Module 3 - Models for Stationary Time Series

MATH1318 Time Series Analysis

Prepared by Dr Haydar Demirhan based on the textbook by Cryer and Chan, Time Series Analysis with R, Springer.

Nomenclature

μ: Mean of series.
Ψ_i: Coefficients of general linear process.
$\{e_t\}$: Sequence of uncorrelated rvs.
ϕ: Coefficients of general linear process defined in between -1 and 1.
θ_i: Coefficients of moving average process.
MA(q): Moving average process of order q.
ϕ_i: Coefficients of autoregressive process.
AR(p): Autoregressive process of order q.
ARMA(p,q): Autoregressive moving average process of order p and q.

Introduction

The autoregressive moving average (ARMA) models constitute a wide class of parametric time series models. These models have great importance and field of applicability in real-world problems.

In this module, we will study

- general linear processes,
- moving average processes, and
- autoregressive processes.

These three types of processes constitute the logic behind the ARIMA models. Then we will focus on

- the mixed Autoregressive Moving Average (ARMA) model and
- invertibility.

General Linear Processes

Throughout the module, we will denote observed times series with $\{Y_t\}$, and let $\{e_t\}$ represent an unobserved white noise series, that is, a sequence of identically distributed, zero-mean, independent random variables. Also, the assumption of independence could be replaced by the weaker assumption that the $\{e_t\}$ are uncorrelated random variables, but we will not pursue that slight generality.

A *general linear process*, $\{Y_t\}$, is one that can be represented as a weighted linear combination of present and past white noise terms as

$$Y_t = e_t + \Psi_1 e_{t-1} \Psi_2 e_{t-2} + \ldots$$

provided that

$$\sum_{1}^{\infty} \Psi_i^2 < \infty.$$

An important nontrivial example to which we will return often is the case where the Ψ's form an exponentially decaying sequence

$$\Psi_j = \phi^j$$

where $\phi \in [-1, 1]$. Then,

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

For this example,

$$E(Y_t) = E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) = 0$$

so that $\{Y_t\}$ has a constant mean of zero. Also,

$$Var(Y_t) = \frac{\sigma_e^2}{1 - \phi^2}$$

$$Cov(Y_t, Y_{t-1}) = \frac{\phi\sigma_e^2}{1 - \phi^2}$$

$$Corr(Y_t, Y_{t-1}) = [\frac{\phi\sigma_e^2}{1 - \phi^2}]/[\frac{\sigma_e^2}{1 - \phi^2}] = \phi$$

And also,

$$Corr(Y_t, Y_{t-k}) = \phi^k$$

Because the autocovariance structure depends only on time lag and not on absolute time, this process is stationary.

For general linear processes of the form

$$Y_t = e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \ldots$$

we have the following results:

$$E(Y_t) = 0,$$

$$\gamma_k = Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=1}^{\infty} \Psi_i \Psi_{i+k}, k \geq 0.$$

where $\Psi_0 = 1$. Note that we assume zero mean here. One can add μ to obtain a nonzero mean.

Moving Average Processes

When only a finite number of the Ψ-weights are nonzero, the general linear process is called a moving average process and it is written as

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

This moving average process has an **order q** and abbreviated as MA(q).

The First-Order Moving Average Process

The first-order moving average, MA(1), process is one of the important moving average processes. The MA(1) model is

$$Y_t = e_t - \theta e_{t-1}.$$

We have $E(Y_y) = 0$, $Var(Y_t) = \sigma_e^2(1 + \theta^2)$,

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\ &= Cov(-\theta e_{t-1}, e_{t-1}) \\ &= -\theta \sigma_e^2 \end{aligned}$$

and

$$\begin{aligned} Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) \\ &= 0 \end{aligned}$$

Similarly, $Cov(Y_t, Y_{t-k}) = 0$ for $k \geq 2$.

This means that the process has no correlation beyond lag 1.

We will use this fact to identify MA(1) process in practice.

In summary, for MA(1) model $Y_t = e_t - \theta e_{t-1}$,

$$\begin{aligned} E(Y_t) &= 0 \\ \gamma_0 &= Var(Y_t) = \sigma_e^2(1 + \theta^2) \\ \gamma_1 &= -\theta \sigma_e^2 \\ \rho_1 &= -\theta/(1 + \theta^2) \\ \gamma_k &= \rho_k = 0, \text{ for } k \geq 2. \end{aligned}$$

The following table shows autocorrelations corresponding to various values of θ parameter in MA(1) process.

θ	$\rho_1 = -\theta/(1+\theta^2)$	θ	$\rho_1 = -\theta/(1+\theta^2)$
0.1	–0.099	0.6	–0.441
0.2	–0.192	0.7	–0.470
0.3	–0.275	0.8	–0.488
0.4	–0.345	0.9	–0.497
0.5	–0.400	1.0	–0.500

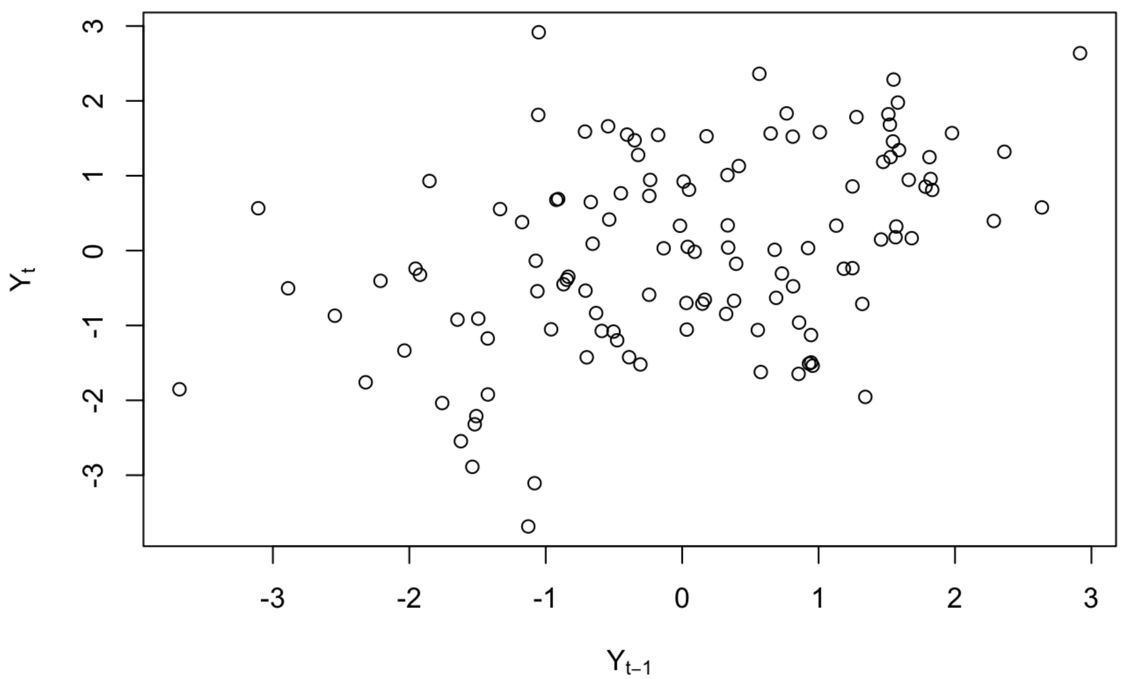
Please keep in mind that

> In a MA(1) process, we expect to see strong correlations between consecutive observations of a series.

For instance, in a MA(1) series with $\theta = 0.8$, autocorrelation at the first lag will be $\rho_1 = -0.488$.

It is also possible to observe the autocorrelation at lag 1 by plotting Y_t versus Y_{t-1} as in Figure 1:

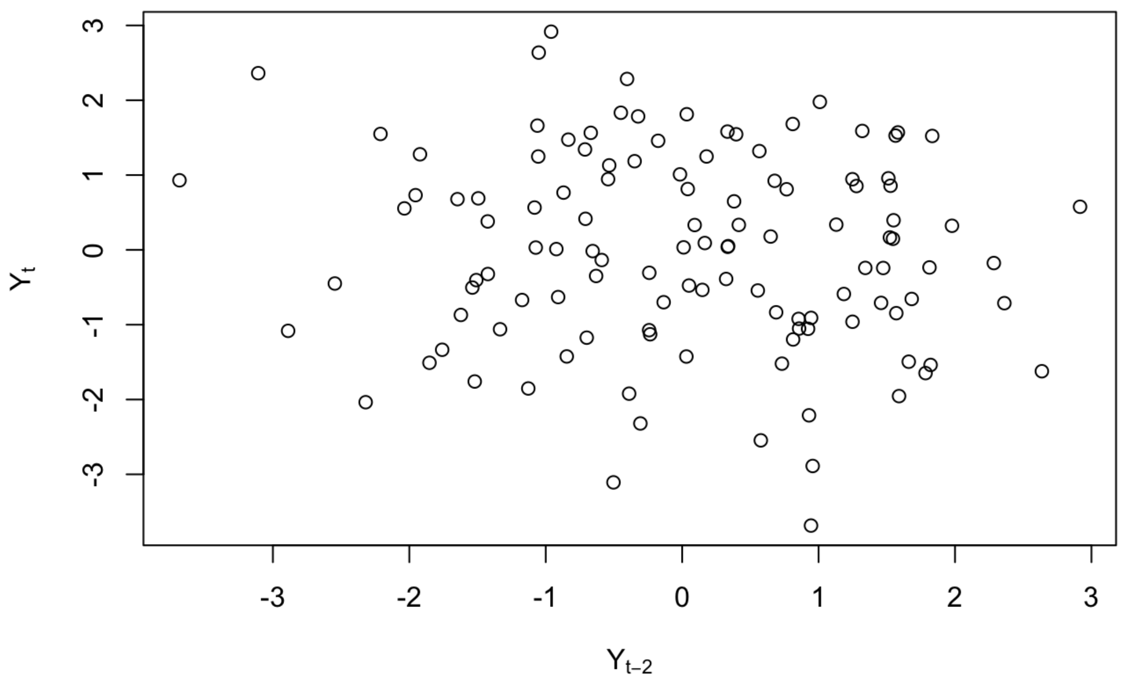
Figure 1. Time series plot for simulated MA(1) process.



And, we can check the pattern of autocorrelation at lag 2 with the plot in Figure 2 which is a strong visualization of the zero autocorrelation at lag 2 for MA(1) model.

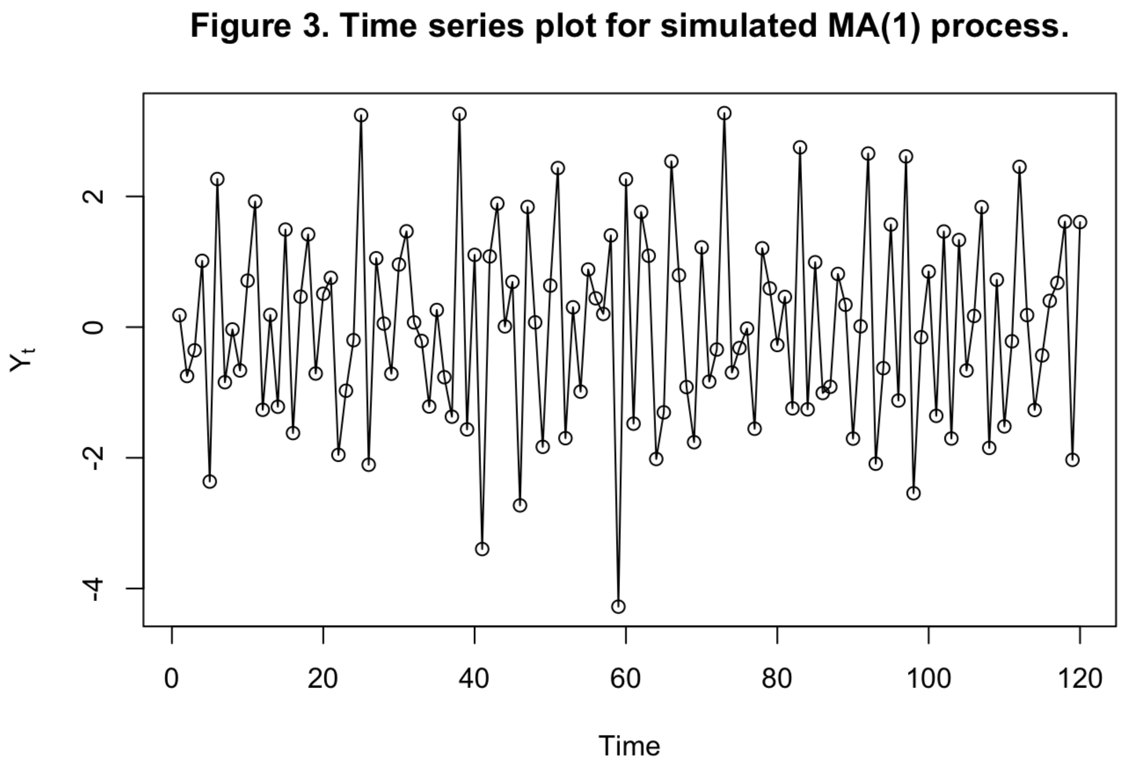
```
data(ma1.2.s)
plot(y=ma1.2.s,x=zlag(ma1.2.s,2),ylab=expression(Y[t]),
     xlab=expression(Y[t-2]),type='p',main="Figure 2. Scatter plot for successive obse
        rvations
     of simulated MA(1) process.")
```

Figure 2. Scatter plot for successive observations of simulated MA(1) process.



The time series plot in Figure 3 visualises an MA(1) series simulated with $\theta = 0.9$, correspondingly $\rho_1 = -0.497$. So, there is a moderately strong negative correlation at lag 1.

```
data(ma1.1.s)
plot(ma1.1.s,ylab=expression(Y[t]),type='o',main="Figure 3. Time series plot for simul
         ated MA(1) process.")
```

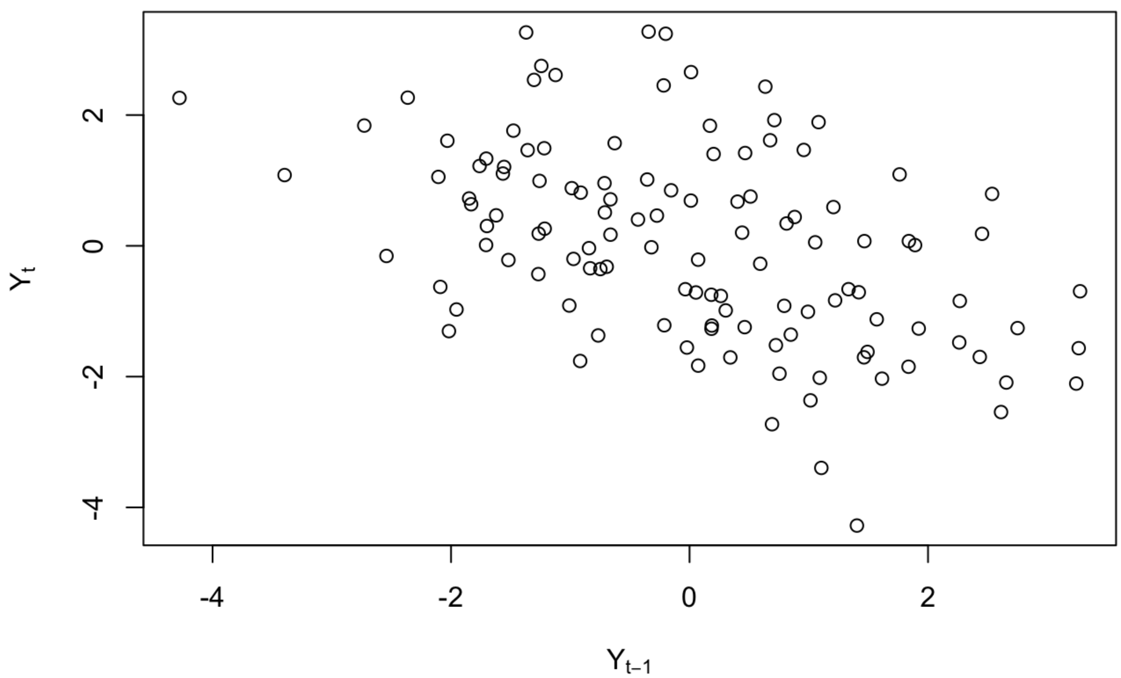


This correlation can be seen in the plot of the series since consecutive observations tend to be on opposite sides of the zero mean. If an observation is above the mean level of the series, then the next observation tends to be below the mean.

The negative lag 1 autocorrelation is even more apparent in Figure 4:

```
plot(y=ma1.1.s,x=zlag(ma1.1.s),ylab=expression(Y[t]), xlab=expression(Y[t-1]),
     type='p',main="Figure 4. Scatter plot for successive observations of
     simulated MA(1) process.")
```

Figure 4. Scatter plot for successive observations of simulated MA(1) process.



Notice that MA(1) processes have no autocorrelation beyond lag 1, but by increasing the order of the process, we can obtain higher-order correlations.

The Second-Order Moving Average Process

The following model is a MA(2) model:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

This model has the following properties:

$$\begin{aligned}
\gamma_0 &= Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&= (1 + \theta_1^2 + \theta_2^2)\sigma_e^2 \\
\gamma_1 &= Cov(Y_t, Y_{t-1}) = (-\theta_1 + \theta_1\theta_2)\sigma_e^2 \\
\gamma_2 &= Cov(Y_t, Y_{t-2}) = -\theta_2\sigma_e^2
\end{aligned}$$

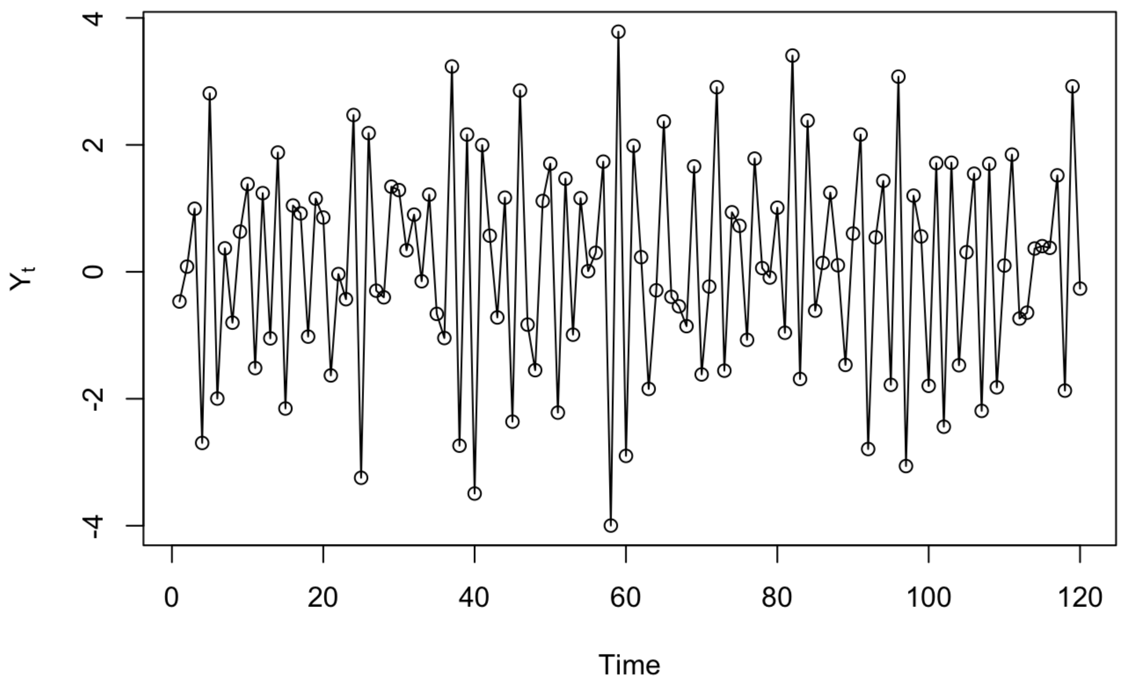
Thus, for an MA(2) process,

$$\begin{aligned}
\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2} \\
\rho_2 &= \frac{-\theta_2}{1+\theta_1^2+\theta_2^2} \\
\rho_k &= 0, \text{ for } k = 3, 4, \ldots
\end{aligned}$$

A time series plot of a simulation of this MA(2) process is shown in Figure 5:

```
data(ma2.s)
plot(ma2.s,ylab=expression(Y[t]),type='o',main="Figure 5. Time series plot of simulate
        d MA(2) process.")
```

Figure 5. Time series plot of simulated MA(2) process.

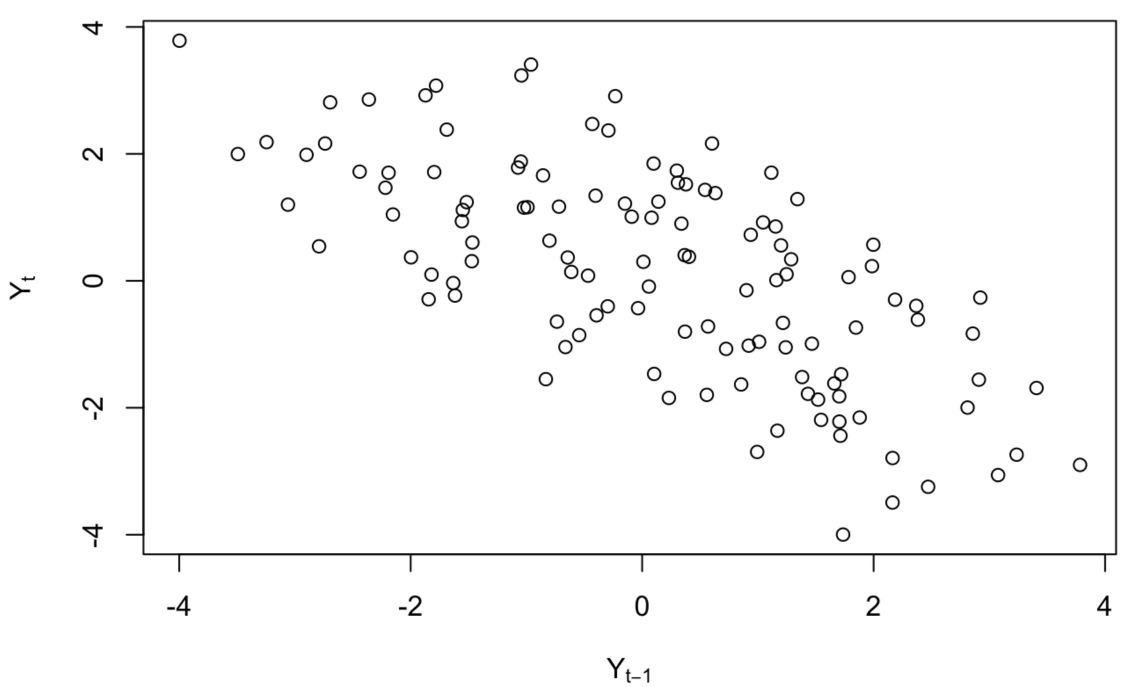


The series tends to move back and forth across the mean in one time unit.

This reflects the fairly strong negative autocorrelation at lag 1. This information is useful in identifying MA(2) series in practice.

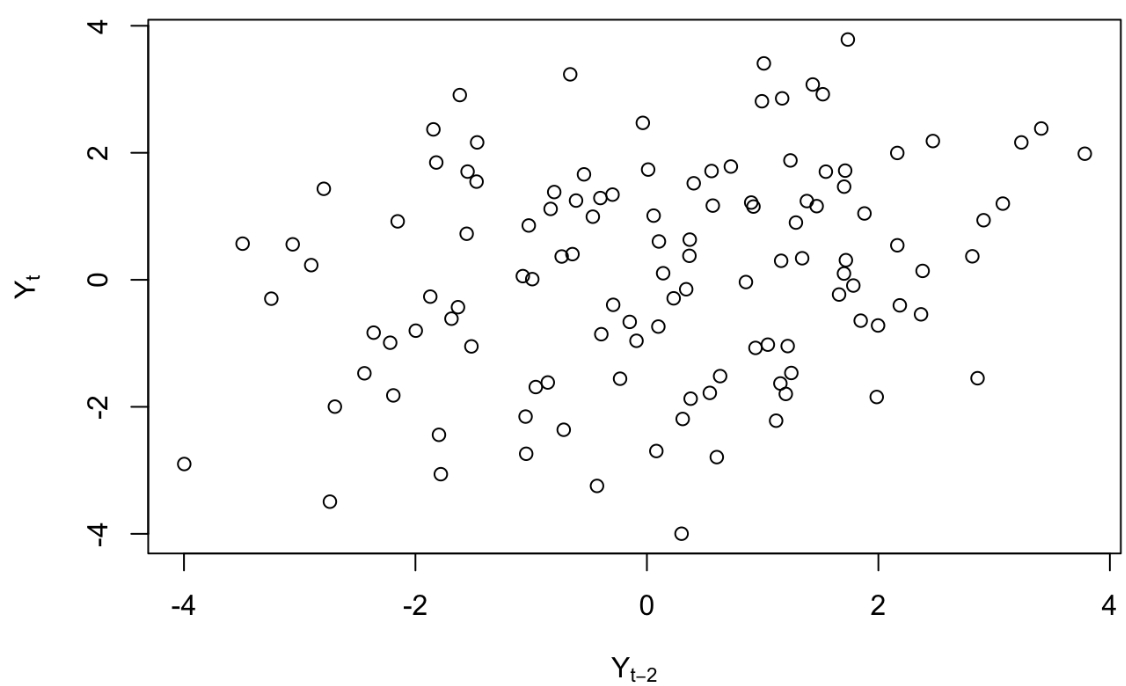
We observe strong negative correlation between Y_t and Y_{t-1} from Figure 6.

Figure 6. Scatter plot of MA(2) process versus its first lag.



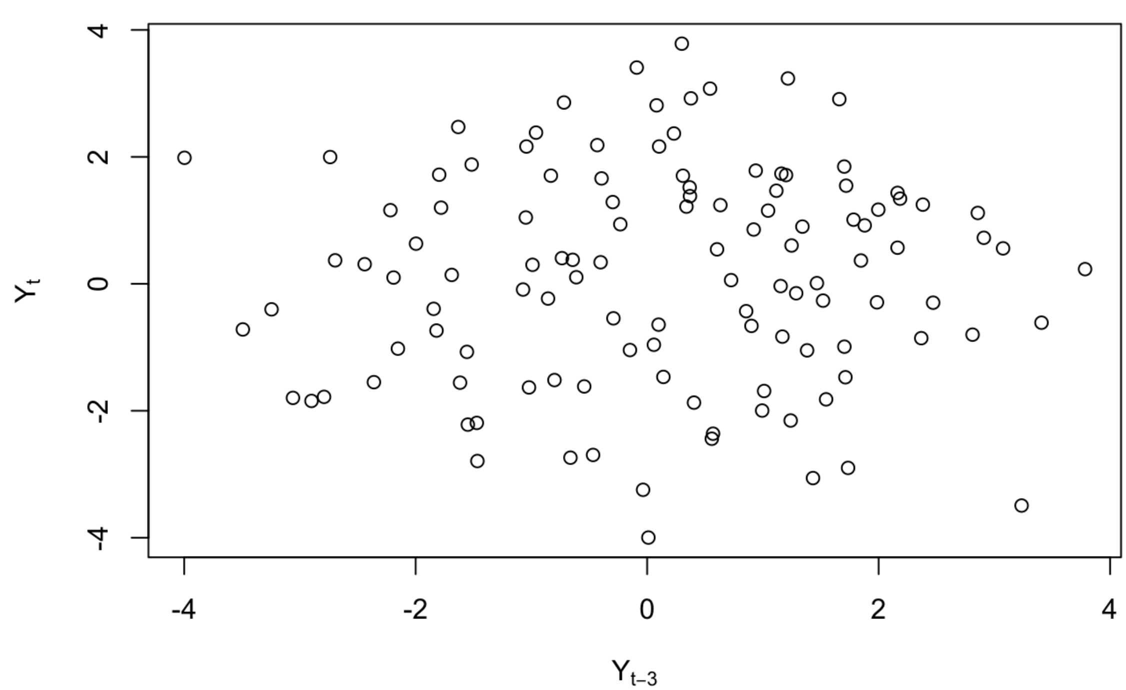
Also, we observe weak positive correlation between Y_t and Y_{t-2} from Figure 7.

Figure 7. Scatter plot of MA(2) process versus its second lag.



And, there is nearly no correlation between Y_t and Y_{t-3} from Figure 8 and so forth.

Figure 8. Scatter plot of MA(2) process versus its third lag.



The General MA(q) Process

The formulation of general MA(q) process is as follows:

The following model is a MA(2) model:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.$$

We have the following autocorrelation result for MA(q) process:

$$\begin{aligned}
\rho_k &= \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \\
\rho_k &= 0, \text{ for } k > q
\end{aligned}$$

The autocorrelation function “cuts off” after lag q; that is, it is zero.

The following App displays time series, ACF and partial-ACF plots for general MA(q) process up to $q = 4$.

App.3.1: Time series, ACF, and PACF plots for MA(q) processes.

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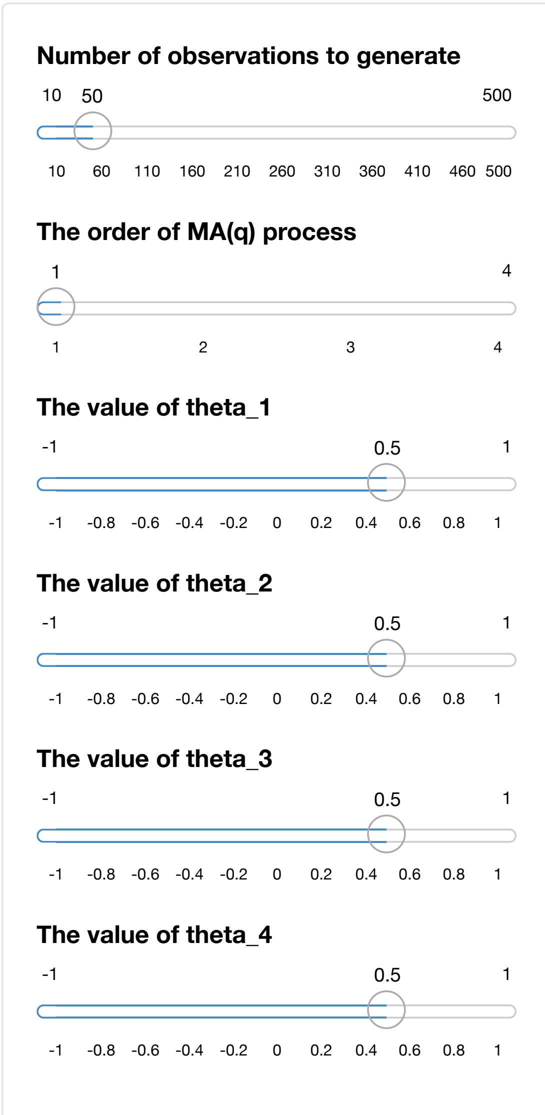
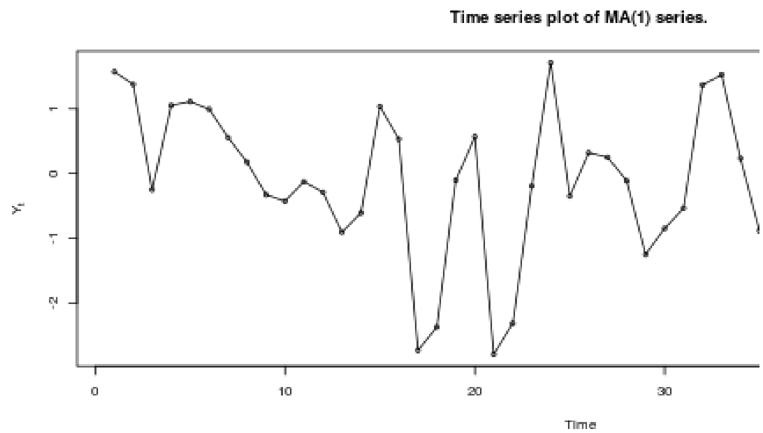
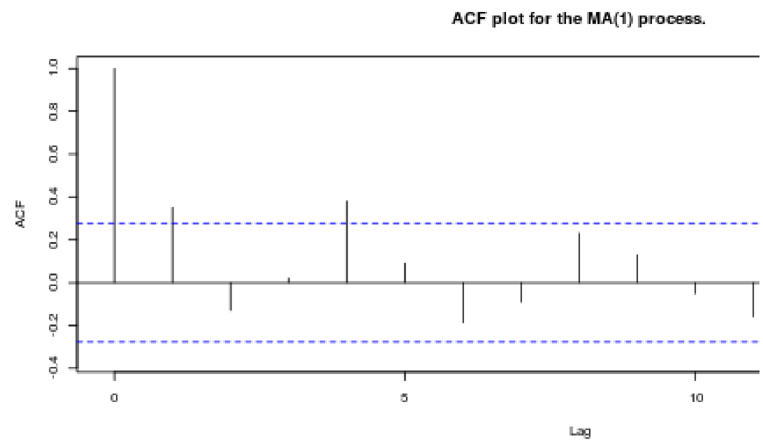
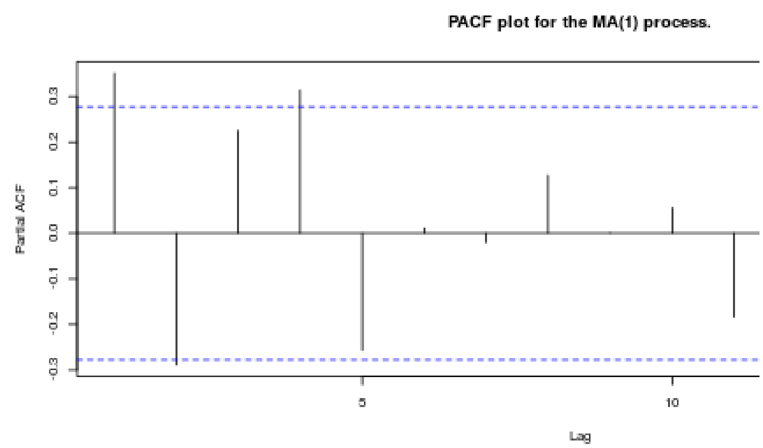








Autoregressive Processes

The autoregressive (AR) processes provide models for alternative autocorrelation patterns than the MA(q) process can handle. AR processes put regression on themselves. A pth-order autoregressive process $\{Y_t\}$ satisfies the following equation:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

The current value of the series Y_t is a linear combination of the p most recent past values of itself plus an error term e_t that incorporates everything new in the series at time t that is not explained by the past values.

The First-Order Autoregressive Process

The model formulation of the first-order AR, namely AR(1), model is

$$Y_t = \phi Y_{t-1} + e_t.$$

Here we assume that the series mean is zero. So, we obtain the following:

$$\begin{aligned} \gamma_0 &= \tfrac{\sigma_e^2}{1-\phi^2}, \\ \gamma_k &= \phi^k \tfrac{\sigma_e^2}{1-\phi^2} \end{aligned}$$

and

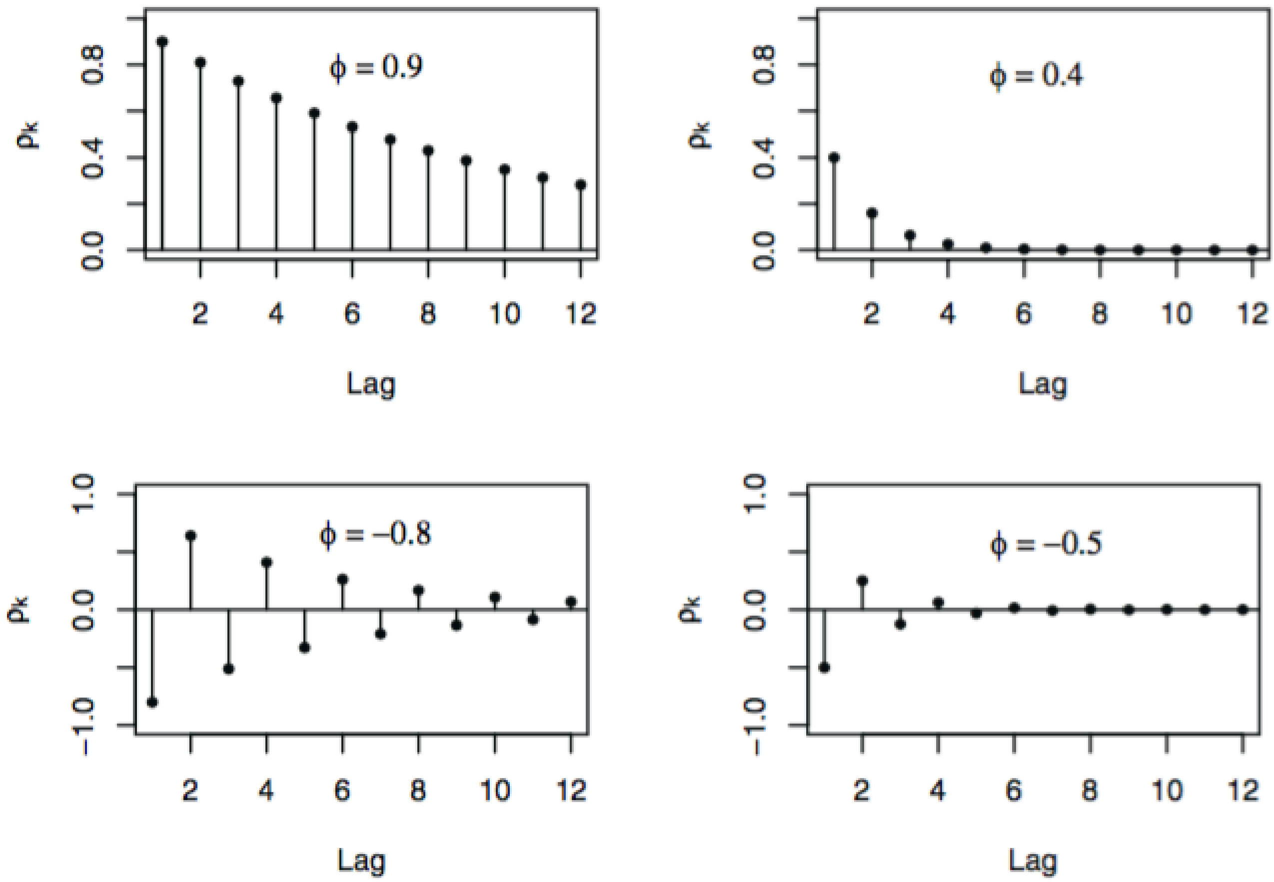
$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k,$$

for $k = 1, 2, 3, \ldots.$

Since $\phi < 1$, the magnitude of the autocorrelation function decreases exponentially as the number of lags, k, increases. If $0 < \phi < 1$, all correlations are positive; if $-1 < \phi < 0$, the lag 1 autocorrelation is negative ($\rho_1 = \phi$) and the signs of successive autocorrelations alternate from positive to negative, with their magnitudes decreasing exponentially.

ACF plots of several autocorrelations for AR(1) process are shown in Figure 9:

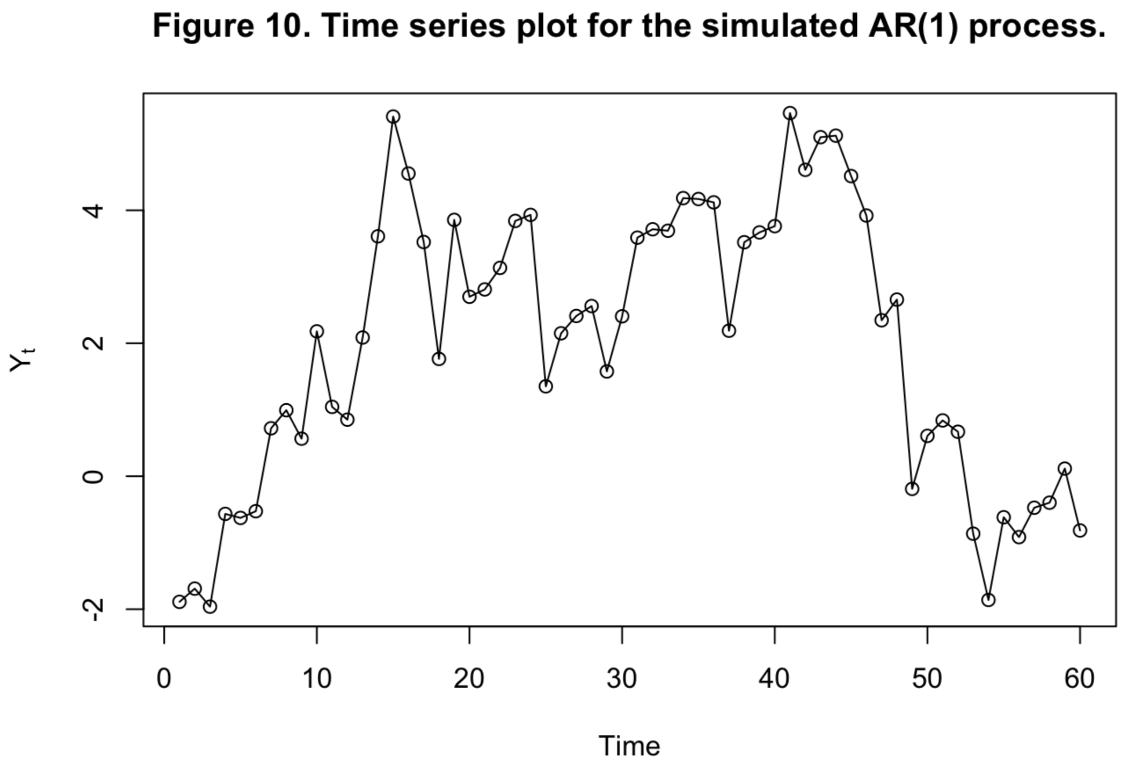
Figure 9. ACF plots for various AR(1) processes.



Notice that for ϕ near ± 1, the exponential decay is quite slow, but for smaller ϕ, the decay is quite rapid. With ϕ near ± 1, the strong correlation will extend over many lags and produce a relatively smooth series if ϕ is positive and a very jagged series if ϕ is negative.

We have time series plot of a simulated AR(1) process with $\phi = 0.9$ in Figure 10.

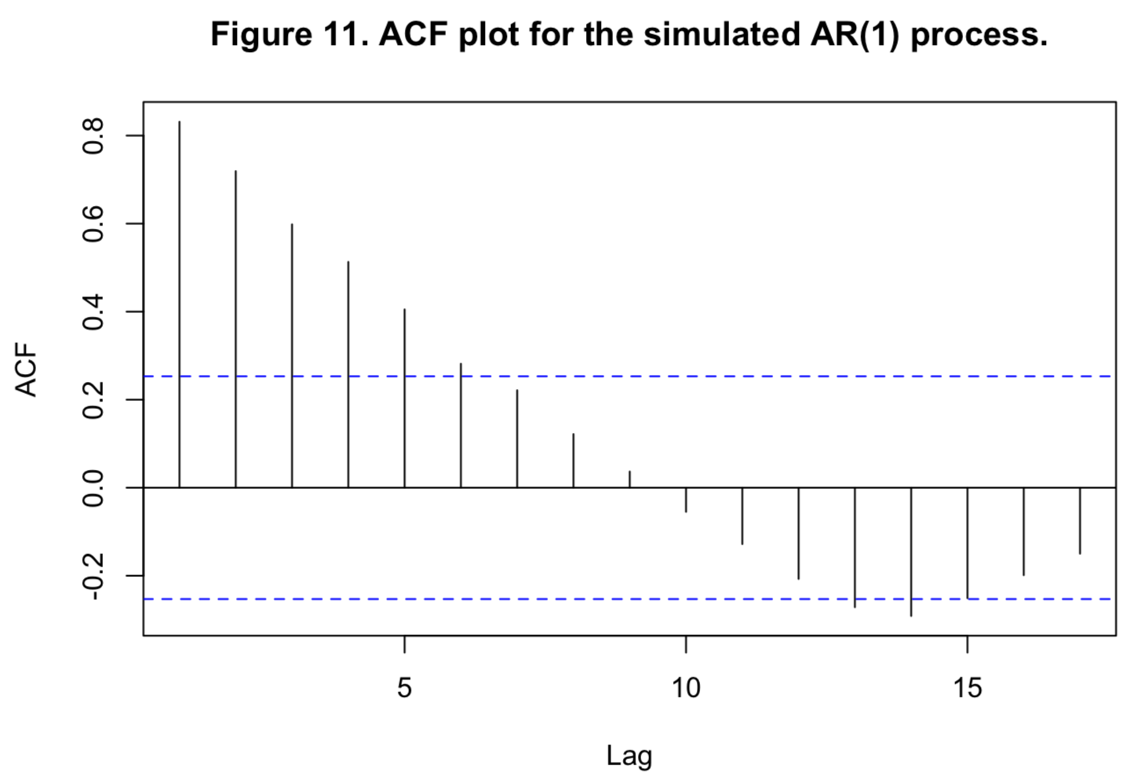
```
data(ar1.s)
plot(ar1.s,ylab=expression(Y[t]),type='o',
     main="Figure 10. Time series plot for the simulated AR(1) process.")
```



The series rarely crosses its theoretical mean of zero and it seems that there are several trends in this series. This is due to the strong autocorrelation of neighboring values of the series.

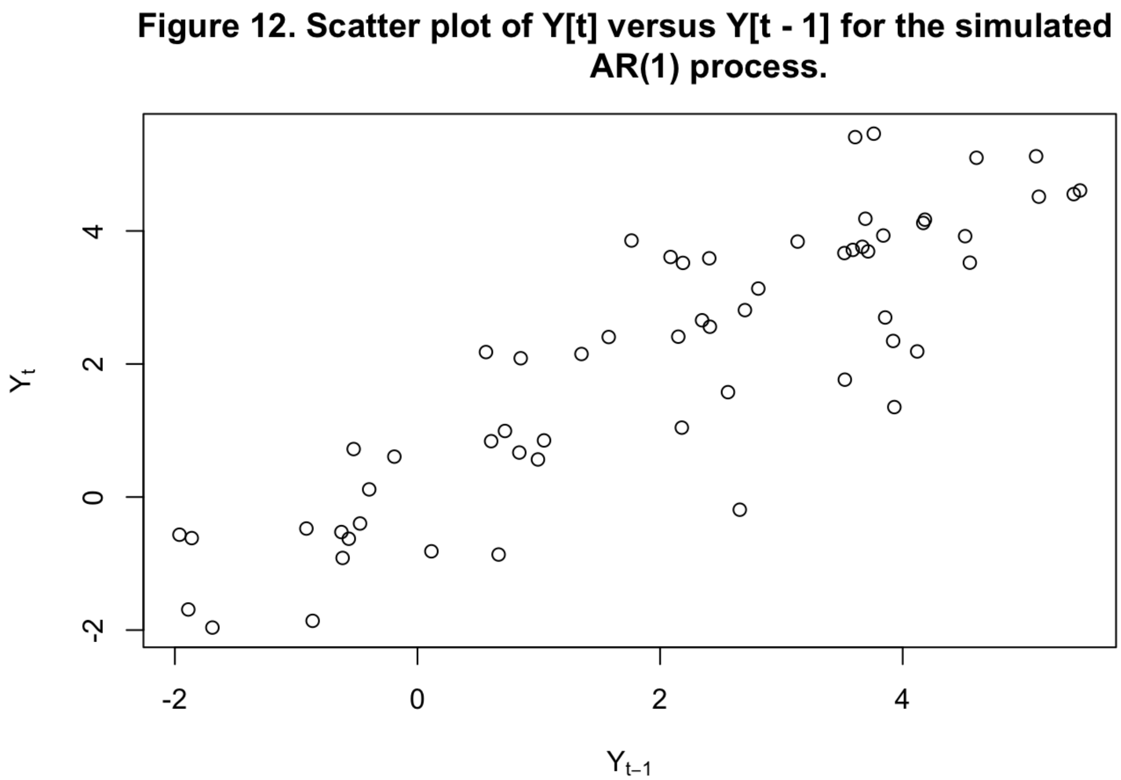
The ACF for this series is shown in Figure 11:

```
acf(ar1.s,main="Figure 11. ACF plot for the simulated AR(1) process.")
```



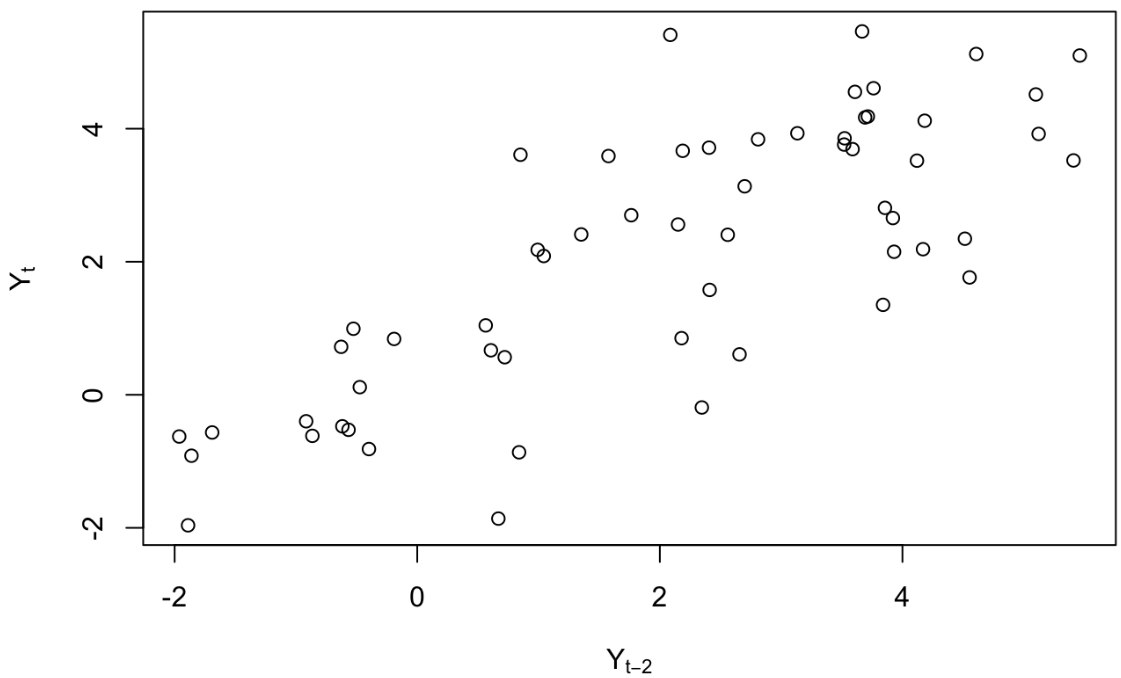
We observe the expected exponential decay in the ACF plot of this series. Also, we have high correlation between Y_t and Y_{t-1}, Y_t and Y_{t-2}, and Y_t and Y_{t-3}. So, AR(1) has autocorrelation at lags 1, 2, 3, and so on as visualised in Figures 12 - 14.

```
plot(y=ar1.s,x=zlag(ar1.s),ylab=expression(Y[t]),xlab=expression(Y[t-1]),type='p',
     main=paste0("Figure 12. Scatter plot of ", expression(Y[t]) , " versus ", express
         ion(Y[t-1]) , " for the simulated
                   AR(1) process."))
```



At lag 2, $\rho_2 = 0.9^2 = 0.81$.

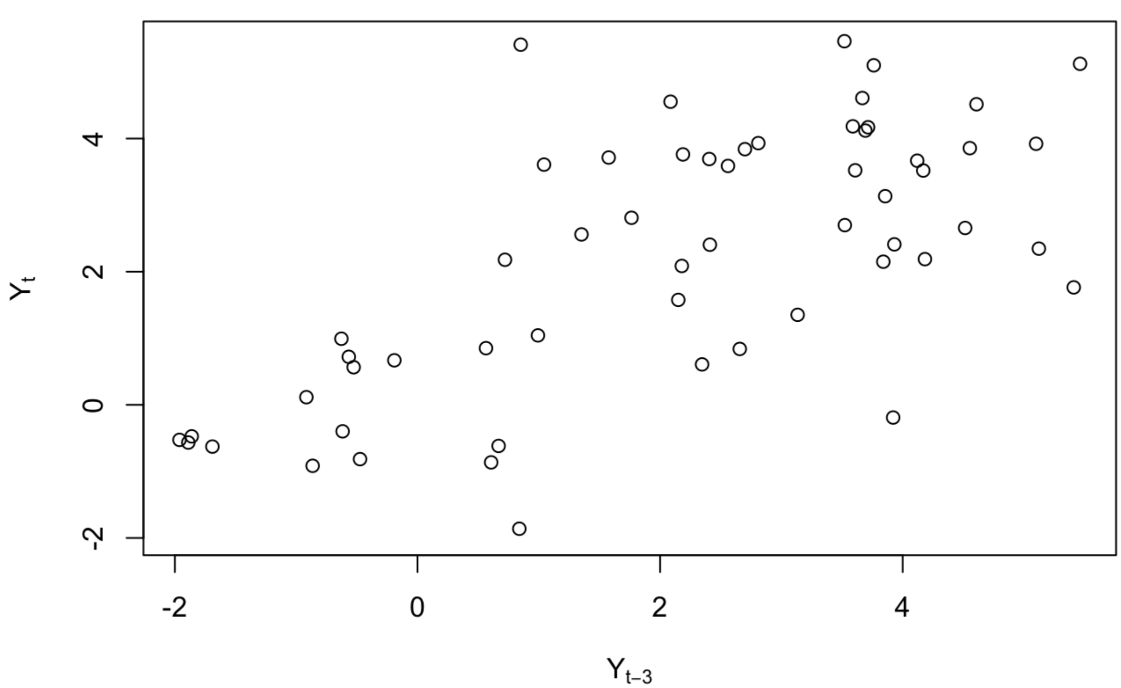
Figure 13. Scatter plot of Y[t] versus Y[t - 2] for the simulated AR(1) process.



At lag 3, it is still high $\rho_3 = 0.9^3 = 0.729$.

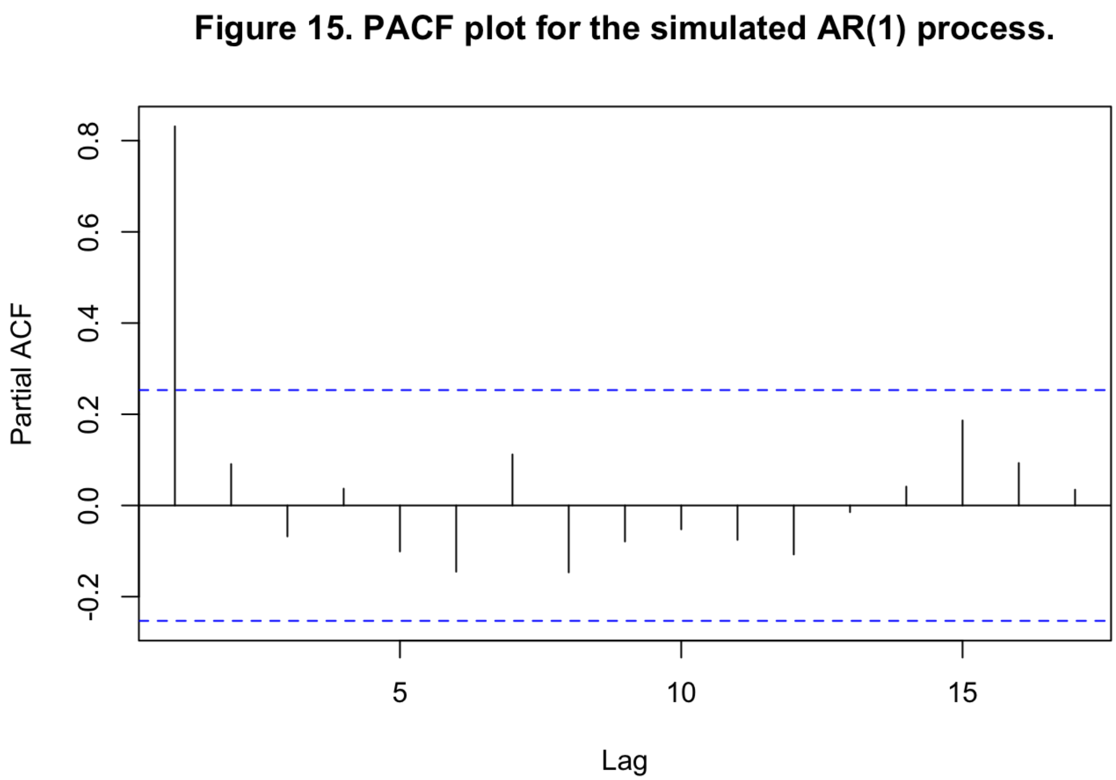
```
plot(y=ar1.s,x=zlag(ar1.s,3),ylab=expression(Y[t]),xlab=expression(Y[t-3]),type='p',
     main=paste0("Figure 14. Scatter plot of ", expression(Y[t]) , " versus ", express
         ion(Y[t-3]) , " for the simulated
                    AR(1) process."))
```

Figure 14. Scatter plot of Y[t] versus Y[t - 3] for the simulated AR(1) process.



The PACF for this series is shown in Figure 15:

```
pacf(ar1.s,main="Figure 15. PACF plot for the simulated AR(1) process.")
```



PACF of AR(1) process has a positive or negative spike at lag 1 depending on the sign of ϕ then cuts off.

The Second-Order Autoregressive Process

Model formulation of a second order AR, namely AR(2), process is as the following:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

where we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$. The autocovariance and autocorrelation functions of an AR(2) process are

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$

and

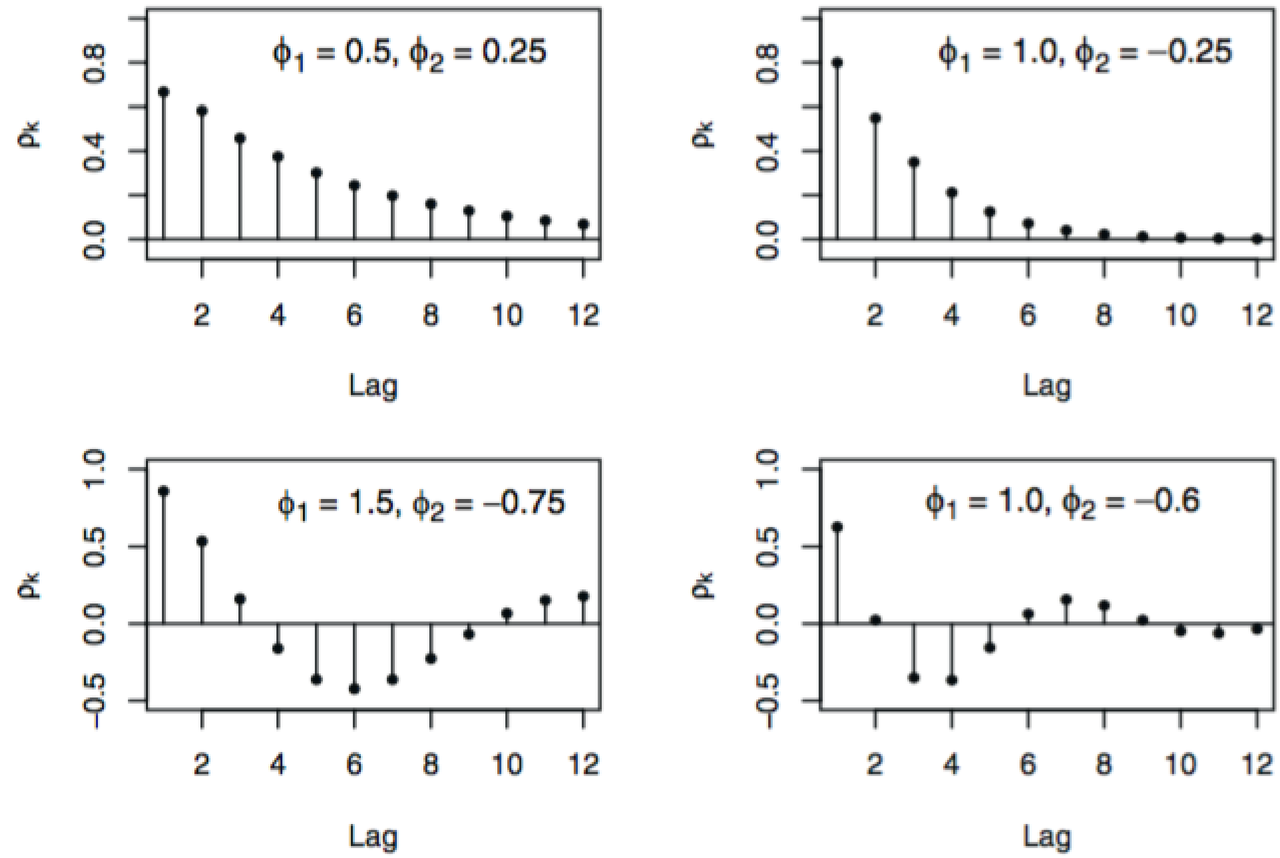
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}.$$

These equations are called the Yule-Walker equation. Due to the nature of these equations, the autocorrelation function of AR(2) process can assume a wide variety of shapes.

In all cases, the magnitude of ρ_k dies out exponentially fast as the lag k increases.

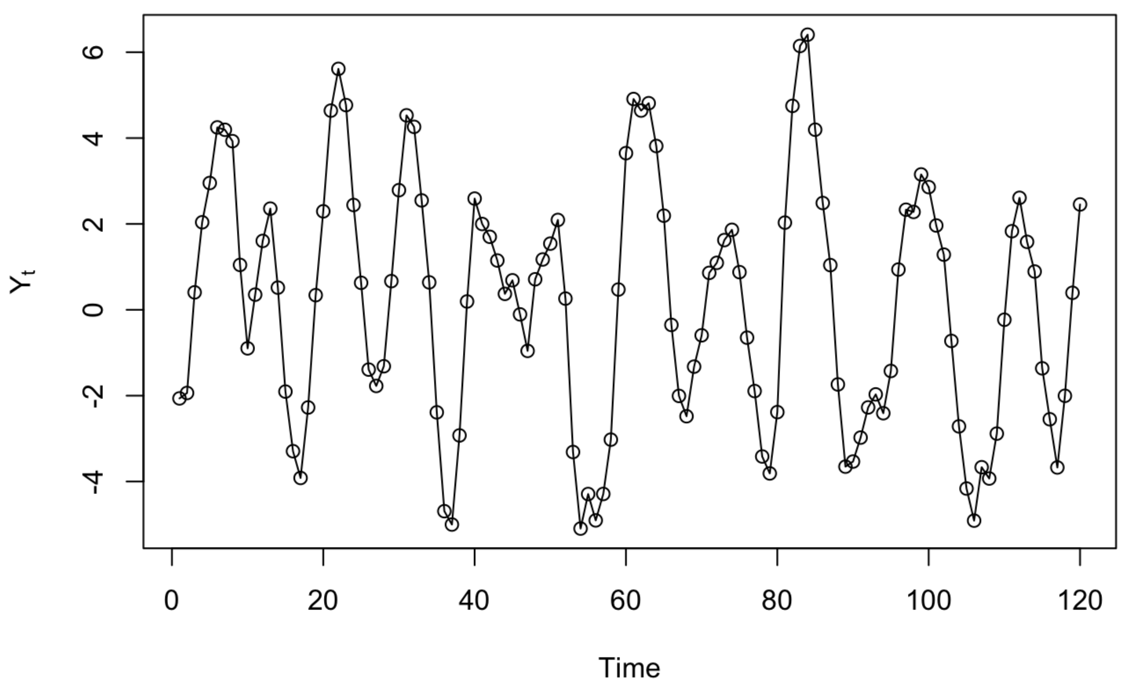
Illustrations of the possible shapes are given in Figure 16:

Figure 16. ACF plots for various AR(2) processes.



The time series plot of a simulated AR(2) series with $\phi_1 = 1.5$ and $\phi_2 = -0.75$ is shown in Figure 17:

Figure 17. Time series plot of AR(2) series with phi[1]=-1.5 and phi[2]=-0.



The periodic behavior of AR(2) process is clearly seen in this plot.

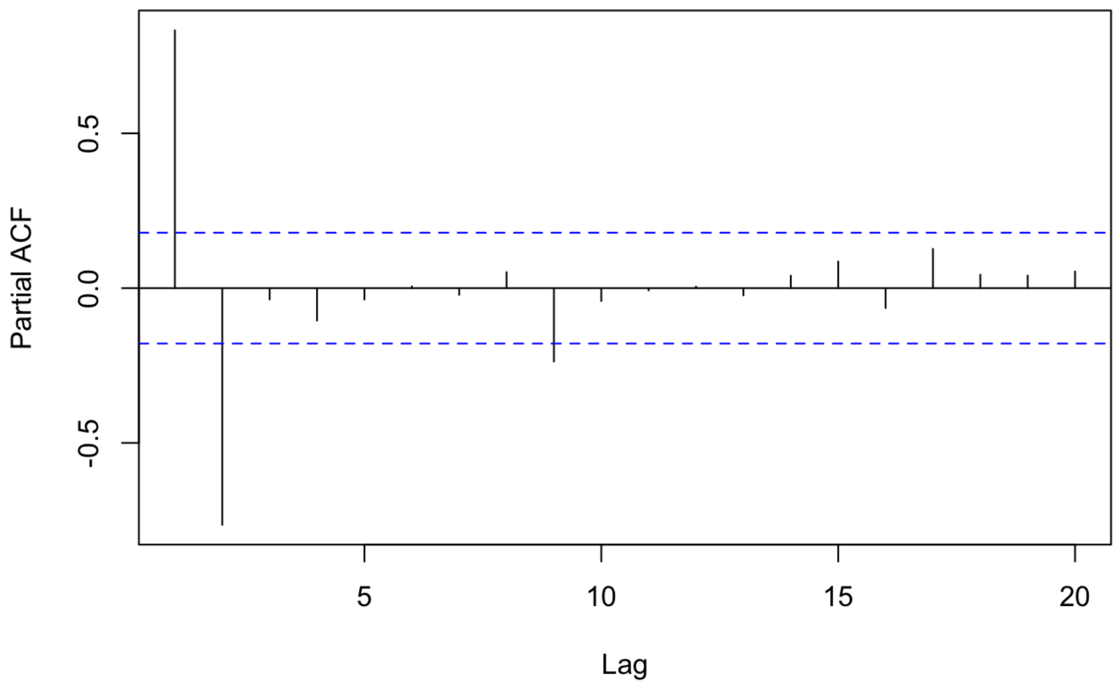
The variance for AR(2) process is as follows:

$$\gamma_0 = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_e^2}{(1-\phi_2)^2-\phi_1^2}.$$

PACF of AR(2) process is shown in Figure 18:

```
pacf(ar2.s,main=paste0("Figure 18. PACF plot of AR(2) series with ",
                        expression(phi[1]), "=-1.5 and ", expression(phi[2]),"=-0.75."
         ))
```

Figure 18. PACF plot of AR(2) series with phi[1]=-1.5 and phi[2]=-0.75.



PACF of AR(2) process cuts off after lag 2.

The General Autoregressive Process

Model formulation of a general AR, namely AR(p), process is as the following:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

where we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$.

The general Yule-Walker equations for this process are

$$\begin{aligned}
\rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \cdots + \phi_p\rho_{p-1} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \cdots + \phi_p\rho_{p-2} \\
&\vdots \\
\rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \cdots + \phi_p
\end{aligned}$$

Given numerical values for $\phi_1, \phi_2, \ldots, \phi_p$, these linear equations can be solved to obtain numerical values for $\rho_1, \rho_2, \ldots, \rho_p$. And the following equation is used to find variance of general AR(p) process:

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}.$$

The autocorrelation at lag k will be a linear combination of exponentially decaying terms and damped sine wave terms.

The following App displays time series, ACF and partial-ACF plots for general AR(p) process up to $p = 4$.

App.3.2: Time series, ACF, and PACF plots for AR(p) processes.

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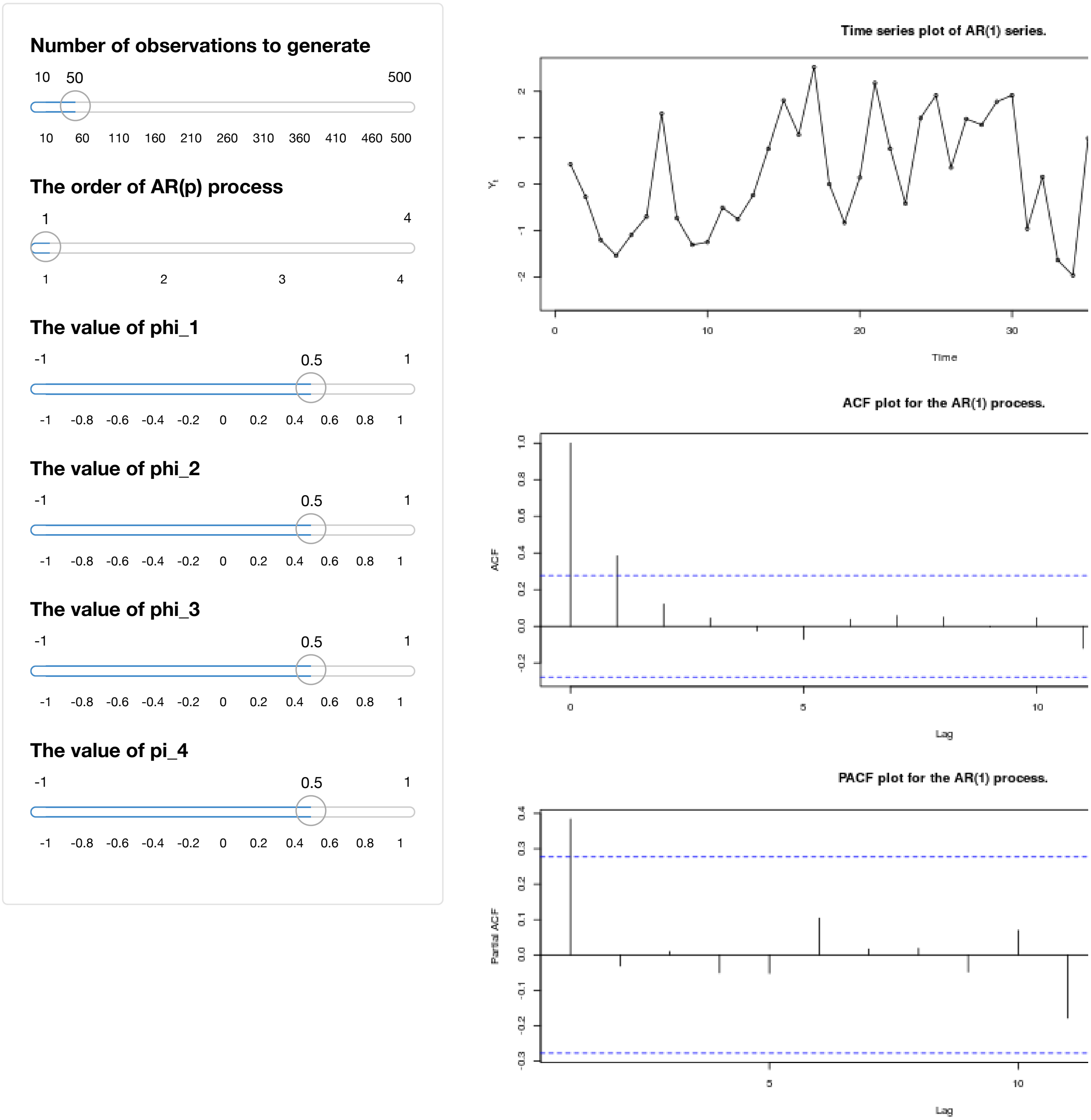


ACF of AR(p) process tails of as a mixture of exponential decays or damped sine waves, which appear if some roots of difference equation are complex.

PACF of AR(p) process will vanish after lag p.

The Autoregressive Moving Average Model

When we assume that the series is partly autoregressive and partly moving average, we obtain a quite general ARMA(p,q) time series model.

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.$$

An ARMA(p.q) process is an autoregressive moving average process of orders p and q.

The ARMA(1,1) Model

When we set $p = 1$ and $q = 1$ in the general formulation of ARMA(1,1) model, we get the following model:

$$Y_t = \phi Y_{t-1} + e_t - \theta_1 e_{t-1}.$$

We have the following characteristics for the AMRA(1,1) model:

$$\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k &= \phi\gamma_{k-1}, \text{ for } k \geq 2. \end{aligned}$$

Correspondingly, we obtain the following for variance of the process

$$\gamma_0 = \frac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_e^2$$

and for the autocorrelation at lag k

$$\rho_k = \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\phi\theta + \theta^2}\phi^{k-1}$$

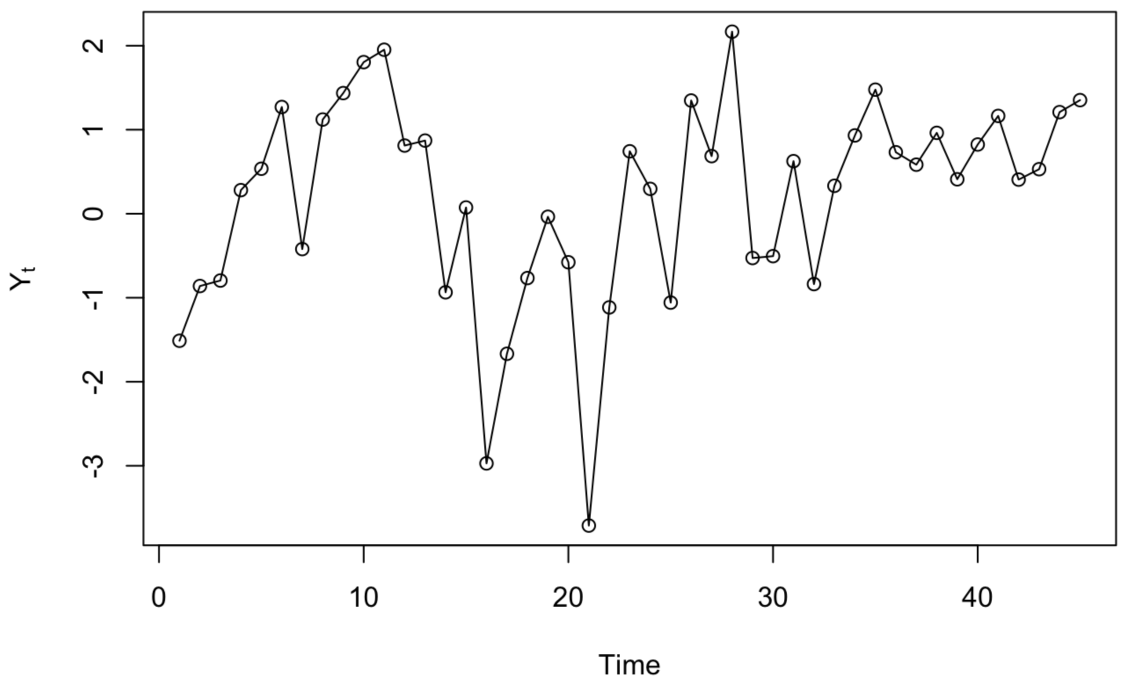
for $k \geq 1$.

The autocorrelation function of ARMA(1,1) process decays exponentially as the lag k increases. The damping factor is ϕ, but the decay starts from initial value ρ_1, which also depends on θ.

Figure 19 and 20 respectively show the time series and the ACF plots of a simulated ARMA(1,1) process with $\phi = 0.7$ and $\theta = -0.2$.

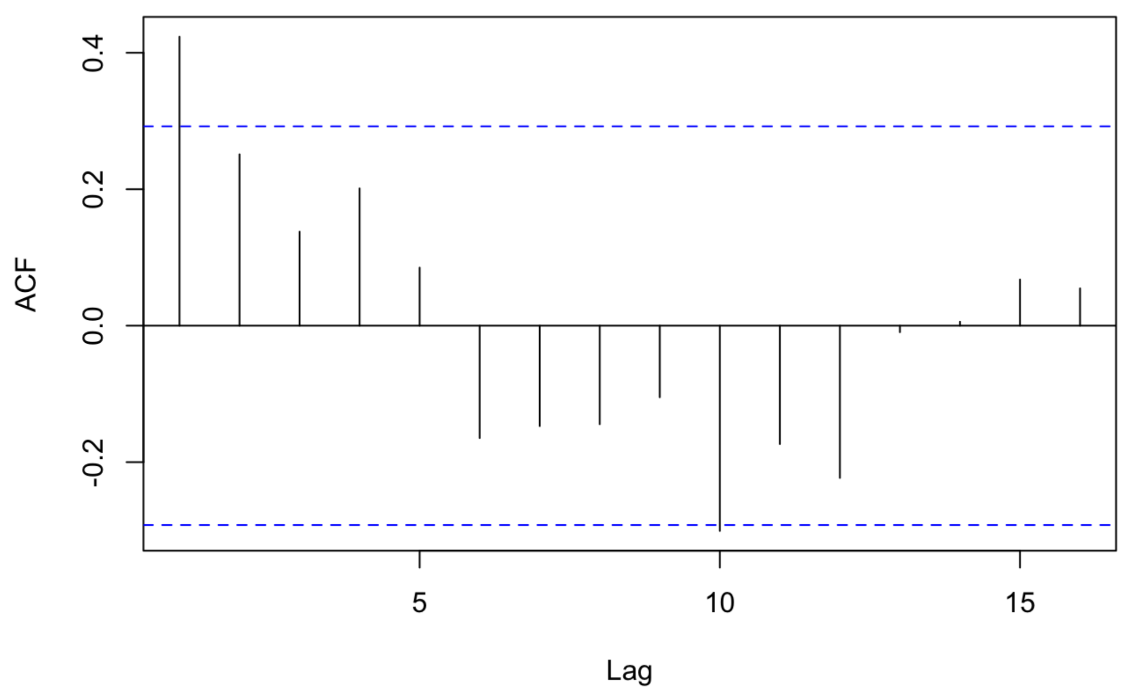
```
sim.data = arima.sim(list(order = c(1,0,1), ar = 0.7, ma =-0.2 ), n = 45)
arma1.1 = ts(sim.data)
plot(arma1.1,ylab=expression(Y[t]),type='o',main=paste0("Figure 19. Time series plot o
        f ARMA(1,1) series."))
```

Figure 19. Time series plot of ARMA(1,1) series.




```
acf(arma1.1,main="Figure 20. ACF plot for the simulated ARMA(1,1) process.")
```

Figure 20. ACF plot for the simulated ARMA(1,1) process.



The following App displays time series, ACF and partial-ACF plots for general ARMA(p,q) process up to $p = 4$ and $q = 4$.

App.3.3: Time series, ACF, and PACF plots for ARMA(p,q) processes.

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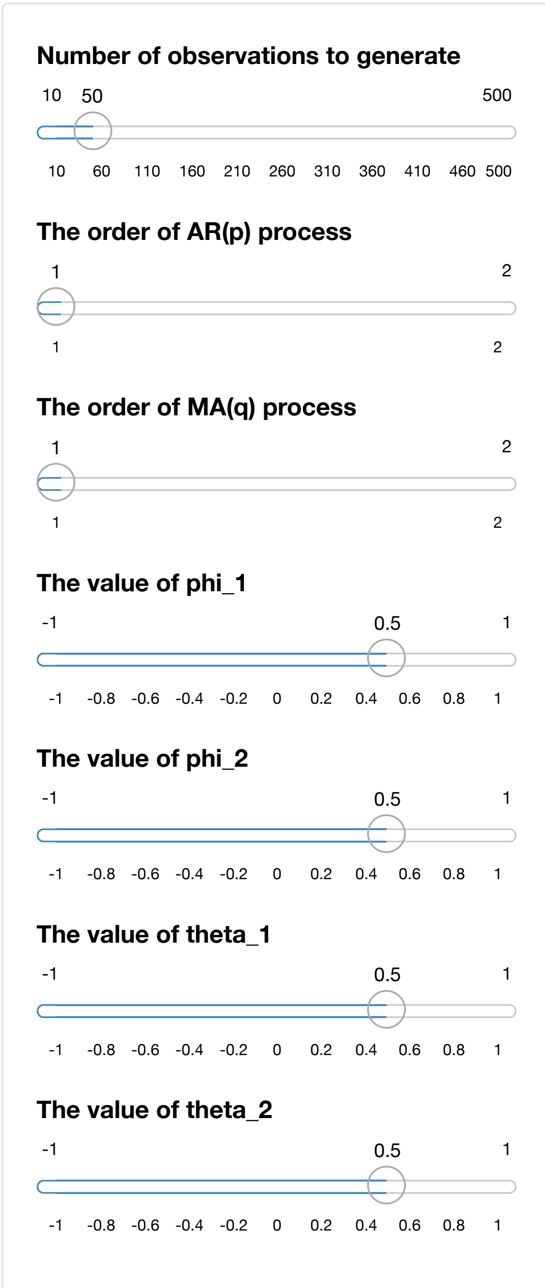

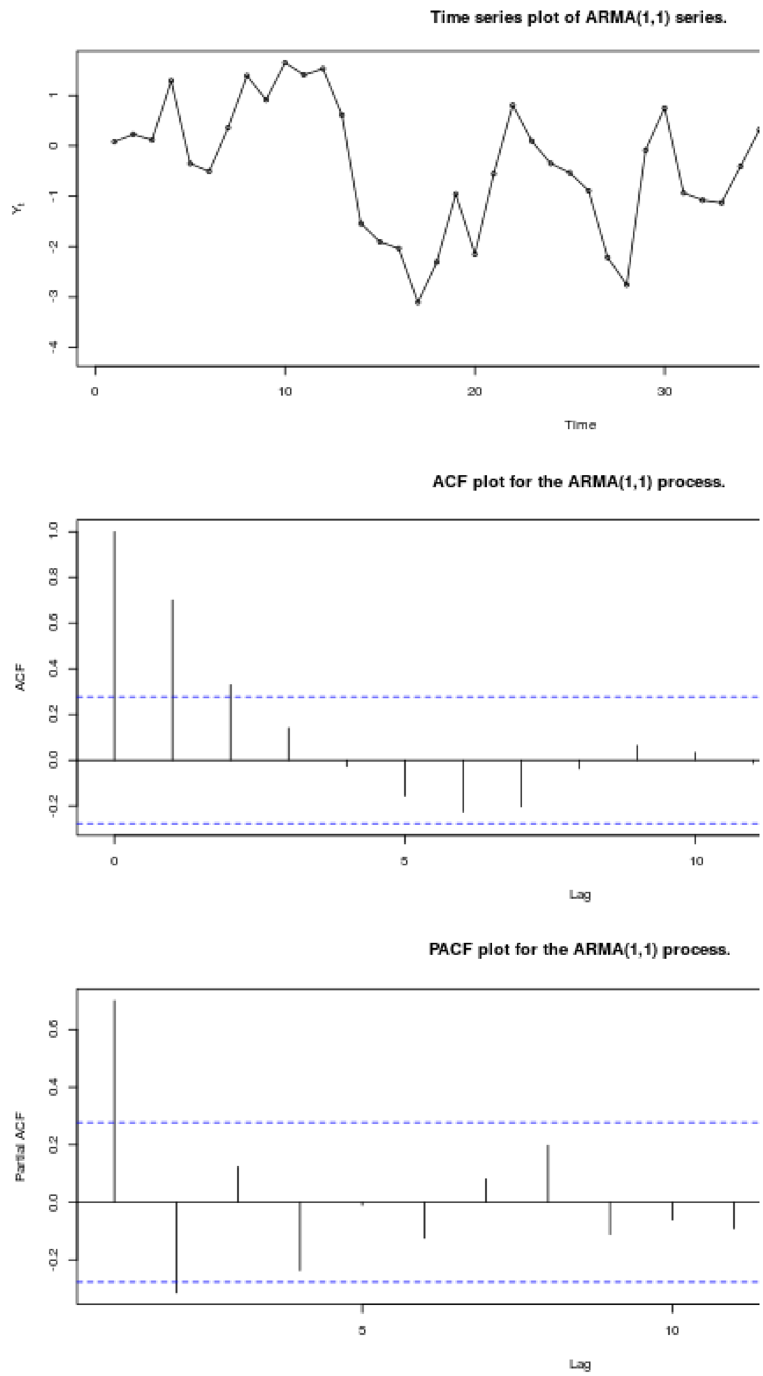

Summary

In this module, we studied basic autoregressive (AR), moving average (MA) and mixed autoregressive moving average (ARMA) processes and their main characteristics. It is important to understand the autocorrelation properties of these models and the various representations of the models to identify models in practice.