CHAPTER - 14
APPLICATION OF DIFFERENTIATION

JEE MAIN - SECTION I

1. 2

$$\frac{dV}{dt} = 40$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{10}{\pi r^2}$$

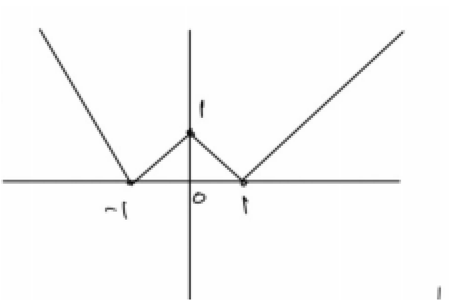
$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$= 8\pi \times 8 \times \frac{10}{\pi \times 8 \times 8}$$

$$= \underline{\underline{10 \text{ cm}^2/s}}$$

2. 1



3. 1

$$(1)\ h'(x) = f'(x) - 2f(x)f'(x) + 3[f(x)]^2 f'(x)$$

$$= f'(x)\underbrace{\left(3(f(x))^2 - 2f(x) + 1\right)}_{+ve}$$

$h(x)\ \uparrow$ when $f(x)\ \uparrow$

4. 2

By Rolle's Theorem in $[0, \alpha]$

5. 1

Let nearest point be $P(x, x^2+2)$

Nearest Distance $= \sqrt{(x-3)^2 + x^4}$

Let $f = (x-3)^2 + x^4$

$f' = 0$ & $f'' > 0$ at $x = 1$

$\therefore$ Nearest Distance $= \sqrt{5}$

6. 1

$\alpha + \beta = a-2, \quad \alpha\beta = -(a+1)$

$\alpha^2 + \beta^2 = (a-2)^2 + 2(a+1)$

Let $f(a) = a^2 - 2a + 6$

$f'(a) = 0$ & $f''(a) > 0$ at $a = 1$

$\therefore$ minimum value $= 5$

7. 4

$$V = 5x - \frac{x^2}{6} \Rightarrow \frac{dV}{dt} = 5\frac{dx}{dt} - \frac{x}{3}.\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\frac{dV}{dt}}{\left(5 - \frac{x}{3}\right)} \Rightarrow \left(\frac{dx}{dt}\right)_{x=2} = \frac{5}{5 - \frac{2}{3}} = \frac{15}{13}\, cm/\sec .$$

8. 1

Displacements $s = -4t^2 + 2t$

Now velocity $v = -8t + 2$ and its acceleration $a = -8$

So $\left(\frac{ds}{dt}\right)_{t=1/2} = -8 \times \frac{1}{2} + 2 = -2$ and $\left(\frac{d^2s}{dt^2}\right)_{t=1/2} = -8$.

9. 1

$f(x) = x^3 - 3x^2 - 24x + 5$

For increasing, $f(x) > 0$, $3x^2 - 6x - 24 > 0$

$\Rightarrow x^2 - 2x - 8 > 0$

$x^2 - 4x + 2x - 8 > 0 \Rightarrow (x+2)(x-4) > 0$

$x \in (-\infty, -2) \cup (4, \infty)$.

10. 2 Here $\frac{f(b)-f(a)}{b-a} = f'(c)$

$\Rightarrow \frac{e^b-e^a}{b-a} = f'(c) \Rightarrow \frac{e-1}{1-0} = e^c \Rightarrow c = \log(e-1)$.

11. 3 $x = e^{2t} + 2e^{-t}, y = e^{2t} + e^t$

$At \quad t = \ln 2 \qquad x = 4 + 1 = 5, \; y = 4 + 2 = 6$

$\frac{dy}{dx} = \frac{2e^{2t}+e^t}{2e^{2t}-2e^{-t}} = \frac{8+2}{8-1} = \frac{10}{7} \Rightarrow$ *equation of tangent is* $y - 6 = \frac{10}{7}(x-5)$

$7y - 42 = 10x - 50$ *or* $10x - 7y = 8$

12. 3 $y^3 = 27x \qquad \Rightarrow 3y^2\frac{dy}{dt} = 27\frac{dx}{dt}$

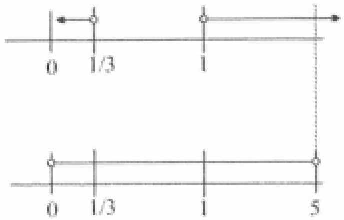
But $\frac{|dx/dt|}{|dy/dt|} < 1 \Rightarrow \frac{y^2}{9} < 1 \Rightarrow -3 < y < 3$ for $y \in (-3,3)$, $x \in (-1,1) \Rightarrow$ (C)

13. 3 Since f is defined on $(0, \infty)$

$\therefore \quad 2a^2 + a + 1 > 0$ which is True as $D < 0$

also $\quad 3a^2 - 4a + 1 > 0$

$(3a-1)(a-1) > 0 \qquad \Rightarrow$



as f is increasing hence

$f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$

$\Rightarrow \quad 2a^2 + a + 1 > 3a^2 - 4a + 1$

$0 > a^2 - 5a$

$\therefore \quad a(a-5) < 0 \Rightarrow (0, 5) \qquad \Rightarrow$

hence $\quad a \in (0, 1/3) \cup (1, 5)$ **Ans.**

14. 3

$x^4 - 10x^2 + 9 \le 0$

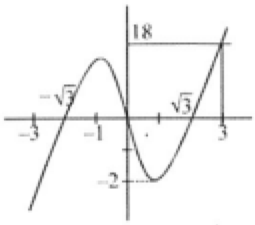
$(x^2 - 9)(x^2 - 1) \le 0$

hence $-3 \le x \le -1$ or $1 \le x \le 3$

now $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3 = 0$

$x = \pm 1$



maximum occurs when $x = 3$

$f(3) = 18$

15. 2

$x^2 + y^2 = r^2$

$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = r \cdot \frac{dr}{dt}$

$0.8 \times \frac{dx}{dt} + 0.6 \times -60 = 1 \times 20$

$\frac{dx}{dt} = 70 \text{ kms/hr}$

16. 1 Use graph

17. 3

Let thickness of ice be 'h'.

Vol. of ice $= v = \frac{4\pi}{3}((10+h)^3 - 10^3)$

$$\frac{dv}{dt} = \frac{4\pi}{3}(3(10+h)^2)\cdot\frac{dh}{dt}$$

Given, $\frac{dv}{dt} = 50cm^3/\text{min}$ and $h = 5cm$

$$\Rightarrow 50 = \frac{4\pi}{3}(3(10+5)^2)\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{50}{4\pi\times 15^2} = \frac{1}{18\pi} \text{ cm/min}$$

18. 2

Let a is first term and d is common difference then, $a+5d=2$ (given) .....(1)

$f(d) = (2-5d)(2-2d)(2-d)$

$f'(d) = 0 \Rightarrow d = \frac{2}{3}, \frac{8}{5}$

$f''(d) < 0$ at $d = 8/5$

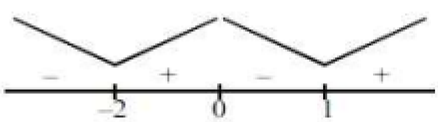
$\Rightarrow d = \frac{8}{5}$

19. 1

$f(x) = 9x^4 + 12x^3 - 36x^2 + 25$

$f'(x) = 36x^3 + 36x^2 - 72x$

$= 36x(x^2+x-2) = 36x(x-1)(x+2)$



Points of minima $= \{-2, 1\} = S_1$

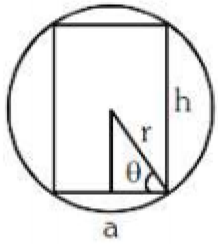
Points of maxima $= \{0\} = S_2$

20. 1 $h = 2r\sin\theta$, $a = 2r\cos\theta$,

$v = \pi(r\cos\theta)^2(2r\sin\theta)$, $v = 2\pi r^3\cos^2\theta\sin\theta$

$\frac{dv}{d\theta} = \pi r^3(-2\cos\theta\sin^2\theta + \cos^3\theta) = 0$ or $\tan\theta = \frac{1}{\sqrt{2}}$

$\because\ h = 2\times 3\times\frac{1}{\sqrt{3}} = 2\sqrt{3}$.



SECTION II (NUMERICAL)

21. 320 $f'(x) = 3x^2 + 6x(\lambda - 7) + 3(\lambda^2 - 9)$

For +ve point of maxima both roots of $f'(x) = 0$ must be +ve, have $\lambda \in (-\infty, -3)\cup\left(3, \frac{29}{7}\right)$

$\therefore \beta + 11v + 70\delta = 320$

By graphically we can obtain points of local extrema is 5

22. –3 Slope of the given line $= \frac{-3}{2}$.

First of all, we try to locate the points on the curve at which the tangent is parallel to the given line.

So, differentiating both sides with respect to x of $3x^2 - 4y^2 = 72$, we get

$\frac{dy}{dx} = \frac{3x}{4y} = \frac{-3}{2}$ (given) $\Rightarrow \frac{x}{y} = -2$

Now, $3\left(\frac{x}{y}\right)^2 - 4 = \frac{72}{y^2} \Rightarrow y^2 = 9 \Rightarrow y = -3, 3$

So, points are $(-6, 3)$ and $(6, -3)$.

Now, distance of $(-6, 3)$ from the given line $= \frac{|-18+6+1|}{\sqrt{13}} = \frac{11}{\sqrt{13}}$

and distance of $(6, -3)$ from the given line $= \frac{|18-6+1|}{\sqrt{13}} = \frac{13}{\sqrt{13}}$

$\therefore$ Clearly, the required point is $M(-6, 3) \equiv (x_0, y_0)$ (given)

So, $x_0 = -6$, $y_0 = 3$.

Hence, $(x_0 + y_0) = -6 + 3 = -3$. **Ans.**

23. 0.17 Let $P(x)$ be a polynomial of degree 5 having extremum at $x = -1, 1$ and $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$. If M and m are the maximum and minimum value of the function $y = P'(x)$ on the set $A = \{x \mid x^2 + 6 \le 5x\}$ then find $\frac{M}{m}$

Consider $P(x) = ax^5 + bx^4 + 6x^3$

$\Rightarrow \quad P'(x) = 5ax^4 + 4bx^3 + 18x^2$

Now, $\quad P'(-1) = 0$ gives $5a - 4b = -18$

and $\quad P'(1) = 0$ gives $5a + 4b = -18$

$\therefore \quad$ On solving, we get

$$a = \frac{-18}{5}, b = 0$$

Hence $P(x) = \frac{-18}{5}x^5 + 6x^3$

$\Rightarrow \quad P'(x) = -18x^4 + 18x^2 = 18(x^2 - x^4)$

and $\quad P''(x) = 18(2x - 4x^3) = 36(x - 2x^3)$

$\Rightarrow \quad P''(x) = 36x(1 - 2x^2)$

Also $\quad A = \{x \mid x^2 + 6 \le 5x\}$

gives $\quad x \in [2, 3]$

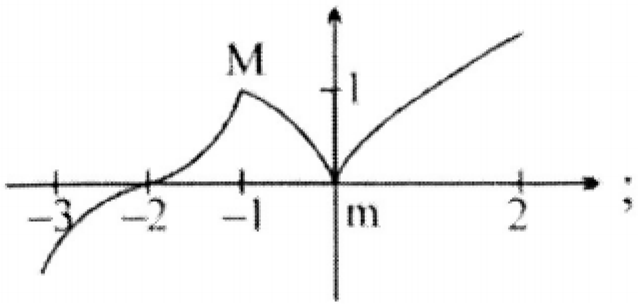
Clearly $P''(x) < 0 \; \forall \; x \in [2, 3]$

So, $\quad y = P'(x)$ is decreasing function in $[2, 3]$

$\therefore \quad M = P'_{max}(x = 2) = 18(4 - 16) = -18 \times 12$

and $\quad m = P'_{min}(x = 3) = 18(9 - 81) = -18 \times 72$

24. 1.5 Graph of f(x)



$x = -1$ maxima and $x = 0$ minima

$x = -1, 0$ are non differentiable points

$$A + \; x = -2, \frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0 \therefore \text{ Inflexion at } x = -2$$

25. 5049 if $b = 1$

$f(x) = 8x^3 + 4ax^2 + 2x + a$

$f'(x) = 24x^2 + 8ax + 2$ or $2(12x^2 + 4ax + 1)$

for non monotonic $f'(x) = 0$ must have distinct roots

hence $D > 0$ i.e. $16a^2 - 48 > 0 \Rightarrow a^2 > 3;$ $\therefore$ $a > \sqrt{3}$ or $a < -\sqrt{3}$

$\therefore$ $a \in 2, 3, 4, ...$

sum $= 5050 - 1 = 5049$ Ans.

JEE ADVANCED LEVEL

SECTION III

26. D Consider a tangent common to both the curves $y = \frac{x^2}{3}$ and the circle whose centre at (15,-3) at $P(x_1, y_1)$ $\therefore$ slope of the tangent at $P(x_1, y_1) = \frac{dy}{dx}|_p = \frac{2x_1}{3}$

$\therefore$ slope of the normal at $P(x_1, y_1)$ is $= \frac{-3}{2x_1}$

$\therefore \frac{-3}{2x_1} = \frac{y_1 + 3}{x_1 - 15} \Rightarrow 2x_1y_1 + 9x_1 - 45 = 0$

$\Rightarrow 2x_1^3 + 27x_1 - 135 = 0$

$\Rightarrow x_1 = 3 \ \& \ y_1 = 3 \Rightarrow$ radius $= 6\sqrt{5}$

27. A Given curve is $y = \sin x$

Let the tangent to the curve at $P(\alpha, \beta)$ be $y - \beta = \cos\alpha(x - \alpha)$....(1)

since (1) passes through (0,0), $-\beta = \cos\alpha(-\alpha)$

ie, $\cos\alpha = \frac{\beta}{\alpha}$....(2)

since P lies on $y - \sin x, \beta = \sin\alpha \rightarrow (3)$

$(2)^2 + (3)^2 \Rightarrow 1 = \frac{\beta^2}{\alpha^2} + \beta^2 \Rightarrow (\alpha, \beta)$ lies on $\frac{1}{x^2} - \frac{1}{y^2} + 1 = 0$

28. C $\frac{dy}{dx} = -\left(\frac{3x+2y}{2x+5y}\right) \Rightarrow \frac{dy}{dx}|_P = 0 \& \frac{dy}{dx}|_Q = \alpha$

$\Rightarrow$ Tangenets at P & Q one $\perp^r$ to each other

29. A From the question, $\left|\frac{dx}{dt}\right| > \left|\frac{dy}{dt}\right| \Rightarrow \left|\frac{dx}{dy}\right| > 1$. Differentiating $x^3 = 12y$ w.r.t. y, we get

$\Rightarrow 3x^2\frac{dx}{dy} = 12 \quad \Rightarrow \frac{dx}{dy} = \frac{4}{x^2} \quad \therefore \frac{4}{x^2} > 1 \quad \left(\because \frac{dx}{dy} > 1\right)$

$\Rightarrow x^2 - 4 < 0 \quad \Rightarrow -2 < x < 2$

30. C Let $y = \cos x \quad \Rightarrow \frac{dy}{dx} = -\sin x$

Now, $\cos 60° 2' = \cos 60° + \Delta y$

$$\Delta y = \left(\frac{dy}{dx}\right)_{x=60°} . \Delta x = -\frac{\sqrt{3}}{2}.\Delta x = -\frac{\sqrt{3}}{2}.1' = -\frac{\sqrt{3}}{2} \times \frac{2\alpha}{60}$$

$\therefore \cos 60° 2' = \frac{1}{2} - \frac{\alpha\sqrt{3}}{60}$

31. C $f'(x) = -(1+3x^2) \to f(x)$ decreasing

Then $1 - f(x) - f^3(x) > f(1-5x) \to f(f(x)) > f(1-5x)$

$\Rightarrow 1 - x - x^3 > 1 - 5x$

$\Rightarrow x^3 - 4x > 0 \Rightarrow x \in (-2,0) \cup (2,\infty)$

32. C Statement 1 is true & statement -2 is true

SECTION IV (More than one correct)

33. A,C,D Using graph of f(x) and using Leibnitz Rule

34. A,B,C,D $f'(x) = (x^2 - x + 2)(x+3)(x+2)(x+1)(x-2)(x-3)(x-4)$

since $f'(-2) = 0 \Rightarrow x + 2 = 0$ in the equation of normal at $x = -2$

Also f(x) has local maximum at

$x = -3, -1, 3 \Rightarrow$ sum $= -1$

35. A, C $f'(x) = 3x^2 + 2ax + b + 5\sin 2x$

f(x) increases always, so $f'(x) > 0\ \forall\ x \in R$

$\Rightarrow 3x^2 + 2ax + b + 5\sin 2x > 0$

which will be true if $3x^2 + 2ax + b - 5 > 0$, always if $D < 0$

SECTION V - (Numerical type)

36. 1 Let $f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3(x+1)(x-1)$

$\therefore f(x)$ is increasing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$

Since $f(-2)f(-1) < 0 \Rightarrow$ one root lies in $(-2, -1)$

$f(0)f(1) < 0 \Rightarrow$ one root lies in (0,1)

$f(1)f(2) < 0 \Rightarrow$ one root lies in $(1, 2)$

$\Rightarrow [x_1] + [x_2] + [x_3] = -1$, where x_1, x_2, x_3 are the roots of $f(x) = 0$

$\{x_1\} + \{x_2\} + \{x_3\} = 1$, since $x_1 + x_2 + x_3 = 0$

37. 5 Since $f'(x) > 0, \forall x \in R, f(x)$ is increasing function

Now, $f\left(f\left(f(x) - 2x^3\right)\right) \ge f\left(f\left(2x^3 - f(x)\right)\right)$ (given)

$\Rightarrow f(x) \ge 2x^3 \Rightarrow 7x^2 - 26x - 8 \le 0 \Rightarrow x \in \left[\frac{-2}{7}, 4\right]$

38. 12 $g(x) = \frac{d}{dx}\left(f'(x).f''(2)\right)$

Also $f'(x) = -f'(6-x)$

$f'(0) = f'(6) = f'(u) = f'(5) = f'(1) = f'(3) = 0$

$f'(x)$ has atleast 7 roots

$\therefore f''(x)$ has atleast 6 roots

$\Rightarrow g(x)$ has atleast 12 roots

39. 2.83

Let $y_1 = \sqrt{2-x_1^2}$ *and* $y_2 = \frac{9}{x_2} \Rightarrow x_1^2 + y_1^2 = 2$ *and* $x_2\ y_2 = 9$

Hence given expression represents the distance between points $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ lying on the curves $x^2 + y^2 = 2$ and $xy = 9$ respectively in the first quadrant.

Thus in order to find the least value of the given expression we must find the least distanc between the indicated curves.

For $xy = 9$, $\frac{dy}{dx} = -\frac{y}{x} = -\frac{9}{x^2}$.

Hence slope of normal to $xy = 9$ at $P_2(x_2, y_2)$ *is* $\frac{x_2^2}{9}$ and the equation of normal at P_2 is;

$(y - y_2) = \frac{x_2^2}{9}(x - x_2.)$

It must pass through the origin (as we are interested in common normal)

$\Rightarrow 0 - \frac{9}{x_2} = \frac{x_2^2}{9}(0 - x_2) \Rightarrow x_2^4 = 81$

$\Rightarrow\ x_2 = 3 \Rightarrow y_2 = 3.$

Thus least distance between the curves is $\sqrt{9+9} - \sqrt{2} = 2\sqrt{2}.$

SECTION VI - (Matrix match type)

40. A-(Q); B(QS); C-(QRS); D-(T)

$$g(x) = \begin{cases} f(x) & -2 \le x < -1 \\ f(-1) & -1 \le x < 0 \\ f(0) & 0 \le x < 1 \\ f(x) & 1 \le x \le 3 \end{cases}$$

$$f(x) = \begin{cases} x^2 + 2x & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ x^2 - 2x & 1 \le x \le 3 \end{cases}$$

(a) f(x) not continuous at x = 0

(b) g(x) not continuous at x = 0, 1 and not differentiative at 0, 1.

(c) No point exist for local extrema

(d) Absolute maxima occurs at x = 3