CHAPTER - 13
CONTINUITY, DIFFERENTIABILITY AND DERIVATIVES

CONTINUITY AT A POINT

1. Continuity at x = a

The real valued function y = f(x) is said to be continuous at x = a if

i) f(x) is defined at x = a i.e. f(a) should exist and is finite

ii) $\underset{x\to a}{\text{Lt}}\, f(x) = f(a)$

A function y = f(x) can be discontinuous in 3 different ways

a) $\underset{x\to a^+}{\text{Lt}}\, f(x) = \underset{x\to a^-}{\text{Lt}}\, f(x) \neq f(a)$. This is known as removable discontiuity

b) $\underset{x\to a^+}{\text{Lt}}\, f(x) \neq \underset{x\to a^-}{\text{Lt}}\, f(x)$. This is known as non removable discontinuity of first type

c) Either $\underset{x\to a^+}{\text{Lt}}\, f(x)$ or $\underset{x\to a^-}{\text{Lt}}\, f(x)$ or both do not exist

2. Continuous function

A function is said to be continuous if it is continuous at every point in the domain definition

3. Results

a) The sum, difference and product of two continuous functions is continuous where as $\frac{f(x)}{g(x)}$ is continuous if $g(x) \neq 0$

b) Composition of continuous functions is also continuous

3) A function is said to be everywhere continuous if it is continuous in $(-\infty, +\infty)$

c) The constant function, the modulus function, the identity function and the exponential function are everywhere continuous where as the logarithmic function is not everywhere continuous

d) If a function is continuous at x = a, then $\underset{x\to a}{\text{Lt}}\, f(x)$ exist, but the converse is not true

e) If f(x) has a first type dicontinuity at x = a, then there will be a break in the graph of the function at x = a

DIFFERENTIATION

Let y = f(x) be a real valued function

4. Right hand derivative at x = a

$$Rf'(a) = f'(a^+) = \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} \text{ where } h > 0$$

5. Left hand derivative at x = 0

$$Lf'(a) = f'(a^-) = \underset{h\to 0}{Lt} \frac{f(a-h)-f(a)}{-h} \text{ where } h > 0$$

6. Differentiability at x = a

The function y = f(x) is said to be differentiable at x = a. If it is defined in some neighbourhood (nbd) of 'a' and R f′(a) = L f′(a). If $Rf'(a) \neq Lf'(a)$ the function is not differentiable at x = a

7. Result

All differentiable functions are continuous but all continuous function are not necessarily differentiable. For example f(x) = |x| is continuous but not differentiable at x = 0

8. Derivative at a point

The function y = f(x) is said to have a derivative at any point 'x' iff it is defined in some neighbourhood of that point 'x' and

$\underset{\Delta x\to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x\to 0}{Lt} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists and is finite . The value of this limit is called the derivative and is denoted by f′(x) or $\frac{dy}{dx}$ etc.

Note

Δx is the increment in x and Δy be the corresponding increment in y and

$$f(x+\Delta x)-f(x) = y+\Delta y - y = \Delta y$$

9. Meaning of differentiability at a point

If y = f(x) is differentiable at a point x = a, then there exist a unique tangent at x = a. Thus when f(x) is differentiable at x = 0, then there should not be a corner point of the curve at x = a

10. Differentiability in closed intercal [a, b]

The function y = f(x) is differentiable in closed intervel [a, b] if

i) it is differentiable at every point in (a, b)

ii) R f′(a) and L f′(b) should exist

11. Results

a) All polynomial functions are differentiable

b) f(x) = a^x, a > 0 is differentiable

c) constant functions are differentiable

d) logarithmic function, trigonometric functions, inverse trigonometric functions are differentiable in their domain

e) The sum, difference and product of differentiable functions are differentiable

f) Composition of differentiable functions are differentiable

12. Parametric functions

Let y = f(t) and x = f(t) where t is the parameter

Then $\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

13. Logarithmic differentiation

For getting the derivative of $\left[f(x)\right]^{g(x)}$ or the function which is the product of many functions logarithmic differentiation may be used

14. Differentiation of a function w.r.t another function

Derivative of V = f(x) w.r.t. u = g(x) is given by $\frac{f'(x)}{g'(x)}$ where $f'(x)=\frac{du}{dx}$ and $g'(x)=\frac{dv}{dx}$

15. Higher order derivatives

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$; $\quad \frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and so on

$\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}$ etc. are functions of x

16. Derivative of a composite function

$$\frac{d}{dx}f\left[g(x)\right]=f'\left[g(x)\right]g'(x)$$

17. Derivative of standard functions

a) $\frac{d}{dx}(K)=0$

b) $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

c) $\frac{d}{dx}(\log x)=\frac{1}{x}$

d) $\frac{d}{dx}\left(a^{x}\right)=a^{x}\log a$

e) $\frac{d}{dx}\left(e^{x}\right)=e^{x}$

f) $\frac{d}{dx}\left(\frac{1}{x^{n}}\right)=\frac{-n}{x^{n+1}}$

g) $\frac{d}{dx}(\sin x)=\cos x$

h) $\frac{d}{dx}(\cos x)=-\sin x$

i) $\frac{d}{dx}(\tan x)=\sec^{2} x$

j) $\frac{d}{dx}(\sec x)=\sec x \tan x$

k) $\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x \cot x$

l) $\frac{d}{dx}(\cot x)=-\operatorname{cosec}^{2} x$

m) $\frac{d}{dx}\left(\sin^{-1} x\right)=\frac{1}{\sqrt{1-x^{2}}}$

n) $\frac{d}{dx}\left(\cos^{-1} x\right)=\frac{-1}{\sqrt{1-x^{2}}}$

o) $\frac{d}{dx}\left(\tan^{-1} x\right)=\frac{1}{1+x^{2}}$

p) $\frac{d}{dx}\left(\cot^{-1} x\right)=\frac{-1}{1+x^{2}}$

q) $\frac{d}{dx}\left(\sec^{-1} x\right)=\frac{1}{x\sqrt{x^{2}-1}}$

r) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right)=\frac{-1}{x\sqrt{x^{2}-1}}$

s) $\frac{d}{dx}\frac{1}{x^n}=\frac{-n}{x^{n+1}}$

t) $\frac{d}{dx}\log_a^x=\frac{1}{x\log a}$

PART I - (JEEMAIN)

SECTION - I - Straight objective type questions

1. In order that the function $f(x)=(x+1)^{1/x}$ is continuous at $x=0$, $f(0)$ must be define

(1) $f(0)=0$ (2) $f(0)=e$ (3) $f(0)=1/e$ (4) $f(0)=1$

2. If $f(x)=\begin{cases}(1+2x)^{1/x}, & \text{for } x\neq 0\\ e^2, & \text{for } x=0\end{cases}$, then

(1) $\lim\limits_{x\to 0+} f(x)=e$ (2) $\lim\limits_{x\to 0-} f(x)=e^2$

(3) $f(x)$ is discontinuous at $x=0$ (4) None of these

3. Let $f(x)=\begin{cases}\frac{x^4-5x^2+4}{|(x-1)(x-2)|}, & x\neq 1,2\\ 6, & x=1\\ 12, & x=2\end{cases}$.

Then f(x) is continuous on the set

(1) R (2) $R-\{1\}$ (3) $R-\{2\}$ (4) $R-\{1,2\}$

4. If $x^y=e^{x-y}$, then $\frac{dy}{dx}$ =:

1) $\frac{\log x}{1+\log x}$ 2) $\frac{\log x}{1-\log x}$ 3) $\frac{\log x}{(1+\log x)^2}$ 4) $\frac{1+\log x}{\log x}$

5. If $y=2^{ax}$ and $\frac{dy}{dx}=\log 256$ at $x=1$, then a =:

1) 2 2) 1 3) 0 4) 3

6. $y=\cot^{-1}\tan\left(\frac{x}{2}\right)$. Then $\frac{dy}{dx}=$

1) $-1/2$ 2) 0 3) $x/2$ 4) $\frac{x}{2}+\frac{1}{2}$

7. $Y = e^{\frac{1}{2}\log\left(1+\tan^2 x\right)}$; then dy/dx =

1) $\frac{1}{2}\sec^2 x$ 2) $\sec^2 x$ 3) sec x tan x 4) $e^{\frac{1}{2}\log\left(1+\tan^2 x\right)}$

8. When |x| < 1

$$\frac{d}{dx}\left[x+\frac{x^3}{3}+\frac{x^5}{5}+.....\right]$$

1) $\frac{-1}{1-x^2}$ 2) $\frac{1}{1-x^2}$ 3) $\frac{1}{(1-x)^2}$ 4) $\frac{-1}{(1-x)^2}$

9. $f(x)=\begin{cases} x^2e^{2x-2}\text{when } 0\le x\le 1 \\ a\,\text{sig}(x+1)\cos(2x-2)+bx^2 \text{ when } 1<x\le 2\end{cases}$

If f(x) is continuous at x = 1 Then a+b =

1) 1 2) 3 3) –1 4) –3

10. f(x) is a continuous function such that f(1) = 2 and $f'(x)=f(x)$. It is given that $h(x)=f(f(x))$. Then $h'(1)=$

1) e 2) 2e 3) 3e 4) 4 e

11. f(x) is a differentiable function such that $|f(x)-f(y)|\le 2|x-y|^{\frac{3}{2}}$. Given f(0) = 1. Then $\int_0^1 f^2(x)dx =$

1) 0 2) 1 3) –1 4) $\frac{1}{2}$

12. $f(x)=\text{Mini}\left(1\ \ x^2\ \ x^3\right)$ then $f(x)$ is

1) continuous and differentiable for all $x\in R$

2) continuous when $x\ge 0$ only

3) continuous for all $x\in R$ but not differentiable at x = 1

4) continuous for all x but not differentiable when $x=-1, x=0$ and x = 1

13. The function $f(x)=\frac{\tan(\pi[x-\pi])}{1+[x]^2}$ where

[.] stands for GIV function is

1) Discontinuous at some ‘x’

2) Continuous only is the first quadrant

3) Continuous at all points

4) Continuous in the first quadrant but discontinuous at a finite number of points in 3rd quadrant

14. If g is the inverse of f and $f'(x)=\frac{1}{1+x^n}$, then $g'(x)=$

1) 1 + g (x) 2) 1 - g (x) 3) 1 + [g (x)]^n 4) 1 - g [(x)]^n

15. $\frac{d}{dx}\left[(x+a)(x^2+a^2)(x^4+a^4)(x^8+a^8)\right]$ =

1) $\frac{15x^4+16a^{15}+a^{12}}{(x-a)^2}$ 2) $\frac{15x^{16}-16x^{15}a+a^{16}}{(x-a)^2}$

3) $\frac{15x^{16}+16x^{15}a-a^{16}}{(x-a)^2}$ 4) $\frac{15x^{16}+16x^{15}a+a^{16}}{(x-a)^2}$

16. x = 3 tan t y = 3 sec t

Then $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$

1) $\frac{1}{6\sqrt{2}}$ 2) $6\sqrt{2}$ 3) $\frac{-1}{6\sqrt{2}}$ 4) $-6\sqrt{2}$

17. $f:[0\ 1]\to R$ such that $f(xy)=f(x)f(y)$ and $f(0)\neq 0$. Also given y(x) is a function where $\frac{dy}{dx}=f(x)$ with y(0) = 0. Then $y\left(\frac{1}{4}\right)+y\left(\frac{3}{4}\right)=$

1) 0 2) $\frac{1}{2}$ 3) –1 4) 1

18. $y=\log^n x$ where log^n means log.log.log...... (repeated n times) then $x\log x\log^2 x\log^3 x......\log^n x\frac{dy}{dx}=$

1) log x 2) x 3) $\frac{1}{\log x}$ 4) log^{n}x

19. $f:[0\ 2]\to R$ is differentiable and f(0) = 1 $F(x)=\int_0^{x^2} f(\sqrt{t})dt$. Also $F'(x)=f'(x)$, The F (2) =

1) e^4-1 2) $1-e^4$ 3) $\frac{1}{e^4}$ 4) e^4

20. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} \dfrac{x^3}{(1-\cos 2x)^2}\log_e\left(\dfrac{1+2xe^{-2x}}{(1-xe^{-x})^2}\right), & x \neq 0 \\ \alpha & , x = 0 \end{cases}$$

If f is continuous at x = 0, then α is equal to

1) 1 2) 3 3) 0 4) 2

21. Statement 1: $f(x) = x + \log x - x\log x$ where $0 < x < \infty$. Then $f'(x) \neq 0$ for every x in (0 1)

Statement 2: In $0 < x < 1$ then $\dfrac{1}{x} > \log x$

1) Statement I is true, statement II is true, statement II is a correct explanaiton for statement 1

2) Statement I is true, statement II is true, statement II is not a correct explanation for statement 1

3) Statement I is true, statement II is false

4) Statement I is false, statement II is true

SECTION - II

Numerical Type Questions

22. If $y = y(x)$ is an implicit function of x such that $\log_e(x+y) = 4xy$, then $\dfrac{d^2y}{dx^2}$ at $x = 0$ is equal to

23. If $y^{\frac{1}{4}} + y^{\frac{-1}{4}} = 2x$, and $(x^2-1)\dfrac{d^2y}{dx^2} + \alpha x\dfrac{dy}{dx} + \beta y = 0$, then $|\alpha - \beta|$ is equal to.....

24. $f(x) = \sin^2\cot^{-1}\sqrt{\dfrac{1+x}{1-x}}$ Then $f'\left(\dfrac{1}{2}\right) = -k$. Then k =

25. $f(x) = \dfrac{(e^x-1)^2}{\sin\left(\dfrac{x}{a}\right)\log\left(1+\dfrac{x}{4}\right)}$ when $x \neq 0$ and f(0) = 12. If f(x) is continuous at x = 0. Then a =

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. Let $f:[0,\infty)\to[0,3]$ be a funciton defined by $f(x)=\begin{cases}\max\{\sin t:0\le t\le x\} & 0\le x\le \pi \\ 2+\cos x, & x>\pi\end{cases}$

Then which of the following is true?

A) f is continuous everywhere but not differentiable exactly at one point in $(0,\infty)$

B) f is differentiable everywhere in $(0,\infty)$

C) f is not continuous exactly at two points in $(0,\infty)$

D) f is continuous everywhere but not differentiable exactly at two points in $(0,\infty)$

27. $f(x)\begin{cases}=\dfrac{\pi}{4}+\tan^{-1}x \text{ when } |x|\le 1 \\ =\dfrac{1}{2}(|x|-1) \text{ when } |x|>1\end{cases}$

Which of the following is true for f(x)

A) Continuous on $R-\{1\}$ and differentiable in $R-\{-1,1\}$

B) Both continuous and differentiable in $R-\{1\}$

C) Both continuous and differentiable in $R-\{-1\}$

D) Continuous in R

28. $f(x)=\begin{cases}\dfrac{1-\cos\left(1-\cos\dfrac{x}{2}\right)}{2^m x^n}, & \text{when } x\ne 0 \\ 1, \text{ when } x=0 & \end{cases}$ where 'm' and 'n' are integers. Then $n-m=$

29. $g(x)=\underset{m\to\infty}{Lt}\dfrac{x^m f(1)+h(x)+1}{2x^m+3x+3}$ is continuous at x = 1. Given $g(1)=\underset{x\to 1}{Lt}(\log ex)^{\frac{2}{\log x}}$. Then the value of $2g(1)+2f(1)-h(1)$ is (given f(x) and h(x) are continuous at x = 1)

30. $f:(1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1)=0$ and $6\int_1^x f(t)\,dt = 3xf(x)-x^3$, for all $x\ge 1$ and $f(1)=2$, then $f'(2)=$

A) 2 B) 3 C) 4 D) 5

31. $y^{y^{y^{\cdot^{\cdot^{\infty}}}}} = \log(x+\log(x+\log(x+....)))$. Then $\frac{dy}{dx}$ at $x=e^2-2$ and $y=\sqrt{2}$

A) $\dfrac{\log\left(\frac{e}{2}\right)}{2\sqrt{2}\left(e^2-1\right)}$ B) $\dfrac{\log 2}{2\sqrt{2}\left(e^2-1\right)}$ C) $\dfrac{\sqrt{2}\log\left(\frac{e}{2}\right)}{e^2-1}$ D) $\dfrac{\sqrt{3}\log e}{e^2+1}$

SECTION - IV (More than one correct answer)

32. $f_1:\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\to R$ $f_2:\left[-1,e^{\frac{\pi}{2}}-2\right)\to R$ and $f_3:R\to R$ such that

$$f_1(x)\left.\begin{matrix}\\ \\ \end{matrix}\right\} \begin{matrix} =\dfrac{|\sin x|}{\tan^{-1}x} & x\neq 0 \\ =1 & x=0\end{matrix} \quad \text{where } \tan^{-1}x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$$

$$f_2(x)=[\sin\log(x+2)] \text{ where } [.]=\text{GIV} \qquad f_3(x) \text{ and}\left.\begin{matrix}\\ \\ \end{matrix}\right\} \begin{matrix} =x^2\sin\dfrac{1}{x} & x\neq 0 \\ =0 & x=0\end{matrix}$$

Which of the following are true

A) $f_1(x)$ is continuous and differentiable at x = 0

B) $f_2(x)$ is continuous and differentiable at x = 0 and its derivative is also differentiable at x = 0

C) f_3 is not continuous at x = 0

D) f_3 is differentiable at x = 0, but its derivative is not continous at x = 0

33. Let $g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)};0<x<2$, m and n are integers, $m\neq 0, n>0$, and let p be the left hand derivative of $|x-1|$ at $x=1$. If $\lim_{x\to 1^+} g(x)=p$, then

A) n = 1 B) m = –1 C) n = 2 D) m = 2

34. Let $(a\ \ b)\in R$ and $f:R\to R$ such that $f(x)=a\cos|x^3-x|+b|x|\sin|x^3+x|$. Then 'f' is

A) Differentiable at x = 0 if a = 0 b = 1 B) Differentiable at x = 1 if a = 1 b = 0

C) not differentiable at x = 0 if a = 1 b = 0 D) Not differentiable at x = 1 if a = 1 b = 1

SECTION - V (Numerical Type)

35. $f:R\to R$ be a differentiable function such that f(0) = 1 and satisfying the equation $f(x+y)=f(x)f'(x)f(y)$ for all $x\in R$ and $y\in R$ then $\frac{1}{4}\log f(4)=$

36. $f:[0,2]\to R$ be a continuous function in [0,2] and differentiable in (0,2) with f(0)=1. Given $F(x)=\int_0^{x^2} f(\sqrt{t})dt$. Also given $F'(x)=f'(x)$ Then

Then $\frac{e^4-F(2)+F(0)}{4}=$

37. If $x+\cos\theta=\sec\theta$ and $y+\cos^8\theta=\sec^8\theta$. Then $\frac{1}{10}\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2=$

38. If $f(x)=\begin{cases}\frac{e^x-1}{\sqrt{(1+x^2)-\sqrt{1-x^2}}} & \text{if } x\neq 0\\ \sqrt{\frac{2}{3}} & \text{if } x=0\end{cases}$;Let $\underset{x\to 0^+}{\text{Lt}} f(x)=m$ and $\underset{x\to 0^-}{\text{Lt}} f(x)=n$. Then $-\frac{1}{2}\left(\frac{m}{n}\right)=$

39. Let $a,b\in R, b\neq 0$ define a function $f(x)=\begin{cases} a\sin\left(\frac{\pi}{2}(x-1)\right), & \text{for} x\leq 0\\ \frac{\tan 2x-\sin 2x}{bx^3} & \text{for} x>0\end{cases}$;it f is continuous at $x=0$

Then $\frac{10-ab}{10}=$

SECTION VI - (Matrix match type)

40. The functions in set A is defined in $-1<x<1$ and one or more statements in set B are valid for them

Set A	Set B
a) $\lvert x\rvert x$	p) ontinuous in (–1 1)
b) $\sqrt{\lvert x\rvert}$	q) Differentiable (–1 1)
c) $x+[x]$ where	r) strictly increasing in (–1 1)
d) $\lvert x-1\rvert+\lvert x+1\rvert$	s) not differentiable at least one point in (–1 1)

A) $\left[(a\quad p,q,r)(b,p,s)(c\ \ rs)(d\ pq)\right]$

B) $(a\quad ps)(b\ \ pr)(c\ \ rs)(d\ pq)$

C) $(a\quad pqr)(b\ \ pr)(c\ \ pq)(d\ rs)$

D) $(a\quad pqs)(b\ \ ps)(c\ \ rs)(d\ pqs)$