CHAPTER - 16

DIFFERENTIAL EQUATIONS

JEE MAIN - SECTION I

1. 2

$y^2 = \pm 4a(x-h)$

$\Rightarrow 2y\,y_1 = \pm 4a \Rightarrow yy_1 = \pm 2a \Rightarrow y_1^2 + yy_2 = 0$

Hence degree = 1, order = 2.

2. 1

Given $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4} = 0$

Taking cube, $\left[\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4}\right]^3 = 0$

Order of highest derivative $= 2$

Degree of highest derivative $= 3$.

3. 3

Given family of curves is,

$x^2 + y^2 - 2ay = 0$ .....(i)

$\therefore\ 2x + 2yy' - 2ay' = 0$ .....(ii)

Putting the value of 2a from (ii) in (i), we get

$2x + 2yy' - \frac{x^2 + y^2}{y}y' = 0$

$\Rightarrow 2xy + (y^2 - x^2)y' = 0$

$\Rightarrow (x^2 - y^2)y' = 2xy$.

4. 3

$\frac{dy}{dx} = -\frac{1+\cos 2y}{1-\cos 2x} \Rightarrow \frac{dy}{dx} = -\frac{2\cos^2 y}{2\sin^2 x} \Rightarrow \sec^2 y dy = -\text{cosec}^2 x dx$

On integrating both sides, we get $\tan y = \cot x + c \Rightarrow \tan y - \cot x = c$.

5. 2

It can be written in the form of homogeneous equation $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Now solve it by putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

6. 2 It is a homogenous equation, solve it by putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

7. 3

$$\frac{dy}{dx} + y\tan x = \sec x$$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

8. 2

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$\Rightarrow \sin^{-1} y = -\sin^{-1} x + \sin^{-1} c$$

$$\Rightarrow \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] = \sin^{-1} c$$

$$\Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} = c.$$

9. 2

$$\frac{dy}{dx} = 1 + x + y + xy \Rightarrow \frac{dy}{dx} = (1+x) + y(1+x)$$

$$\Rightarrow \frac{dy}{dx} = (1+x)(1+y) \Rightarrow \frac{dy}{(1+y)} = dx(1+x)$$

Integrating both sides, $\int \frac{dy}{(1+y)} = \int dx(1+x)$

$$\log(1+y) = x + \frac{x^2}{2} + \log c$$

$$y = ce^{x+(x^2/2)} - 1$$

$$\Rightarrow y(-1) = ce^{-1+(1/2)} - 1 = 0$$

$$\therefore ce^{-1/2} = 1 \Rightarrow c = e^{1/2}$$

$$\therefore y = e^{1/2}e^{x+\frac{x^2}{2}} - 1,\ y = e^{\frac{(x+1)^2}{2}} - 1.$$

10. 1 Rearranging the terms, $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$

$$\text{I.F.} = e^{\int -\frac{t}{1+t}dt} = e^{-t}.(1+t)$$

$\therefore$ Solution is $ye^{-t}.(1+t) = \int (1+t).e^{-t}\frac{1}{(1+t)} + c$

$$ye^{-t}(1+t) = -e^{-t} + c$$

Also, $y(0) = -1 \Rightarrow c = 0 \Rightarrow y(1) = \frac{-1}{2}$.

11. 3

$x\frac{dy}{dx}=y-x\tan\left(\frac{y}{x}\right)$ or $\frac{dy}{dx}=\frac{y}{x}-\tan\left(\frac{y}{x}\right)$

It is homogeneous equation, hence put $y=vx$

we get, $v+x\frac{dv}{dx}=v-\tan v$

$\Rightarrow \int \cot v dv=-\int\frac{dx}{x} \Rightarrow \log(x\sin v)=\log c$

$\Rightarrow x\sin\left(\frac{y}{x}\right)=c$.

12. 2

$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

$(1+y^2)\frac{dx}{dy}+x=e^{\tan^{-1}y}$

$\frac{dx}{dy}+\frac{x}{(1+y^2)}=\frac{e^{\tan^{-1}y}}{(1+y^2)}$

I.F. $=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

$\Rightarrow x\left(e^{\tan^{-1}y}\right)=\int\frac{e^{\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy$

$\Rightarrow x\left(e^{\tan^{-1}y}\right)=\frac{e^{2\tan^{-1}y}}{2}+c$,

$\therefore 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$.

13. 1

Here $\frac{dy}{dx}=\frac{y}{x}\left(\log\frac{y}{x}+1\right)$ .....(i)

It is homogeneous equation

So now put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, then the equation (i) reduces to $\frac{dv}{v\log v}=\frac{dx}{x}$

On integrating, we get $\log(\log v)=\log x+\log c$

$\Rightarrow \log\left(\frac{y}{x}\right)=cx \Rightarrow y=xe^{cx}$.

14. 1

$\frac{dy}{dx}=e^{x-y}+x^2e^{-y}=e^{-y}(e^x+x^2)$

$\Rightarrow e^y dy=(x^2+e^x)dx$

Now integrating both sides, we get $e^y=\frac{x^3}{3}+e^x+c$

15. 3 Given equation is $\frac{dy}{dx} = -\left(\frac{x+y-1}{2x+2y-3}\right)$

Put $x + y = t \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$\therefore \frac{dy}{dx} = \frac{1-t}{2t-3} \Rightarrow \frac{dt}{dx} - 1 = \frac{1-t}{2t-3} \Rightarrow \frac{dt}{dx} = \frac{t-2}{2t-3}$

$\Rightarrow \frac{2t-3}{t-2}dt = dx$. Integrating both sides, we get

$\int \frac{2t-4}{t-2}dt - \int \frac{3-4}{t-2}dt = \int 1dx$

$\Rightarrow 2t + \log(t-2) = x + c$

$\Rightarrow 2(x+y) + \log(x+y-2) = x + c$

$\Rightarrow 2y + x + \log(x+y-2) = c$.

16. 1 $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$

It is linear equation of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{2x}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = (1+x^2)$

Therefore, solution is given by

$y.(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)dx + c = \frac{4x^3}{3} + c$.

But it passes through (0,0) therefore $c = 0$,

Hence the curve is $3y(1+x^2) = 4x^3$.

17. 2 $(x^2 - y^2)dx + 2xy\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Solving we get, $\int \frac{2v}{v^2+1}dv = \int -\frac{dx}{x}$

$\ln(v^2+1) = -\ln x + C$

$(y^2 + x^2) = Cx$

$1 + 1 = C \Rightarrow \mathrm{C} = 2$

$y^2 + x^2 = 2x$

18. 4

$$x-y=t \Rightarrow \frac{dy}{dx}=1-\frac{dt}{dx}$$

$$\Rightarrow 1-\frac{dt}{dx}=t^2 \Rightarrow \int\frac{dt}{1-t^2}=\int 1\,dx$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)=x+\lambda$$

$$\Rightarrow \frac{1}{2}\ln\left(\frac{1+x-y}{1-x+y}\right)=x+\lambda \text{ given } y(1)=1$$

$$\Rightarrow \frac{1}{2}\ln(1)=1+\lambda \Rightarrow \lambda=-1$$

$$\Rightarrow \ln\left(\frac{1+x-y}{1-x+y}\right)=2(x-1)$$

$$\Rightarrow -\ln\left(\frac{1-x+y}{1+x-y}\right)=2(x-1)$$

19. 4

$e^y\frac{dy}{dx}-e^y=e^x$. Let $e^y=t$

$$\Rightarrow e^y\frac{dy}{dx}=\frac{dt}{dx} \Rightarrow \frac{dt}{dx}-t=e^x$$

$$\text{I.F.}=e^{\int -dx}=e^{-x}$$

$$te^{-x}=x+c \Rightarrow e^{y-x}=x+c$$

$$y(0)=0 \Rightarrow c=1$$

$$e^{y-x}=x+1 \Rightarrow y(1)=1+\log_e 2$$

20. 3

Given, $\frac{dy}{dx}=\frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$

$\int\frac{dy}{y\sqrt{y^2-1}}=\int\frac{dx}{x\sqrt{x^2-1}}$

$\Rightarrow \sec^{-1}y=\sec^{-1}x+c$.

At $x=2, y=\frac{2}{\sqrt{3}}, \frac{\pi}{6}=\frac{\pi}{3}+c \Rightarrow c=-\frac{\pi}{6}$

Now, $y=\sec\left(\sec^{-1}x-\frac{\pi}{6}\right)=\cos\left[\cos^{-1}\frac{1}{x}-\cos^{-1}\frac{\sqrt{3}}{2}\right]$

$=\cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x}+\sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$.

$y=\frac{\sqrt{3}}{2x}+\frac{1}{2}\sqrt{1-\frac{1}{x^2}}$.

SECTION II (NUMERICAL)

21. 2

$\frac{2+\sin x}{y+1}\cdot\frac{dy}{dx}=-\cos x, y>0$

$\Rightarrow \frac{dy}{y+1}=\frac{-\cos x}{2+\sin x}dx$

By integrating both sides:

$\ln|y+1|=-\ln|2+\sin x|+\ln K$

$\Rightarrow y+1=\frac{k}{2+\sin x}(y+1>0)$.

$\Rightarrow y(x)=\frac{k}{2+\sin x}-1$

Given $y(0)=1 \Rightarrow \mathrm{K}=4$

So, $y(x)=\frac{4}{2+\sin x}-1$

$a=y(\pi)=1$

$b=\left.\frac{dy}{dx}\right]_{x=\pi}=\left.\frac{-\cos x}{2+\sin x}(y(x)+1)\right]_{x=\pi}=1$

So, (a, b) = (1, 1).

22. 16

$$\frac{dy}{dx}+\left(\frac{2x}{x^2+1}\right)y=\frac{1}{(x^2+1)^2} \text{ (Linear differential equation)}$$

$\therefore$ I.F. = $e^{\ln(x^2+1)}=(x^2+1)$

So, general solution is $y(x^2+1)=\tan^{-1}x+c$

As $y(0)=0 \Rightarrow c=0$

$$\therefore\ y(x)=\frac{\tan^{-1}x}{x^2+1}$$

As, $\sqrt{a}, y(1)=\dfrac{\pi}{32}$

$$\Rightarrow \sqrt{a}=\frac{1}{4} \Rightarrow a=\frac{1}{16}.$$

23. 4

$$\frac{dy}{dx}=\frac{y}{x}=\ln x$$

$$e^{\int\frac{1}{x}dx}=x$$

$$xy=\int x\ln x+C$$

$$\ln x\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}$$

$xy=\dfrac{x}{2}\ln x-\dfrac{x^2}{4}+C$, for $2y(2)=2\ln 2-1$

$$\Rightarrow \mathrm{C}=0,\ y=\frac{x}{2}\ln x-\frac{x}{4}$$

$$\Rightarrow\ y(e)=\frac{e}{4}$$

24. 8

$$\frac{dy}{dx}+2y\tan x=\sin x$$

I.F. = $e^{\int 2\tan x\,dx}=e^{2\ln\sec x}$

I.F. = $\sec^2 x$

$$y\cdot(\sec^2 x)=\int \sin x\cdot\sec^2 x\,dx$$

$$y\cdot(\sec^2 x)=\sec x+C$$

$$x=\frac{\pi}{3},y=0\ \Rightarrow\ C=-2$$

$$\Rightarrow\ y=\frac{\sec x-2}{\sec^2 x}=\cos x-2\cos^2 x$$

$$y=t-2t^2\ \Rightarrow\ \frac{dy}{dt}=1-4t=0\ \Rightarrow\ t=\frac{1}{4}$$

$$\therefore\ \max=\frac{1}{4}-\frac{1}{8}=\frac{2-1}{8}=\frac{1}{8}.$$

25. 8

We have $\frac{dy}{dx}=\frac{x\left(\frac{y}{x}\tan\frac{y}{x}-1\right)}{x\tan\frac{y}{x}}\ \Rightarrow\ \frac{dy}{dx}=\frac{y}{x}-\cot\left(\frac{y}{x}\right)$.

Put $\frac{y}{x}=v\ \Rightarrow\ y=vn\ \Rightarrow\ \frac{dy}{dx}=v+\frac{ndv}{dx}$

Now, we get $v+n\frac{dv}{dx}=v-\cot(v)\ \Rightarrow\ \int(\tan)dv=-\int\frac{dx}{x}$

$\Rightarrow\ \ln\left|\sec\left(\frac{y}{x}\right)\right|=-\ln|x|+c$. As $\left(\frac{1}{2}\right)=\left(\frac{y}{x}\right)\ \Rightarrow$ C = 0 $\Rightarrow\ \sec\left(\frac{y}{x}\right)=\frac{1}{x}$

$\Rightarrow\ \cos\left(\frac{y}{x}\right)=x\ \Rightarrow\ y=x\cos^{-1}(x)$.

So, required bounded are = $\int_0^{1/\sqrt{2}}\underset{II}{x}(\underset{I}{\cos^{-1}x})\,dx=\left(\frac{\pi-1}{8}\right)$.

26. D $\frac{d^3y}{dx^3}=8c_1e^x+c_2e^{-x}-c_3e^{-x}$, Putting into the given differential equation.

We get,8 + 4a + 2b + c = 0, 1 + a + b + c = 0
-1 + a - b + c = 0 $\Rightarrow$ a = -2, b = -1, c = 2.

Thus $\frac{a^3+b^3+c^3}{abc}=-\frac{1}{4}$.

27. A $(x-0)^2+(y-k)^2=k^2 \Rightarrow x^2+(y-k)^2=k^2$; $2x+2(y-k)\frac{dy}{dx}=0$

$y-k=-\frac{xdx}{dy}k=y-\frac{xdx}{dy} \Rightarrow x^2+\left(y\left(y-\frac{xdx}{y}\right)\right)^2=\left(y-\frac{xdx}{dy}\right)^2$

$\Rightarrow x^2+x^2\left(\frac{dx}{dy}\right)^2=y^2+x^2\left(\frac{dy}{dx}\right)^2-\frac{2xydx}{dy}$; $x^2=y^2-\frac{2xydx}{dy}(x^2+y^2)\frac{dy}{dx}-2xy=0$

28. B $(xy^4+y)dx=xdy\ \frac{dy}{dx}=\frac{xy^4+y}{x}$

$\frac{dy}{dx}-\frac{y}{x}=y^4$; $\frac{1}{y^4}\frac{dy}{dx}-\frac{1}{y^3}\frac{1}{x}=1$ Substitute $\left(\frac{1}{y^3}=V\right)$

29. A We have, $\frac{dy}{dx}=\frac{\sin 2y}{x+\tan y} \Rightarrow \frac{dx}{dy}-\frac{x}{\sin 2y}=\frac{\tan y}{\sin 2y}$; $I.F.\ I.F.=e^{-\int\frac{dy}{\sin 2y}}=e^{\log\sqrt{\cot y}}=\sqrt{\cot y}$

Hence the solution is $x\sqrt{\cot y}=\int\frac{\tan y}{\sin 2y}.\sqrt{\cot y}dy+c$

$=\int\frac{1}{2}\frac{\sec^2 y}{\sqrt{\tan y}}dy+c=\sqrt{\tan y}+c$

Since the curve passes through $\left(1,\frac{\pi}{4}\right)$, therefore 1 = 1 + c $\Rightarrow$ c = 0

Thus, the equation of curve is $x=\tan y$

30. A Given equation can be rewritten as $2y\frac{dy}{dx}+y^2\cot x=2\cos x$

Put $y^2=v=2y\frac{dy}{dx}=\frac{dv}{dx}$

$\frac{dv}{dx}+v\cot x=2\cos x$

If $\int_e\frac{\cos x}{\sin x}dx=e^{\log\sin x}=\sin x$

solution is $v\sin x + \int 2\cos x \sin x dx + c$

$y^2 \sin x = \sin^2 x + c$; when $x = \frac{\pi}{2}, y = 1$

$1 = 1 + c = 0$; $y^2 = \sin x$.

31. A Given, $\lim\limits_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1 \Rightarrow x^2 f'(x) - 2xf(x) + 1 = 0$

$\Rightarrow \frac{x^2 f'(x) - 2xf(x)}{(x^2)^2} + \frac{1}{x^4} = 0 \Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x^2}\right) = -\frac{1}{x^4}$

On integrating both sides, we get $f(x) = cx^2 + \frac{1}{3x}$

Also f(1) = 1, $c = \frac{2}{3}$ Hence, $f(x) = \frac{2}{3}x^2 + \frac{1}{3x}$

SECTION IV (More than one correct)

32. AC $\frac{dy}{dx} = \frac{e^{2x}}{e^y} - e^x \Rightarrow e^y \frac{dy}{dx} + e^x e^y = e^{2x}$

Put $e^y = v$ $e^y \frac{dy}{dx} = \frac{dv}{dx}$; $\frac{dv}{dx} + v e^{x} = e^{2x} \Rightarrow I.F\ is\ \int\limits_e e^x dx = e^{e^x}$

$y e^{e^x} = \int e^{e^x}.e^{2x} dx = \int e^{e^x} e^x.e^x dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$y e^{e^x} = \int e^t.dt = t.e^t - e^t + C$

$y e^{e^x} = e^x.e^{e^x} - e^{e^x} + C \Rightarrow y = (e^x - 1) + Ce^{-e^x}$

33. AD $\because \frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y \sin y \Rightarrow \cot y \operatorname{cosec} y \frac{dy}{dx} + \operatorname{cosec} y.\frac{1}{x} = \frac{1}{x^2}$

Putting $\operatorname{cosec} y = v$ we get linear differential equation whose solution is

$v\left(\frac{1}{x}\right) = \int\left(\frac{-1}{x^2}\right)\left(\frac{1}{x}\right) dx = \frac{1}{2x^2} + c \Rightarrow \frac{1}{x\sin y} = \frac{1}{2x^2} + c \Rightarrow 2x = \sin y(1 + 2cx^2)$

or $2x = \sin y(1 + cx^2)$

34. AD Put y = vx

$$\frac{dy}{dx}=3(A+Bx)e^{3x}+Be^{3x}\Rightarrow\frac{dy}{dx}+my=(3+m)(A+Bx)e^{3x}+Be^{3x}$$

$$\Rightarrow\frac{d^2y}{dx^2}+m\frac{dy}{dx}+ny=(9+3m+n)(A+Bx)e^{3x}+B(6+m)e^{3x}=0$$

$\Rightarrow 3m+n+9=0\; and\; m+6=0\Rightarrow$ m=-6 and n=9

35. A,C Given, $y^2=2c\left(x+\sqrt{c}\right)$ On differentiating w.r.t.x, we get $2y\frac{dy}{dx}=2c\Rightarrow c=y\frac{dy}{dx}$

On putting this value of c in Equation (1) we get $y^2=2y\frac{dy}{dx}\left(x+\sqrt{y\frac{dy}{dx}}\right)\Rightarrow y=2\frac{dy}{dx}.x+2y^{1/2}\left(\frac{dy}{dx}\right)^{3/2}$

$\Rightarrow\left(y-2x\frac{dy}{dx}\right)^2=4y\left(\frac{dy}{dx}\right)^3$ Therefore, order of this differential equation is 1 and degree is 3.

SECTION V - (Numerical type)

36. 1 Equations of normal at the point P(x, y) is $Y-y=\frac{dx}{dy}(X-x)$,

Let, $m=\frac{dx}{dy}\Rightarrow mY-my+X-x=0\Rightarrow X+mY-(x+my)=0$

37. 0 $\frac{x\,dy-ydx}{y^2}=\frac{dy}{y}\Rightarrow-d\left(\frac{x}{y}\right)=\frac{dy}{y}$

$\Rightarrow-\frac{x}{y}=\log y\Rightarrow e^{-x/y}=cy\quad\Rightarrow ye^{-x/y}=c$

at x = 0, y = 1, c = 1 $\quad y.e^{x/y}=1$

At y = e $\quad e.e^{x/y}=1\quad e^{x/y}=e^{-1}\quad\Rightarrow$ x = -e

a = -b, b = e $\therefore$ a + b = 0

38. 6 Differentiate both sides with respect to x.

$$f^1(x)=\frac{f(x)+x^2}{x}\Rightarrow\frac{dy}{dx}-\frac{y}{x}=x\Rightarrow y=x(x+1)$$

39. 6 I.F $=\frac{x}{x-1}$ Solution is $\frac{xy}{x-1}=\frac{x^3}{3}+C$ and the answer is 6

SECTION VI - (Matrix match type)

40. A

(A) Given

$$\frac{x+y\frac{dy}{dx}}{y-x\frac{dy}{dx}}=x^2+2y^2+\frac{y^4}{x^2}=\frac{\left(x^2+y^2\right)^2}{x^2}\Rightarrow\frac{d\left(x^2+y^2\right)}{\left(x^2+y^2\right)}=2\frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2}$$

Integrating, we get $-\frac{1}{x^2+y^2}=\frac{-2}{x/y}+c\Rightarrow c=\frac{2y}{x}-\frac{1}{x^2+y^2}$

(B) $\frac{dy}{dx}=e^{-y}\left(e^x+x^2\right)\Rightarrow e^y dy=\left(e^x+x^2\right)dx\Rightarrow\int e^y dy=\int\left(e^x+x^2\right)dx$; $e^y=e^x+\frac{x^3}{3}+c$

(C) $xdy+\left(x+y\right)dx=0\Rightarrow\left(x,dy+ydx\right)+x\,dx=0\Rightarrow d\left(xy\right)+x\,dx=0\Rightarrow xy+\frac{x^2}{2}=c$

$\Rightarrow 2xy+x^2=2c$

(D) $\frac{dy}{dx}=\frac{x}{1+x^2}\Rightarrow dy=\frac{1}{2}.\frac{2x}{1+x^2}\Rightarrow\int dy=\frac{1}{2}\int\frac{2x\ dx}{1=x^2}=\frac{1}{2}\log\left(1+x^2\right)+c\Rightarrow y=\frac{1}{2}\left(1+x^2\right)+c$