CHAPTER - 05

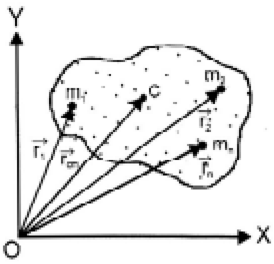
CENTRE OF MASS, CONSERVATION OF MOMENTUM & COLLISIONS

CENTER OF MASS

Every physical system has associated with it a certain point whose motion characterises the motion of the whole system. When the system moves under some external forces, then this point moves as if the entire mass of the system is concentrated at this point and also the external force is applied at this point for translational motion. This point is called the center of mass of the system.

CENTER OF MASS OF A SYSTEM OF 'N' DISCRETE PARTICLES

Consider a system of N point masses $m_1, m_2, m_3, \ldots\ldots\ldots\ldots m_n$ whose position vectors from origin O are given by $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots\ldots\ldots\ldots \vec{r}_n$ respectively. Then the position vector of the center of mass C of the system is given by.



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots\ldots + m_n} \; ; \; \vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$$\vec{r}_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i$$

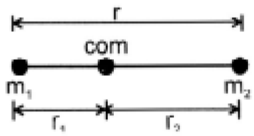
where, $m_i \vec{r}_i$ is called the moment of mass of the particle w.r.t O.

$M = \left(\sum_{i=1}^{n} m_i \right)$ is the total mass of the system.

Note: If the origin is taken at the center of mass then $\sum_{i=1}^{n} m_i \vec{r}_i = 0$. hence, the COM is the point about which the sum of "mass moments" of the system is zero.

POSITION OF COM OF TWO PARTICLES

Center of mass of two particles of masses m_1 and m_2 separated by a distance r lies in between the two particles. The distance of center of mass from any of the particle (r) is inversely proportional to the mass of the particle (m)



i.e. $\quad r \propto 1/m$

or $\quad \dfrac{r_1}{r_2} = \dfrac{m_2}{m_1}$

or $\quad m_1 r_1 = m_2 r_2$

or $\qquad r_1 = \left(\dfrac{m_2}{m_2+m_1}\right) r$ and $r_2 = \left(\dfrac{m_1}{m_1+m_2}\right) r$

Here, $\qquad r_1$ = distance of COM from m_1

and $\qquad r_2$ = distance of COM from m_2

From the above discussion, we see that

$r_1 = r_2 = 1/2$ if $m_1 = m_2$, i.e., COM lies midway between the two particles of equal masses.

Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_2 < m_1$, i.e., COM is nearer to the particle having larger mass.

CENTER OF MASS OF A CONTINUOUS MASS DISTRIBUTION

For continuous mass distribution the center of mass can be located by replacing summation sign with an integral sign. Proper limits for the integral are chosen according to the situation

$$x_{cm} = \frac{\int x\,dm}{\int dm},\ y_{cm} = \frac{\int y\,dm}{\int dm},\ z_{cm} = \frac{\int z\,dm}{\int dm}$$

$\int dm = M$ (mass of the body)

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm.$$

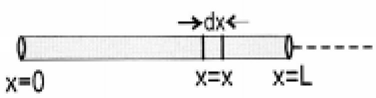
Note: If an object has symmetric mass distribution about x axis then y coordinate of COM is zero and vice-versa

CENTER OF MASS OF A UNIFORM ROD

Suppose a rod of mass M and length L is lying along the x-axis with its one end at $x = 0$ and the other at $x = L$. Mass per unit length of the rod $= \dfrac{M}{L}$

Hence, dm, (the mass of the element dx situated at $x = x$ is) $= \dfrac{M}{L}dx$

The coordinates of the element dx are (x, 0, 0). Therefore, x-coordinate of COM of the rod will be



$$x_{COM} = \frac{\int_0^L x\,dm}{\int dm}$$

$$= \frac{\int_0^L (x)\left(\frac{M}{L}dx\right)}{M}$$

$$= \frac{1}{L}\int_0^L x\,dx = \frac{L}{2}$$

The y-coordinate of COM is

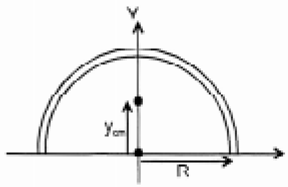
$$y_{COM} = \frac{\int y\,dm}{\int dm} = 0$$

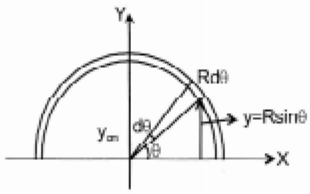
Similarly, $z_{COM} = 0$

i.e., the coordinates of COM of the rod are $\left(\frac{L}{2}, 0, 0\right)$, i.e. it lies at the center of the rod.

CENTER OF MASS OF A SEMICIRCULAR RING

Figure shows the object (semi circular ring). By observation we can say that the x-coordinate of the center of mass of the ring is zero as the half ring is symmetrical about y-axis on both sides of the origin. Only we are required to find the y-coordinate of the center of mass.





To find y_{cm} we use $\quad y_{cm} = \frac{1}{M}\int dm\,y \quad$ ...(i)

Here for dm we consider an elemental arc of the ring at an angle θ from the x-direction of angular width $d\theta$. If radius of the ring is R then its y coordinate will be $R \sin\theta$, here dm is given as

$$dm = \frac{M}{\pi R} \times R\,d\theta$$

So from equation ---(i), we have

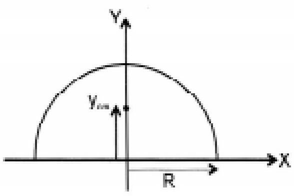
$$y_{cm} = \frac{1}{M}\int_0^{\pi}\frac{M}{\pi R}R\,d\theta\,(R\sin\theta) \qquad = \frac{R}{\pi}\int_0^{\pi}\sin\theta\,d\theta$$

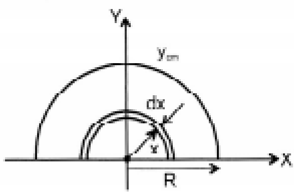
$$y_{cm} = \frac{2R}{\pi} \qquad \text{...(ii)}$$

CENTER OF MASS OF SEMICIRCULAR DISC

Figure shows the half disc of mass M and radius R. Here, we are only required to find the y-coordinate of the center of mass of this disc as center of mass will be located on its half vertical diameter. Here to find y_{cm}, we consider a small elemental ring of mass dm of radius x on the disc (disc can be considered to be made up such thin rings of increasing radii) which will be integrated from 0 to R. Here dm is given as

$$dm = \frac{2M}{\pi R^2}(\pi\,x)dx$$





Now the y-coordinate of the element is taken as $\frac{2x}{\pi}$, as in previous section, we have derived that the center of mass of a semi circular ring is concentrated at $\frac{2R}{\pi}$

Here y_{cm} is given as $\quad y_{cm} = \frac{1}{M}\int_0^R dm\ \frac{2x}{\pi} \quad = \frac{1}{M}\int_0^R \frac{4M}{\pi R^2} x^2\,dx$

$$y_{cm} = \frac{4R}{3\pi}$$

CENTER OF MASS OF A SOLID HEMISPHERE

The hemisphere is of mass M and radius R. To find its center of mass (only y-coordinate), we consider an element disc of width dy, mass dm at a distance y from the center of the hemisphere. The radius of this elemental disc will be given as

$$r = \sqrt{R^2 - y^2}$$

The mass dm of this disc can be given as

$$dm = \frac{3M}{2\pi R^3} \times \pi r^2\,dy$$

$$= \frac{3M}{2R^3}\,(R^2 - y^2)\,dy$$

y_{cm} of the hemisphere is given as

$$y_{cm} = \frac{1}{M}\int_0^R dm\,y = \frac{1}{M}\int_0^R \frac{3M}{2R^3}\,(R^2 - y^2)\,dy\ y$$

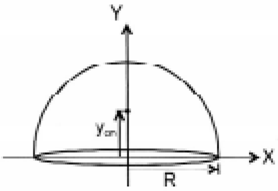
$$= \frac{3}{2R^3}\int_0^R (R^2 - y^2)\,y\,dy$$

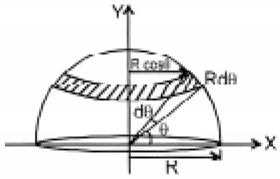
$$y_{cm} = \frac{3R}{8}$$

CENTER OF MASS OF A HOLLOW HEMISPHERE

A hollow hemisphere of mass M and radius R. Now we consider an elemental circular strip of angular width $d\theta$ at an angular distance θ from the base of the hemisphere. This strip will have an area.

$$dS = 2\pi R\cos\theta\ R d\theta$$





Its mass dm is given as

$$dm = \frac{M}{2\pi R^2}\ 2\pi R\cos\theta\ Rd\theta$$

Here y-coordinate of this strip of mass dm can be taken as R sinθ. Now we can obtain the center of mass of the system as.

$$y_{cm} = \frac{1}{M}\int_0^{\frac{\pi}{2}} dm\ R\sin\theta \quad = \frac{1}{M}\int_0^{\frac{\pi}{2}}\left(\frac{M}{2\pi R^2}\ 2\pi R^2\cos\theta\ d\theta\right) R\sin\theta$$

$$= R\int_0^{\frac{\pi}{2}} \sin\theta\ \cos\theta\ d\theta \qquad \Rightarrow \qquad y_{cm} = \frac{R}{2}$$

CENTER OF MASS OF A SOLID CONE

A solid cone has mass M, height H and base radius R. Obviously the center of mass of this cone will lie somewhere on its axis, at a height less than H/2. To locate the center of mass we consider an elemental disc of width dy and radius r, at a distance y from the apex of the cone. Let the mass of this disc be dm, which can be given as

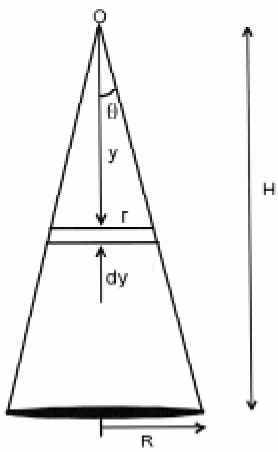
$$dm = \frac{3M}{\pi R^2 H} \times \pi r^2\,dy$$

here y_{cm} can be given as

$$y_{cm} = \frac{1}{M}\int_0^H y\,dm$$

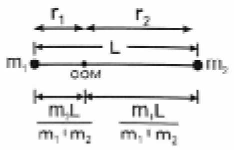
$$= \frac{1}{M}\int_0^H \left(\frac{3M}{\pi R^2 H}\pi\left(\frac{Ry}{H}\right)^2 dy\right) y$$

$$= \frac{3}{H^3}\int_0^H y^3\,dy = \frac{3H}{4}$$



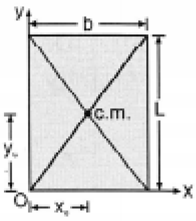
CENTER OF MASS OF SOME COMMON SYSTEMS

⇒ A system of two point masses $m_1 r_1 = m_2 r_2$
The center of mass lies closer to the heavier mass.



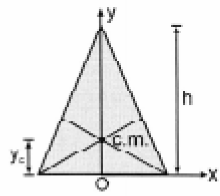
⇒ Rectangular plate (By symmetry)

$x_c = \frac{b}{2}$ $\quad\quad$ $y_c = \frac{L}{2}$



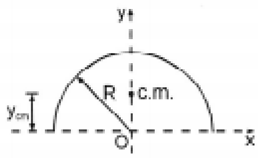
⇒ A triangular plate (By qualitative argument)

at the centroid : $y_c = \frac{h}{3}$



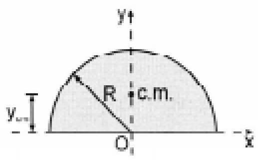
⇒ A semi-circular ring

$y_c = \frac{2R}{\pi}$ $\quad\quad$ $x_c = 0$



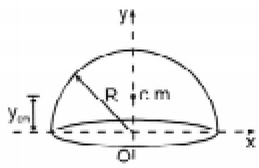
⇒ A semi-circular disc

$y_c = \frac{4R}{3\pi}$ $\quad\quad$ $x_c = 0$



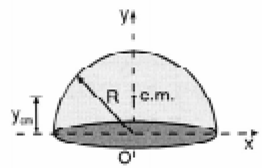
⇒ A hemispherical shell

$y_c = \frac{R}{2}$ $\quad\quad$ $x_c = 0$



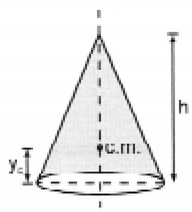
⇒ A solid hemisphere

$y_c = \frac{3R}{8}$ $\quad\quad$ $x_c = 0$

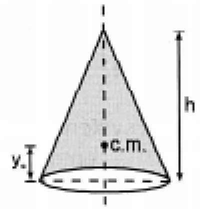


⇒ A circular cone (solid)

$y_c = \frac{h}{4}$



$\Rightarrow$ A circular cone (hollow)



$$y_c = \frac{h}{3}$$

MOTION OF CENTER OF MASS AND CONSERVATION OF MOMENTUM :

Velocity of center of mass of system

$$v_{cm} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt}.............+m_n\frac{d\vec{r}_n}{dt}}{M} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3.........+m_n\vec{v}_n}{M}$$

Here numerator of the right hand side term is the total momentum of the system i.e., summation of momentum of the individual component (particle) of the system

Hence velocity of center of mass of the system is the ratio of momentum of the system to the mass of the system

$\therefore \quad \vec{P}_{System} = M\, v_{cm}$

Acceleration of center of mass of system

$$\vec{a}_{cm} = \frac{m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt}.............+m_n\frac{d\vec{v}_n}{dt}}{M} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3.........+m_n\vec{a}_n}{M}$$

$$= \frac{\text{Net force on system}}{M} = \frac{\text{Net External Force} + \text{Net internal Force}}{M} = \frac{\text{Net External Force}}{M}$$

($\because$ action and reaction both of an internal force must be within the system. Vector summation will cancel all internal forces and hence net internal force on system is zero)

$\therefore \quad \vec{F}_{ext} = M\vec{a}_{cm}$

where $\vec{F}_{ext}$ is the sum of the 'external' forces acting on the system. The internal forces which the particles exert on one another play absolutely no role in the motion of the center of mass.

If no external force is acting on a system of particles, the acceleration of center of mass of the system will be zero. If $a_c = 0$, it implies that v_c must be a constant and if v_{cm} is a constant, it implies that the total momentum of the system must remain constant. It leads to the principal of conservation of momentum in absence of external forces.

If $\vec{F}_{ext} = 0$ then $\vec{v}_{cm}$ = constant

"If resultant external force is zero on the system, then the net momentum of the system must remain constant".

Motion of COM in a moving system of particles:

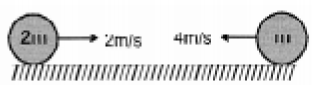
(1) COM at rest :

If $F_{ext} = 0$ and $V_{cm} = 0$, then COM remains at rest. Individual components of the system may move and have non-zero momentum due to mutual forces (internal), but the net momentum of the system remains zero.

(i) All the particles of the system are at rest.

(ii) Particles are moving such that their net momentum is zero.

example:



(iii) A bomb at rest suddenly explodes into various smaller fragments, all moving in different directions then, since the explosive forces are internal & there is no external force on the system for explosion therefore, the COM of the bomb will remain at the original position and the fragment fly such that their net momentum remains zero.

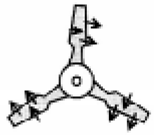
(iv) Two men standing on a frictionless platform, push each other, then also their net momentum remains zero because the push forces are internal for the two men system.

(v) A boat floating in a lake, also has net momentum zero if the people on it changes their position, because the friction force required to move the people is internal of the boat system.

(vi) Objects initially at rest, if moving under mutual forces (electrostatic or gravitation)also have net momentum zero.

(vii) A light spring of spring constant k kept compressed between two blocks of masses m_1 and m_2 on a smooth horizontal surface. When released, the blocks acquire velocities in opposite directions, such that the net momentum is zero.

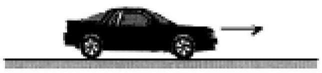
(viii) In a fan, all particles are moving but COM is at rest



(2) COM moving with uniform velocity :

If $F_{ext} = 0$, then V_{cm} remains constant therefore, net momentum of the system also remains conserved. Individual components of the system may have variable velocity and momentum due to mutual forces (internal), but the net momentum of the system remains constant and COM continues to move with the initial velocity.

(i) All the particles of the system are moving with same velocity.
e.g.: A car moving with uniform speed on a straight road, has its COM moving with a constant velocity.



(ii) Internal explosions / breaking does not change the motion of COM and net momentum remains conserved. A bomb moving in a straight line suddenly explodes into various smaller fragments, all moving in different directions then, since the explosive forces are internal & there is no external force on the system for explosion therefore, the COM of the bomb will continue the original motion and the fragment fly such that their net momentum remains conserved.

(iii) Man jumping from cart or buggy also exert internal forces therefore net momentum of the system and hence, Motion of COM remains conserved.

(iv) Two moving blocks connected by a light spring on a smooth horizontal surface. If the acting forces is only due to spring then COM will remain in its motion and momentum will remain conserved.

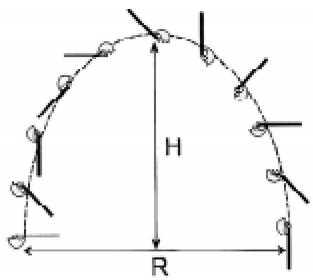
(v) Particles colliding in absence of external impulsive forces also have their momentum conserved.

(3) COM moving with acceleration :

If an external force is present then COM continues its original motion as if the external force is acting on it, irrespective of internal forces.

Example:

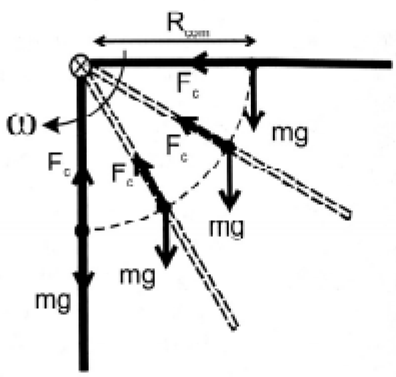
Projectile motion : An axe thrown in air at an angle θ with the horizontal will perform a complicated motion of rotation as well as parabolic motion under the effect of gravitation



The motion of axe is complicated but the COM is moving in a parabolic motion.

$$H_{com} = \frac{u^2 \sin^2\theta}{2g}$$

$$R_{com} = \frac{u^2 \sin 2\theta}{g} \qquad T = \frac{2u\sin\theta}{g}$$



Example:

Circular Motion : A rod hinged at an end, rotates, than its COM performs circular motion. The centripetal force (F_c) required in the circular motion is assumed to be acting on the COM.

$$F_c = m\omega^2 R_{COM}$$

Momentum Conservation :

The total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its center of mass. $\vec{P} = M\,\vec{v}_{cm}$

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

If $\quad \vec{F}_{ext} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \quad ; \quad \vec{P} = \text{constant}$

When the vector sum of the external forces acting on a system is zero, the total linear momentum of the system remains constant.

$$\vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \ldots\ldots\ldots\ldots + \vec{P}_n = \text{constant.}$$

IMPULSE

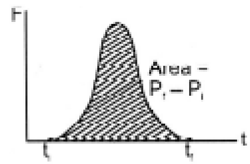
Impulse of a force $\vec{F}$ acting on a body for the time interval $t = t_1$ to $t = t_2$ is defined as :-

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}dt \qquad \Rightarrow \qquad \vec{I} = \int \vec{F}dt = \int m\frac{d\vec{v}}{dt}dt = \int m d\vec{v}$$

$$\vec{I} = m(\vec{v}_2 - \vec{v}_1) = \Delta\vec{P} = \text{change in momentum due to force } \vec{F}$$

Also, $\vec{I}_{Res} = \int_{t_1}^{t_2} \vec{F}_{Res}\,dt = \Delta\vec{P}$ **(impulse - momentum theorem)**

Note: Impulse applied to an object in a given time interval can also be calculated from the area under force time (F-t) graph in the same time interval.



Instantaneous Impulse :

There are many cases when a force acts for such a short time that the effect is instantaneous, e.g., a bat striking a ball. In such cases, although the magnitude of the force and the time for which it acts may each be unknown but the value of their product (i.e., impulse) can be known by measuring the initial and final momenta. Thus, we can write.

$$\vec{I} = \int F\,dt = \Delta P = P_f - P_i$$

Important Points :

(1) It is a vector quantity.

(2) Dimensions = $[MLT^{-1}]$

(3) SI unit = kg m/s

(4) Direction is along change in momentum.

(5) Magnitude is equal to area under the F-t. graph.

(6) $\vec{I} = \int \vec{F} dt = \vec{F}_{av} \int dt = \vec{F}_{av} \Delta t$

(7) It is not a property of a particle, but it is a measure of the degree to which an external force changes the momentum of the particle.

Impulsive force :

A force, of relatively higher magnitude and acting for relatively shorter time, is called impulsive force. An impulsive force can change the momentum of a body in a finite magnitude in a very short time interval. **Impulsive force** is a relative term. There is no clear boundary between an impulsive and Non-Impulsive force.

Note: Usually colliding forces are impulsive in nature.
Since, the application time is very small, hence, very little motion of the particle takes place.

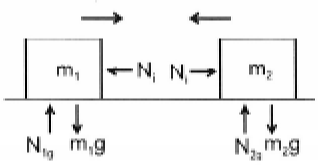
Important points :

1. Gravitational force and spring force are always non-Impulsive.

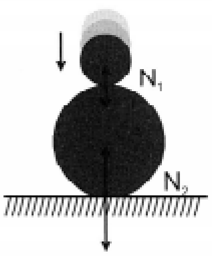
2. Normal, tension and friction are case dependent.

3. An impulsive force can only be balanced by another impulsive force.

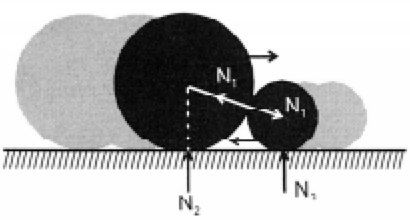
1. **Impulsive Normal :** In case of collision, normal forces at the surface of collision are always impulsive

eg. 

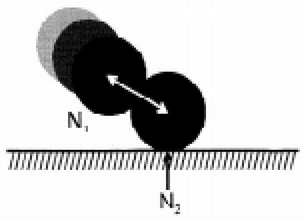
N_i = Impulsive; N_g = Non-impulsive



Both normals are Impulsive

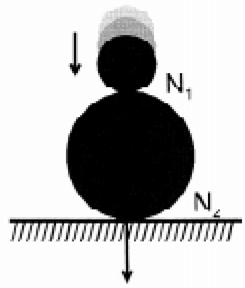


N_1, N_3 = Impulsive; N_2 = non-impulsive

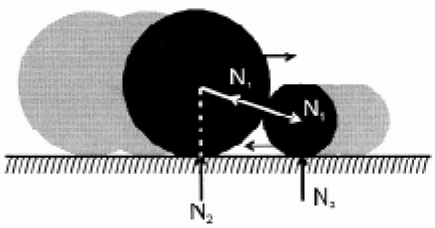


Both normals are Impulsive

2. **Impulsive Friction :** If the normal between the two objects is impulsive, then the friction between the two will also be impulsive.



Friction at both surfaces is impulsive

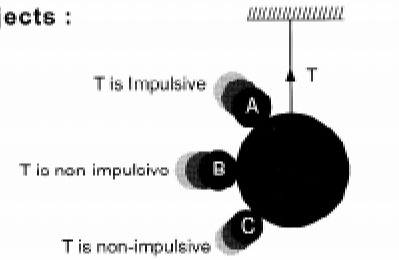


Friction due to N_2 is non-impulsive and due to N_3 and N_1 are impulsive

3. **Impulsive Tensions :** When a string jerks, equal and opposite tension act suddenly at each end. Consequently equal and opposite impulses act on the bodies attached with the string in the direction of the string. There are two cases to be considered.

(a) **One end of the string is fixed :**
The impulse which acts at the fixed end of the string cannot change the momentum of the fixed object there. The object attached to the free end however will undergo a change in momentum in the direction of the string. The momentum remains unchanged in a direction perpendicular to the string where no impulsive forces act.

(b) **Both ends of the string attached to movable objects :**
In this case equal and opposite impulses act on the two objects, producing equal and opposite changes in momentum. The total momentum of the system therefore remains constant, although the momentum of each individual object is changed in the direction of the string. Perpendicular to the string however, no impulse acts and the momentum of each particle in this direction is unchanged.



All normal are impulsive but tension T is impulsive only for the ball A

For this example:
In case of rod, Tension is always impulsive and in case of spring, Tension is always non-impulsive

COLLISION OR IMPACT

Collision is an event in which an impulsive force acts between two or more bodies for a short time, which results in change of their velocities.

Note : (a) In a collision, particles may or may not come in physical contact.

(b) The duration of collision, Δt is negligible as compared to the usual time intervals of observation of motion.

(c) In a collision the effect of external non impulsive forces such as gravity are not taken into a account as due to small duration of collision (Δt) average impulsive force responsible for collision is much larger than external forces acting on the system.

The collision is infact a redistribution of total momentum of the particles. Thus, law of conservation of linear momentum is indispensable in dealing with the phenomenon of collision between particles.

Line of Impact

The line passing through the common normal to the surfaces in contact during impact is called line of impact. The force during collision acts along this line on both the bodies.

Direction of Line of impact can be determined by:

(a) Geometry of colliding objects like spheres, discs, wedge etc.

(b) Direction of change of momentum.

If one particle is stationary before the collision then the line of impact will be along its motion after collision.

Classification of collisions

(a) On the basis of line of impact

(i) **Head-on collision** : If the velocities of the colliding particles are along the same line before and after the collision.

(ii) **Oblique collision** : If the velocities of the colliding particles are along different lines before and after the collision.

(b) On the basis of energy :

(i) **Elastic collision :** In an elastic collision, the colliding particles regain their shape and size completely after collision. i.e., no fraction of mechanical energy remains stored as deformation potential energy in the bodies. Thus, kinetic energy of system after collision is equal to kinetic energy of system before collision. Thus in addition to the linear momentum, kinetic energy also remains conserved before and after collision.

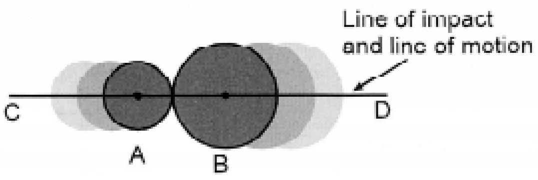
(ii) Inelastic collision : In an inelastic collision, the colliding particles do not regain their shape and size completely after collision. Some fraction of mechanical energy is retained by the colliding particles in the form of deformation potential energy. Thus, the kinetic energy of the particles after collision is not equal to that of before collision. However, in the absence of external forces, law of conservation of linear momentum still holds good.

(iii) Perfectly inelastic : If velocity of separation along the line of impact just after collision becomes zero then the collision is perfectly inelastic. Collision is said to be **perfectly inelastic** if both the particles stick together after collision and move with same velocity,

Note : Actually collision between all real objects are neither perfectly elastic nor perfectly inelastic, its inelastic in nature.

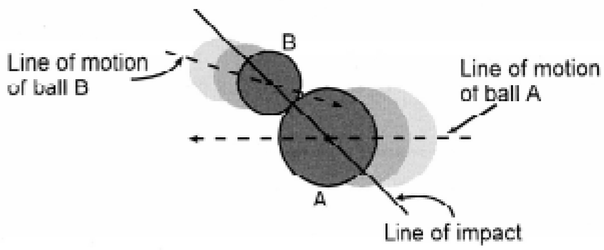
Examples of line of impact and collisions based on line of impact

(i) Two balls A and B are approaching each other such that their centers are moving along line CD.



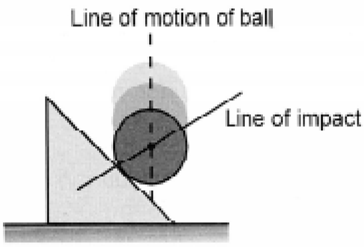
Head on Collision

(ii) Two balls A and B are approaching each other such that their center are moving along dotted lines as shown in figure.



Oblique Collision

(iii) Ball is falling on a stationary wedge.



Oblique Collision

COEFFICIENT OF RESTITUTION (e)

The coefficient of restitution is defined as the ratio of the impulses of reformation and deformation of either body.

$$e = \frac{\text{Impulse of reformation}}{\text{Impulse of deformation}} = \frac{\int F_r\,dt}{\int F_d\,dt}$$

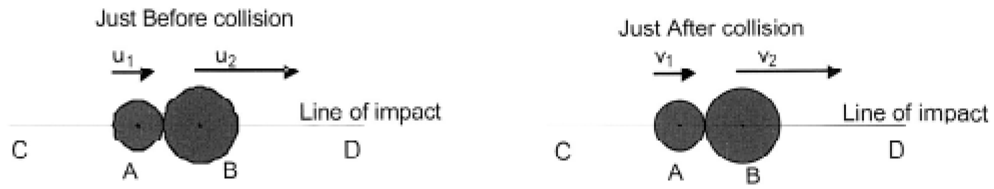
$$= \frac{\text{Velocity of seperation along line of impact}}{\text{Velocity of approach along line of impact}}$$

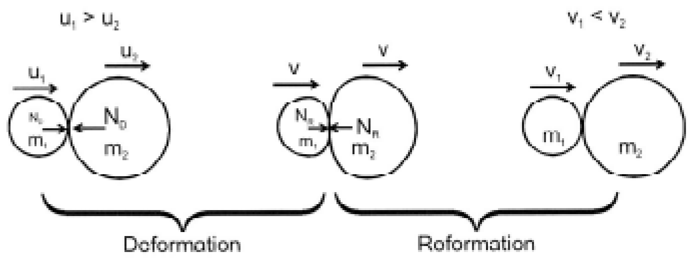
The most general expression for coefficient of restitution is

$$e = \frac{\text{velocity of separation of points of contact along line of impact}}{\text{velocity of approach of point of contact along line of impact}}$$

Example for calculation of e

Two smooth balls A and B approaching each other such that their centers are moving along line CD in absence of external impulsive force. The velocities of A and B just before collision be u_1 and u_2 respectively. The velocities of A and B just after collision be v_1 and v_2 respectively.





$\because$ $F_{ext} = 0$ momentum is conserved for the system.

$\Rightarrow$ $m_1u_1 + m_2u_2 = (m_1 + m_2)v = m_1v_1 + m_2v_2$

$$\Rightarrow \quad v = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \qquad ....(1)$$

Impulse of Deformation :

J_D = change in momentum of any one body during deformation.

$= m_2(v - u_2)$ for m_2

$= m_1(-v + u_1)$ for m_1

Impulse of Reformation :

J_R = change in momentum of any one body during Reformation.

$= m_2(v_2 - v)$ for m_2

$= m_1(v - v_1)$ for m_1

$$e = \frac{\textbf{Impulse of Reformation } (\vec{J}_R)}{\textbf{Impulse of Deformation } (J_D)} = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\textbf{Velocity of separation along line of impact}}{\textbf{Velocity of approach along line of impact}}$$

Note : e is independent of shape and mass of object but depends on the material.
The coefficient of restitution is constant for a pair of materials.

(a) $e = 1$
- Impulse of Reformation = Impulse of Deformation
- Velocity of separation = Velocity of approach
- Kinetic energy of particles after collision may be equal to that of before collision.
- Collision is elastic.

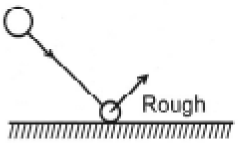
(b) $e = 0$
- Impulse of Reformation = 0
- Velocity of separation = 0
- Kinetic energy of particles after collision is not equal to that of before collision.
- Collision is perfectly inelastic .

(c) $0 < e < 1$
- Impulse of Reformation < Impulse of Deformation
- Velocity of separation < Velocity of approach
- Kinetic energy of particles after collision is not equal to that of before collision.
- Collision is Inelastic.

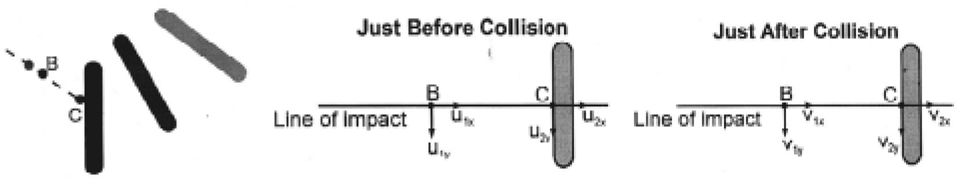
Note : In case of contact collisions e is always less than unity.
$\therefore \quad 0 \le e \le 1$

Important Point :

In case of elastic collision, if rough surface is present then
$k_f < k_i$ (because friction is impulsive)
Where, k is Kinetic Energy.

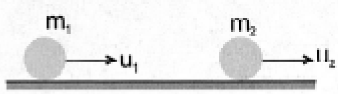


A particle 'B' moving along the dotted line collides with a rod also in state of motion as shown in the figure. The particle B comes in contact with point C on the rod.
To write down the expression for coefficient of restitution e, we first draw the line of impact. Then we resolve the components of velocities of points of contact of both the bodies along line of impact just before and just after collision.



Then $\quad e = \dfrac{v_{2x} - v_{1x}}{u_{1x} - u_{2x}}$

Collision in one dimension (Head on)



(a)
Before Collision
$u_1 > u_2$

(b)
After Collision
$v_2 > v_1$

$$e = \frac{v_2 - v_1}{u_1 - u_2} \quad \Rightarrow \quad (u_1 - u_2)e = (v_2 - v_1)$$

By momentum conservation,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$v_2 = v_1 + e(u_1 - u_2)$$

and $$v_1 = \frac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$$

$$v_2 = \frac{m_1u_1 + m_2u_2 + m_1e(u_1 - u_2)}{m_1 + m_2}$$

Special Case :

(1) $e = 0$

$\Rightarrow$ $v_1 = v_2$

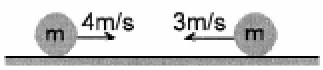
$\Rightarrow$ for perfectly inelastic collision, both the bodies, move with same vel. after collision.

(2) $e = 1$

and $m_1 = m_2 = m$,

we get $v_1 = u_2$ and $v_2 = u_1$

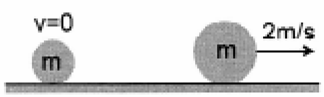
i.e., when two particles of equal mass collide elastically and the collision is head on, they exchange their velocities., e.g.



Before Collision

After Collision

Before Collision



After Collision

(3) $m_1 >> m_2$

$$m_1 + m_2 \approx m_1 \text{ and } \frac{m_2}{m_1} \approx 0$$

$\Rightarrow$ $v_1 = u_1$ No change

and $v_2 = u_1 + e(u_1 - u_2)$

Collision in two dimension (oblique)

1. A pair of equal and opposite impulses act along common normal direction. Hence, linear momentum of individual particles do change along common normal direction. If mass of the colliding particles remain constant during collision, then we can say that linear velocity of the individual particles change during collision in this direction.

2. No component of impulse act along common tangent direction. Hence, linear momentum or linear velocity of individual particles (if mass is constant) remain unchanged along this direction.

3. Net impulse on both the particles is zero during collision. Hence, net momentum of both the particles remain conserved before and after collision in any direction.

4. Definition of coefficient of restitution can be applied along common normal direction, i.e., along common normal direction we can apply

   Relative speed of separation = e (relative speed of approach)

VARIABLE MASS SYSTEM :

If a mass is added or ejected from a system, at rate μ kg/s and relative velocity $\vec{v}_{rel}$ (w.r.t. the system), then the force exerted by this mass on the system has magnitude $\mu|\vec{v}_{rel}|$.

Thrust Force ($\vec{F}_t$)

$$\vec{F}_t = \vec{v}_{rel}\left(\frac{dm}{dt}\right)$$

Suppose at some moment t = t mass of a body is m and its velocity is $\vec{v}$. After some time at t = t + dt its mass becomes (m – dm) and velocity becomes $\vec{v}+d\vec{v}$. The mass dm is ejected with relative velocity $\vec{v}_r$. Absolute velocity of mass 'dm' is therefore ($\vec{v} + \vec{v}_r$). If no external forces are acting on the system, the linear momentum of the system will remain conserved, or

$$\vec{P}_i = \vec{P}_f$$

or $\quad m\vec{v} = (m - dm)(\vec{v} + d\vec{v}) + dm(\vec{v} + \vec{v}_r)$

or $\quad m\vec{v} = m\vec{v} + md\vec{v} - (dm)\vec{v} - (dm)(d\vec{v}) + (dm)\vec{v} + \vec{v}_r\, dm$

The term (dm) ($d\vec{v}$) is too small and can be neglected.

$\therefore \quad md\vec{v} = -\vec{v}_r\, dm$

or $\quad m\left(\frac{d\vec{v}}{dt}\right) = \vec{v}_r\left(-\frac{dm}{dt}\right)$

Here, $m\left(-\frac{d\vec{v}}{dt}\right)$ = thrust force $(\vec{F}_t)$

and $\quad -\frac{dm}{dt}$ = rate at which mass is ejecting

or $\quad \vec{F}_t = \vec{v}_r\left(\frac{dm}{dt}\right)$

Problems related to variable mass can be solved in following four steps

1. Make a list of all the forces acting on the main mass and apply them on it.
2. Apply an additional thrust force $\vec{F}_t$ on the mass, the magnitude of which is $\left|\vec{v}_r\left(\pm\frac{dm}{dt}\right)\right|$ and direction is given by the direction of $\vec{v}_r$ in case the mass is increasing and otherwise the direction of $-\vec{v}_r$ if it is decreasing.
3. Find net force on the mass and apply

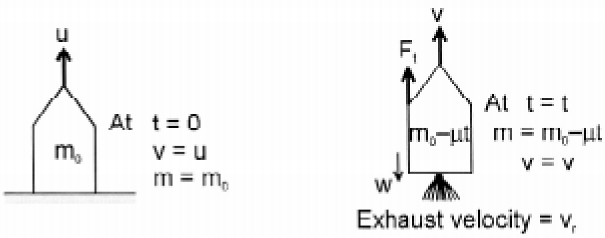
   $$\vec{F}_{net} = m\frac{d\vec{v}}{dt} \qquad \text{(m = mass at the particular instant)}$$

4. Integrate it with proper limits to find velocity at any time t.

Note : Problems of one-dimensional motion (which are mostly asked in JEE) can be solved in easier manner just by assigning positive and negative signs to all vector quantities. Here are few example in support of the above theory.

Rocket propulsion :

Let m_o be the mass of the rocket at time t = 0. m its mass at any time t and v its velocity at that moment. Initially, let us suppose that the velocity of the rocket is u.



Further, let $\left(\frac{-dm}{dt}\right)$ be the mass of the gas ejected per unit time and v_r the exhaust velocity of the gases with respect to rocket. Usually $\left(\frac{-dm}{dt}\right)$ and v_r are kept constant throughout the journey of the rocket. Now, let us write few equations which can be used in the problems of rocket propulsion. At time t = t,

1.	Thrust force on the rocket	$F_t = v_r\left(\frac{-dm}{dt}\right)$	(upwards)
2.	Weight of the rocket	$W = mg$	(downwards)
3.	Net force on the rocket	$F_{net} = F_t - W$	(upwards)

or $\quad F_{net} = v_r\left(\frac{-dm}{dt}\right) - mg$

4. Net acceleration of the rocket $\quad a = \frac{F}{m}$

or $\quad \frac{dv}{dt} = \frac{v_r}{m}\left(\frac{-dm}{dt}\right) - g$

or $\quad dv = \frac{v_r}{m}(-dm) - g\,dt$

or $\quad \int_u^v dv = v_r \int_{m_0}^{m} \frac{-dm}{m} - g\int_0^t dt$

Thus, $\quad v = u - gt + v_r \ln\left(\frac{m_0}{m}\right) \qquad \ldots(i)$

Note : 1. $F_t = v_r\left(-\frac{dm}{dt}\right)$ is upwards, as v_r is downwards and $\frac{dm}{dt}$ is negative.

2. If gravity is ignored and initial velocity of the rocket u = 0, Eq. (i) reduces to $v = v_r \ln\left(\frac{m_0}{m}\right)$.

LINEAR MOMENTUM CONSERVATION IN PRESENCE OF EXTERNAL FORCE.

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

$\Rightarrow$ $\vec{F}_{ext}\,dt = d\vec{P}$

$\Rightarrow$ $d\vec{P} = \vec{F}_{ext}\,)_{impulsive}\,dt$

$\therefore$ If $\vec{F}_{ext}\,)_{impulsive} = 0$

$\Rightarrow$ $d\vec{P} = 0$

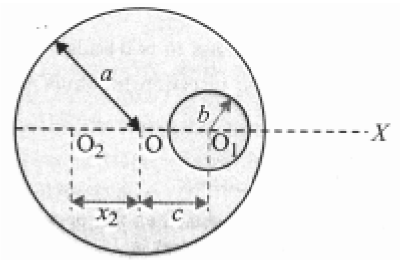
or $\vec{P}$ is constant

Note: Momentum is conserved if the external force present is non impulsive. eg. gravitation or spring force

PART - I (JEEMAIN)

SECTION-I

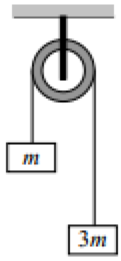
1. A uniform circular disc of radius a is taken. A circular portion of radius b has been removed from it as shown in fig. If the centre of hole is at a distance c from the centre of the disc, the distance x_2 of the centre of mass of the remaining part from the initial centre of mass O is given by



(1) $\frac{\pi b^2}{(a^2 - b^2)}$ (2) $\frac{cb^2}{(a^2 - b^2)}$

(3) $\frac{\pi c^2}{(a^2 - b^2)}$ (4) $\frac{\pi a^2}{(a^2 - b^2)}$

2. If the system is released, then the acceleration of the centre of mass of system is



(1) g/4 (2) g/2 (3) g (4) 2g

3. Two skaters A and B, having masses 50 kg and 70 kg respectively, stand facing each other 6 m apart on a horizontal smooth surface. They pull on a rope stretched between them. How far does each move before they meet ?

(1) both moves 3 m

(2) A moves 2.5 m and B moves 3.5 m

(3) A moves 3.5 m and B moves 2.5 m

(4) none of the above

4. A shell is fired from a gun with a muzzle velocity *u* m/sec at an angle θ with the horizontal. At the top of the trajectory, the shell explodes into two fragments P and Q of equal mass. If the speed of the fragment P immediately after explosion becomes zero, at what horizontal distance from the point of projection, will the centre of mass of the fragments hit the ground?

(1) $\dfrac{u^2 \sin^2\theta}{g}$ (2) $\dfrac{u^2 \sin 2\theta}{g}$

(3) $\dfrac{u^2 \sin^2\theta}{2g}$ (4) $\dfrac{u \sin\theta}{g}$

5. An isolated particle of mass *m* is moving in horizontal plane *(x- y),* along the x-axis. At a certain height above the ground, it suddenly explode into two fragment of masses *m/4* and 3 *m/4.* An instant later, the smaller fragment is at *y* = + 15 cm. The larger fragment at this instant is at

(1) y = – 5 (2) y = + 20 cm

(3) y = + 5 cm (4) y = –20 cm

6. A shell is fired from a cannon with a velocity v at an angle e with the horizontal. At the highest point, it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. The speed of the other piece immediately after the explosion is

(1) 3 v cos θ (2) 2 v cos θ (3) 3/2 cos θ (4) v cos θ

7. A ball of mass m_1 collides head-on and elastically with an identical ball of mass m_2 initially at rest. The transfer of energy will be maximum when

(1) $m_1 = m_2$ (2) $m_1 = m_2/2$

(3) $m_1 = 2\ m_2$ (4) none of the above

8. Two *equal* masses m_1 and m_2 moving along same straight line with velocities + 3 m/s and – 5 *m/s* respectively collide elastically. Their velocities after collision will be respectively

(1) + 4 m/s for both (2) – 3 m/s and + 5 m/s

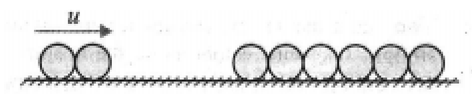
(3) – 4 m/s and +4 m/s (4) – 5 m/s and + 3 m/s

9. A bullet of mass *m* moving with velocity *v* strikes a suspended wooden block of mass *M* at rest. If the collision is perfectly inelastic and the block rises to a height *h,* the initial velocity *v* of the bullet will be

(1) $\sqrt{(2gh)}$ (2) $\dfrac{(M+m)}{m}\sqrt{(2gh)}$

(3) $\dfrac{m}{(M+m)}\sqrt{(2gh)}$ (4) none of these

10. As shown in fig. six steel balls of identical size are lined up along a straight groove (frictionless). Two similar balls moving with speed *u* along the groove collide elastically with this row on the extreme left end. Then,



(1) one ball from the right end will move on with speed 2 *u,* all the remaining balls will be at rest

(2) two balls from extreme right end will move on with speed *u* each and the remaining balls will be at rest

(3) all the balls will start moving to the right with speed *u/8* each.

(4) all the six balls originally at rest will move on with the speed *u/6* each and the two incident balls will come to rest

11. A body is dropped from height h. After the third rebound, it rises to $h/2$. What is the coefficient of restitution?

(1) 1/2 (2) $(1/2)^{1/2}$ (3) $(1/2)^{1/4}$ (4) $(1/2)^{1/6}$

12. A smooth steel ball strikes a fixed smooth steel plate at an angle a with the normal. If coefficient of restitution is e, the angle at which the rebounce will take place is

(1) θ (2) $\tan^{-1}\left[\frac{\tan\theta}{e}\right]$ (3) e tan θ (4) $\tan^{-1}\left[\frac{e}{\tan\theta}\right]$

13. Two identical blocks A and B, each of mass m resting on smooth horizontal floor are connected by a light spring of natural length L and spring constant k, with the spring in its natural length. A third identical block of mass m moving with a speed v along the line joining A and B collides with A elastically. The maximum compression in the spring is

(1) $v\sqrt{\frac{m}{2k}}$ (2) $m\sqrt{\frac{v}{2k}}$ (3) $\sqrt{\frac{mk}{v}}$ (4) $\frac{mv}{2k}$

14. A ball strikes a horizontal floor at an angle $\theta = 45°$. The coefficient of restitution between the ball and the floor is $e = 1/2$. The fraction of its kinetic energy lost in collision is

(1) 5/8 (2) 3/8 (3) ¾ (4) 1/4

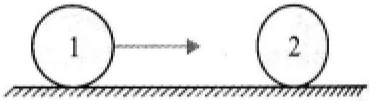
SECTION-II

Numerical Type Questions

15. A boat of length 10 m and mass 450 kg is floating without motion in still water. A man of mass 50 kg standing at one end of it walks to the other end of it and stops. The magnitude of the displacement of the boat in meters relative to ground is

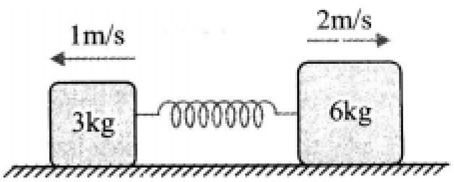
16. A mass of 20 kg moving with a speed of 10 m/s collides with another stationary mass of 5 kg. As a result of the' collision, the two masses stick together. The K.E. of the composite mass will be

17. Ball 1 collides with an another identical ball 2 at rest as shown in fig. 5.78. For what value of coefficient of restitution *e*, the velocity of second ball becomes two times that of 1 after collision?



18. A non-uniform thin rod of length L is placed along x-axis such that its one end is at the origin. The linear mass density of rod is $l = l_0 x$. The distance of centre of mass of rod from the origin is

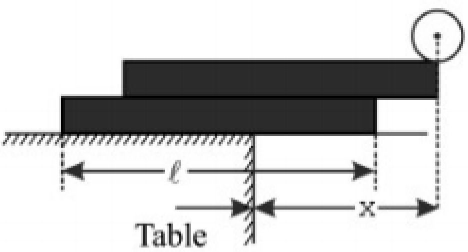
19. Two blocks of mass 3 kg and 6 kg respectively are placed on a smooth horizontal surface. They are connected by a light spring of force constant k = 200 N/m. Initially, the spring is unstretched. The indicated velocities are imparted to the blocks. The maximum extension of the spring will the



PART - II (JEE ADVANCED)

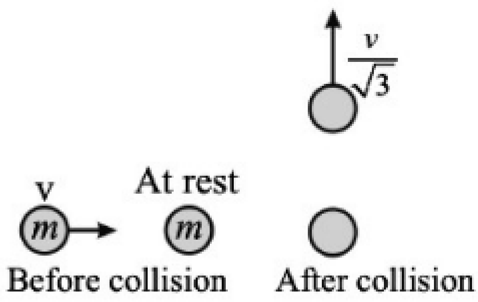
SECTION - III (One correct answer)

20. Two identical uniform rectangular blocks (with longest side ℓ) and a solid sphere of radius *R* are to be balanced at the edge of a heavy table such that the centre of the sphere remains at the maximum possible horizontal distance from the vertical edge of the table without toppling as indicated in the figure. If the mass of each block is *M* and of the sphere is $\frac{M}{2}$, the maximum distance x that can be achieved is:



A) $\frac{8\ell}{15}$ B) $\frac{5\ell}{6}$ C) $\frac{3\ell}{4}+R$ D) $\frac{7\ell}{15}+R$

21. A mass m moves with a velocity v and collides inelastically with another identical mass. After collision the 1st mass moves with velocity $\frac{v}{\sqrt{3}}$ in a direction perpendicular to the initial direction of motion. Find the speed of the 2nd mass after collision:



A) $\frac{2}{\sqrt{3}}v$ B) $\frac{v}{\sqrt{3}}$ C) v D) $\sqrt{3}v$

22. A particle of mass m is projected from the ground with an initial speed u_0 at an angle α with the horizontal. At the highest point of its trajectory, it makes a completely inelastic collision with another identical particle, which was thrown vertically upward from the ground with the same initial speed u_0. The angle that the composite system makes with the horizontal immediately after the collision is

A) $\frac{\pi}{4}$ B) $\frac{\pi}{4}+\alpha$ C) $\frac{\pi}{2}-\alpha$ D) $\frac{\pi}{2}$

SECTION - IV (More than one correct answer)

23. A man of mass m is stationary on a stationary flat car. The car can move without friction along horizontal rails. The man starts walking with velocity v relative to the car. Work done by him:

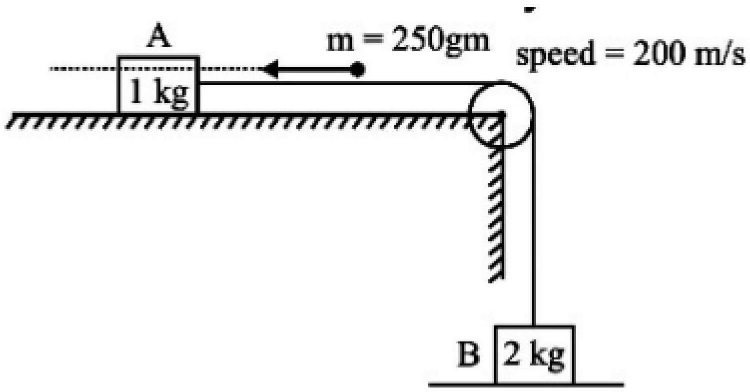
A) is less than $\frac{1}{2}mv^2$, if he walks along the rails

B) is equal $\frac{1}{2}mv^2$, if he walks normal to rails

C) can never be less than $\frac{1}{2}mv^2$

D) is greater than $\frac{1}{2}mv^2$, if he walks along the rails

24. Block A (1kg) is placed on smooth horizontal surface abd connected with a block B (2kg), as shown in the figure, by an inextensible string. A bullet of mass 250gm, strikes the block A horizontally with speed 200 m/s. The bullet penetrates through the block A and comes out with velocity 100 m/s.



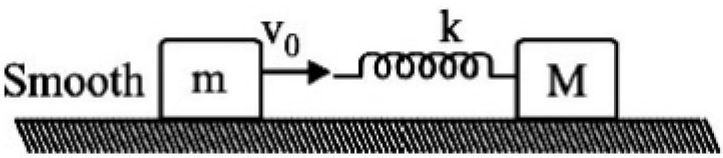
A) Velocity of 2kg block just after bullet comes out of block A is $\frac{25}{3}\text{m/s}$

B) Impulse produced by string on block B is $\frac{50}{3}\text{N}-\text{s}$

C) Maximum displacement of block A in left direction is approximately (g = 10 m/s^2)

D) Maximum displacement of block A in left direction is approximately (g = 10 m/s^2) is 5.2 m

25. A block of mass m moving with a velocity v_0 collides with a stationary block of mass M at the back of which a spring of spring constant k is attached, as shown in the figure. Select the correct alternativest(s)



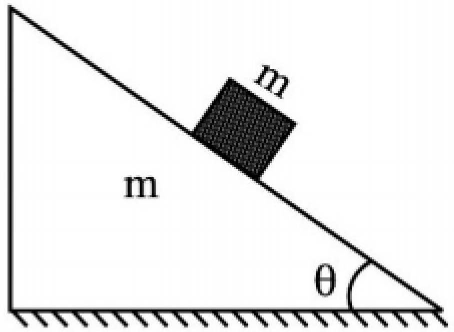
A) The velocity of centre of mass is $\left(\frac{m}{m+M}\right)v_0$

B) The initial kinetic energy of the system in the centre of mass frame is $\frac{1}{4}\left(\frac{mM}{M+m}\right)v_0^2$

C) The maximum compression in the spring is $v_0\sqrt{\frac{mM}{(M+m)}\frac{1}{k}}$

D) When the spring is in the state of maximum compression the kinetic energy in the centre of mass frame is zero.

26. A block of mass m slides down an inclined wedge of same mass m shown in figure. Friction is absent everywhere.



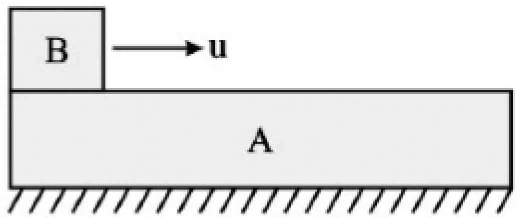
A) The acceleration of block vertically downwards is $\frac{g\cos^2\theta}{(1+\sin^2\theta)}$

B) The acceleration of centre of mass is $\frac{g\sin^2\theta}{(1+\sin^2\theta)}$

C) The acceleration of centre of mass is $\frac{g\cos\theta}{1+\sin^2\theta}$

D) The acceleration of block vertically downwards is $\frac{2g\sin^2\theta}{(1+\sin^2\theta)}$

27. A horizontal block A is at rest on a smooth horizontal surface. A small block B, whose mass is half of A, is placed on A at one end and project along other end with some velocity u. The coefficient of friction between blocks is μ. Then:



A) the blocks will reach a final common velocity u/3

B) the work done against friction is two-third of the initial kinetic energy of B

C) before the blocks reach a common velocity, the acceleration of A relative to B is $(2/3)\mu g$

D) before the block reach a common velocity, the acceleration of A relative to B is $(3/2)\mu g$

SECTION - V (Numerical Type - Upto two decimal place)

28. A particle of mass 2m is projected at an angle of 45^0 horizontal with a velocity of $20\sqrt{2}m/s$. After 1 second explosion takes place and the particle is broken into two equal pieces. As a result of explosion one part comes to rest. The maximum height from the ground attained by the other part is
($g = 10\ m/s^2$)

29. A rocket of mass 40kg has 160kg fuel. The exhaust velocity of the fuel is 2.0 km/s. The rate of consumption of fuel is 4 kg/s. Calculate the ultimate vertical speed gained by the rocket. ($g = 10\ m/s^2$) in km/s

SECTION - VI (Matrix Matching)

30. Two identical balls A and B are kept on a smooth table as shown. B collides with A awith speed v. For different conditions mentioned in column I, match with speed of A after collision given in column II.

Column I	Column II
a) Elastic collision	p) $\frac{3}{4}v$
b) Perfectly inelastic collision	q) $\frac{5}{8}v$
c) Inelastic collision with $e = \frac{1}{2}$	r) v
d) Inelastic collision with $e = \frac{1}{4}$	s) $\frac{v}{2}$

A) $a \to r, b \to p, c \to p, d \to s$
B) $a \to s, b \to p, c \to p, d \to r$
C) $a \to r, b \to p, c \to s, d \to s$
D) $a \to q, b \to p, c \to r, d \to s$