CHAPTER - 6
SEQUENCE AND SERIES

Sequence

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range

Series

If $a_1, a_2, a_3, a_4 \ldots\ldots\ldots\, a_n \ldots\ldots\ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + \ldots\ldots + a_n + \ldots\ldots$ is a series. A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite

Progression

Those sequences whose terms follow certain patterns are called progressions.

1. Arithmetic Progression (A.P.)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same In other words a sequence $a_1, a_2, a_3, \ldots\ldots\, a_n \ldots\ldots$ is called an AP if $a_{n+1} - a_n$ = constant (= d) for all $n \in N$.

The constant difference, generally denoted by 'd' is called the common difference.

eg. 1,4,7,10..... is an AP whose first term is 1 and the common difference is 3

Arithmetic Progression (AP) - Standard form a, a + d, a + 2d, -----

If T_n is the n^{th} term

$T_1 = a$

$T_2 = a + d$

$T_3 = a + 2d$

$T_4 = a + 3d$

$\vdots$

$T_n = a + (n-1)d = T_1 + (n - 1)\, d$

(1) $T_n = T_1 + (n - 1)\, d$

(2) $d = \dfrac{T_n - T_1}{n - 1}$

(3) $n = \frac{(T_n - T_1)}{d} + 1$

(4) Sum to n terms of an AP $S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left[T_1 + T_n\right]$

(5) If a,b,c are in A.P, then b - a = c -b $\Rightarrow$ 2b = a + c

(6) If $T_p = q$ and $T_q = p$, $T_r = p + q - r$ and $T_{p+q} = 0$

(7) If $m.T_m = n.T_n$ then $T_{m+n} = 0$

SELECTIONS OF TERMS OF AN A.P – When the **sum is given**, the following way is adopted in selecting certain number of terms:

Number of terms	**terms to be taken**
3	a – d, a , a + d
4	a – 3d, a – d, a + d, a + 3d
5	a – 2d, a – d , a, a + d, a+ 2d

when the **sum is not given**, then the following way is adopted in selection of terms

Number of terms	**terms to be taken**
3	a, a + d, a + 2d
4	a, a + d, a + 2d, a + 3d
5	a, a + d, a + 2d, a + 3d, a + 4d

Arithmetic Mean

If three numbers a, A, b are in AP then A is called the arithmetic mean between a and b

So, if a, A, b are in AP then

A-a = b - A

2A = a + b $\Rightarrow$ $A = \frac{a+b}{2}$

If a, x_1, x_2, x_3 ---- x_n , b are in AP, x_1, x_2, x_3 ---- x_n are the n arithmetic means between a and b and its sum = $x_1 + x_2 + x_3 + \text{----} + x_n = \left(\frac{a+b}{2}\right)n$ and the common difference $d = \frac{b-a}{n+1}$

Properties

(1) If the terms of a sequence in AP, increased, decreased, multiplied or divided by a non-zero constant, resulting sequence will be also in AP.

(2) A sequence is an A.P. iff its n^{th} term is a linear expression in n i.e. $a_n = An + B$, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.

(3) A sequence is an A.P. iff the sum of the first n terms is of the form $An^2 + Bn$ where A, B are constants, independent of n. In such case the common difference is 2A ie. 2 times the coefficient of n^2.

(4) In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.

(5) If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

(6) In an A.P. the sum of **m** terms is equal to **n** and the sum of **n** terms is equal to **m**, then the sum of **(m + n)** terms is **–(m + n)**

(7) If the sum of **m** terms of an A.P. is the same as the sum of its **n** terms, then the sum of its **(m + n)** terms

is zero

(8) If the sum S_n of n terms of a sequence is given, then n^{th} term T_n of the sequence can be determined by the formula $T_n = S_n - S_{n-1}$

2. Geometric Progression (GP):

A sequence of non-zero numbers is called a geometric progression (G.P.) if the ratio of a term and the term preceeding to it is always a constant quantity.

In other words, a sequence $a_1, a_2, a_3, ...a_n .....$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$

The constant ratio is called the common ratio of the G.P.

Eg. 4,12,36,108, ... is a GP whose first term is 4 and common ratio is 3

Standard Form: a, ar, ar^2, ------

If T_n is the n^{th} term

$T_1 = a$

$T_2 = ar$

$T_3 = ar^2$

$T_4 = ar^3$

$\vdots$

$T_n = ar^{n-1}$

(i) n^{th} term $T_n = a\, r^{n-1}$

(ii) sum of the first n terms $S_n = a\left(\frac{1-r^n}{1-r}\right) = a\left(\frac{r^n-1}{r-1}\right)$. If l is the last term of GP, then $l = ar^{n-1}$

$$\therefore S_n = a\frac{\left(1-r^n\right)}{1-r} = \frac{a-ar^n}{1-r},\ S_n = \frac{a-\left(ar^{n-1}\right)r}{1-r} = \frac{a-lr}{1-r} = \frac{lr-a}{r-1},\ r \neq 1$$

(iii) If three numbers a,b,c are in GP then $\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac$

(iv) Geometric mean between 'a' and 'b' = $\sqrt{ab}$

Let $G_1, G_2, ...G_n$ be n geometric means between two given numbers a and b. Then

a. $G_1, G_2, ..., G_n$, b is a G.P. consisting of (n + 2) terms. Let r be the common ratio of this G.P. Then,

b = $(n + 2)^{th}$ term = $ar^{n+1} \Rightarrow r^{n+1} = \frac{b}{a} \Rightarrow r = (b/a)^{\frac{1}{n+1}}$

$$\therefore G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}, G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)} ....$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}$$

If n geometric means are inserted between two quantities, then the product of n geometric means is the n^{th} power of the single geometric mean between the two quantities

G_1G_2......... Gn = $\left(\sqrt{ab}\right)^n = G^n$

(v) If $|r| < 1$, $a + ar + ar^2 + ---- ¥ = S_¥ = \frac{a}{1-r} = \frac{\text{first term}}{(\text{1-common ratio})}$

(vi) Geometric mean of $x_1\, x_2\, x_3 ---- x_n = (x_1.x_2.x_3---- x_n)^{1/n}$
(vii) If a, x_1, x_2, x_3 ---- x_n b are in GP, x_1, x_2, x_3 ---- x_n are the n geometric means between a and b.
(viii) If a_1, a_2, a_3 ---- a_n ----- is a G.P. of non-zero non-negative terms, then log a_1, log a_2 ---- log a_n ---- are an AP and vice versa.

Some Important properties

(1) If all the term of a G.P be multiplied or divided by the same non-zero constant, then it remains a G.P with the same common ratio.

(2) The reciprocals of the terms of a given G.P form a G.P.

(3) If each term of a G.P be raised to the same power, the resulting sequence also forms a G.P.

(4) In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last terms.

(5) If the terms of a given G.P are chosen at regular intervals, then the new sequence so formed also forms a G.P.

(6) If A and G are respectively arithmetic and geometric means between two posititive numbers **a** and **b**, then A > G

(7) If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$

(8) If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$

(9) In a GP $(m + n)^{th}$ term is p and (m - n)th term is q, its m^{th} term is equal to $\sqrt{pq}$

SELECTION OF TERMS OF G.P – When the **product is given**, the following way is adopted in selecting certain number of terms:

Number of terms	terms to be taken
3	$\frac{a}{r}$, a, ar
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

when the **product is not given**, then the following way is adopted in selection of terms

Number of terms	terms to be taken
3	a, ar, ar^2
4	a, ar, ar^2, ar^3
5	a, ar, ar^2, ar^3, ar^4

3. Harmonic Progression: (HP) - A sequence is said to in harmonic progression (H.P.) when the reciprocals of the terms form an Arithmetic progression.

The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$....... is a HP because the sequence 1,3,5,7,9 .... is an AP

Standard form $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, --$; $T_n = \frac{1}{a+(n-1)d}$

(i) Harmonic mean between 'a' and 'b' $= \frac{2ab}{a+b}$

(ii) The Harmonic mean of x_1, x_2, x_3 --- x_n = $\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+-----+\frac{1}{x_n}}$

(iii) If a_1, a_2, a_3 ----- a_n are in H.P., then the expression $a_1a_2 + a_2a_3 +$ ---- $+a_{n-1}\, a_n = (n-1)\, a_1a_n$

4. For two positive numbers a and b:

   (i) Arithmetic mean A = (a + b)/2, geometric mean $G = \sqrt{ab}$ and Harmonic mean $H = \frac{2ab}{a+b}$

   (ii) A > G > H and (iii) $G^2 = A \times H$
   (iii) The AM, GM and HM of two positive numbers form a decreasing GP

5. $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ becomes A, G, H of a & b when n takes the values 0, -1/2, -1 respectively.

6. a, b, c are in A.P, G.P or H.P according as $\frac{a-b}{b-c} = \frac{a}{a}$ or $\frac{a}{b}$ or $\frac{a}{c}$ respectively

7. (i) Sum of first 'n' natural numbers

   $Sn = 1+2+3+....+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

   (ii) Sum of the squares of first 'n' natural numbers

   $Sn = 1^2 + 2^2 + ...+ n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

   (iii) Sum of the cubes of first 'n' natural numbers

   $Sn = 1^3+2^3+3^3+.....+n^3 = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 = \left[\sum_{k=1}^{n} k\right]^2 = \frac{n^2(n+1)^2}{4}$

   (iv) The sum of first 'n' odd natural numbers = 1 + 3 + 5 + ----------- upto n terms = n^2
   (v) The sum of first 'n' even natural numbers = 2 + 4 + 6 + -----------upto n terms = n (n+1)

8. If n^{th} term of a sequence $T_n = an^3 + bn^2 + cn + d, S_n = a\sum_1^n r^3 + b\sum_1^n r^2 + c\sum_1^n r + nd$

9. **Arithmetico geometric progression: (AGP)**: A sequence in which each term is the product of the corresponding terms of an arithmetic and a geometric progression is called an arithmetico-geometric progression (A.G.P.)

   Eg. $1, 3x, 5x^2, 7x^3$..... is an arithmetico - geometric sequence whose corresponding AP are 1,3,5,7,.... and GP are $1, x, x^2, x^3$.......

   Standard form a, (a + d)r, (a + 2d) r^2 , ---- $[a+(n-1)d]r^{n-1}$

$$S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]\,r^n}{(1-r)}$$

If $|r| < 1$, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

PART I - (JEEMAIN)

SECTION - I - Straight objective type questions

1. An A.P. consists of 23 terms, if the sum of the three terms in the middle is 141 and the sum of the last three terms is 261, then the first term is

   1) 6 2) 5 3) 4 4) 3

2. Let $x_1, x_2, \ldots, x_n$ be in an A.P. If $x_1 + x_4 + x_9 + x_{11} + x_{20} + x_{22} + x_{27} + x_{30} = 272$, then $x_1 + x_2 + x_3 + \ldots + x_{30}$ is equal to

   1) 1020 2) 1200 3) 716 4) 2720

3. If $\sum_{i=1}^{21} a_i = 693$ where $a_1, a_2, \ldots a_{21}$ in A.P. then the value of $\sum_{r=0}^{10} a_{2r+1}$ is

   1) 360 2) 363 3) 265 4) 286

4. An A.P has the property that the sum of first ten terms is half the sum of next ten terms. If the second term is 13, then the common difference is

   1) 3 2) 2 3) 5 4) 4

5. If the sum of the first n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\sqrt{507}+\ldots\ldots$ is $435\sqrt{3}$, then n equals

   1) 18 2) 15 3) 13 4) 29

6. If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in A.P. then their common difference will be:

   1) ± 1 2) ± 2 3) ± 5 4) ± 3

7. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11} \ldots$ are in an AP with common difference –2, then the time taken by him to count all notes is

   1) 34 minutes 2) 125 mintues 3) 135 mintues 4) 24 minutes

8. Let $a_1, a_2, \ldots\ldots, a_{10}$ are in G.P., If $\frac{a_3}{a_1} = 25$, then $\frac{a_9}{a_5}$ equals:

   1) $2(5^2)$ 2) $4(5^2)$ 3) 5^4 4) 5^3

9. If $x = 1 + a + a^2 + \ldots\ldots\infty$ and $y = 1 + b + b^2 + \ldots\ldots\infty$ where a and b are proper fractions, then $1 + ab + a^2 b^2 + \ldots\ldots\infty$ equals

5) $xy/_{y+x-1}$ 6) $\frac{x+y}{x-y}$ 7) $\frac{x^2+y^2}{x-y}$ 4) $\frac{x}{x-y}$

10. Let **a** be a positive number such that the arithmetic mean of **a** and 2 exceeds their geometric mean by 1. Then the value of **a** is

1) 3 2) 5 3) 9 4) 8

11. If a_1, a_2, a_3 ------ a_{50} are in G.P., then $\frac{a_1 - a_3 + a_5 - \ldots\ldots + a_{49}}{a_2 - a_4 + a_6 - \ldots\ldots + a_{50}} =$

1) 0 2) 1 3) $\frac{a_1}{a_2}$ 4) $\frac{a_{25}}{a_{24}}$

12. If the arithmetic mean of two numbers a and b, a>b>0, is five times their geometric mean, then $\frac{a+b}{a-b}$ is equal to

1) $\frac{\sqrt{6}}{2}$ 2) $\frac{3\sqrt{2}}{4}$ 3) $\frac{7\sqrt{3}}{12}$ 4) $\frac{5\sqrt{6}}{12}$

13. In a sequence of 9 terms, the first 5 terms are in A.P whose common difference is 2 and the last 5 terms are in G.P whose common ratio is 1/2. If the middle terms of the A.P and G.P are equal, then the middle term of the G.P is

1) 1/3 2) 4/3 3) 5/3 4) 7/3

14. Let a_n be the n^{th} term of a G.P. of positive integers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n+1} = \beta$ such that $\alpha \neq \beta$. Then the common ratio is

1) $\frac{\alpha}{\beta}$ 2) $\frac{\beta}{\alpha}$ 3) $\left(\frac{\alpha}{\beta}\right)^{1/2}$ 4) $\left(\frac{\beta}{\alpha}\right)^{1/2}$

15. A ball is dropped from a height of 48 meters and rebounds 2/3 of the distance it falls. If it continues to fall and rebound in this way the distance the ball travels before coming to rest is (in meters)

1) 144 2) 120 3) 240 4) 96

16. Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man started the job with one of the end stones by carrying them in succession. In carrying all the stones, the man covered a total distance of 3 km. Then the total number of stones is

1) 23 2) 25 3) 27 4) 29

17. If $\sum_{k=1}^{n} k(k+1)(k-1) = pn^4 + qn^3 + tn^2 + sn$, where p,q,t and s are constants, then the value of s is equal to

1) $-\frac{1}{4}$ 2) $-\frac{1}{2}$ 3) $\frac{1}{2}$ 4) $\frac{1}{4}$

18. 99[th] term of the series 2 + 7+ 14 + 23 + 34 + ... is

1) 9997 2) 9999 3) 10000 4) 9998

19. Statement I: If a,b,c are in A.P. then $2a+b, 2b+c, 2c+a$ are also in A.P.

Statement II: If a,b,c,d are in A.P. then $a+b, b+c, c+d$ are in A.P.

1) Statement 1 is true, statement II is true, statement II is a correct explanation for statement I

2) Statement I is true, statement II is true, statement II is not a correct explanation for statement I

3) Statement I is true, statement II is false

4) Statement I is false, statement II is true.

20. Statement I: If G_1, G_2,....G_{50} are the 50 geometric means inserted between $\frac{1}{5}$ and 5 then $G_1.G_{50} + G_2\ G_{49} + ... + G_{25}G_{26} = 25$

Statement II: If $g_1, g_2, ...g_n$ are in G.P., then $g_1\ g_n = g_2\ g_{n-1} =$ ......

1) Statement 1 is true, statement II is true, statement II is a correct explanation for statement I

2) Statement I is true, statement II is true, statement II is not a correct explanation for statement I

3) Statement I is true, statement II is false

4) Statement I is false, statement II is true

SECTION II (NUMERICAL)

21. Given an A.P.whose terms are all positive integers. The sum of its first nine terms is greater than 200 and less than 220. If the second term in it is 12, then its 4[th] term is

22. The sum of the series: $(2)^2 + 2(4)^2 + 3(6)^2 + ...$ upto 10 terms is :

23. The value of $\sum_{r=16}^{30} (r+2)(r-3)$ is equal to

24. Let A be the sum of the first 20 terms and B be sum of the first 40 terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ If $B - 2A = 100\ \lambda$, then λ is equal to :

25. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + ...$ is $\frac{16}{5}m$ then m is equal to

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. The sum of all odd numbers of four digits which are divisible by 9, is

A) 2754000 B) 2753000 C) 2752000 D) 2755000

27. The sum of 10 terms of the series $5+7+13+31+85+\ldots\ldots$ is equal to

A) 29564 B) 29563 C) 29654 D) 39564

28. The sum of the two numbers is $2\frac{1}{6}$. An even numbers of arthmetic means are inserted between them and their sum exceeds their number by 1. Then the number of means inserted is

A) 6 B) 8 C) 12 D) 15

29. The value of the sum $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{k}{2^{n+k}}$ is equal to

A) 5 B) 4 C) 3 D) 2

30. Let $S=2+\frac{6}{7}+\frac{12}{7^2}+\frac{20}{7^3}+\frac{30}{7^4}+\ldots..$ then 4S is equal to

A) $\left(\frac{7}{3}\right)^2$ B) $\frac{7^3}{3^2}$ C) $\left(\frac{7}{3}\right)^3$ D) $\frac{7^2}{3^3}$

SECTION - IV (More than one correct answer)

31. All the term of an A.P. are natural numbers and the sum of the first 20 terms is greater than 1072 and less than 1162. If the sixth term is 32 then

A) first term 12 B) first term is 7 C) common diference is 4 D) common difference is 5

32. Consider the series $\frac{8}{5}+\frac{16}{65}+\frac{24}{325}+\ldots\ldots$ If S_n is the sum of first n terms and a_n is the n^{th} term of the series, then

A) $S_\infty=2$ B) $a_5=\frac{40}{2601}$ C) $s_{10}=\frac{440}{221}$ D) $s_n=\frac{4n^2+4n}{n^2+2n+2}$

33. For the two positive numbers a,b if a,b and $\frac{1}{18}$ are in a geometric progression, while $\frac{1}{a}$, 10 and $\frac{1}{b}$ are in an arithmetic progression, then

A) $a=\frac{1}{12}$ B) $b=\frac{1}{12}$ C) $b=\frac{1}{8}$ D) 16 a +12 b =3

34. Let $s_n=(1)(5)+(2)(5^2)+(3)(5^3)+\ldots.+(n)(5^n)=\frac{1}{16}\left[(4n-1)5^a+b\right]$, then

A) $a=n+1$ B) $a=n$ C) $b=5$ D) $b=25$

SECTION - V (Numerical Type)

35. If the sum $5\sum_{n=1}^{\infty}\frac{2^{n+2}}{4^{n-2}}$ is equal to $320-k$, then k equals to.....

36. The sums of n terms of two arithmetic progressions are in the ratio $(7n+1):(4n+17)$. Then the ratio of their n^{th} term is $\frac{14n-6}{8n+\lambda}$. The numerical quantity $\frac{\lambda}{13}$ must be equal to

37. Given $\alpha=\sum_{k=1}^{\infty}\frac{1}{k^4}$ and $\beta=\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$, then $\frac{16\beta}{\alpha}-10$ is

38. Let $\{a_n\}$ be a sequence of integers in G.P. where $a_6:a_4=4:1$ and $a_5+a_7=340$, then a_3 is

39. The number of terms common to the A.P's 3,7,11,....247 and 2,9,16,.....142 is

SECTION VI - (Matrix match type)

40. Match the following

Column-I	Column-II
a) If $\sum n=210$, then $\sum n^2$ is divisible by the greatest prime number which is greater than	p) 16
b) Between 4 and 2916 is inserted odd number $(2n+1)$G.M's. Then the $(n+1)^{th}$ G.M is divisible by greatest odd integer which is less than	q) 10
c) In a certain progression, four consecutive terms are 40, 30, 24, 20 then the integel part of the next term of the progression is more than	r) 34
d) $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+.....\text{to }\infty=\frac{a}{b}$, where H.C.F (a,b)=1, then a-b is less than	s) 30

A) A-PQRS; B-RS; C-PQ; D-RS
B) A-PQRS; B-PQ; C-PQ; D-RS
C) A-PQRS; B-RS; C-P; D-RS
D) A-PS; B-RS; C-PQ; D-RS