CHAPTER - 12
LIMITS AND REAL FUNCTIONS

JEE MAIN - SECTION I

1. 2 Put $u = \frac{b}{a^x}$ when $x \to \infty \Rightarrow a^x \to \infty$

2. 2 $\cot 4x = \frac{1}{\tan 4x}$ $\frac{1}{\cot^2 2x} = \tan^2 2x$

3. 3 Use L - Hospital's Rule.

4. 2 $\sum x^k = x + x^2 + x^3 + \cdots + x^n$

   $\therefore \frac{0}{0}$ and hence use LHR

5. C Use L. Hospital's Rule

   $\frac{d}{dx} a^x = a^x \log_a$

   $\frac{d}{dx} a^{f(x)} = a^{f(x)} \log_a \frac{d}{dx} f(x)$

6. 3 $\underset{x \to 0}{Lt} \frac{e^{2x} - 2x - 1}{x^2}$

   $= \underset{x \to 0}{Lt} \frac{1 + 2x + \frac{(2x)^2}{2} - 2x - 1}{x^2} = \underline{\underline{2}}$

7. 3 $\sin(\pi - \theta) = \sin\theta$

   $\therefore$ Write $\sin^2(\pi \cos^4\theta) = \sin^2(\pi - \pi\cos^4\theta)$

8. 1 $\underset{x\to\frac{\pi}{2}}{Lt} \dfrac{a^{\cos x}\left(a^{\cot x - \cos x} - 1\right)}{(\cot x - \cos x)}$

9. 3 $\cos A \cos 2A \ldots \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \sin A}$

10. 1 $\underset{x\to 1}{Lt} \dfrac{2}{x-1} \displaystyle\int_{4}^{f(x)} t\,dt$

Integrate and use LHR

or

use Leibnitz Rule.

11. 4 Rationalize and apply limit

12. 4 $\underset{x\to 0}{Lt} \left[1 + \dfrac{a^x + b^x + c^x}{3} - 1\right]^{\frac{1}{x}}$

13. 3,4 use the result $\underset{x\to 0}{Lt} \left[1 + f(x)\right]^{\frac{1}{g(x)}} = e^{\underset{x\to a}{Lt} \frac{f(x)}{g(x)}}$

14. 1 $x \to 3^+ \Rightarrow [x] = 3 \Rightarrow [x]^2 = 9$

$x \to 3^+ \Rightarrow x > 3 \Rightarrow x^2 > 9$

15. 2

$$\underset{m\to\infty}{Lt}\ \frac{m\left[\frac{m(m+1)(2m+1)}{6}\right]}{\frac{m^2(m+1)^2}{4}}$$

16. B

$$f(1^-) = f(1^+) \text{ and}$$

$$1 < x \leq 2 \Rightarrow sig(x+1) = 1$$

17. 2

Use

$$\underset{x\to 0}{Lt}\ \frac{e^x-1}{x} = 1,\ \underset{x\to 0}{Lt}\ \frac{\sin x}{x} = 1 \text{ and}$$

$$\underset{x\to 0}{Lt}\ \frac{\log(1+x)}{x} = 1$$

18. 2

$$x \to 0^- \Rightarrow |x| = -x \text{ and } [x] = -1$$

$$= \frac{x(-1-x)\sin(-1)}{-x}$$

$$= \frac{\cancel{-x}(1+x)\,\overline{\sin 1}}{\cancel{-x}} = -\sin 1 \text{ when}$$

x tends to zero

19. 4

$$\log\left(\frac{a}{b}\right) = \log a - \log b \text{ and}$$

$$\underset{x\to 0}{Lt}\ \frac{\log(1+x)}{x} = 1$$

20. 2

$$= \underset{t \to 0}{Lt} \left(n^{\frac{1}{\sin^2 t}}\right)^{\sin^2 t} \left[\frac{1^{\frac{1}{\sin^2 t}}}{n^{\frac{1}{\sin^2 t}}} + \frac{2^{\frac{1}{\sin^2 t}}}{n^{\frac{1}{\sin^2 t}}} + \dots + 1 \right]^{\sin^2 t}$$

$$= \underset{t \to 0}{Lt} \; n \left[\left(\frac{1}{n}\right)^{\frac{1}{\sin^2 t}} + \left(\frac{2}{n}\right)^{\frac{1}{\sin^2 t}} + \dots + 1 \right]^{\sin^2 t}$$

$$= n \left[\left(\frac{1}{n}\right)^{\infty} + \left(\frac{2}{n}\right)^{\infty} + \dots + 1 \right]^{\sin^2 t}$$

$$= n \left[0 + 0 + \dots + 1 \right]^{\sin^2 0} = n \times 1^0 = \underline{\underline{n}}$$

SECTION II (NUMERICAL)

21. 3

$$\underset{x \to 0}{Lt} \left[1 + 1 - \cos x \sqrt{\cos 2x} \right]^{\frac{1}{\left(\frac{x^2}{x+2}\right)}}$$

$$\text{use } \underset{x \to a}{Lt} \left[1 + f(x) \right]^{\frac{1}{g(x)}} = e^{\underset{x \to a}{Lt} \frac{f(x)}{g(x)}}$$

22. 3

Use when $x \to 0 \Rightarrow \sin^2 x = x^2$

when $x \to 0$

$$\frac{\alpha x\left(1 + x + \frac{x^2}{2}\right) - \beta\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + \gamma x^2 (1 - x)}{x^3}$$

$\alpha - \beta = 0$ $\quad \alpha + \frac{\beta}{2} + \gamma = 0$ $\quad \frac{\alpha}{2} - \frac{\beta}{3} - \gamma = 10$

$\alpha = \beta$ $\quad 2\alpha + \beta + 2\gamma = 0$ $\quad 3\alpha - 2\beta - 6\gamma = 60$

$3\alpha + 2\gamma = 0$ $\qquad 3\alpha + 2\gamma = 0$

$\alpha - 6\gamma = 60$ $\qquad \underline{3\alpha - 18\gamma = 180}$

$\qquad\qquad\qquad 20\gamma = -180$

$3\alpha - 18 = 0 \Rightarrow \alpha = 6$ $\qquad \gamma = -9$

$\alpha + \beta + \gamma = 6 + 6 - 9 = \underline{\underline{3}}$

23. 7 Find $f(1)$ and $f'(1)$
Also use LHR

24. 1 Use L. Hospital's Rule twice

25. 1.2

$$f(a) = \underset{x\to 1}{Lt} \frac{x^a - ax + a - 1}{(x-1)^2} = \frac{0}{0}$$

$$= \underset{x\to 1}{Lt} \frac{ax^{a-1} - a}{2(x-1)} = \frac{0}{0}$$

$$= \underset{x\to 1}{Lt} \frac{a(a-1)x^{a-2}}{2} = \frac{a(a-1)}{2}$$

$$f(a) = \frac{a(a-1)}{2} \quad \therefore f(4) = \frac{4\times 3}{2} = 6$$

$$\frac{f(4)}{5} = \frac{6}{5} = 1.2$$

JEE ADVANCED LEVEL
SECTION III

26. B

$$\underset{h\to 0}{Lt} \frac{1}{2h}\left(\left(1+\frac{h}{8}\right)^{1/3} - 1\right)$$

Now use expansion Method

27. C Use the expansion

$$(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots\right)$$

28. C

$$\underset{x\to 0}{Lt} \frac{a(1+x^2) + b\left(1 - \frac{x^2}{2}\right)}{x^2} = \frac{1}{2}$$

$$\underset{x\to 0}{Lt} \frac{a+b}{x^2} + \frac{\left(a - \frac{b}{2}\right)x^2}{x^2} = \frac{1}{2}$$

$$\therefore \underline{\underline{a+b=0}}$$

29. 3 $\displaystyle \underset{x\to 0}{Lt} \frac{8}{x^8}\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)$

30. 3 Equation with root α and β is

$(x-\alpha)(x-\beta)=0$. Use expansion Method

$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots$

31. D $x\to 0^+ \Rightarrow [x]=0$

$x\to 0^- \Rightarrow [x]=-1$

$\text{RHL} = \underset{x\to 0^+}{Lt} f(x)=0$

$\text{LHL} = \underset{x\to 0^-}{Lt} f(x)=\dfrac{\sin[x]}{[x]}=\dfrac{\sin 1}{-1}=-\sin 1$

SECTION IV (More than one correct)

32. A,B By LHR

$$\alpha = \underset{x\to\frac{\pi}{4}}{Lt} \frac{3\tan^2 x\sec^2 x-\sec^2 x}{-\sin\left(x+\frac{\pi}{4}\right)}$$

$$=\frac{3\times 1\times 2-2}{-\sin\frac{\pi}{2}}=-4$$

$$\beta = e^{\frac{\cos x-1}{\tan x}} = e^{\underset{x\to\infty}{Lt}\frac{-\sin x}{\sec^2 x}} = e^{-\sin 0}=e^0=1$$

$\alpha=-4 \quad \beta=1$ are roots

$(x+4)(x-1)=0$

$x^2+3x-4=0$

$a = 1, b = 3$

33. A,B,C $\frac{|f(x)|}{f(x)}\left\{\begin{array}{l} =1 \text{ when } f(x) > 0 \\ =-1 \text{ when } f(x) < 0 \\ = \text{Does not exist when } f(x) = 0 \end{array}\right.$

34. A,C $\underset{x\to 0}{\text{Lt}} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, a > 0$ is L

$\underline{\text{A) } a = 2 \quad \text{B) } a = 1 \quad \text{C) } L = \frac{1}{64} \quad \text{D) } L = \frac{1}{32}}$

$$\underset{x\to 0}{\text{Lt}} \frac{a - a\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x^2}{4}}{x^4} = L(\text{Finite})$$

$$\underset{x\to 0}{\text{Lt}}\ a - a\left[1 - \frac{1}{2}\frac{x^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\times 2}\frac{x^4}{a^4} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^6}{1\times 2\times 3 \quad a^6} + \ldots\ldots\right] - \frac{x^2}{4} = L$$

$$\underset{x\to 0}{\text{Lt}} \frac{a - a + \frac{1}{2}\frac{x^2}{a} - \frac{\frac{1}{2}\times -\frac{1}{2}}{2}\frac{x^4}{a^3} + \frac{\frac{1}{2}\times\frac{-1}{2}\times -\frac{1}{2}}{6}\frac{x^6}{a^5} \ldots \frac{x^2}{4} = L}{x^4}$$

$$\frac{x^2\left(\frac{1}{2a} - \frac{1}{4}\right)}{x^4} + \frac{1}{8a^3} + (\;)\frac{x^2}{a^5} \to 0 \quad = L$$

Limit is finite $\Rightarrow \frac{1}{2a} - \frac{1}{4} = 0 \Rightarrow a = 2$

When $a = 2$

$$L = \underset{x\to 0}{\text{Lt}} \frac{1}{8a^3} = \frac{1}{8\times 2^3} = \frac{1}{64}$$

35. A,B $f(x) = \frac{a+b}{x^3} + \frac{1+a-b+c}{x^3}x + \frac{\left(\frac{a}{2} + \frac{b}{2} - \frac{c}{2}\right)x^2}{x^3}$

$$a+b=0$$
$$1+a-b+c=0$$
$$\frac{a}{2}+\frac{b}{2}-\frac{c}{2}=0$$
$$a=\frac{-1}{2}\ b=\frac{1}{2}\ c=0$$

$$\text{Limit}=\frac{1}{6}\left[-1-\frac{1}{2}-\frac{1}{2}\right]$$

SECTION V - (Numerical type)

36. 2 $$\underset{x\to 0}{\text{Lt}}\frac{e\left(e^{\cos\alpha^n-1}\right)}{\alpha^m}=\frac{-e}{2}$$

$$\underset{x\to 0}{\text{Lt}}\frac{\left(e^{\cos\alpha^n}-1\right)}{\left(\cos\alpha^n-1\right)}\frac{\left(\cos\alpha^n-1\right)}{\alpha^m}=\frac{-1}{2}$$

$$\underset{x\to 0}{\text{Lt}}\frac{\left(1-\cos\alpha^n\right)}{\alpha^m}=\frac{1}{2};\frac{1-\cos x}{x^2}$$

$$a^m=\left(\alpha^n\right)^2;\ a^m=\alpha^{2n}$$

$$m=2n$$

$$\frac{m}{n}=2$$

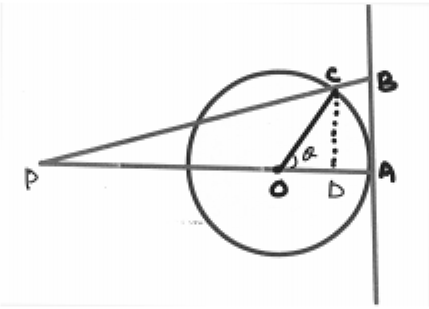
37. 11 $$\underset{x\to 0}{\text{Lt}}\frac{1-\cos\left(1-\cos\frac{x}{2}\right)}{\left(1-\cos\frac{x}{2}\right)^2}\frac{\left(1-\cos\frac{x}{2}\right)^2}{2^m x^n}=1$$

$$\underset{x\to 0}{\text{Lt}}\frac{1}{2}\left(\frac{1-\cos\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2\frac{\left[\left(\frac{x}{2}\right)^2\right]^2}{2^m x^n}=1;\ \underset{x\to 0}{\text{Lt}}\frac{1}{2}\left(\frac{1}{2}\right]^2\frac{x^4}{2^4}\frac{1}{2^m x^n}=1$$

$$\underset{x\to0}{\mathrm{Lt}}\ \frac{1}{2}\ \frac{1}{2^2}\ \frac{x^4}{2^4}\ \frac{1}{2^m x^n}=1;\ \underset{x\to0}{\mathrm{Lt}}\ \frac{x^4}{x^n}=2^{m+7}$$

$$\underset{x\to0}{\mathrm{Lt}}\ x^{4-n}=2^{m+7}\quad \therefore n=4\Rightarrow m+7=0\ \Rightarrow m=-7;\ n-m=4-(-7)=11$$

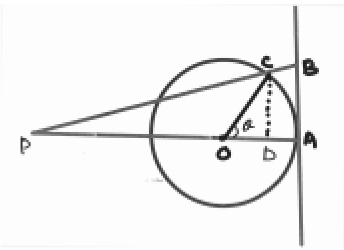
38. C



ΔPAB and ΔPBC are similar

$CD=\sin\theta\ OB=\cos\theta\ \ AB=\theta$

$$\therefore PA=f(\theta)=\frac{\theta(\cos\theta-1)}{(\sin\theta-\theta)}$$



ΔPAB and ΔPBC are similar

$CD=\sin\theta\ OB=\cos\theta\ \ AB=\theta$

$$\therefore PA=f(\theta)=\frac{\theta(\cos\theta-1)}{(\sin\theta-\theta)}$$

$$\lim_{\theta\to0}PA=\lim_{\theta\to0}\frac{\theta(\cos\theta-1)}{(\sin\theta-\theta)}$$

$$=\lim_{\theta\to0}\frac{\theta(\sin\theta)+(\cos\theta-1)}{\cos\theta-1}$$

$$=\lim_{\theta\to 0}\frac{\theta\sin\theta}{1-\cos\theta}+1$$

$$=\frac{1}{\frac{1}{2}}+1=3$$

39. A $\quad \lim_{\theta\to 0}\theta^2\frac{PC}{BC}=\lim_{\theta\to 0}\theta^2\frac{\sin 0}{\theta\sin 0}$

$$=\lim_{\theta\to 0}\left(\frac{\frac{\theta^2\sin 0}{\theta}}{a-\sin 0}\right)$$

$$\lim_{\theta\to 0}\frac{\theta^3}{\theta-\sin\theta}=6$$

SECTION VI - (Matrix match type)

40. A

A) $e^2 \simeq 7.8$

$\therefore [e^2] = 7 \qquad [-e^2] = -8$

$\boxed{A \to q}$

B) When $x \to 0$

$\left[\frac{\sin x}{x}\right] = 0$ and $\left[\frac{\tan x}{x}\right] = 1$

$\boxed{B \to r}$

c) Use L. Hospital's Rule

$\boxed{C \to S}$

D) $x \to 0 \implies \sin^2 x = x^2$

$\boxed{D \to P}$