FUNCTIONS, LIMIT, DIFFERENTIATION AND INTEGRATION

Functions

Consider two variables x any y. Whenever there is a change in ‘x’ if there is a corresponding change in y we say the variable y is a function of the variable x.

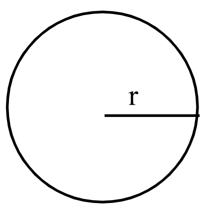
It is denoted by y = f(x) (we read it as ‘y’ is a function of ‘x’)

Here ‘x’ is called the **independent** variable and ‘y’ is called the **dependent** variable

Thus the function $y = f(x)$ means when ever there is a change in the independent variable 'x' there is a corresponding change in the dependent variable y

For example we know that the area of circle is Area of circle is $\text{Area} = \pi r^2$ where ‘r is the radius. Whenever there is a **change in radius ‘r’ there is a corresponding change in Area**

$$\therefore \boxed{\begin{array}{l}\text{Independent variable} = r \\ \text{Dependent variable} = \text{Area}\end{array}}$$



Hence we say Area of a circle is a function of its radius and is denoted by $\boxed{A = f(r)}$

Example 2 : The **mark** of a student is a function of **Hard work** . ie

Independent variable = Hard work

Dependent variable = Mark

$\therefore$ **Mark = f (Hard work)**

Univariate function

A dependent variable depending on one independent variable . In case of a circle

$$\boxed{\begin{array}{l} \text{Area} = f(\text{radius}) \\ \text{Displacement} = f(\text{time}) \end{array}}$$ are univariate function

Bivariate function

A dependent variable 'u' depends on two independent variables 'x' and 'y'.

$$\boxed{u = f(x, y) \text{ is a Bivariate function}}$$

Ex: The area of Triangle is given by $A = \frac{1}{2} bh$

b = base b = Altitude

Area depends on base and Altitude

$\therefore \boxed{\text{Area of } \Delta = f(b, h)}$

Is Marks of students a Bivariate function?

Function as a production process

A function can be regarded as a production process in which

Input = Independent variable

Output = Dependent variable

whenever you give an input 'x', the production process makes some work and gives you the output y

$$y = f(x)$$

$$\xrightarrow{\text{Input }(x)} \boxed{\begin{array}{l}\text{Production}\\ \text{Process}\end{array}} \xrightarrow{\text{Output }(y)}$$

For example whenever you give an input radius (r) of a circle the process makes the work $\pi\ r^2$ and gives you the output Area of circle

$$A = f(r)$$

$$\xrightarrow{\text{Input }(r)} \boxed{\begin{array}{l}\text{Process}\\ \pi r^2\end{array}} \xrightarrow{\text{Area (Output)}}$$

Domain	**Range**
Set of values of the independent variable 'x' **or input**	Set of values of the dependent variable y **or output**

Increasing and Decreasing functions

y = f(x) is an **increasing function** if dependent variable 'y' increases when there is increase in independent variable

r = radius of circle A = Area of circle

$\therefore \boxed{A = f(r)}$ is an increasing function

$\boxed{\text{Marks} = f\left(\text{Hard work}\right)}$ is an increasing function.

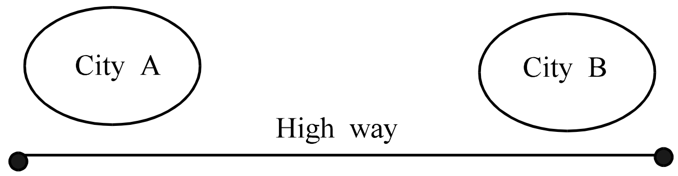
Decreasing function. y = f(x) is a decreasing function if the **dependent variable decreases** as the **independent variable increases**

For example at constant temperature, as pressure of gas increases, the volume of gas decreases

$\therefore \boxed{\text{Volume} = f(\text{pressure})}$ is a decreasing function

Also $\boxed{\text{Marks} = f\left(\text{laziness}\right)}$ is a decreasing function

Example



As **speed increases**, the **time taken to travel** from city A to cityB decreases

$\boxed{\text{Travel time} = f(\text{speed})}$ is a decreasing function

Graph of a function

Consider the function $y = f(x)$. x = Independent variable and y = dependent variable. Corresponding to every **value of x** there is unique value of y so that we get a **set of ordered pairs (x,y).** These points are plotted on a graph paper and are joined by a smooth curve. It is called the graph of that function.

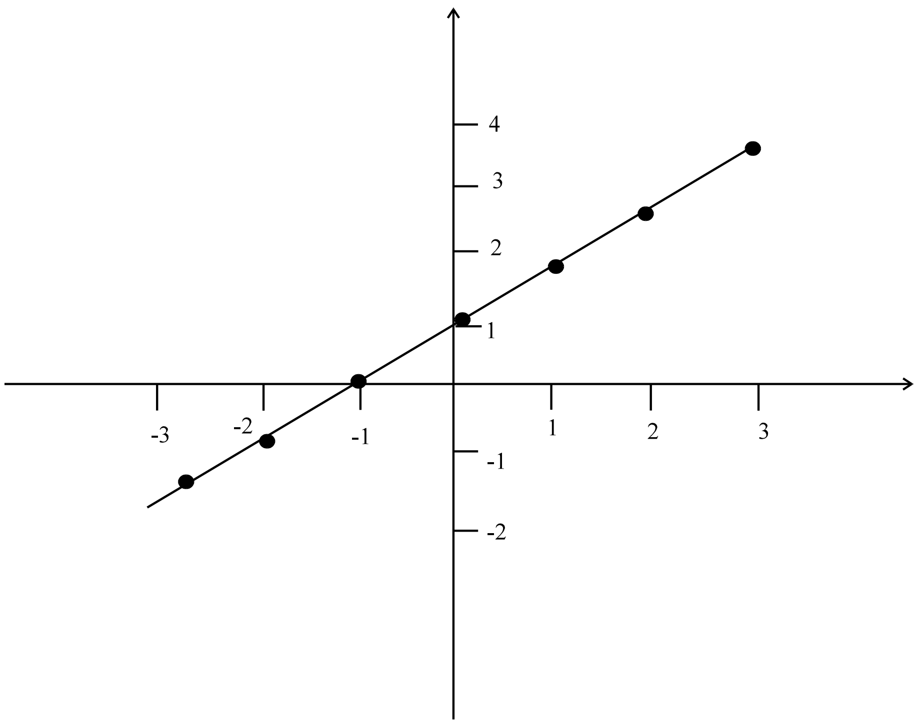
$$\boxed{\begin{array}{l} X - \text{axis} - \text{Independent variable} \\ Y - \text{axis} - \text{dependent variable} \end{array}}$$

For example consider the function $f(x) = x + 1$

Here $y = f(x) = x + 1 \rightarrow$ corresponding to every x, the value of y = x +1. The values of x and y

x	-3	-2	-1	**0**	1	2	3
y=x+1	-2	1	0	**1**	2	3	4

Graph of Y = x+1



Constant function

$y = f(x) = k$

$y = f(x) = 0$ is $x - \text{axis}$

$y = x$ is identify function

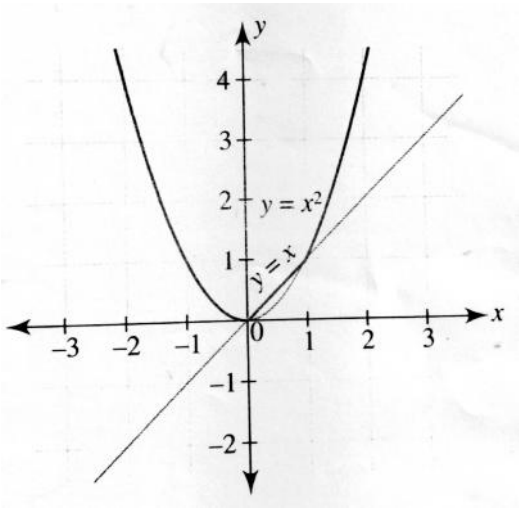
y = -x

Q.2 Draw the graph of $f(x) = x^2$; $y = f(x) = x^2$

Y is the square of x

$x \longrightarrow \boxed{\text{square}} \xrightarrow{y=x^2}$

x	-2	-1	0	1	2
y=f(x)=x2	4	1	0	1	4



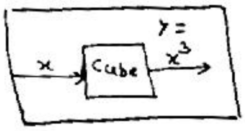
$\boxed{\text{Graph } f(x) = x^2}$

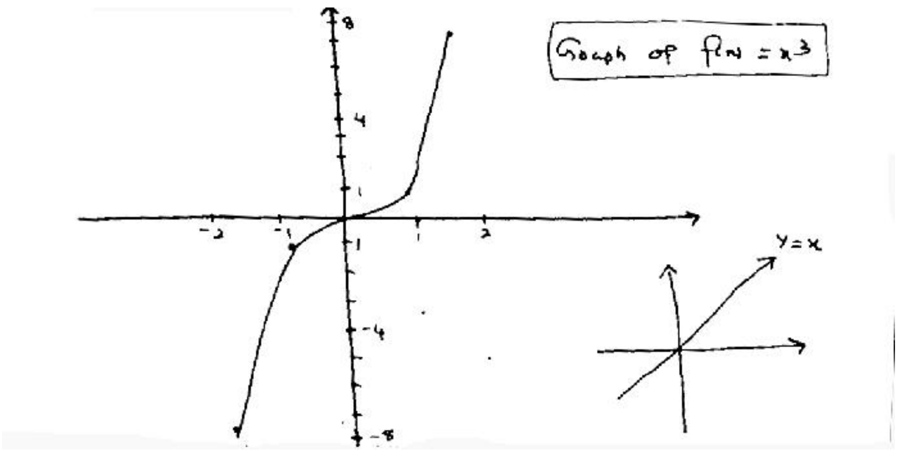
This shape is called parabola

The graph of $f(x) = x^2$ is a parabola

Q.3 Draw graph of y=f(x) = x3

x	−2	−1	0	1	2
$y = x^3$	−8	−1	0	1	8

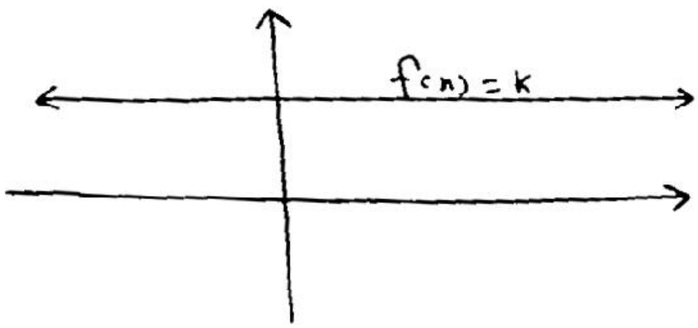




Constant function

f(x) = k

x	−2	−1	0	1	2
$f(x) = k$	k	k	k	k	k



Note (x-axis)

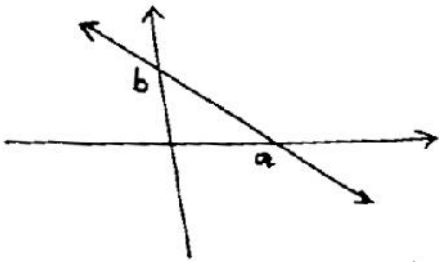
f(x) = 0 or y = 0 is a constant function and it is the x-axis

Linear function

f(x) = ax +b or y = ax +b

Intercept form of a straight line

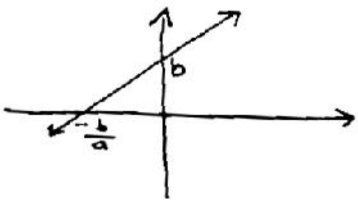
$$\frac{x}{a}+\frac{y}{b}=1$$



$\therefore f(x) = y = ax + b$ can be converted in to intercept form y = ax +b

$$-ax + y = b$$

$$\frac{x}{\left(-\frac{b}{a}\right)}+\frac{y}{b}=1$$



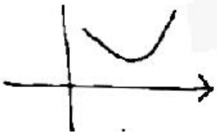
Quadratic function

$f(x) = ax^2 + bx + c$

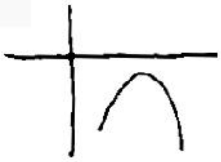
$a > 0$ concave up 

$a < 0$ concave down 

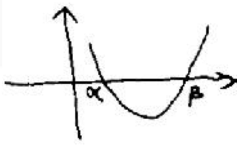
1) $f(x) = ax^2 + bx + c \quad a > 0 \quad b^2 - 4ac < 0$



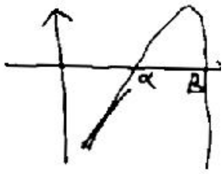
2) $f(x) = ax^2 + bx + c \quad a < 0 \quad b^2 - 4ac < 0$



3) $f(x) = ax^2 + bx + c \quad a > 0 \quad b^2 - 4ac > 0$

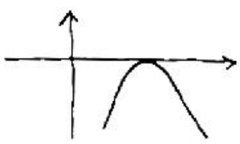


4) $f(x) = ax^2 + bx + c \quad a < 0 \quad b^2 - 4ac > 0$



5) $f(x) = ax^2 + bx + c \quad a > 0 \quad b^2 - 4ac = 0$

6) $f(x) = ax^2 + bx + c \quad a < 0 \quad b^2 - 4ac = 0$



Before having the graphs of some other functions we may introduce $+\infty$ and $-\infty$

The concept plus infinity $(+\infty)$

Consider the function $f(x) = 2^x$

$f(0)=1 \quad f(1)=2 \quad f(2)=2^2=4 \quad f(3)=2^3=8$

$f(0)=2^{10}=1024$. As x increases $f(x)=2^x$ will increase much faster than the increase in x

Now consider the function $f(x)=10^x$

x	0	1	2	3	4	5	6	9	12
$y\text{-}f(x)10^x$	1	10	100	1000	10000	100000	10^6=million	10^9 billion	10^{12} Trillion

It can be seen that as x increases $f(x)=10^x$, increases much much faster than x. Hence when **x is very big** number $f(x)=10^x$ **tends to a very, very** big number and it is denoted by $+\infty$ (Read as + infinity or positive ∞)

The concept -ve (-) infinity $(-\infty)$

consider f(x) = -10^x

$f(0)=-10^0=-1 \quad f(1)=-10^1=-10 \quad f(2)=-10^2=-100$

$f(3)=-10^3=-1000 \quad f(4)=-10^4=-10000$

$f(5)=-10^5=-100,000 \quad f(6)=-10^6=-(\text{million})=-1000000(-10 \text{ lakhs})$

$f(9)=-10^9=-\text{Billion} \quad f(12)=-10^{12}=-\text{Trillion}$

$f(100)=-10^{100}=-\text{Googol}=\text{verysmall number}$

x	0	1	2	3	4	5	6	9	12
$y\text{-}f(x)10^x$	-1	-10	-100	-1000	-10000	-100,000	- million	- billion	- Trillion

As x increases -10^x decreases much faster than the increase in x. When ‘x’ is a very big number -10^x will be a very very small number which can not be visulized, which can not be writtern on paper and which can not be operated. This very very small number is represented by $-\infty$ and is called -ive or minus infinity

Is infinity a number?

No, infinity (∞) is not a real number. It is only a concept, an idea. It can not be measured. Even the far away galaxies can not comepte with infinity.

Since ∞ is not a number the mathematical operations, Algebraic laws, laws of exponents etc are not valid in ∞

Limit of a function:

Consider the function $y = f(x)$. When the independent variable 'x' approaches or x tends to a constant value 'a' (denoted by $x \to a$) if the dependent variable y approaches to another constant value 'k' (denoted by y or $f(x) \to k$) we say, the limit of y = f(x) when x tends to a $(x \to a)$ is k. It is denoted by \

$$\boxed{\underset{x \to a}{\text{Lt}} \; f(x) = k}$$

Here (i) The variable 'x' may or may not become exactly equal to 'a'.

ii) f (x) may or may not take the value k

Example -1 consider the function f(x) = x^2

x	1.5	1.8	1.9	1.999	2	2.1	2.2	2.5
$f(x) = x^2$	2.25	3.24	3.61	3.996001	4	4.41	4.84	6.25

From table when $x \to 2$ from either side the value of $f(x) = x^2 \to 4$ and we write

$$\boxed{\underset{x \to 2}{\text{Lt}} \; f(x) = \underset{x \to 2}{\text{Lt}} \; x^2 = 4}$$

Here 'x' takes the value 2 and $f(x) = x^2$ takes the value 4

<u>Right Hand Limit (RHL) and Left Hand Limit (LHL)</u>

From the table it can be seen that when $x \to 2$ from $x < 2$, the value of $f(x) = x^2$ tends to 4 and it is denoted by

$$\boxed{\underset{x \to 2^-}{\text{Lt}} \; f(x) = \underset{x \to 2^-}{\text{Lt}} \; x^2 = 4 \;\; \text{and is called the LHL}}$$

Also from table when $x \to 2$ from $x > 2$ then also $f(x) = x^2 \to 4$ and it is denoted by

$$\boxed{\underset{x \to 2^+}{\text{Lt}} \; f(x) = \underset{x \to 2^+}{\text{Lt}} \; x^2 = 4 \;\; \text{and is called RHL}}$$

Note

1) $\underset{x\to 2}{\text{Lt}}\ f(x) = \underset{x\to 2}{\text{Lt}}\ x^2 = 4$ RHL = LHL = 4
In general $\underset{x\to a}{\text{Lt}}\ f(x) = k \Rightarrow \underset{x\to a^+}{\text{Lt}}\ f(x) = \underset{x\to a^-}{\text{Lt}}\ f(x) = k$
2) If RHL $\neq$ LHL $\Rightarrow \underset{x\to a}{\text{Lt}}\ f(x)$ Does not exist

Now consider the function $f(x) = \frac{x^2-1}{x-1}$

x	0.99	0.999	1	1.01	1.1
$f(x) = \frac{x^2-1}{x-1}$	1.99	1.999	$\frac{1-1}{1-1} = \frac{0}{0}$ not defined	2.01	2.1

From table when $x \to 1$ from $x < 0$ the value of $f(x) = \frac{x^2-1}{x-1} \to 2$ and we write

$$\text{LHL} = \underset{x\to 1^-}{\text{Lt}}\ f(x) = \underset{x\to 1^-}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2$$

Also when $x \to 2$ from $x > 1$, then also the value of $f(x) = \frac{x^2-1}{x-1} \to 2$ and we write

$$\text{RHL} = \underset{x\to 2+}{\text{Lt}}\ f(x) = \underset{x\to 2+}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2$$

The RHL = LHL ie $\underset{x\to 2-}{\text{Lt}}\ \frac{x^2-1}{x-1} = \underset{x\to 2+}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2$

$$\boxed{\therefore \underset{x\to 2}{\text{Lt}}\ \frac{x^2-1}{x-1} = 2}$$

Objective of limit

$$f(x) = \frac{x^2 - 1}{x - 1} \Rightarrow f(1) = \frac{1-1}{1-1} = \frac{0}{0}$$

$\frac{0}{0}$ is undefined (Not a finite quantity / Exact value) The concept limit gives you the expected value (not exact value) of $f(x) = \frac{x^2 - 1}{x - 1}$ and the expected value is 2

So the <u>objective of limit</u> is to find the expected value (and the exact value) of a function at a point where the <u>direct subtitution</u> results in an <u>undefined value</u>

$$f(x) = \frac{\sin x}{x} \Rightarrow f(0) = \frac{\sin 0}{0} = \frac{0}{0} \text{undefined}$$

x	-0.2	-0.05	0	0.01	0.03
$\frac{\sin x}{x}$	.993347	.999583	$\frac{\sin 0}{0}$	.999983	.99985

$$\underset{x \to 0-}{\text{Lt}} \frac{\sin x}{x} = 1 \Leftrightarrow \underset{x \to 0+}{\text{Lt}} \frac{\sin x}{x} = 1$$

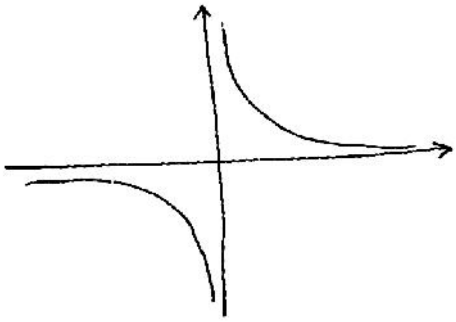
$$\boxed{\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1}$$

Result

If $\text{RHL} \neq \text{LHL}$ at $x = a$ then $\underset{x \to a}{\text{Lt}} f(x)$ does not exist

<u>Reciprocal function (Rectangular Hyperbola)</u>

$$f(x) = \frac{1}{x}$$



$$\underset{x\to\infty}{\mathrm{Lt}}\ \frac{1}{x}=\frac{1}{\infty}=0$$

$$\underset{x\to-\infty}{\mathrm{Lt}}\ \frac{1}{x}=\frac{1}{-\infty}=0$$

v.Important

$$\boxed{\begin{array}{l}\underset{x\to 0+}{\mathrm{Lt}}\ \dfrac{1}{x}=\dfrac{1}{0+}=\infty\\ \underset{x\to 0-}{\mathrm{Lt}}\ \dfrac{1}{x}=\dfrac{1}{0-}=-\infty\end{array}}$$

x	.001	.0001	−.00001	0	0.00001	.0001	.001	.01	.1
$f(x)=\frac{1}{x}$	−1000	−10000	−100000	$\frac{1}{0}$ undefined	100000 10^5	10.000 10^4	1000	100	10

x	10	100	1000	$10,000$	......	$-\infty$
$y=\frac{1}{x}$	0.1	0.01	0.001	0.0001		$1/\infty=0$

$$\boxed{\underset{x\to\infty}{\mathrm{Lt}}\ \frac{1}{x}=\frac{1}{\infty}=0}$$

x	-10	-100	-1000	-10000	$-\infty$
$y=\frac{1}{x}$	-0.1	-0.01	$-.001$	$-.0001$	$\frac{1}{-\infty}=0$

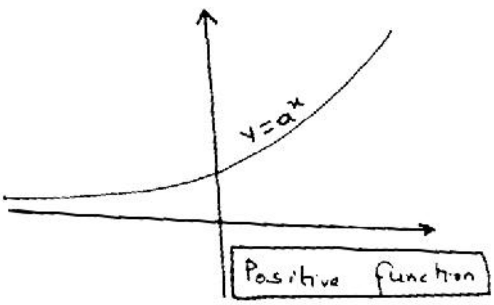
$$\boxed{\underset{x\to-\infty}{\mathrm{Lt}}\ \frac{1}{x}=\frac{1}{-\infty}=0}$$

Exponential function

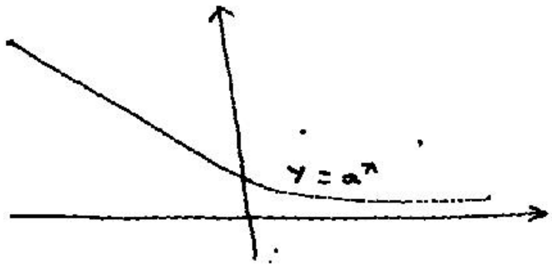
$$f(x)=a^x \quad \left(y=2^x,\ \ y=3^x,\ \ y=10^x\ \ ....\right)$$

$$\boxed{\begin{array}{l} a^{\infty}=\infty \\ a^{-\infty}=0 \end{array}}$$

$$2^{\infty}=\infty \qquad 2^{-\infty}=0$$
$$3^{\infty}=\infty \qquad 3^{-\infty}=0$$



case 2 $\boxed{f(x) = a^x \quad 0 < a < 1}$

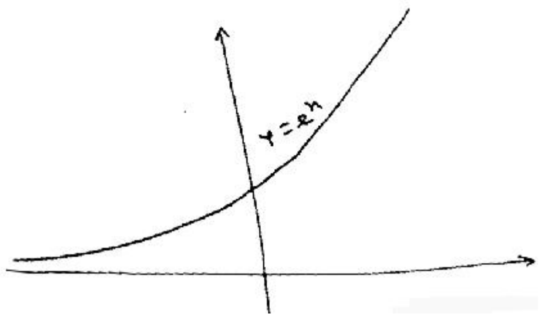


$$\boxed{\begin{array}{l} a^{\infty} = 0 \\ a^{-\infty} = \infty \end{array}}$$

$$\left(\frac{1}{2}\right)^{\infty} = 0 \quad \left(\frac{1}{2}\right)^{-\infty} = \infty$$

Natural exponential function

$$f(x) = e^x$$



$$\boxed{\begin{array}{l} e^{\infty} = \infty \\ e^{-\infty} = 0 \end{array}}$$

$$\underset{x\to\infty}{\mathrm{Lt}}\left(\left(\frac{4}{5}\right)^{x} + 1\right)^{\frac{1}{x}} = (0+1)^0 = 1^0 = 1$$

Find $\boxed{\underset{x\to 0^+}{\text{Lt}} \dfrac{1}{3-2^{\frac{1}{x}}}}$ $\boxed{\underset{x\to 0^-}{\text{Lt}} \dfrac{1}{3-2^{\frac{1}{x}}}}$ $\boxed{\underset{x\to\infty}{\text{Lt}} \left(4^x+5^x\right)^{\frac{1}{x}}}$

Logarithms

$2^3=8$ Then we say $\log_2 8 = 3$

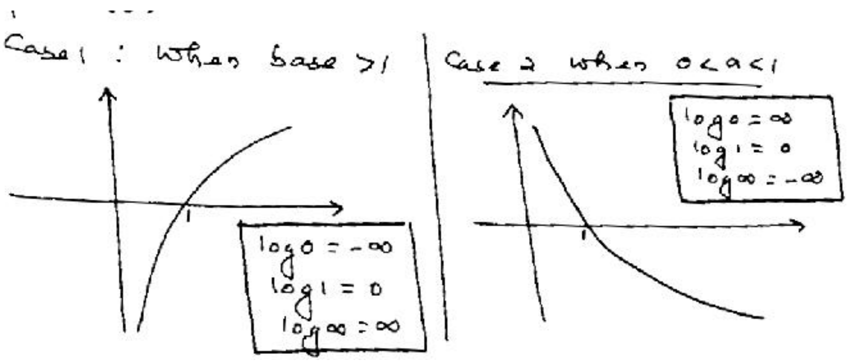
$3^4=81$ Then we say $\log_3 81 = 4$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8} \Rightarrow \log_{\left(\frac{1}{2}\right)}\left(\frac{1}{8}\right) = 3$

In general $a^m = k \Rightarrow \log_a k = m$ (Read it as logarithm of k to the base 'a' is m)

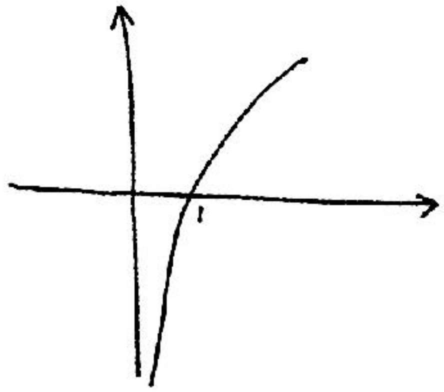
<u>logarithmic function</u>

$f(x) = \log_a x$ where x is a +ive real no. and a > 0 and $a \neq 1$ is called the logarithmic function



Natural logarithmic function

When base $a = e \simeq 2.72$



Properties

$$\log(ab) = \log a + \log b$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\log a^{m} = m\ \log a$$

$$-\log a = \log\left(\frac{1}{a}\right)$$

Series expansion of functions

$\boxed{5! = 1 \times 2 \times 3 \times 4 \times}$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + ...$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^{2}}{2!} - \frac{x^{3}}{3!} + ...$$

$$\log\left(1 + x\right) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - ....$$

$$\log\left(1 - x\right) = -x - \frac{x^{2}}{2} - \frac{x^{3}}{2} - .\frac{x^{4}}{4}........$$

$$\tan\ x = x + \frac{x^{3}}{3} + \frac{2x^{5}}{15} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ......$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....$$

$$a^x = 1 + \frac{x \log a}{1!} + \frac{(x \ \log a)^2}{2!}$$

$$(1+x)^x = 1 + nx + \frac{n(n-1)}{1\times 2}\ x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3} x^3 + ...\ \ \text{when} |x| < 1$$

$$\left.\begin{array}{l} \underset{x\to 0}{\mathrm{Lt}} \dfrac{\sin x}{x} = 1 \\ \underset{x\to 0}{\mathrm{Lt}} \dfrac{\tan x}{x} = 1 \\ \underset{x\to 0}{\mathrm{Lt}} \dfrac{a^x - 1}{x} = \log a \\ \underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(1+x)}{x} = 1 \end{array}\right\} \textbf{Prove using expansion method}$$

Some important limits

1) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\sin x}{x} = \underset{x\to 0}{\mathrm{Lt}} \dfrac{x}{\sin x} = 1$

2) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\tan x}{x} = \underset{x\to 0}{\mathrm{Lt}} \dfrac{x}{\tan x} = 1$

3) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{x^n - a^n}{x - a} = n\ a^{n-1}$

4) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{e^x - 1}{x} = 1$

5) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{\log(1+x)}{x} = 1$

6) $\underset{x\to 0}{\mathrm{Lt}} \dfrac{a^x - 1}{x} = \log\ a$

7) $\underset{x \to 0}{\mathrm{Lt}} \frac{1-\cos x}{x^2} = \frac{1}{2}$

Note : In all these limits the direct substitution is undefined. Hence Limit gives us the expected value of the fn when $x \to 0$ or a etc

Questions find

1) $\underset{x \to 0}{\mathrm{Lt}} \frac{\sin 5x}{x}$

2) $\underset{x \to 0}{\mathrm{Lt}} \frac{\tan 3x}{\tan 5x}$

3) $\underset{x \to 0}{\mathrm{Lt}} \frac{x^3 - 8}{x - 2}$

4) $\underset{x \to 0}{\mathrm{Lt}} \frac{\log(1+2x)}{x}$

5) $\underset{x \to 0}{\mathrm{Lt}} \frac{2^x - 1}{x}$

Limits of Rational functions

$$\underset{x \to \infty}{\mathrm{Lt}} \frac{a_0 x^n + a_1 x^{n-1} + ...an}{b_0 n^m + b_1 n^{m-1} + ...bm} \left. \begin{matrix} \frac{a_0}{b_0} & f \;\; m = n \\ = 0 & m > n \\ = \infty & n > m \end{matrix} \right.$$

1) $\underset{x \to \infty}{\mathrm{Lt}} \frac{5x^3 + 2x^2 + 1}{4x^3 - 3x + 7} = \frac{5}{4}$

2) $\underset{x \to \infty}{\mathrm{Lt}} \frac{5x^2 + 7x + 1}{4x^3 + 3x^2 + 2} = 0$

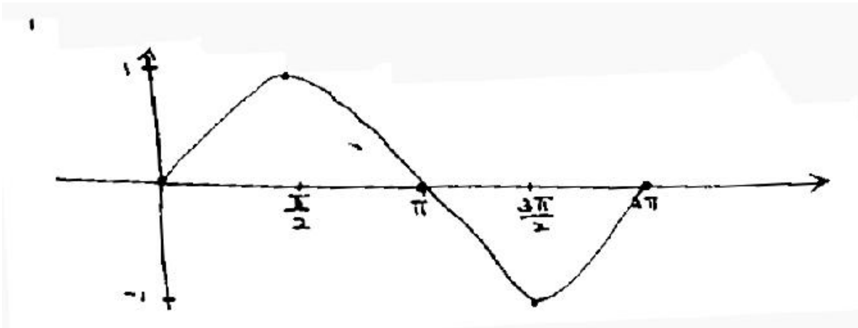
3) $\underset{x \to \infty}{\mathrm{Lt}} \frac{2x^3 + 3x - 1}{4x^3 - 2x + 7} = \infty$

Sin<u>e function : f(x) = sin x</u>

$$180^0 = \pi \text{ radians} \quad \Rightarrow 90^0 = \frac{\pi}{2} \text{radians}$$

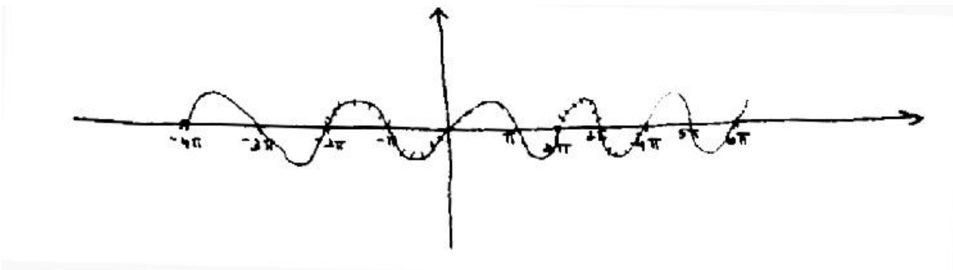
$$2\pi = 2 \times 180 = 360^0 \quad \text{and so on}$$

x	0	$\frac{\pi}{2} = 90$	$180 = \pi$	$270 = 3\frac{\pi}{2}$	$360 = 2\pi$
$f(x) = \sin x$	0	1	0	-1	0



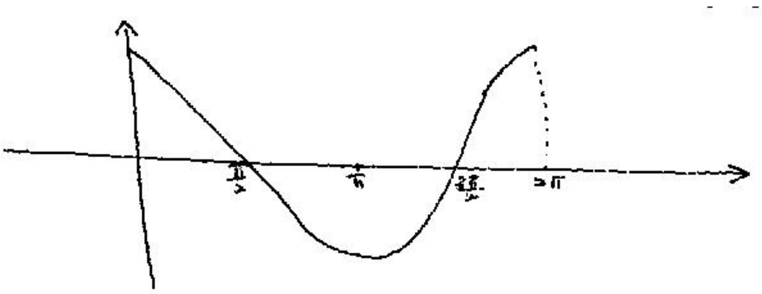
f(x) sin x is periodic with period = 2π

Now cut and paste



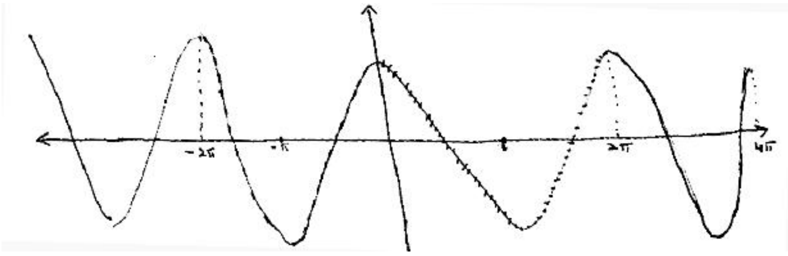
cosine function $\Rightarrow f(x) = \cos x$

0	$\frac{\pi}{2} = 90$	$\pi = 180$	$\frac{3\pi}{2} = 270$	$2\pi = 360$
1	0	-1	0	1



f(x) = cos x is periodic with period = 2π

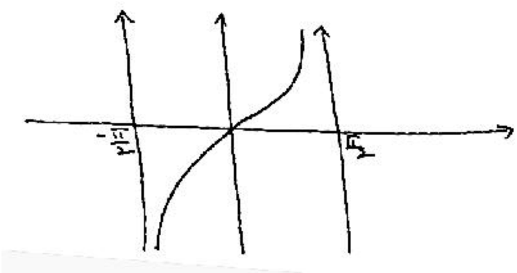
so cut and pase



Tangent function

$$f(x) = \tan x \qquad \tan 0 = 0 \qquad \tan\frac{\pi}{2} = \infty$$

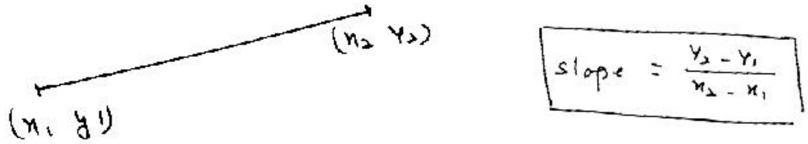
$$\tan\left(-\frac{\pi}{2}\right) = -\infty$$



$$f(x) = \tan x \qquad \tan 0 = 0 \qquad \tan\frac{\pi}{2} = \infty$$

$$\tan\left(-\frac{\pi}{2}\right) = -\infty$$

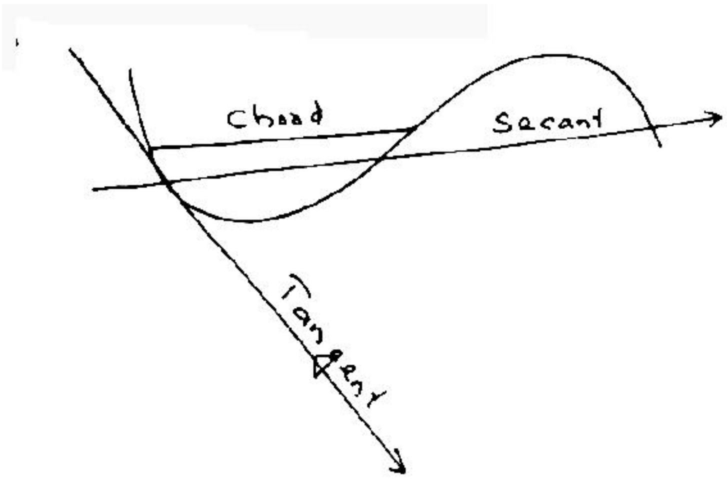
Differentiation



Chord : Line segment joining exactly two points

Secant : Line segment joining two or more points

Tangent : **Limiting line of a secant**

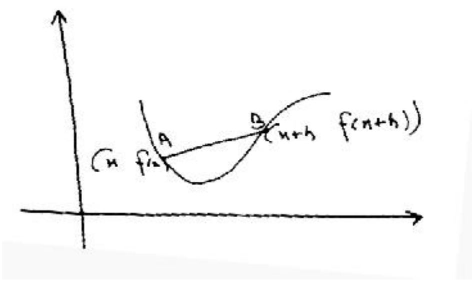


Derivative or Differential Coefficient

Consider the function y = f (x) ; $\text{Let}A\left(x \quad f(x)\right)$ and $B\left(x+h \quad f\left(x+h\right)\right)$ be two points on the graph of f(x) as shown below

Slope of secant AB

$$\frac{f(x+h)-f\left(x\right)}{x+h-x}$$



Slope of secant $AB = \dfrac{f\left(x+h\right)-f\left(x\right)}{h}$

$\therefore$ Slope of tangent at $A = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

This limit, if it exists, is called the derivative or differential coefficient of $y = f(x)$ w.r.t. x and is called the ab initio derivative or the derivative from first principles. It is denoted by $\frac{dy}{dx}$ or $f^1(x)$

The process of finding the Derivative is called differentiation.

<u>Result -1</u>

In Geometrical sense $\frac{dy}{dx}$ or $f^1(x)$ is the slope of tangent at the point $(x \quad f(x))$

Result -2

In physical sense $\frac{dy}{dx}$ is the rate of change of y w.r.t.x

Questions Find the at-initio Derivative of

1) $f(x) = k$ 2) $f(x) = x^2$ 3) $f(x) = x^3$

4) $f(x) = \frac{1}{x}$ 5) $f(x) = e^x$

List of standard Derivatives

1) $\frac{d}{dx} \; k = 0$

2) $\frac{d}{dx} \; x = 1$

3) $\frac{d}{dx} \; x^n = n \; x^{n-1}$

4) $\frac{d}{dx} \; \frac{1}{x} = -\frac{1}{x^2}$

5) $\frac{d}{dx} \; \sqrt{x} = -\frac{1}{2\sqrt{x}}$

6) $\frac{d}{dx} \; \log x = \frac{1}{x}$

7) $\frac{d}{dx} e^x = e^x$

8) $\frac{d}{dx} a^x = a^x \log a$

9) $\frac{d}{dx} \sin x = \cos x$

10) $\frac{d}{dx} \cos x = -\sin x$

11) $\frac{d}{dx} \tan x = \sec^2 x$

12) $\frac{d}{dx} \sec x = \sec x \tan x$

13) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

14)) $\frac{d}{dx} \frac{1}{x^n} = \frac{-n}{x^{n+1}}$

Algebra of Derivatives

1) $\frac{d}{dx} k f(x) = k \frac{d}{dx} f(x)$

2) $\frac{d}{dx} f(x) + g(x) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

3) Product Rule

$$\frac{d}{dx} f(x) g(x) = f(x) g^1(x) + g(x) f^1(x)$$

4) Quotien Rule

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) f^1(x) - f(x) g^1(x)}{(g(x))^2}$$

5) Power Rule $\frac{d}{dx}\left(f\left(x\right)\right)^{n} = n\left(f\left(x\right)\right)^{n-1}\ \frac{d}{dx}\ f\left(x\right)$

6) Reciprocal Rule

$$\frac{d}{dx}\quad \frac{1}{f\left(x\right)} = \frac{-1}{\left(f\left(x\right)\right)^{2}}\ \frac{d}{dx} f\left(x\right)$$

Function of a function and chain Rule

$f\left[g\left(x\right)\right], g\left[f\left(x\right)\right]$ etc are function of functions

1) $\frac{d}{dx} f\left[g\left(x\right)\right] = f\left(g\left(x\right)\right) g^{1}\left(x\right)$

2) $\frac{d}{dx} g\left[f\left(x\right)\right] = g^{1}\left(f\left(x\right)\right) f^{1}\left(x\right)$

Derivative of f(x) w.r.t. another variable 't'

$$\boxed{\frac{d}{dt} f\left(x\right) = f^{1}\left(x\right)\frac{dx}{dt}}$$

1) $\frac{d}{dt} x^{2} = 2x\frac{dx}{dt}$

2) $\frac{d}{dt}\sin\ x = \cos x\ \frac{dx}{dt}$

3) $\frac{d}{dt}\sin^{2}\ x = 2\sin x\ \cos x\frac{dx}{dt}$

4) $\frac{d}{dx}\sin y = \cos y\ \frac{dy}{dt}$

5) $\frac{d}{du}\log t = \frac{1}{t}\ \frac{dt}{du}$

Parametric Differentiation

$$x = f\left(t\right)\ \text{and}\ y = \phi\left(t\right)$$

y = f(x) is a parametric
function in parameter 't'

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

Ex:1 $x = 2t^2 \quad y = 4t$

$$\frac{dx}{dt} = 4t \quad \frac{dy}{dt} = 4$$

$$\frac{dy}{dx} = \frac{4}{4t} = \frac{1}{t}$$

2) $x = a\cos t \quad y = a\sin t$

$$\frac{dx}{dt} = -a\sin t \quad \frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{a\cos t}{-a\sin t} = -\cot t$$

Physical Application of Derivatives

$$S = f(t) \Rightarrow \boxed{\frac{ds}{dt} = \text{Velocity}} \quad \boxed{\frac{dv}{dt} = \text{Acceleration}}$$

Q.1) A particle is protected vertically upwards satisfies $S = 60t - 16t^2$. What is the velocity when t = 0

$$S = 60t - 16t^2 \Rightarrow \frac{dy}{d}$$

$$V = \frac{ds}{dt} = 60 - 32t \ \text{ when } t = 0 \quad \Rightarrow \boxed{v = 60}$$

Q.2) Velocity $v = ks^2$. Then the acceleration is

$$a = \frac{dv}{dt} = 2ks\frac{ds}{dt} = 2ks\left(ks^2\right) = 2k^2s^3$$

Q.3) A circular plate is heated uniformly and its area exponds 3c times as fast as its radius. What is the value of 'c' when r = 6

$\text{Area} = A = \pi r^2$ Diff.w.r.t 't'

$$\frac{dA}{dt} = 2\pi \frac{dr}{dt} \quad \text{Given} \quad \frac{dA}{dt} = 3c\ \frac{dx}{dt}$$

$$3c\ \frac{dr}{dt} = 2\pi r\ \frac{dr}{dt} \quad \text{Given} \quad r = 6$$

$$3c = 2\pi \times 6 \Rightarrow c = \frac{2\pi 6}{3} = 4\pi$$

Geometrical Applications

Q.1) What is the slope of tangent at (1 4) to the curve $f\left(x\right) = 3x^2 - 5x + 6$

$$f^1\left(x\right) = 6x - 5$$

slope at (1 4) = $f^1\left(1\right) = 6 - 5 = 1$

Q.2) Slope of tangent of $f(x) = x^2 - \frac{1}{x^2}$ at $\left(1, 0\right)$

$$f^1\left(x\right) = 2x - \left(\frac{-2}{x3}\right) = 2x + \frac{2}{x^3}$$

$$f^1\left(1\right) = 2 + 2 = 4$$

Q.3) Slope of tangent at $\left(-1\ \ -3\right)$ to the curve

$$y^2\ e^y = 9e^{-3}x^2 \quad \text{ie} \quad y^2e^y = 9e^{-3}x^2$$

$$y^2e^y\ \frac{dy}{dx} + e^y\ \ 2y\ \frac{dy}{dx} = 9e^{-3}2x \quad \text{put}\ \ x = -1\ \ y = 3$$

$$9\ e^{-3}\ \frac{dy}{dx} = 6e^{-3}\ \frac{dy}{dx} = -18\ e^{-3}$$

$$\frac{dy}{dx}\left(9 - 6\right) = -18 \Rightarrow \frac{dy}{dx} = \frac{-18}{3} = -6$$

Increasing and decreasing function

$f^1(x) > 0 \Rightarrow f(x)$ is strictly ←

$f^1(x) < 0 \Rightarrow f(x)$ is strictly –

1) $f(x) = x^2 \Rightarrow f^1(x) = 2x \begin{cases} > 0 & \text{when } x > 0 \\ < 0 & \text{when } x < 0 \end{cases}$

$f(x) = x^2 \quad S\uparrow \quad \text{when } x > 0 \quad \text{and } S\downarrow \quad \text{when } x < 0$

2) $f(x) = x^3$

$f^1(x) = 3x^2 \; > 0 \quad \text{for all } x$

$\therefore f(x) = x^3 \quad \text{is} \quad S\uparrow \text{ for all } x$

INTEGRATION

list of Integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ c: Integrating constant

$\int dx = x + c$

$\int k\,dx = kx + c; k$ is a constant

$\int \sin x \; dx = -\cos x + c$

$\int \cos x \; dx = \sin x + c$

$\int \sec x \; \tan x\,dx = \sec x + c$

$\int \operatorname{cosec} x \; \cot x\,dx = -\operatorname{cosec} x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$\int \frac{1}{x} dx = \log|x| + c$

$$\int e^x dx = e^x + c$$

Examples

1. $\int \sin 2x\, dx = \frac{-\cos 2x}{2} + c$ [Divide by co-efficient of x

2. $\int \cos 3x\, dx = \frac{\sin 3x}{3} + c$

3. $\int (3x^2 - 5x + 8) dx = 3 \times \frac{x^3}{3} - 5 \times \frac{x^2}{2} + 8x + c$

$= x^3 - \frac{5x^2}{2} + 8x + c$

4. $\int (3\sin x - 4\sin 2x) dx = 3 \times \cos x - 4 \times \frac{-\cos 2x}{2} + c$

5. $\int (3\sin 2x - 6\cos 4x + e^{2x}) dx$

$= 3 \times \frac{-\cos 2x}{2} - 6 \times \frac{\sin 4x}{4} + \frac{e^{2x}}{2} + c$

$= \frac{-3\cos 2x}{2} - \frac{3\sin 4x}{2} + \frac{e^{2x}}{2} + c$

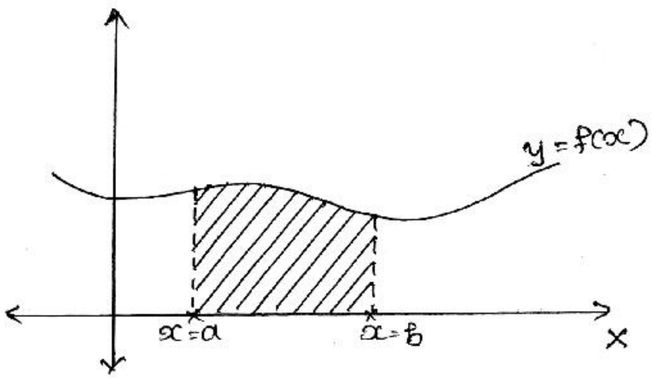
6. $\int \left(x^2 - x - \frac{4}{x} \right) dx = \frac{x^3}{3} - \frac{x^2}{2} - 4 \times \log|x| + c$

7. $\int (x^3 - 4x^2 + e^{3x}) dx$

$= \frac{x^4}{4} - 4 \times \frac{x^3}{3} + \frac{e^{3x}}{3} + c$

$= \frac{x^4}{4} - \frac{4x^3}{3} + \frac{e^{3x}}{3} + c$

Definite integrals -Area under the curve



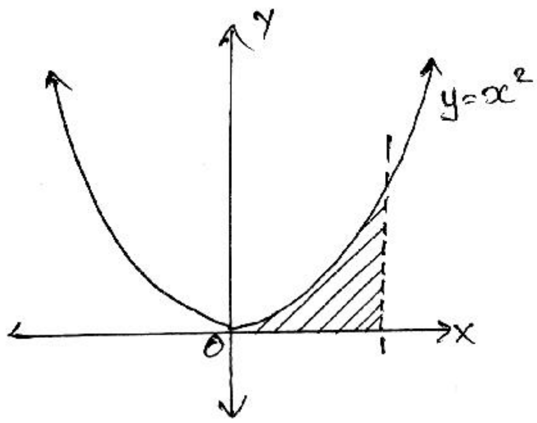
Area under the curve y = f(x) from x = a to x = b is

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$

a: lower limit

b: upper limit

1. Find the area under the curve $y = x^2$ from x = 0 to x = 1 solution



Shaded portion is the required area

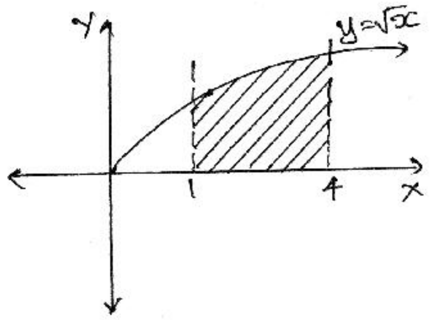
$$\text{Area} = \int_0^1 y\,dx = \int_0^1 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 \quad \left[\begin{array}{l}\text{No need to write integrating} \\ \text{constant on definite integrals}\end{array}\right.$$

$$=\left[\frac{1}{3}\right]-\left[\frac{0^3}{3}\right]\quad\left(\begin{array}{l}\text{Put upper limit I}^{st}\\ \text{and lower limit}\end{array}\right.$$

$$=\frac{1}{3}\text{sq.units}$$

2. Find the area under the curve $y=\sqrt{x}$ from $x=1$ to $x=4$



Required area = $\int_1^4 \sqrt{x}\,dx \;=\int_1^4 x^{1/2}dx \;=\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$

$$=\left.\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4 \qquad =\left.\frac{2}{3}\times x^{3/2}\right]_1^4$$

$$=\frac{2}{3}\times 4^{3/2}-\frac{2}{3}\times 1^{3/2}$$

$$=\frac{2}{3}\times 8-\frac{2}{3}\times 1$$

$$=\frac{16}{3}-\frac{2}{3}=\frac{14}{3}\text{ sq.units}$$

3) $\int_{-\pi}^{\pi} \cos 2x \, dx = \left.\frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$ $\pi : 180°$

$$= \frac{\sin 2\pi}{2} - \frac{\sin(-2\pi)}{2}$$

$$= 0 - 0 = 0$$

4) $\int_{0}^{\frac{\pi}{2}} \sin 2x \, dx = \left.\frac{-\cos 2x}{2}\right]_{0}^{\frac{\pi}{2}}$

$$= \left[\frac{-\cos 2\times\frac{\pi}{2}}{2}\right] - \left[\frac{-\cos 2\times 0}{2}\right]$$

$$= \frac{\cos \pi}{2} + \frac{\cos 0}{2} = -\frac{-1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

5) $\int_{-1}^{1} \left(x^2 - 4x + 1\right) dx$

$$= \left.\frac{x^3}{3} - 4\times\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \left[\frac{1}{3} - 4\times\frac{1}{2} + 1\right] - \left[\frac{-1}{3} - 4\times\frac{1}{2} - 1\right]$$

$$= \frac{1}{3} - \cancel{2} + 1 + \frac{1}{3} + \cancel{2} + 1$$

$$= \frac{2}{3} + 2$$

$$= \frac{8}{3}$$