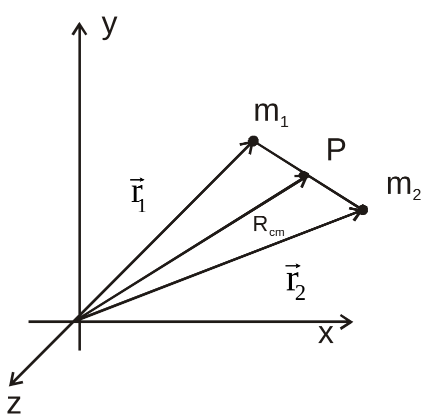
CHAPTER -

CENTRE OF MASS

If a system consists of a large no of particles, having different types of motion, it would be quite complicated and laborious to describe the motion of particles of the system. Rather, it will be more convenient to describe the motion of the entire system, by reducing it to an equivalent single particle, located at a characteristic geometric point of the system, called the centre of mass of the system of particles.

Centre of mass of a system of particles is a geometrical point, where the entire mass of the system of particles is assumed to be concentrated and external forces on the system of particles appear to be applied at that point.

C.M of a two Particle System



Consider a system of 2-particles of mass m_1 and m_2 located at position vectors $\vec{r}_2$ and $\vec{r}_2$ respectively.

The system of 2-particles can be reduced to a single particle of mass $m = m_1 + m_2$ located at ‘p’ whose position vector is

$$\boxed{\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$$

✶ The C.M of a 2-particle system lies on the straight line joining the two particles at a location in between the two particles.

✶ C.M is located near the heavier particle.

✶ If the particles are of same mass, then $R_{cm} = \frac{\vec{r}_1 + \vec{r}_2}{2}$ ie C.M is located at the centre of the line joning the two masses.

✶ The term $m_1\vec{r}_1$ is called the moment of mass of particle 1 about the origin of co-ordinate system and $m_2\vec{r}_2$ is the moment of mass of particle 2 about the origin.

✶ The geometric locations of the C.M does not depend upon the choice of co-ordinate system. Even if we select any other point as the origin. We will obtain the same point as the C.M of the 2 particle system.

✶ If the origin 'O' of the coordinate system is shifted to the centre of mass, then $R_{cm} = 0 \Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 = 0$.

The sum of moments of all masses of a system, about the centre of mass of a system, is zero.

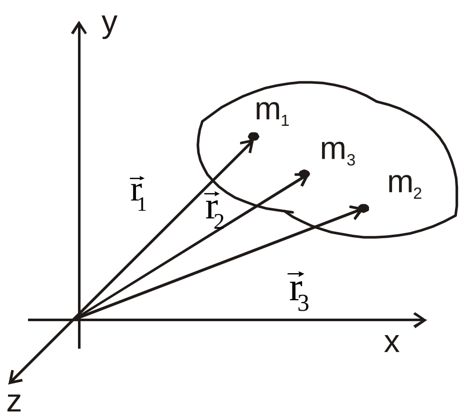
✶ If the origin of the co-ordinate system coincides with the position of one of the particle say m_1

Then $\vec{R}_{cm} = \frac{m_2\vec{r}_2}{m_1 + m_2}$

C.M. of a Multiparticle System

Consider a system of particles of masses m_1, m_2, ... m_n whose position vectors are $\vec{r}_1, \vec{r}_2, ...\vec{r}_n$. Then position vector of C.M of system of particles ie. $\boxed{\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}}$

$\vec{r}$ is the position vector of each particle which has x, y and z components. Similarly C.M also has three components. X_{cm}, Y_{cm} and Z_{cm}



$$\boxed{\begin{aligned} X_{cm} &= \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n} \\ Y_{cm} &= \frac{m_1y_1 + m_2y_2 + ... + m_ny_n}{m_1 + m_2 + ... + m_n} \\ Z_{cm} &= \frac{m_1z_1 + m_2z_2 + ... + m_nz_n}{m_1 + m_2 + ... + m_n} \end{aligned}}$$

$$\boxed{R_{cm} = \left|\vec{R}_{cm}\right| = \sqrt{X_{cm}^2 + Y_{cm}^2 + Z_{cm}^2}}$$

Rigid Body

A rigid body can be considered as a continuous distribution of system of particles. In a rigid body, the particles are arranged so that the distance between the particles is fixed. A rigid body does not undergoes changes in shape or size under the action of external forces. However, there is no such thing as a perfectly rigid body. The entire mass of the body may be considered to be concentrated at the centre of mass of the body.

Locating C.M of a Rigid Body

For continuous distribution of mass (ie for a rigid body) the position of the C.M can be determined by integration method.

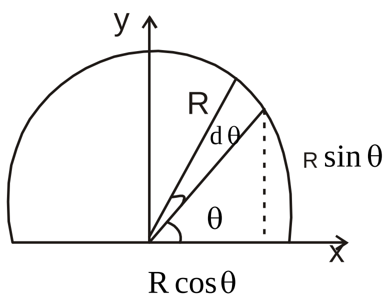
$$\vec{R}_{cm} = \frac{\text{moments of the masses with respect to the origin}}{\text{total mass}}$$

$\boxed{\vec{R}_{cm} = \dfrac{\int \vec{r}\,dm}{M}}$ Where M = mass of rigid body $\vec{r}$ is the position vector of an elemental mass ‘dm’.

The co-ordinates of C.M can be written as

$$\boxed{X_{cm} = \frac{1}{M}\int x\,dm,\ Y_{cm} = \frac{1}{M}\int y\,dm,\ Z_{cm} = \frac{1}{M}\int z\,dm}$$

Centre of Mass of a Uniform Semicircular Wire



Let M be the mass and K the radius of a uniform semicircular wire. Take centre as the origin mass per unit length of the wire $=\frac{M}{\pi R}$

Consider a small element of length, $d\ell = Rd\theta$

$$\therefore dm = \frac{M}{\pi R} \times R\, d\theta = \frac{M}{\pi} d\theta$$

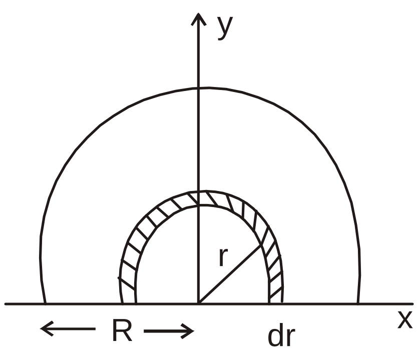
The coordinates of the centre of mass are

$$\boxed{X = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^{\pi} R\cos\theta \frac{M}{\pi} d\theta = 0}$$

$$\boxed{Y = \frac{1}{M}\int y\, dm = \frac{1}{M}\int_0^{\pi} R\sin\theta \frac{M}{\pi} d\theta = \frac{2R}{\pi}}$$

$\therefore$ The centre of mass is at $\left(\theta, \frac{2R}{\pi}\right)$

Centre of Mass of a Uniform Semicircular Disc



Let M be the mass and R be the radius of semicircular disc. Consider a semicircular ring of radius r and thickness dr. Then mass of semicircular element is

$$dm = \frac{M}{\pi \frac{R^2}{2}} \times \pi r\, dr = \frac{2M}{R^2} r\, dr$$

The y-coordinate of the elemental wire is at $\frac{2r}{\pi}$

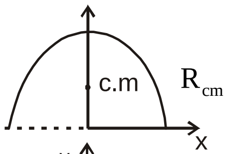
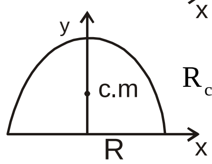
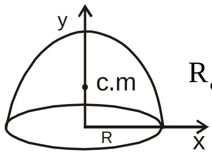
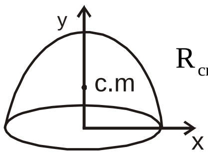
$\therefore$ The y-coordinate of the C.M of the plate is

$$Y = \frac{1}{M}\int_0^R \left(\frac{2r}{\pi}\right)\frac{2M}{R^2} r\, dr = \frac{1}{M} \times \frac{4MR^3}{\pi R^2 \times 3} = \frac{4R}{3\pi}$$

x-coordinate of C.M is zero by symmetry.

$\therefore$ co-ordinate of C.M is $\left(\theta, \frac{4R}{3\pi}\right)$

C.M of Rigid Bodies

Shape of body	Position of C.M
1. Uniform rod	Midpoint of the rod
2. Circular Ring	Centre of ring
3. Circular disc	Centre of disc
4. Square/rectangular lamina	At the point of intersection of diagonals
5. cubical block	At the point of intersection of diagonals
6. Cylinder	Centre of cylinder
7. Triangular lamina	At the point of intersection of medians
8. Solid cone	3/4th of height of cone from apex, on its axis (h/4 from base)
9. Hollow cone	2/3rd of height of cone from apex, on its axis (h/3 from base)
10. Semicircular ring	 $R_{cm} = \left(0, \frac{2R}{\pi}\right)$
11. Semicircular disc	 $R_{cm} = \left(0, \frac{4R}{3\pi}\right)$
12. hollow-hemisphere	 $R_{cm} = \left(0, \frac{R}{2}\right)$
13. Solid hemisphere	 $R_{cm} = \left(0, \frac{3R}{8}\right)$

Motion of C.M of a System of Particles

Consider a system of n particles of masses m_1, m_2, m_3, ... m_n at position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots \vec{r}_n$ respectively. Let $F_{1ext}, F_{2ext}, F_{3ext} \ldots F_{n\,ext}$ be the external forces of system $f_{1int}, f_{2int}, f_{3int} \ldots f_{n\,int}$ be the internal forces of system on particles 1,2,3, ... n respectively. The velocities of particles are $\vec{V}_1, \vec{V}_2, \vec{V}_3, \ldots \vec{V}_n$, their linear momentum are $\vec{P}_1, \vec{P}_2, \vec{P}_3 \ldots \vec{P}_n$ and acceleration are $\vec{a}_1, \vec{a}_2, \vec{a}_3 \ldots \vec{a}_n$ respectively.

Position vector of c.m $\vec{R}_{cm} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{m_1 + m_2 + ... + m_n}$

$$\boxed{\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n}{M}}$$

$$\therefore M\vec{R}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n \quad \rightarrow (1)$$

Hence sum of the moments of masses about the origin is equal to the moment of the total mass of the system, placed at the c.m.

Taking first derivative of (1) with respect to time, we get $\dfrac{d}{dt}\left(M\vec{R}_{cm}\right) = \dfrac{d}{dt}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n\right)$

$$M\frac{d\vec{R}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + ... + m_n\frac{d\vec{r}_n}{dt}$$

$$\boxed{M\vec{V}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n} \quad ----(2)$$

$$\boxed{\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{M}} \quad ----(3)$$

If $\vec{P}_{system}$ is the total linear momentum of the system, then

$$\vec{P}_{system} = \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n = m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n \quad ------ (4)$$

Comparing eq. (2) and (4)

$$\boxed{M\,V_{cm} = \vec{P}_{system}}$$

Taking the first derivative of equation (2) with respect to time, we get

$$\frac{d}{dt}\left(M\vec{V}_{cm}\right) = \frac{d}{dt}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n\right)$$

$$M\frac{d\,V_{cm}}{dt} = m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + ... + m_n\frac{d\vec{v}_3}{dt}$$

$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n$$

$$\boxed{\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{M}}$$

but $m_1\vec{a}_1 =$ total force on particle $1 = F_{1ext} + f_{1int}$

m_2a_2 = total force on particle $2 = F_{2ext} + f_{2int}$

and so on.

$$\therefore\ m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n a_n = F_{1_{ext}} + f_{1_{int}} + F_{2_{ext}} + f_{2_{int}} + ... + F_{n_{ext}} + f_{n_{int}}$$

$$= \left(F_{1_{ext}} + F_{2_{ext}} + ... + F_{n_{ext}}\right) + \left(f_{1_{int}} + f_{2_{int}} + ... + f_{n_{int}}\right)$$

$$= F_{(\text{Net external})} + 0$$

$$\because f_{1_{int}} + f_{2_{int}} + ... + f_{n_{int}} = 0$$

$$\therefore\ \boxed{F_{ext} = M\vec{a}_{cm}}$$

This Newton's 2nd law of motion for the system of particles, (this law holds good only for a closed system ie a system of invariant mass)

When there is no addition or removed of mass from the system of particles.

If $\vec{F} = 0$

Then $\dfrac{dP_{system}}{dt} = 0$

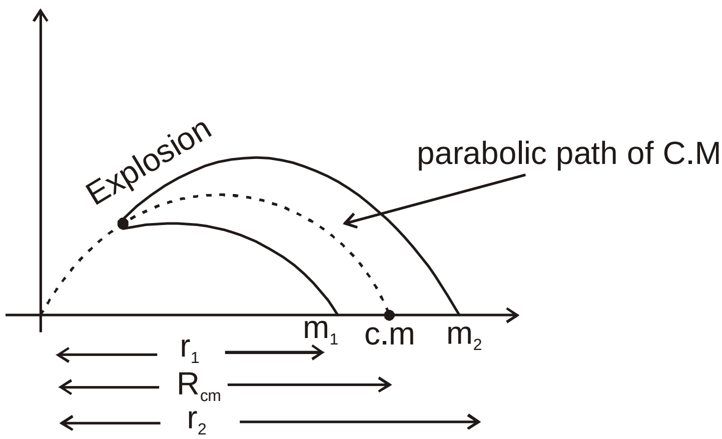
$\Rightarrow \vec{P}_{system} = a$ constant

This is called law of conservation of linear momentum for a system of particles. Hence, if the net external force on a system of particles is zero, the c.m of the system is either at rest or moves with constant velocity.

If $KE_1 = \frac{1}{2}m_1v_1^2,\ KE_2 = \frac{1}{2}m_2v_2^2, ...KE_n = \frac{1}{2}m_nv_n^2$ are the K.E of the particles of the system, total KE of the system $KE = KE_1 + KE_2 + ... + KE_n$.

Thus only if all the particles of the system are not in motion (ie v = 0 for all particles), then only the K.E of the system will be zero. Even if one particle of a system is in motion, the K.E of the system of particles cannot be zero. If all particles of a system are at rest, total K.E and total linear momentum of the system are zero. However, if the total linear momentum of a system is zero, it is not necessary that the K.E of the system is zero.

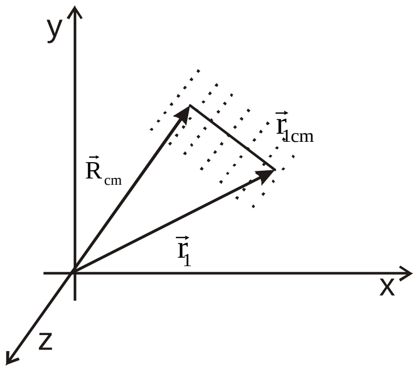
eg: when a projectile explodes, the trajectory of the center of mass of the fragments remains the same as the original parabolic trajectory of the projectile.



C-Frame

A frame of reference, whose origin is at the centre of mass of a system is called a C-frame or centre of mass frame of reference. In this frame $\vec{R}_{cm} = 0$ and hence $\vec{V}_{cm} = 0$. Hence the C.M is at rest in the frame and consequently $\vec{P}_{cm} = MV_{cm} = 0$ in this frame. ie total linear momentum of the system in the C-frame is zero.

If the net external force acting on a system of particles is zero w.r. to any inertial frame, then the C-frame also will be inertial. However, if the net external force acting on a system of particle is not zero w.r. to an inertial frame, then the C-frame will be non-inertial.



From figure $\vec{r}_1 = \vec{R}_{cm} + \vec{r}_{1cm}$

$$\vec{v}_1 = \vec{v}_{cm} + \vec{v}_{1cm}$$

$$\vec{v}_{1cm} = \vec{v}_1 - \vec{v}_{cm}$$

If K is the total K.E of system of particles about an inertial frame,

$$K = \frac{1}{2}m_1|\vec{v}_1|^2 + \frac{1}{2}m_2|\vec{v}_2|^2 + ... + \frac{1}{2}m_n|\vec{v}_n|^2$$

K_c = Kinetic energy of a system of particles with respect to C-frame

$$=\frac{1}{2}m_1v_{1cm}^2+\frac{1}{2}m_2v_{2cm}^2+...+\frac{1}{2}m_nv_{ncm}^2$$

$\overline{P}_{system}$ = total linear momentum of the system about an inertial frame

$$\therefore \boxed{K=K_c+\frac{P_{system}^2}{2M}}$$

✶ If total linear momentum of a system about a inertial frame is zero (ie P_{system} = 0) the K = K_c is the total K.E of the system of particles about an inertial frame is equal to the K.E of system of particles about C-frame.

ie total K.E of a system of particles will be minimum with respect to C-frame.

✶ K.E of a system of particles about an inertial frame is zero, only if the K.E of the system of particles about C-frame is zero and the total linear momentum of system about the inertial frame is zero. ie K = 0, only if K_c = 0 and $\overline{P}_{system}=0$.

<u>Reduced mass of a two particle system</u>

Consider a two-particle system of masses m_1 and m_2 having velocities $\overline{v}_1$ and $\overline{v}_2$ respectively with respect to an inertial frame. Then

$$\vec{V}_{cm}=\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}$$

$$\vec{V}_{1cm}=\vec{v}_1-\vec{v}_{cm}$$

$$=\vec{v}_1-\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}$$

$$=\frac{m_{12}\left(\vec{v}_1-\vec{v}_2\right)}{m_1+m_2}$$

$$\vec{v}_{2cm}=\vec{v}_2-\vec{v}_{2cm}=\frac{m_1\left(\vec{v}_2-\vec{v}_1\right)}{m_1+m_2}$$

Linear momentum of 1 w.r. to C.M

$$\vec{P}_{1cm}=m_1\vec{v}_{1cm}=\frac{m_1m_2\left(\vec{v}_1-\vec{v}_2\right)}{m_1+m_2}$$

Then, $\frac{m_1m_2}{m_1+m_2}$ is called the reduced mass of the system $\boxed{\mu=\frac{m_1m_2}{m_1+m_2}}$

$\left|\vec{P}_{1cm}\right| = P_{1cm} = \mu v_{rel}$, whose $v_{rd} = \left|\vec{v}_1 - \vec{v}_2\right|$ is the magnitude of the relative velocity of the particles.

Similarly linear momentum of particle 2 w.r. to C.M

$$\vec{P}_{2cm} = m_2\vec{v}_{2cm} = \frac{m_1 m_2\left(\vec{v}_2 - \vec{v}_1\right)}{m_1 + m_2}$$

$\left|\vec{P}_{2cm}\right| = P_{2cm} = \mu v_{rel}$ where $v_{rel} = \left|\vec{v}_2 - \vec{v}_1\right|$

$$\therefore \vec{P}_{1cm} = -\vec{P}_{2cm}$$

K.E of particle 2 w.r. to C.M

$$K_{2cm} = \frac{1}{2}m_2\left|\vec{v}_{2cm}\right|^2 = \frac{P_{2cm}^2}{2m_2}$$

Hence total K.E of system in C-frame

$$K_C = K_{1cm} + K_{2cm}$$

$$= \frac{P_{1cm}^2}{2m_1} + \frac{P_{2cm}^2}{2m_2} = \frac{\left(\mu v_{rel}\right)^2}{2m_1} + \frac{\left(\mu v_{rel}\right)^2}{2m_2}$$

$$= \frac{\mu^2 v_{rel}^2\left(m_1 + m_2\right)}{2m_1 m_2} = \frac{1}{2}\mu v_{rel}^2$$

$$\therefore \boxed{K_C = \frac{1}{2}\mu v_{rel}^2}$$

Centre of Gravity

The centre of gravity of a rigid body is a point, on the body or outside the body at which the entire weight of the body is considered to be concentrated.

✶ In a region of uniform gravitational field, the centre of mass and centre of gravity of a rigid body will coincide.

✶ If the gravitational field is not uniform, for a uniform rigid body its centre of mass and centre of gravity may not coincide.

✶ The centre of gravity of a rigid body has no meaning in a region where there is no effective gravitational field. However c.m of the rigid body has a definite meaning even in such regions.

Extra Points

For laminar type body the formula for finding the position of c.m are

$$\vec{R}_{cm} = \frac{A_1\vec{r}_1 + A_2\vec{r}_2 + ... + A_n\vec{r}_n}{A_1 + A_2 + ... + A_n}$$

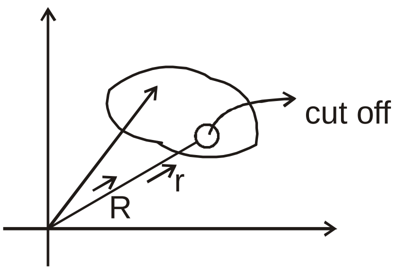
$$X_{cm} = \frac{A_1x_1 + A_2x_2 + ... + A_nx_n}{A_1 + A_2 + ... + A_n}$$

$$Y_{cm} = \frac{A_1y_1 + A_2y_2 + ... + A_ny_n}{A_1 + A_2 + ... + A_n}$$

$$Z_{cm} = \frac{A_1z_1 + A_2z_2 + ... + A_nz_n}{A_1 + A_2 + ... + A_n}$$

C.M. of residual body

Consider a body of mass M whose c.m is at a position vector $\vec{R}$. Let a small portion of mass m is removed from the body whose c.m is at position vector $\vec{r}$. Then the location of c.m of r_{cm} assing portion is



$$\boxed{\vec{r}_{cm} = \frac{M\vec{R} - m\vec{r}}{M - m}}$$

or

$$\boxed{\vec{r}_{cm} = \frac{A\vec{R} - a\vec{r}}{A - a}}$$

$A \rightarrow$ Area of complete body

$a \rightarrow$ Area of cut – off portion

Law of conservation of linear momentum

The law of conservation of linear momentum states that the total linear momentum of a system is constant if the net external force on the system is zero. ie The total linear momentum of an isolated system remains constant.

Law of conservation of linear momentum is independent of the frame of reference though linear momentum depends on the frame of reference.

Conservation of linear momentum is equivalent to Newton's III law of motion. For a system of two particles in absence of external force by law of conservation of linear momentum.

$$\vec{P}_1 + \vec{P}_2 = \text{constant}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}$$

Differentiating with respect to time

$$m_1\frac{d\vec{v}_1}{dt}+m_2\frac{d\vec{v}_2}{dt}=0$$

$$m_1\vec{a}_1+m_2\vec{a}_2=0$$

$$\vec{F}_1+\vec{F}_2=0$$

$$\vec{F}_1=-\vec{F}_2$$

Applications of law conservation of linear momentum

(i) Recoil of gun

When a gun is fired the bullet moves forward with high velocity and the gun moves back with a small velocity. This is called recoil of gun.

Initially the gun and bullet inside are at rest; initial linear momentum of the system is zero. On releasing the bullet from rest, the bullet (mass m) leaves the muzzle with velocity $'\upsilon'$ and the gun (mass M) recoils with a velocity 'v' relative to ground, then the absolute velocity of the bullet (velocity of bullet respect to ground) is $\vec{V}_{BE}=\vec{V}_{BG}+\vec{V}_{GE}$

$V_{BE}\rightarrow$ velocity of bullet with respect to earth

$V_{BG}\rightarrow$ velocity of bullet with respect to gun

$V_{GE}\rightarrow$ velocity of gun with respect to earth.

$$\therefore\left|\vec{V}_{BE}\right|=\upsilon-V$$

$$0=m\left(\upsilon-V\right)-MV$$

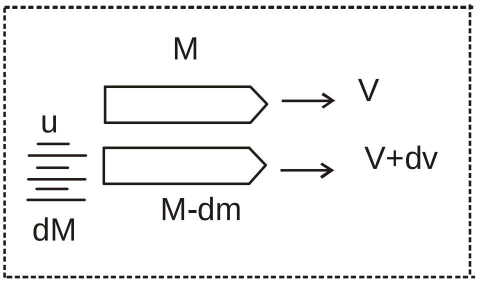
$$\boxed{V=\frac{m\upsilon}{M+m}}$$

System of variable mass

Rocket Propulsion

Let M be the mass of the rocket and V its velocity at an arbitrary time 't'. After a time interval dt, the rocket has a velocity V + dv and mass M - dM where the change in mass is dM. The exhaust products released by the rocket during the time interval dt have mass 'dM' and velocity 'V' with respect to inertial frame. For an observer in an inertial frame such as earth's surface the mass ejected dM will appear to be moving in the same direction as the rocket.

The system is closed and isolated, therefore no external force. Therefore linear momentum is conserved.



Initial linear momentum of system = Mv

Final linear momentum of system = (M - dM) (V + dV) + dM.U

$$\therefore MV = (M - dM)(V + dV) + dm.U$$

If v_{rel} = speed with which mass is ejected out relative to rocket

then $v_{rel} = U - (v + dV)$

$$U = (V + dV) + v_{rel}$$

$$MV = (M - dM)(V + dV) + dM(V + dV) + dM.v_{rel}$$

$$MV = MV + M\,dV + dM.v_{rel}$$

$$-dM\,v_{rel} = M\,dV$$

$$\Rightarrow \boxed{\frac{-dM}{dt} v_{rel} = M\frac{dV}{dt}}$$

Replace $\frac{-dM}{dt}$ (the rate at which the rocket losses mass)

by R where R is the rate of find consumption and $\frac{dv}{dt}$ as acceleration of rocket.

$\boxed{R\,V_{rel} = Ma}$ (first rocket equation)

$R\,V_{rel}$ is called the thrust of the rocket engine $F_{thrust} = Ma$

If an external force $\vec{F}_{ext}$ is acting on the rocket, in addition to the thrust, then the equation becomes.

$$\boxed{F_{ext} + F_{thrust} + M\frac{dv}{dt}}$$

$$-dM.V_{rel} = M\,dV$$

$$dV = -v_{rel}\frac{dM}{M}$$

$$\int_{v_i}^{V_f} dV = -v_{rel}\int_{m_i}^{m_f}\frac{dM}{M}$$

$$\boxed{V_f = v_i = v_{rel}\ln\frac{M_i}{M_f}}$$ (second rocket equation)

Impulse

Impulse of a force F acting on a body is defined as $J = \int_0^t F\,dt = \int F\,dt = \int m\frac{dV}{dt}.dt = \int mdV$

It is also defined as change in momentum.

$\boxed{\vec{J} = \Delta\vec{P}}$ (Impulse-momentum theorem)

Instantaneous Impulse

When a force acts for such a short time that effort is instantaneous eg:- bat striking a ball. In such cases although the magnitude of the force and the time for which it acts may be unknown, the value of there product can be known by measuring the initial and final momenta.

$$\boxed{\vec{J} = \int \vec{F}dt = \Delta\vec{P} = P_f - P_i}$$

Impulsive Force

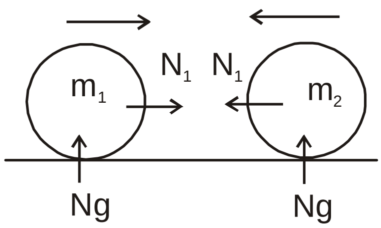
A force of relatively higher magnitude and acting for relatively shorter time is called impulsive force.

An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval.

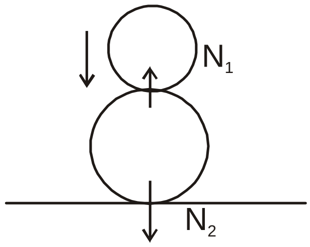
$$\begin{Bmatrix}\text{Gravitational force and spring force are always non} - \text{impulsive} \\ \text{Normal, tension and friction are case dependent}\end{Bmatrix}$$

Impulsive Normal

In case of collision, normal forces at the surface of collision are always impulsive.



N_1 is impulsive; N_g is non-impulsive



N_1 and N_2 are impulsive

Impulsive Friction

If the normal between two objects is impulsive, then the friction between the two will be impulsive.

Impulsive Tension

When string jerks, equal and opposite tension acts suddenly at each end. Consequently, equal and opposite impulses acts on the bodies attached with the string, in the direction of the string.

Case 1

✶ **One end of the string is fixed:**

The impulse which acta at the fixed end of the string cannot change the momentum of the fixed object. The object attached to the free end, however, will undergo a change in momentum in the direction of the string. The momentum remains unchanged in direction perpendicular to the string where no impulsive for us act.

✶ **Both ends of string are attached to movable objects**

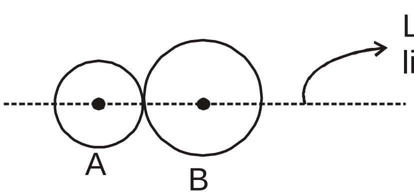
In this case, equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system, therefore remains constant, although the momentum of each individual object is changed in the direction of the string. However, no impulse acts perpendicular to the string and the momentum of each particle in this direction remain unchanged.

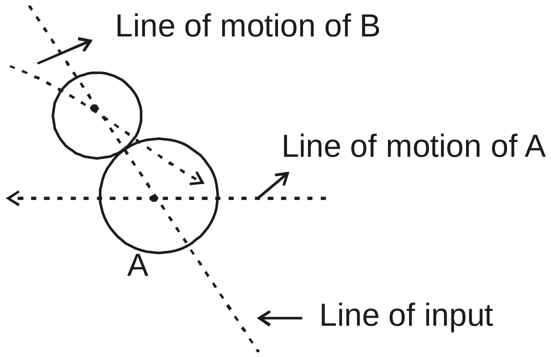
In case of rod, tension is always impulsive. In case of spring, tension is always non-impulsive.

Collisions

Line of impact

The line passing through the common normal to the surfaces in contact during the impact is called line of impact. The force during collision acta along this line on both the bodies.





Classification of Collisions

On the basis of line of impact

1. Head-on collision

   The velocities of the particles are along the same line before and after collision.

2. Oblique collision

   The velocities of the particles are along different lines before and after collision.

On the bases of Energy

Elastic Collision

In an elastic collision, the particles regain their shape and size completely after collision. That is, no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus kinetic energy of a system after collision is equal to kinetic energy of system before collision.

Inelastic Collision

In an inelastic collision, the particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is returned by the colliding particles in the form of deformation potential energy. Thus K.E of the particles no longer remains conserved.

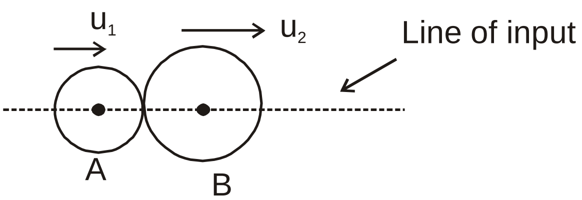
Perfectly inelastic Collision

If velocity of separation just after collision becomes zero, then collision is perfectly inelastic. Collision is said to be perfectly inelastic if both the particles stick together after collision and move with the same velocity.

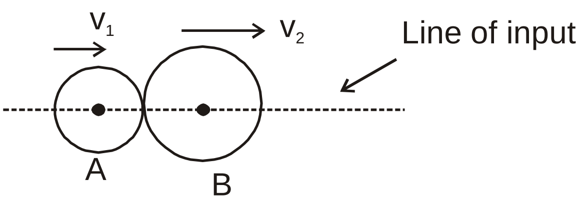
Co-efficient of Restitution

The co-efficient of restitution is defined as the ratio of the impulse of reformation to the impulse of deformation of either body.

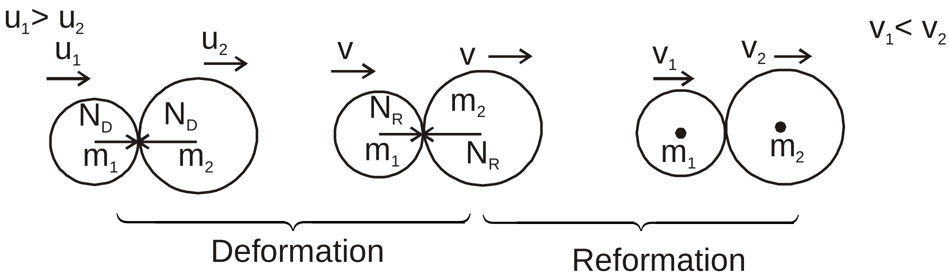
$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r dt}{\int F_d dt}$$



just before collision



just after collision



$$m_1u_1 + m_2u_2 = (m_1 + m_2)v = m_1v_1 + m_2v_2$$

$$\boxed{V = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}}$$

Impulse of Deformation

J_D = Change in momentum of any body during deformation

$$= m_2(v - u_2) \text{ for } m_2$$

$$= m_1(-v + u_1) \text{ for } m_1$$

Impulse of Reformation

J_R = change in momentum of any body during reformation

$$= m_2(v_2 - v) \text{ for } m_2$$

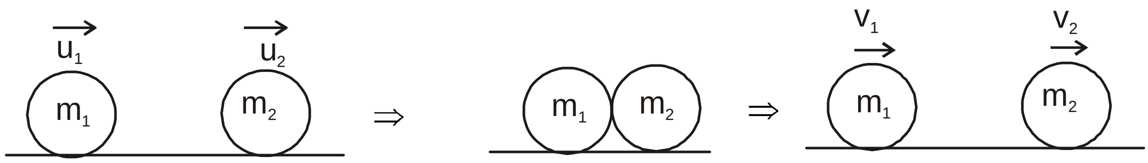
$$= m_1(v - v_1) \text{ for } m_1$$

$$\boxed{e = \frac{J_R}{J_D} = \frac{v_2 - v}{v - u_2} - \frac{v_2 - v_1}{u_1 - u_2}}$$

$$\therefore \boxed{e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along the line of impact}}}$$

One dimensional Collision

One dimensional collision means the velocities of the bodies before and after collision are along the same straight line. The change of momentum vector for each body will be along the same line, which means final momentum vector of each body will be along the same line. After collision $v_2 > v_1$



The momentum conservation equation is $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

For elastic collision mechanical energy is conserved $\therefore \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

hence $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad ------(1)$

$$m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2 ------(2)$$

(1) $\Rightarrow$ $m_1u_1 - m_1v_1 = m_2v_2 - m_2u_2$

$$m_1(u_1 - v_1) = m_2(v_2 - u_2) -----(3)$$

(2) $\Rightarrow$ $m_1u_1^2 - m_1v_1^2 = m_2v_2^2 - m_2u_2^2$

$$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) -----(4)$$

$\frac{(4)}{(3)} \Rightarrow$ $u_1 + v_1 = v_2 + u_2$

$$v_2 - v_1 = u_1 - u_2$$

$$\frac{v_2 - v_1}{u_1 - u_2} = 1 = e$$

However in all type of collisions, linear momentum of the system is conserved.

ie $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

For in elastic collisions mechanical energy is not conserved.

$$e = \frac{v_2 - v_1}{u_1 - u_2}$$

$$v_2 = v_1 + e(u_1 - u_2)$$

$$\therefore m_1u_1 + m_2u_2 = m_1v_1 + m_2[v_1 + eu_1 - eu_2]$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_1 + m_2eu_1 - m_2eu_2$$

$$(m_1 + m_2)v_1 = m_1u_1 - m_2eu_1 + m_2u_2 + m_2eu_2$$

$$(m_1 + m_2)v_1 = (m_1 - em_2)u_1 + m_2(1+e)u_2$$

$$\therefore \boxed{v_1 = \frac{(m_1 - em_2)}{m_1 + m_2}u_1 + \frac{(1+e)m_2u_2}{m_1 + m_2}}$$

Similarly $v_1 = v_2 - e(u_1 - u_2)$

And we get $\boxed{v_2 = \frac{(m_2 - em_1)u_2}{m_1 + m_2} + \frac{(1+e)m_1u_1}{m_1 + m_2}}$

hence for elastic collision e = 1

$$\therefore \boxed{\begin{array}{l} v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \frac{2m_2u_2}{m_1 + m_2} \\ v_2 = \frac{(m_2 - m_1)u_2}{m_1 + m_2} + \frac{2m_1u_1}{m_1 + m_2} \end{array}}$$

Special Cases

(i) $m_1 = m_2 = m$ and $u_2 = 0, u = u$

(A body collides with body of equal mass at rest)

$$v_1 = \frac{m - m}{2m}.u + \frac{2m}{m} \times 0 = 0$$

$$v_2 = \frac{m - m}{2m}.0 + \frac{2m}{2m}u = u$$

Body 1 comes to rest and body 2 takes off at the same limited speed of 1.

(ii) $m_1 = m_2 = m$ and u_1 = u, u_2 = -u

$$v_1 = \frac{m - m}{2m} \times u + \frac{2m}{2m}(-u) = -u$$

$$v_2 = \left(\frac{m-m}{2m}\right)(-u) + \frac{2m}{2m}.u = u$$

Both turn back with what ever velocities they came.

(iii) $m_1 = m_2 = m,\ u_1 = u,\ u_2 = u_2$ and negative

$$v_1 = \frac{m-m}{2m}u_1 + \frac{2m}{2m}(-u_2) = -u_2$$

$$v_2 = \frac{m-m}{2m}(-u_2) + \frac{2m}{2m}u_1 = u_1$$

Both turn back but with exchanged velocities.

(iv) $m_1 = m_2 = m,\ u_1 = u,\ u_2 = u_2$ (same sign)

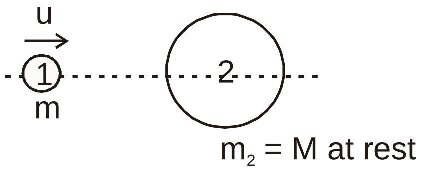
$$v_1 = \frac{m-m}{2m}.u_1 + \frac{2m}{2m}u_2 = u_2$$

$$v_2 = \frac{m-m}{2m}.u_2 + \frac{2m}{2m}u_1 = u_1$$

They simply exchange their velocities.

(v) $m_1 = m,\ m_2 = m >> m$

$u_1 = u,\ u_2 = 0$



$$v_1 = \frac{m-M}{m+M}.u + \frac{2M}{m+M}.0 \approx \frac{-M}{M}u + 0 \approx -u$$

Body 1 turned back with approximately same speed.

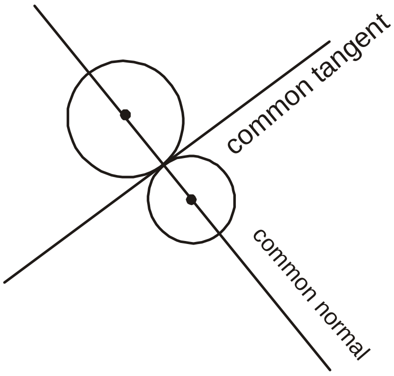
Oblique collisions in case of smooth surface

Common Normal

Force is excerted in common normal direction only. Both bodies excert equal and opposite forces (action and reaction) on each other.

Hence, momentum and velocities change accordingly in CN direction.

Common Tangent (CT)



F = 0 (in case of smooth surface)

$F = \mu N$ (in case of rough surface)

Neither momentum nor velocity changes in CT direction.

Collisions in two Dimensions

1) A pair of equal and opposite impulses acts along common normal direction. Hence linear momentum of individual particles do change along common normal direction.

2) No component of impulse acts along common tangent direction. Hence linear momentum or linear velocity of individual particles remains unchanged along this direction.

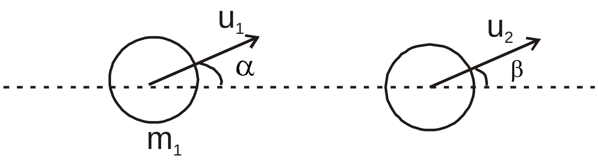
3) Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remains conserved before and after collision in any direction.

4) Co-efficient of restitution can be along common normal direction.

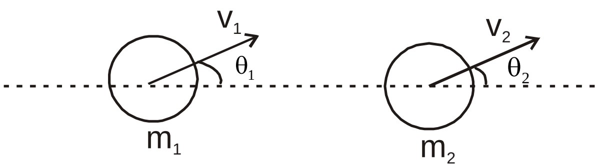
ie along common normal direction

Relative speed of separation = e × relative speed of approach

If the velocities of colliding masses are not linear, then it is known as oblique collision.



Before input



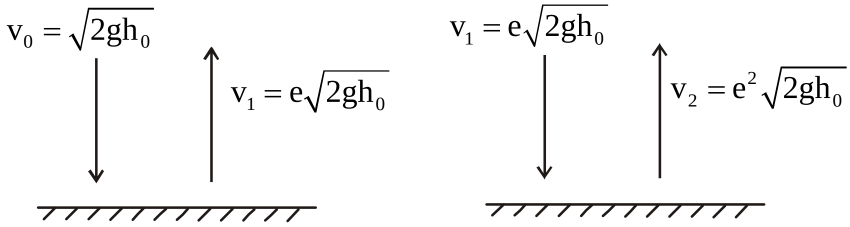
After input

Along X-axis $m_1u_1\cos\alpha + m_2u_2\cos\beta = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$

Along Y-axis $m_1u_1\sin\alpha + m_2u_2\sin\beta = m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2$

Rebounding of ball after collision with ground

A ball is dropped from a height ho on an inelastic floor.



After first collision After second collision

The ball strikes the ground for the first time with velocity $\sqrt{2gh_0}$ after time $t_0 = \sqrt{\frac{2h_0}{g}}$. The ball returns with the same velocity to the ground. Rebound velocity after second impact is $e^2\sqrt{2gh_0}$.

Thus, velocity after nth impact, $\boxed{v_n = e^n\sqrt{2gh_0}}$

Time of flight after first impact $t_1 = \frac{2v_1}{g} = \frac{2e\sqrt{2gh_0}\,g}{}$

Time of flight after nth impact $\boxed{t_n = \frac{2v_n}{g} = \frac{2e^n\sqrt{2gh_0}}{g}}$

Total time of flight $= t_0 + t_1 + ...t_n$

$$= t_0 + \frac{2e\sqrt{2gh_0}}{g} + \frac{2e^2\sqrt{2gh_0}}{g} + ... + \frac{2e^n\sqrt{2gh_0}}{g}$$

$$= \sqrt{\frac{2h_0}{g}}\left[1 + 2e\left(1 + e + e^2 + ...\right)\right]$$

For large no of collisions we can apply sum of geometric stries with n tends to infinity.

$$= \sqrt{\frac{2h_0}{g}}\left[1 + 2e\left(\frac{1}{1-e}\right)\right]$$

$$\boxed{T = \sqrt{\frac{2h_0}{g}}\left[\frac{1+e}{1-e}\right]}$$