MATRICES AND DETERMINANTS

An ordered rectangular array of numbers (real or complex) or functions is called a matrix.

The horizontal lines of elements are called rows and the vertical lines of elements are called columns.

eg
$$\begin{array}{cc} \begin{bmatrix}1 & 2 & 3\\ 5 & 6 & 0\end{bmatrix} & \begin{array}{l}\rightarrow \text{First Row}\\ \rightarrow \text{II Row}\end{array} \\ \begin{array}{ccc}\downarrow & \downarrow & \downarrow\\ \text{I Column} & \text{II Column} & \text{III Column}\end{array} & \end{array}$$

Order: If a matrix has m rows and n columns, then its order is defined as $m \times n$

In general a matrix is denoted by A,B,C .... and its elements (entries) are denoted by a,b,c ....

In general an $m \times n$ matrix has the following rectangular array :

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$$

or $A = \left[a_{ij}\right]_{m\times n},\ 1 \le i \le m,\ \ 1 \le j \le n,$ for all $i, j \in N.$

Thus the rth row consists of the elements $a_{i1,} a_{i2}\ a_{i3} \ldots\ldots a_{in},$ while jth column consists of the elements $a_{1j,} a_{2j}\ a_{3j} \ldots\ldots a_{mj}$

In general a_{ij} is an element lying in the rth row and jth column

We can also call it as the $(r, j)^{th}$ element of A.

The number of elements in an $m \times n$ matrix will be equal to mn

i) If a matrix has elements, what are the possible orders it can have

$1\times 8,\ 8\times 1,\ 2\times 4,\ 4\times 2$

ii) Construct a 3×2 matrix whose elements are given by

$a_{ij} = (i + j)^2$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$; given $a_{ij} = (i+j)^2$

$$\therefore A = \begin{bmatrix} 4 & 9 \\ 9 & 16 \\ 16 & 25 \end{bmatrix}$$

iii) Construct a 2×2 matrix, $a_{ij} = |i-j|$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Types of matrices

i) Column Matrix

A matrix is said to be a column matrix if it has only one column

$$A = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, 3\times 1 \quad A = \left[a_{ij}\right]_{m\times 1}$$

ii) Row matrix

A matrix is said to be Row matrix if it has only one row. $A = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}_{1\times 3}$, $A = \left[a_{ij}\right]_{1\times n}$

iii) Square matrix:

A matrix in which the number of rows is equal to the number of columns, is said to be a square matrix. Thus an $m\times n$ is known as a square matrix of order n (n×n) (if $m = n$, then it is rectangular)

$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 5 & -2 & 7 \end{bmatrix}$ is a square matrix of order 3.

If $A = \left[a_{ij}\right]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, a_{33}, ...a_{nn}$ are said to constitute the diagonal of the matrix A

Thus, If $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$. Then the elements of the diagonal of A are 1,4,6

iv) Diagonal matrix

A square matrix $B = \left[b_{ij}\right] n \times n$ is said to be a diagonal matrix if all its non diagonal elements are zero,

ie., a matrix $B = \left[b_{ij}\right] n \times n$ is said to be a diagonal matrix if $b_{ij} = 0$, where $i \neq j$

For example $A = [3], B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \text{dia}(1, 0, -2)$$

$$\text{dia}(1, -1, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

v) Scalar matrix

A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix $B = \left[b_{ij}\right]_{n \times n}$ is said to be a scalar matrix if

$b_{ij} = 0$ when $i \neq j$

$b_{ij} = k$ when $i = j$ for some constant k

example

$$A = [3], B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} C = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

vi) Identity matrix :

A square matrix in which elements in the diagonal are all 1(one) and rest are all zero is called an identify (unit matrix)

$$aij = 1 \quad \text{if } i = j$$
$$= 0 \quad \text{if } i \neq j$$

eg: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

vii) Zero matrix (null matrix)

A matrix is said to be zero matrix or null matrix if its elements are zero

For example $[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, [0, 0] = 0$ are all zero matrices . Denoted by O

Equality of matrices:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

i) They are of the same order

ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j

For example $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, A = B$

If $\begin{bmatrix} x & y \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ a & b \end{bmatrix}$ then $x = -1, y = 0, a = 3$ and $b = 2$

EXERCISE 3.1

1) $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$

i) order of A = 3×4

ii) The number of elements is $A = 3 \times 4 = 12$

iii) $a_{13} = 19, a_{21} = 35, a_{33} = -5, a_{24} = 12, a_{23} = 5/2$

2) If a matrix has 24 elements, what are the possible orders it can have, what if it has 13 elements

$24 \Rightarrow 1\times 24, 24\times 1, 2\times 12, 12\times 2, 3\times 8, 8\times 3, 4\times 6, 6\times 4$

$13 \Rightarrow 1\times 13,\ 13\times 1$ only

4) Construct a 2×2 matrix , $A = \left[a_{ij}\right]$ whose elements are given by

i) $aij = \dfrac{(i+j)^2}{2}$

ii) $aij = \dfrac{i}{j}$

iii) $aij = \dfrac{(i+2j)^2}{2}$

(i) $A = \begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

5) Construct a 3×4 matrix whose elements are given by

i) $a_{ij} = -3i + j$ ii) $a_{ij} = 2i - j$

i) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ -5 & -4 & -3 & -2 \\ -8 & -7 & -6 & -5 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

7) Find the value of a,b,c and d from the equation

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$a-b=-1-(1)$ $\quad$ $2a-b=0-(3)$ $\quad$ $(3)-(1) \Rightarrow a=1$ $\quad$ $2+c=5, c=3$

$2a+c=5-(2)$ $\quad$ $3c+d=13-(4)$ $\quad$ $-1-b=-1, b=2$ $\quad$ $9+d=13, d=4$

8) $A=\left[a_{ij}\right]_{m\times n}$ is a square matrix, if m = n (c)

The number of all possible matrices of order 3×3 with each entry 0 or 1 is $2^9=512$ (Each element having 2 ways)

Operations on Matrices

1) Addition of matrices : Two matrices A and B are said to be conformable for addition if A and B are of the same order. If A and B are m×n matrices then A+B of order m×n

Example $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} B=\begin{bmatrix} x & y \\ w & z \end{bmatrix}$

$$A+B=\begin{bmatrix} a+x & b+y \\ c+w & d+z \end{bmatrix}$$

If $A=\left[a_{ij}\right]_{m\times n,}$ $B=\left[b_{ij}\right]_{m\times n}$ then

$$A+B=C \Rightarrow \left[a_{ij}+b_{ij}\right]=\left[c_{ij}\right]_{m\times n}$$

2) Multiplication of a matrix by a scalar

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $kA=\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

If $A=\left[a_{ij}\right]_{m\times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k

If $A=\begin{bmatrix} 3 & 1 & 4 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then $4A=\begin{bmatrix} 12 & 4 & 16 \\ 4\sqrt{5} & 28 & -12 \\ 8 & 0 & 20 \end{bmatrix}$

Negative of a matrix: The negative of a matrix is denoted by-A. We define $-A=(-1)A$

If $A=\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then $-A=\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$

Difference of matrices: If $A=\left[a_{ij}\right], B=\left[b_{ij}\right]$ are two matrices of the same order, say $m\times n,$ then difference A-B is defined as a matrix $D=\left[d_{ij}\right]$, where $a_{ij}=a_{ij}-b_{ij}$ for all value of i and j. In other words, $D=A-B=A+(-1)B,$ that is sum of the matrix A and the matrix -B

Properties of matrix addition

i) Commutative Law : If $A=\left[a_{ij}\right], B=\left[b_{ij}\right]$ are matrices of the same order, say m×n, then $A+B=B+A$

ii) Associative Law: For any three matrices A,B,C of the same order, say $m\times n,\ (A+B)+C=A+(B+C)$

iii) Existence of additive identity : Let $A=\left[a_{ij}\right]$ be an $m\times n$ matrix and θ be an m×n zero matrix, then $A+\theta=\theta+A=A$

In other words, θ is the additive identity for matrix addition

iv) The existence of additive inverse;

Let $A=\left(a_{ij}\right)_{m\times n}$ be any matrix, then $-A=\left(a_{ij}\right)_{m\times n}$ such that $A+(-A)=(-A)+A=0$. So $-A$ is the additive inverse of A or negative of A.

Properties of scalar multiplication of a matrix

If A and B be two matrices of the same order, say m×n and k and l are scalars, then

(i) $k(A+B)=kA+kB$

(ii) $(k+\ell)A=kA+\ell A$

example

i) If $A=\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 2\\ -2 & 4\end{bmatrix}$ then find $2A+3B$

$$2A+3B=2\begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}+3\begin{bmatrix}-1 & 2\\ -2 & 4\end{bmatrix}=\begin{bmatrix}4 & 0\\ 2 & 6\end{bmatrix}+\begin{bmatrix}-3 & 6\\ -6 & 12\end{bmatrix}=\begin{bmatrix}1 & 6\\ -4 & 18\end{bmatrix}$$

ii) Find X and Y if $X+Y=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$

$$X+Y+X-Y=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}+\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$$

$$2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}, X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

EXERCISE 3.2

4) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$

then compute (A+B) and $(B-C)$. Also, verify that $A+(B-C)=(A+B)-C$

$$A+B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 5 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$$

$$A+(B-C) = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} - (1)$$

$$(A+B)-C = \begin{bmatrix} 4 & 1 & 5 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 9 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} - (2)$$

RHS of (1) = (2) $\therefore A+(B-C)=(A+B)-C$

5) If $A=\begin{bmatrix}\frac{2}{3} & 1 & \frac{5}{3}\\ 1 & 2 & \frac{4}{3}\\ \frac{7}{3} & 2 & \frac{2}{3}\end{bmatrix}$ and $B=\begin{bmatrix}\frac{2}{5} & \frac{3}{5} & 1\\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5}\\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5}\end{bmatrix}$, then compute $3A-5B$.

$$3A-5B=\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 7 & 6 & 2\end{bmatrix}-\begin{bmatrix}2 & 3 & 5\\ 1 & 2 & 4\\ 7 & 6 & 2\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

7) i) Find x and y if $x+y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $x-y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

$$x+y+x-y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}+\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

$$2x=\begin{bmatrix}10 & 0\\ 2 & 8\end{bmatrix},\ x=\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}$$

$$y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}-\begin{bmatrix}5 & 0\\ 1 & 4\end{bmatrix}=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$$

8) Find x, if $y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2x+y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$

$$\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}-\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}-2 & -2\\ -4 & -2\end{bmatrix},x=\begin{bmatrix}-1 & -1\\ -2 & -1\end{bmatrix}$$

12) Given $3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}=\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$ find x, y, z and w.

$$\begin{bmatrix}3x & 3y\\ 3z & 3w\end{bmatrix}=\begin{bmatrix}x+4 & 6+x+y\\ -1+2+w & 2w+3\end{bmatrix}$$

$$3x=x+4, 2x+4, x=2$$

$3y = 6 + x + y,\ 2y = 6 + 2 = 8, y = 4$

$3w = 2w + 3, w = 3$

$3z = -1, z + w, 2z = -1 + 3 = 2, z = 1$

Multiplication of Matrices

Two matrices A and B are said to be conformable for multiplication if the number of columns of A is equal to the number of rows of B

That is , If $A_{m\times n}$ and $B_{n\times p}$ are matrices, then $AB_{m\times p}$

$(A)_{m\times n} .B_{(n\times p)} = (AB)_{m\times p}$

let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 0 \end{bmatrix}$. Find AB

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 2\times5+1\times7 & 2\times6+1\times0 \\ 3\times5+4\times7 & 3\times6+4\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 10+7 & 12+D \\ 15+28 & 18+B \end{bmatrix} = \begin{bmatrix} 17 & 12 \\ 43 & 18 \end{bmatrix}$$

Matrix multiplication is not commutative $AB \neq BA$

If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix. Properties of multiplication of matrices:

i) Associative law: For any three matrices A, B and C, we have $(AB)C = A = (BC)$

ii) Distributive law : (i) $A(B + C) = AB + AC.$ (ii) $(A + B)C = AC + BC$

(iii) The existence of multiplicative identity , $AI = IA = A$ where I is the identity matrix and A is a square matrix

EXERCISE 3:2

(3) Compute the products

i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \times \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 \neq b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2+0+12 & -6+6+0 & 10+12+20 \\ 3+0+15 & -9+8+0 & 15+16+25 \\ 4+0+18 & 12+10+0 & 20+20+30 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & 22 & 70 \end{bmatrix}$

vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6-1+9 & -9+0+3 \\ -2+0+6 & 3+0+2 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

14. (ii) show that $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$\text{LHS} = \begin{bmatrix} -1+0+6 & 1-2+9 & 0+2+12 \\ 0+0+0 & 0-1+0 & 0+1+0 \\ -1+0+0 & 1-1+0 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 14 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{RHS} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1+0+0 & -2+1+0 & -3+0+0 \\ 0+0+1 & 0-1+1 & 0+0+0 \\ 2+0+4 & 2+3+4 & 6+0+0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 6 & 9 & 6 \end{bmatrix}$$

$\text{LHS} \neq \text{RHS}$ Hence the result

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 6 & 3 \\ 1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} (2+1) & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix}$$

$$-5A = \begin{bmatrix} -10 & 0 & -15 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$$

$$A^2 - 5A + 6I = \begin{bmatrix} 3 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{bmatrix} 3-10+6 & -1 & 2-5 \\ 9-10 & -1 & -10 \\ -5 & 4 & 8 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 8 \end{bmatrix}$$

17) If $A=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}, I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ find k so that $A^2 = ka - 2I$

$$A^2=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}\times\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}=\begin{bmatrix}9-8 & -6+4\\12-8 & -8+4\end{bmatrix}=\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}$$

$$A^2 = kA - 2I$$

$$\begin{bmatrix}1 & -2\\4 & -4\end{bmatrix}=k\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}-\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}, k\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}=\begin{pmatrix}1 & -2\\4 & -4\end{pmatrix}+\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

$$k\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}=\begin{pmatrix}3 & -2\\4 & -2\end{pmatrix}, k=1$$

Transpose of a matrix

If $A=\left[a_{ij}\right]$ is an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of **A** is called the transpose of A. Transpose of the matrix A is denoted by A^1 or $\left(A^T\right)$. In other words, if $A=\left[a_{ij}\right]_{m\times n}$, then $A^1=\left[a_{ji}\right]_{n\times m}$

Example $A=\begin{bmatrix}1 & 2 & 3\\0 & -1 & 4\end{bmatrix}_{2\times 3}, A^T=\begin{bmatrix}1 & 0\\2 & -1\\3 & 4\end{bmatrix}_{3\times 2}$

Properties of transpose of the matrices

(i) $\left(A^T\right)^T = A$ ii) $(kA)^T = kA^T$ iii) $(A+B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$ v) $(A-B)^T = A^T - B^T$

EXERCISE 3:3

(i) Let $A=\begin{bmatrix}5\\\frac{1}{2}\\-1\end{bmatrix}, A^T=\begin{bmatrix}5 & \frac{1}{2} & -1\end{bmatrix}$

2) (i) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ then verify that

1) $(A+B)' = A' + B'$

$$A+B = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}, \quad \text{LHS}(A+B)^T = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} - (1)$$

$$\text{RHS} = A' + B' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} - (2)$$

$(1) = (2)$

$\therefore (A+B)' = A' + B'$

5) For the matrices A and B, verity that $(AB)' = B'A'$, where

$$A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = [-1\ 2\ 1], \ (AB)' = B'A'$$

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix} - (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} - (1)$$

$$B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} - (2) \quad (1) = (2) \quad (AB)' = B'A'$$

6) i) if $A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$, then verify that $A'A=I$

LHS = $A'A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}\cos^2\alpha+\sin^2\alpha & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

Symmetric and skew symmetric matrices

A square matrix A is said to be symmetric if $A^T=A$

example $A=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 5\end{bmatrix}, A^T=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 5\end{bmatrix}$

A square matrix A is said to be skewsymmetric if $A^T=-A$

eg $A=\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & 3\\ -2 & -3 & 0\end{bmatrix}$ is skewsymmetric

For symmetric matrix, $a_{ij}=a_{ji}$ for all i and j

For skewsymmetric matrix $a_{ij}=-a_{ji}$ for $i\neq j$

$a_{ij}=0$, for $i=j$

1) If A is a square matrix, then

1) $A+A^T$ is a symmetric matrix

2) $A-A^T$ is a skewsymmetric matrix

i) $\left(A+A^T\right)^T=A^T+\left(A^T\right)^T=A^T+A=A+A^T$

$\therefore A+A^T$ is symmetric matrix

ii) $\left(A-A^T\right)^T=A^T-\left(A^T\right)^T=A^T+A=-\left(A-A^T\right)$

$\therefore A-A^T$ is skew symmetric matrix

2) Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix

$$A=\frac{1}{2}\left(A+A^{T}\right)+\frac{1}{2}\left(A-A^{T}\right)-(1)$$

$$\left(\frac{1}{2}\left(A+A^{T}\right)^{T}\right)=\frac{1}{2}\left(A+A^{T}\right)^{T}=\frac{1}{2}\left(A^{T}+A^{TT}\right)=\frac{1}{2}\left(A^{T}+A\right)=\frac{1}{2}\left(|\,A+A^{T}\right)$$

$\therefore \frac{1}{2}\left(A+A^{T}\right)$ is symmetric matrix - (2)

$$\left(\frac{1}{2}\left(A-A^{T}\right)^{T}\right)=\frac{1}{2}\left(A^{T}-A^{TT}\right)=\frac{1}{2}\left(A^{T}-A\right)=-\frac{1}{2}\left(A-A^{T}\right)-(2)$$

$\therefore \frac{1}{2}\left(A-A\right)^{T}$ is skew symmetric matrix from (2) and (1), Hence the result

EXERCISE 3:3

7) i) show that the matrix $A=\begin{bmatrix}1 & -1 & 5\\ -1 & 2 & 1\\ 5 & 1 & 3\end{bmatrix}$ is a symmetric matrix

$A=\begin{bmatrix}1 & -1 & 5\\ -1 & 2 & 1\\ +5 & 1 & 3\end{bmatrix} A^{T}=\begin{bmatrix}1 & -1 & 5\\ -1 & 2 & 1\\ 5 & 1 & 3\end{bmatrix} \quad \therefore A=A^{T},$ A is symmetric

ii) show that the matrix $A=\begin{bmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}$ is a skew symmetric matrix

$$A=\begin{bmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}, A^{T}=\begin{bmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{bmatrix}=-\begin{bmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}$$

$A^{T}=-A. \quad \therefore A$ is skew symmetric

8) For the matrix $A=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}$ verify that

i) $A+A'$ is a symmetric matrix

ii) $A-A'$ is a skew symmetric matrix

$$A+A'=\begin{bmatrix}1 & 5\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 6\\ 5 & 7\end{bmatrix}=\begin{bmatrix}2 & 11\\ 11 & 14\end{bmatrix}=\therefore a_{12}=a_{21}$$

$\therefore A+A'$ is symmetric

$$A-A'=\begin{pmatrix}1 & 5\\ 6 & 7\end{pmatrix}+\begin{pmatrix}1 & 6\\ 5 & 7\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ a_{12}=-a_{21}$$

$$a_{11}=a_{22}=0$$

$\therefore A-A'$ is a skew symmetric matrix

10) Express the following matrices as the sum of a symmetric and a skew symmetric matrix

i) $\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$ ii) $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$

i) Let $A=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$ $A^T=\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}$

$$A=\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$$

$$A=\frac{1}{2}\left(\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}\right)+\frac{1}{2}\left(\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}-\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}\right)$$

$$A=\frac{1}{2}\begin{bmatrix}6 & 6\\ 6 & -2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}$$

$$A=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}+\begin{bmatrix}0 & 4\\ -2 & 0\end{bmatrix}$$

A= symme tric + shew symmetric

ii) Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

Symmetric + skew symmetric

$$A = \frac{1}{2}\left[\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right] + \frac{1}{2}\left[\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right]$$

$$A = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

symmetric + skewsymmetric

Elementary operation (Transformation) of a matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns,

which are known as elementary operations or transformation

1) The interchange of any two rows or columns

$R_i \Leftrightarrow R_j$

or

$C_i \Leftrightarrow C_j$

2) The multiplication of the elements of any row or column by a non zero number.

$R_i \to kR_i$ or $C_i \to kC_i$

3) The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Invertible matrices

If A is a square matrix of order m and if there exists another square matrix B of the same order m, such that $AB = BA = I$ (I is unit matrix) then B is is called the inverse of matrix A and it is denoted by $A^{-1}$and A is the inverse of B.

Inverse of square matrix, if it exists, is unique

Let B and C are inverse of A

$$\therefore AB = BA = I$$

$$AC = CA = I$$

$$B = BI = B(AC) = (BA)C = IC = C$$

$$\therefore B = C \quad \text{unique}$$

If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$

$$(AB)(AB)^{-1} = I$$

$$A^{-1}(AB)(AB^{-1}) = A^{-1}I$$

$$(A^{-1}A)B(AB)^{-1} = A^{-1}$$

$$I \quad B(AB)^{-1} = A^{-1}$$

$$B(AB)^{-1} = A^{-1}$$

$$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$I(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

To find A^{-1} by elementary row transformation $A = IA$

To find A^{-1} by elementary column transformation, A= AI

EXERCISE 3.4

Find the inverse of the matrices by using elementary row transformations

i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

A = I A

$$\begin{bmatrix} +1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A,\ R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A,\ \ R_2 \to \frac{R_2}{5}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix} A, R_1 \to R_1 + R_2$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

3) $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

$A = IA$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A, \left(R_2 \to R_2 - 2R_1\right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A, \ \left(R_1 \to R_1 - 3R_2\right)$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

4) $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

A = IA

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \ \left(R_2 \to R_2 - 2R_1\right)$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} A\left(R_1 \leftrightarrow R_2\right)$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & 2 \end{bmatrix} A\left(R_1 \to R_1 - R_2\right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} A\left(R_1 \to R_1 - R_2\right)$$

$$A^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

15) $$A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$$

$$A = IA$$

$$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 3 & -2 & 2 \\ 2 & 2 & 3 \\ 2 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A, \left(R_1 \leftrightarrow R_3\right)$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 2 & -3 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A,\ R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 0 & -2 \end{bmatrix} A, \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ \frac{2}{5} & \frac{1}{5} & \frac{-2}{5} \\ \frac{3}{5} & 0 & \frac{-2}{5} \end{bmatrix} A, \begin{array}{l} R_2 \to \frac{R_2}{5} \\ R_3 \to \frac{R_3}{5} \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{3}{5} & 0 & -\frac{2}{5} \end{bmatrix} A, \ R_2 \to R_2 + R_3$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} A, \ R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix} A, \ R_1 \to R_1 + R_3$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{-2}{5}&0&\frac{3}{5}\\-\frac{1}{5}&\frac{1}{5}&0\\\frac{2}{5}&\frac{1}{5}&-\frac{2}{5}\end{bmatrix}A \quad R_1\to R_5-R_2$$

$$I=A^{-1}A$$

$$A^{-1}=\begin{bmatrix}\frac{4}{5}&0&\frac{3}{5}\\-\frac{1}{5}&\frac{1}{5}&0\\\frac{2}{5}&\frac{1}{5}&\frac{-2}{5}\end{bmatrix}$$

Miscellareous (3)

1) Let $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$,show that $(aI+bA)^n=a^nI+na^{n-1}bA,$

where I is the identity matrix of order 2 and $n\in N$

$$LHS=(aI+bA)^n=(aI)^n+nC_1(aI)^{n-1}bA+nl_2(aI)^{n-2}(bA)^2+......$$

$$=a^nI^n+na^{n-1}I^{n-1}bA+nl_2(aI)^{n-2}(bA)^2+.....(bA)^n$$

$$A^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}=0$$

$$\therefore LHS=a^nI+nb\ a^{n-1}A \quad RHS$$

3) If $A=\begin{bmatrix}3&-4\\1&-1\end{bmatrix}$, then prove that $A^n=\begin{bmatrix}1+2n&-4n\\n&1-2n\end{bmatrix}$

where n is any positive integer

using P.M.I

$n=1$

$$A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$$

$\therefore$ The result in true for n= 1

Let p(r) = A^r be true

$$A^r = \begin{bmatrix} 1+2r & 4r \\ r & 1-2r \end{bmatrix} - (1)$$

$$A^{r+1} = A^r A = \begin{bmatrix} 1+2r & -4r \\ r & 1-2r \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6r+4r & -4-8r+4r \\ 3r+1-2r & -4r-1+2r \end{bmatrix}$$

$$= \begin{bmatrix} 3+2r & -4-4r \\ r+1 & -2r-1 \end{bmatrix} = \begin{bmatrix} 1+2(r+1) & -4(r+1) \\ (r+1) & 1-2(r+1) \end{bmatrix}$$

$\therefore P(r+1)$ is true whenever $p(r)$ is true

Hence by induction the result is true for all natural numbers

5) Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric

$$(B'AB)' = (B'(AB))' = (AB)'(B')'$$

$$= B'A'B = B'AB \qquad (\therefore A' = A)$$

$= -B'AB$ is shew symmetric

$\therefore$ If A is shew symmetric then $B'AB$ is also show symmetric

11) Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

$$\text{let} \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4\alpha & 2c+5\alpha & 3c+6\alpha \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{array}{lll} a+4b & =-7 & -(1) \\ 2a+5b & =-8 & -(2) \\ 3a+6b & =-9 & -(3) \\ \hline \end{array}$$

$(1)+(2)+(3)\;\; 3b=-6$

$b=-2, a=1$

$$\begin{array}{lll} c+4d & =2 & -(4) \\ 2a+5d & =4 & -(5) \\ 3c+6d & =6 & -(6) \\ \hline \end{array}$$

$(4)+(5)+(6)\, 3d=0, d=0\;\; c=2$; $\quad \therefore x = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

DETERMINANTS

To every square matrix $A = \left[a_{ij}\right]$ of order n, we can associate a number (real or complex) called, determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A

Determinant of A is denoted by $|A|$ or det A

Determinant of a matrix of order one

Let $A = [-k]$ be the matrix of order 1, then determinant of $A = |A| = |-k| = -k$

Determinant of a matrix of order two

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2, then the determinant of A is defined as

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant of a matrix of order 3

Determinant of a matrix of order three can be determined by expressing it in terms of seconds order determined by expressing it in terms of seconds order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows ($R_{1,}R_2$ and R_3) and three columns (C_1,C_2 and C_3) giving the same value

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)$$

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$

EXERCISE 4.1

Evaluate the determinants

i) $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} = 2\times -1 - 4\times -5 = -2 + 20 = 18$

2 (i) $\begin{vmatrix} \cos\theta - & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos\theta\cos\theta - \sin\theta(-\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

4) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

$$\text{LHS} = |3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix} = 3\times 3\times 12 = 108$$

$$\text{RHS} = 27|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix} = 27\times 1\times 1\times 4 = 108$$

$\therefore |3A| = 27|A|$

5) i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 3(-5) -^{-} 1(0 -^{-} 3) - 2(0-0)$

$= -15 + 3 = -12$

7) i) Find the value of x if (i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$2 - 20 = 2x^2 - 24$

$2x^2 = 2 + 24 - 20 = 6, x^2 = 3, x = \pm\sqrt{3}$

8) If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

$$x^2 - 36 = 36 - 36$$

$$x^2 = 36, \ x = \pm 6$$

Properties of Determinants

1) The value of the determinant remains unchanged its rows and columns are interchanged ie $|A| = |A^T|$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2) If any two rows (or columns) of a determinant are interchanged then sign of determinant changes

ie $\begin{vmatrix} a_2 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (R_1 \Leftrightarrow R_2)$

3) If any two rows (or columns) of a determinant are identical (all corresponding elements are same) then value of determinant is zero

ie $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad R_1 \propto R_2$

4) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets maltiplied by k

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5) If some or all elements of a row on column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

ie $$\begin{vmatrix} a_1+k & a_2+p & a_3+q \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k & p & q \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6) If to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, the value of determinant remain same if we apply the operation

$R_1 \to R_1 + kR_2$ or $C_1 \to C_1 + kC_3$

EXERCISE 4.2

5) $$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\text{L.H.S} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Now applying $R_1 \to R_1 + R_2 + R_3$

$$\text{L.H.S} = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= 2\begin{vmatrix} a & p & x \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix} R_1 \to R_1 + R_2 + R_3$$

$$= 2(-1)(-1)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{RHS}$$

Hence the result

6) $$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix} \quad \therefore |A| = |A^T|$$

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = (-1)(-1)(-1)\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$\Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = -\Delta$$

$$\Delta + \Delta = 0$$

$$\therefore \Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$ Hence the result

8) i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\text{LHS} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{matrix} R_1 \to R_1 - R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} R_1 \to R_1 - R_2$$

Expanding along $R_1, (a-b)(b-c)(a-c)(0-1) = (c-a)(a-b)(b-c)$

$\therefore \text{LHS} = (a-b)(b-c)(c-a) = \text{RHS}$

8) ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

$$\text{LHS} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix} C_1 \to C_1 - C_2$$

Expanding along R_1

$$LHS = (a-b)(b-c)1(c^2 - a^2 + bc - ab)$$

$$(a-b)(b-c)((c-a)(c+a)+b(c-a))$$

$$= (a-b)(b-c)(c-a)(c+a+b) = RHS$$

9) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} (x-y)(y-z)(z-x)(xy+yz+zx)$$

$$LHS = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}$$

$$= \begin{vmatrix} x-y & x^2-y^2 & -z(x-y) \\ y-z & y^2-z^2 & x(y-z) \\ z & z^2 & xy \end{vmatrix} \begin{matrix} R_1 \to R_1 \to R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 1 & x+y & -z \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 0 & -(z-x) & -(z-x) \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix} R_1 \to R_1 - R_2$$

$$= (x-y)(y-z)(z-x)\begin{vmatrix} 0 & -1 & -1 \\ 1 & y+z & -x \\ z & z^2 & xy \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\begin{vmatrix}0 & 0 & -1\\ 1 & x+y+z & -x\\ z & z^2-xy & xy\end{vmatrix} C_2 \to C_2 - C_3$$

Expanding along R_1

$$\text{LHS}=(x-y)(y-z)(z-x)(-1)\left(z^2-xy-z(x+y+z)\right)$$

$$(x-y)(y-z)(z-x)\left[-z^2+xy+zx+zy+z^2\right]$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)=\text{RHS}$$

ii) $$\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=(a+b+c)^3$$

$$\text{LHS}=\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

$$=\begin{vmatrix}a+b+c & a+b+c & a+b+c\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix} \text{Applying } R_1 \to R_1 + R_2 + R_3$$

$$=(a+b+c)\begin{vmatrix}1 & 1 & 1\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & 0 & 0\\ 2b & -(b-c-a) & 0\\ 2c & 0 & (a+b+c)\end{vmatrix}\begin{matrix}C_2 \to C_2 - C_1\\ C_3 \to C_3 - C_1\end{matrix}$$

$$=(a+b+c)1(a+b+c)^2 \text{ expanding along } R_1$$

$$=(a+b+c)^3$$

11. $$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$$

$$\text{L.H.S.}=\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$=\begin{vmatrix} 2(x+y+z) & x & c_1 \to c_1+c_2+c_3 \\ 2(x+y+z) & 2x+y+z & y \\ 2(x+y+z) & & z+x+2y \end{vmatrix}, C_1 \to C_1+C_2+C_3$$

$$=\begin{vmatrix} 2(x+y+z) & x & y \\ 0 & (x+y+z) & 0 \\ 0 & 0 & (x+y+z) \end{vmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$=2(x+y+z)^3 = \text{RHS}$$

12. $$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

$$\text{L.H.S} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} C_1 \to C_1+C_2+C_3$$

$$=(1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

$$=\left(1+x+x^2\right)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x & x(1-x)\\ 0 & -x(1-x) & (1-x)\end{vmatrix}\begin{matrix}R_2\to R_2-R_1\\ R_3\to R_3-R_1\end{matrix}$$

$$=\left(1+x+x^2\right)(1-x)(1-x)\begin{vmatrix}1 & x & x^2\\ 0 & 1 & x\\ 0 & -x & 1+x\end{vmatrix}$$

$$=\left(1-x^3\right)(1-x)\begin{vmatrix}1 & x & x^2\\ 0 & 1 & x\\ 0 & -x & 1+x\end{vmatrix}$$ expanding along R_1

$$=\left(1-x^3\right)(1-x)\left[x+\left(1+x^2\right)\right]=\left(1-x^3\right)\left(1-x^3\right)=\left(1-x^3\right)^2=\mathrm{RHS}$$

13. $$\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}=\left(1+a^2+b^2\right)^3$$

$$\mathrm{L.H.S}=\begin{vmatrix}1+a^2-b^2 & 2ab & -2b\\ 2ab & 1-a^2+b^2 & 2a\\ 2b & -2a & 1-a^2-b^2\end{vmatrix}$$

$$=\begin{vmatrix}1+a^2+b^2 & 0 & -2b\\ 0 & 1+a^2+b^2 & 2a\\ +b\left(1+a^2+b^2\right) & -a\left(1+a^2+b^2\right) & 1-a^2-b^2\end{vmatrix}\begin{matrix}C_1\to C_1-bC_3\\ C_2\to C_2+aC_3\end{matrix}$$

$$=\left(1+a^2+b^2\right)^2\begin{vmatrix}1 & 0 & -2b\\ 0 & 1 & 2a\\ b & -a & 1-a^2-b^2\end{vmatrix}$$

$$=\left(1+a^2+b^2\right)^2\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 2a\\ b & -a & 1-a^2+b^2\end{vmatrix}\begin{matrix}C_3\to C_3+2bC_1\\ \\ \text{Expanding along } R_1\end{matrix}$$

$$=\left(1+a^2+b^2\right)\left(1-a^2+b^2+2a^2\right)=\left(1+a^2+b^2\right)^3=\text{RHS}$$

14. $$\begin{vmatrix}a^2+1 & ab & ac\\ ab & b^2+1 & bc\\ ca & cb & c^2+1\end{vmatrix}=1+a^2+b^2+c^2$$

$$\text{LHS}=\frac{1}{abc}\begin{vmatrix}a^3+a & ab^2 & ac^2\\ a^2b & b^3b & bc^2\\ ca^2 & cb^2 & c^3+c\end{vmatrix}$$

$$=\frac{1}{abc}abc\begin{vmatrix}a^2+1 & b^2 & c^2\\ a^2 & b^2+1 & c^2\\ a^2 & b^2 & c^2+1\end{vmatrix}=\begin{vmatrix}a^2+1 & b^2 & c^2\\ a^2 & 1+b^2 & c^2\\ a^2 & b^2 & 1+c^2\end{vmatrix}$$

$$=\begin{vmatrix}1+a^2+b^2+c^2 & b^2 & c^2\\ 1+a^2+b^2+c^2 & b^2+1 & c^2\\ 1+a^2+b^2+c^2 & b^2 & c^2+1\end{vmatrix}C_1\to C_1+C_2+C_3$$

$$=\left(1+a^2+b^2+c^2\right)\begin{vmatrix}1 & b^2 & c^2\\ 1 & 1+b^2 & c^2\\ 1 & b^2 & 1+c^2\end{vmatrix}$$

$$=\left(1+a^2+b^2+c^2\right)\begin{vmatrix}1 & b^2 & c^2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}\begin{matrix}R_2\to R_1-R_1\\ R_3\to R_3-R_1\end{matrix}$$

$$=\left(1+a^2+b^2+c^2\right)=\text{RHS}\quad\text{expanding along } R_1$$

Area of triangle

The area of a triangle whose vertices are (x_1, y_1) (x_2, y_2) and (x_3,y_3) is Δ, $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

The area of a triangle formed by three collinear points is zero

Exercise 4.3

1. Find area of the triangle with vertices

(i) (1,0), (6,0),(4,3)

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[1(-3) - 0 + 1(18)\right) = 7.5\text{sq.units}$$

(ii) $(-2,-3),(3,2),(-1,-8)$

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} = \frac{1}{2}\left(-2(2+8) + 3(3+1) + 1(-24+2)\right)$$

$$= \frac{1}{2}(-20+12-22) = \frac{-30}{2} = 15 \text{ sq.units}$$

2. Show that points $A(a,b+c), B(b,c+a), C(c,a+b)$ are collinear

$$\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \Rightarrow C_2 \to C_2 + C_1$$

$$\begin{vmatrix} a & a+b & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} = 0 \because C_3 \propto C_2$$

$\therefore$ The given points are collinear

3. (i) Find equation of the line joining (1,2) and (3,6) using determinants

Let (x,y) be any point on the line joing (1,2) and (3,6)

$$\therefore \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$x(2-6) - y(1-3) + 1(6-6) = 0$$

$$-4x + 2y = 0, \Rightarrow y = 2x$$

$\therefore$ Equation of line is $2x - y = 0$

Minors and cofactors

Minor: Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ii} is denoted by M_{ij}

Example : Minor of element 3 in $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix}$ is

$$\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3$$

Cofactor : Cofactor of an element a_{ij} is denoted by A_{ij} or C_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}

Exercise 44

Write minors and cofactors of the elements of following determinants

1) (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ $M_{11} = |3| = 3 \quad M_{21} = |-4| = -4$
$M_{12} = |0| = 0 \quad M_{22} = |2| = 2$

2) (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $\text{cofactors} \Rightarrow A_{11} = 3, A_{12} = 0, A_{21} = 4; A_{22} = 2$
$M_{11} = 1, M_{12} = 0, M_{13} = 0$

$$A_{11} = 1, A_{12} = 0 \ A_{13} = 0$$

$$M_{21} = 0, M_{22} = 1, M_{23} = 0$$

$$A_{21} = 0 \ A_{22} = 1 \ A_{23} = 0$$

$$M_{31} = 0, M_{32} = 0, M_{33} = 1$$

$A_{31}=0\ A_{32}=0\ A_{33}=1$

3) Using cofactors of elements of third column, evaluate

$$\Delta=\begin{vmatrix}1 & x & yz\\ 1 & y & zx\\ 1 & z & xy\end{vmatrix} M_{31}=\begin{vmatrix}1 & y\\ 1 & z\end{vmatrix}=z-y, M_{32}=\begin{vmatrix}1 & x\\ 1 & z\end{vmatrix}=z-x\ M_{33}=\begin{vmatrix}1 & x\\ 1 & y\end{vmatrix}=y-x$$

$$\Delta=a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}$$

$$=yz(z-y)-zx(z-x)+xy(y-x)$$

$$=yz^2-y^2z=xz^2+x^2z+xy^2-x^2y$$

$$=(x-y)(y-z)(z-x)$$

$$\left[\because(x-y)(y-z)(z-x)\right.$$

$$=(x-y)\left[y_2-xy-z^2+xz\right)$$

$$=xyz-x^2y-xz^2+x^2z$$

$$-y^2z+xy^2+yz^2-xyz$$

$$=-x^2y-x^2-y^2z$$

$$\left.+x^2z+xy^2+yz^2\right]$$

4) $$\Delta=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

D) $\Delta=a_{11}\ A_{11}+a_{21}A_{21}+a_{31},A_{31}$

$\because$ if elements of a row (a column) are multiplied with cofactors of any other row (or column), then their sum is zero. Sum of products elements of any row (column) with their corresponding cofactors = Δ

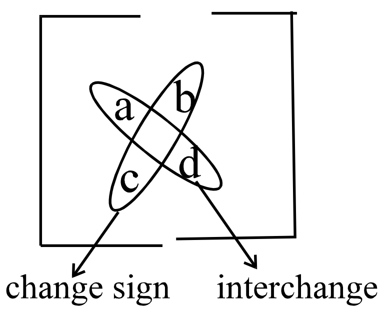
Adjoint of matrix

The adjoint of a square matrix $A=\left[a_{ij}\right]_{n\times n}$ is defined as the transpose of the matrix $\left[A_{ij}\right]_{n\times n}$. Where

A_{ij} is the cofactor of the element a_{ij} . Adjoint of the matrix A is denoted by adjA

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\text{adj } A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

adj of 

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A(\text{adj } A) = (\text{adj} A)A = |A| I$$

Singular matrix : A square matrix A is said to be singular if $|A| = 0$

A square matrix A is said to non singular if $|A| \neq 0$, $|AB| = |A||B|$ where A and B are square matrices of the same order

$|\text{adj } A| = |A|^{n-1}$ where n is the order of A (n×n) A square matrix A is invertible if and only if A is nonsingular matrix

Inverse of $A = \frac{\text{adj} A}{|A|}$, where A is a square matrix. If $AB = BA = I,$ then B is the inverse of A or A is the inverse of B. Where A and B are square matrices of the same order. Inverse of A is denoted by A^{-1}, Inverse of B is $B^{-1}, AA^{-1} = I$

$$A(\text{adj} A) = |A| I = (\text{adj} A) A$$

$$\div |A| \quad A\frac{(\text{adj} A)}{|A|} = I = \frac{(\text{adj } A)}{|A|} A$$

$$A\, A^{-1} = I = A^{-1} A$$

$\therefore A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$, Provided $|A| \neq 0$

EXERCISE 4.5

Find adjoint of (1) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

(i) adjoint of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

$A_{11} = 3, A_{12} = -(2+10) = -12 \ A_{13} = 6$

$A_{21} = -(-1) = 1, A_{22} = 1+4 = 5, A_{23} = -(-2) = 2$

$A_{31} = (-5-6) = -11, A_{32} = -(5-4) = -1 \ A_{33} = 3+2 = 5$

$$\text{adj of } A = \operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

4) Verify $a(\operatorname{adj} A) = (\operatorname{adj} A)A = |A|I, A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & +2 \\ 1 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{....(1)}$$

$$(\text{adj } A)A = \begin{bmatrix} 0 & 3 & -2 \\ -7 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \text{....(2)}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 1(0) + 1(9-2) + 2(0) = 7$$

$$A(\text{adj } A) = (\text{adk } A)A = 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 7I$$

$$\therefore A(\text{adj } A) = (\text{adj} A)A = |A|I$$

Find the inverse of (5) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ (10) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

5) Let $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \frac{\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}}{6+8} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

10) Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}$

$$\operatorname{adj}A=\begin{bmatrix}2 & 0 & -1\\ -9 & -2 & 3\\ -6 & -1 & 2\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{vmatrix}$$

$$=1(8-6)+1(0+9)+2(0-6)$$

$$=2+9-12=-1$$

$$A^{-1}=\frac{\operatorname{adj}A}{|A|}=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$$

12) Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$ verify that $(AB)^{-1}=B^{-1}A^{-1}$

$$\text{LHS}=(AB)^{-1}=\left(\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\times\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}\right)^{-1}$$

$$=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}^{-1}=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}^{-1}$$

$$=\frac{\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}}{67\times 61-47\times 87}=\frac{\begin{bmatrix}67 & -87\\ -47 & 67\end{bmatrix}}{-2\quad ...(1)}$$

$$\text{RHS}=B^{-1}A^{-1}=\left(\frac{\operatorname{adj}B}{|B|}\right)\left(\frac{\operatorname{adj}A}{|A|}\right)=\frac{\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}}{-2}\times\frac{\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}}{1}$$

$$= -\frac{1}{2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix} = \frac{\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}}{-2} \text{....(2)}$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{1}$

14) For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$

$$A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

$$A^2 + aA + bI = 0$$

$$\begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = 0$$

$$11 + 3a + b = 0 \text{....(1)}$$

$$8 + 2a = 0 \text{....(2)}$$

$$a = -4$$

$$b = -11 - 3a$$

$$b = -11 + 12 \Rightarrow b = 1$$

$$a = -4 \text{ and } b = 1$$

Applications of Determinants and Matrices:

Consistent system : A system of equations is said to be consistent it its solution (one or more) exists.

Inconsistent system : A system of equations is said to be inconsistent it its solution does not exist.
Solution of system of linear equations using inverse of a matrix

Consider the system of equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, $AX = B$, ie

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B \quad (\because |A| \neq 0)$$

$$X = A^{-1}B$$

If A is a non singular matrix, thenthe system is consistent and unique solution, x = $A^{-1}B$

If A is a singular matrix, then $|A| = 0$. In this case, find (adj) B

If $(\text{adj}A)B \neq 0,$ then solution does not exist and the system of equations is called inconsistent (no solution) If $(\text{adj}A)B = 0,$ then the system is consistent and infinitely many solution

EXERCISE 4.6

(1) Examine the consistency of the system of equations and find its solution

(1) $\begin{matrix} x + 2y = 2 \\ 2x + 3y = 3 \end{matrix}$ $|A| \neq 0$ consistent find its solution

$$AX = B, X = A^{-1}B$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}, |A| = 3 - 4 = -1, \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}}{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = , x = 0, y = +1$$

4) $x + y + z = 1$

$$2x + 3y + 2z = 2$$

$$ax + ay + 2az = 4$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}}{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{\begin{bmatrix} 6 & -6 \\ -4 & +3 \end{bmatrix}}{-1} = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{-1} \Rightarrow x = 0, y = 1$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{vmatrix} = 1(6a-2a) - 1(4a-2a) + 1(2a-3a) = 4a + 2a - a = 5a \neq 0$$

$\therefore$ The system is consistent

Solve system of linear equations, using matrix method

7) $5x + 2y = 4$

$7x + 3y = 5$

$AX = B; X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}}{1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x = 2, y = -3$

11) Solve $2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}, X = A^{-1}B$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{vmatrix} = 2(10+3) - 1(-5) + 1(3) = 26 + 5 + 3 = 34$$

$$\text{adj } A = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \\ 9 \end{bmatrix}$$

$$= \frac{1}{34} \begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34} \begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-3}{2} \end{bmatrix} x = 1,\ y = \frac{1}{2}, z = \frac{-3}{2}$$

14) $x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|A| = 1(12-5) + 1(9+10) + 2(-3-8)$$

$$= 7 + 19 - 22 = 4$$

$$\text{adj } A = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix} \because X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}}{4} \times \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$x = 2, y = 1, z = 3$

15) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ find A^{-1}. Using A^{-1} solve the system of equations $2x-3y+5z=11$, $3x+2y-4z=-5$ and $x+y-2z=-3$

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$|A| = 2(-4+4)+3(-6+4)+5(3-2)$$

$$= 0-6+5=-1$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{(\text{adj } A)}{|A|}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} \times \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}}{-1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1\begin{bmatrix} 0+5-6 \\ 22+45-69 \\ 11+25-39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x=1, y=2, z=3$

Miscellaneous exercises

(2) With out expanding the determinant, prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$$LHS = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$= \frac{1}{abc} abc \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^3 \end{vmatrix} C_1 \Leftrightarrow C_3$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \begin{matrix} C_2 \Leftrightarrow C_3 \\ = RHS \end{matrix}$$

3) Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along second row (R_2)

$$\Delta = (-1)(-\sin\beta)\begin{vmatrix} \cos\alpha\sin\beta & -\sin\alpha \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} + \cos\beta\begin{vmatrix} \cos\alpha\cos\beta & -\sin\alpha \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix}$$

$$= \sin\beta\left(\cos^2\alpha\sin\beta + \sin^2\alpha\sin\beta\right) + \cos\beta\left(\cos^2\alpha\cos\beta + \sin^2\alpha\cos\beta\right)$$

$$= \sin\beta\sin\beta\left(\cos^2\alpha + \sin^2\alpha\right) + \cos\beta\,\cos\beta\left(\cos^2\alpha + \sin^2\alpha\right)$$

$$= \sin^2\beta.1 + \cos^2\beta.1 = \sin^2\beta + \cos^2\beta = 1$$

5) Solve the equation $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

$$C_1 \to C_1 + C_2 + C_3$$

$$\begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$$

$$\begin{vmatrix} 3x+a & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0 \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$(3x+a)a^2 = 0$$

$$\therefore 3x + a = 0$$

$$3x = -a$$

$$x = \frac{-a}{3}$$

4) a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ a+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$ show that $a+b+c=0$ or a = b = c

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

$$C_1 \to C_1 + C_2 + C_3$$

$$\begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0$$

$$2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 0$$

$$2(a+b+c)\begin{vmatrix}1 & c+a & a+b\\0 & b-c & c-a\\1 & b-a & c-b\end{vmatrix}\begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-R_1\end{matrix}$$

$$2(a+b+c)(b-c)(c-b)-(b-a)(c-a)=0$$

Expanding along R_1

$$2(a+b+c)\left[bc-b^2-c^2+bc-bc+ab+ac-a^2\right]=0$$

$$-2(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)=0$$

$$-(a+b+c)\left(2a^2-2ab+2b^2-2bc+2c^2-2ca\right)=0$$

$$-(a+b+c)\left(a^2-2ab+2b^2+b^2-2bc+c^2+c^2-2ca+a^2\right)=0$$

$$-(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)=0$$

$$\therefore a+b+c=0 \text{ an } (a-b)^2=0,(b-c)^2=0,(c-a)^2=0$$

$$a-b=0, b-c=0, c-a=0$$

$$a=b, b=c, c=a$$

is $a+b+c=0$ or $a=b=c$

12) Show that $\begin{vmatrix}x & x^2 & 1+px^3\\y & y^2 & 1+py^3\\z & z^2 & 1+pz^3\end{vmatrix}=(1+pxyz)(x-y)(y-z)(z-x)$

L H S = $\begin{vmatrix}x & x^2 & 1\\y & y^2 & 1\\z & z^2 & 1\end{vmatrix}+\begin{vmatrix}x & x^2 & px^3\\y & y^2 & py^3\\z & z^2 & pz^3\end{vmatrix}$

$$=\begin{vmatrix}x & x^2 & 1\\y & y^2 & 1\\z & z^2 & 1\end{vmatrix}+xyz\begin{vmatrix}1 & x & px^2\\1 & y & py^2\\1 & z & py^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & x^2 & x\\1 & y^2 & y\\1 & z^2 & z\end{vmatrix}+pxyz\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}$$

$C_1 \Leftrightarrow C_3$

$$=1\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}+pxyz\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}$$

$$=\left(1+pxyz\right)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}$$

$$=\left(1+pxyz\right)\begin{vmatrix}0 & x-y & x^2-y^2\\0 & y-z & y^2-z^2\\1 & z & z^2\end{vmatrix}\begin{matrix}R_1\to R_1-R_2\\R_2\to R_2-R_3\end{matrix}$$

$$=\left(1+pxyz\right)\left(\left(x-y\right)\left(y^2-z^2\right)-\left(y-z\right)\left(x^2,y^2\right)\right)$$

Expanding along C_1

$$=\left(1+pxyz\right)\left(\left(x-y\right)\left(y-z\right)\left(y+z\right)-\left(y-z\right)\left(x-y\right)\left(x+y\right)\right)$$

$$=\left(1+pxyz\right)\left(x-y\right)\left(y-z\right)\left(y+z-x-y\right)$$

$$=\left(1+pxyz\right)\left(x-y\right)\left(y-z\right)\left(z-x\right)=RHS$$

14) Show that $\begin{vmatrix}1 & 1+p & 1+p+q\\2 & 3+2p & 4+3p+2q\\3 & 6+3p & 10+6p+3q\end{vmatrix}=1$

LHS= $\begin{vmatrix}1 & 1+p & 1+p+q\\0 & 1 & 2+p\\0 & 3 & 7+3p\end{vmatrix} R_2\to R_2-2R_1, R_3\to R_3-3R_1$

$$=7+3p-6-3p=1\ RHS\left(\text{expanding along } R_1\right)$$

17) Show that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

$$\text{LHS} = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0 = \text{RHS}\ C_3 \to C_3 - \cos\delta C_2 + \sin\delta C_1$$

16) Solve the equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4 \quad \text{put } \frac{1}{x} = X$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \quad \frac{1}{y} = y$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{2} = 2 \quad \frac{1}{z} = X$$

$$2X + 3Y + 10Z = 4$$

$$4X - 6Y + 5Z = 1$$

$$6X + 9Y - 20Z = 2$$

$$X = A^{-1}B$$

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{adj} A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$$

$$= 150 + 330 + 720 = 1200$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 450 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$x = \frac{1}{2} = \frac{1}{x} \quad \therefore x = 2$$
$$y = \frac{1}{3} = \frac{1}{y} \qquad y = 3$$
$$z = \frac{1}{5} = \frac{1}{z} \qquad z = 5$$

17) If a,b,c are in A.P then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$ is

given a,b, c are in A.P. $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

$\therefore 2b = a + c \qquad R_1 \to R_1 + R_3 - 2R_2$

$$\begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

18) If x, y, z are non zero real numbers then that inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

$$A^{-1} = \frac{\text{adj } A}{|A|}, |A| = xyz$$

$$A^{-1} = \frac{\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}}{xyz} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$