CHAPTER - 12
LIMITS OF REAL FUNCTIONS

LIMIT OF A FUNCTION

Consider the function f(x). As $x \to a$, if $f(x) \to l$ we say that $\underset{x\to a}{\mathrm{Lt}}\, f(x) = \ell$

<u>Left Limit</u>

The left hand limit of f at x = a is defined as $\underset{x\to a^-}{\mathrm{Lt}}\, f(x) = \underset{h\to 0}{\mathrm{Lt}}\, f(a-h)$, h > 0

<u>Right limit</u>

The right limit of f at x = a is defined as $\underset{x\to a^+}{\mathrm{Lt}}\, f(x) = \underset{h\to 0}{\mathrm{Lt}}\, f(a+h)$, h>0

Note

If $\underset{x\to a^-}{\mathrm{Lt}}\, f(x) = \underset{x\to a^+}{\mathrm{Lt}}\, f(x)$ then we say that $\underset{x\to a}{\mathrm{Lt}}\, f(x)$ exists

Algebra of Limits

a) $\underset{x\to a}{\mathrm{Lt}}\, K = K$ where K, is a real number

b) $\underset{x\to a}{\mathrm{Lt}}\,(f(x) \pm g(x)) = \underset{x\to a}{\mathrm{Lt}}\, f(x) \pm \underset{x\to a}{\mathrm{Lt}}\, g(x)$

c) $\underset{x\to a}{\mathrm{Lt}}\,(f(x).g(x)) = \underset{x\to a}{\mathrm{Lt}}\, f(x).\underset{x\to a}{\mathrm{Lt}}\, g(x)$

d) $\underset{x\to a}{\mathrm{Lt}}\, \dfrac{f(x)}{g(x)} = \dfrac{\underset{x\to a}{\mathrm{Lt}}\, f(x)}{\underset{x\to a}{\mathrm{Lt}}\, g(x)}, \underset{x\to a}{\mathrm{Lt}}\, g(x) \neq 0$

e) $\underset{x\to a}{\mathrm{Lt}}\,(f(x))^n = \left(\underset{x\to a}{\mathrm{Lt}}\, f(x)\right)^n$

f) $\underset{x\to a}{\mathrm{Lt}}\,|f(x)| = \left|\underset{x\to a}{\mathrm{Lt}}\, f(x)\right|$

Sandwich theorem or Squeeze principle

If f,g,h are functions such that f(x) $\le$ g(x) $\le$ h(x) for all x in some neighbourhood of a and if $\underset{x\to a}{\mathrm{Lt}}\, f(x) = \underset{x\to a}{\mathrm{Lt}}\, h(x) = \ell$ then $\underset{x\to a}{\mathrm{Lt}}\, g(x) = \ell$

Some important expansions

a) $(1+x)^n = 1 + nc_1 x + nc_2 x^2 + \dots\dots\dots\dots\dots\dots \; |x| < 1$

b) $(x + a)^n = x^n + nc_1 x^{n-1}a + nc_2 x^{n-2}a^2 + \dots\dots\dots\dots nc_n a^n$, neN

c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$

d) $a^x = 1 + \frac{x\log a}{1!} + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \dots\dots |x| < 1$

e) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots |x| < 1$

f) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots\dots |x| < 1$

g) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots\dots\dots |x| < 1$

h) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\dots$

Theorems on limits

a) $\underset{x\to a}{\mathrm{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$

b) $\underset{\theta\to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1$

c) If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, then $\lim\limits_{x\to a} (1 + f(x))^{1/g(x)} = e^{\lim\limits_{x\to a}\frac{f(x)}{g(x)}}$

Particular Cases

a) $\lim\limits_{x\to 0} (1+x)^{1/x} = e$

b) $\lim\limits_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

c) $\lim\limits_{x\to 0} (1+\lambda x)^{1/x} = e^\lambda$

d) $\lim\limits_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

DE' L' Hospitals Rule

It f (x) and g (x) be two functions of x such that

(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or ∞

(ii) both are continous and differentiable at x = a

(iii) f′ (x) and g′ (x) are continous at x = a

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},\ g'(a)\neq 0$$

Newton -Leibnitz formula

$$\frac{d}{dx}\left(\int_{h(x)}^{g(x)} f(t)\,dt\right)=f(g(x))g'(x)-f(h(x))h'(x)$$

PART I - (JEEMAIN)

SECTION - I - Straight objective type questions

1. If $a>1$ then $\underset{x\to\infty}{\mathrm{Lt}}\ a^x\sin\frac{b}{a^x}=$

1) a 2) b 3) $\frac{a}{b}$ 4) $\frac{b}{a}$

2. $\underset{x\to 0}{\mathrm{Lt}}\frac{x\cot 4x}{\sin^2 x\cot^2 2x}$

1) 0 2) 1 3) –1 4) $\frac{1}{2}$

3. The value of $\lim_{x\to 0}\left(\frac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$ is equal to

1) 0 2) 4 3) –4 4) –1

4. $\underset{x\to 1}{\mathrm{Lt}}\frac{\sum_{k=1}^{100}x^k-100}{x-1}=$

1) 0 2) 5050 3) 4550 4) –5050

5. $\underset{x\to 2}{\mathrm{Lt}}\frac{3^x+3^{3-x}-12}{3^{\frac{-x}{2}}-3^{1-x}}=$

1) $\frac{1}{12}$ 2) 12 3) 36 4) $\frac{1}{36}$

6. $\underset{x \to 0}{Lt} \dfrac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$

1) 0 2) 1 3) 2 4) –2

7. $\lim\limits_{x \to 0} \dfrac{\sin^2(\pi \cos^4 x)}{x^4}$ is equal to

1) π^2 2) $2\pi^2$ 3) $4\pi^2$ 4) 4π

8. $\underset{x \to \frac{\pi}{2}}{Lt} \dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} =$

1) log a 2) log 2 3) a 4) log x

9. The value of $\underset{n \to \infty}{Lt} \cos\dfrac{x}{2}.\cos\dfrac{x}{4}.\cos\dfrac{x}{8}.....\cos\dfrac{x}{2^n} =$

1) 1 2) 0 3) $\dfrac{\sin x}{x}$ 4) cos x

10. Let $f: R \to R$ such that $f(1) = 4$ and $f'(1) = 2$. Then $\underset{x \to 1}{Lt} \int_4^{f(x)} \dfrac{2t}{(x-1)} dt =$

1) 16 2) 8 3) 4 4) 2

11. $\underset{x \to \infty}{Lt} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right] =$

1) 0 2) 1 3) –1 4) 1/2

12. $\underset{x \to 0}{Lt} \left[\dfrac{a^x + b^x + c^x}{3}\right]^{1/x} =$

1) abc 2) log abc 3) $\dfrac{\log abc}{3}$ 4) $(abc)^{1/3}$

13. If $\lim_{x\to 0}\left[1+x\ell n\left(1+b^2\right)\right]^{\frac{1}{x}}=2b\sin^2\theta, b>0$ and $\theta\in(-\pi,\pi)$, then the value of θ is

1) $\pm\frac{\pi}{4}$ 2) $\pm\frac{\pi}{3}$ 3) $-\frac{\pi}{2}$ 4) $\frac{\pi}{2}$

14. Statement 1: $\underset{x\to 3^+}{\mathrm{Lt}}\frac{[x]^2-9}{x^2-9}=0$ where stands for greatest integer value

Statement 2: $[x]=3$ when $3\le x<4$ where $[.]$ stands for GIV function

1) Statement I is true, statement II is true, statement II is a correct explanation for statement I
2) Statement I is true, statement II is true, statement II is not a correct explanation for statement I
3) Statement I is true, statement II is false
4) Statement I is false, statement II is true

15. $\underset{m\to\infty}{\mathrm{Lt}}\frac{m\left(1+2^2+3^2+.....+m^2\right)}{1+2^3+3^3+.....+m^3}=$

1) 2/3 2) 4/3 3) 0 4) 1

16. $f(x)\begin{cases} =x^2e^{2x-2}\text{when } 0\le x\le 1 \\ =a\,\mathrm{sig}(x+1)\cos(2x-2)+bx^2 \text{ when } 1<x\le 2\end{cases}$

$\underset{x\to 1}{\mathrm{Lt}}f(x)$ exists. Then $a+b=$

1) 0 2) 1 3) –1 4) 2

17. $\lim_{x\to 0}\frac{\left(4^x-1\right)^3}{\sin\left(\frac{x}{p}\right)\ell n\left(1+\frac{x^2}{3}\right)}$ is equal to

1) $9p(\ell n4)$ 2) $3p(\ell n4)^3$ 3) $12p(\ell n4)^3$ 4) $27p(\ell n4)^2$

18. $\underset{x\to 0^-}{\mathrm{Lt}}\frac{x([x]+|x|)\sin[x]}{|x|}$

where [.] stands for GIV function is

1) sin 1 2) –sin 1 3) cos 1 4) –cos 1

19. $\underset{x\to 0}{\text{Lt}} \frac{1}{x}\log\left(\frac{1+3x}{1-2x}\right)=$

1) 1 2) -2 3) -1 4) 5

20. $\underset{t\to 0}{\text{Lt}}\left(\frac{1}{1^{\sin^2 t}}+\frac{1}{2^{\sin^2 t}}+\frac{1}{3^{\sin^2 t}}+....+\frac{1}{n^{\sin^2 t}}\right)^{\sin^2 t}=$

1) n^2+n 2) n 3) $\frac{n(n+1)}{2}$ 4) n^2

SECTION - II

Numerical Type Questions

21. If the value of $\lim_{x\to 0}\left(2-\cos x\sqrt{\cos 2x}\right)^{\left(\frac{x+2}{x^2}\right)}$ is equal to e^a, then a is equal to

22. If $\lim_{x\to 0}\frac{\alpha xe^x-\beta\log_e(1+x)+\gamma x^2e^{-x}}{x\sin^2 x}=10,\alpha,\beta,\gamma\in R$, then the value of $\alpha+\beta+\gamma$ is

23. Let $f(x)=x^6+2x^4+x^3+2x+3, x\in R$. Then the natural number n for which $\lim_{x\to 1}\frac{x^n f(1)-f(x)}{x-1}=44$

24. $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}=$

25. $\underset{x\to 1}{\text{Lt}}\left(\frac{x^a-ax+a-1}{(x-1)^2}\right)=f(a)$ Then $\frac{f(4)}{5}$

PART - II (JEE ADVANCED)

SECTION - III (Only one option correct type)

26. $\underset{h\to 0}{\text{Lt}}\left(\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right)$

A) $\frac{1}{48}$ B) $-\frac{1}{48}$ C) 48 D) -48

27. The value of $\underset{x\to 0}{\text{Lt}} \dfrac{e-(1+x)^{\frac{1}{x}}}{\tan x}$

A) 1 B) –e C) $\dfrac{e}{2}$ D) $\dfrac{-e}{2}$

28. $\underset{x\to 0}{\text{Lt}} \dfrac{ae^{x^2}+b\cos x}{x^2}=\dfrac{1}{2}$

Then $a+b=$

A) $\dfrac{1}{3}$ B) $-\dfrac{1}{3}$ C) 0 D) $\dfrac{1}{2}$

29. The value of $\lim\limits_{x\to 0}\dfrac{8}{x^8}\left(1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right)$

1) 0 2) 1 3) $\dfrac{1}{32}$ 4) Does not exist

30. If α,β are the distinct roots of $x^2+bx+c=0$, then $\lim\limits_{x\to\beta}\dfrac{e^{2(x^2+bx+c)}-1-2(x^2+bx+c)}{(x-\beta)^2}$ is equal to

1) b^2+4c 2) $2(b^2+4c)$ 3) $2(b^2-4c)$ 4) b^2-4c

31. $f(x)\begin{cases} =\dfrac{\sin[x]}{[x]} \text{ when } [x]\neq 0 \\ =0 \text{ when } [x]=0 \text{ where } [.] \text{ is GIV of } x \end{cases}$

Then $\underset{x\to 0}{\text{Lt}} f(x)=$

A) 1 B) 0 C) –1 D) Does not exist

SECTION - IV (More than one correct answer)

32. If $\alpha=\underset{x\to\frac{\pi}{4}}{\text{Lt}}\dfrac{\tan^3 x-\tan x}{\cos\left(x+\dfrac{\pi}{4}\right)}$ and $\beta=\underset{x\to 0}{\text{Lt}}(\cos x)^{\cot x}$ are the roots of the equation $ax^2+bx-4=0$ then which of the following are true

A) a = 1 B) b = 3 C) $a+b=5$ D) $a-b=7$

33. Let α,β be the roots of $ax^2+bx+c=0$, where $1<\alpha<\beta$ and $\lim\limits_{x\to x_0}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$, then

A) $a>0$ and $x_0<1$ B) $a>0$ and $x_0>\beta$

C) $a<0$ and $\alpha<x_0<\beta$ D) $a<0$ and $x_0<1$

34. Let $L=\lim_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}, a>0$. If L is finite, then

A) a = 2 B) a = 1 C) $L=\dfrac{1}{64}$ D) $L=\dfrac{1}{32}$

35. Consider $f(x)=\dfrac{\sin x+ae^x+be^{-x}+c\ell n(1+x)}{x^3}$ where a,b,c are real numbers

A) If $\lim_{x\to 0^+} f(x)$ is finite, then the value of $a+b+c$ is O

B) If $\lim_{x\to 0} f(x)=\ell(\text{finite})$, then the value of ℓ is $-\dfrac{1}{3}$

C) The value of $a=0$ $b=\dfrac{1}{3}$ and $c=-\dfrac{1}{3}$

D) The value of $a=-\dfrac{1}{3} b=0$ and $c=\dfrac{1}{3}$

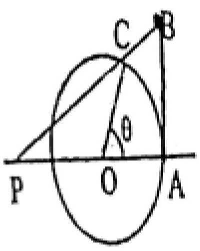
SECTION - V (Numerical Type)

36. Let and n be two positive integers greater than 1. If $\lim_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\dfrac{e}{2}\right)$ then the value of $\dfrac{m}{n}$ is

37. $f(x)=\dfrac{1-\cos\left(1-\cos\dfrac{x}{2}\right)}{2^m x^n}$. Given $\underset{x\to 0}{\mathrm{Lt}}\, f(x)=1$. Where m and n are positive integers. Then n-m=

Question stem

A tangent line is drawn to a circle of radius unity at point A and a segment AB is laid off whose length is equal to that of arc AC. A straight line BC is drawn to intersect the extension of radius AO at point P as shown fig.



Based on the stem answer the following

38. $\lim_{\theta\to 0} PA$ is equal to

A) 6 B) 2 C) 3 D) 4

39. $\underset{\theta\to 0}{\text{Lt}}\ \theta^2 \dfrac{PC}{BC} =$

A) 6　　B) 2

C) 3　　D) 4

SECTION VI - (Matrix match type)

40. $\lim\limits_{x\to 0} f(x)$, where f(x) is as in column –1, is

	Column I		Column II
A)	$f(x)=\dfrac{\tan\left[e^2\right]x^2-\tan\left[-e^2\right]x^2}{\sin x}$ where [.] integer function	p)	$\dfrac{\sqrt{2}}{8}$
B)	$f(x)=\left[\left(\min\left(t^2+4t+6\right)\right)\dfrac{\sin x}{x}\right]$	q)	15
C)	$f(x)=\dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x}$	r)	1
D)	$f(x)=\dfrac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$	s)	1/2

A) Aq, Br, Cs, DP　　B) Ar, Bq, Cs, Dp

C) Aq, Br, Cp, Ds　　D) Ar, Bp, Cs, Dq