COMPLEX NUMBERS

1. Imaginary number - square root of a –ve number

2. $$\left.\begin{array}{r} i^{4n} = 1 \\ i^{4n+1} = i = i^{4n-3} \\ i^{4n+2} = -1 = i^{4n-2} \\ i^{4n+3} = -i = i^{4n-1} \end{array}\right\} \begin{array}{l} i^{n} = i^{r}, \\ i^{-n} = i^{4-r} \end{array} \quad \begin{array}{c} \text{where r is} \\ \text{the remainder} \\ \text{obtained when} \\ \text{n is divided by 4} \end{array}$$

3. **Complex number**

   A number of the form $a + ib$, where $a, b \in R$

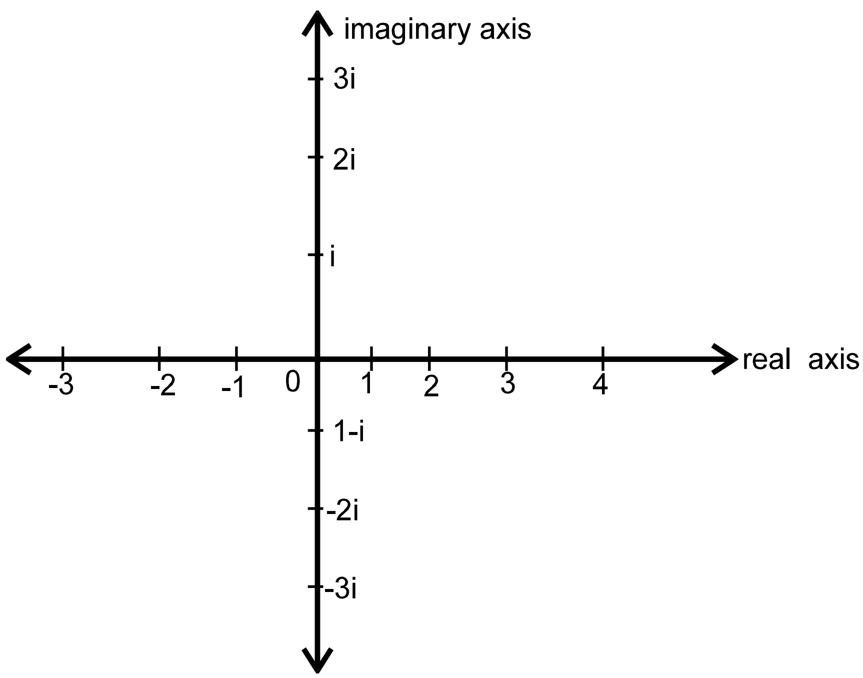
   Real part of z, $\mathrm{Re}(z) = a$ &

   Imaginary part of z, Im(z) = b

   Note : $\mathrm{Re}(z) = 0 \Leftrightarrow$ is purely imaginary CN

   Im(z) = 0 $\Leftrightarrow$ z is purely real CN

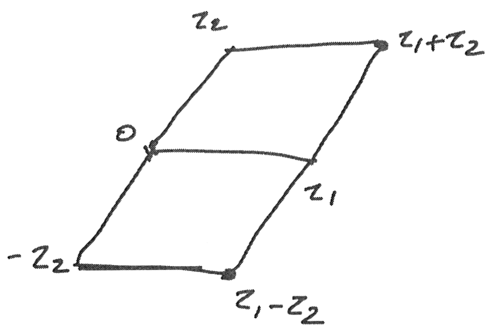
4 **Argand plane** (Complex plane)



5. **Set of complex numbers (C)**

$C = \left\{x + iy / x.y \in R, \sqrt{-1} = i\right\}$ & $C \supset R$

6. **Geometrical interpretation of complex addtion & subtraction**



7. **Conjugate of a complex number**

Congugate of $z = x + iy$ is $\bar{z} = x = iy$, which is obtained by replacing i by –i. Geometrically conjugate represents the refleciton of z about real axis

Properties of conjugate

1. $\overline{\left(\bar{z}\right)} = z$
2. $\left(\bar{z}\right) = z \Leftrightarrow z$ is purely real CN
3. $\left(\bar{z}\right) = -z \Leftrightarrow z$ is purely imaginary CN
4. $z + \bar{z} = 2\operatorname{Re}(z)$
5. $z - \bar{z} = 2i\ 1m(z)$
6. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
7. $\overline{nz} = n\bar{z}$
8. $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$
9. $\overline{z^n} = \left(\bar{z}\right)^n$
10. $\left(\frac{\bar{z}_1}{z_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$
11. $z\bar{z} = \left(\operatorname{Re}(z)\right)^2 + \left(1m(z)\right)^2$
12. $\overline{az_1 + bz_2} = a\bar{z}_1 + b\bar{z}_2$ where $a, b \in R$

8. Modulus of a complex number (Magnitude)

Let $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$, which is a non-negative real value. Geometrically, it represents the distance taken form origin

Properties of modulus

1. $z\bar{z} = |z|^2$
2. $|z| = 0 \Leftrightarrow z = 0$
3. $|z| = |\bar{z}|$
4. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\left(z_1\bar{z}_2\right)$
5. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}\left(z_1\bar{z}_2\right)$
6. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$
7. $|z_1 z_2| = |z_1|.|z_2|$
8. $|z^n| = |z|^n$
9. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, |z_2| \neq 0$
10. $\left||z_1| - |z_2|\right| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
11. $\left||z_1| - |z_1|\right| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
12. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$
13. If $|z| = 1 \Rightarrow \frac{1}{Z} = \bar{z}$

9. Distance formula

Distance between two complex numbers z_1 & z_2 in complex plane is $|z_1 - z_2|$

10. $|z - z_1| = r$, represents a circle with centre z_1 & radius r

11. $|z - z_1| \leq r$, represents the interior and the circumference of a circle with centre z_1 and radius r

12. $|z - z_1| \geq r$, represents the exterior and the circumference of a circle with centre z_1 and radius r

13. $\left|\frac{z - z_1}{z - z_2}\right| = 1$, here locus of z represents the $\perp^r$ bisector of the line joining the 2 fixed points z_1 & z_2

14. Locus of z of $\left|\frac{z-z_1}{z-z_2}\right|=\lambda$, where $\lambda \neq 0,1$ is a circle

15. General equation of a circle is $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+c=0$ where centre is '$-\alpha$' and radius is $\sqrt{|\alpha|^2-c}$

16. Locus of z of $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$ is a circle

17. Locus of z of $|z-z_1|+|z-z_2|=2a$

1) Where $|z_1-z_2|<2a$, is an ellipse with foci z_1 & z_2 and length of major axis 2a

2) Where $|z_1-z_2|=2a$, is a line segment joining z_1 & z_2

3) Where $|z_1-z_2|>2a$, represents no locus

18. Locus of z of $\big||z-z_1|-|z-z_2|\big|=2a$

1) Where $|z_1-z_2|>2a$ is a hyperbola with foci z_1 & z_2 and length of transverse axis 2a

2) Where $|z_1-z_2|=2a$, represents two opposite open rays with end points z_1 & z_2

3) Where $|z_1-z_2|<2a$, represents no locus

19. Argument or amplitude of a complex number z is the angle made by ray z in anticlockwise direction about the origin from the +ve direction of real axis and it is denoted by arg(z) or amp (z)

20. arg z is +ve if the rotation is in anti-clockwise direction and it is –ve if the rotation is in clockwise direction

21. Principal argument of z lies in $(-\pi,\pi]$ is, $-\pi<\arg Z\leq\pi$

22. Argument of a complex number z is

$=\alpha,$ if z in $\mathrm{I^{st}}$ quadarnt

$=\pi-\alpha,$ if z in 2^{nd} quadrant

$=-(\pi-\alpha)$ if z in 3^{rd} quadrant

$=-\alpha$ if z in 4^{th} quadrant

where $\alpha=\tan^{-1}\left(\left|\frac{1m(z)}{re(z)}\right|\right)$

23. Argument of z is

$=0,$ if z lies on +ve real axis

$=\pi,$ if z lies on –ve real axis

$=\frac{\pi}{2}$ if z lies on +ve imaginary axis

$=\frac{-\pi}{2}$ if z lies on –ve imaginary axis

24. Arg z is not defined when z = 0

25. Polar form of complex number $z = r(\cos\theta + i\ \sin\theta)$, where $r = |z|\ \&\ \theta = \arg(z)$

26. Polar form of a complex numbers z in the product of a non negative real number and a unimodular complex number $\cos\theta + i\sin\theta$

27. Eules function is $e^{i\theta} = \cos\theta + \sin\theta$

28. Eulerian form of a complex number z is $z = re^{i\theta}$, where $r = |z|\ \&\ \theta = \arg(z)$

29. $e^{i\theta} = \cos\theta + i\sin\theta$

$\overline{e}^{i\theta} = \cos\theta - i\sin\theta$

$e^{i\pi/2} = i,\ e^{-i\pi/2} = -i,\ e^{i0} = 1,\ e^{i\pi} = -1$

30. **Properties of arguments**

1) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

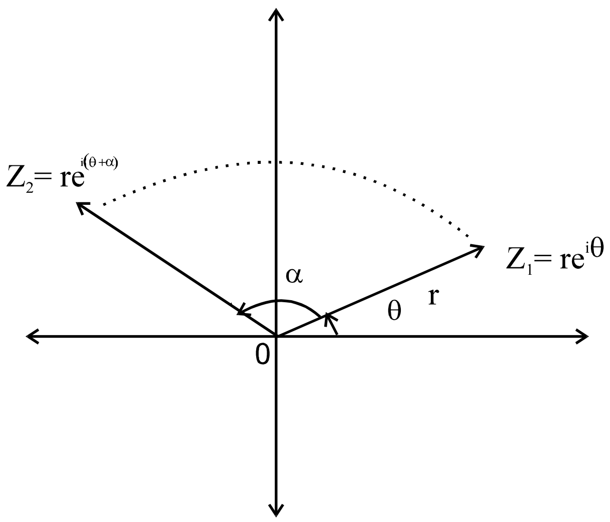
2) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

3) $\arg(z^n) = \cap\arg(z)$

4) $\arg(\overline{z}) = -\arg(z)$

5) $\arg\left(\dfrac{1}{z}\right) = -\arg(z)$

31. Circular Rotation in complex plane about origin



When $z_1 = re^{i\theta}$ rotates an angle α about the origin in anticlockwise direction, then $z_2 = z_1 e^{i\alpha}$ and if the rotation is in clockwise direction then $z_2 = z_1 e^{-i\alpha}$

Some particular cases

Let $z = re^{i\alpha}$

1) $kz = kr\ e^{i\theta}$, where k is +ve, then z streches k times in the direction of z.

2) $kz = kr\ e^{i\theta}$, where k is –ve, the z streches k times in the direction opposite to z.

3. Multiplying by i

$$iz = ire^{i\theta} = re^{i\theta}.e^{i\pi/2} = re^{i\left(\theta+\pi/2\right)}$$

ie; z rotates an angle $\pi/2$ in anticlockwise direction about the origin.

4. Division by i (Multiplying by –i)

$$\frac{z}{i} = \frac{re^{i\theta}}{i} = ire^{i\theta} = re^{i\theta}.e^{-i\pi/2}$$

$$\text{ie; } \frac{z}{i} = re^{i\left(\theta-\pi/2\right)}$$

ie; z rotates an angle $\frac{\pi}{z}$ in clockwise direction about the origion

5. Multiplication by ω

$$z\omega = re^{i\theta}.e^{i\frac{2\pi}{3}} = re^{i\left(\theta+\frac{2\pi}{3}\right)}$$

ie; Z rotates an angle $\frac{2\pi}{3}$ in anticlockwise direction about the origin.

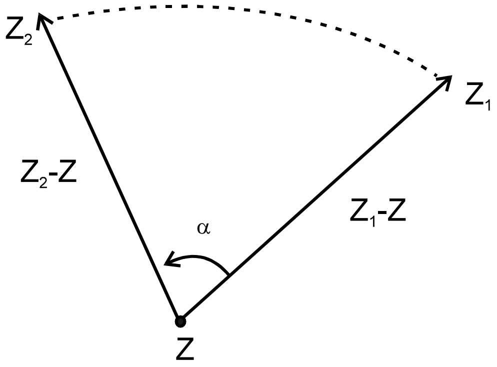
6. Division by ω (Multiplication by ω^2)

$$\frac{Z}{\omega} = \frac{re^{i\theta}}{\omega} = \omega^2 re^{i\theta} = re^{i\theta}.e^{-i\frac{2\pi}{3}}$$

$$\frac{z}{\omega} = z\omega^2 = re^{i\left(\theta-2\frac{\pi}{3}\right)}$$

ie; z rotates an angle $\frac{2\pi}{3}$ in clockwise direction about the origin.

32. Circular rotation about z



$$z_2 - z = (z_1 - z)e^{i\alpha}$$

When $z_1 - z$ rotates an angle α about z in anticlockwise direction, then $z_2 - z = (z_1 - z)e^{i\alpha}$

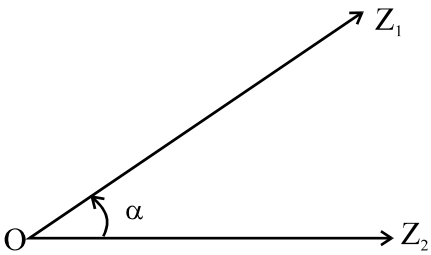
33. Complex division

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2.e^{i\theta_2}$ be two complex numbers then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \Rightarrow \frac{z_1 - 0}{z_2 - 0} = \frac{|z_1 - 0|}{|z_2 - 0|} e^{i\alpha}$$

where $\theta_1 - \theta_2 = \alpha$

The division geometrically represents the rotation of z_2 an angle α about the origin in anticlock wise direction and reaches z_2, if α in positive.

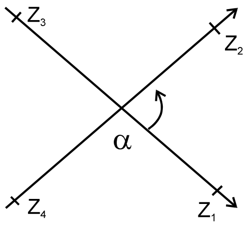


34. Rotation about z

Let z, z_1, z_2 be complex number in complex plane, then

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|} e^{i\alpha},$$ where α in the angle of rotation.

35.



Let z_1, z_2, z_3, z_4 be 4 complex numbers in complex plane, then $\frac{z_2 - z_4}{z_1 - z_3} = \frac{|z_2 - z_4|}{|z_2 - z_3|} e^{i\alpha}$, which represents the rotation of $z_1 - z_3$ an angle α (positive) and reaches z_1- z_4 in anticlockwise direction.

36. Locus related to argument

1) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) = 0 \Rightarrow$ locus of z represents 2 open opposite rays with end points z_1 and z_2.

2) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \pi \Rightarrow$ Locus of z represnets a line segment with end points z_1 and z_2

3) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pm\pi}{2}, \Rightarrow$ Locus of z represents a circle with diameters joining z_1 & z_2

4) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) =$ an acute angle , $\alpha \Rightarrow$ Locus of z represents a major arc with end points $z_1 \& z_2$

5) $\operatorname{Arg}\left(\frac{z - z_1}{z - z_2}\right) =$ an obtuse angle, $\alpha \Rightarrow$ Locus of z represents a minor arc with end points z_1 & z_2

37. Cube roots of unity

1. $\sqrt[3]{1} = 1, \omega, -\omega^2$, when $\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ $\quad \omega^2 = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

2. $\sqrt[3]{-1} = -1, -\omega, -\omega^2$, when $-\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2} - \omega^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

3. $\omega^3 = 1 \Rightarrow \begin{cases} \omega^{3m} = 1 \\ \omega^{3m+1} = \omega \\ \omega^{3m+2} = \omega^2 \end{cases}$

4. $1 + \omega + \omega^2 = 0 \begin{cases} 1 + \omega = -\omega^2 \\ 1 + \omega^2 = -\omega \\ \omega + \omega^2 = -1 \end{cases}$

5. $1+\omega^n+\omega^{2n}=\begin{cases}0, n\neq 3m\\ 3, n=3m\end{cases}$

6. $|\omega|=|\omega^2|=1$

7. $\arg(\omega)=\frac{2\pi}{3}, \arg(\omega^2)=\frac{4\pi}{3}$ or $\frac{-2\pi}{3}$

   $\arg(-\omega)=-\pi/3 \quad \arg(-\omega^2)=\pi/3$

8. $\omega=e^{i2\pi/3}, \omega^2=e^{-i\frac{2\pi}{3}}, -\omega=e^{-i\pi/3}, -\omega^2=e^{i\pi/3}$

9. $(\overline{\omega})=\omega^2$ & $(\overline{\omega}^2)=\omega, (-\overline{\omega})=-\omega^2, (-\overline{\omega}^2)=-\omega$

10. $(\omega)^2=\omega^2$ & $(\omega^2)^2=\omega$

11. $\frac{1}{\omega}=\omega^2$ & $\frac{1}{\omega^2}=\omega$

12. $x^2+x+1=(x-\omega)(x-\omega^2)$

    $x^2-x+1=(x+\omega)(x+\omega^2)$

13. $x^3-1=(x-1)(x-\omega)(x-\omega^2)$

    $x^3+1=(x+1)(x+\omega)(x+\omega^2)$

14. 1, ω, ω^2 lies on a unit circle $|z|=1$

15. 1, ω, ω^2 divides the circumference of $|z|=1$ into 3 equal segments.

16. 1, ω, ω^2 are the vertices of an equilateral triangle

38. **n[th] roots of unity**

1) $\sqrt[n]{1}=e^{i\frac{2k\pi}{n}}, k=0,1,2,3......(n-1)=1,\alpha,\alpha^2,\alpha^3....\alpha^{n-1}$, where $\alpha=e^{i\frac{2\pi}{n}}$

2) $1+\alpha+\alpha^2+.....+\alpha^{n-1}=0$

3) $\alpha^n=1$

4) $x^n-1=(x-1)(x-\alpha)(x-\alpha^2).....(x-\alpha^{n-1})$

   $\frac{x^n-1}{x-1}=(x-\alpha)(x-x^2).....(x-\alpha^{n-1})$

5) $(x-\alpha)(x-\alpha^2)........(x-\alpha^{n-1})=1+x+x^2+x^3+.....+x^{n-2}+x^{n-1}$

6) n^{th} roots of unity divides the circumferene of $|z|=1$ into n equal segments.

7) n^{th} roots of unity are the vertices of an n- sided regular polygon.

8) $\alpha^r=\dfrac{1}{\alpha^{n-r}}$ or $\alpha^{n-r}=\dfrac{1}{\alpha^r}$

9) $\bar{\alpha}=\dfrac{1}{\alpha}=\alpha^{n-1}$

10) $\sqrt[n]{-1}=-1,-\alpha,-\alpha^2.......,-\alpha^{n-1}$ where $-\alpha=-e^{i\frac{2\pi}{n}}$

11) $x^n+1=(x+1)(x+\alpha)(x+\alpha^2)............(x+\alpha^{n-1})$