CHAPTER - 15

INTEGRATION AND ITS APPLICATION

JEE MAIN - SECTION I

$$\int\sqrt{1+\sin\left(\frac{x}{4}\right)}dx = \int\sqrt{\left(\sin^2\frac{x}{8}+\cos^2\frac{x}{8}\right)+\left(2\sin\frac{x}{8}\cos\frac{x}{8}\right)}dx \qquad = \int\sqrt{\left(\sin\frac{x}{8}+\cos\frac{x}{8}\right)^2}dx = \int\left(\sin\frac{x}{8}+\cos\frac{x}{8}\right)dx$$

1. 1

$$= \frac{-\cos\frac{x}{8}}{\left(\frac{1}{8}\right)}+\frac{\sin\frac{x}{8}}{\left(\frac{1}{8}\right)}+c = 8\left(\sin\frac{x}{8}-\cos\frac{x}{8}\right)+c$$

2. 2

$$\int\frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}dx$$

$$= \int\frac{\sqrt{1+x}[\sqrt{1+x}+\sqrt{x}]}{(\sqrt{x}+\sqrt{1+x})}dx$$

$$= \int\sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}+c.$$

3. 4

$$\int\sqrt{\frac{a+x}{a-x}}dx.$$ Put $x = a\cos\theta$

$\Rightarrow dx = -a\sin\theta\, d\theta,$ then it reduces to

$$-a\int\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}(\sin\theta)d\theta = -2a\int\sqrt{\frac{2\cos^2(\theta/2)}{2\sin^2(\theta/2)}}.\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= -a\int(1+\cos\theta)d\theta = -a\left[\cos^{-1}\frac{x}{a}+\sqrt{\frac{a^2-x^2}{a}}\right]+c$$

$$= -a\cos^{-1}\frac{x}{a}-\sqrt{a^2-x^2}+c.$$

4. 2

$$\int\frac{1}{x^2(x^4+1)^{3/4}}dx = \int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Put $1+\frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5}dx = dt,$ then it reduces to

$$-\frac{1}{4}\int\frac{dt}{t^{3/4}} = -\frac{1}{4}\frac{4}{1}t^{1/4}+c = -t^{1/4}+c$$

$$= -\left(1+\frac{1}{x^4}\right)^{1/4}+c = -\frac{(x^4+1)^{1/4}}{x}+c.$$

5. 1

$$\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \int \frac{dx}{\cos^3 x\sqrt{4\sin x\cos x}} = \frac{1}{2}\int \frac{dx}{\cos^{7/2} x\ \sin^{1/2} x}$$

$$= \frac{1}{2}\int \frac{\sec^4 x}{\sqrt{\tan x}}dx = \frac{1}{2}\int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}}dx = \frac{1}{2}\int \frac{1+t^2}{\sqrt{t}}dt \quad (\text{Put } \tan x = t,\ \therefore \sec^2 x\,dx = dt)$$

$$= \frac{1}{2}\int t^{-1/2}dt + \frac{1}{2}\int t^{3/2}dt = t^{1/2} + \frac{t^{5/2}}{5} + c$$

$$= \sqrt{\tan x} + \frac{1}{5}\tan^{5/2} x + c\ .$$

6. 1

$$\int e^{2x}\left(\frac{\sin 4x - 2}{1-\cos 4x}\right)dx = \int \frac{e^{2x}\sin 4x}{1-\cos 4x}dx - 2\int \frac{e^{2x}}{1-\cos 4x}dx$$

$$= \int e^{2x}\cot 2x\,dx - \int e^{2x}\operatorname{cosec}^2 2x\,dx$$

$$= \frac{e^{2x}\cot 2x}{2} + \int 2\frac{e^{2x}}{2}\operatorname{cosec}^2 2x\,dx - \int e^{2x}\operatorname{cosec}^2 x\,dx$$

$$= \frac{1}{2}(e^{2x}\cot 2x) + c.$$

7. 2

$$I = \int \frac{x^2-1}{x^4+x^2+1}dx = \int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left[\left(x+\frac{1}{x}\right)^2-1\right]}dx$$

$$\text{Put } \left(x+\frac{1}{x}\right) = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$$

$$I = \int \frac{dt}{t^2-1} = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + c$$

$$\therefore\ I = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right| + c \Rightarrow a = \frac{1}{2}, b = \frac{1}{2}\ .$$

8. 2

$$\int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \left[\frac{4x^3}{(x^4-1)} - \frac{4}{x}\right]dx$$

$$= \frac{1}{4}[\log(x^4-1) - 4\log x] + c = \frac{1}{4}\log\frac{x^4-1}{x^4} + c\ .$$

9. 2

$$\int \frac{dx}{\cos(x-a)\cos(x-b)}$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a).\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \left\{\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)}\right\}dx$$

$$= \operatorname{cosec}(a-b)\log\frac{\cos(x-a)}{\cos(x-b)} + c\ .$$

10. 3

$$I=\int e^x \sin 2x\,dx=\sin 2x\,.e^x-2\int \cos 2x\,.e^x dx$$

$$=\sin 2x\,.e^x-2\cos 2x\,.e^x-4\int e^x \sin 2x\,dx$$

$$\Rightarrow 5I=e^x(\sin 2x-2\cos 2x)+\text{constant}$$

Equating the given value, we get $K=5$.

11. 3

$$\int\frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}dx=\int\frac{\cos x-\sin x}{\sqrt{9-(\sin x+\cos x)^2}}dx$$

Let $\sin x+\cos x=t$

$$\int\frac{dt}{\sqrt{9-t^2}}=\sin^{-1}\frac{t}{3}+c=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$$

So, $a=1, b=3$.

12. 4

$$I=\int\frac{dx}{x^4+1}\Rightarrow I=\frac{1}{2}\int\frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$=\frac{1}{2}\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx-\frac{1}{2}\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2}dx.$$

$$I=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{2}\right)+\frac{1}{2}\cdot\frac{1}{2}\log\left(\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right)+C$$

$\therefore$ Statement-I is false

13. 13. 3 $\int_0^{2\pi}\sqrt{1+\sin\frac{x}{2}}dx=\int_0^{2\pi}\left|\sin\frac{x}{4}+\cos\frac{x}{4}\right|dx=4\left[\sin\frac{x}{4}-\cos\frac{x}{4}\right]_0^{2\pi}$

$=4[1-0-0+1]=8$.

14. 1

$$I=\int_0^1\sqrt{\frac{1-x}{1+x}}dx=\int_0^1\sqrt{\frac{1-x}{1+x}}.\frac{\sqrt{1-x}}{\sqrt{1-x}}dx$$

$$=\int_0^1\frac{1-x}{\sqrt{1-x^2}}dx=\int_0^1\frac{dx}{\sqrt{1-x^2}}-\int_0^1\frac{x}{\sqrt{1-x^2}}dx$$

$$I=[\sin^{-1}x]_0^1+[\sqrt{1-x^2}]_0^1=\frac{\pi}{2}-1.$$

15. 1 $I=\int_{-1}^{3}\left[\tan^{-1}\left(\frac{x}{x^2+1}\right)+\cot^{-1}\left(\frac{x}{x^2+1}\right)\right]dx$

$=\int_{-1}^{3}\left(\frac{\pi}{2}\right)dx=\left[\frac{\pi x}{2}\right]_{-1}^{3}=2\pi$, $\left(\because \tan^{-1}(x)+\cot^{-1}(x)=\frac{\pi}{2}\right)$.

16. 2 Let $f(x)=x|x|$. Then $f(-x)=-x|-x|=-x|x|=-f(x)$

Therefore $\int_{-1}^{1}x|x|\,dx=0$, (By the property of definite integral).

17. 4 $I=\int_0^{\pi/2}\frac{dx}{1+\tan^3 x}=\int_0^{\pi/2}\frac{\cos^3 x}{\sin^3 x+\cos x^3}dx$ .....(i)

$=\int_0^{\pi/2}\frac{\sin^3 x}{\cos^3 x+\sin^3 x}dx$ .....(ii)

Adding (i) and (ii), we get

$2I=\int_0^{\pi/2}dx \Rightarrow I=\frac{\pi}{4}$.

18. 3 $I=\int_0^{\sqrt{2}}[x^2]\,dx=\int_0^1[x^2]dx+\int_1^{\sqrt{2}}[x^2]\,dx$

$=\int_0^1 0\,dx+\int_1^{\sqrt{2}}dx=[x]_1^{\sqrt{2}}=\sqrt{2}-1$.

19. 2 $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}e^{\frac{r}{n}}=\int_0^1 e^x dx=[e^x]_0^1=e-1$.

20. 2 Since, $f(x)=\int_0^x t\sin t\,dt$ .Now, according to Leibnitz's rule,

$f'(x)=x\sin x.(1)-0=x\sin x$.

21. 3 $I=\int_0^{\pi/2}\frac{dx}{2+\cos x}=\int_0^{\pi/2}\frac{dx}{2\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$

$=\int_0^{\pi/2}\frac{dx}{\sin^2\frac{x}{2}+3\cos^2\frac{x}{2}}=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$

Put $t=\tan\frac{x}{2}\Rightarrow dt=\frac{1}{2}\sec^2\frac{x}{2}dx$, then

$I=2\int_0^1\frac{dt}{3+t^2}=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

22. 4 $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)dx$

Put $x=\cos\theta$, then $\sin\left[2\tan^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right]$

$=\sin\left[2\tan^{-1}\left(\cot\frac{\theta}{2}\right)\right]$

$=\sin\left[2\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]\right]=\sin\left[2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$

$=\sin(\pi-\theta)=\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-x^2}$

Now, $\int_0^1 \sin\left(2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right)dx=\int_0^1\sqrt{1-x^2}\,dx$

$=\left[\frac{1}{2}x\sqrt{1-x^2}\right]_0^1+\frac{1}{2}[\sin^{-1}x]_0^1=\frac{\pi}{4}.$

23. 4 $I=\int_2^3\frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ .....(i)

Using the property $I=\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

i.e., change in $x=(2+3-x)=5-x$ or $dx=-dx$

$\therefore I=\int_3^2\frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}(-dx)=\int_2^3\frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$ .....(ii)

Adding (i) and (ii), $2I=\int_2^3\frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx=\int_2^3 1\,dx$

$=[x]_2^3=3-2=1\Rightarrow I=\frac{1}{2}.$

24. 1

$$I=\frac{1}{2}\int_0^{\pi}[\sin(m+n)x-\sin(m-n)x]dx$$

$$=-\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n}-\frac{\cos(m-n)x}{m-n}\right]_0^{\pi}$$

$$=-\frac{1}{2}\left[\left\{\frac{(-1)^{m+n}}{m+n}-\frac{(-1)^{m-n}}{m-n}\right\}-\left\{\frac{1}{m+n}-\frac{1}{m-n}\right\}\right]$$

Since $n-m$ is odd, therefore $n+m$ must be oc

so $(-1)^{m+n}=(-1)^{m-n}=-1$.

Also, since $|m|\neq|n|$, $m+n\neq 0, m-n\neq 0$

$$\therefore\ I=\frac{1}{m+n}-\frac{1}{m-n}=\frac{m+n-m-n}{m^2-n^2}=\frac{2n}{n^2-m^2}.$$

25. 2

$$I=\int_0^1\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx=\int_0^1\tan^{-1}\left(\frac{x+(x-1)}{1-x(x-1)}\right)dx$$

$$I=\int_0^1(\tan^{-1}x+\tan^{-1}(x-1))dx$$

$$I=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(x-1)dx$$

$I=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(1-x-1)dx$, {Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ in second integral}

$$I=\int_0^1\tan^{-1}x\,dx+\int_0^1\tan^{-1}(-x)dx$$

$$I=\int_0^1\tan^{-1}x\,dx-\int_0^1\tan^{-1}x\,dx=0.$$

26. 4

$$I=\int_{-\pi}^{\pi}(\cos ax-\sin bx)^2dx$$

$$I=\int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx-2\cos ax\ \sin bx)\,dx$$

$$I=\int_{-\pi}^{\pi}(\cos^2 ax+\sin^2 bx)\,dx-\int_{-\pi}^{\pi}2\cos ax\sin bx\ \ dx$$

$$I=2\int_0^{\pi}(\cos^2 ax+\sin^2 bx)\,dx-0$$

$$I=2\int_0^{\pi}\left(\frac{1+\cos 2ax}{2}+\frac{1-\cos 2bx}{2}\right)dx$$

$$I=\int_0^{\pi}(2+\cos 2ax-\cos 2bx)dx=2\pi.$$

27. 2

Let $S = \lim_{n\to\infty} \frac{1}{1^3+n^3} + \frac{4}{2^3+n^3} + \dots + \frac{1}{2n}$

$= \lim_{n\to\infty} \frac{1}{1^3+n^3} + \frac{4}{2^3+n^3} + \dots + \frac{n^2}{n^3+n^3}$

$\therefore S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{r^3+n^3} = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3\left(\frac{r^3}{n^3}+1\right)}$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{\left(\frac{r}{n}\right)^2}{\left[1+\left(\frac{r}{n}\right)^3\right]}$

Applying the formula,

we get $A = \int_0^1 \frac{x^2}{1+x^3} dx$

$= \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3} dx = \frac{1}{3}[\log_e(1+x^3)]_0^1 = \frac{1}{3}\log_e 2.$

28. 1

$I = \int_0^1 x \tan\left(\frac{1}{1+x^2(x^2-1)}\right) dx$

$I = \int_0^1 x \tan\left(\frac{1}{1+x^2(x^2-1)}\right) dx$

$x^2 = t \Rightarrow 2x\,dx = dt \quad I = \frac{1}{2}\int_0^1 (\tan^{-1} t - \tan^{-1}(t-1))dx$

$= \frac{1}{2}\int_0^1 \tan^{-1} t\,dt - \frac{1}{2}\int_0^1 \tan^{-1}(t-1)dt = \frac{1}{2}\int_0^1 \tan^{-1} t\,dt - \frac{1}{2}\int_0^1 \tan^{-1} dt = \int_0^1 \tan^{-1} dt$

$\tan^{-1} t = \theta \Rightarrow t = \tan\theta,\ dt = \sec^2\theta\,d\theta$

$\int_0^{\pi/4} \theta \cdot \sec^2\theta\,d\theta$

$I = (\theta\cdot\tan\theta)\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan\theta\,d\theta$

$= \left(\frac{\pi}{4}-0\right) - \ln(\sec\theta)\Big|_0^{\pi/4} = \frac{\pi}{4} - (\ln\sqrt{2}-0)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

29. 4 Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ .... (1)

$\therefore\ I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$.

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx$ ..... (2)

On adding eqs. (1) and (2), we get

$2I = \int_{\pi/6}^{\pi/3} dx \Rightarrow 2I = [x]_{\pi/6}^{\pi/3} dx$.

$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$

Statement-I is false

But $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ is a true statement by property of definite integrals.

30. 2 $y^2 = 8x$ and $y = x \Rightarrow x^2 = 8x \Rightarrow x = 0, 8$

$\therefore$ Required area $= \int_0^8 (2\sqrt{2}\sqrt{x} - x)dx$

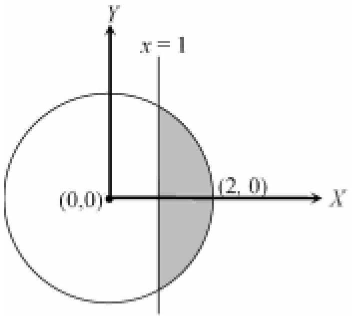
$= \left[\frac{4\sqrt{2}}{3}x^{3/2} - \frac{x^2}{2}\right]_0^8 = \frac{128}{3} - \frac{64}{2} = \frac{32}{3}$ *sq.* unit.

31. 2 $\int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{3}$.

32. 2 Area of smaller part $= 2\int_1^2 \sqrt{4-x^2}\,dx$

$= 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_1^2 = 2\left[2.\frac{\pi}{2} - \left[\frac{\sqrt{3}}{2} - 2.\frac{\pi}{6}\right]\right]$

$= 2\left[\pi - \left[\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right]\right] = \frac{8\pi}{3} - \sqrt{3}$.



33. 1 $x = \frac{\pi}{4}$ is the point of intersection of both curve

$\therefore$ Required area $= \int_0^{\pi/4} (\cos x - \sin x)\,dx$

$= [\sin x + \cos x]_0^{\pi/4} = \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right]$

$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$.

34. 4 $x^2 + y^2 = 1, x + y = 1$ meet when

$x^2 + (1-x)^2 = 1 \Rightarrow x^2 + 1 + x^2 - 2x = 1$

$\Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$

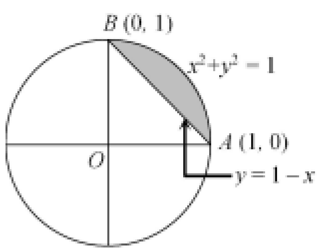
$\Rightarrow x = 0, x = 1$

$\Rightarrow y = 1, y = 0$, i.e., $A(1,0); B(0,1)$

Required area $= \int_0^1 [\sqrt{1-x^2} - (1-x)]\,dx$

$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x - x + \frac{x^2}{2}\right]_0^1$

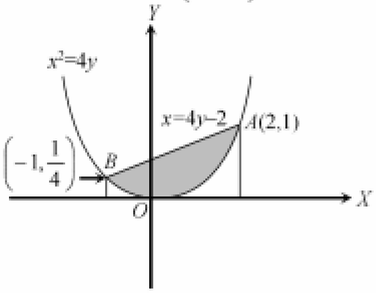
$= \frac{1}{2}.\frac{\pi}{2} - 1 + \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$.



35. 2

Solving the equations $x^2 = 4y$ and $x = 4y - 2$ simultaneously.

The points of intersection of the parabola and the line are $A(2,1)$ and $B\left(-1,\frac{1}{4}\right)$.



$\therefore$ The required area = shaded area

$= \left[\int_{-1}^{2} y\,dx_{\,(\text{for } x=4y-2)}\right] - \left[\int_{-1}^{2} y\,dx_{\,(\text{for } x^2=4y)}\right]$

$= \int_{-1}^{2} \frac{1}{4}(x+2)dx - \int_{-1}^{2} \frac{1}{4}x^2 dx$

$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{8}$ sq. unit.

SECTION II (NUMERICAL)

36. 7

$$\int \frac{1}{[(x-1)^3(x+2)^5]^{1/4}}dx = \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}dx$$

$$= \frac{1}{3}\int \frac{1}{t^{3/4}}dt, \quad \left\{\because \frac{x-1}{x+2} = t \Rightarrow \frac{3}{(x+2)^2}dx = dt\right\}$$

$$= \frac{1}{3}\left(\frac{t^{1/4}}{1/4}\right) + c = \frac{4}{3}t^{1/4} + c = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c.$$

37. 22

$$m + n = -\frac{3}{7} + \left(\frac{-11}{7}\right) = -2 \qquad (-ve \text{ integer})$$

$$I = \int \cos^{-3/7} x\left(\sin^{(-2+3/7)} x\right)dx = \int \cos^{-3/7} x \sin^{-2} x \sin^{3/7} x\,dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\left(\frac{\cos^{3/7} x}{\sin^{3/7} x}\right)}dx = \int \frac{\operatorname{cosec}^2 x\,dx}{\cot^{3/7} x}$$

Put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\,dx = dt$

$$I = -\int \frac{dt}{t^{3/7}} = -\frac{t^{-\frac{3}{7}+1}}{-\frac{3}{7}+1} + c = -\frac{7}{4}t^{4/7} + c$$

$$= -\frac{7}{4}\cot^{4/7} x + c = -\frac{7}{4}\tan^{-4/7} x + c.$$

38. 18

$$\int_{\pi}^{10\pi} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{10\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx$$

$$= \int_0^{10\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx$$

$$= 10\int_0^{\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx = 9\int_0^{\pi} \sin x\, dx$$

$[\because |\sin x|$ is periodic with period π and in $[0,\pi], \sin x \ge 0]$

$$= 9[-\cos x]_0^{\pi} = 9(-\cos\pi + \cos 0) = 9(1+1) = 18 .$$

39. 8

$$I_n = \int_0^{\pi/4} (\sec^2\theta - 1)\tan^{n-2}\theta\, d\theta$$

$$I_n = \int_0^{\pi/4} \sec^2\theta \tan^{n-2}\theta\, d\theta - \int_0^{\pi/2} \tan^{n-2}\theta\, d\theta$$

$$I_n = \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} - I_{n-2} \Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$$

Hence $I_8 + I_6 = \frac{1}{8-1} = \frac{1}{7}$.

40. 4

If $x \ge 1$, then $3 - x = x - 1 \Rightarrow x = 2$

If $0 \le x \le 1$, then $3 - x = 1 - x$, which is not possible.

Also if $x < 0$, then $3 + x = 1 - x$ i.e., $x = -1$

Thus required area $= \int_{-1}^{2} (3 - |x| - |x-1|)\, dx$

$$= \int_{-1}^{0} [3 + x - (1-x)]dx + \int_0^1 [(3-x) - (1-x)]dx + \int_1^2 [(3-x) - (x-1)]dx$$

$= 1 + 2 + 1 = 4\, sq$. unit.

JEE ADVANCED LEVEL

SECTION III

41. B

On integrating both functions, we get

$$= \frac{1}{2}\left|\log(1+t^2)\right|_{1/e}^{\tan x} + \left|\left\{\log t - \frac{1}{2}\log(1+t^2)\right\}\right|_{1/e}^{\cot x}$$

$$= \frac{1}{2}\left[\log\sec^2 x - \log\left(1 + \frac{1}{e^2}\right)\right] + \log\cot x - \log\left(\frac{1}{e}\right)$$

$$-\frac{1}{2}\left\{\log(\mathrm{cosec}^2 x) - \log\left(1 + \frac{1}{e^2}\right)\right\} = -\log\left(\frac{1}{e}\right) = \log e = 1 .$$

42. D $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ .....(i)

$=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)\cos x\sin x}{\sin^4 x+\cos^4 x}$ .....(ii)

By adding (i) and (ii), we get $2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\cos x\sin x}{\cos^4 x+\sin^4 x}dx$

$\Rightarrow I=\frac{\pi}{4}\int_0^{\pi/2}\frac{\tan x\sec^2 x}{1+\tan^4 x}dx$

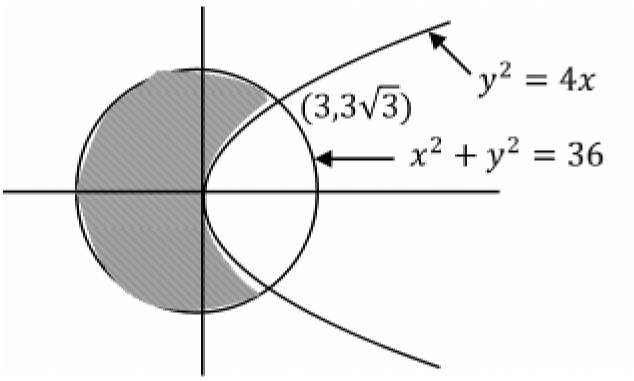
Now, Put $\tan^2 x=t$, we get

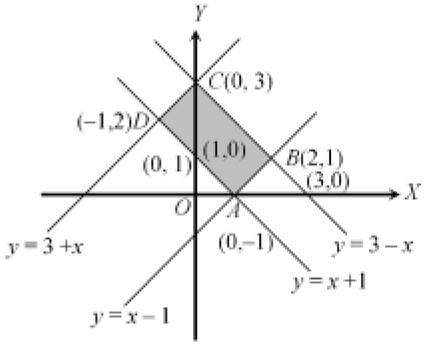
$I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}=\frac{\pi}{8}[\tan^{-1}t]_0^{\infty}=\frac{\pi^2}{16}$.

43. C Required area = $\pi\times(6)^2-2\int_0^3\sqrt{9}x\,dx-\int_3^6\sqrt{36-x^2}dx$

$=36\pi-12\sqrt{3}-2\left(\frac{x}{2}\sqrt{36-x^2}+18\sin^{-1}\frac{x}{6}\right)_3^6$

$=36\pi-12\sqrt{3}-2\left(9\pi-3\pi-\frac{9\sqrt{3}}{2}\right)=24\pi-3\sqrt{3}$.





44. D $\quad I=\int (\cos x)^{-2005} \cos ec^2 x dx - 2005\int \frac{dx}{\cos^{2005} x}$

$I=(\cos x)^{-2005}-(\cot x)-\int(-2005)(\cos x)^{-2006}-(-\sin x)(-\cot x)dx-2005\int\frac{dx}{\cos^{2005}x}$

$I=-\frac{\cot x}{(\cos x)^{2005}}+c$

45. B $\quad \int (x^{14}+x^9+x^4)(2x^{15}+3x^{10}+6x^5)^{1/5}dx \qquad$ put $2x^{15}+3x^{10}+6x^5=t$

$=\frac{1}{30}\int t^{1/5}dt=t^{6/5}+c=\frac{1}{36}(2x^{15}+3x^{10}+6x^5)^{6/5}+c$

46. C $\quad$ Put secx+tanx=t

47. A $\quad f(x)=\frac{x}{(1+x^n)^{1/n}}$; $g(x)=(\underbrace{fofo....of}_{n\ times})(x)=\frac{x}{(1+nx^n)^{1/n}}$

Now $I=\int x^{n-2}g(x)\,dx=\int\frac{x^{n-1}}{(1+nx^n)^{1/n}}dx$

Put $1+nx^n=t^n$ we get $I=\frac{1}{n(n-1)}(1+nx^n)^{\frac{n-1}{n}}+K$.

48. D $\quad \sin x < x \quad \forall x>0$

$\frac{\sin x}{\sqrt{x}}<\frac{x}{\sqrt{x}};\int_0^1\frac{\sin x}{\sqrt{x}}<\int_0^1\sqrt{x}dx$

$I<\left.\frac{2}{3}x^{3/2}\right|_0^1 \qquad I<\frac{2}{3}$

$$J=\int_0^1 \frac{\cos x}{\sqrt{x}}dx \qquad \cos x<1$$

$$\int_0^1 \frac{\cos x}{\sqrt{x}}dx<\int_0^{\pi}\frac{dx}{\sqrt{x}} \qquad J<\left(2x^{\frac{1}{2}}\right)_0^1=2$$

49. B $\quad I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(x^2+\log\left(\frac{1-x}{1+x}\right)\right)\cos x dx$ as, $\int_{-a}^{a} f(x)dx=0$, when $f(-x)=-f(x)$

$$\therefore I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^2\cos x dx=\left(\frac{\pi^2}{2}-4\right)$$

50. A $\quad$ Let $x^2=t$ and use

$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$, then $I=\frac{1}{4}\ln\left(\frac{3}{2}\right)$

51. B

SECTION IV (More than one correct)

52. B,C $\quad I=\int\frac{\cos x+\sin 2x}{(2-\cos^2 x)(\sin x)}dx$; $I=\int\frac{(1+2\sin x)\cos x}{(1+\sin^2 x)(\sin x)}dx$

$I=\int\frac{1+2t}{(1+t^2)t}dt$, (Put sin x=t)

Use partial fractions $\frac{1+2t}{t(1+t^2)}=\frac{1}{t}+\frac{(-t+2)}{1+t^2}$

53. A,B,C,D $\quad$ Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx=\int\frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx$

$$=\frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}dx-x+c...(i)$$

Now, $\int\sin^{-1}\sqrt{x}dx$

Put $x=\sin^2\theta\Rightarrow dx=\sin 2\theta=\int\theta.\sin 2\theta d\theta=-\frac{\theta\cos 2\theta}{2}+\int\frac{1}{2}\cos 2\theta d\theta$

$=-\frac{\theta\cos 2\theta}{2}+\frac{1}{4}\sin 2\theta \qquad =-\frac{1}{2}\theta(1-2\sin^2\theta)+\frac{1}{2}\sin 2\theta\sqrt{1-\sin^2\theta}$

$=-\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x)+\frac{1}{2}\sqrt{x}\sqrt{1-x}$ ....(ii)

From Eqs. (i) and (ii), we get

$$I=\frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}\right]-x+c$$

$$=\frac{2}{\pi}\left[\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}\right]-x+c$$

54. ABD $\quad I_1=\int_1^e \frac{1+x}{x}\left(x+\log_e^x\right)^{100}.xdx$

$$\therefore I_1=\frac{e(1+e)^{101}-1}{101}-\int_1^e \frac{(x+\log_e x)^{101}}{101}dx$$

$$I_2=\int_{\frac{1}{e}}^{1}(\log et+et)^{101}dt=\frac{1}{e}\int_1^e(\log x+x)^{101}dx$$

55. A,D $\quad I_2=\int_1^{1/x}\frac{dt}{1+t^2}$ put t=1/y $=-\int_1^x\frac{dy}{1+y^2}=\int_x^1\frac{dt}{1+t^2}=I_1$

So, choice (a) is true and (b) and (c) are ruled out on integrating

$$\int_x^{1/x}\frac{dt}{1+t^2}=\tan^{-1}\left(\frac{1}{x}\right)-\frac{\pi}{4}=\cot^{-1}(x)-\frac{\pi}{4} \qquad =\frac{\pi}{2}-\tan^{-1}(x)-\frac{\pi}{4}$$

Hence choice (d) is true

Thus correct choice are (a), (d).

56. A $\quad$ Integrate by parts taking t^m as 2nd function and $(1+t)^n$ as 1st function.

$$I(m,n)=\frac{t^{m+1}}{m+1}(1+t^n)\Big|_0^1-\frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}dt \qquad I(m,n)=\frac{2^n}{m+1}-\frac{n}{m+1}I(m+1,n-1)$$

SECTION V - (Numerical type)

57. 1.33 $\quad \int\frac{\sin^2 x\cos^2 xdx}{(\sin x+\cos x+1)^2}=\frac{1}{4}\int\frac{\left((\sin x+\cos x)^2-1\right)^2}{(\sin x+\cos x+1)^2}dx=\frac{1}{4}(\sin x+\cos x-1)^2dx$

on simplifying a+b+c=–4

58. 2 $\quad I=\int\frac{(x^3+1)-x^3}{(x^3+1)^2}dx=I-\int x.\frac{x^2}{(x^3+1)^2}dx=I-\frac{1}{3}\int x.\frac{3x^2}{(x^3+1)^2}dx=I-\frac{1}{3}\int xd\left(\frac{-1}{x^3+1}\right)$

$$=I-\frac{1}{3}\left[\frac{-x}{x^3+1}+I\right]$$

59. 1 $\quad I_1=\int_0^2\frac{x\sin^2\pi x}{(x-1)^2+2}dx=\int_{-1}^1\frac{(t+1)\sin^2\pi t}{t^2+2}dt=2\int_0^1\frac{\sin^2\pi t}{t^2+2}$; now get $I_2=\int_0^1\sin^2\pi t\;dt-I_1$

$$\therefore I_1+I_2=\int_0^1\sin^2\pi t\;dx=\int_0^1\frac{1-\cos 2\pi t}{2}dt=\frac{1}{2}\quad\therefore k=1$$

SECTION VI - (Matrix match type)

60. A-S,B-Q,C-Q,D-Q

A) When $0 \le x \le \frac{\pi}{2} \frac{1+2\sin^2 x}{1+\sin^2 x} \in \left[1, \frac{3}{2}\right]$

B) When $\frac{-\pi}{2} \le x \le \frac{\pi}{2} \cos x - \cos^2 x \in \left[0, \frac{1}{4}\right]$

C) When $\frac{\pi}{6} \le x \le \frac{\pi}{4}, \therefore f(x) \in \left[\sqrt{2}-1, 1-\frac{1}{\sqrt{3}}\right]$

Then $\left[\frac{1}{1+\sin+\cos x}\right] = 0$

D) In $\frac{\pi}{6} \le x \frac{\pi}{4}. \frac{\cot x - \tan x}{2} = \cot 2x$; $0 \le \cot 2x \le \frac{1}{\sqrt{3}}$ $\quad [\cot 2x] = 0$