RELATIONS, FUNCTIONS AND BINARY OPERATION

Types of relation

(1) Reflexive

A relation on set A is said to be reflexive if $a \in A \Rightarrow (a, a) \in R$

or If $a \in A \Rightarrow aRa$

Let $A = \{1, 2, 3\}$ Relations are subsets of $A \times A$

$A \times A = \{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)\ (3,1)\ (3,2)\ (3,3)\}$

$R_1 = \{(1,1)(2,2)(3,3)\}$ is reflexive

$R_2 = \{(1,1)(2,2)(1,2)\}$ is not reflexive because $3 \in A$ but $(3,3) \notin R_2$

$R_3 = \{(1,2)(2,2)(3,3)(1,2)\}$ is reflexive

$R_4 = \phi$ is not reflexive

$R_5 = A \times A$ is reflexive

Note

If n(A) = n then number of reflexive relation on $A = 2^{n^2-n}$

2. Symmetric

If $(a,b) \in R \Rightarrow (b,a) \in R$

or

If $a\,R\,b \Rightarrow b\,R\,a$

Let $A = \{1,2,3\}$

$R_1 = \{(1,2)(2,1)(2,3)(3,2)(1,3)(3,1)(1,1)\}$ is symmetric

$R_2 = \{(1,2)(2,1)(2,2)\}$ is symmetric

$R_3 = \{(1,2)(2,1)(2,3)(3,3)\}$ is not symm because $(2,3) \in R_3$ but $(3,2) \notin R_3$

$R_4 = \phi$ is symmetric

$R_5 = A \times A$ is symmetric

Note

(1) If R is symmetric then $R = R^{-1}$

(2) If $n(A) = n$ the number of symmetric relation is $2^{\frac{n(n+1)}{2}}$

(3) Transitive relation

If (a,b) and $(b,c) \in R \Rightarrow (a,c) \in R$ then R is transitive

Let $A = \{1,2,3\}$

$R_1 = \{(1,2)(2,3)(1,3)\}$ is transitive

$R_2 = \{(2,3)(3,1)\}$ is not transitive because $(2,1) \notin R_2$

$R_3 = \{(1,2)\}$ is transitive

$R_4 = \phi$ is transitive

$R_5 = \{(1,1)(2,2)(3,3)\}$ is transitive

$R_5 = \{(1,1)(2,2)(3,3)(1,2)\}$ is transitive

Equivalence relation

A relation which is reflexice symmetric and transitive is called an equivalence relation

Note

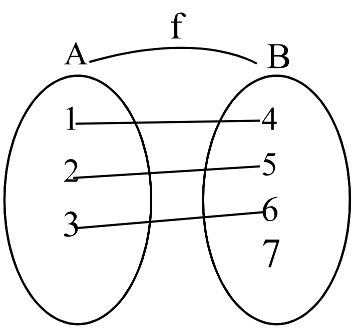
If $n(A) = n+1$ then number of equivalence relation (number of partition on a set) can be find by using the following reccurring formula

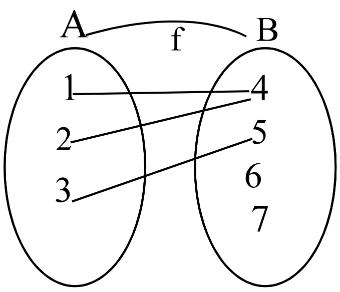
$$P_{n+1} = \sum_{r=0}^{n} nC_r . P_r \quad \text{where } P_o = 1, P_1 = 1, P_2 = 2$$

Types of function

1) Into function

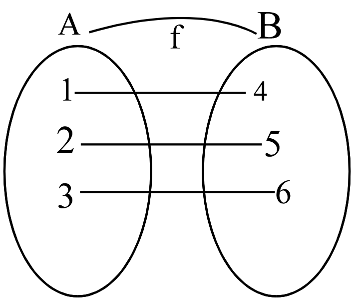
A function $f : A \to B$ is said to be an into function then $\operatorname{ran}(f) \subset$ codom (f)

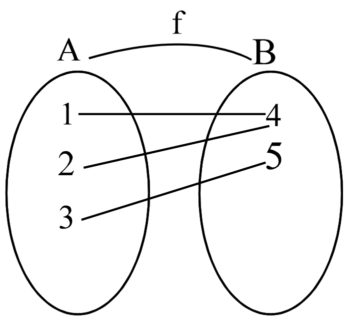
(1) 

(2) 

2) Onto function (surjection)

If $f : A \to B$ is said to be an onto function then ran (f) = codom (f)

(1) 

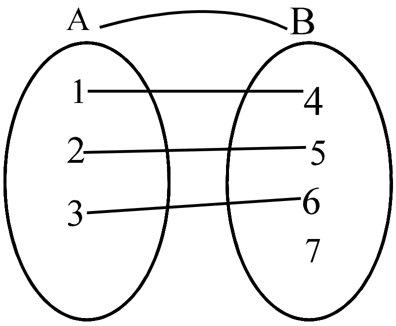
(2) 

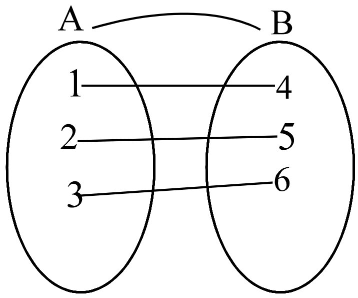
3) One one function (Injection)

If $f : A \to B$ is said to be a one one function then different elements in A have different images in B

Or

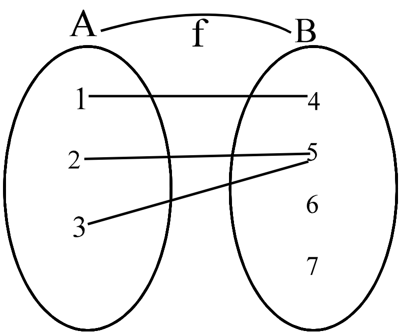
If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

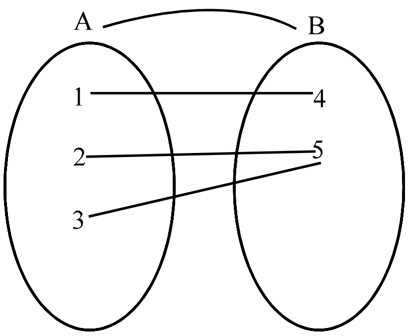




Many one function

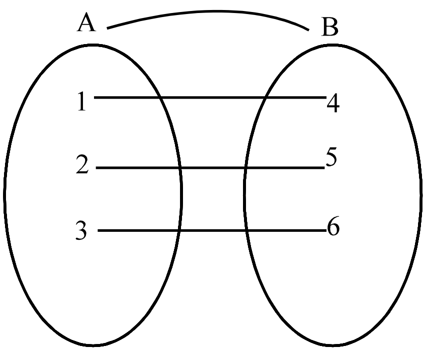
At least two different elements in A have same image in B





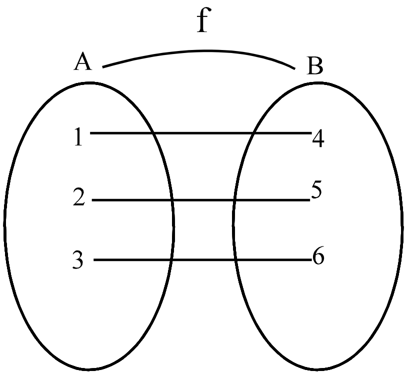
Bijection

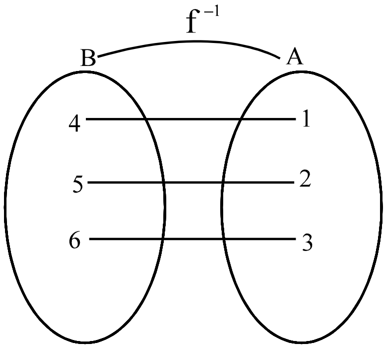
A function which is one one and onto is called bijection



Inverse function

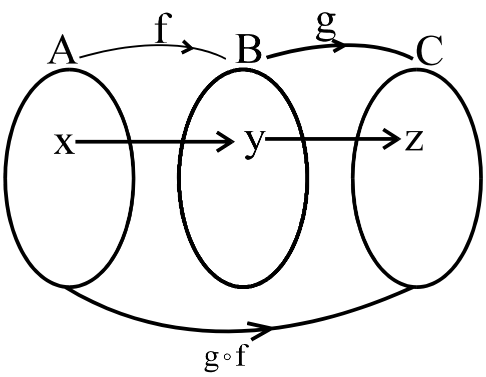
If $f : A \to B$ is a bijection then f^{-1} exists and $f^{-1} : B \to A$





Composition of function

If $f : A \to B$ and $g : B \to C$ are two functions then $g \circ f \ A \to C$ is called the composition of functions and is defined as $g \circ f(x) = g(f(x))$



From the figure $f(x) = y \to (1)$

$g(y) = z \to (2)$

Sub (1) in (2)

$g(f(x)) = z \Rightarrow g \circ f(x) = z$

$g \circ f : A \to C$

Note

(1) $g \circ f$ is defined only when the dom (g) = the codom (f)

(2) $\text{Dom}(g \circ f) = \text{Dom}(f)$ and $\text{codom}(g \circ f) = \text{codo}(g)$

(3) $g \circ f \neq f \circ g$

(4) $g\circ(f\circ h)=(g\circ f)\circ h$

(5) $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$

(6) $(g\circ h\circ f)^{-1}=f^{-1}\circ h^{-1}\circ g^{-1}$

Number of functions

(1) If $n(A)=m,n(B)=n$ then number of functions from $A\text{ to }B=n^{m}$

(2) If $n(A)=m,n(B)=n$ then number of one one functions from A to B $=\begin{cases}{}^{n}P_{m} & \text{if } n\geq m\\ 0 & \text{if } n<m\end{cases}$

(3) If $n(A)=m=n(B)$ then number of bijections from A to B = m!

(4) If $n(A)=m,n(B)=n$ then number of onto functions from A to B

$$\begin{cases}\sum_{r=1}^{n}(-1)^{n-r}\,nC_{r}r^{m} & \text{if } m\geq n\\ 0 & \text{if } m<n\end{cases}$$

Note

$$\sum_{r=1}^{n}(-1)^{n-r}\,n\,C_{r}r^{m}=nC_{n}\,(n)^{m}-n_{n-1}C(n-1)^{m}+nC_{n-2}\,(n-2)^{m}+....+nC_{1}\,(1)^{m}$$

Binary operation

Binary operation on a set A is a function from $A\times A$ to A

It is denoted by the symbols $*,\oplus,\otimes...$

Note

Addition and multiplication are functions from $N\times N\to N\;\therefore$ they are binary operations on N But substraction and division are not functions from $N\times N\to N\;\therefore$ they are not binary operation on N

Properties of binary operation

(1) Commutative

If $a*b=b*a$ then $*$ is commutative

(2) Associative

If $a*(b*c)=(a*b)*c$ then $*$ is associative

(3) Identity

If $a*e=a=e*a,\ a,e\in A$ then e is called the identity

(4) Inverse

If $a * a' = e = a' * a, a, a', e \in A$ then a' is called then inverse of a

Note

(1) n(A) = n then number of binary operation on $A = n^{n^2}$

(2) If n(A) = n then number of commutative binary operation on $A = n^{\frac{n(n+1)}{2}}$

Even and odd function

(1) If $f(-x) = f(x)$ then $f(x)$ is even function

Example

(1) $f(x) = x^2$, $(2) f(x) = |x|$

(3) $f(x) = \cos x$ (4) $f(x) = \sin^2 x$ (5) $f(x) = \cos^2 x$

The graphs of even functions are symmetrical about Y axis

(2) If $f(-x) = -f(x)$ then f is an odd function

Example

(1) f(x) =x, (2) f(x) = x^3, (3) $f(x) = \sin x$ (4) $f(x) = \tan x$ (5) $f(x) = \sin^3 x$

The graphs of odd function are symmetrical about origin

Periodic function

If $f(T + x) = f(x)$ then f is periodic function the least (+) ve value of T is called the fundamental period of the function

$f(x) = \sin x, \text{period} = 2\pi$

$f(x) = \cos x, \text{period} = 2\pi$

$f(x) = \sin^2 x, \text{period} = \pi$

$f(x) = |\sin x| + |\cos x|$

Period = $\frac{\pi}{2}$