CHAPTER - 4

COMPLEX NUMBERS

Important Points

1. The imaginary number $\sqrt{-1}$ is denoted by 'i' and is defined by the equation $i^2 = -1$
2. $\sqrt{-1}$ is called 'iota' and is written as i
3. If a, b $\in$ R, then $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ provided, a, b are not both negative
4. If x and y are two real numbers, then x + iy is called a complex number

   In Z = x + iy, the real numbers x and y are respectively called the real part [Re (z)] and imaginary part (Im(z))
5. A complex number z = a + ib is called purely real if Im(z) = 0, ie b = 0

   It is purely imaginary if Re(z) = 0 ie. a = 0
6. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

   Then $z_1 = z_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$
7. If z = x + iy is a complex number, then the complex number x – iy denoted by $\overline{z}$ is called the conjugate of the complex number ie $\overline{z} = x - iy$

Properties

(i) $\overline{(\overline{z})} = z$

(ii) $z + \overline{z} = 2\text{Re}(z) = 2\,\text{Re}(\overline{z})$

(iii) $z - \overline{z} = 2i\,\text{Im}\,(z)$

(iv) $z = \overline{z}$ iff z is purely real

(v) $z = -\overline{z}$ iff z is purely imaginary

(vi) $\left(\overline{z_1 \pm z_2}\right) = \overline{z}_1 \pm \overline{z}_2$

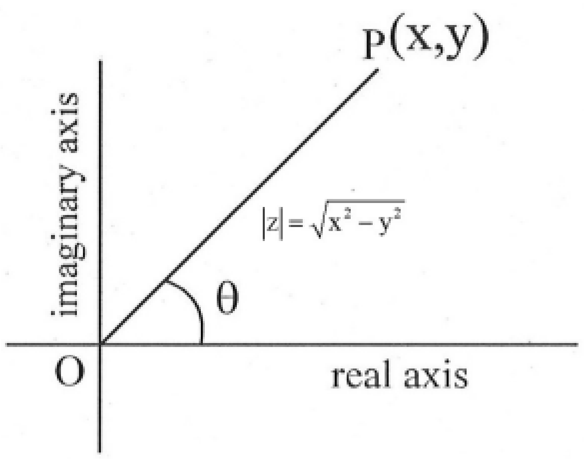
(vii) $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$ provided $z_2 \neq 0$

(ix) $z . \overline{z} = [Re(z)]^2 + [Im (z)]^2$ ie if $z = x + iy$, then $z. \overline{z} = x^2 + y^2$

(x) $\overline{(z^n)} = (\overline{z})^n$

Geometrical Interpretation of a Complex Number



A complex number $z = x + iy$ represents a point p (x, y) in the Argand plane. In the Argand diagram length OP ie the non negative real number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number x + iy written as $|x + iy|$ and the angle between OP and positive direction of x axis called the argument or amplitude of z.

Properties

(i) $z. \overline{z} = |\overline{z}|^2 = |z|^2 \therefore |z| = |\overline{z}|$

(ii) $z = 0$ iff $|z| = 0$

(iii) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2 |z_1||z_2| \cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2 \pm 2 \, Re(z_1 \overline{z}_2)$

(iv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2 \{|z_1|^2 + |z_2|^2\}$

(v) $|z_1 z_2| = |z_1| \, |z_2|$, $|z^n| = (|z|)^n$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}; |z_2| \neq 0$

(vii) $-|z| \leq \text{Re}(z) \leq |z|$

$-|z| \leq \text{Im}(z) \leq |z|$

(viii) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ix) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(x). The order relations are not defined on the set of complex numbers. But =, $\neq$ are defined.

$Z_1 < Z_2$, $Z_1 > Z_2$ etc have no meaning

8. The multiplicative inverse of the nonzero complex number z is denoted by z^{-1} and is given by $z^{-1} = \frac{\bar{z}}{|z|^2}$

9. **Polar representation**

The complex number z = x + iy when expressed in the form $z = r(\cos\theta + i\sin\theta)$ is called the polar form or modulus amplitude form or trigonometric form of the complex number, where $r = \sqrt{x^2 + y^2} = |z|$ and the value of θ obtained by solving the equations $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ & $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$ is called the argument (amplitude) of z denoted by arg z. If $-\pi < \theta \leq \pi$ then θ is called the principal argument of z. argument of zero is not defined.

10. $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$, $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. If $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$ then

(i) $z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)}$ (ii) $\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$

(ii) arg (z_1z_2) = arg (z_1) + arg (z_2) and arg $\left(\frac{Z_1}{Z_2}\right)$ = arg z_1 – arg (z_2)

(iii) arg $\bar{z} = -\arg z$

11. De - Moivre's Theorem

If 'n' is any integer, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

12. Distance between two points having affixes Z_1 and Z_2 is $|Z_2 - Z_1|$, the complex number z = x + iy is known as the affix of the point P.

13. The area of the triangle formed by z, iz and z + iz is equal to $\frac{1}{2}|z|^2$

14. The triangle with vertices z_1, z_2, z_3 is equilateral iff $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Cube roots of unity

The roots of the equation $z^3 - 1 = 0$ are called cube roots of unity

$\therefore z^3 - 1 = (z-1)(z^2 + z + 1) = 0$

$\Rightarrow z = 1, w, w^2$ are roots, where

$$w = \frac{-1 + i\sqrt{3}}{2}, w^2 = \frac{-1 - i\sqrt{3}}{2}$$

(i) $1 + w + w^2 = 0$ and $w^3 = 1$

(ii) $w^{3n} = 1, w^{3n+1} = w$

$w^{3n+2} = w^2$ and $w^{3n} + w^{3n+1} + w^{3n+2} = 0, n \in N$

(iii) $w^2 = \frac{1}{w}, w = \frac{1}{w^2}$

(iv) $\overline{w} = w^2, \overline{w}^2 = w$

(v) $\sqrt{w^2} = \pm w, \sqrt{w} = \pm w^2$

(vi) Cube roots of unity lie on a circle $|z| = 1$ and divide its circumference into three equal parts

(vii) In the Argand plane cube roots of unity form an equilateral triangle with area $\frac{3\sqrt{3}}{4}$ sq,units

PART I - (JEEMAIN)

SECTION - I - Straight objective type questions

1. If $(1+i)(1+2i)(1+3i)....(1+ni)=a+ib$, then $2\times5\times10\times....\times(1+n^2)$ is equal to

1) a^2+b^2 2) $\sqrt{a^2+b^2}$ 3) $\sqrt{a^2-b^2}$ 4) a^2-b^2

2. If z is a complex number such that $z+|z|=8+12i$, then the value of $|z^2|$ is equal to

1) 228 2) 144 3) 121 4) 169

3. If $e^{i\theta}=\cos\theta+i\sin\theta$ and $a=e^{i\alpha},b=e^{i\beta},c=e^{i\gamma}$ and $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=1$, then $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=$

1) $\frac{3}{2}$ 2) $-\frac{3}{2}$ 3) 0 4) 1

4. If O,Z_1,Z_2 form the vertices of an equilateral triangle then Z_1^2, Z_1Z_2, Z_2^2 will be vertices of

(1) an equilateral triangle with centre at O
(2) an isosceles triangle
(3) a right angled triangle
(4) none of these

5. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+.....+a_{2n}x^{2n}$, then $a_0+a_3+a_6+.....=$

1) 0 2) 1 3) 3^n 4) 3^{n-1}

6. If $z_1=1+i$, $z_2=-1+\sqrt{3}i$ and z is a complex number lying between the line segment joining z_1 and z_2 then arg(z) can be

(1) $-\frac{3\pi}{4}$ (2) $-\frac{\pi}{6}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

7. The imaginary part of $(3+2\sqrt{-54})^{1/2}-(3-2\sqrt{-54})^{1/2}$ can be:

(1) $-2\sqrt{6}$ (2) 6 (3) $\sqrt{6}$ (4) $-\sqrt{6}$

8. Let $A=\left\{0\in\left(-\frac{\pi}{2},\pi\right);\frac{3+2i\sin\theta}{1-2i\sin\theta}\text{is purely imaginary}\right\}$. Then the sum of the elements in A is:

(1) $\frac{5\pi}{6}$ (2) $\frac{2\pi}{3}$ (3) $\frac{3\pi}{4}$ (4) π

9. The value of $\left(\dfrac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is:

(1) $\frac{1}{2}(\sqrt{3}-i)$ (2) $-\frac{1}{2}(\sqrt{3}-i)$ (3) $-\frac{1}{2}(1-i\sqrt{3})$ (4) $\frac{1}{2}(1-i\sqrt{3})$

10. Let $\alpha=\dfrac{-1+i\sqrt{3}}{2}$. If $a=(1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b=\sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation:

(1) $x^2-102x+101=0$ (2) $x^2+101x+100=0$

(3) $x^2-101x+100=0$ (4) $x^2+102x+101=0$

11. The minimum value of $|Z-1+2i|+|4i-3-Z|$ is

(1) $\sqrt{5}$ (2) 5 (3) $2\sqrt{13}$ (4) $\sqrt{15}$

12. If $|Z_i|=\lambda$, $i=1,2,3,....,n$ then $\left|\dfrac{Z_1^{-1}+Z_2^{-1}+....+Z_n^{-1}}{Z_1+Z_2+....+Z_n}\right|$ is equal to

(1) λ^2 (2) $\dfrac{1}{\lambda^2}$ (3) 1 (4) none of these

13. If $Z_1=1+i$, $Z_2=1-i$, Z and origin are four concyclic points then the maximum value of |Z| is

(1) 1 (2) 2 (3) 3 (4) 4

14. If α,β and γ are the roots of $x^3-3x^2+3x+7=0$, ω is a non real cube root of unity then $\dfrac{\alpha-1}{\beta-1}+\dfrac{\beta-1}{\gamma-1}+\dfrac{\gamma-1}{\alpha-1}$ is

(1) $\dfrac{3}{\omega^2}$ (2) ω^2 (3) $2\omega^2$ (4) $3\omega^2$

15. If $Z+\dfrac{1}{Z}=-1$, then $\sum_{r=1}^{5}\left(Z^r+\dfrac{1}{Z^r}\right)^2=$

(1) 8 (2) 10 (3) 12 (4) 15

16. If a,b,c are distinct integers the minimum value of $|a+bw+cw^2|+|a+bw^2+cw|$, where $w=e^{i2\frac{\pi}{3}}$ is

(1) 2 (2) $2\sqrt{2}$ (3) $2\sqrt{3}$ (4) $2\sqrt{6}$

17. If Z lies on the circle $|Z-2i|=2\sqrt{2}$ then the value of $\arg\left(\frac{Z-2}{Z+2}\right)$ is equal to

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

18. The area of the region of the Argand plane described by complex numbers Z satisfying $\frac{\pi}{6}<\arg(Z)<\frac{2\pi}{3}$ and $3<|Z|<5$ is (in sq. units)

(1) 17π (2) 16π (3) $\frac{16\pi}{3}$ (4) 4π

19. Assertion & Reasoning

(1) If both Statement-I and Statement-II are true and the reason is the correct explanation of the statement-I.

(2) If both Statement-I and Statement-II are true but reason is not the correct explanation of the statement-I.

(3) If Statement-I is true but Statement-II is false.

(4) If Statement-I is false but Statement-II is true.

Consider z_1 and z_2 are two complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$

Statement-I: $\text{amp}(z_1)-\text{amp}(z_2)=0$

Statement-II: The complex numbers z_1 and z_2 are collinear with origin

20. Let z_1, z_2 be two complex numbers satisfying $|z|=\sqrt{2}$ and $|z-3-3i|=2\sqrt{2}$ respectively. Then

Statement-I: $\min|z_1-z_2|=0$ and $\max|z_1-z_2|=6\sqrt{2}$

Statement-II: Two curves $|z|=\sqrt{2}$ and $|z-3-3i|=2\sqrt{2}$ touch each other externally

SECTION - II

Numerical Type Questions

21. If z_1 and z_2 be two variable complex numbers such that $|z_1|^2\le 169$ and $|z_2+3-4i|^2\le 25$, then the maximum value of $|z_1-z_2|$ is $15+p$. The value of p is

22. If $\left|z-\frac{3}{z}\right|=2$, then the greatest value of $|z|$ is

23. Let $\left(-2-\frac{1}{3}i\right)^3=\frac{x+iy}{27}(i=\sqrt{-1})$, where x and y are real numbers, then $y-x$ equals:

24. If z_1,z_2,z_3 are distinct non-zero complex numbers and $a,b,c\in R^+$ such that $\frac{a}{|z_1-z_2|}=\frac{b}{|z_2-z_3|}=\frac{c}{|z_3-z_1|}$ then $\frac{a^2}{z_1-z_2}+\frac{b^2}{z_2-z_3}+\frac{c^2}{z_3-z_1}$ is always equal to

25. If the equation $z^2+(a+ib)z+(c+id)=0$ (a,b,c,d are real and $bd\neq 0$) has a real root, then $d^2-abd+b^2c$ is equal to ______

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. If $z(1+a)=b+ic, a^2+b^2+c^2=1$, then $\frac{1+iz}{1-iz}=$

A) $\frac{(a-ib)}{1+c}$ B) $\frac{(a+ib)}{1+c}$ C) $\frac{(a+ib)}{1-c}$ D) $\frac{(a-ib)}{1-c}$

27. If $|z_1|=2, |z_2|=3$, then $|z_1+z_2+5+12i|$ is less than or equal to

A) 8 B) 18 C) 10 D) 5

28. If z is a complex number satisfying $z^4+z^3+2z^2+z+1=0$, then the set of possible values of $|z|$ is

A) $\{1,2\}$ B) $\{1\}$ C) $\{1,2,3\}$ D) $\{1,2,3,4\}$

29. If $|z-25i|\leq 15$, then $|\max.\arg(z)-\min.\arg(z)|=$

A) $2\cos^{-1}\frac{3}{5}$ B) $2\cos^{-1}\frac{4}{5}$ C) $\frac{\pi}{2}+\cos^{-1}\frac{3}{5}$ D) $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{3}{5}$

30. Let z, w be complex number such that $\bar{z}+i\bar{w}=0$ and $\arg zw=\pi$. Then arg z equals

A) $\frac{\pi}{4}$ B) $\frac{\pi}{2}$ C) $\frac{3\pi}{4}$ D) $\frac{5\pi}{4}$

31. The complex number 3+4i is rotated (+ve) about origin by an angle of $\frac{\pi}{4}$ and then stretched 2-times. The complex number corresponding to new position is

A) $\sqrt{2}(-3+4i)$ B) $\sqrt{2}(-1+7i)$ C) $\sqrt{2}(3-4i)$ D) $\sqrt{2}(-1-7i)$

32. The mirror image of the curve $\arg\left(\frac{z-3}{z-i}\right)=\frac{\pi}{6}$ in the real axis is

A) $\arg\left(\frac{z+3}{z+i}\right)=\frac{\pi}{6}$ B) $\arg\left(\frac{z-3}{z+i}\right)=\frac{\pi}{6}$ C) $\arg\left(\frac{z+i}{z+3}\right)=\frac{\pi}{6}$ D) $\arg\left(\frac{z+i}{z-3}\right)=\frac{\pi}{6}$

33. The value of $1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$ is

A) 0 B)1 C) -1 D) i

SECTION - IV (More than one correct answer)

34. Points A, B and C with affixes z_1, z_2 and $(1-i)z_1+iz_2$ are the vertices of

A) an isosceles triangle B) an equilateral triangle

C) a right triangle D) an obtuse angled triangle

35. Given that the two curves $\arg(z)=\frac{\pi}{6}$ and $\left|z-2\sqrt{3}i\right|=r$ intersect in two distinct points then

A) $r>3$ B) r = 6 C) $0<r<3$ D) $[r]\neq 2$

36. Let a,b,x and y be real numbers such that a-b = 1 and $y\neq 0$. If the complex number z = x+iy satisfies $\operatorname{Im}\left(\frac{az+b}{z+1}\right)=y$, then which of the following is (are) possible value (s) of x?

A) $1-\sqrt{1+y^2}$ B) $-1-\sqrt{1-y^2}$ C) $1+\sqrt{1+y^2}$ D) $-1+\sqrt{1-y^2}$

37. If from a point P representing the complex numebr z_1 on the curve $|z|=2$, pair of tangents are drawn to the curve $|z|=1$, meeting at point $Q(z_2)$ and $R(z_3)$, then

A) complex number $\frac{z_1+z_2+z_3}{3}$ will on the curve |z| = 1

B) $\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$

C) $\arg\left(\frac{z_2}{z_3}\right)=\frac{2\pi}{3}$

D) orthocentre and circumcentre of ΔPQR will coincide

SECTION - V (Numerical Type - Upto two decimal place)

38. If z_1 lies on the circle $|z|=3$ and $x+iy=z_1+\frac{1}{z_1}$ then $\frac{x^2}{100}+\frac{y^2}{64}=\frac{1}{k}$ then k is equal to

39. If z is any complex number satisfying $|z-3-2i|\le 2$, then the minimum value of $|2z-6+5i|$ is

SECTION VI - (Matrix match type)

40.

Column - I	**Column-II**
A) The curve represented by $\text{Re}(z^2)=4$ is	p) a straight line
B) The curve represented by $z^2+\bar{z}^2=2$ is	q) an ellipse
C) The curve represented by $\left\| \|z-z_1\|-\|z-z_2\| \right\|=\lambda, \lambda<\|z_1-z_2\|$ is	r) a hyperbola
D) The curve represented by $\text{Im}\left(\frac{2z+1}{iz+1}\right)=-2$ is	s) a circle

A) A-R, B-R, C-R, D-P
B) A-P, B-R, C-R, D-P
C) A-R,R, B-R, C-R, D-P
D) A-R, B-R, C-R, D-P, P