ラプラス変換

ラプラス変換の定義

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$f(t)$ を原関数，$F(s)$ を像関数という.

・$f(t)$ のラプラス変換を行うことを，$\mathcal{L}[f(t)] = F(s)$ と書く.

原関数（時間の関数）

$$f(t) \xrightarrow{\text{ラプラス変換}} F(s)$$

像関数

s は複素数

$s = \alpha + i\beta$

・ラプラス変換の利点
- 難しい関数を簡単な関数に変換できる.
  - ↳ 微分方程式を解くのが楽になる.
- 制御系や電気系の専門科目では必須!!

次ページから重要な公式の一覧と，その導出を行います.

・ $f(t)=1$ のラプラス変換

$$F(s)=\mathcal{L}[f(t)]=\mathcal{L}[1]$$

$$=\int_0^\infty 1\cdot e^{-st}\,dt$$

$$=\left[-\frac{1}{s}\cdot e^{-st}\right]_0^\infty$$

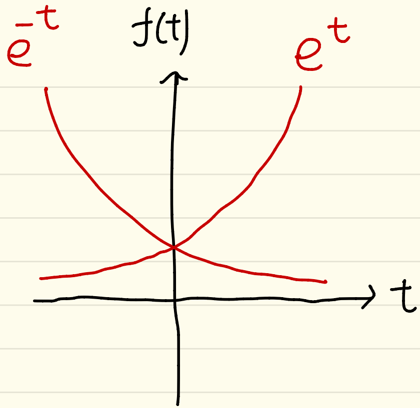
$$=\left(-\frac{1}{s}\cdot e^{-\infty}\right)-\left(-\frac{1}{s}\cdot e^{0}\right)$$

ここで、$\lim_{t\to\infty} e^{-t}=0$

したがって、与式 $=-\left(-\frac{1}{s}\cdot e^{0}\right)=\frac{1}{s}$

(2)

$\mathcal{L}[t]$



※まとめ　$\mathcal{L}[1]=\frac{1}{s}$

・ $f(t)=t$ のラプラス変換

$$F(s)=\mathcal{L}[f(t)]=\mathcal{L}[t]$$

$$=\int_0^{\infty} t\cdot e^{-st}\,dt$$

$$=\int_0^{\infty} t\cdot\left(-\frac{1}{s}e^{-st}\right)'dt$$

部分積分

$$=\underbrace{\left[-\frac{t}{s}e^{-st}\right]_0^{\infty}}_{=0}-\int_0^{\infty}-\frac{1}{s}\cdot e^{-st}\,dt$$

ここで、$\displaystyle\lim_{t\to\infty} t\,e^{-st}=\lim_{t\to\infty}\frac{t}{e^{st}}=\lim_{t\to\infty}\frac{1}{s\,e^{st}}=0$ //

ロピタルの定理 →

したがって、(与式) $\displaystyle=+\frac{1}{s}\int_0^{\infty} e^{-st}\,dt$

$$=+\frac{1}{s}\mathcal{L}[1]$$

$$=\frac{1}{s^2}$$ //

※まとめ

$$\mathcal{L}[t]=\frac{1}{s^2}$$ //

・ $f(t) = e^{\alpha t}$ のラプラス変換

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[e^{\alpha t}]$$

$$= \int_0^\infty e^{\alpha t} \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-\alpha)t}\, dt$$

ここで、$\int_0^\infty e^{-st} dt = \frac{1}{s}$,

つまり、$\int_0^\infty e^{-(s-0)t} dt = \frac{1}{s-0}$ より、

$$(与式) = \int_0^\infty e^{-(s-\alpha)t}\, dt = \frac{1}{s-\alpha}$$

※ まとめ

$$\mathcal{L}[e^{\alpha t}] = \frac{1}{s-\alpha}$$

$$\int (2x-1)e^x\,dx$$
$$= \int (2x-1)\cdot(e^x)'\,dx$$
$$= (2x-1)e^x - \int 2e^x\,dx$$
$$= (2x-1)e^x - 2e^x$$
$$= e^x(2x-3)$$

$$\int (x+1)\log x\,dx$$
$$= \int \left(\tfrac{1}{2}x^2+x\right)'\cdot \log x\,dx$$
$$= \left(\tfrac{1}{2}x^2+x\right)\log x - \int \left(\tfrac{1}{2}x^2+x\right)\cdot\frac{1}{x}\,dx$$
$$= \log x\left(\tfrac{1}{2}x^2+x\right) - \int \tfrac{1}{2}x+1\,dx$$
$$= \log x\left(\tfrac{1}{2}x^2+x\right) - \tfrac{1}{4}x^2+x$$

$$\int \frac{\log x}{x^2}\,dx = \int \frac{1}{x^2}\cdot \log x\,dx$$
$$= \int \left(-\frac{1}{x}\right)' \log x\,dx$$
$$= -\frac{\log x}{x} - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx$$
$$= -\frac{\log x}{x} + \int \frac{1}{x^2}\,dx$$
$$= -\frac{\log x}{x} - \frac{1}{x} = -\frac{(\log x+1)}{x}$$

$$\int x^2(\log x)^2\,dx$$
$$= \int \left(\tfrac{1}{3}x^3\right)'(\log x)^2\,dx$$
$$= \tfrac{1}{3}x^3(\log x)^2 - \int \tfrac{1}{3}x^3\cdot 2(\log x)\cdot\frac{1}{x}\,dx$$
$$= \tfrac{1}{3}x^3(\log x)^2 - \tfrac{2}{3}\int x^2\cdot \log x\,dx$$

もう一回部分積分

sint のラプラス変換

$$\mathcal{L}[\sin t] = \boxed{\int_0^\infty \sin t \, e^{-st} dt}$$

$$\mathcal{L}[\sin t] = F(s)$$

$$F(s) = \int_0^\infty \sin t \left(-\frac{1}{s}e^{-st}\right)' dt$$

$$F(s) = \left[\sin t \cdot -\frac{1}{s}e^{-st}\right]_0^\infty - \int_0^\infty \cos t \cdot -\frac{1}{s}e^{-st} dt$$

$$〃 = \frac{1}{s}\int_0^\infty \cos t \cdot e^{-st} dt$$

$$〃 = \frac{1}{s}\int_0^\infty \cos t \cdot \left(-\frac{1}{s}e^{-st}\right)' dt$$

$$〃 = \frac{1}{s}\left\{\left[\cos t \cdot -\frac{1}{s}e^{-st}\right]_0^\infty + \frac{1}{s}\boxed{\int_0^\infty \sin t \cdot e^{-st} dt}\right\}$$

$$〃 = \frac{1}{s}\left\{\frac{1}{s} + \frac{1}{s}F(s)\right\}$$

$$F(s) = \frac{1}{s}\left(\frac{1}{s} + \frac{1}{s}F(s)\right),$$

$$F(s) = \frac{1}{s^2} + \frac{1}{s^2}F(s)$$

$$F(s) - \frac{1}{s^2}F(s) = \frac{1}{s^2}$$

$$\frac{s^2-1}{s^2}F(s) = \frac{1}{s}$$

$$F(s) = \frac{1}{s^2-1}$$

$\cos t$

$\mathcal{L}[\cos t]$

$$F(s) = \int_0^\infty \cos t \cdot e^{-st}\, dt$$

$$= \int_0^\infty \cos t \cdot \left(-\frac{1}{s} e^{-st}\right)' dt$$

$$= \left[\cos t \cdot -\frac{1}{s} e^{-st}\right]_0^\infty + \frac{1}{s}\int \sin t\, e^{-st}\, dt$$

$$= \frac{1}{s} + \frac{1}{s}\left(\int_0^\infty \sin t \cdot \left(-\frac{1}{s} e^{-st}\right) dt\right)$$

$$= \frac{1}{s} + \frac{1}{s}\left(\left[\sin t \cdot -\frac{1}{s} e^{-st}\right]_0^\infty - \frac{1}{s}\int \cos t\, e^{-st}\, dt\right)$$

$$F(s) = \frac{1}{s} + \frac{1}{s}\left(-\frac{1}{s} F(s)\right)$$

$$F(s) = \frac{1}{s} - \frac{1}{s^2} F(s), \quad F(s) + \frac{1}{s^2} F(s) = \frac{1}{s}$$

$$\frac{s^2+1}{s^2} F(s) = \frac{1}{s}$$

$$F(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}[\sin wt]$$

$$= \int_0^\infty \sin wt \cdot e^{-st}\, dt$$

$$= \int_0^\infty \sin wt \cdot \left(-\frac{1}{s} e^{-st}\right)' dt$$

$$= \left[\sin wt \cdot \left(-\frac{1}{s} e^{-st}\right) \right]_0^\infty - \int_0^\infty w \cos wt \cdot -\frac{1}{s} e^{-st}\, dt$$

$$= + \frac{w}{s} \int_0^\infty \cos wt \cdot e^{-st}\, dt$$

$$= \frac{w}{s} \left\{ \int_0^\infty \cos wt \cdot \left(-\frac{1}{s} e^{-st}\right)' dt \right\}$$

$$= \frac{w}{s} \left\{ \left[\cos wt \cdot -\frac{1}{s} e^{-st} \right]_0^\infty - \int_0^\infty -w \sin wt \cdot -\frac{1}{s} e^{-st}\, dt \right\}$$

$$= \frac{w}{s} \left\{ +\frac{1}{s} + \frac{w^2}{s} - \int_0^\infty \sin wt \cdot e^{-st}\, dt \right\}$$

$$= \frac{w}{s} \left(\frac{1}{s} - \frac{w^2}{s} F(s) \right)$$

$$F(s) = \frac{w}{s^2} - \frac{w^2}{s^2} F(s)$$

$$F(s) + \frac{w^2}{s^2} F(s) = \frac{w}{s^2}$$

$$\left(\frac{s^2 + w^2}{s^2} \right) F(s) = \frac{w}{s^2}$$

$$F(s) = \frac{w}{s^2 + w^2}$$

$$\mathcal{L}[e^{\alpha t}\sin\beta t]$$

$$F(s) = \int_0^\infty e^{-(s-\alpha)t}\sin\beta t \, dt$$

$$= \int_0^\infty \left\{-\frac{1}{(s-\alpha)} e^{-(s-\alpha)t}\right\}' \sin\beta t \, dt$$

$$= \left[-\frac{1}{(s-\alpha)} e^{-(s-\alpha)t} \sin\beta t\right]_0^\infty + \frac{\beta}{s-\alpha}\int_0^\infty \cos\beta t \cdot e^{-(s-\alpha)t} dt$$

$$= 0 + \frac{\beta}{s-\alpha}\int_0^\infty \left\{-\frac{1}{(s-\alpha)} e^{-(s-\alpha)t}\right\}' \cos\beta t \, dt$$

$$= \frac{\beta}{s-\alpha}\left\{\left[-\frac{1}{(s-\alpha)} e^{-(s-\alpha)t}\cos\beta t\right]_0^\infty - \frac{\beta}{(s-\alpha)}\int_0^\infty \sin\beta t \cdot e^{-(s-\alpha)t} dt\right\}$$

$$F(s) = \frac{\beta}{s-\alpha}\left\{\frac{1}{s-\alpha} - \frac{\beta}{s-\alpha}F(s)\right\}$$

$$F(s) = \frac{\beta}{(s-\alpha)^2} - \frac{\beta^2}{(s-\alpha)^2}F(s)$$

$$F(s) + \frac{\beta^2}{(s-\alpha)^2}F(s) = \frac{\beta}{(s-\alpha)^2}$$

$$\frac{(s-\alpha)^2+\beta^2}{(s-\alpha)^2}F(s) = \frac{\beta}{(s-\alpha)^2}$$

$$F(s) = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

※ まとめ

$$\mathcal{L}[e^{\alpha t}\sin\beta t] = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

$$\mathcal{L}[e^{\alpha t}\cos\beta t]$$

$$\mathcal{L}[e^{\alpha t}f(t)] = F(s-\alpha)$$

$$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$\mathcal{L}[\cos t] = \frac{s}{s^2+1}$$

$$\mathcal{L}[\cos\beta t] = \frac{1}{\beta}\cdot\frac{\frac{s}{\beta}}{\left(\frac{s}{\beta}\right)^2+1}$$

$$= \frac{\frac{s}{\beta}}{\frac{s^2}{\beta^2}+1}\cdot\frac{1}{\beta}$$

$$= \frac{s}{s^2+\beta^2} \,/\!/$$

$$\mathcal{L}[e^{\alpha t}\cos\beta t]$$

$$= \frac{s-\alpha}{(s-\alpha)^2+\beta^2} \,/\!/$$

※まとめ

$$\mathcal{L}[e^{\alpha t}\cos\beta t]$$

$$= \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$$

$\mathcal{L}[t\sin\omega t]$

$$\mathcal{L}[tf(t)] = -F'(s)$$

① $\mathcal{L}[\sin\omega t] = \dfrac{\omega}{s^2+\omega^2}$

$$\mathcal{L}[t\sin\omega t] = -\left(\frac{\omega}{s^2+\omega^2}\right)' = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\left(\frac{\omega}{s^2+\omega^2}\right)' = \frac{0-2s\omega}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$\mathcal{L}[t \cos \omega t]$

① $\mathcal{L}[\cos \omega t] = \dfrac{s}{s^2+\omega^2}$

$$\left(\frac{s}{s^2+\omega^2}\right)' = \frac{s^2+\omega^2-2s^2}{(s^2+\omega^2)^2} = \frac{\omega^2-s^2}{(s^2+\omega^2)^2}$$

② $\mathcal{L}[t f(t)] = -F'(s)$ より

$$\mathcal{L}[t \cos \omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

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(1) $3t^2-4t+5$

$$\mathcal{L}[3t^2-4t+5]$$

$$= 3\mathcal{L}[t^2]-4\mathcal{L}[t]+5\mathcal{L}[1]$$

$$= \frac{3\cdot 2!}{s^3} - 4\cdot\frac{1}{s^2} + \frac{5}{s}$$

$$= \frac{6}{s^3} - \frac{4}{s^2} + \frac{5}{s}$$

(2) te^{2t}

※ $\mathcal{L}[tf(t)] = -F'(s)$

$\mathcal{L}[e^{2t}] = \dfrac{1}{s-2}$ より、

微分してマイナス

$\mathcal{L}[te^{2t}] = \dfrac{+1}{(s-2)^2}$

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$f(t)$ をラプラス変換せよ

→ $\mathcal{L}[f(t)] = F(s)$

$F(s)$ を逆ラプラス 〃

→ $\mathcal{L}^{-1}[F(s)] = f(t)$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$

$\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos t$

① 例題12

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(1) $\mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}$

(2) $\mathcal{L}^{-1}\left[\frac{1}{(s-\alpha)^2}\right] = t\,e^{\alpha t}$

$\mathcal{L}[e^{2t}] = \frac{1}{s-2}$, $\mathcal{L}[t] = \frac{1}{s^2}$

$\mathcal{L}[t \cdot e^{2t}] = \frac{1}{(s-2)^2}$

$$\frac{1}{(s-a)(s-b)} = \frac{1}{b-a}\left(-\frac{1}{s-a} + \frac{1}{s-b}\right)$$

(3) $\mathcal{L}^{-1}\left[\frac{1}{(s-\alpha)(s-b)}\right]$

$$\frac{1}{(s-\alpha)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

部分分数分解

$$= \frac{A(s-b)+B(s-a)}{(s-a)(s-b)}$$

$$= \frac{As - Ab + Bs - aB}{(s-a)(s-b)}$$

$$= \frac{\overset{0}{(A+B)}s + (-Ab - Ba)}{(s-a)(s-b)}$$

$$\begin{cases} A+B=0 \\ -Ab-Ba=1 \end{cases}$$

$$\begin{array}{rl} & Ab + Bb = 0 \\ + & -Ab - Ba = 1 \\ \hline & Bb - Ba = 1 \end{array}$$

$(b-a)B = 1$

$B = \frac{1}{b-a}$, $A = -\frac{1}{b-a}$

$(与式) = \frac{1}{b-a}\mathcal{L}^{-1}\left[\frac{1}{s-b} - \frac{1}{s-a}\right]$

$= \frac{1}{b-a}\left(e^{bt} - e^{at}\right)$

$= \frac{1}{a-b}\left(e^{at} - e^{bt}\right)$

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(4) $\mathcal{L}^{-1}\left[\dfrac{1}{(s-a)^2+b^2}\right]$

$\mathcal{L}[\sin t] = \dfrac{1}{s^2+1}$

$\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$

$\dfrac{1}{b^2} \cdot \dfrac{1}{\dfrac{(s-a)^2}{b^2}+1}$

$\mathcal{L}[f(bt)] = \dfrac{1}{b} F\left(\dfrac{s}{b}\right)$

$\Downarrow$

$\dfrac{1}{b} \cdot \boxed{\dfrac{1}{b} \cdot \dfrac{1}{\dfrac{(s-a)^2}{b^2}+1}}$

逆ラプラス

$e^{at} \sin bt$

$\underline{\dfrac{1}{b} e^{at} \sin bt}$

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(1) $$\frac{1}{s^2-s-2} = \frac{1}{(s-2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1}$$

$$= \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

$$= \frac{As+A+Bs-2B}{(s-2)(s+1)}$$

$$= \frac{(A+B)s+(A-2B)}{(s-2)(s+1)}$$

$$\begin{cases} A+B=0 \\ A-2B=1 \end{cases}$$

$$3B = 1$$

$$B = \frac{1}{3}$$

$$A = -\frac{1}{3}$$

$$\frac{1}{s^2-s-2} = +\frac{1}{3}\left(\frac{1}{s+1} - \frac{1}{s-2}\right)$$

$$\mathcal{L}^{-1}\left[\frac{1}{3}\left(\frac{1}{s+1} - \frac{1}{s-2}\right)\right]$$

$$= \frac{1}{3}\left(e^{-t} - e^{2t}\right)$$

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(1) $\dfrac{s-1}{s^2-2s+5} = \dfrac{(s-1)}{(s-1)^2+(2)^2} = \dfrac{\left(\frac{s-1}{2}\right)}{\left(\frac{s-1}{2}\right)^2+1} \cdot \dfrac{1}{2}$ より

$s-1 \Rightarrow e^t$

形 $\Rightarrow \cos 2t$

$\mathcal{L}^{-1}[\quad]$

$= e^t \cos 2t$

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部分分数分解

(2) $$\frac{s-3}{s^2-8s+16} = \frac{s-3}{(s-4)^2} = \frac{A}{s-4} + \frac{B}{(s-4)^2}$$

$$= \frac{As-4A+B}{(s-4)^2}$$

$$= \frac{As+(B-4A)}{(s-4)^2}$$

$A = 1$

$B-4A = -3$

$B-4 = -3$

$B = 1$

$$\frac{1}{s-4} + \frac{1}{(s-4)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-4} + \frac{1}{(s-4)^2}\right] = e^{4t} + te^{4t}$$

$$= (1+t)e^{4t}$$ //

$$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$= \frac{A(s-1)(s-2) + B(s-2)s + C(s-1)s}{s(s-1)(s-2)}$$

$$= \frac{A(s^2-3s+2) + B(s^2-2s) + C(s^2-s)}{''}$$

$$= \frac{As^2 - 3As + 2A + Bs^2 - 2Bs + Cs^2 - Cs}{''}$$

$$= \frac{(A+B+C)s^2 + (-3A-2B-C)s + (2A)}{''}$$

$$\begin{cases} A+B+C = 0 \\ -3A-2B-C = 0 \\ 2A = 1 \end{cases}$$

$$\underline{A = \frac{1}{2}}_{/\!/}$$

$$B + C = -\frac{1}{2}$$

$$-2B - C = \frac{3}{2}$$

$$+ \overline{\quad -B = 1 \quad}$$

$$\underline{B = -1}_{/\!/}$$

$$-1 + C = -\frac{1}{2}$$

$$C = -\frac{1}{2} + 1$$

$$\underline{= \frac{1}{2}}_{/\!/}$$

$$\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s-2}$$

$$\mathcal{L}^{-1}$$

$$\frac{1}{2} - e^{t} + \frac{1}{2}e^{2t}$$

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+7=0$$

微分方程式 → ラプラス変換 → 代数方程式

$$s^3+2s^2+5s+4=0$$

- 変数分離
- 同次形
- 特性方程式

3年生

特殊解

$$x=5t$$

逆ラプラス変換

解く

$$\frac{5}{s^2}$$

$$\frac{dx}{dt} + x = e^t, \quad x(0) = 1$$

$$\mathcal{L}[x(t)] = X(s), \quad \mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0)$$

$$sX(s) - x(0) + X(s) = \frac{1}{s-1}$$

$$(s+1)X(s) - x(0) = \frac{1}{s-1}$$

$$(s+1)X(s) = \frac{1}{s-1} + 1$$

$$〃 = \frac{1}{s-1} + \frac{s-1}{s-1}$$

$$〃 = \frac{s}{s-1}$$

$$X(s) = \frac{s}{(s+1)(s-1)}$$

部分分数分解

$$X(s) = \frac{A}{s+1} + \frac{B}{s-1}$$

$$= \frac{As - A + Bs + B}{(s+1)(s-1)}$$

$$= \frac{(A+B)s + (B-A)}{(s+1)(s-1)}$$

$$A + B = 1,$$

$$+\ \ -A + B = 0$$

$$2B = 1$$

$$, \quad B = \frac{1}{2}$$

$$A = \frac{1}{2}$$

$$X(s) = \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s-1}\right)$$

逆ラプラス変換可能な形

⇓ 実行

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2}\left(e^{-t} + e^{t}\right)$$

・ラプラス変換を用いて微分方程式を解く

問1

(1) $\frac{dx}{dt} = 2x$, $x(0) = 1$

・変数分離形を用いて解く!

$\frac{dx}{dt} = 2x$, $dx = 2x\ dt$

↓ 2xで両辺を割る

$\frac{1}{2x}\ dx = 1\ dt$ (この書き方は正しくない)

$\int \frac{1}{2x} dx = \int 1\ dt$

$\frac{1}{2} \int \frac{1}{x} dx = \int 1\ dt$

$\frac{1}{2} \lg x = t + c$ (cは定数)

$\lg x = 2t + C$

$x = e^{2t+C} = e^{2t} \cdot e^{C}$

$= A e^{2t}$ (Aは定数)

一般解

$x(0) = 1$ より

$x(t) = Ae^{2t}$

$x(0) = A$

$= 1$

$A = 1$

$x(t) = e^{2t}$

特殊解

・ラプラス変換を用いて解く!

$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(0)$

$\mathcal{L}[f'(t)] = sF(s) - f(0)$

$\frac{dx}{dt} = 2x$

$\mathcal{L}\left[\frac{dx}{dt}\right] = 2\mathcal{L}[x]$

$sX(s) - 1 = 2X(s)$

$(s-2)X(s) = 1$

$X(s) = \frac{1}{s-2}$ となる。

したがって、微分方程式の解は、

$\mathcal{L}^{-1}[X(s)]$

$x(t) = e^{2t}$

利点

一発で特殊解が出る

$$\frac{d^2x}{dt^2} + 4x = e^{-t} \quad \left(t=0,\ x=0,\ \frac{dx}{dt}=0\right)$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] + 4\mathcal{L}[x] = \mathcal{L}[e^{-t}]$$

$$s^2X(s) - x(0)s - x'(0) + 4X(s) = \frac{1}{s+1}$$

$$s^2X(s) + 4X(s) = \frac{1}{s+1}$$

$$(s^2+4)X(s) = \frac{1}{s+1}$$

$$X(s) = \frac{1}{(s^2+4)(s+1)} \quad //$$

⇓ 部分分数分解

$$X(s) = \frac{As+B}{s^2+4} + \frac{C}{s+1} = \frac{(As+B)(s+1)+Cs^2+4C}{(s^2+4)(s+1)}$$

$$= \frac{As^2+As+Bs+B+Cs^2+4C}{(s^2+4)(s+1)}$$

$$\frac{(A+C)s^2+(A+B)s+(B+4C)}{(s^2+4)(s+1)}$$

$$\begin{cases} A+C=0, & A=-C \\ A+B=0, & A=-B \\ B+4C=1 \end{cases} \qquad B=C$$

$$5C=1,\quad C=\frac{1}{5}$$

$$B=\frac{1}{5} \qquad A=-\frac{1}{5}$$

$$X(s) = \frac{-\frac{1}{5}s+\frac{1}{5}}{s^2+4} + \frac{\frac{1}{5}}{s+1}$$

逆ラプラス変換できる形に変形する

$$= \frac{1}{5}\left(\frac{1-s}{s^2+4} + \frac{1}{s+1}\right)$$

$$= \frac{1}{5}\left(\frac{1}{2}\cdot\frac{2}{s^2+4} - \frac{s}{s^2+4} + \frac{1}{s+1}\right)$$

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{5}\left(\frac{1}{2}\sin 2t - \cos 2t + e^{-t}\right) \quad //$$

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(1) $\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = e^t$

$$s^2X(s) - x(0)s - x'(0) - 5sX(s) - 5x(0) + 6X(s) = \frac{1}{s-1}$$

$$(s^2 - 5s + 6)X(s) = \frac{1}{s-1}$$

① $X(s) = \frac{1}{(s-2)(s-3)(s-1)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} + \frac{C}{(s-1)}$

$$= \frac{A(s-1)(s-3) + B(s-1)(s-2) + C(s-2)(s-3)}{(s-1)(s-2)(s-3)}$$

$$= \frac{A(s^2-4s+3) + B(s^2-3s+2) + C(s^2-5s+6)}{}$$

$$= \frac{As^2 - 4As + 3A + Bs^2 - 3Bs + 2B + Cs^2 - 5Cs + 6C}{}$$

$$= \frac{(A+B+C)s^2 + (-4A-3B-5C)s + (3A+2B+6C)}{}$$

$$\begin{cases} A+B+C=0 \Rightarrow C=-A-B \\ -4A-3B-5C=0 \\ 3A+2B+6C=1 \end{cases}$$

$$\begin{cases} -4A-3B+5A+5B=0 \\ 3A+2B-6A-6B=1 \end{cases}$$

$$\begin{cases} A+2B=0 \\ -3A-4B=1 \end{cases} \qquad 2B=1 \qquad B=\frac{1}{2}$$

$$\begin{cases} 2A+4B=0 \\ -3A-4B=1 \end{cases} \qquad C = 1-\frac{1}{2} = \frac{1}{2}$$

$$-A = 1 \qquad A = -1$$

$$\frac{-1}{s-2} + \frac{1}{2}\frac{1}{s-3} + \frac{1}{2}\frac{1}{s-1}$$

$$\mathcal{L}^{-1}[F(s)]$$

$$= -e^{2t} + \frac{1}{2}e^{3t} + \frac{1}{2}e^t$$

2

(2) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 5x = 0$

$s^2F(s) - sf(0) - f'(0) - 2sF(s) - f(0) + 5F(s) = 0$

$(s^2 - 2s + 5)F(s) - 1 = 0$

$F(s) = \frac{1}{s^2 - 2s + 5}$

$\mathcal{L}[\sin t] = \frac{1}{s^2 + 1}$

$\mathcal{L}[e^{at}\sin t] = \frac{1}{(s-a)^2 + 1}$

$\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2 + \omega^2}$

平方完成

変形してラプラス変換ができるように

$\frac{1}{(s-1)^2 + 4}$

$F(s) = \frac{1}{(s-1)^2 + 2^2} = \frac{1}{2} \cdot \frac{2}{(s-1)^2 + 2^2}$

$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2} \cdot e^t \sin 2t$

例題3 $\frac{d^2x}{dt^2}-x=0$, $x(0)=0$, $x(1)=1$

[解き方] ① $\frac{dx}{dt}=\alpha$ とおく.

② 両辺をラプラス変換

$$s^2F(s)-sf(0)-f'(0)-F(s)=0$$

$$s^2F(s)-\alpha-F(s)=0$$

$$(s^2-1)F(s)=\alpha$$

$$F(s)=\frac{\alpha}{s^2-1}=\frac{\alpha}{(s-1)(s+1)}=\alpha\left(\frac{A}{s-1}+\frac{B}{s+1}\right)$$

$$=\alpha\left(\frac{As+A+Bs-B}{(s-1)(s+1)}\right)$$

$$=\alpha\left(\frac{(A+B)s+(A-B)}{(s-1)(s+1)}\right)$$

$A+B=0$, $A-B=1$

$2A=1$

$A=\frac{1}{2}$, $B=-\frac{1}{2}$

$$F(s)=\alpha\left(\frac{1}{2}\cdot\frac{1}{s-1}-\frac{1}{2}\cdot\frac{1}{s+1}\right)$$

$$=\frac{\alpha}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

③ $\mathcal{L}^{-1}[F(s)]=\frac{\alpha}{2}\mathcal{L}^{-1}\left[\frac{1}{s-1}-\frac{1}{s+1}\right]$

$$=\frac{\alpha}{2}(e^t-e^{-t})$$

$f(t)=\frac{\alpha}{2}(e^t-e^{-t})$

$f(0)=0$, $f(1)=1$ を代入

$0=\frac{\alpha}{2}(1-1)=0$

$1=\frac{\alpha}{2}\left(e-\frac{1}{e}\right)$, $\frac{\alpha}{2}=\frac{1}{e-e^{-1}}$

$\alpha=\frac{2}{e-e^{-1}}$

④ 解

$$f(t)=\frac{\alpha}{2}(e^t-e^{-t})$$

$$=\frac{e^t-e^{-t}}{e-e^{-1}}$$

問3

(1) $\frac{d^2x}{dt^2} + \frac{dx}{dt} = 0$, $x(0)=2$, $x(1) = 1+e^{-1}$

$$s^2X(s) - s\,x(0) - x'(0) + sX(s) - x(0) = 0$$
$$s^2X(s) - 2s - \alpha + sX(s) - 2 = 0$$
$$(s^2+s)X(s) = 2s+2+\alpha$$

$$X(s) = \frac{2(s+1)+\alpha}{s(s+1)}$$
$$= \frac{2}{s} + \frac{\alpha}{s(s+1)}$$
$$= \frac{2}{s} + \alpha\left(\frac{1}{s(s+1)}\right) = \frac{2}{s} + \alpha\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$\frac{A}{s} + \frac{B}{s+1} = \frac{As+A+Bs}{s(s+1)} = \frac{(A+B)s+A}{s(s+1)}$$

$$A+B=0$$
$$\begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\mathcal{L}^{-1}[X(s)]$$
$$= x(t) = 2+\alpha(1-e^{-t})$$

$$\boxed{\begin{array}{ll} x(0)=2 \text{ より} & x(1) = 1+e^{-1} \text{ より} \\ 2 = 2+\alpha(1-1) & 1+e^{-1} = 2+\alpha(1-e^{-1}) \\ & \alpha(1-e^{-1}) = e^{-1}-1 \\ & \alpha = -1 \end{array}}$$

$$x(t) = 2-(1-e^{-t})$$
$$= 2-1+e^{-t}$$
$$= 1+e^{-t}$$
//

問3

(2) $\frac{d^2x}{dt^2}+x=1$, $x(0)=0$, $x\left(\frac{\pi}{2}\right)=0$

$$s^2X(s)-x(0)-x'(0)+X(s)=\frac{1}{s}$$

$$(s^2+1)X(s)=\frac{1}{s}+\alpha$$

$$X(s)=\frac{1}{s(s^2+1)}+\frac{\alpha}{s^2+1}$$

$$=\frac{A}{s}+\frac{Bs+C}{s^2+1}+\alpha\left(\frac{1}{s^2+1}\right)$$

$$=\frac{As^2+A+Bs^2+Cs}{s(s^2+1)}+\alpha\left(\frac{1}{s^2+1}\right)$$

$$=\frac{(A+B)s^2+Cs+A}{s(s^2+1)}+\alpha\left(\frac{1}{s^2+1}\right)$$

$A+B=0$

$C=0$

$\begin{cases} A=1 \\ B=-1 \end{cases}$

$$X(s)=\frac{1}{s}-\frac{s}{s^2+1}+\alpha\left(\frac{1}{s^2+1}\right)$$

$$x(t)=1-\cos t+\alpha\sin t$$

$$x\left(\frac{\pi}{2}\right)=1-\cos\frac{\pi}{2}+\alpha\sin\frac{\pi}{2}=0$$

$$=1-0+\alpha=0$$

$$\alpha=-1$$

$$x(t)=1-\cos t-\sin t \quad //$$