RIGID BODY DYNAMICS

Motion of Rigid body

For translational motion of a rigid body, we can assume that entire mass of the body is concentrated at a point called centre of mass.

When a force is applied on a rigid body, it may either move as a whole is any direction, or may turn or rotate or may undergo both motions simultaneously.

For translation motion, the force must be applied at the centre of mass of the body. If the force acts at a point other than the cm, the resulting motion will be either rotational or both translational and rotational.

When rigid body is in translation motion, all elements (or particles) of the body move with same velocity. When a rigid body is in rotational motion, different elements of the body move with different speeds but all elements move with same angular velocity. The paths of the particles are concentric circles.

Moment of Inertia

The opposition of a particle / system of particle/ rigid body to the state of rotational is called rotational inertia rotational inertia depends on

1) Axis of rotation

2) mass/ masses of particle which comprise the system

3) The distribution of masses of the particles of the system with respect to the axis of rotation (ie, shape of the rotating body)

The rotational inertia of a body about an axis of rotation is called moment of inertia of that body about that axis of rotation.

For a particle of mass 'm' M.I is $I = mr^2$

where 'r' is the distance of particle from the axis of rotation.

For system of particles, $I = \sum_{i=1}^{n} m_i r_i^2$

For a rigid body or continuous mass distribution $I = \int r^2 dm$

Where 'd'm is a small mass element located at a perpendicular distance 'r' from the axis of rotation

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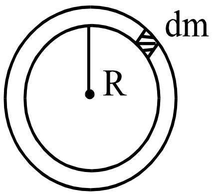
S.I unit - kgm^2

The M.I is rotational motion is analogus to mass in translation motion.

M.I of some regular bodies

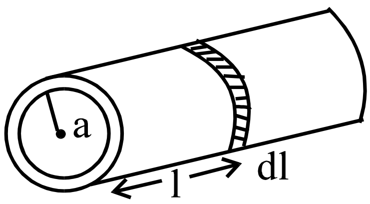
M.I of a ring about the axis passing through its centre and $\perp$ to its plane.



Consider a mass element dm on the ring of radius R

M.I $= \int R^2 dm = R^2 \int dm = MR^2$; $\quad I = MR^2$

M. I of a thin hollow cylinder about its axis



Consider a ring of width dl at a distance l from one end of a thin hollow cylinder of mass and length L. Its mass is $\frac{M}{L} dl$

$$I = \int \frac{M}{L} d\ell\, R^2 = \frac{MR^2}{L} \int d\ell = \frac{MR^2}{L} \times L$$

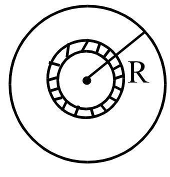
$$\boxed{I = MR^2}$$

M.I is same as that of ring

M.I of a thin disc about an axis passing through its centre and $\perp$ to its plane

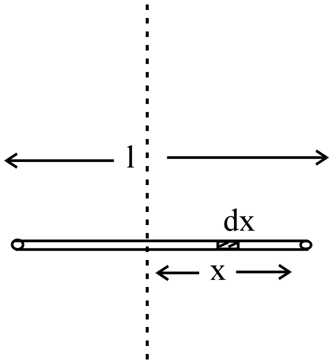
Consider a ring of width de at a radius 'a' of a disc of mass M and radius its mass is $\frac{M}{\pi R^2}(2\pi r dr)$

$$I = \int \frac{M}{\pi R^2}(2\pi r dr).r^2 = \frac{2M}{K^2}\int_0^R r^3 dr = \frac{2M}{R^2} \times \frac{R^4}{4}$$



$$\boxed{I = \frac{MR^2}{2}}$$

M. I of thin rod of length L and mass M about an axis passing through it mid point.

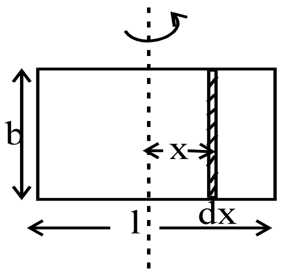


Consider an element of length dx at a distance x from the axis. Its mass is $\frac{m}{\ell}dx$

$$I = \int_{-\ell/2}^{\ell/2} \frac{m}{L} dx.x^2 = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\ell/2}^{\ell/2}$$

$$\boxed{I = \frac{ML^2}{12}}$$

M.I of a uniform rectangular lamina passing through its centre and parallel to its breadth.



mass per unit area = $\frac{M}{\ell b}$

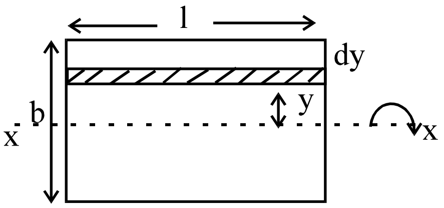
mass of strip = $\frac{M}{\ell b} \times bdx$

Since each particle on the differential element is at the same distance x from the axis of rotation.

M.IK, $I = dmx^2 = \left(\frac{M}{\ell b} \times bdx\right)x^2$

$$I = \int_{-1/2}^{1/2} \frac{M}{\ell} x^2 dx = \frac{M}{\ell}\left(\frac{x^3}{3}\right)_{-1/2}^{1/2} = \frac{M\ell^2}{12}$$

Axis in the plane lamina passing through its centre and parallel to its length.



M.I of strip about the axis xx^1

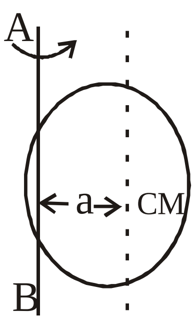
$$dI = \frac{M}{\ell b} \times \ell dy.y^2 = \frac{M}{b} y^2 dy$$

$$\therefore \quad I = \int_{-1/2}^{1/2} \frac{M}{b} y^2 dy = \frac{M}{b} \times \left(\frac{y^3}{3}\right)_{-b/2}^{b/2} = \frac{M}{b} \times \frac{b^3}{12}$$

$$\boxed{I = \frac{Mb^2}{12}}$$

Parallel Axis Theorem

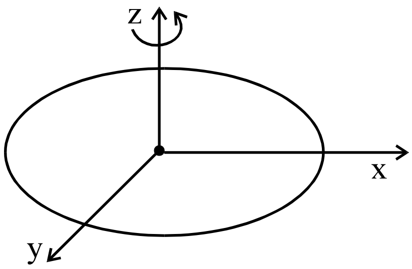
The M.I of a body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of mass and the product of mass of the body and the square of the distance between them.



$$\boxed{I_{AB} = I_{cm} + Ma^2}$$

Perpendicular Axis Theorem

The M.I of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia of the lamina about two axes at right angles to each other in the plane of the lamina and passing through the point where the perpendicular axis intersects the lamina.



$\boxed{I_z = I_x + I_y}$

Radius of gyration

If 'I' is the M.I of a body of mass M about an axis of rotation such that $I = MK^2$, then K is called the radius of gyration of the body about that axis of rotation radius of gyration depends.

1) Axis of rotation

2) The distribution of mass about the axis of rotation

Radius of gyration of a rigid body is independent of mass of the body.

Consider a body of mass M to be made of 'n' identical particles each of mass m.

$\therefore\ M = nm$

Let r_1, r_2, ... r_n be the distances of these particles from the axis of rotation. Then $I_1 = mr_1^2, I_2 = mr_2^2, ...I_n = mr_n^2$ are the M.I of these particles the axis of rotation.

The total M.I of the body about the axis of rotation is

$$I = mr_1^2 + mr_2^2 + ...+ mr_n^2 = m\left(r_1^2 + r_2^2 + ...+ r_n^2\right)$$

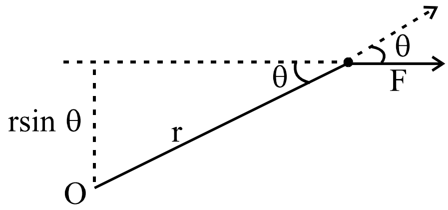
$$I = MK^2 = nmk^2$$

$$nmk^2 = m\left(r_1^2 + r_2^2 + ...+ r_n^2\right)$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + ...r_n^2}{n}}$$

Torque

The rotating effect of a force about a point (axis of rotation) is called torque. It is the rotational analogue of force in translational motion.



Torque about the point O is

$T = r\sin\theta \times F$

$\boxed{T = rF\sin\theta}$

$\boxed{\vec{T} = \vec{r} \times \vec{F}}$

If the point is on the line of action of force, the torque exerted by the force about that point will be zero. ie, The radial component of a force cannot produce any torque. Torque is a vector and its s.I unit is Nm.

Direction of torque can be obtained from right hand rule.

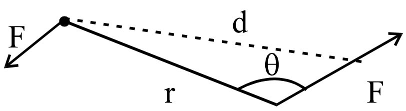
Couple

Two force $\vec{F}$ and $\vec{F}$ having the same magnitude parallel lines of action and opposite senses are said to form a couple.

A couple does not exert a net force on a body but exerts a torque.

The moment of a couple is defined as $\boxed{\overrightarrow{M} = \vec{r} \times \vec{F}}$

Where $\vec{r}$ is the vector joining the points of application of the two forces constituting the couple. The direction of moment of couple can be obtained from right hand rule.



magnifide of $\overrightarrow{M}$ is M = Fd

where 'd' is the $\perp$ distance between the force vectors.

Equilibrium of rigid bodies

Consider a rigid body in static equilibrium under the action of several forces $\overrightarrow{F_1}, \overrightarrow{F_2}, ...F_n$

Then $\sum \overrightarrow{F_i} = 0$ (for translational equilibrium)

$\sum \overrightarrow{T_i} = 0$ ((for rotational equilibrium)

Where Ti are the torques of the forces $\overrightarrow{Fi}$ with respect to an arbitrary point.

When a body is in rotational equilibrium but not in translational equilibrium, the net torque will be zero only about the cm of the body and not about any other point.

Relation between torque and angular acceleration

A force acting on a particle produces linear acceleration. Similarly, a torque acting on a particle, about an axis of rotation produces an angular acceleration $(\bar{\alpha})$.

$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta$ (θ angle between $\vec{r}$ and $\vec{F}$)

$\therefore T = rFt \left(F_t = \tan\text{gential component of } F = F\sin\theta\right) = rma_t$

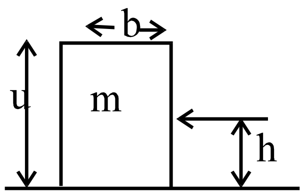
$a_t = r\alpha$

$\therefore T = rm(r\alpha) = mr^2\alpha$

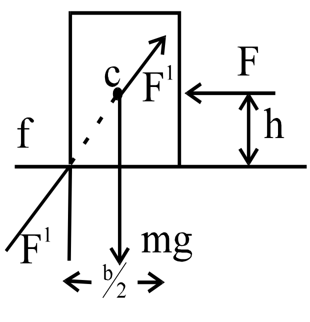
$\boxed{\bar{\tau} = I\bar{\alpha}}$

Condition for sliding / toppling of a rigid body on surface

Consider a uniform rectangular block of mass m, width b and height H. Let an external F be applied on one face of the block, at height 'h' from the base as shown.



In the absence of any external force in horizontal direction the normal reaction N passes through the centre of mass of the block. When force F is applied normal reaction shifts in the direction of applied force F.



Since right part of body is having tending to lift from surface, at the instant of tipping over about the edge the normal reaction passes through edge. From conditions of equilibrium.

In horizontal direction $f = F = \mu_x mg$

In vertical direction N = mg

Balancing torque about edge $Fh = mg \times b/2$

$$\boxed{h = \frac{mgb}{2F} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k}}$$

This is the greater height 'h' at which the force 'F' can be applied to that the block will slide with out tipping over.

As the point of application of force is raised higher the location of the line of action of normal reaction N moves to the left. In the limiting case $x = b/2$ the normal reaction passes through the edge.

If $h < h_{max}$, $F < \mu mg$ neither toppling nor sliding occurs.

If $h < h_{max}$, $F = \mu mg$ no toppling, sliding motion impending

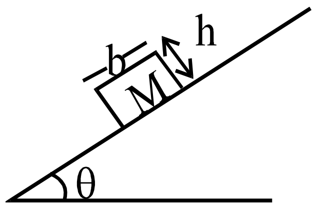
If $h < h_{max}$, $F > \mu mg$ no toppling, sliding occurs.

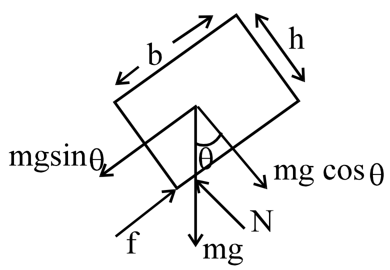
If $h = h_{max,}$ $F = \mu mg$, both sliding and toppling are impending

If $h > h_{max}$ The body will tip over for any value of weight of the body.

Motion of black in an inclined plane

A block of length b and height h is placed at rest m a rough inclined plane of inclination θ with the horizontal.





The print of application of normal reaction is displaced through x in the downward direction. It has to produce a clockwise torque about the c.m. that may balance the anticlockwise torque produced by friction.

There are two tendencies of the block

1) To slide down

2) To relate (or topple) about point A

For translational equilibrium $\sum F = 0$

$f = mg\sin\theta$

$n = Mg\cos\theta$

For rotational equilibrium $\sum \tau = 0$

$N \times x = f \times h/2$

$Mg\cos\theta\, x = \dfrac{fh}{z}$

$f = \dfrac{2Mg\cos\theta \times x}{h}$

If the block topples over about A, $x = b/2$

$$\boxed{f = \frac{2Mg\cos\theta \times b}{2h} = \frac{Mg\cos\theta\, b}{h}}$$

If block does not slide down $f < \mu N$

$\mu g\sin\theta < \mu \times Mg\ \cos\ \theta$, $\tan\theta \le \mu$

$\therefore$ if the block slides $\mu < \tan\theta$

If the block topples before sliding $f \le \mu\ (Mg\cos\theta)$

$\dfrac{b\,Mg\cos\theta}{h} \le \mu mg\ \cos\theta$; $\dfrac{b}{h} \le \mu$

if the block topples before sliding $\mu \ge b/h$

The block has a tending to slide down before toppling if $\mu < b/h$

The block has a tending to topple before sliding if $\mu > b/h$

The block slides down if $\mu < \tan\theta$, where θ is the angle of inclination of the incline with respect to the horizontal.

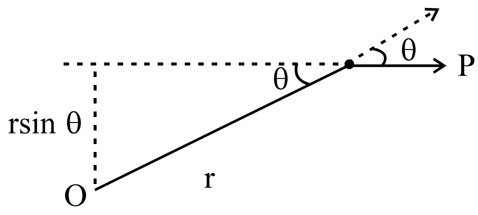
The both topples over of $\mu > \tan\theta$ and $\mu > b/h$

Angular Momentum

Angular momentum is associated with any particle is motion, which need not be rotational motion. The angular momentum of a particle is measured with respect to a fixed reference point and it is defined as

$\vec{L} = \vec{r} \times \vec{P}$

$\vec{r}$ - vector connecting the position of the particle with the reference point about which angular momentum is measured and $\vec{p}$ is the linear momentum vector of the particle.



Angular momentum about point 'θ' is $L = r\sin\theta \times p = rp\sin\theta$

$\boxed{\vec{L} - \vec{r} \times \vec{p}}$

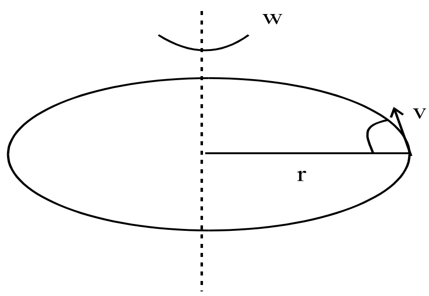
Angular momentum is zero for any point along the direction of linear momentum (or velocity)

Angular momentum is a vector quantity and its direction is determined by right hand rule.

S.I unit kgm^{2}s$^{-1|}$

Angular momentum of a rotating particle about the centre of rotation.

For a particle rotating about point 'O' with radius 'r' and angular velocity ω, its velocity $\vec{v}$ is always $\perp$ to position vector $\vec{r}$, so that $\theta = 90^o$, and $\sin\theta = 1,\ v = r\omega$



$\therefore$ Angular momentum of a rotating particle is given by $L = rp = mvr = m(rw)r = mr^2\omega$

For a particle I = mr^2

$\therefore \boxed{L = I\omega}$

Rotation between Torque and Angular momentum

For a rigid body, $\tau = I\alpha$

$$\tau = I\frac{d\omega}{dt} = \frac{d}{dt}(I\omega)$$

$$\boxed{\vec{T} = \frac{dL}{dt}}$$

Law of conservation of Angular momentum

If the net external torque acting in a rigid body or system of particles is zero, the total angular momentum of the rigid body /system of particles is conserved.

if $\tau = 0 \quad \frac{dL}{dt} = 0$

ie L = a instant

ie; $\boxed{I\omega = \text{a constant}}$

Relation between angular momentum (L) and kinetic energy of rotating body.

K. E, $K = \frac{1}{2}I\omega^2$

$L = I\omega$

$\omega = L/I$

$\therefore K = \frac{1}{2}I\frac{L^2}{I^2} = \frac{L^2}{2I}$

$I \rightarrow$ M.I of the body

$L \rightarrow$ angular momentum

Angular impulse

$\vec{T} = \frac{dL}{dt} \Rightarrow \vec{\tau}dt = d\vec{L}$

The term $\vec{\tau}dt$ is called angular impulse.

$d\vec{L}$ = change in angular momentum

$$\int_{t_1}^{t_2} \tau lt = \int_{t_1}^{t_2} d\vec{L} = \vec{L_2} - \vec{L_1}$$

Hence angular impulse = change in angular momentum.

Comparison of linear and rotational motion

Linear motion	Rotational motion
Linear displacement	Angular displacement $= \theta$
Linear velocity $v = \frac{dx}{dt}$	Angular velocity $w = \frac{d\theta}{dt}$
Linear acceleration $a = \frac{dv}{dt}$	Angular acceleration $\alpha = \frac{d\omega}{dt}$
Linear momentum $\vec{p} = m\vec{v}$	Angular momentum $\vec{L} = I\vec{\omega}$
Force $F = ma$	Torque, $\tau = I\alpha$
work done $w = \int \bar{F}.d\bar{d}$	work done $w = \int \bar{T}.d\bar{v}$
Power $P = \bar{F}.\bar{V}$	Power $P = \bar{\tau}.\bar{\omega}$
Kinetic energy $K = \frac{1}{2}mv^2$	Kinetic energy $k = \frac{1}{2}Iw^2$
Kinetic equations $V = u + at$ $S = ut + \frac{1}{2}at^2$ $V^2 = u^2 + 2as$ $Sn = u + \frac{a}{2}(2n-1)$	Kinetic equations $\omega = \omega_o + \alpha t$ $\theta = \omega_o t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$ $\theta_n = \omega_o + \frac{\alpha}{2}(2n-1)$

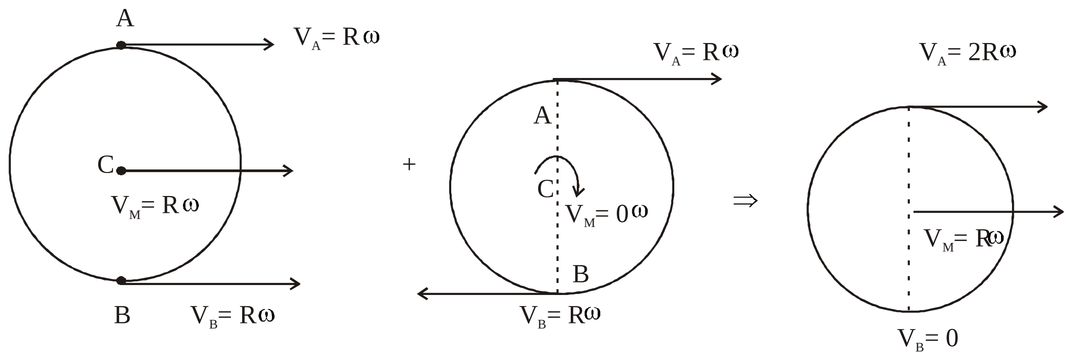
Rolling Motion

The motion of a rigid body undergoing rotation about an axis with the axis of rotation having a translation motion, is called rolling motion.

Pure rolling

In this use, there is no relative motion between the contact points, ie, the contact point between the body and surface is at relative rest.

Pure rolling motion is a combination of pure translation and pure rotation.



Pure translation Pure rotation

In pure rolling motion $V_{cm} = R\omega$

$\because$ K of rolling , $KE_{rolling} = KE_{translation} + KE_{relation} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

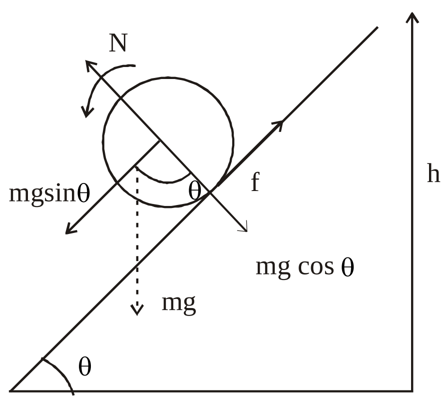
$$= \frac{1}{2}mv^2 + \frac{1}{2}mk^2\omega^2 \left(\because I = mk^2\right) = \frac{1}{2}mv^2 + \frac{1}{2}mk^2\frac{v^2}{R^2}$$

$$\boxed{KE_{rolling} = \frac{1}{2}mv^2\left(1 + k^2/R^2\right)}$$

$$\boxed{KE_{rolling} = KE_{translation}\left(1 + k^2/r^2\right)}$$

Pure rolling on a ramp

A round object rolls down a ramp with out slipping. The tending for the body slide down is prevented by the force of friction acting up the ramp.



Translated motion $Mg\sin\theta - fs = Ma_{cm}$

$$Rf_s = I_{cm}\frac{a_{cm}}{R}\ ; \quad Rf_s = mk^2\frac{a_{cm}}{R}\ ; \quad fs = \frac{Mk^2 a_{cm}}{R^2}$$

$$\therefore\ Mg\sin\theta - \frac{mk^2 a_{cm}}{R^2} = Ma_{cm}\ ;\ Mg\sin\theta = Ma_{cm} + \frac{Mk^2 a_{cm}}{R^2}\ ;\ g\sin\theta = a_{cm}\left(1+k^2/R^2\right)$$

$$\boxed{a_{cm} = \frac{g\sin\theta}{1+k^2/R^2}}$$

Velocity at the bottom of the ramp $V^2 = 2as$

$$V^2 = 2\frac{g\sin\theta}{1+k^2/R^2}\times\frac{h}{\sin\theta}$$

$$\boxed{V = \sqrt{\frac{2gh}{1+k^2/R^2}}}$$

taken to reach the bottom $t = \dfrac{V}{a} = \dfrac{\sqrt{\dfrac{2gh}{1+k^2/R^2}}}{\dfrac{g\sin\theta}{1+k^2/R^2}}$

$$\boxed{t = \frac{1}{\sin\theta}\sqrt{\frac{2h\left(1+k^2/R^2\right)}{g}}}$$

friction $f = \dfrac{mk^2}{R^2}\dfrac{g\sin\theta}{\left(1+k^2/R^2\right)}$

$$\boxed{f = \frac{mg\sin\theta}{1+R^2/K^2}}$$

Co- efficient of friction $\mu = \dfrac{f}{N} = \dfrac{\dfrac{Mg\sin\theta}{1+R^2/K^2}}{Mg\cos\theta} = \dfrac{\tan\theta}{1+R^2/k^2}$

$$\boxed{\mu = \frac{\tan\theta}{1+R^2/K^2}}$$