THEORY OF PROBABILITY PART -1

Random experiment: An experiment whose outcomes can not be predicted in advance and it satisfies the following conditions (i) It has more than one possible outcomes (ii) It is not possible to predict the outcomes in advance (iii) The outcomes of the experiment should vary irregularly (iv) when we repeat the experiment it should result in one of its different possibilities

Sample space: The set of all possible outcomes of the random experiment. For example when a coin is tossed the sample space $S=\{H,T\}$. When two coins are tossed the sample space

$S=[HH,HT,TH,TT]$

Event: Any finite sub set of the sample space is called an event

Let $S=\{1,2,3,4,5,6\}$

A = Even face = $\{2,4,6\}$ and $B=\{1,3,5\}$ etc are events

$$\boxed{\begin{array}{l}\text{Let } n(s)=n\\ \text{number of events}=2^n\\ S=\text{sure event}\\ \phi=\text{Impossible event}\end{array}}$$

Algebra of events

i) Complement of event A $\left(A' \text{ or } \overline{A} \text{ or } A^c\right)$

The set of all outcomes in S, but not in event A is called complement of A

ie $A'=S-A$ and $\boxed{A\cup A'=S,\ A\cap A'=\phi}$

ii) Union of two events: The union of two even A and B are the set of outcomes either in A or B

ie $\boxed{A\cup B=A \text{ or } B}$

III) Intersection of two events : The intersection of two events A and B is the set of outcomes in both A and B

ie $\boxed{A\cap B=A \text{ and } B}$

iv) $'A \text{ but not } B'$:It is the event A–B

$\boxed{\text{A but not B}=A-B=A\cap B'}$

v) $'\text{B but not A}'$: It is B–A

$\boxed{\text{B but not A=B-A=B}\cap\text{A}'}$

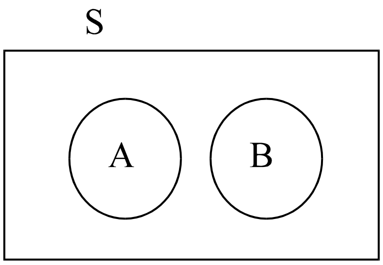
Type of Events: There are 3 different types of events namely equally likely events, mutually exclusive events and exhaustive events.

i) Equally likely events: Events are equally likely if they have the same chance to occur. For example when a fair coin is tossed $\{H\}$ and $\{T\}$ are equally likely events

ii) Mutually exclusive events

Events are mutually exclusive if they can not occur at the same time. For example when a fair coin is tossed $\{H\}$ and $\{T\}$ are mutually exclusive

$\boxed{\text{A and B are mutually exclusive} \Rightarrow A\cap B=\phi}$



$A\cap B=\phi\Rightarrow A$ and B are mutually exclusive

iii) Exhaustive events : Events are exhaustive if their union is the sample space

$$\boxed{\begin{array}{l}\text{A and B are exhaustive} \Rightarrow A\cup B=S\\ \text{A, B and C are exhaustive} \Rightarrow A\cup B\cup C=S\end{array}}$$

Probability: Probability is defined as a numerical measure of chance of future events. There are 3 different schools of thought on the concept of probability. They are classical or mathematical probability, statistical or empirical probability and the axiomatic probability

1) Mathematical or classical probability

In classical definition probability of an event 'A' is defined as

$$\boxed{P(n)=\frac{m}{n}}$$

Where n is the total numbers of outcomes in sample space and 'm' is the number of favourable outcomes in event A

The classici definition can be used only when 'n' is finite and the various outcomes in the sample space are equally likely

Examples

1) A fair die is thrown. The sample space is $S=\{1,2,3,4,5,6\}\therefore n=6$

Let A = Prime faces = $\{2,3,5\}\therefore m=3$

$$\therefore P(A)=P(\text{Prime face})=\frac{m}{n}=\frac{3}{6}=\frac{1}{2}$$

2) Three coins are tossed. What is the probability of gelting i) Exactly two heads ii) At least two heads and iii) At most two heads

$$S=\{HHH, HHT, THH, HTH, TTH, HTT, THT, TTT]$$

$$n=n(s)=8$$

i) $P(A)=P[\text{Exactly two heads})$

$$=P(HHT, THH, HTH)=\frac{3}{8}$$

ii) P(A) = P(At least two heads)

$$=P(HHT, THH, HTH, HHH)=\frac{4}{8}=\frac{1}{2}$$

iii) P(A) = P(At most two heads)

$$=P(TTT, TTH, HTT, THT, HHT, THH, HTH)=\frac{7}{8}$$

3) $A=\{1,2,3,4,5\}$ $B=\{x, y\}$. A relation is selected from set A to B. What is the probability that it is a function.

$$n(A)=m=5 \quad n(B)=n=2$$

Total number of relations = 2^{mn}=2^{10}
Number of functions from A to B = n^m

$$=n^m=2^5$$

$$P[\text{Relation is a function}]=\frac{2^5}{2^{10}}=\frac{1}{32}$$

4) ‘a’ and ‘b’ are obtained by throwing a pair of dice. What is the probability that $\underset{x\to 0}{\mathrm{Lt}}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}=6$

Answer
Throwing a pair of dice means total number of outcomes = 36

$$\underset{x\to 0}{\mathrm{Lt}}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}=\underset{x\to 0}{\mathrm{Lt}}\left(1+\frac{a^x+b^x}{2}-1\right)^{\frac{1}{x}}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{1}{\left(\frac{x}{2}\right)}}$$

$$= e^{\underset{x\to 0}{Lt}\frac{\left(\frac{a^x+b^x-2}{2}\right)}{\left(\frac{x}{2}\right)}} = e^{\underset{x\to 0}{Lt}\frac{a^x+b^x-2}{x}}$$

$$= e^{\underset{x\to 0}{Lt}\frac{a^x-1}{x}+\frac{b^x-1}{x}} = e^{\log a+\log b}$$

$$= e^{\log ab} = ab$$

Given limit = 6

$$\therefore P(\text{Limit}=6) = P(ab=6)$$

'a' and 'b' are obtained by throwing a pair of dice

$$\therefore P(ab=6) = P\{(1,6)(6,1)(2,3)(3,2)\}$$

$$= \frac{4}{36} = \frac{1}{9}$$

5) The letters of the word 'SLEEPLESSNESS' are arranged at random. What is the probability that all the Ss come together

Answer : SLEEPLESSNESS

Total number of letters = 13

Number of S = 5 number of E = 4 number of L = 2

Total number of arrangements = $\frac{13!}{5!\times 2!\times 4!}$

When Ss are together

$$\overset{1}{\boxed{SSSSS}}\ \overset{2}{E}\ \overset{3}{E}\ \overset{4}{E}\ \overset{5}{E}\ \overset{6}{L}\ \overset{7}{L}\ \overset{8}{N}\ \overset{9}{P}$$

Number of favourable arrangements = $\frac{9!}{4!\times 2!}$

$$P(\text{Ss together}) = \frac{\left(\frac{9!}{4!\times 2!}\right)}{\left(\frac{(13!)}{5!\times 2!4!}\right)}$$

$$= \frac{9!\times 5!}{13!} = \frac{1\times 2\times 3\times 4\times 8}{10\times 11\times 12\times 13} = \frac{1}{143}$$

Statistical or Empirical probability

Let a random experiment be repeated ‘n’ times and let an event ‘A’ occurs ‘r’ times. Then $\left(\frac{r}{n}\right)$ is called the frequency Ratio. In statistical or empirical definition probability of event is defined as $P(A)=\underset{n\to\infty}{Lt}\left(\frac{r}{n}\right)$ where the limit exists and is finite

Axiomatic probability

In Axiomatic theory probability is a real valued set function from the power set of sample space to the set of real nos in [0 1] satisfying the following anions

1) $P(A)\geq 0$

2) $P(A)\leq 1$

3) $P(S)=1\Rightarrow P(\text{sure event})=1$

4) $P(\phi)=0\Rightarrow P(\text{Impossible event})=0$

5) $P(A\cup B)=P(A)+P(B)\text{ if }A\cap B=\phi$ ie if A and B are mutually exclusive

As a function

Domain of probability = Power set of S
Range of probability = [0 1]

Show that $P(A')=1-P(A)$ where A’ is the complement of A

Answer : $A\cup A'=S$ and $A\cap A'=\phi\therefore$ A and A’ are mutually exclusive and exhaustive

$$P(A\cup A')=P(S)$$

$$P(A\cup A')=1\quad(\text{By axiom } 3)$$

$$\boxed{\begin{array}{l}P(A')=1-P(A)\\ P(A)=1-P(A')\end{array}}$$

Addition theorem on Probability

If A and B are any two events the addition theorem states that

$$\boxed{P(A\cup B)=P(A)+P(B)-P(A\cap B)}$$

But $A\cup B=A\text{ or }B\quad A\cap B=A\text{ and }B$

Hence the addition theorem can also be written as

$$\boxed{P(A\text{ or }B)=P(A)+P(B)-P(A\text{ and }B)}$$

Now suppose that A,B are C are any three event. The addition theorem states that

$$P\left(A\cup B\right)=P(A)+P(B)+P(C)-P(A\cap B)$$

$$-P\left(B\cap C\right)-P\left(A\cap C\right)+P\left(A\cap B\cap C\right)$$

Questions

1) The probability that a contractor may get an electric contract is $\frac{1}{2}$ and that he may get a plumbing contract is $\frac{1}{3}$. The probability that he will get both the contracts is $\frac{1}{4}$. What is the probability that he will get at least one contract.

A = Electric contract B = Plumbing contract

$$P(A)=\frac{1}{2}\ \ P(B)=\frac{1}{3}\ P(A\text{ and }B)=P\left(A\cap B\right)=\frac{1}{4}$$

$$P\left[\text{At least once}\right]=P\left[A\text{ or }B\right]=P\left(A\cup B\right)$$

$$P\left(A\cup B\right)=P\left(A\right)+P\left(B\right)-P\left(A\cap B\right)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}$$

2) An integer is taken at random from the first 200 natural numbers. What is the probability that it is divisible by 6 or 8.

$A=\text{Divisible by }6$

$B=\text{Divisible by }8$

$$n\left(A\right)=\left[\frac{200}{6}\right]=33\ \ \text{where}\left[.\right]=\text{GIV}$$

$$n\left(B\right)=\left[\frac{200}{8}\right]=25$$

$$n\left(A\cap B\right)=n\left(\text{Divisible by }6\text{ and }8\right)=\left[\frac{200}{\text{LCM of }6\text{ and }8}\right]$$

$$=\left[\frac{200}{24}\right]=8$$

$P(A)=\frac{33}{200}P(B)=\frac{25}{200}P(A\cap B)=\frac{8}{200}$

$P(\text{Disible by 6 or 8})=P(A \text{ or } B)$

$=P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=\frac{33}{200}+\frac{25}{200}-\frac{8}{200}=\frac{50}{200}=\frac{1}{4}$

Odds in favour and against of an event A

Let n(A) = number of outcomes in favour of an event A and

$n(A')$ = number of outcomes against an event A

$$\boxed{\begin{array}{l}\text{Odds in favour of } A=\dfrac{n(A)}{n(A')}\\ \text{Odds against } A=\dfrac{n(A')}{n(A)}\end{array}}$$

Question

A party of n persons sit around a table. What are the odds against two persons sitting next to each other

Total number of persons = n

Total number of arrangements = $(n-1)!$

n(A) = number of arrangements in favour of 2 persons together = $(n-2)!\times 2!$

$\therefore n(A')=\text{Total}-n(A)$

$=(n-1)!-2(n-2)!$

$=(n-2)!(n-1-2)=(n-2)!(n-3)$

Odds against $A=\frac{n(A')}{n(A)}=\frac{(n-2)!(n-3)}{2(n-2)!}=\frac{n-3}{2}$

Problems based on packet of playing cards

The details regarding the packet of playing cards are given below

Sl. No.	Spade (Black)	Club (Black)	Hearts (Red)	Diamond (Red)	Total
1	KING	KING	KING	KING	4
2	QUEEN	QUEEN	QUEEN	QUEEN	4
3	JACK	JACK	JACK	JACK	4
4	ACE	ACE	ACE	ACE	4
5	2	2	2	2	4
6	3	3	3	3	4
7	4	4	4	4	4
8	5	5	5	5	4
9	6	6	6	6	4
10	7	7	7	7	4
11	8	8	8	8	4
12	9	9	9	9	4
13	10	10	10	10	4
TOTAL	**13**	**13**	**13**	**13**	**52**

Total number of cards = $13 \times 4 = 52$
Number of Red cards = 26
Number of Black cards = 26
Number of Kings / Queens/ Jack / Ace cards = 4
Number of Spade / Clubs / Hearts / Diamonds = 13
Face cards | court cards: King + Queen + Jack = 12 cards

1) A card is taken from a packet of cards. What is the probability that it is a spade or Ace

$P(\text{spade})=\frac{13}{52} \quad P(\text{Ace})=\frac{4}{52}$

$P(\text{Spade and Ace})=\frac{1}{52}$

$P(\text{Spade or Ace})= P(\text{spade})+P(\text{Ace})-P[\text{ Spade and Ace}]$-(Addition Theorem)

$=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$

2) Two cards are taken from a packet of cards. What is the probability that both are queens if cards are taken i) at a time ii) one by one with replacement iii) one by one without replacement

i) At a time

P (Both Queens)= $\frac{4C_2}{52C_2}=\frac{4\times 3}{52\times 51}=\frac{1}{221}$

ii) with Replacement

P (Both Queens) $=\frac{4C_1}{52C_1}\times\frac{4C_1}{52C_1}=\frac{1}{13}\times\frac{1}{13}=\frac{1}{169}$

iii) with out replacement

P (Both Queens) $P(\text{Both}=\frac{4C_1}{52C_1}\times\frac{3C_1}{51C_1}=\frac{4}{52}\times\frac{3}{51}=\frac{1}{221}$

3) A card is taken from packet of cards and gambler bets that it is a spade or Ace. What are the odds against his winning the bet.

Answer: By Addition Theorem

$P(\text{spade or Ace})=P(\text{spade})+P(\text{Ace})-P(\text{spade and Ace})$

$=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}$

$P(\text{wins bet})=\frac{4}{13}$

$\therefore \text{Total}=13 \quad n(A)=4 \quad n(A')=13-4=9$

Odds against $=\frac{n(A')}{n(A)}=\frac{9}{4}$

4) What is the probability that in a hand of 7 cards drawn from a packet of 52 cards will contain

i) All kings

ii) 3 kings

iii) At least 3 kings

i) $P(\text{All kings}) = \frac{4C_4 \times 48C_3}{52C_7} = \frac{1}{7735}$

ii) $P(3 \text{ kings}) = \frac{4C_3 \times 48C_4}{52C_7} = \frac{45}{7735}$

iii) $P[\text{At least 3 kings}] = P(\text{All}) + P(3)$

$= \frac{1}{7735} + \frac{45}{7735} = \frac{46}{7735}$