PRINCIPLE OF MATHEMATICAL INDUCTION AND LINEAR INEQUALITY

Deduction: Deduction is the method of deducing a particular result from a general result
Induction: Induction is the method of obtaining a general result from particular results
Principle of Mathematical induction
Statement : Let p(n) be a statement involving natural number n such that
i) p(1) is true
ii) p(m+1) is true, whenever p(m) is true. Then, p(n) is true for every natural number n
Prove the following by using the principle of mathematical induction for all $n \in N$

(1) $1+3+3^2+....+3^{n-1}=\frac{(3^n-1)}{2}$

Let $p(n)=1+3+3^2+.....+3^{n-1}=\frac{3^n-1}{2}$

For n = 1, p(1) = 1

LHS of p(1) = 1, RHS of p(1) = $\frac{3^1-1}{2}=1$

Thus, p(1) is true
Let the statement p(k) be true

p(k) = $1+3+3^2+.....+3^{k-1}=\frac{3^k-1}{2}$

Now we shall prove that p(k+1) is true

LHS of p(k+1) = $1+3+3^2+...+3^{k-1}+3^k$

$$=\frac{3^{k-1}}{2}+3^k=\frac{3^k-1+2.3^k}{2}=\frac{3.3^k-1}{2}=\frac{3^{k+1}-1}{2}=\text{RHS}$$

$\therefore p(k+1)$ is true
Thus p(k+1) is true whenever p(k) is true
Hence, by the principle of mathematical induction p(n) is true for every $n \in N$

(2) $1^3+2^3+3^3+.....+n^3=\left(\frac{n(n+1)}{2}\right)^2$

Let $p(n):1^3+2^3+3^3+.....+n^3=\left(\frac{n(n+1)}{2}\right)^2$

LHS of p(1) = 1^3 = 1

RHS of p(1) = $\left(\frac{1(1+1)}{2}\right)^2=1$

$\therefore$ LHS of p(1) = RHS of p(1). So p(1) is true

Assume that p (m) is true

$$p(m)=1^3+2^3+3^3+....+m^3=\left(\frac{m(m+1)}{2}\right)^2$$

Now, we shall prove that p(m+1) is true

LHS of $p(m+1)=1^3+2^3+.....+m^3+(m+1)^3$

$$=\left(\frac{m(m+1)}{2}\right)^2+(m+1)^3=(m+1)^2\left(\frac{m^2}{4}+(m+1)\right)$$

$$=\frac{(m+1)^2(m^2+4m+4)}{4}=\frac{(m+1)^2(m+2)^2}{4}=\frac{(m+1)(m+1+2)^2}{2}$$

$\therefore p(m+1)\text{is true}$

This p(m+1) is true whenever p(m) is true

Hence, by the principle of mathematical induction p(n) is true for every $n\in N$

(3) $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+....+\frac{1}{(1+2+3+...+n)}=\frac{2n}{(n+1)}$

Let p(n) : $1+\frac{1}{1+2}+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+....+n}=\frac{2n}{(n+1)}$

LHS of p(1) = 1

RHS of p(1) = $\frac{2\times 1}{1+1}=1$

$\therefore$ LHS of p(1) = KHS of p(1)

So p(1) is true

Assume that p(m) is true

$$p(m) = 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + ... + \frac{1}{1+2+3+...+m} = \frac{2m}{m+1}$$

Now, we shall prove that p(m+1) is true

LHS of $p(m+1) = 1 + \frac{1}{1+2} + ... + \frac{1}{1+2+3+....+m} + \frac{1}{1+2+.....+m+1}$

$$= \frac{2m}{m+1} + \frac{1}{\frac{(m+1)(m+2)}{2}} = \frac{2m}{m+1} + \frac{2}{(m+1)(m+2)}$$

$$= \frac{2}{m+1} + \left(m + \frac{1}{(m+2)}\right) = \frac{2}{(m+1)}\left(\frac{m^2+2m+1}{m+2}\right)$$

$$= \frac{2}{(m+1)} \times \frac{(m+1)^2}{(m+2)} = \frac{2(m+1)^2}{(m+1)(m+1+1)} = \frac{2(m+1)}{(m+1)+1} = \text{RHS of } p(m+1)$$

$\therefore p(m+1)$ is true

Thus p(m+1) true whenever p(m) is true

Hence, by P.M.I p(n) is true for every $n \in N$

(4) 1.2.3+2.3.4+3.4.5+.....+n(n+1)(n+2) = $\frac{n(n+1)(n+2)(n+3)}{4}$

Let $p(n) = 1.2.3 + 2.3.4 + ... + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

LHS of p(1) = 1.2.3=6

RHS of p(1) = $\frac{1(1+1)(1+2)(1+3)}{4} = 6$

$\therefore$ LHS of p(1) = RHS of p(1)

So p(1) is true

Assume that p(m) is true

$$p(m) = 1.2.3 + 2.3.4..... + m(m+1)(m+2) = \frac{m(m+1)(m+2)(m+3)}{4}$$

Now LHS of $p(m+1) = 1.2.3 + 2.3.4 + m(m+1)(m+2) + (m+1)(m+2)(m+3)$

$$= \frac{m(m+1)(m+2)(m+3)}{4} + (m+1)(m+2)(m+3)$$

$$=(m+1)(m+2)(m+3)\left(\frac{m}{4}+1\right)$$

$$=\frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

$$=\frac{(m+1)(m+1+1)(m+1+2)(m+1+3)}{4}$$

=RHS of p(m+1)

$=\therefore$ p(m+1) is true

Thus p(m+1) is true whenever p(m) is true

Hence, by PMI p(n) is true for every natural numbers

(5) $1.3+2.3^2+3.3^3+.....+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$

Let $p(n)=1.3+2.3^2+3.3^3+....+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$

LHS of p(1) = 1.3=3

RHS of p(1) = $\frac{(2\times 1-1)3^{1+1}+3}{4}=\frac{9+3}{4}=3$

$\therefore$ LHS of p(1) = RHS of p(1)

So p(1) is true

Assume that p(m) is true

$$p(m)=1.3+2.3^2+3.3^3+....+m.3^m=\frac{(2m-1)(3)^{m+1}+3}{4}$$

Now, we shall prove that p(m+1) is true

LHS pf $p(m+1)=1.3+2.3^2+.....+m.3^m+(m+1)3^{m+1}$

$$=\frac{(2m-1)3^{m+1}+3}{4}+(m+1)3^{m+1}$$

$$=\frac{(2m-1)3^{m+1}+3+4(m+1)3^{m+1}}{4}$$

$$=\frac{3^{m+1}[2m-1+4m+4]+3}{4}$$

$$=\frac{3^{m+1}(6m+3)+3}{4}$$

$$= \frac{3.3^{m+1}\left(2m+1\right)+3}{4}$$

$$= \frac{\left(2\left(m+1\right)-1\right)3^{m+1+1}+3}{4} = \text{RHS of } p(m+1)$$

$\therefore$ p(m+1) is true

Thus p(m+1)is true whenever p(m) is true. Hence by PMI p(n) is true for every natural number

(6) $1.2+2.3+3.4+.....+n(n+1) = \frac{n(n+1)(n+2)}{3}$

Let $p\left(n\right) = 1.2+2.3+.....+n\left(n+1\right) = \frac{n\left(n+1\right)\left(n+2\right)}{3}$

LHS of p(1) =1.2=2

RHS of p(1) = $\frac{1\left(1+1\right)\left(1+2\right)}{3} = 2$

$\therefore$ LHS of p(1) = RHS of p(1)

So p(1) is true

Assume that p(m) is true

$$p(m) = 1.2+2.3+.....+m\left(m+1\right) = \frac{m\left(m+1\right)\left(m+2\right)}{3}$$

Now, we shall prove that p(m+1) is true

$$p(m+1) = 1.2+2.3+.....+m\left(m+1\right)+\left(m+1\right)\left(m+2\right)$$

$$= \frac{m\left(m+1\right)\left(m+2\right)}{3}+\left(m+1\right)\left(m+2\right)$$

$$= \left(m+1\right)\left(m+2\right)\left(\frac{m}{3}+1\right) = \frac{\left(m+1\right)\left(m+2\right)\left(m+3\right)}{3}$$

$$= \frac{\left(m+1\right)\left(m+1+1\right)\left(m+1+2\right)}{3} = \text{RHS of } p(m+1)$$

$\therefore p(m+1)$ is true

Thus, p(m+1) is true whenever p(m) is true

Hence, by PMI, p(n) is true for all $n \in N$

7. $1.3+3.4+5.7+.....+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$

Let $p(n)=1.3+3.5+.....+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$

LHS of p(1) = 1.3=3

RHS of $p(1)=\frac{1(4.1^2+6.1-1)}{3}=\frac{9}{3}=3$

$\therefore \text{LHS of } p(1)=\text{RHS of } p(1)$

So p(1) is true

Let p(m) be true

$$p(m)=1.3+3.5+.....+(2m-1)(2m+1)=\frac{m(3m^2+6m-1)}{3}$$

Now, we shall prove that p(n+1) is true

LHS of p(m+1) = 1.3+3.5+....+(2m–1)(2m+1)+(2(m+1)–1) (2(m+1)+1)

$$=\frac{m(4m^2+6m-1)}{3}+(2m+1)(2m+3)$$

$$=\frac{m(4m^2+6m-1)+3(4m^2+8m+3)}{3}$$

$$=\frac{4m^3+6m^2-m+12m^2+24m+9}{3}$$

$$=\frac{4m^3+18m^2+23m+9m^2}{3}=\frac{(m+1)(4m^2+14m+9)}{3}$$

$=\frac{(m+1)(4(m+1)^2+6(m+1)-1)}{3}=$ = RHS of p(m+1)

$\therefore$ p(m+1) is true. Thus p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for every $n \in N$

(8) $1.2+2.2^2+3.2^3+.....+n.2^n=(n-1)2^{n+1}+2$

Let $p(n)=1.2+2.2^2+3.2^3+.....+n.s^n=(n-1)2^{n+1}+2$

LHS of p(1) = 1.2=2

RHS of p(1) = 0+2=2

$\therefore$ LHS of p(1) = RHS of p(1), Sp p(1) is true

Assume that p(m) is true

$$p(m)=1.2+2.2^2+3.2^3+.......+m\,2^m=(m-1)2^{m+1}+2$$

Now, we shall prove that p(m+1) is true

$$p(m+1)=1.2+2.2^2+.....+m.2^m+(m+1)2^{m+1}$$

$$=(m-1)2^{m+1}+2+(m+1)2^{m+1}$$

$$=2^{m+1}(m-1+m+1)+2=2^{m+1}(2m)+2$$

$$=2((m+1)-1)2^{m+1}+2=(m+1-1)2^{m+1+1}+2=\text{RHS of }p(m+1)$$

$\therefore p(m+1)$ is true

Thus, p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for every natural number

(9) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=1-\frac{1}{2^n}$

Let $p(n)=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=1-\frac{1}{2^n}$

LHS of $p(1)=\frac{1}{2}$, RHS of $p(1)=1-\frac{1}{2}=\frac{1}{2}$

$\therefore$ LHS of p(1) = RHS of p(1)

So p(1) is true

Assume that p(m) is true, $p(m)=\frac{1}{2}+\frac{1}{4}+.....+\frac{1}{2^n}=1-\frac{1}{2^m}$

Now, we shall prove that p(m+1) is true

LHS of $p(m+1)=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^m}+\frac{1}{2^{m+1}}=1-\frac{1}{2^m}+\frac{1}{2^{m+1}}$

$$=1-\frac{1}{2^m}+\frac{1}{2.2^m}=1-\frac{1}{2}.\frac{1}{2^m}=1-\frac{1}{2^{m+1}}=\text{RHS of }p(m+1)$$

$\therefore p(m+1)$ is true

Thus p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for every natural number n

10. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+....+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

Let $p(n)=\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

LHS of $p(1)=\frac{1}{2.5}=\frac{1}{10}$, RHS of $p(1)=\frac{1}{6\times1+4}=\frac{1}{10}$

$\therefore$ LHS of p(1) = RHS of p(1), so p(1) is true

Assume that p(m) is true, $p(m)=\frac{1}{2.5}+\frac{1}{3.8}+\frac{1}{8.11}+....+\frac{1}{(3m-1)(3m+2)}=\frac{m}{6m+4}$

Now, we shall prove that p(m+1) is true

LHS of $p(m+1)=\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+....+\frac{1}{(3m-1)(3m+2)}=\frac{1}{\left(3(m+1)+\left(3(m+1)+4\right)\right)}$

$$=\frac{m}{6m+4}+\frac{1}{(3m+3-1)(3m+5)}$$

$$=\frac{m}{6m+4}+\frac{1}{(3m+2)(3m+5)}$$

$$=\frac{1}{(3m+2)}\left(\frac{m}{2}+\frac{1}{3m+5}\right)=\frac{1}{(3m+2)}\left(\frac{3m^2+5m+2}{2(3m+5)}\right)$$

$$=\frac{1}{(3m+2)}\frac{(3m+2)(m+1)}{2(3m+5)}=\frac{m+1}{6m+10}=\frac{m+1}{6(m+1)+4}$$

$\therefore p(m+1)$ is true

Thus p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for all natural numbers n

11. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

Let $p(n)=\frac{1}{1.4}+\frac{1}{4.7}+.....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

LHS of $p(1)=\frac{1}{1.4}=\frac{1}{4}, \text{RHS of } p(1)=\frac{1}{3\times1+1}=\frac{1}{4}$

$\therefore$ LHS of p(1) = RHS of p(1), so p(1) is true

Assume that p(m) is true, $p(m)=\frac{1}{1.4}+\frac{1}{4.7}+....+\frac{1}{(3m-2)(3m+1)}=\frac{m}{3m+1}$

Now, we shall prove that p(m+1) is true

$$p(m+1)=\frac{1}{1.4}+\frac{1}{4.7}+.....+\frac{1}{(3m-2)(3m+1)}+\frac{1}{\left(3(m+1)-2\right)\left(3(m+1)+1\right)}$$

$$=\frac{m}{(3m+1)}+\frac{1}{(3m+4)(3m+1)}=\frac{1}{(2m+1)}\left[m+\frac{1}{3m+4}\right]$$

$$= \frac{1}{(3m+1)}\left[\frac{3m^2+4m+1}{3m+4}\right]$$

$$= \frac{1}{(3m+1)}\frac{(3m+1)(m+1)}{(3m+4)} = \frac{m+1}{3(m+1)+1} \text{RHS of } p(m+1)$$

Thus p(m+1) is true whenever p(m) is true

Hence by PMI, p(n) is true for every natural number n

18. $1+2+3+....+n < \frac{1}{8}(2n+1)^2$

For n = 1, $p(1): 1 < \frac{1}{8}(2\times 1+1)^2 = \frac{9}{8}$ is true

$1 < \frac{9}{8}$ is true

$\therefore$ p(1) is true

Let p(m) be true

$$p(m) = 1+2+3+....+m < \frac{1}{8}(2m+1)^2$$

Now $p(m+1) = 1+2+....+m+m+1 < \frac{1}{8}(2m+1)^2 + (m+1)$

$$p(m+1) < \frac{(2m+1)^2+8(m+1)}{8}$$

$$\frac{< 4m^2+4m+1+8m+8}{8}$$

$$\frac{< 4m^2+12m+9}{8}$$

$$\frac{<(2m+3)^2}{8} = \frac{(2(m+1)+1)^2}{8}$$

$\therefore p(m+1)$ is true

Thus p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for all natural numbers

19. n(n+1) (n+5) is a multiple of 3

Let $p(n) = n(n+1)(n+5) = 3k$, where $k \in \text{Integer}$ for n = 1 p(1) = 1(2) (6) = 12 = 3.4 multiple A?

$\therefore$ p(1) is true?

Let p(m) be true

$p(m) = m(m+1)(m+5) = 3p, p \in I$

Now $p(m+1) = (m+1)(m+1+1)(m+1+5)$

$= (m+1)(m+2)(m+6)$

$= (m+1)(m^2+8m+12) = (m+1)(m^2+5m+3m+16)$

$= (m+1)m(m+5) + (m+1)(3)(m+4)$

$= 3p + 3(m+1)(m+4)$

$= 3(p+(m+1)(m+4)) = 3n$ where $n \in 2$

$\therefore p(m+1)$ is true

Thus, p(m+1) is true whenever p(m) is true

Hence by PMI p(n) is true for all natural numbers

20. $10^{2n-1}+1$ is divisible by 11

Let $p(n) = 10^{2n-1}+1 = 11k, k \in I$

$p(1) = 10^{2\times1-1}+1 = 10+1 = 11$

$\therefore$ p(1) is true

Let p(m) between, $p(m) = 10^{2m-1}+1 = 11p, p \in I$

Now $p(m+1) = 10^{2(m+1)-1}+1 = 10^{2m+2-1}+1$

$= 10^2(10^{2m-1})+1$

$= 100(10^{2m-1}+1) - 99$

$= 11p - 11\times9 = 11(p-9) = 11z \qquad z \in I$

$\therefore$ p(m+1) is true

Thus p(m+1) is true whenever p(m) is true

Hence by PMI, the result p(n) is true for all $n \in N$

21. $x^{2n} - y^{2n}$ is divisible by x+y

Let $p(n) = x^{2n} - y^{2n} = (x+y)k$, where $k \in z$

$p(1) = x^2 - y^2 = (x+y)(x-y)$

$\therefore$ p(1) is true

Let p(m) between, $p(m) = x^{2m} - y^{2m} = (x+y)p, p \in z$

Now $p(m+1) = x^{2(m+1)} - y^{2(m+1)} = x^{2m}y^2 - y^{2m}y^2$

$= x^2(x^{2m} - y^{2m}) + x^2y^{2m} - y^{2m}y^2$

$= x^2\left(x^{2m} - y^m\right) + y^{2m}\left(x^2 - y^2\right)$

$= x^2\left(x + y\right)p + y^{2m}\left(x + y\right)\left(x - y\right)$

$= \left(x + y\right)\left[x^2p + y^{2m}\left(x - y\right)\right]$ divisible by x+y

$\therefore p\left(m + 1\right)$ is true

Thus p(m+1) is true whenever p(m) is true
Hence by PMI, p(n) is true for all $n \in N$

22. $3^{2x+2} - 8^{n-9}$ is divisible by 8

Let $p\left(n\right) = 3^{2n+2} - 8n - 9$ is divisible by 8

$p\left(1\right) = 3^4 - 8.1 - 9 = 81 - 17 = 64 = 8 \times 8$

$\therefore$ p(1) is true

Let p(m) between, $p\left(m\right) = 3^{2m+2} - 8m - 9 = 8k, k \in z$

Now $p\left(m + 1\right) = 3^{2(m+1)+2}$

$= 3^{2m+2}3^2 - 8m - 8 - 9$

$= 9\left(3^{2m+2} - 8m - 9\right) + 64m + 64$

$= 9.8k + 64\left(m + 1\right) = 8\left(9k + 8\left(m + 1\right)\right)$ is divisible by 8

$\therefore$ p(m+1) is true
Thus p(m+1) is true whenever p(m) is true
Hence by PMI p(n) is true for all $n \in N$

23. $41^n - 14^n$ is a multiple of 27

Let $p\left(n\right) = 41^n - 14^n$ is a multiple of 27

$p\left(1\right) = 41 - 14 = 27 \;\therefore p\left(1\right)$ is true

Let p(m) between, $p\left(m\right) = 41^m - 14^m = 27k, n \in z$

Now $p\left(m + 1\right) = 41^{m+1} - 14^{m+1}$

$41^m.41 - 14^m.14$

$= 41\left(41^m - 14^m\right) + 41.14^m - 14.14^m$

$= 14.27k + 14^m 27 = 27\left(14k + 14^m\right)$

$\therefore p\left(m + 1\right)$ is true

Thus p(m+1) is true whenever p(m) is true
Hence by PMI the result p(n) is true for all $n \in N$

24. $(2n+7)<(n+3)^2$

Let $p(n)=(2n+7)<(n+3)^2$

$p(1)=9<16, 1.p(1)\text{is true}$

Let p(m) be true, $p(m)=(2m+7)<(m+3)^2$

Now p(m+1) to prove $2(m+1)+7<(m+1+3)^2$

We have $(2m+7)<(m+3)^2\ 2m+7+2<(m+3)^2+2$

$2(m+1)+7<m^2+6m+11$

$2(m+1)+7<(m+4)^2$

$2(m+1)+7<(m+1+3)^2$

$\therefore$ p(m+1) is true

Thus p(m+1) is true whenever p(m) is true

Hence, by PMI p(n) is true for all $n \in N$

Linear inequality

Two real numbers or two algebraic expressions related by the symbol $\leq, \geq, <\text{ or }>$ form an inequality 3<5, 5<8 are the examples of numerical inequalities

$x<5; y\geq 2\ x\leq 4\ x>4$ are some examples of literal inequality

$x>5, y<3$ are strict inequalities

$x\leq 3, y\geq 4$ are slack inequalities

$ax^2+bx+c<0, a\neq 0$ is a quadratic inequality. Equal numbers may be added (subtracted) both sides of an inequality without affecting the sign of inequality

Both sides of an inequality can be multiplied (divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is reversed

If $a<b \Rightarrow -2a>-2b$

If $x>y \Rightarrow 3x>3y$

If $x<y \Rightarrow 2+x<2+y$

If $x<y \Rightarrow x-3<y-3$

If $a<b \Rightarrow \frac{a}{2}<\frac{b}{2}$

If $x\leq y \Rightarrow \frac{x}{-3}\geq \frac{y}{-3}$

If $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

Ex:6.1

(1) Solve 24x<100, when (i) x is a natural number (ii) x is an integer

$$24x < 100, x < \frac{100}{24} (i) 1,2,3,4$$

(i) $x = \{1,2,3,4\}$ (ii) $x = \{......-2,-1,0,1,2,3,4\}$

(2) Solve $-12x > 30$, when (i) $x \in N$ (ii) $x \in z$

$$-12x > 30$$

$$x < \frac{-30}{12} \quad (i) x = \{\ \} (ii) \{.........-2,-3\}$$

$$x < -2.5$$

(3) Solve $5x - 3 < 7$ (i) $x \in Z$ (ii) $x \in R$

$$5x - 3 < 7, 5x < 10, x < 2$$

(i) $x = \{.......-2,-1,0,1\}$ (ii) $x \in (-\infty, 2)$

(4) Solve $3x + 8 > 2$ when (i)$x \in Z$ (ii) $x \in R$

$$3x + 8 > 2,\ 3x > -6,\ x > -2$$

(i) $x = \{-1,0,1,2.....\}$ (ii) $(-2, \infty)$

Solve the inequalities in Ex. 5 to 16 for real x

(5) $4x + 3 < 5x + 7$

$$4x - 5x < 7 - 3 \Rightarrow -x < 4, x > -4$$

$$x \in (-4, \infty)$$

(6) $3x - 7 > 5x - 1$

$$-7 + 1 \ > 5x - 3x$$

$$2x < -6 \Rightarrow x < -3 \ x \in (-\infty, -3)$$

(7) $3(x-1) \le 2(x-3)$

$$3x - 3 \le 2x - 6 \Rightarrow x \le 3 \ x \in (-\infty, -3)$$

(8) $3(2-x) \ge 2(1-x)$

$$6 - 3x \ge 2 - 2x \Rightarrow -x \ge -4, x \le 4, x \in (-\infty, 4]$$

(9) $x+\frac{x}{2}+\frac{x}{3}<11$

$6x+3x+2x<66 \Rightarrow 11x<66, x<6, x\in(-\infty,6)$

(10) $\frac{x}{3}>\frac{x}{2}+1$

$2x>3x+6, -x>6, x<-6 \;\; x\in(-\infty,-6)$

(11) $\frac{3(x-2)}{5}\le\frac{5(2-x)}{3} \Rightarrow 9x-18\le 50-25x$

$34x\le 68, x\le 2 \;\; x\in(-\infty,2]$

(12) $\frac{1}{2}\left(\frac{3x}{5}+4\right)\ge\frac{1}{3}(x-6)$

$\frac{3x}{10}+2\ge\frac{x}{3}-2$

$9x+60\ge 10x-60$

$120\ge x \;\; \therefore x\in(-\infty,120]$

(13) $2(2x+3)-10<6(x-2)$

$4x+6-10<6x-12 \Rightarrow 8<2x \Rightarrow 2x>8 \;\; x\in(4,\infty)$

(14) $37-(3x+5)\ge 9x-8(x-3)$

$37-3x-5\ge 9x-8x+24 \Rightarrow 5x \Rightarrow 8\ge 4x, x\le 2 \;\; x\in(-\infty,2]$

(15) $\frac{x}{4}<\frac{(5x-2)}{3}-\frac{(7x-3)}{5}$

$15x<100x-40-84x+36$

$4<x \quad x\in(4,\infty)$

(16) $\frac{2x-1}{3}\ge\frac{3x-2}{4}-\left(\frac{2-x}{5}\right)$

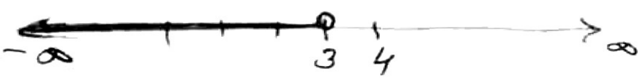
$40x-20\ge 45x-30-24+12x$

$34\ge 17x, \;\; x\le 2 \;\; x\in(-\infty,2]$

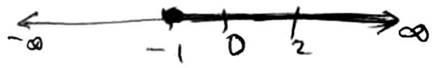
Solution on number line

(17) $3x-2<2x+1$

$x<3$



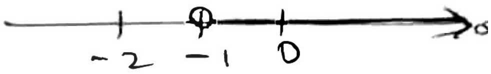
(18) $5x-3 \geq 3x-5, 2x \geq -2, x \geq -1$



(19) $3(1-x) < 2(x+4)$

$3-3x < 2x+8$

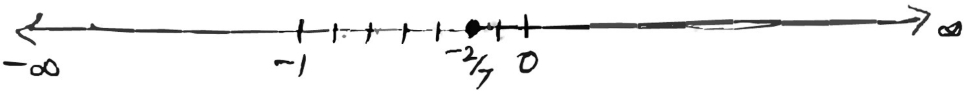
$-5x < 5, -x < 1, x > -1$



(20) $\dfrac{x}{2} \geq \dfrac{5x-2}{3} - \left(\dfrac{7x-3}{5}\right)$

$15x \geq 50x - 20 - 42x + 18$

$7x \geq -2,\ x \geq \dfrac{-2}{7}$



(21) Ravi obtained 70 and 75 marks is first two unit test. Find the minimum marks be should get in the third test to have an average of at least 60 marks

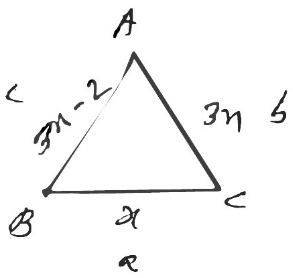
Let x be marks obtained in third test

Average marks = $\dfrac{70+75+x}{3} \geq 60(\text{given})$

$145 + x \geq 180 \ \therefore x \geq 35$

So minimum marks in III test = 35

(25) The longest side of a triangle is 3 times the shortest side and the third side is 2cm shorter than the longest side. If the perimeter of the triangle is at least 61cm, find the minimum length of the shortest side



Perimeter, $a+b+c \geq 61$

$3x + 3x - 2 + x \geq 61$

$7x \geq 63,\ x \geq 9$

Minimum length of the shortest side is 9 cm

(26) A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board of the third piece is to be at least 5 cm longer than the second

Let x be the length of the shortest board

Thus, $x, x+3, 2x$ are the length of all the pieces

$x + x + 3 + 2x \leq 91$ and $2x \geq x + 3 + 5$

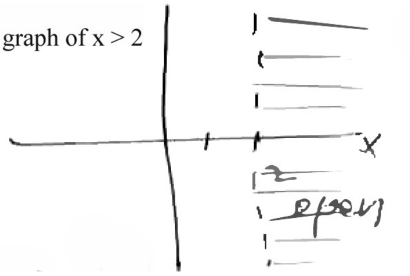
$4x \leq 88$ and $x \geq 8$

$x \leq 22$

$\therefore 8 \leq x \leq 22$

Graph of a strick inequality is an open half plane and graph of slack inequation is a closed half plane.

graph of $x > 2$



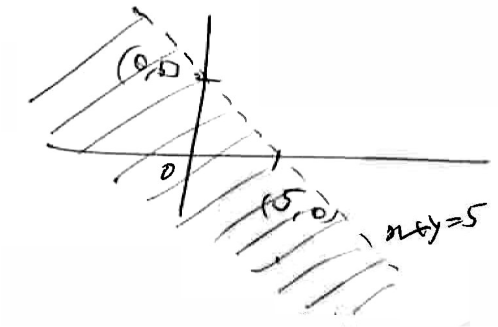
graph of $x \leq 2$m

Ex.6.2 Solve graphically

(1) $x + y < 5$

$x + y = 5$

x	0	5
y	5	0

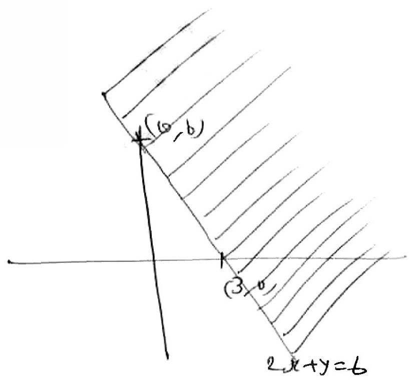


$0(0,0) \Rightarrow 0 + 0 < 5$

True (origin side)

(2) $2x + y \geq 6$

x	x	3
y	6	0

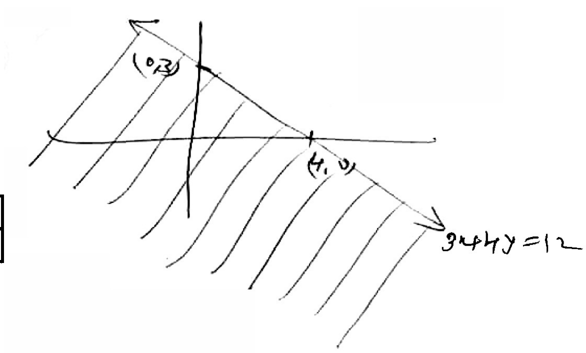


Sub (0,0) $0 + 0 \geq 6$ false

(True orgin side)

(3) $3x + 4y \leq 12$

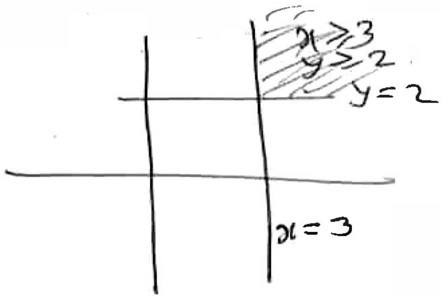
x	0	4
y	3	0



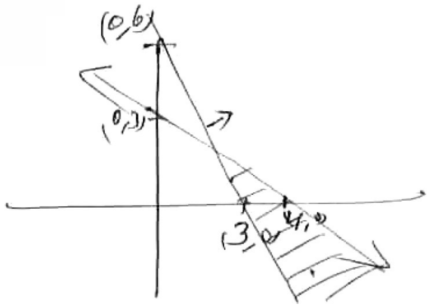
$(0,0)$ $0 + 0 \leq 12$ True

Ex 6.3 Solve graphically

(1) $x \geq 3, y \geq 2$



$2x + y \geq 6, 3x + 4y \leq 12$

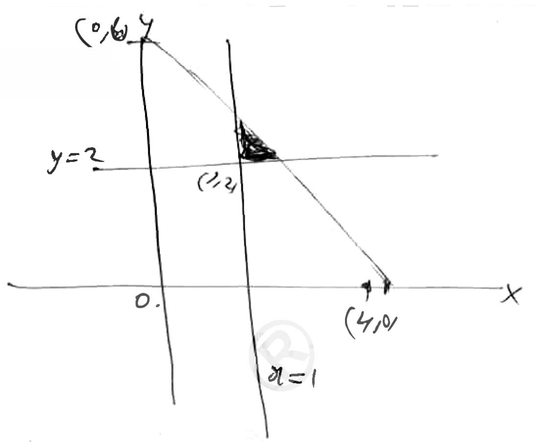


x	0	3
y	6	0

x	0	4
x	3	0

$0 + 0 \geq 6$ false

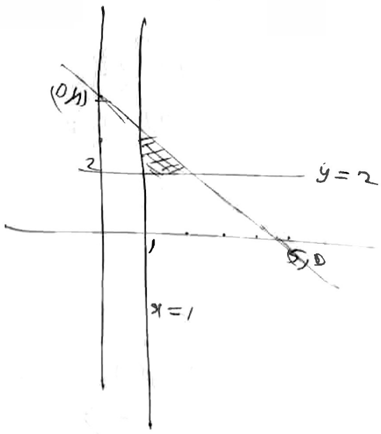
(2) $3x + 2y \leq 12, x \geq 1, y \geq 2$



x	0	4
y	6	0

(9) $5x + 4y \le 20, x \ge 1, y \ge 2$

x	0	4
y	5	0



Miscellaneous 6

Solve

(1) $2 \le 3x - 4 \le 5$

$6 \le 3x \le 9$

$2 \le x \le 3$

(2) $6 \le -3(2x-4) < 12$

$6 \le -6x + 12 < 12$

$6 - 12 \le -6x \le 12 - 12$

$-6 \le -6x \le 0$

$1 \ge x \ge 0, x \in [0,1]$

(4) $-15 < \dfrac{3(x-2)}{5} \le 0 \Rightarrow -75 < 3x - 6 \le 0$

$-69 < 3x \quad x > -23$

(5) $-12 < 4 - \dfrac{3x}{-5} \le 2 \Rightarrow -60 < 20 + 3x \le 10$

$-80 < 3x < -10 \qquad \dfrac{-80}{3} < x < \dfrac{-10}{3}$

(6) $-7 \le \dfrac{3x+11}{2} \le 11 \Rightarrow -14 \le 3x + 11 \le 22$

$-25 \le 3x \le 11 \Rightarrow \dfrac{-25}{3} \le x \le \dfrac{11}{3}$

(7) $5x+1>-24,\quad 5x-1<24$ graphically on a number line

$5x>-25 \qquad 5x<25$

$x>-5-(1) \qquad x<5-(2)$

From (1) and (2) $x\in(-5,\infty)\cap(-\infty,5)\,x\in(-5,5)$

(8) $2(x-1)<x+5 \quad 3(x+2)>2-x$

$2x-2<x+5 \qquad 3x+6>2-x$

$x<7 \qquad 4x>-4$

$x>-1$

$x\in(-\infty,7)\cup(-1,\infty)$

$x\in(-1,7)\,x\in(-1,7)$

(9) $3x-7>2(x-6) \quad 6-x>11-2x$

$3x-7>2x-12 \qquad 6-x>11-2x$

$x>-5-(1) \qquad x>5-(1)$

$x\in(-5,\infty)\cap(5,\infty)$

$x\in(+5,\infty)$

(10) $5(2x-7)-3(2x+3)\le 0, \quad 2x+19\le 6x+47$

$10x-35-6x-9\le 0,$

$4x\le 44 \qquad 19-47\le 6x-2x$

$x\le 11-(1) \qquad -28\le 4x$

$x\in(-\infty,11]\cup[-7,\infty) \qquad x\ge -7$

$x\in[-7,11]$

(1) $|x| < a \Rightarrow -a < x < a$ or $x \in (-a, a)$ where $a \in R$

(2) $|x| \le a \Rightarrow -a \le x \le a$ or $x \in [-a, a]$

(3) $|x| > a \Rightarrow x \in (-\infty, -a) \cup (a, \infty)$ or $(x < -a$ or $x > a)$

(4) $|x| \ge a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$ or $(x \le -a$ or $x \ge a)$

(5) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$ where $a < b\, a, b \in R$

(6) $(x-a)(x-b) \le 0 \Rightarrow x \in [a, b]$

(7) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$

(8) $(x-a)(x-b) \ge 0 \Rightarrow x \in (-\infty, a] \cup [b, \infty)$

(9) $|x| = a \Rightarrow x = \pm a$

(10) If a,b are +ve numbers then $A.M \ge G.M$ ie $\frac{a+b}{2} \ge \sqrt{ab}$

(11) If a<b then $\frac{1}{a} > \frac{1}{b}, -a > -b$

(12) If $a, b, c \in R$ such that $b^2 - 4ac < 0$ then

(i) $a > 0 \Rightarrow ax^2 + bx + c > 0$ for are real $x \in R$

(ii) $a < 0 \Rightarrow ax^2 + bx + c < 0$ for all real $x \in R$

Partial fraction

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$\Rightarrow 1 = A(x+1)(x+2) + Bx(x+2) + c(x+1)x$$

Put x = 0, I=2A, $A = \frac{1}{2}$

Put x = –1, $1 = B(-1)(-1+2) \Rightarrow B = -1$

Put $x = -2, 1 = c(-2)(-2+1) = 2c, c = \frac{1}{2}$

$$\therefore \frac{1}{x(x+1)(x+2)} = \frac{\frac{1}{2}}{x} - \frac{1}{x+1} + \frac{\frac{1}{2}}{x+2}$$

(1) $\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$

$I = A(x-1)(x-2) + B(x)(x-2) + c(x)(x-1)$

Put x = 0, I = 2A, $A = \frac{1}{2}$

Put x = 1, $I = B(1)(1-2) = -B, B = -1$

Put x=2, $1 = C(2)(2-1) = 2C, C = \frac{1}{2}$

$$\frac{1}{x(x-1)(x-2)} = \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}$$

LEVEL I

1. B Let $S = 1.3 + 2.3^2 + 3.3^3 + ... + n.3^n .....(1)$

$$3S = 1.3^2 + 2.3^3 + ... + (n-1)3^n + nB^{n+1} .....(2)$$

$$(1)-(2) \Rightarrow -2S = 3 + 3^2 + 3^3 + ... + 3^n - n.3^{n+1}$$

$$-2S = \frac{3(3^n - 1)}{3-1} - n.3^{n+1}$$

$$-2S = \frac{3^{n+1} - 3 - 2n.3^{n+1}}{2}\ ;\ S = \frac{(2n-1)3^{n+1} + 3}{4}$$

2. B $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ... + \frac{1}{n(n+1)(n+2)} =$

$$t_n = \frac{1}{n(n+1)(n+2)} = \frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2}$$

$$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)\frac{-1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n} - \frac{1}{n+1}\right)\frac{-1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{n+2} - \frac{1}{n+1}\right)$$

$$= \frac{1}{2}\left(\frac{n^2 + 3n + 2 + 2n + 2 - 2n - 4}{2(n+1)(n+2)}\right) = \frac{1}{4}\frac{n(n+3)}{(n+1)(n+2)}$$

3. B $\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right).....\left(1 + \frac{2n+1}{n^2}\right)$

$$= \left(\frac{4}{1}\right)\left(\frac{9}{4}\right)\left(\frac{16}{9}\right).....\left(\frac{n^2 + 2n + 1}{n^2}\right)$$

$$= n^2 + 2n + 1 = (n+1)^2$$

4. B $|3-4x| \geq 9$

$3-4x \leq -9,\quad 3-4x \geq 9$

$12 \leq 4x \qquad -6 \geq 4x$

$x \leq \frac{-3}{2} \qquad \text{Solution}\left(-\infty, \frac{-3}{2}\right] \cup [3, \infty)$

$x \geq 3$

5. A $A.M \geq G.M$

$\frac{a^2+b^2}{2} \geq \sqrt{a^2b^2}, \frac{b^2+c^2}{2} \geq \sqrt{b^2c^2}, \frac{c^2+a^2}{2} \geq \sqrt{a^2c^2}$ adding

$\frac{2\left(a^2+b^2+c^2\right)}{2} \geq ab+bc+ca$

$\therefore \frac{a^2+b^2+c^2}{2} \geq \frac{ab+bc+ca}{2}$; $a^2+b^2+c^2 \geq ab+bc+ca$

6. A $\frac{x+4}{x-3} - 2 < 0, \frac{x+4-2x+6}{x+3} < 0$

$\frac{10-x}{x-3} < 0 \Rightarrow \frac{x-10}{x-3} > 0 \left(\frac{x-10}{x-3}\right)^{(x-3)^2 > 0 (x-3)^2}$

$(x-10)(x-3) > 0 \Rightarrow x < 3 \text{ or } x \geq 10$

$x \in (-\infty, 3) \cup (10, \infty)$

7. D $5x+2 < 3x+8 \qquad \frac{x+2}{x-1} < 4,\ \frac{x+2}{x-1} - 4 < 0$

$2x < 6 \qquad \frac{x+2-4x+4}{x-1} < 0 \Rightarrow \frac{6-3x}{x-1} < 0, \div(-3), \frac{x-2}{x-1} > 0$

$x < 3 \qquad (x-2)(x-1) > 0\ x \in (-\infty, 1) \cup (2, \infty)......(2)$

Solution $(2,3) \cup (-\infty, 1)$

8. C $0<|3x+1|<\frac{1}{3} \Rightarrow |3x+1|<\frac{1}{3}$

$-\frac{1}{3}<3x+1<\frac{1}{3} \Rightarrow \frac{-4}{3}<3x<\frac{-2}{3}$

$\frac{-4}{9}<x<\frac{-2}{9}$ $x \in \left(\frac{-4}{9},\frac{-2}{9}\right)-\left\{-\frac{1}{3}\right\}$

9. C

$-3 \le x-1 \le 3,$ $|x-1| \ge 1$

$-2 \le x \le 4...(1),$ $x-1 \le -1$ or $x-1 \ge 1$

$x \in [-2,4]...(1)$ $x \le 0$ or $x \ge 2$

$x \in (-\infty,0] \cup [2,\infty)$

$x \in [-2,0] \cup [2,4]$

10. D $\left|\frac{1}{x}-2\right|<4, \Rightarrow -4<\frac{1}{x}-2<4 \Rightarrow -2<\frac{1}{x}<6$

$-\frac{1}{2}>x>\frac{1}{6}$

$-\frac{1}{2}>x, \left(-\infty,-\frac{1}{2}\right).....(1)$

$x>\frac{1}{6}\left(\frac{1}{6},\infty\right).....(2)$ $\left(-\infty,-\frac{1}{2}\right) \cup \left(\frac{1}{6},\infty\right)$

LEVEL II

11. B $\frac{1}{5}(n^5)+\frac{1}{3}(n^3)+\frac{7}{15}n=\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=1$

12. B Let S(K) = 1+3+5+.....+(2K-1)=3+k^2

$S(K) \Rightarrow S(K+1)$

13. D $p(n)=n^2-n+41$ is prime

p(3)=9–3+41=47 is prime

p(5) = 25–5+41= 61 is prime

14. C $2^{3n}-7n-1, n=1$ $8-7-1=0$

$n=2,$ $64-14-1=49$

15. D $n^3 + 2n = 3$

$8 + 4 = 12$

16. C $10^n + 3\left(4^{n+2}\right) + 5 \Rightarrow 10 + 3\left(64\right) + 5 = 207 \div 9$

17. A $a^{2n-1} + b^{2n-1} = a + b$

18. C $\dfrac{mx^2 - x + 1}{x} \geq 0 \Rightarrow \left(mx^2 - x + 1\right)x \geq 0$ Assume x $(+ve)$

$mx^2 - x + 1 \geq 0$

Quadratic, $m > 0, b^2 - 4a < 0$

$1 - 4m \leq 0,\ 1 \leq 4m \ \ m \geq \dfrac{1}{4}$

Thus minimum value of $m = \dfrac{1}{4}$

19. B $\dfrac{|x-2|-1}{|x-2|-2} \leq 0$ is put $|x-2| = k$

$\dfrac{k-1}{k-2} 0\ (k-1)(k-2) \leq 0\ \ k \in [1,2]$

$|\leq|x - 2 |\leq 2$ $\qquad$ $|x-2| \leq 2$

$|x-2| \geq 1$ $\qquad$ $-2 \leq x - 2 \leq 2$

$x - 2 \leq -1, x - 2 \geq 1$ $\qquad$ $0 \leq x \leq 4. x \in [0,4]$

$x \leq 1, x \geq 3$

$x \in (-\infty, 1] \cup [3, \infty)$ $\qquad$ $x \in [0,4]$

$x \in [0,1] \cup [3,4]$

20. C $1 \leq |x-2| \leq 3$ $\quad$ $|x-2| \leq 3,\ -3 \leq x - 2 \leq 3 - 1 \leq x \leq 5[-1,5]...(1)$

$|x-2| \geq 1\ \ x - 2 \leq -1$ or $x - 2 \geq 1$

$x \leq 1$ or $x \geq 3$

$(-\infty,1] \cup [3,\infty).....(2)$ from (1) and (2) $[-1,1] \cup [3,5]$

21. C $(x-1)(x^2-5x+7)<(x-1)$

$(x-1)(x^2-5x+6)<0$

$(x-1)(x-2+6)(x-3)<0 \quad 1,2,3$

$(-\infty,1)\cup(2,3) \qquad (-\infty\downarrow,1),\uparrow(1,2),(2,\downarrow 3),\uparrow(3,\infty)$

22. A $Am \geq Gm$

$$\frac{bcx+cay+abz}{3} \geq (bcx\ cay\ abz)^{\frac{1}{3}}$$

$$bcx+cay+abz \geq 3(b^2c^2a^2abc)^{\frac{1}{3}} \quad (xyz = abc\,\text{given})$$

$$\geq 3abc$$

23. C Solution of the $\sin^4\left(\frac{x}{3}\right)+\cos^4\left(\frac{x}{3}\right)>\frac{1}{2}$

$$\left(\sin^2\frac{x}{3}+\cos^2\frac{x}{3}\right)^2-2\sin^2\frac{x}{3}\cos^2\frac{x}{3}>\frac{1}{2}$$

$$1-\frac{1}{2}\left(2\sin\frac{x}{3}\cos\frac{x}{3}\right)^2>\frac{1}{2}$$

$$2-\left(\sin^2\frac{2x}{3}\right)>1$$

$$1-\sin^2\frac{2x}{3}$$

$$\sin^2\frac{2x}{3}<1 \Rightarrow \frac{2x}{3}\in R-(2x+1)\frac{\pi}{2};x\in R-\frac{3}{2}(2x+1)\frac{\pi}{2}$$

$$x\in R-\frac{3}{2}(2n+1)\frac{\pi}{2}$$

$$x\in R-\left(\frac{3n\pi}{2}+\frac{3\pi}{4}\right),n\in I$$

24. A $4x+6-10<6x-12$

$8<2x,\ x>4,\ (4,\infty)$......(1)

$\frac{2x-3}{4}+6\geq 2+\frac{4x}{3}$

$6x-9+72\geq 8+16x$, $63-8\geq 10x\Rightarrow \frac{55}{10}\geq x$ $\left(-\infty,\frac{55}{10}\right)$ infinite

solution $\left(4,\frac{55}{10}\right]$

25. A $\frac{\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}}{6}\geq\left(\frac{b}{a}\frac{c}{a}\frac{c}{b}\frac{a}{b}\frac{a}{c}\frac{b}{c}\right)^{\frac{1}{6}}=1$

$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$

26. A $\log_7\left(\frac{x-2}{x-3}\right)<0,\ \therefore 0<\frac{x-2}{x-3}<1$

$\frac{x-2}{x-3}<1\Rightarrow\frac{x-2}{x-3}-1<0$

$\Rightarrow\frac{x-2-x+3}{x-3}<0\Rightarrow\frac{1}{x-3}<0\Rightarrow\frac{1(x-3)^2}{x-3}<0$

$(x-3)<0\Rightarrow x<3$......(1)

$\frac{x-2}{x-3}>0,\Rightarrow(x-2)(x-3)>0$

$x\in(-\infty,2)\cup(3,\infty)$.....(2)

from (1) and (2) $x\in(-\infty,2)$

27. C $\left(\frac{5}{13}\right)^x+\left(\frac{12}{13}\right)^x\geq 1$ $\frac{5}{13}=\sin\alpha,\frac{12}{13}=\cos\alpha$

$(\sin\alpha)^x+(\cos\alpha)^x\geq 1\Rightarrow x=2,$ it is true

x<2 also true

$\therefore$ solution $(-\infty,2]$, x>2 false

28. A.M $\geq$ G.M., $\dfrac{\dfrac{\cos^3 x}{\sin x}+\dfrac{\sin^3 x}{\cos x}}{2} \geq \sqrt{\dfrac{\sin^3 x \cos^3 x}{\sin x \cos x}}$

$$\frac{\cos^3 x}{\sin x}+\frac{\sin^3 x}{\cos x} \geq 2\sin x \cos x$$

$$\frac{\cos^3 x}{\sin x}+\frac{\sin^3 x}{\cos x} \geq \sin 2x$$

$$\frac{\cos^3 x}{\sin x}+\frac{\sin^3 x}{\cos x} \geq 1 \quad \because \sin 2x \leq 1$$

$\therefore$ minimum $= 1$

29. C $n = 3$ $3! > 2^2$ true

30. B $a_2, a_3, a_4, ...a_n \in (-\pi, \pi)$

$0 \leq \sin^2 a_i \leq 1, ai \in (-\pi, \pi)$ and $\sin ai \neq 0$

when $\sin^2 a_2 = \sin^2 a_3 = ... = \sin^2 a_n = 1, a_2, a_3......a_n = -\dfrac{\pi}{2}$ or $\dfrac{\pi}{2}$

LHS = 2,3,4.....n = n!

RHS n!; $n! \leq n!$

When $\sin^2 a_2$, $\sin^2 a_3$, ...,$\sin^2 a_n < 1$

then LHS > n! (false)

$\therefore a_2, a_3.....a_n = -\dfrac{\pi}{2}$ or $\dfrac{\pi}{2}$

Number of solution = 2.2.2....2=2^{n-1}