CHAPTER - 11

MATRICES AND DETERMINANTS

JEE MAIN - SECTION I

1. 1 Put $\lambda = 0$, $\lambda = 1$, $\lambda = -1$

$$\Rightarrow \lambda = 0 \Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = C \qquad .....(1)$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} = A + B + C \qquad .... (2)$$

2. 3 $A^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ sub in $A^2 + aA + bI = 0$

3. 1 $A^2 = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 12 & 12 \\ 12 & 12 & 12 \end{bmatrix}$, $A^3 = \begin{bmatrix} 72 & 72 & 72 \\ 72 & 72 & 72 \\ 72 & 72 & 72 \end{bmatrix} \Rightarrow A^3 - 35A = A$

4. 2 $A^2 = \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix}$, Now $f(A) = A^2 - 3A + 7$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix}$$

$$\therefore f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. 1 $A^2 = \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = 0$

$\Rightarrow A^3 = A \cdot A^2 = 0$ and $A^n = 0$ for all $n \geq 2$

6. 1 Put $n = 3 \Rightarrow A^2 = 2A - I \Rightarrow A^3 = A(A^2) = A(2A - I) = 2A^2 - A$

$= 2(2A - I) - A = 3A - 2IA^4 = 4A - 3I \Rightarrow A^n = nA - (n-1)I$

7. 2 $AB=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix}a & 2b\\ 3a & 4b\end{pmatrix}$, $BA=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}a & 2a\\ 3b & 4b\end{pmatrix}$

$AB = BA \Rightarrow a = b$.

8. 2 $P^2=\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 8 & 1 & 0\\ 48 & 8 & 1\end{bmatrix}$, $P^3=\begin{bmatrix}1 & 0 & 0\\ 8 & 1 & 0\\ 48 & 8 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 12 & 1 & 0\\ 96 & 12 & 1\end{bmatrix}$

$$\therefore\ P^n=\begin{bmatrix}1 & 0 & 0\\ 4n & 1 & 0\\ 8(n^2+n) & 4n & 1\end{bmatrix}, \therefore\ P^{50}=\begin{bmatrix}1 & 0 & 0\\ 200 & 1 & 0\\ 8\times 50(52) & 200 & 1\end{bmatrix} \Rightarrow P^{50}-Q=I$$

$\therefore$ Equating we get $q_{21}=200$

$\Rightarrow q_{31}=400\times 51 \Rightarrow q_{32}=200$

$$\Rightarrow \frac{q_{31}+q_{32}}{q_{21}}=\frac{400\times 51+200}{200}=2(51)+1=103$$

9. 4 For infinite many solutions

$D=D_1=D_2=D_3=0$

Now, $D=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & \lambda\end{vmatrix}=0$

$1\cdot(2\lambda-9)-1\cdot(\lambda-3)+1\cdot(3-2)=0$

$\therefore\ \lambda=5$

Now, $D_1=\begin{vmatrix}2 & 1 & 1\\ 5 & 2 & 3\\ \mu & 3 & 5\end{vmatrix}=0$

$2(10-9)-1(25-3\mu)+1(15-2\mu)=0$,

$\mu=8$.

10. 4 $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1(1+1)+k(-k+1)-(k+1)=0$

11. 4 If A is non-singular matrix of order $n \times n$, then $|\text{adj } A| = |A|^{n-1}$

Hence, statement-I is false and statement-II is true

12. 4 $\therefore \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(1+\lambda)^2 - 1 - \lambda - 1 - \lambda = 0.$

$\Rightarrow \lambda^3 + \lambda^2 + \lambda + 1 = 0$

$\Rightarrow A^3 + A^2 + A + I = 0$

$\Rightarrow A^3 + A^2 + A = -I$

Statement-I is false but statement-II is true.

13. 2 $|A^{2015} - 6A^{2014}| = |A|^{2014} |A - 6I| = 2^{2014} \begin{vmatrix} 0 & 11 \\ 2 & -2 \end{vmatrix} = (-22)2^{2014} = -11(2)^{2015}$

14. 2 Given, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$; Operating

15. 1 $A(\text{adj } A) = |A| I = 6I = 2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

16. 1 Since, $a_1, a_2, \ldots, a_n$ are in G.P., $a_n = a_1 r^{n-1}$

$\Rightarrow \log a_n = \log a_1 + (n-1)\log r, a_{n+1} = a_1 r^n$

$\Rightarrow \log a_{n+1} = \log a_1 + n \log r \Rightarrow a_{n+2} = a_1 r^{n+1}$

$\Rightarrow \log a_{n+2} = \log a_1 + (n+1)\log r$ and so on

17. 3 Using $[C_1 \to C_1 + C_2 + C_3]$

$$\Delta = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix} = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$.

18. 4

$$A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix} (b > 0)$$

$$|A| = 2(2b^2 + 2 - b^2) - b(2b - b) + 1(b^2 - b^2 - 1)$$

$$|A| = 2(b^2 + 2) - b^2 - 1$$

$$|A| = b^2 + 3$$

$$\frac{|A|}{b} = b + \frac{3}{b} \Rightarrow \frac{b + \frac{3}{b}}{2} \geq \sqrt{3}$$

$$\therefore\ b + \frac{3}{b} \geq 2\sqrt{3}$$

19. 4

$$f(x) = \begin{vmatrix} 1 + (a^2 + b^2 + c^2 + 2)x & (1+b^2)x & (1+c^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & 1 + b^2x & (1+c^2)x \\ 1 + (a^2 + b^2 + c^2 + 2)x & (1+b^2)x & (1+c^2)x \end{vmatrix} \text{[Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\therefore\ f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} (\because a^2 + b^2 + c^2 + 2 = 0) = (x-1)^2$$. Hence, degree is 2.

20. 1 Statement-II is always true for statement-I

$$\cos\left(x + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \sin\left(\frac{\pi}{4} - x\right) = -\sin\left(x - \frac{\pi}{4}\right).$$

$$\cot\left(\frac{\pi}{4} + x\right) = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - x\right)\right) = \tan\left(\frac{\pi}{4} - x\right) = -\tan\left(x - \frac{\pi}{4}\right).$$

Also, $\ln\left(\frac{y}{x}\right) = -\ln\left(\frac{x}{y}\right)$.

Therefore, determinant given in statement-I is skew-symmetric and hence its value is zero. Hence, both statements are true and statement-II is a correct explanation of Statement-I

SECTION II (NUMERICAL)

21. 2020 $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B=\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$,

AB = B

$\Rightarrow (A-I)B=O \Rightarrow |A-I|=O$

Since $B \neq O$

$$\begin{vmatrix} (a-1) & b \\ c & (d-1) \end{vmatrix}=0$$

$ad-bc=2020$.

22. 6 $P=\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$$5I-8P=\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}-\begin{bmatrix} 16 & -8 \\ 40 & -24 \end{bmatrix}=\begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}.$$

$$P^2=\begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix},\ P^3=\begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix},\ P^6=\begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}=P^n$$

$\Rightarrow n=6$.

23. 180 $\Delta=\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 2x-3 & 3x-4 & 4x-5 \\ 3x-5 & 5x-8 & 10x-17 \end{vmatrix}=Ax^3+Bx^2+Cx+D$. $[R_2 \to R_2-R_1, R_3 \to R_3-R_2]$

$$\Delta=\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-1 & x-1 & x-1 \\ x-2 & 2(x-2) & 6(x-2) \end{vmatrix}=(x-1)(x-2)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

$=-3(x-1)^2(x-2)=-3x^3+12x^2-15x+6$

$\therefore\ B \times C=12 \times 15=180$

24. 2 $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10 \Rightarrow \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$

$\Rightarrow x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$;

25. 1 $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$.Hence. det A= $\sec^2 x$

$\therefore\ \det A^T = \sec^2 x$

Now, $f(x) = \det(A^T A^{-1}) = (\det A^T)(\det A^{-1})$

$= (\det A^T)(\det A)^{-1} = \dfrac{(\det A^T)}{(\det A)} = 1$.

Hence, $f(x) = 1$

JEE ADVANCED LEVEL

SECTION III

26. A $Q^{2012} = (PAP^T)(PAP^T).......(PAP^T)(2012\,times)$

$= PAAP^T .... = PA^{2012}P^T$

$P^T PA^{2012} P^T P = A^{2012}$ But

$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\therefore A^{2012} = \begin{bmatrix} 1 & 2012 \\ 0 & 1 \end{bmatrix}$

$\therefore \alpha = 1, \beta = 0$

27. A We have, $A^2 = O$, $A^k = O$, $\forall k \ge 2$

Thus, $(A+I)^{50} = I + 50A \Rightarrow a = 1, b = 0$, $c = 0, d = 1$

28. B $\sum_{k=0}^{n-1} 3^k = 1 + 3 + 3^2 + ... + 3^{n-1} = \dfrac{3^n - 1}{3-1} = \dfrac{3^n - 1}{2}$

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)(k+2)}=\sum_{k=0}^{n-1}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\frac{n}{n+1}$$

$$\sum_{k=0}^{n-1}\cos(k+1)\theta=1+\cos\theta+....+\cos(n-1)\theta$$

$$=\mathrm{Re}\left[1+e^{i\theta}+e^{i2\theta}+.....+e^{i(n-1)\theta}\right] \qquad =\mathrm{Re}\left(\frac{1-e^{in\theta}}{1-e^{i\theta}}\right)=\mathrm{Re}\left(\frac{(1-e^{in\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)$$

$$=\mathrm{Re}\left[\frac{1-e^{-i\theta}-e^{in\theta}+e^{i(n-1)\theta}}{2-(e^{i\theta}+e^{i\theta})}\right]=\frac{1-\cos\theta-\cos\theta+\cos(n-1)\theta}{2(1-\cos\theta)}$$

$$=\frac{2\sin^2\frac{\theta}{2}+2\sin\left(n-\frac{1}{2}\right)\theta\sin\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}+\sin\left(n-\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}}=\frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}$$

29. D $x=Ap^{R-1}, y=Aq^{R-1}, z=Ar^{R-1}$

$\log x=\log A+(R-1)\log p$

$\log y=\log A+(R-1)\log q$

$\log z=\log A+(R-1)\log r$ and substitute

30. C The given system is consistent. Therefore, $\Delta=\begin{vmatrix}1 & 1 & -1\\ 2 & -1 & -c\\ -b & 3b & -c\end{vmatrix}=0$

or $c+bc-6b+b+2c+3bc=0$ or $3c+4bc-5b=0$ or $c=\dfrac{5b}{4b+3}$

Now, $c<1 \Rightarrow \dfrac{5b}{4b+3}<1$ or $\dfrac{5b}{4b+3}-1<0$ or $\dfrac{b-3}{4b+3}<0 \Rightarrow b\in\left(-\dfrac{3}{4},3\right)$

SECTION IV (More than one correct)

31. AD $\det(adJ\ P)=(\det P)^2 \Rightarrow \det P=2\ or -2$

32. ABC $A^2=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} \Rightarrow A^3=\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 6 & 1\end{bmatrix}$

$A^{2012}=\begin{bmatrix}1 & 0\\ 4024 & 1\end{bmatrix} \Rightarrow a=d, a+b+c+d=4026$

33. A,B $A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$

34. A,B,C $f(x)=2+\sin 2x$ (by simplifying)& $\alpha=3$ (maximum value of $f(x)$)

$\beta=1$ (minimum value of $f(x)$)

35. C $a+b+c=3a^2bc^2 \Rightarrow abc(ab+bc+ca)=3a^2bc^2 \Rightarrow \frac{1}{c}+\frac{1}{a}=\frac{2}{b}$

SECTION V - (Numerical type)

36. 21 $x+3y+\lambda z-\mu = p(x+y+z-5)+q(x+2y+2z-6)$ on comparing the coefficient,

$p+q=1$ and $p+2q=3$

$\Rightarrow (p,q)=(-1,2)$

Hence, $x+3y+\lambda z-\mu = x+3y+3z-7$

$\Rightarrow \lambda=3, \mu=7$

37. 6 $\frac{D}{(10!)^3}-4$

(10)(2.10² + 8n + 10)

Hence $\frac{D}{(10!)^3}-4=2900$

38. 4 $|A|=(2k+1)^3, |B|=0$

But det (adjA)+det (adj B) = 10^6

$\Rightarrow (2k+1)^6=10^6 \Rightarrow k=\frac{9}{2} \Rightarrow [k]=4$

39. 0 $\begin{vmatrix}\lambda & 1 & 1\\ -1 & \lambda & 1\\ -1 & -1 & \lambda\end{vmatrix}=0 \Rightarrow \lambda=0$

SECTION VI - (Matrix match type)

40. A $A\rightarrow S; B\rightarrow R; C\rightarrow Q; D\rightarrow P$

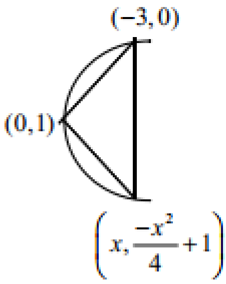
$4a^2f(-1)+4af(1)+f(2)=3a^2+3a$

$4b^2f(-1)+4bf(1)+f(2)=3b^2+3b$

$$4f(-1)(a^2-b^2)+4f(1)(a-b)$$
$$=3(a^2-b^2)+3(a-b)$$
$$4f(-1)(a+b)+4f(1)=3(a+b)+3$$
$$4f(-1)=3 \qquad 4f(1)=3$$
$$f(-1)=\frac{3}{4}, \qquad f(1)=\frac{3}{4}, \qquad f(2)=0$$
$$f(2)=0$$
$$4a+2b+c=0$$
$$a-b+c=\frac{3}{4}$$
$$a+b+c=\frac{3}{4}$$

$$2b=0 \Rightarrow b=0$$
$$4a+c=0$$
$$a+c=\frac{3}{4}$$

$$3a=-\frac{3}{4} \Rightarrow a=-\frac{1}{4}$$
$$c=\frac{3}{4}+\frac{1}{4}=1$$
$$f(x)=-\frac{1}{4}x^2+1$$
$$(\alpha,\beta)=(0,1) \quad A=(-2,0) \Rightarrow P=-2$$



$$\frac{-\frac{x^2}{4}}{x}\times\frac{1}{2}=-1 \Rightarrow x=8 \quad B(8,-15)$$
$$y-0=-\frac{3}{2}(x+2)$$
$$\Rightarrow 2y=-3x-6 \Rightarrow y=-\frac{3}{2}x-3 \Rightarrow 3x+2y+6=0$$
$$y_1-y_2=-\frac{x^2}{4}+1+\frac{3}{2}x+3=-\frac{x^2}{4}+\frac{3x}{2}+4$$
$$\Delta=\frac{\left(\frac{9}{4}-4\times-\frac{1}{4}\times 4\right)^{3/2}}{6\times\frac{1}{16}}=\frac{125}{8}\times\frac{8}{3}$$