SYSTEM OF PARTICLES AND RIGID BODY ROTATION

Rigid Body

A body is said to be rigid only if it possess a definite size and definite shape. This is due to the fact that the interparticle distance between various particles of the rigid body remains the same even under the application of external forces.

The force of interaction between these particles are called internal forces. The internal forces between any two particles are always equal and opposite. Hence such forces cancel out in pairs.

Motion if a rigid body

1. **Pure translational motion**

   In pure translation the instantaneous velocity of every particles remains the same thought the motion.

   eg: sliding motion of a block on an inclined plane.

2. **Pure rotational motion**

   In pure rotational motion, a rigid body rotates about a fixed axis, every particle describes a circular path with centre at the axis and plane of the circle perpendicular to axis of rotation.

✶ Pure rotation is possible about a fixed axis or about a fixed point.

eg: motion of ceiling fan, Giant wheel in circus

3. **Combined motion of translation and rotation**

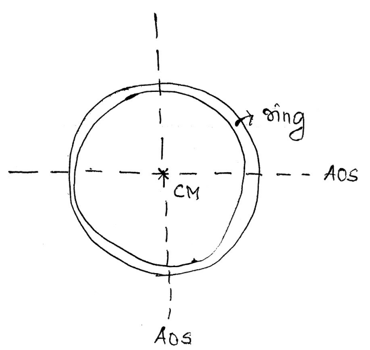
   In combined motion, the instantaneous velocity of every particles of the body are different throughout the motion.

   eg: Rolling motion

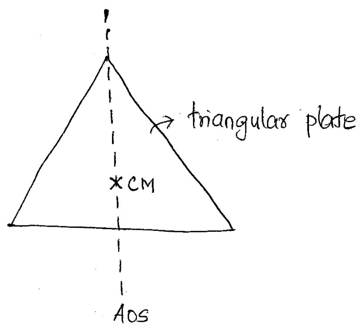
Centre of Mass

Centre of mass is a point at which the entire mass of body or system of bodies are assumed to be concentrated.

✶ if a body posses at least two axis of symmetry, then the CM will lie on the point of intersection



✶ if a body posses only one axis of symmetry, then the CM will lie on the axis depending upon mass distribution of the body.

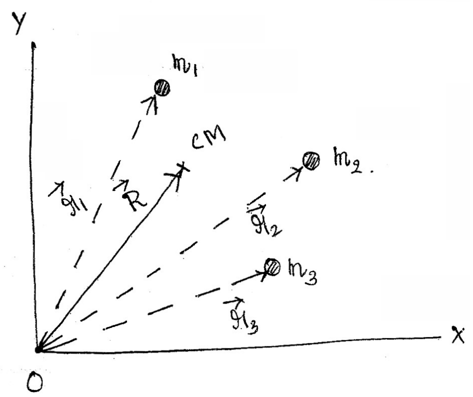


✶ The position of the CM of a s/m is independent of the choice of coordinate s/m.

✶ The position of CM depends on the shape and size of the body and distribution of its mass. Hence it may lie within or outside the material of the body.

✶ The position of CM changes only under the translatory motion. There is no effect of rotatory motion on CM of body.

Centre of mass for a s/m of point masses



in general, $\vec{R} = \frac{\sum m_1 \vec{r}_i}{\sum m_i}$

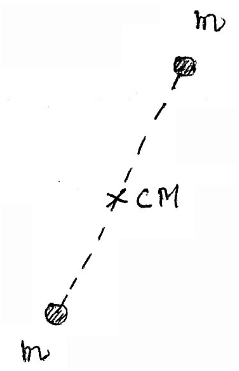
Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{r}_i = r_i\hat{i} + y_i\hat{j} + z_i\hat{k}$

$$\boxed{\therefore\ x = \frac{\sum m_i x_i}{\sum m_i}\ y = \frac{\sum m_i y_i}{\sum m_i}\ z = \frac{\sum m_i z_i}{\sum m_i}}$$

✶ For a 2 particle s/m and if $m_1 = m_2$.

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{x_1 + x_2}{2}$$

$$y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{y_1 + y_2}{2}$$



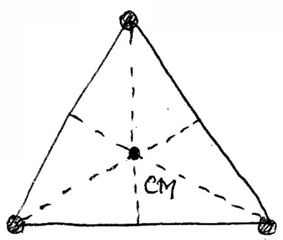
CM of the s/m will lie on the midpoint of the line joining the two point masses.

✶ for 3 particle system and if $m_1 = m_2 = m_3$

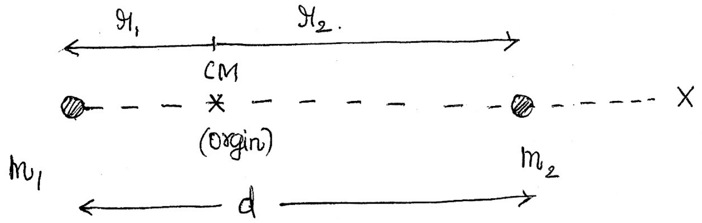
$$x = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{x_1 + x_2 + x_3}{3}$$

$$y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{y_1 + y_2 + y_3}{3}$$

ie the CM of the system will lie on the centroid of the triangle formed by three equal masses



Analysis of two particle system in the frame of CM



$$x = \frac{\sum m_i x_i}{\sum m_i} = 0 \Rightarrow \boxed{\sum m_i x_i = 0} - - - - - (1)$$

$$m_1x_1 + m_2x_2 = 0$$

$$m_1x - r_1 + m_2r_2 = 0$$

$$\boxed{m_1r_1 + m_2r_2} - - - - - (2)$$

ie mr = constant

$$\boxed{r \propto \frac{1}{m}} - - - - (3)$$

✶ From equation (1) and (2) the net mass moment about center of mass of the system of particles is always zero.

✶ From equation (3) it is clear that centre of mass of the system lies closer to the heavier particles always.

✶ From equation (2)

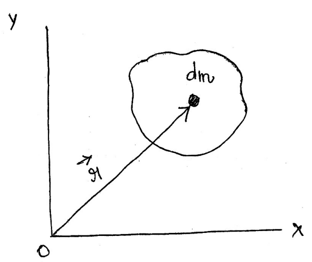
$m_1r_1 = m_2r_2$ Similarly,

$$m_1r_1 = m_2(d - r_1) \quad \boxed{r_2 = \frac{m_1 d}{m_1 + m_2}} \; ; \; \boxed{r_1 = \frac{m_2 d}{m_1 + m_2}}$$

✶ We know that $\vec{R} = \frac{\sum m_i \vec{r}_1}{\sum m_i} = 0$ [CM is at origin] ; $m_1\vec{r}_1 + m_2\vec{r}_2 = 0$

$\boxed{\vec{r}_2 = \frac{-m_1}{m_2}\vec{r}_1}$ ie CM divides internally the line joining the two particles in the inverse ratio of masses.

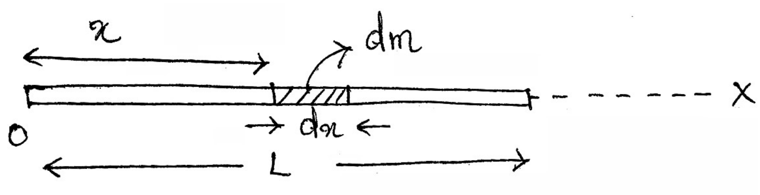
<u>Centre of mass of a rigid body</u>



For uniform mass distribution, $\vec{R} = \frac{\int \vec{r}\, dm}{\int dm}$

ie $\boxed{x = \frac{\int x dm}{\int dm} \quad y = \frac{\int y dm}{\int dm}}$

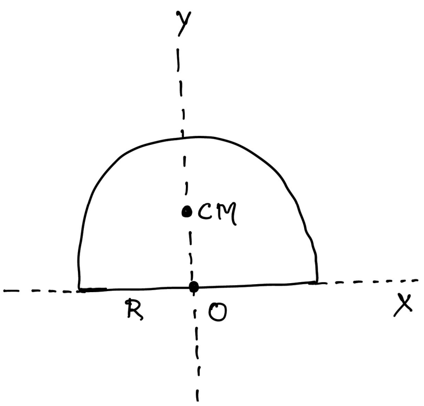
1. **Uniform rod**



Let $\lambda = \dfrac{\text{Total mass}}{\text{Total length}}$; $\lambda = \dfrac{M}{L}$

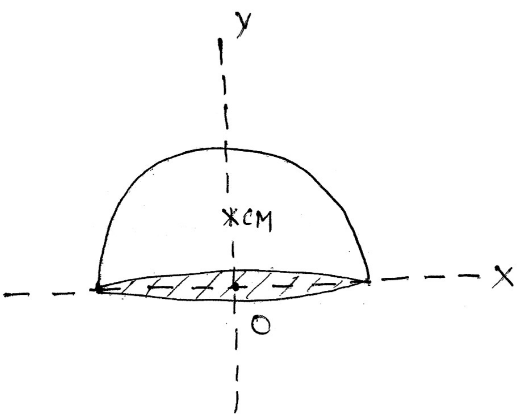
$$x = \frac{\int x\,dm}{\int dm} = \frac{\int_0^L x\,\lambda dx}{\int_0^L \lambda dx} = \frac{L}{2}$$

2. **Uniform semicircular disc**



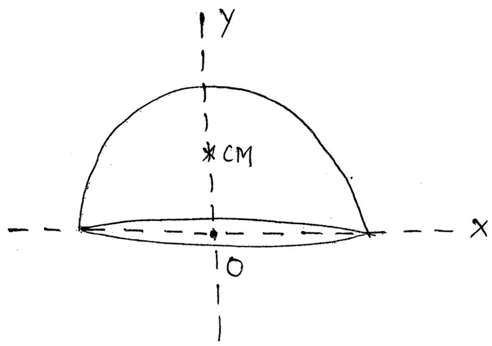
$$\boxed{Y = \frac{4R}{3\pi}}$$

Solid hemisphere



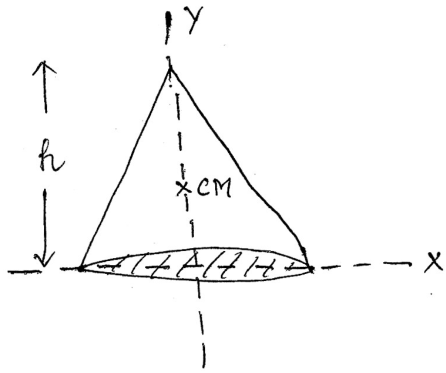
$$\boxed{\begin{aligned} x &= 0 \\ y &= \frac{3R}{8} \end{aligned}}$$

Hollow hemisphere



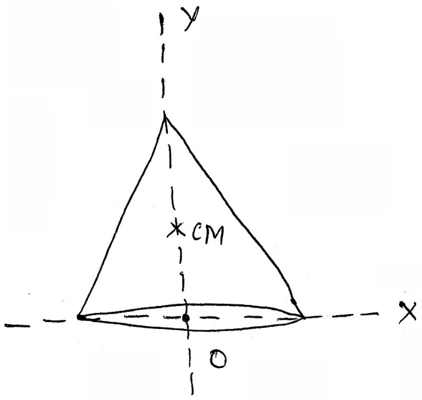
$$\boxed{\begin{aligned}x &= 0\\ y &= \frac{R}{2}\end{aligned}}$$

Solid cone



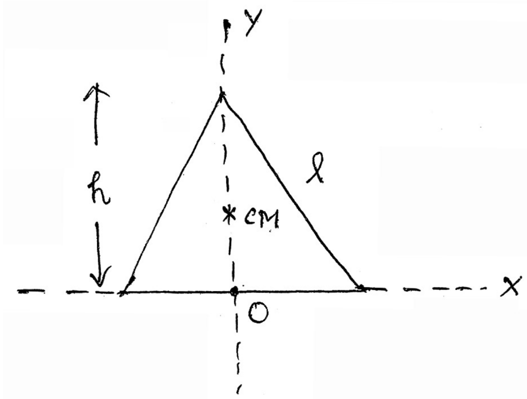
$$\boxed{\begin{aligned}x &= 0\\ y &= \frac{h}{4}\end{aligned}}$$

Hollow cone



$$\boxed{\begin{aligned}x &= 0\\ y &= \frac{h}{3}\end{aligned}}$$

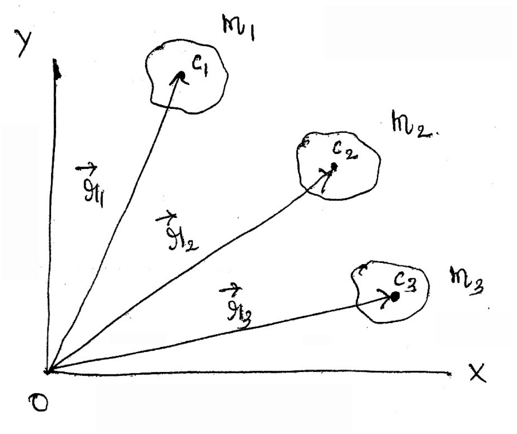
Triangular lamina



$$\boxed{\begin{aligned}x &= 0\\ y &= \frac{h}{3}\end{aligned}}$$

if it is equilateral triangle the $\boxed{y = \frac{\ell}{2\sqrt{3}}}$

Centre of mass for a system of rigid bodies



$$\boxed{\begin{aligned} x &= \frac{\sum m_i x_i}{\sum m_i} \\ y &= \frac{\sum m_i y_i}{\sum m_i} \end{aligned}}$$

Velocity and acceleration of CM

For a system consisting of 'n' number of particles the velocity of CM of the system is given by

$$\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_1\vec{v}_1}{\sum m_i}$$

The acceleration of CM of the system is given by

$$\vec{A}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{m_1 + m_2 + ... + m_n} = \frac{\sum m_i\vec{a}_i}{\sum m_i}$$

Linear momentum of a system

The total linear momentum of a system can be obtained by

$$\vec{P}_{s/m} = \vec{p}_1 + \vec{p}_2 + ... + \vec{p}_n$$

$$= m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n = \sum m_i\vec{v}_i$$

$$= \vec{v}_{cm} \times \sum m_i$$

Let $\sum m_i = M$ [Total mass of the system]

$$\boxed{\therefore \vec{p}_{s/m} = M\vec{V}_{cm}}$$

Motion of CM

We know that $\vec{A}_{cm} = \dfrac{\sum m_i \vec{a}_i}{\sum m_i}$

$$M\vec{A}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n$$

$$= \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \vec{F}_{total} = \vec{F}_{int} + \vec{F}_{ext}$$

$$\left[\text{but } \vec{F}_{int} = 0\right]$$

$$\boxed{\therefore M\vec{A}_{cm} = \vec{F}_{ext}}$$

That is the centre of mass of the system of particles behaves as a point object which is under the action of external forces applied to the system.

Whatever be the system of particles or whatever be the motion of system of particles, the CM always moves according to $\vec{F}_{ext} = M\vec{A}_{cm}$.

Conservation of linear Momentum

$$\vec{F}_{extdt} = \frac{d}{dt}\left(\vec{P}_{s/m}\right) = \frac{d}{dt}\left(M\vec{V}_{cm}\right)$$

if $\vec{F}_{ext} = 0$

$$M\vec{V}_{cm} = \text{constant} \Rightarrow \boxed{\vec{V}_{cm} = \text{constant}}$$

✶ When a bomb at rest explodes, the fragments move in opposite direction such that, the centre of mass will remain at rest [explosion is due to internal forces only]

✶ When a fire cracker explodes in mid air, the individual fragments flies off in different directions, in such a way that the center of mass will continue to move along initial parabolic path.

✶ When a bullet is fired from a gun, it recoils backwards such that the center of mass will remain at rest.

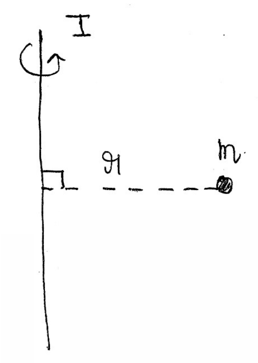
✶ If the center of mass of the system is initially at rest and

if $F_x = 0$ then $\sum m_i \Delta x_i = 0$

if $F_y = 0$ then $\sum m_i \Delta y_i = 0$

Moment of inertia (I)

It is the rotational analogy of mass. Larger the value of moment of inertia, greater will be the tendency of the body to continue in a stable of rest or uniform motion.

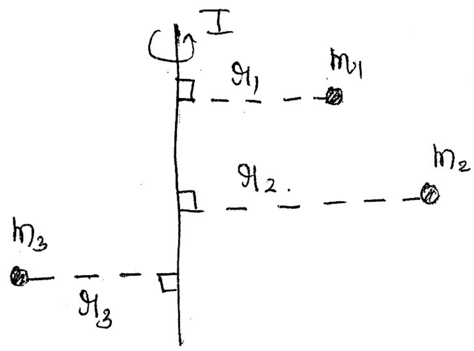


$\boxed{I = mr^2}$

unit $\rightarrow$ kgm^2

it is a tensor quantity

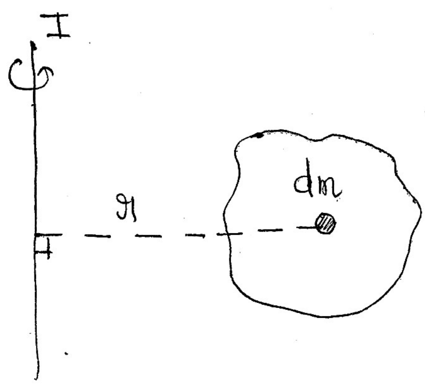
Moment of inertia for a system of point masses



$$I = I_1 + I_2 + I_3 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$\boxed{I = \sum m_i r_i^2}$

Moment of inertia of rigid body



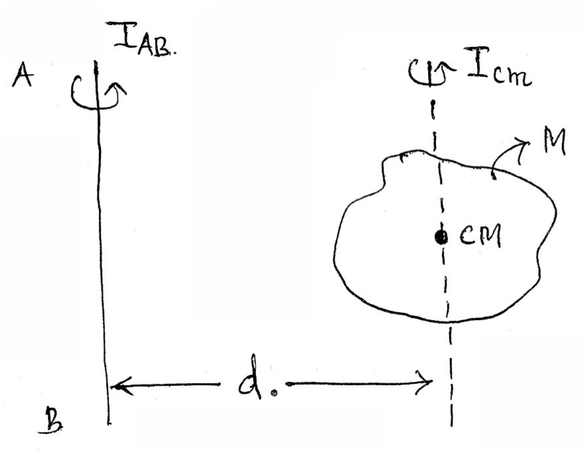
$dI = dmr^2$

$\boxed{I = \int dmr^2}$

Parallel axis theorem

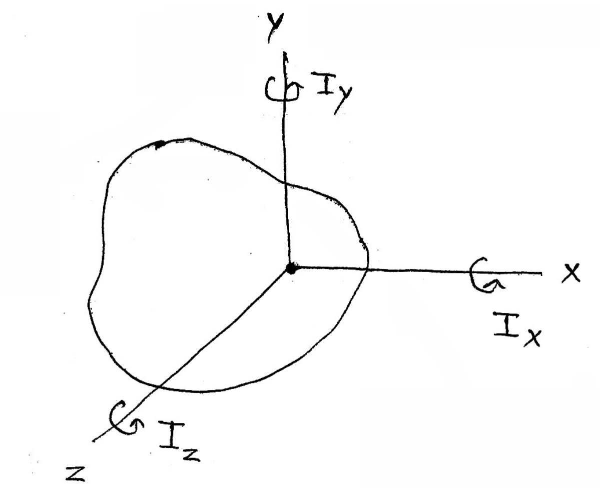
The moment of inertia of a body about any axis equal to moment of inertia about a parallel axis through its center of mass plus the product of mass of the body and the square of its perpendicular distance between the two parallel axis.

$\boxed{I_{AB} = I_{cm} + md^2}$



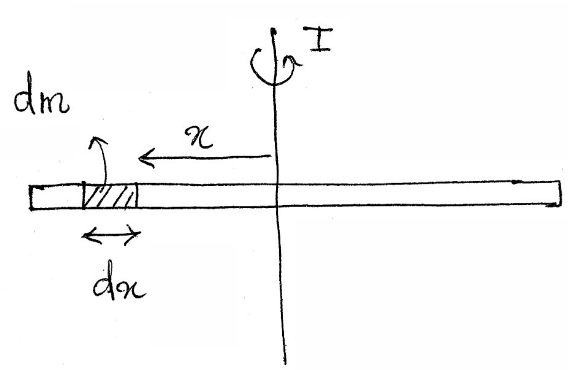
Perpendicular Axis Theorem

The moment of inertia of a plane lamina about an axis perpendicular to plane of the lamina is equal to sum of moment of inertia of the lamina about any two mutually perpendicular axis in the plane of lamina, meeting at a point where the given axis passes through at lamina.



Moment of inertia of a uniform rod

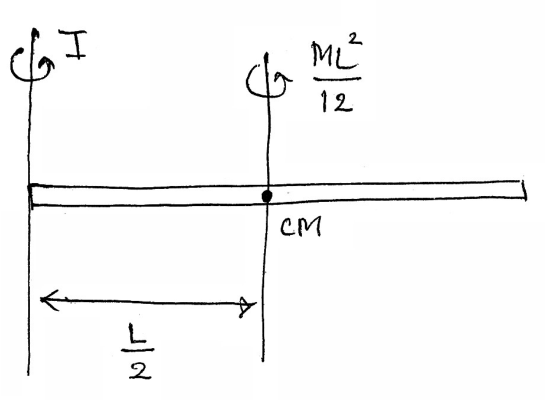
Case I: (about an axis passing through center and perpendicular to rod)



$$dI = dmx^2$$

$$I = \int dmx^2 = \int_{\frac{-L}{2}}^{\frac{+L}{2}} \left(\frac{M}{L}dx\right)x^2 \; ; \; \boxed{L = \frac{ML^2}{12}}$$

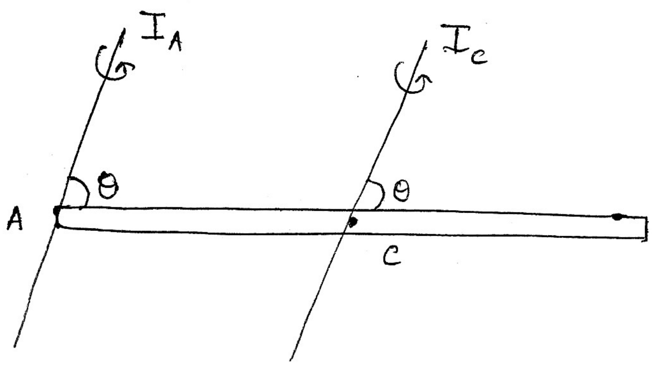
Case II: (about an axis passing through end and perpendicular to rod)



$$I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$$

$$\boxed{I = \frac{ML^2}{3}}$$

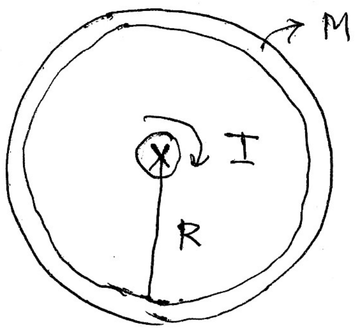
Case III: (axis makes an angle θ with rod]



$$\boxed{\begin{aligned} I_C &= \frac{ML^2}{12}\sin^2\theta \\ I_A &= \frac{ML^2}{3}\sin^2\theta \end{aligned}}$$

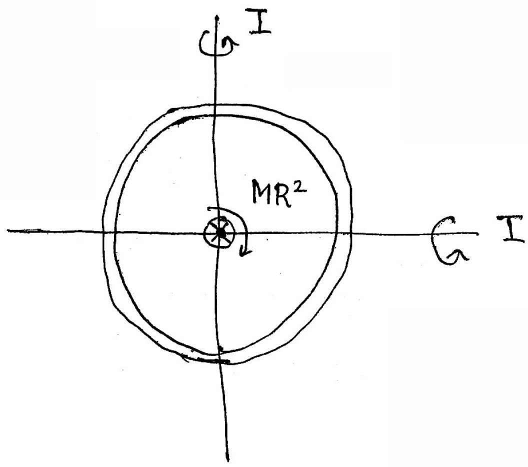
Moment of inertia of ring

Case I: (about an axis passing through center and perpendicular to plane of ring)



$$\boxed{I = MR^2}$$

Case II: (about diameter)

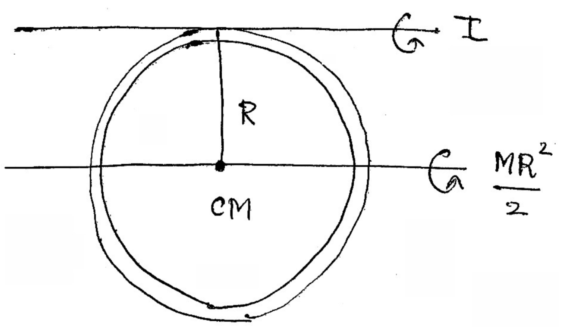


by perpendicular axis theorem.

$$I + I = MR^2$$

$$\boxed{I = \frac{MR^2}{2}}$$

Case III: (about a tangential axis)



$$I = \frac{MR^2}{2} + M(R)^2$$

$$\boxed{I = \frac{3MR^2}{2}}$$

Moment of inertia of a disc

I) About an axis through center and perpendicular to plane of disc

$$\boxed{I = \frac{MR^2}{2}}$$

II) About any diameter, $\boxed{I = \frac{MR^2}{4}}$

III) About any tangent in its plane,

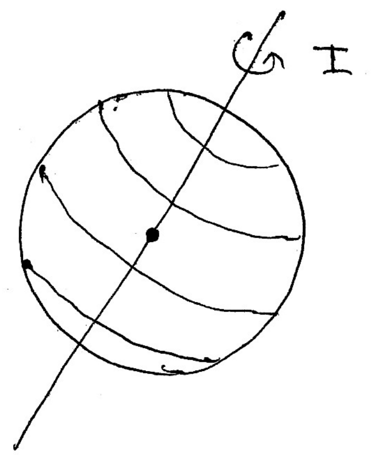
$$\boxed{I = \frac{5MR^2}{4}}$$

IV) About any tangential axis perpendicular to its plane

$$\boxed{I = \frac{3MR^2}{2}}$$

MI of solid sphere

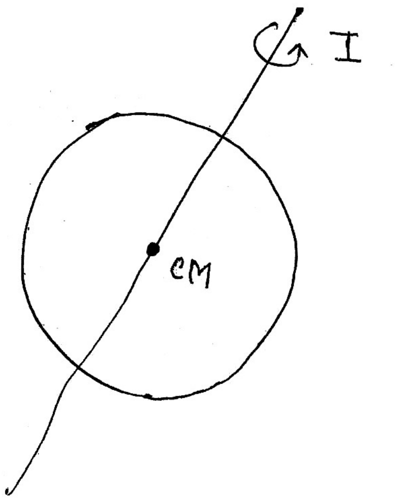
about any diameter.



$$\boxed{I = \frac{2}{5}MR^2}$$

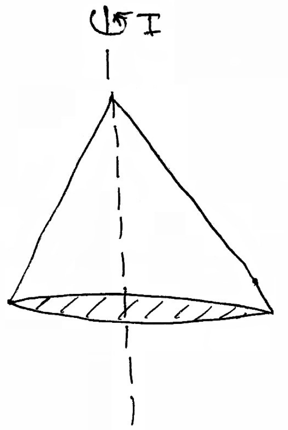
MI of hollow sphere

about any diameter



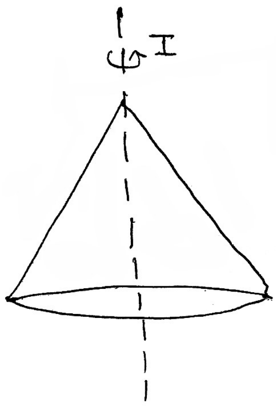
$$\boxed{I = \frac{2}{3}MR^2}$$

MI of solid cone



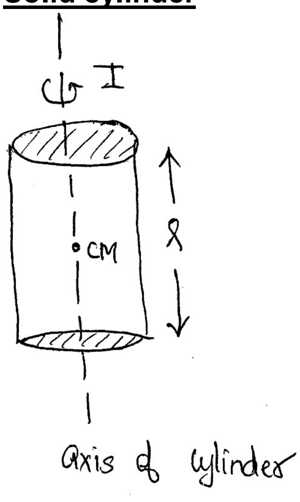
$$\boxed{I = \frac{3MR^2}{10}}$$

MI of hollow cone

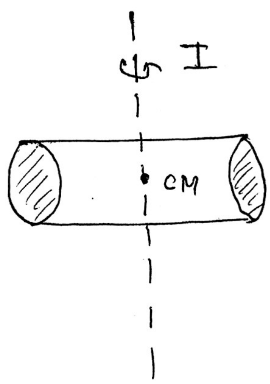


$$\boxed{I = \frac{MR^2}{2}}$$

Solid cylinder

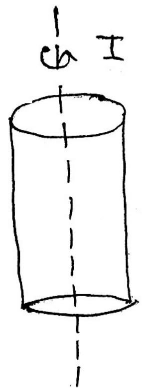


$$\boxed{I = \frac{MR^2}{2}}$$

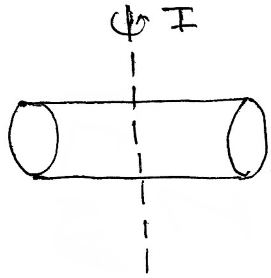


$$I = M\left[\frac{\ell^2}{12} + \frac{R^2}{4}\right]$$

hollow cylinder

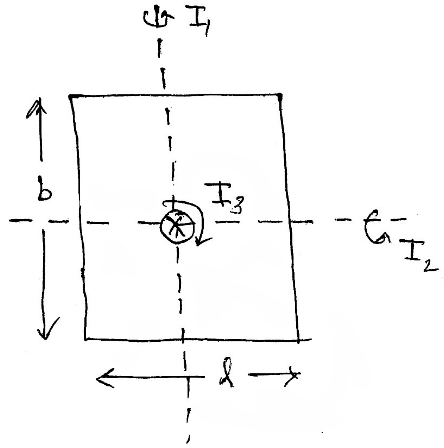


$$\boxed{I = MR^2}$$

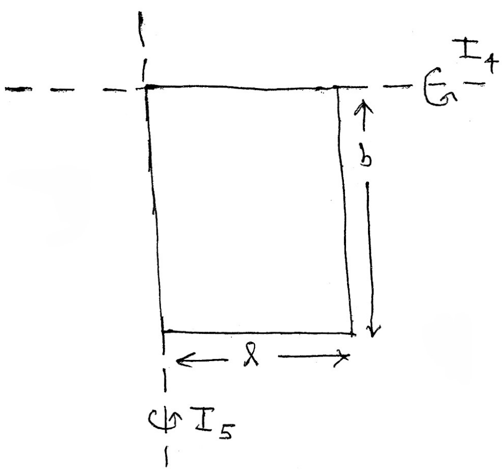


$$\boxed{I = M\left[\frac{\ell^2}{12} + \frac{R^2}{2}\right]}$$

Rectangular lamina

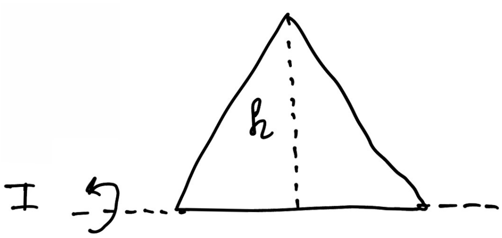


$$\boxed{\begin{aligned} I_1 &= \frac{m\ell^2}{12} \\ I_2 &= \frac{mb^2}{12} \\ I_3 &= \frac{m}{12}\left(\ell^2 + b^2\right) \end{aligned}}$$



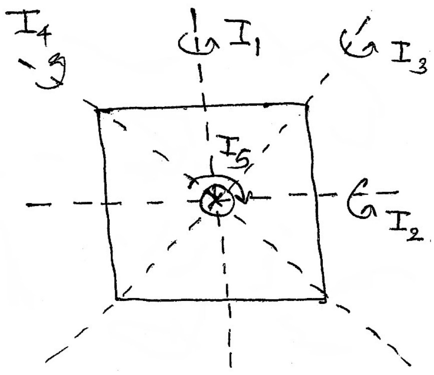
$$\boxed{\begin{aligned} I_4 &= \frac{mb^2}{3} \\ I_5 &= \frac{m\ell^2}{3} \end{aligned}}$$

Triangular lamina



$$\boxed{I = \frac{mh^2}{6}}$$

Square lamina

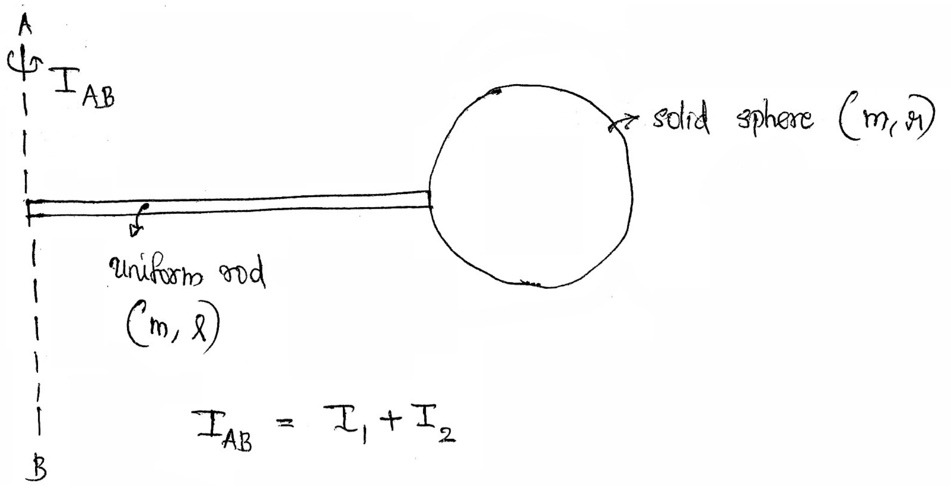


$$\boxed{I_5 = \frac{m\ell^2}{6}}$$

$$\boxed{I_1 = I_2 = I_3 = I_4 = \frac{m\ell^2}{12}}$$

✶ if mass distribution about the axis is same, the moment of inertia also remains the same.

Moment of inertia for a system of rigid bodies

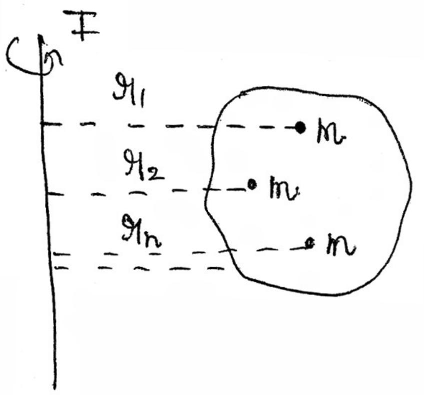


Moment of inertia of rod about AB, $I_1 = \frac{m\ell^2}{3}$

Moment of inertia of sphere about AB, $I_2 = \frac{2}{5}mr^2 + m(\ell + r)^2$

Radius of gyration

Radius of gyration of a body about a given axis of rotation is the perpendicular distance of a point from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass



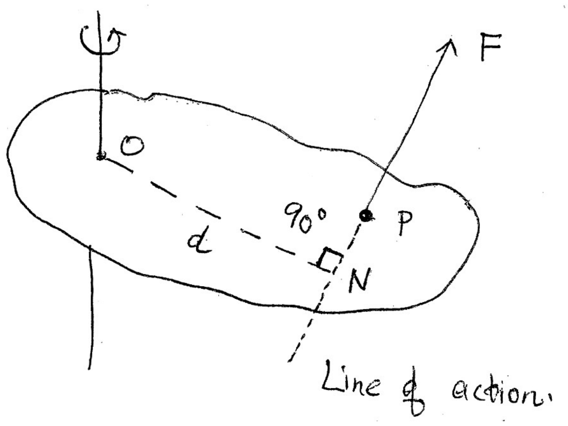
$$I = mr_1^2 + mr_2^2 + \ldots + mr_n^2$$

$$= (nm)\left[\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}\right]$$

$$\boxed{I = MK^2}$$

Torque

The torque or moment of force is the turning effect of the force about the axis of rotation, it is measured as the product of magnitude of the force and the $\perp$ distance between the line of action of the force and the axis of rotation.



$\tau = F \times ON$

$= F \times d$

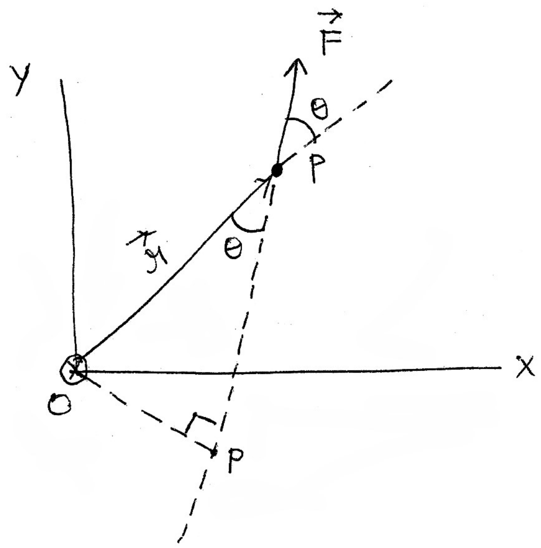
unit $\rightarrow$ Nm

$\rightarrow$ it is a vector quantity

$\rightarrow$ torque is the rotational analogy of force.

✶ Clockwise torque is taken as negative and anticlockwise torque is taken as positive.

Torque acting on a particle

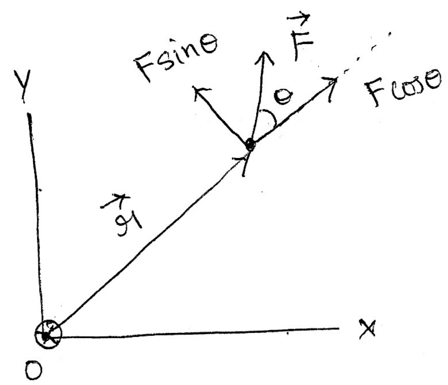


Consider a particle P in xy plane. Suppose its position vector is $\vec{OP}=\vec{r}$. Let $\vec{F}$ be the force acting on the particle. Then the torque action on the particle about origin.

$\tau_0 = F \times OP = F \times r \sin\theta$

$\boxed{\tau_0 = rF\sin\theta}$ $\boxed{\vec{\tau}_0 = \vec{r} \times \vec{F}}$

- ✶ When $\theta = 0^\circ$, the line of action passes through orgin $\sin\theta = 0 \Rightarrow \tau_0 = 0$
- ✶ When $\theta = 90^\circ$ $\tau = rF\sin 90 = rF$ [torque is maximum]



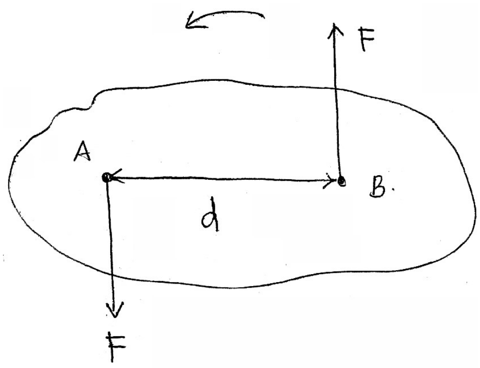
$F_{radial} = F\cos\theta$

$F_{tangential} = F\sin\theta$; $\tau = (F\sin\theta)r$

ie torque is due to tangential component of force.

Force couple

A pair of equal and opposite forces acting along two different lines of action constitute a force couple.



$F_{net} = F - F = 0$

$(\tau_{net})_A = 0 + F \times d = Fd$

$\boxed{\begin{matrix} F_{net} = 0 \\ \tau_{net} \neq 0 \end{matrix}}$ $\therefore$ moment of force = Force × perpendicular distance between two forces.

✶ Torque of a couple is independent of choice of point of rotation. If the net force acting on a system is zero then net torque about any point remains the same.

Conditions for equilibrium of rigid body

1. A rigid body is said to be in translational eqb, if it remains at rest or moving with a constant velocity in a particular direction. For this, the net external force or the vector sum of all the external forces acting on the body must be zero.

   ie $\boxed{\sum \vec{F} = 0}$ it implies $\boxed{\sum F_x = 0 \ \sum F_y = 0 \ \sum F_z = 0}$

2. A rigid body is said to be in rotational eqb, if the body does not rotate or rotates with constant angular velocity. For this, the net external torque or the vector sum of all the torques acting on the body is zero.

   a body is in rotational eqb, $\boxed{\sum \tau_{CM} = 0}$

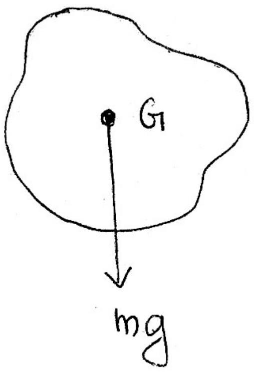
Examples for force couple

✶ When we open the lid of a bottle by turning it, our fingers apply a couple on the lid

✶ When a compass needle is held arbitrarily in any direction in earth's magnetic field, a couple acts on the needle and aligns it along north-south direction.

Centre of Gravity

The centre of gravity of a body is a point where the weight of the body acts and total gravitational torque on the body is zero.



* if the body is extended such that value of g varies from part to past of the body the centre of gravity shall not coincide with centre of mass of the body.

Rotational Kinematic Equations

* $\omega = \omega_0 + \alpha t \; [v = u + at]$

* $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \left[s = ut + \frac{1}{2}at^2\right]$

* $\omega^2 = \omega_0^2 + 2\alpha\theta \left[v^2 = u^2 + 2as\right]$

* $\omega = \frac{d\theta}{dt}\left[v = \frac{ds}{dt}\right]$

* $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$

* $\left[a = \frac{dv}{dt} = \frac{d_s^2}{dt^2} = v\frac{dv}{ds}\right]$

Angular Momentum

It is the rotational analogue of linear momentum. The angular momentum of a particle rotating about an axis is defined as the moment of the linear momentum of the particle about that axis.

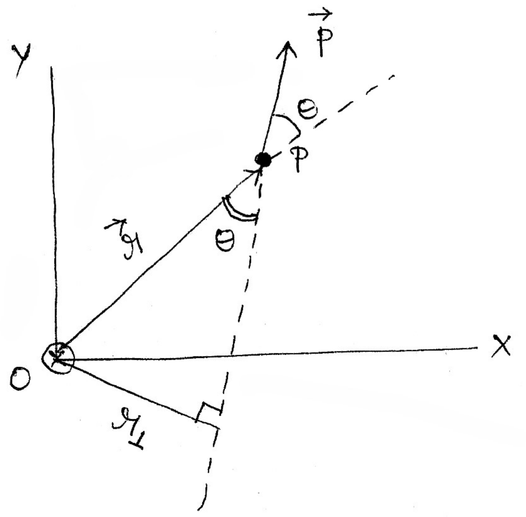
It is measured as the product of linear momentum and $\perp$ distance of its line of action from the axis of rotation

$\boxed{\ell = p \times r_\perp}$ unit $\to$ Kgm^2/s or Js

it is a vector quantity

Angular Momentum in vector form

Consider a particle P of mass m rotating about an axis through O as shown



$\sin\theta = \frac{r_\perp}{r}$ $\qquad$ $r_\perp = r\sin\theta$

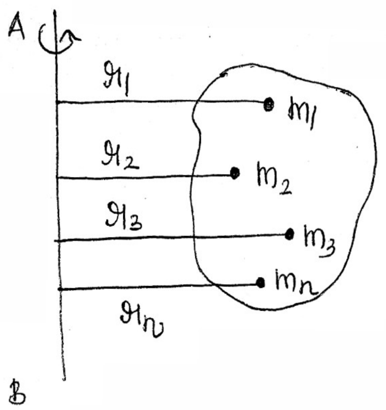
$$\ell_0 = p \times r_\perp = p \times (r\sin\theta)$$

$$\boxed{\vec{\ell}_0 = \vec{r} \times \vec{p}}$$

The direction of angular momentum $\vec{\ell}$ is perpendicular to plane of $\vec{r}$ and $\vec{p}$ in the sense given by right hand thumb rule.

Relation between torque and moment of inertia

Suppose a rigid body consists of n particles of masses m_1, m_2, ... m_n situated at distances r_1, r_2, ...r_n from the axis of rotation AB.



Linear acceleration of first particle $a_1 = r_1\alpha$

Force acting on first particle $F_1 = m_1 a_1 = m_1 r_1 \alpha$

Torque acting on first particle $\tau_1 = F_1 r_1 = m_1 r_1^2 \alpha$

Total torque acting on the rigid body is

$$\tau = \tau_1 + \tau_2 + \tau_3 ... + \tau_n = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + ... + m_n r_n^2 \alpha$$

$$= \left(m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2\right)\alpha = \left(\Sigma m_i r_i^2\right)\alpha$$

$\boxed{\vec{\tau} = I\vec{\alpha}}$ which is analogs to $\vec{F} = m\vec{a}$

Relation between angular momentum and moment of inertia

Consider a rigid body rotating about a fixed axis with uniform angular velocity ω. The body consists of n particles of masses m_1, m_2 ..... m_n.

Linear momentum of first particle, $p_1 = m_1 v_1 = m_1 r_1 \omega$

Angular momentum of first particle $\ell_1 = p_1 r_1 = m_1 r_1^2 \omega$

The angular momentum of rigid body.

$$L = \ell_1 + \ell_2 + ... + \ell_n = m_1 r_1^2 \omega + m_2 r_2^2 \omega + ... + m_n r_n^2 \omega$$

$$L = \left(m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2\right)\omega$$

$\boxed{\vec{L} = I\vec{\omega}}$ which is analogous to $\vec{p} = m\vec{v}$

Newtons law in rotation

Angular momentum of a system

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + ... + \vec{\ell}_n = \Sigma \vec{\ell}_i = \Sigma\left[\vec{r}_i \times \vec{p}_i\right]$$

differentiating w.r.t. time

$$\frac{d\vec{L}}{dt} = \Sigma\left[\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right]$$

$$= \Sigma\left[\vec{v}_i \times m_i \vec{v}_i + \vec{r}_i \times \vec{F}_i\right]$$

$= \Sigma \vec{\tau}_i = \vec{\tau}_{total} = \vec{\tau}_{ext} + \vec{\tau}_{int}$ but $\vec{\tau}_{int} = 0$

$\boxed{\frac{d\vec{t}}{dt} = \vec{\tau}_{ext}}$ which is analogous to $\vec{F}_{ext} = \frac{d\vec{p}}{dt}$

Law of conservation of angular momentum

Suppose the external torque acting on a rigid body due to external is zero. Then

$$\tau = \frac{dL}{dt} = 0$$

$$\boxed{L = \text{constant}}$$

The total external torque acting on a rigid body is zero, the total angular momentum of the body is conserved.

$$\boxed{I_1\omega_1 = I_2\omega_2}$$

Applications of law of conservation of angular momentum

- ✶ A man carrying heavy weights in his hands and standing on a rotating turntable can change the angular speed of the turntable by stretching his arms.
- ✶ A diver jumping from a springboard exhibits somersault in air before touching the water surface.
- ✶ An ice-skater or a ballet dancer can increase her angular velocity by folding her arms and bringing the stretched led close to the other leg.
- ✶ The speed of the inner layers of the whirlwind in a tornado is alarmingly high.

Rotational Kinetic Energy

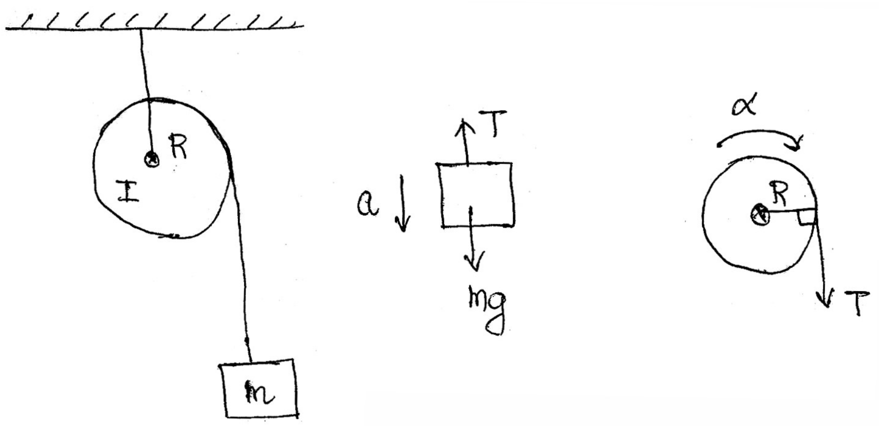
Kinetic energy of first particle $K_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$

kinetic energy of rigid body, $K_r = K_1 + K_2 + ... + K_n$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + ... + \frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}\omega^2\left[m_1r_1^2\omega^2 + m_2r_2^2 + ... + m_nr_n^2\right]$$

$$\boxed{K_r = \frac{1}{2}I\omega^2}$$

1. 

$$mg - T = ma ----(1)$$

Without slipping $\quad T \times R = I\alpha$

$$T = \frac{I\alpha}{R} ----(2)$$

$$a = R\alpha -----(3)$$

$$(2) \Rightarrow mg - ma = \frac{T}{R} \times \frac{a}{R} \Rightarrow \boxed{a = \frac{mg}{m + \frac{I}{R^2}}}$$

Analogy between translational and rotation

$s \to \theta$

$v \to \omega$

$a \to \alpha$

$F \to \tau$

$m \to I$

$P \to L$

$k_t \to k_r$

1. work done, $\omega = \vec{F}.\vec{s}$ $\quad \boxed{\therefore \omega = \vec{\tau}.\vec{\theta}}$

2. Power, $P = \vec{F}, \vec{v}$ $\quad \boxed{P = \vec{\tau}.\vec{\omega}}$

3. Work energy theorem $Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$$\boxed{\tau\theta = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2}$$

Angular impulse

Linear impulse, $\quad \vec{J} = \vec{F}\,\Delta t = m\,\Delta\vec{v} = \Delta p$

Angular impulse, $\quad A.I = \tau\,\Delta t = I\,\Delta\omega$

$$(F \times r_\perp)\Delta t = I\,\Delta\omega$$

$$\Delta p \times r_\perp = I\,\omega$$

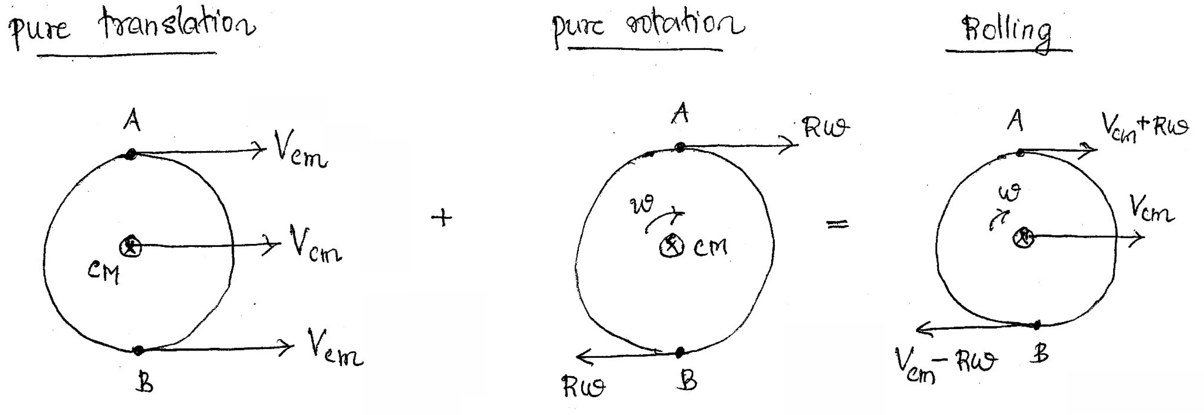
$\boxed{A.I = J \times r_\perp = I\omega}$

Rolling motion (pure rolling motion)

Rolling motion can be regarded as the combination of pure rotation and pure translation. In pure rolling without slipping at any instant of time the contact point of the body is at rest.

ie there is no relative motion between contact point and surface.

pure rolling = pure translation + pure rotation.



For pure rolling the contact point must be at rest

$V_B = 0$

$V_{cm} - R\omega = 0$

$\boxed{V_{cm} = R\omega}$

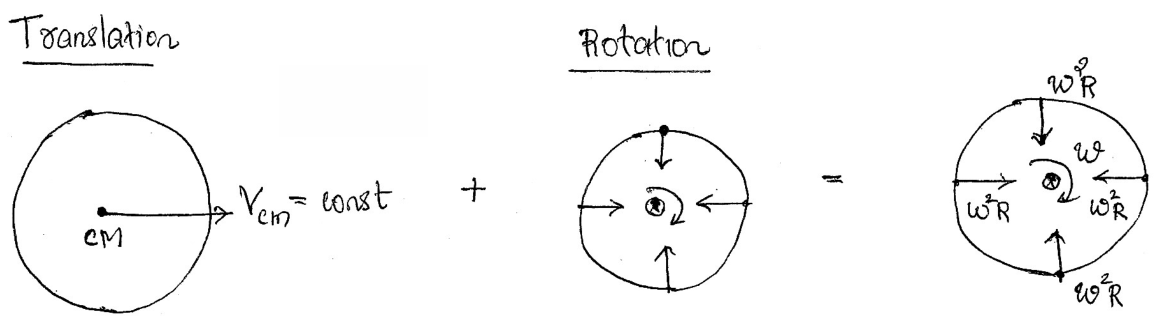
$\therefore V_A = V_{cm} + R\omega = 2R\omega$; $V_B = V_{cm} - R\omega = 0$

✶ if $V_{cm} > R\omega$, the body undergoes forward slipping

✶ if $V_{cm} < R\omega$, the body undergoes backward slipping

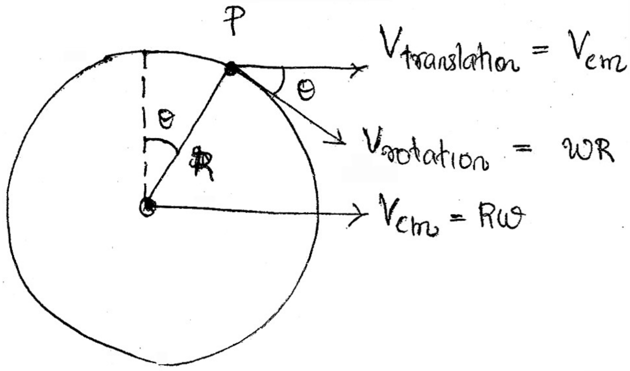
Acceleration of a point in rolling motion

Consider a rolling wheel, each point on the periphery of the wheel is rotating in a circle of radius R, due to which centripetal acceleration of each point on the wheel is $\omega^2 R$.



Velocity of a point in rolling wheel

Consider a point P on a rolling body, the velocity of point P is the vector sum of velocity due to translation and due to rotation.



$$\vec{v}_p = \vec{v}_t + \vec{v}_r$$

$$v_p = \sqrt{v_{cm}^2 + v_{cm}^2 + 2v_{cm}^2 \cos\theta}$$

$$= \sqrt{2}\, v_{cm} \sqrt{1+\cos\theta}$$

$$\boxed{v_p = 2\, v_{cm} \cos\left(\frac{\theta}{2}\right)}$$

Kinetic energy in rolling motion

$$K_{rolling} = K_t + \left(K_r\right)_{cm}$$

$$= \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I_{cm}\, \omega^2 = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} m k^2 \frac{v_{cm}^2}{R^2}$$

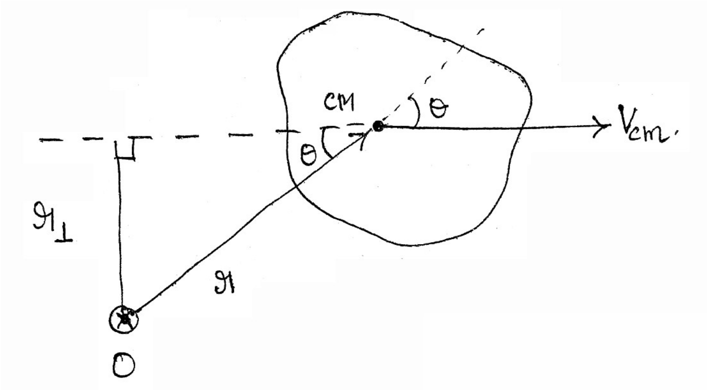
$$\boxed{K_{rolling} = \frac{1}{2} m v_{cm}^2 \left[1 + \frac{K^2}{R^2}\right]}$$

let $\beta = 1 + \frac{K^2}{R^2}$

$$\boxed{\frac{K_t}{K_{rolling}} = \frac{1}{\beta}}$$

$$\boxed{K_{rolling} = K_t \times \beta} \qquad \boxed{\frac{K_r}{K_{rolling}} = 1 - \frac{1}{\beta}}$$

Angular momentum in combined motion

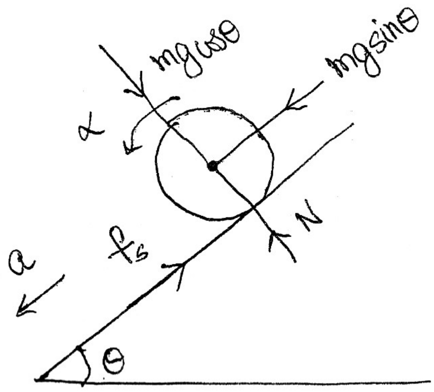


$\boxed{\vec{L}_0 = \vec{L}_t + \left(\vec{L}_r\right)_{cm}}$ $L_t = mv_{cm}r_\perp$ $L_r = I_{cm}\omega$

Rolling motion on an inclined plane

Consider a body of mass m and radius R rolling down an inclined plane of inclination θ. Due to action of $mg\sin\theta$ component the contact point has a tendency to slip downwards as a result the force of static friction acts in the upward direction which provides angular acceleration. So if the inclined plane is sufficiently rough at any constant of the motion $v_{cm} = R\omega$. The body executes pure rolling motion.

Acceleration down an inclined plane



$$mg\sin\theta - f_s = ma ----(1)$$

$$f_s \times R = I\alpha ----(2)$$

$$a = R\alpha \quad -----(3)$$

from (2) $f_s \times R = mk^2 \times \dfrac{a}{R}$

$$\boxed{f_s = ma\frac{k^2}{R^2}} ----(4)$$

(1) $\Rightarrow$ $mg\sin\theta - ma\dfrac{k^2}{R^2} = ma$

$$\boxed{a = \dfrac{g\sin\theta}{\left[1+\dfrac{k^2}{R^2}\right]} = \dfrac{g\sin\theta}{\beta}}$$

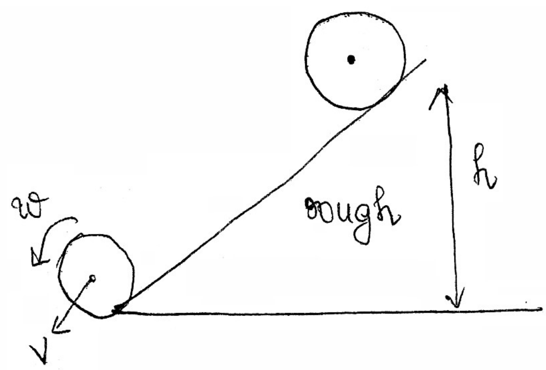
Minimum friction required for pure rolling

$$f_s = ma\dfrac{k^2}{R^2} = \text{In}\ \dfrac{g\sin\theta}{\left[1+\dfrac{k^2}{R^2}\right]}k^2 \qquad \boxed{(f_s)_{min} = \dfrac{mg\sin\theta}{\left[1+\dfrac{R^2}{k^2}\right]}}$$

Minimum coefficient of friction required for pure rolling

$$(\mu_s)_{min} = \dfrac{(f_s)_{min}}{N} = \dfrac{mg\sin\theta}{\left[1+\dfrac{R^2}{k^2}\right]} \times \dfrac{1}{mg\cos\theta} \qquad \boxed{(\mu_s)_{min} = \dfrac{\tan\theta}{1+\dfrac{R^2}{k^2}}}$$

Velocity on reaching bottom



Loss of potential energy = gain in kinetic energy

$$mgh = \dfrac{1}{2}mv^2 \times \beta \qquad \boxed{v = \sqrt{\dfrac{2gh}{\beta}}}$$

Time taken to reach bottom

$$v_{cm} = u_{cm} + a_{cm}t = 0 + a_{cm}t$$

$$t = \dfrac{v}{a} = \dfrac{\sqrt{2gh}}{\sqrt{\beta}} \times \dfrac{\beta}{g\sin\theta}; \qquad \boxed{t = \dfrac{1}{\sin\theta}\sqrt{\dfrac{2h\beta}{g}}}$$