - u_1^2) \rightarrow (2)$$

$$\frac{2}{3} \Rightarrow \frac{m_2 (u_2^2 - v_2^2)}{m_2 (u_2 - v_2)} = \frac{m_1 (v_1^2 - u_1^2)}{m_1 (v_1 - u_1)}$$

$$\frac{(u_2 + v_2)(u_2 - V_2)}{(u_2 - v_2)} = \frac{(v_1 + u_1)(v_1 - u_1)}{(v_1 - u_1)}$$

$$u_2 + v_2 = u_1 + v_1$$

$$v_2 - v_1 = u_1 - u_2 \rightarrow (3)$$

$$\frac{v_2 - v_1}{u_1 - u_2} = 1$$

v_2-v_1 $\rightarrow$ relative velocity of separation after collision along LOI

$$\therefore \frac{v_2 - v_1}{u_1 - u_2} = e\text{, the coefficient of restitution}$$

u_1 - u_2 $\rightarrow$ relative velocity of approach before sion before collision along LOI

Hence for an elastic collision e = 1

from (3); $v_2 = u_1 - u_2 + v_1$

sub in (1), $m_2 [u_2 - (u_1 - u_2 + v_1)] = m_1 (v_1 - u_1)$

$m_2u_2 - m_2u_1 + m_2u_2 - m_2v_1 = m_1v_1 = m_1u_1$

$(m_1 + m_2) v_1 = (m_1 - m_2) u_1 + 2m_2u_2$

$$\boxed{V_1 = \frac{(m_1 - m_2)\ u_1}{m_1 + m_2} + \frac{2m_2u_2}{m_1 + m_2}}$$

Similarly we can set; $\boxed{V_2 = \frac{(m_2 - m_1)\ u_2}{m_1 + m_2} + \frac{2m_1u_1}{m_1 + m_2}}$

Now let us consider some particular cases.

Case I : When masses of two bodies are equal,

ie, $m_1 = m_2 = m$, then

$$v_1 = \frac{(m-m)u_1}{m+m} + \frac{2mu_2}{m+m}$$

$\boxed{v_1 = u_2}$ $\Rightarrow$ Velocity of A = velocity of B

after collision before collision

Similarly ; $v_2 = \frac{(m-m)u_2}{m+m} + \frac{2mu_1}{m+m}$

$\boxed{v_2 = u_1}$ $\Rightarrow$ velocity of B = Velocity of A

after collision before collision

Hence, when two bodies of equal masses undergo elastic head on collision in one dimension their velocities are just inter changed.

Case 2 : Let target (m_2) is initially at rest

$\therefore$ $u_2 = 0$. Here we can consider different situations

(i) <u>Let $m_1 = m_2 = m$</u>

$$v_1 = \frac{(m-m)}{m+m}u_1 + \frac{2m \times 0}{2m} ; v_2 = \frac{(m-m)u_2}{m+m} + \frac{2m\ u_i}{m+m}$$

$\boxed{V_1 = 0_1}$ $\boxed{V_2 = u_1}$

So velocities are interchanged

(ii) let $m_1 >> m_2$ (target rest)

$$v_1 = \frac{(m_1 - m_2)u_1}{m_1 + m_2} + \frac{2m_2 \times 0}{m_1 + m_2}$$ (m_2 can be neglected as $m_2 << m_1$)

$\therefore$ $\boxed{v_1 = u_1}$

$$v_2 = \frac{(m_2 - m_1)}{m_1 + m_2} + \frac{2m_1 u_1}{m_1 + m_2}$$ (m_2 can be neglected)

$\boxed{v_2 = 2u_1}$

Hence when a heavy body A undergoes an elastic head on collision with a light body B at rest, the body A keeps on moving with the same velocity of its own and the body 13 starts moving with double the initial velocity of A

(iii) Let $m_2 >> m_1$ (target rest)

$$v_1 = \frac{(m_1 - m_1)\, u_1}{m_1 + m_2} + \frac{2m_2 \times 0}{m_1 + m_2}$$ (m_1 can be neglected as $m_1 << m_2$)

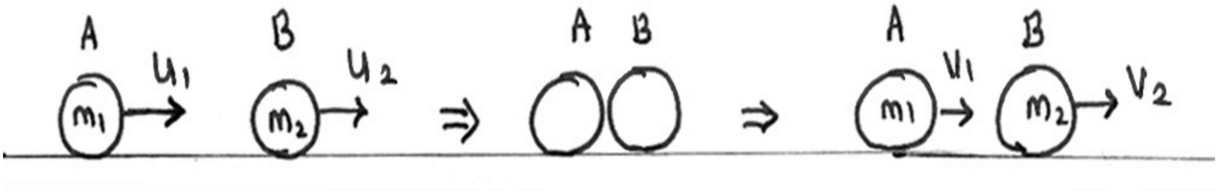
$$\boxed{V_1 = -u_1}$$

$$V_2 = \frac{(m_2 - m_1) \times 0}{m_1 + m_2} + \frac{2m_1 u_1}{m_1 + m_2}$$ (m_1 can be neglected)

$$\boxed{V_2 = 0}$$

Hence when a light body A, undergoes an elastic head on collision with a heavy body B at rest, A rebounds with its own velocity and B continues to be at rest.

HEAD ON INELASTIC COLLISION IN ONE DIMENSION



Here only momentum is conserved and not KE using conservation of linear momentum;

$$m_1u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \rightarrow (1)$$

Let e be the coefficient of restitution, then

$$e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow v_2 - v_1 = e(u_1 - u_2);\; v_2 = v_1 + e(u_1 - u_2) \rightarrow (2)$$

Sub in (2) in (1) ; we get

$$\boxed{\begin{aligned} v_1 &= \frac{(m_1 - em_2)u_1}{m_1 + m_2} + \frac{(m_2 + em_2)u_2}{m_1 + m_2} \\ V_2 &= \frac{(m_2 - em_1)u_2}{m_1 + m_2} + \frac{(m_1 + em_1)\, u_1}{m_1 + m_2} \end{aligned}}$$

There will be a loss in KE given by

$$\text{KE of loss} = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \left[\frac{1}{2}m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right]$$

HEAD ON PERFECTLY INELASTIC COLLISION IN ONE DIMENSION

Since collision is perfectly inelastic, bodies stick together during collision and thereafter moves as a single body with a common velocity v.

Here also only linear momentum is conserved

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

$$\boxed{V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}}$$

Loss in KE due to collision ;

$$\Delta E = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2}(m_1 + m_2) v^2 = = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)\left(\frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}\right)^2$$

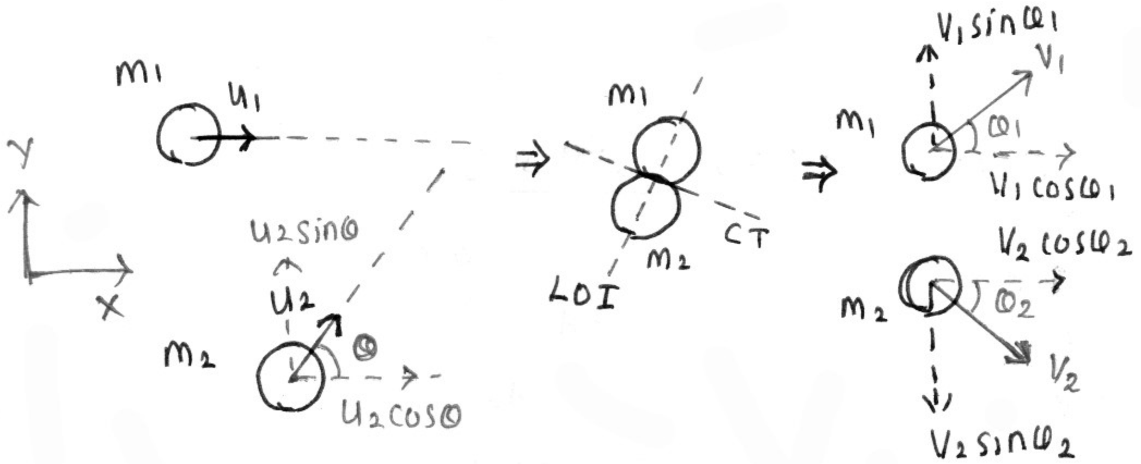
$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left[u_1^2 + u_2^2 - 2u_1 u_2\right] \qquad \boxed{\Delta E = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2}$$

In all these cases of one dimensional head on collisions, u_1 and u_2 must be substituted with proper signs. If both u_1 and u_2 are in the same directions, both are taken as positive. But if they are taken in opposite directions, one is taken as positive and other negative.

COLLISIONS IN TWO DIMENSIONS

Generally, most of the practical collisions are two dimensional. In such a collision, the velocities before and after collosion may lie in a plane. When we use law of conservation of linear momentum to solve such collisions, equations are separately written in the two reference directions forming that plane. This is because, linear momentum is a vector quantity. For example, if collision occurs in the X Y plane, equations for momentum conservation can be written separately for x and y directions.

Consider an example for a two dimensional collision happening in the XY plane.



Let u_1 is exactly in the x direction. Since collision is in the XY plane, we resolve the velocity of bodies before and after collision as components in the X and Y directions. We will prepare equationd using conservation of momentum in both X and Y directions seperately.

Using momentum conservation in x direction

$$(P_x) \text{ initial } = (P_x) \text{ final}$$

$$m_1u_1 + m_2 u_2 \cos\theta = m_1v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \rightarrow (1)$$

using momentum conservation in Y direction

$$(P_y) \text{ initial } = (P_y) \text{ final}$$

$$m_2u_2 \sin\theta = m_1v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 \rightarrow (2)$$

These equations can be solved to obtain the unknown quantities. If collision is elastic, KE will also be conserved as $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^{\ 2} + \frac{1}{2}m_2 v_2^{\ 2}$

CASES OF REBOUNDING

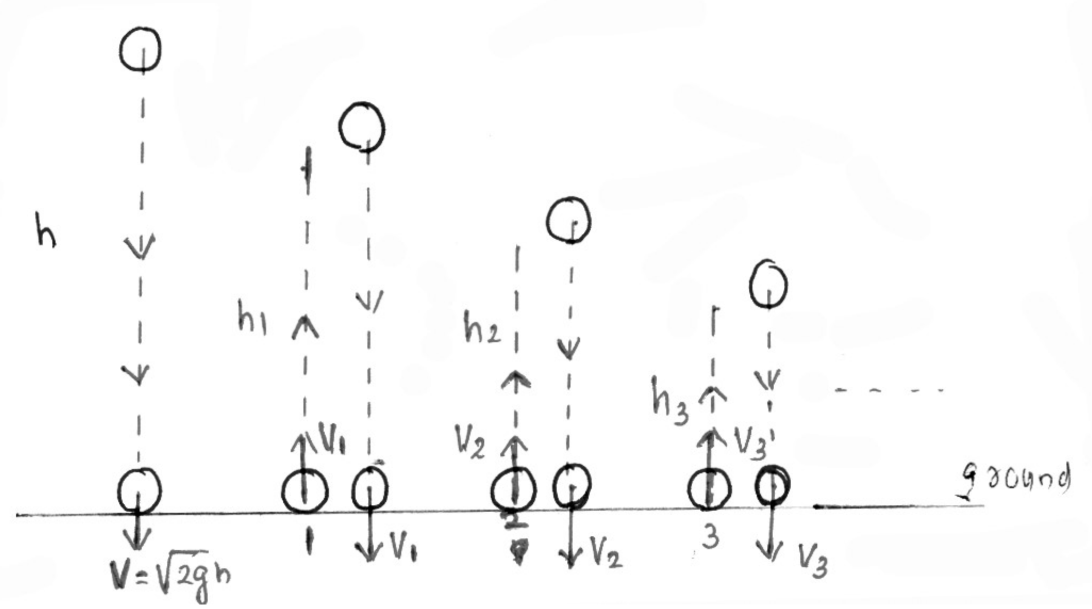
In most of the practical rebounding situations target (like ground, wall etc) will be at rest.

NORMAL REBOUNDING

In normal rebounding, when a body hits a target like ground or wall, the direction of velocity of colliding body will be perpendicular to the surface of the target. Re bounding can be elastic or inelastic. If reubounding is elastic, after rebound the body will return to the same height from where it is dropped. this is because no loss in KE happens during rebound. But if rebounding is inelastic, KE loses and body cannot return to the initial height.

Let a ball is dropped from a height h to a horizontal floor, normally and the ball undergo continuous rebounds from the floor inelastically. Let e be the coefficient of restitution between ball and ground. Since ground is at rest e can be defined as

$$e = \frac{\text{velocity of ball just after rebound}}{\text{velocity of ball just before rebound}}$$



Here V is the velocity with which ball hits the ground. Since collision is inelastic ball suffers a KE loss. Let V_1 be the velocity of ball just after the first rebound. Then

$$e = \frac{v_1}{v} \Rightarrow v_1 = ev$$

Since $v_1 < v$, it will reach a height h_1 (< h) after first rebound. After that it will return and hits the ground with same speed v_1 . Then it again rebounds. Let v_2 be the velocity after the second rebound. Then

$$e = \frac{v_2}{v_1} \Rightarrow v_2 = ev_1 = e(ev)$$

$$\therefore V_2 = e^2 v$$

Since $v_2 < v_1$, after the second rebound, the height reached by it $h_2 < h_1$. So maximum height reached by body goes on decreasing after each rebound. Similarly, velocity after 3rd, 4th etc rebounds can be written as :

$$v_3 = e^3 v, v_4 = e^4 v .....$$

So velocity after the nth rebound is given by;

$\boxed{V_n = e^n v}$ Where v is the velocity with which the body first hits the target (ground)

n = 1, 2, 3 ...... here $V = \sqrt{2gh}$

Maximum heights reached after each rebound can be calculated as ;

$$h_1 = \frac{{v_1}^2}{2g} = \frac{(ev)^2}{2g} = e^2\left(\frac{v^2}{2g}\right)$$

$$v = \sqrt{2gh}$$

$$\frac{v^2}{2g} = h$$

$$\therefore \boxed{h_1 = e^2 h}$$

$$h_2 = \frac{{v_2}^2}{2g} = \frac{(e^2 v)^2}{2g} = e^4 \left(\frac{v^2}{2g}\right) \Rightarrow \boxed{h_2 = e^4 h}$$

Similarly we can write, $h_3 = e^6 h$, $h_4 = e^8 h$ .....

So maximum height reached by the body after nth rebound

$\boxed{h_n = e^{2n} h}$ $h \rightarrow$ initial height from which body is dropped

n = 1, 2, 3.......

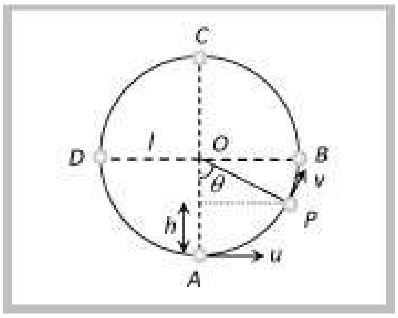
Motion in vertical circle

This is an example of non-uniform circular motion. In this motion body is under the influence of gravity of earth. When body moves from lowest point to highest point. Its speed decrease and becomes minimum at highest point. Total mechanical energy of the body remains conserved and KE converts into PE and vice versa.

(1) **Velocity at any point on vertical loop**:If u is the initial velocity imparted to body at lowest point then. Velocity of body at height h is given by

$$w = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2gl(1-\cos\theta)} \quad [As\ h = 1 - l\cos\theta = 1(1-\cos\theta)]$$

Where l in the length of the string

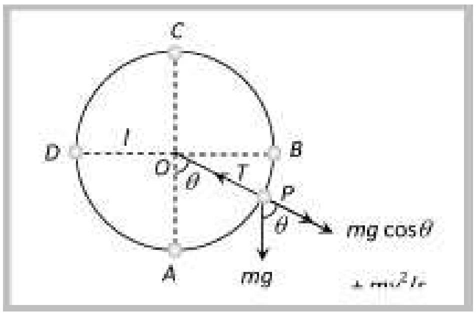


(2) **Tension at any point on vertical loop**: Tension at general point P, According to Newton's second law of motion.

Net force towards center = centripetal force

$$T - mg\cos\theta = \frac{mv^2}{l} \qquad \text{or} \qquad T = mg\cos\theta + \frac{mv^2}{l}$$

$$T = \frac{m}{l}[u^2 - gl(2 - 3\cos\theta)] \qquad [As\ v = \sqrt{u^2 - 2gl(1-\cos\theta)}]$$



3) **Velocity and tension in a vertical loop at different positions**

Position	Angle	Velocity	Tension
A	0°	u	$\frac{mu^2}{l} + mg$
B	90°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$
C	180°	$\sqrt{u^2 - 4gl}$	$\frac{mu^2}{l} - 5mg$
D	270°	$\sqrt{u^2 - 2gl}$	$\frac{mu^2}{l} - 2mg$

It is clear form the table that : $T_A > T_B > T_C$ and $T_P = T_P$

and
$$T_A - T_B = 3mg$$
$$T_A - T_C = 6mg$$
$$T_B - T_C = 3mg$$

5) Critical condition for vertical looping : If the tension at C is zero, then body will just complete revolution in the vertical circle. This state of body is known as critical state. The speed of body in critical state is called as critical speed.

From the above table $T_C = \frac{mv^2}{l} - 5mg = \theta = u = \sqrt{5g\ell}$

It means to complete the vertical circle the body must be projected with minimum velocity of $\sqrt{5gl}$ at the lowest point.

6) Various quantities for a critical condition in a vertical loop at different positions:

(6) **Various quantities for a critical condition in a vertical loop at different positions :**

Quantity	Point A	Point B	Point C	Point D	Point P
Linear velocity (v)	$\sqrt{5gl}$	$\sqrt{3gl}$	$\sqrt{gl}$	$\sqrt{3gl}$	$\sqrt{gl(3 + 2\cos\theta)}$
Angular velocity (ω)	$\sqrt{\frac{5g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}}$	$\sqrt{\frac{3g}{l}}$	$\sqrt{\frac{g}{l}(3 + 2\cos\theta)}$
Tension in String (T)	$6\ mg$	$3\ mg$	0	$3\ mg$	$3mg(1 + \cos\theta)$
Kinetic Energy (KE)	$\frac{5}{2}mgl$	$\frac{3}{2}mgl$	$\frac{1}{2}mgl$	$\frac{3}{2}mgl$	$\frac{mgl}{2}(3 + 2\cos\theta)$
Potential Energy (PE)	0	mgl	$2\ mgl$	mgl	$mgl(1 - \cos\theta)$
Total Energy (TE)	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$	$\frac{5}{2}mgl$

Various conditions for vertical motion :

Velocity at lowest point	Condition
$u_A > \sqrt{5gl}$	Tension in the string will not be zero at any of the point and body will continue the circular motion.
$u_A = \sqrt{5gl}$,	Tension at highest point C will be zero and body will just complete the circle.
$\sqrt{2gl} < u_A < \sqrt{5gl}$,	Particle will not follow circular motion. Tension in string become zero somewhere between points B and C whereas velocity remain positive. Particle leaves circular path and follow parabolic trajectory.
$u_A = \sqrt{2gl}$	Both velocity and tension in the string becomes zero between A and B and particle will oscillate along semi-circular path.
$u_A < \sqrt{2gl}$	velocity of particle becomes zero between A and B but tension will not be zero and the particle will oscillate about the point A.

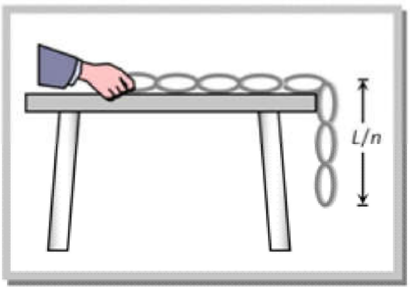
Work Done in Pulling the Chain against Gravity

A chain of length L and mass M is held on a frictionless table with (1/n) th of its length hanging over the edge.

Let $m = \dfrac{M}{L}$ mass per unit length of the chain and y is the length of the chain hanging over the edge.

So the mass of the chain of length y will be ym and the force acting on it due to gravity will be mgy.

The work done in pulling the dy length of the chain on the table.



$dW = F(-dy)$ [As y is decreasing]

i.e. $dW = mgy\,(-dy)$

So the work done in pulling the hanging portion on the table.

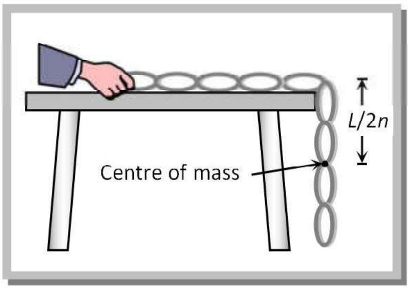
$$W = -\int_{L/n}^{0} mgy\,dy = mg\left[\frac{y^2}{2}\right]_{L/n}^{0} = \frac{mgL^2}{2n^2}$$

$$W = \frac{MgL}{2n^2} \quad \text{[As m = M/L]}$$

Alternative method:

If point mass m is pulled through a height h then work done W = mgh

Similarly for a chain we can consider its centre of mass at the middle point of the hanging part i.e. at a height of L/(2n) from the lower end and mass of the hanging part of chain $\frac{M}{n}$



So work done to raise the centre of mass of the chain on the table is given by

$$W = \frac{M}{n} \times g \times \frac{L}{2n} \qquad \text{[As W = mgh]}$$

or $$W = \frac{MgL}{2n^2}$$