Dr. Babasaheb Ambedkar Marathwada University

Seminar on

A Simple Introduction to Integral Equations

Presented By

Ahmed Abdullah Hamoud Al Selwi

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1. Introduction

Background - What is an Integral Equation?

An *integral equation* is an equation in which the unknown function $u(x)$ appears under an integral sign. An *integral equation* are encountered in various fields of science and numerous applications in elasticity, plasticity, electrodynamics, electrical engineering, heat and mass transfer, economics, approximation theory, filtration theory, fluid dynamics, electrostatics, biomechanics, game theory, control, queuing theory, medicine, etc. The name integral equation was given by du Bois-Reymond in (1888). However, the Volterra integral equations can be derived from initial value problems. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name Volterra integral equation was first coined by Lalesco in 1908. Fredholm integral equations can be derived from boundary value problems. Erik Ivar Fredholm (1866 -1927) is best remembered for his work on integral equations and spectral theory [6, 8, 10]. .

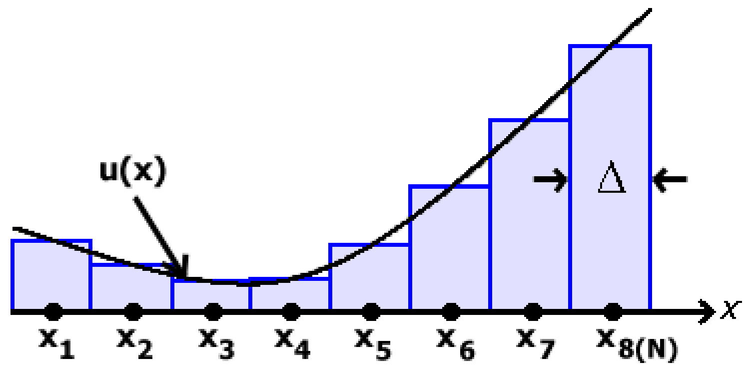
General Form of an Integral Equation

$$\alpha(x)u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt$$

λ	- A constant parameter
$g(x), h(x)$	- Limits determined by physical geometry
$u(x)$	- Unknown function to be determined
$k(x,t)$	- Known "kernel" (also called a *Green's Function*)
$\alpha(x), f(x)$	- Known forcing function

An Approximation Technique

- Difficulty: $u(x)$ possesses infinite number of unknowns
- Solution: discretize $u(x)$ using series of N weighted pulses
- Approximation reduces problem to solving for N unknowns

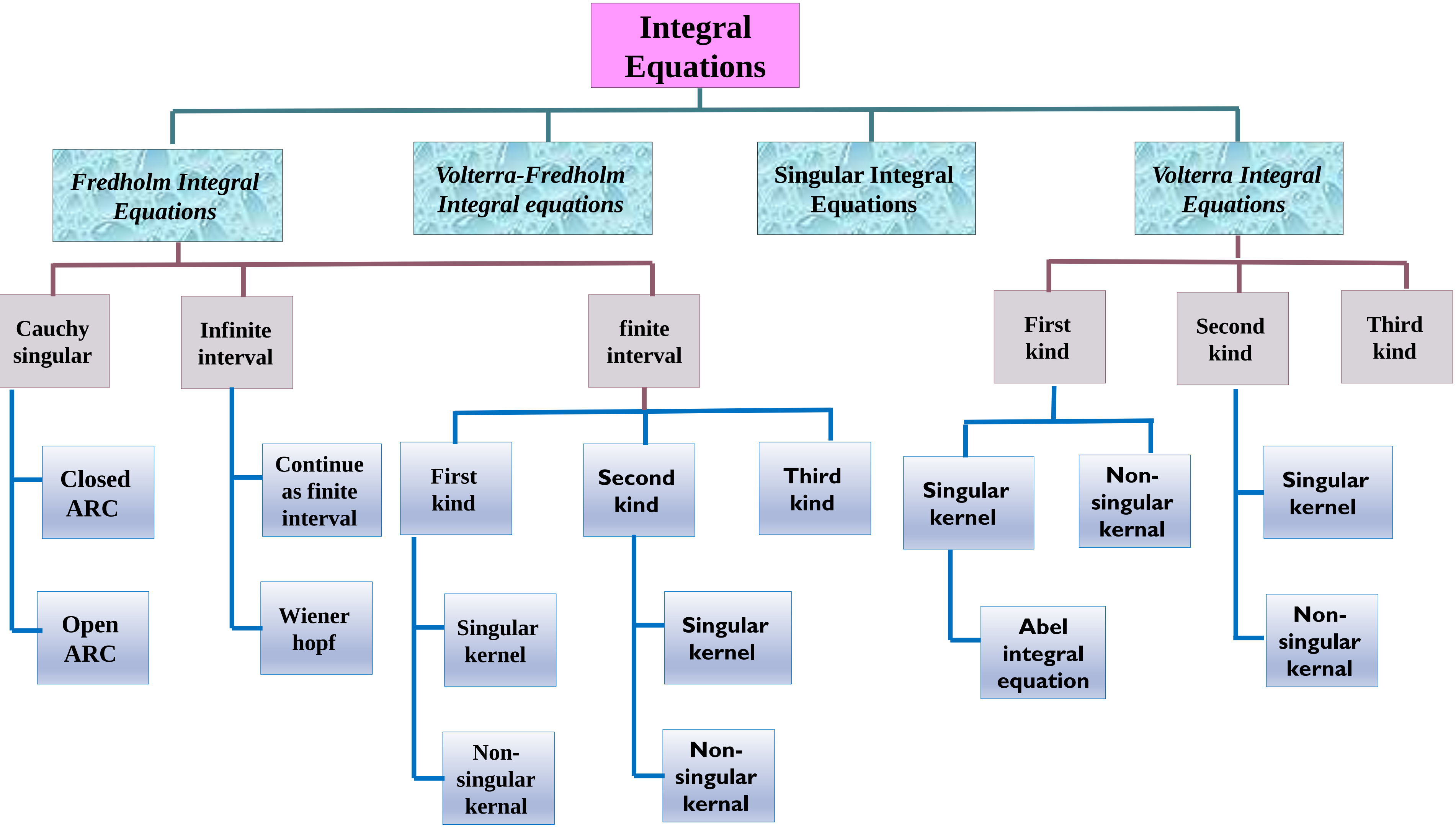


$$\Delta = \frac{\mathbf{b}-\mathbf{a}}{\mathbf{N}} \qquad \Pi_{\mathrm{n}}(\mathbf{x}) = \begin{cases} 1, \ \mathbf{x} \in \left(\mathbf{x}_{\mathrm{n}} - \frac{\Delta}{2}, \mathbf{x}_{\mathrm{n}} + \frac{\Delta}{2}\right) \\ 0, \ \textbf{otherwise} \end{cases}$$

$$\mathbf{u}(\mathbf{x}) \approx \sum_{1}^{\mathrm{N}} \mathbf{U}_{\mathrm{n}} \Pi_{\mathrm{n}}(\mathbf{x})$$

2. Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations [9].



The most general form of a linear integral equation:

$$\alpha(x)u(x) = f(x) + \lambda \int_{a}^{h(x)} k(x,t)u(t)dt$$

The type of integral equation can be determined via the following conditions:

$\alpha(x) = 0$	First Kind
$\alpha(x) = 1$	Second Kind
$\alpha(x) = z(x)$	Third Kind
$f(x) = 0$	Homogeneous
$f(x) \neq 0$	Nonhomogeneous
$h(x) = b$	Fredholm Integral Equation
$h(x) = x$	Volterra Integral Equation

2.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral sign in the form:

$$f(x)=\lambda\int_a^b K(x,\, t)u(t)dt,$$

This is called Fredholm integral equation of the *first kind*. However, for Fredholm integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x)=f(x)+\lambda\int_a^b K(x,\, t)u(t)dt,$$

2.2 Volterra Integral Equations

- In Volterra integral equations, at least one of the limits of integration is a variable. For the *first kind* Volterra integral equations, the unknown function *u*(*x*) appears only inside integral sign in the form:

$$f(x)=\lambda\int_a^x K(x,\ t)u(t)dt,$$

- However, Volterra integral equations of the *second kind*, the unknown function *u*(*x*) appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x)=f(x)+\lambda\int_a^x K(x,\ t)u(t)dt,$$

2.3 Volterra-Fredholm Integral equations

The Volterra-Fredholm integral equations [6,7] arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x)=f(x)+\lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt,$$

and

$$u(x,t) = f(x,t)+\lambda \int_a^t \int_\Omega F\big(x,t,\zeta,\tau,u(\zeta,\tau)\big)d\zeta d\tau$$

where $f(x, t)$ and $F(x, t, \xi, \tau, u(\xi, \tau))$ are analytic functions on $D = \Omega\times[a, T]$, and Ω is a closed subset of R^n, $n = 1, 2, 3$. It is interesting to note that contains disjoint Volterra and Fredholm integral equations, whereas contains mixed Volterra and Fredholm integral equations.

Moreover, the unknown functions $u(x)$ and $u(x, t)$ appear inside and outside the integral signs, this is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs is a first kind integral equation.

2.4. Singular Integral Equations

Volterra integral equations of the first kind or of the second kind are called singular if one of the limits of integration $g(x)$, $h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x, t)$ becomes unbounded at one or more points in the interval of integration.

No	Name of equation	form
1	Fredholm - second kind	$u(x)=f(x)+\lambda\int_a^b K(x,t)u(t)dt,$
2	Fredholm - first kind	$f(x)=\lambda\int_a^b K(x,t)u(t)dt,$
3	Fredholm – third kind	$a(x)u(x)=f(x)+\lambda\int_a^b K(x,t)u(t)dt,$
4	Winer hopf	$u(x)=f(x)+\lambda\int_0^\infty K(x-t)u(t)dt,$
5	volterra - second kind	$u(x)=f(x)+\lambda\int_a^x K(x,t)u(t)dt,$
6	volterra - first kind	$f(x)=\lambda\int_a^x K(x,t)u(t)dt,$
7	Abel equation	$u(x)=\int_0^x \frac{u(s)}{(x-s)^{\alpha}}ds, 0<\alpha<1$

Some Integral Equation Examples

$$y(t) = \int_0^t \frac{2s - y(s)}{1 - s^2} ds$$ Volterra **second** kind

$$y(t) = \sqrt{t} - \int_0^t 2\sqrt{ts}\, y(s)\, ds$$ Volterra **second** kind

$$y(t) = \sqrt{t} - \int_0^1 2\sqrt{ts}\, y(s)\, ds$$ Fredholm **second** kind

$$\int_0^t \frac{y(s)\, ds}{\sqrt{t^2 - s^2}} = t$$ Volterra **first** kind

Solution of an Integral Equation

A solution of a differential or an integral equation arises in any of the following two types:

1). **Exact solution**:

The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions. Examples of exact solutions are as follows:

$$u(x) = x + e^{x},$$
$$u(x) = sinx + e^{2x},$$
$$u(x) = 1 + coshx + tanx,$$

and many others.

2). **Series solution**:

For concrete problems, sometimes we cannot obtain exact solutions. In this case we determine the solution in a series form that may converge to exact soliton if such a solution exists. Other series may not give exact solution, and in this case the obtained series can be used for numerical purposes. The more terms that we determine the higher accuracy level that we can achieve.

Volterra Integral Equations of the Second Kind (VIESKs)

Given $g : \mathbf{R}^3 \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$,

such that

$$y(t) = f(t) + \int_a^t g(t, s, y(s))\, ds$$

Linear: $$y(t) = f(t) + \int_a^t k(t, s)\, y(s)\, ds$$

Separable: $$k(t, s) = p(t)\, q(s)$$

Convolved: $$k(t, s) = r(t - s)$$

Leibnitz Rule for Differentiation of Integrals

One of the methods that will be used to solve integral equations is the conversion of the integral equation to an equivalent differential equation. The conversion is achieved by using the well-known *Leibnitz rule* [4,6,7] for differentiation of integrals. Let $f(x,t)$ be continuous and $\partial f/\partial t$ be continuous in a domain of the $x-t$ plane that includes the rectangle

$a \le x \le b, t_0 \le t \le t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\, dt \qquad (1)$$

then differentiation of the integral in (1) exists and is given by:

$$F'(x) = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}(x,t)\, dt \qquad (2)$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (1.106) reduces to

$$F'(x) = \int_{a}^{b} \frac{\partial f}{\partial x}(x,t)\, dt \qquad (3)$$

Example

Find $\boldsymbol{F}'(x)$ for the following:

$$\boldsymbol{F(x) = \int_{sinx}^{cosx} \sqrt{1 + t^3}\, dt}$$

we can set $g(x)$ = sin$x$ and $h(x)$ = cos$x$. It is also clear that $f(x, t)$ is a function of t only, using Leibnitz rule (1) we find that

$$\boldsymbol{F'(x) = -\sin x\sqrt{1 + cos^3x} - \cos x\sqrt{1 + sin^3x}}$$

Converting IVP to Volterra Integral Equation

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''(x) - y(x) = \sin x \ , y(0) = 0, y'(0) = 0. \qquad (4)$$

Proceeding as before, we set

$$y''(x)=\mathrm{u}(x) \qquad (5)$$

Integrating both sides of (5), using the initial condition $y'(0) = 0$ gives

$$y'(x) = \int_0^x u(t)dt \qquad (6)$$

Integrating (6) again, using the initial condition $y(0) = 0$ yields

$$y(x) = \int_0^x \int_0^x u(t)dt = \int_0^x (x-t)u(t)\,dt \qquad (7)$$

obtained upon using the rule to convert double integral to a single integral. Inserting $(5-7)$ into (4) leads to the following Volterra integral equation:

$$u(x) = sinx + \int_0^x (x-t)u(t)dt$$

Convert BVP to an equivalent Fredholm integral equation

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + 9y(x) = cos(x),\ y(0) = y(1) = 0. \qquad (9)$$

Proceeding as before, we set

$$y''(x) = \mathbf{u}(x) \qquad (10)$$

Integrating both sides of (10),

$$y'(x) = y'(0) + \int_0^x u(t)dt \qquad (11)$$

Integrating (11) again, using the initial condition $y(0) = 0$ yields

$$\mathbf{y(x) = x\,y'(0) + \int_0^x \int_0^x u(t)dt = x\,y'(0) + \int_0^x (x - t)u(t)\,dt} \qquad (12)$$

we set $x = 1$ in $(\mathbf{12})$, using the initial condition $y(1) = 0$ yields

$$y(1) = y'(0) + \int_0^1 (1 - t)u(t)\,dt \qquad (13)$$

$$y'(0) = -\int_0^1 (1 - t)u(t)\,dt = -\int_0^x (1 - t)u(t)\,dt - \int_x^1 (1 - t)u(t)\,dt$$

$$y(x) = -x\int_0^x (1 - t)u(t)\,dt - x\int_x^1 (1 - t)u(t)\,dt + \int_0^x (x - t)u(t)\,dt$$

$$= -\int_0^x t(1 - x)u(t)\,dt - \int_x^1 x(1 - t)u(t)\,dt \qquad (14)$$

Inserting $(10 - 14)$ into (9) leads to the following Fredholm integral equation:

$$u(x) = sinx + \int_0^1 K(x,t)u(t)dt$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & for \quad 0 \le t \le x \\ 9x(1-t), & for \quad x \le t \le 1 \end{cases}$$

Methods For Solving Integral Equations

- The Adomian Decomposition Method
- The Modified Decomposition Method
- The Noise Terms Phenomenon Method
- The Variational Iteration Method
- The Successive Approximations Method
- The Laplace Transform Method
- The Series Solution Method
- The Direct Computation Method
- The Homotopy Perturbation Method

Integral Equations from Differential Equations

First-Order Initial Value Problem $\Rightarrow$ **Volterra** Integral Equation Of the **Second Kind**

$$y'(t) = F(t) + g(t, y(t)), \quad y(a) = y_0$$

$$y(t) - y_0 = \int_a^t F(s)\, ds + \int_a^t g(s, y(s))\, ds$$

Used to **prove theorems** about differential equations.

Used to **derive numerical methods** for differential equations.

Symbolic Solution of Separable VIESKs

Solve $$y(t) = \sqrt{t} - \int_0^t 2\sqrt{ts}\, y(s)\, ds$$

Let $$Y(t) = \int_0^t \sqrt{s}\, y(s)\, ds$$

Then $$y(t) = \sqrt{t} - 2\sqrt{t}\, Y(t)$$

New Problem:

$$Y'(t) = \sqrt{t}\, y = t - 2t\, Y(t), \quad Y(0) = 0$$

Solution: $$y(t) = \sqrt{t}\, e^{-t^2}$$

Volterra Sequences

Given f, g, and y_0, **define** $y_1, y_2, \ldots$ by

$$y_1(t) = f(t) + \int_a^t g(t, s, y_0(s))\, ds$$

$$y_2(t) = f(t) + \int_a^t g(t, s, y_1(s))\, ds$$

and so on.

Volterra Sequences

Given f, g, and y_0, **define** $y_1, y_2, \ldots$ by

$$y_n(t) = f(t) + \int_a^t g(t, s, y_{n-1}(s))\, ds, \quad n = 1, 2, \ldots$$

Does y_n **converge** to some y**?**

If so, does y **solve** the VESK**?**

If so, is this a **useful** iterative method**?**

A Linear Example

$$y(t) = 1 - \int_0^t (t-s)\, y(s)\, ds$$

Let $y_0 = 1$. Then

$$y_1(t) = 1 - \int_0^t (t-s)\, ds = \cdots = 1 - \tfrac{1}{2}t^2$$

$$y_2(t) = 1 - \int_0^t (t-s)(1 - \tfrac{1}{2}s^2)\, ds = \cdots = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{24}t^4$$

$$\downarrow\downarrow$$

$$y_\infty = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{24}t^4 - \tfrac{1}{6!}t^6 + \cdots = \cos t$$

The sequence **converges** to the known **solution**.

Convergence of the Sequence

y_0

y_1

y_2

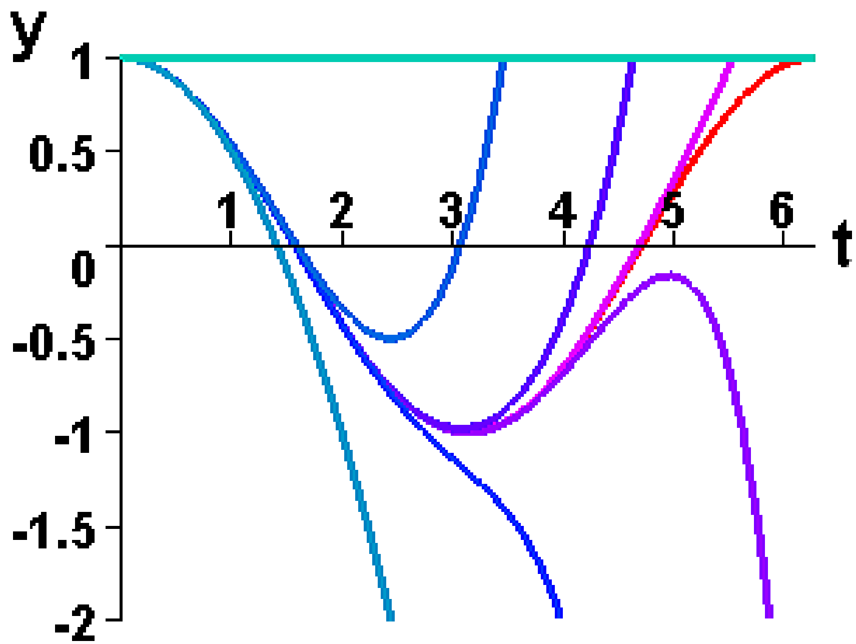
y_3

y_4

y_5

y_6

y



A Nonlinear Example

$$y(t) = \int_0^t \frac{2s - y(s)}{1 - y(s)} ds$$

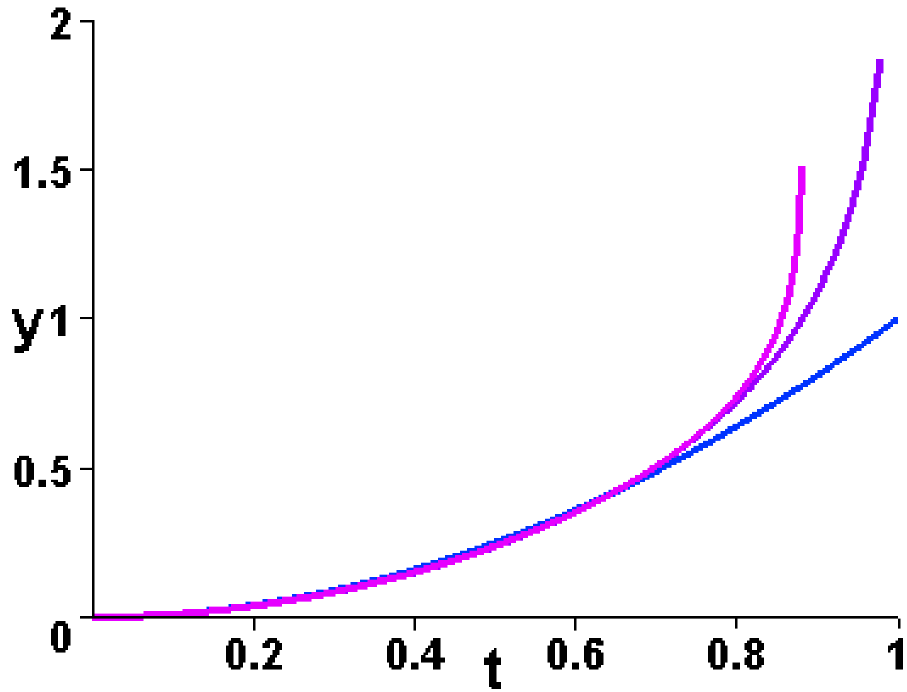
Let $y_0 = 0$. **Then** $y_1(t) = \int_0^t 2s\, ds = t^2$

$$y_2(t) = \int_0^t \frac{2s - s^2}{1 - s^2} ds = t - \tfrac{1}{2}\ln(1-t) - \tfrac{3}{2}\ln(1+t)$$

$$y_3(t) = \int_0^t \frac{\ln(1-s) + 3\ln(1+s) + 2s}{\ln(1-s) + 3\ln(1+s) - 2s + 2} ds$$

Convergence of the Sequence

y_3
y_2
y_1



The Adomian Decomposition Method (ADM)

$$u(x)=\sum_{n=0}^{\infty}u_n(x)$$

s.t

$$u_0(x)=f(x)$$

$$u_n(x)=\lambda\int_0^x k(x,t)u_{n-1}(t)\,dt \qquad , n>0$$

Example: $u(x)=1-\int_0^x u(t)dt$

- Sol:

$$u_0(x)=1$$

$$u_1(x)=-\int_0^x u_0(t)dt=-x$$

$$u_2(x)=-\int_0^x -tdt=\frac{x^2}{2!}$$

$$u_3(x)=-\int_0^x t^2dt=-\frac{x^3}{3!}$$

$$\vdots$$

$$u_n(x)=(-1)^n\frac{x^n}{n!}$$

$$u(x)=e^{-x}$$

is exact solution

The Successive Approximations Method (SAM)

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

s.t

$u_0(x) =$ any selective real valued function

$$u_n(x) = f(x) + \lambda \int_0^x k(x,t) u_{n-1}(t)\, dt \quad , n>0$$

Example: $u(x) = 1 - \int_0^x u(t)dt$

- Sol:

$$u_0(x) = 1$$

$$u_1(x) = 1 - \int_0^x u_0(t)dt = 1 - x$$

$$u_2(x) = 1 - \int_0^x u_1(t)dt = 1 - \int_0^x dt + \int_0^x tdt = 1 - x + \frac{x^2}{2!}$$

$$u_3(x) = -\int_0^x u_2(t)dt = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$$\vdots$$

$$u(x) = e^{-x}$$

is exact solution

The Laplace Transform Method (LTM)

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt$$

$$U(s) = F(s) + \lambda \kappa(s) U(s)$$

$$U(s) = \frac{F(s)}{1 - \lambda \kappa(s)}$$

$$u(x) = \ell^{-1}\left(\frac{F(s)}{1 - \lambda \kappa(s)}\right)$$

Example: $u(x) = 1 - \int_0^x u(t)dt$

- Sol:

$$\ell\{u(x)\} = \ell\{1\} - \ell\{1.u(x)\}$$

$$U(s) = \frac{1}{s} - \frac{1}{s}.U(s)$$

$$U(s)(1+\frac{1}{s}) = \frac{1}{s} \Rightarrow U(s) = \frac{1}{s+1}$$

$$\ell^{-1}\{U(s)\} = \ell^{-1}\{\frac{1}{s+1}\}$$

$$u(x) = e^{-x}$$

is exact solution

The Series Solution Method
(ASM)

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

Example: $u(x) = 1 - \int_0^x u(t)\,dt$

- Sol:

$$\sum_{n=0}^{\infty} a_n x^n = 1 - \int_0^x \sum_{n=0}^{\infty} a_n t^n dt$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = 1 - \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} = 1 - a_0 x - a_1 \frac{x^2}{2} + a_3 \frac{x^3}{3} + \ldots$$

$$a_0 = 1, a_1 = -a_0 \Rightarrow a_1 = -1, a_2 = \frac{a_1}{2} = \frac{1}{2}, a_3 = \frac{-a_2}{3} = -\frac{1}{3!}$$

$$\therefore u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots$$

$u(x) = e^{-x}$ is exact solution

Lemma I

Homogeneous linear Volterra sequences

$$y_n(t) = \int_a^t k(t,s)\, y_{n-1}(s)\, ds, \quad n = 1, 2, \ldots$$

decay to 0 on [a, T] whenever k is bounded.

Proof: (a=0)

Choose constants K, Y and T such that

$$|k(t,s)| \le K, \quad |y_0(t)| \le Y, \quad \forall\, 0 \le t, s \le T$$

Then

$$|y_1(t)| \le \int_0^t \left| k(t,s) y_0(s) \right| ds \le \int_0^t KY\, ds = YKt$$

Given

$$|k(t,s)| \le K, \quad |y_0(t)| \le Y, \quad \forall\, 0 \le t, s \le T$$

$$y_n(t) = \int_0^t k(t,s)\, y_{n-1}(s)\, ds, \quad n = 1, 2, \ldots$$

we have

$$|y_2(t)| \le \int_0^t K(YKs)\, ds = \frac{1}{2} YK^2 t^2$$

$$|y_3(t)| \le \int_0^t K(\tfrac{1}{2} YK^2 s^2)\, ds = \frac{1}{3!} YK^3 t^3$$

In general,

$$|y_n(t)| \le \frac{1}{n!} YK^n t^n \quad \forall\, n$$ so $$\lim_{n\to\infty} y_n = 0$$

Theorem 2

Homogeneous linear VIESK

$$y(t) = \int_a^t k(t,s)\,y(s)\,ds$$

have the unique solution $y \equiv 0$.

Proof:

Let φ be a **solution.**

Choose $y_0 = \varphi$**.**

Then $y_n = \varphi$ **for all** n, **so** $y_n \to \varphi$**.**

But the **lemma requires** $y_n \to 0$**.**

Therefore $\varphi \equiv 0$**.**

Theorem 3

Linear VIESKs have at most one solution.

Proof:

Suppose $$\varphi(t) = f(t) + \int_a^t k(t,s)\,\varphi(s)\,ds$$

and $$\psi(t) = f(t) + \int_a^t k(t,s)\,\psi(s)\,ds$$

Let $y = \varphi - \psi$. Then

$$y(t) = \int_a^t k(t,s)\,y(s)\,ds$$

By Theorem 2, $y \equiv 0$; hence, $\varphi = \psi$.

Theorem 5

Linear VIESKs have a unique solution.

Proof:

By **Lemma 4**, the sequence

$$y_n(t) = f(t) + \int_a^t k(t,s)\, y_{n-1}(s)\, ds, \quad y_0 = f$$

converges to some y. Taking a **limit** as $n \to \infty$:

$$y(t) = f(t) + \int_a^t k(t,s)\, y(s)\, ds$$

The **solution** is **unique** by **Theorem 3.**

Fuzzy Integral Equations

- The topics of fuzzy integral equations which attracted growing interest for some time, in particular in relation to fuzzy control, have been developed in recent years.
- The concept of integration of fuzzy functions was first introduced by Dubois and Prade.
- Usually in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp.
- Fuzzy Linear integral equations arise frequently in physical problems.
- The most important contribution of the theory of fuzzy integral equations consists in the solution of fuzzy initial and boundary value problems. The fuzzy boundary value problems for equations of elliptic type can be reduced to fuzzy Fredholm integral equations while the study of parabolic and hyperbolic fuzzy differential equations leads to fuzzy Volterra integral equations.
- The theory of fuzzy Volterra Integral equations makes it possible to solve an initial value problem for a linear fuzzy ordinary differential equation of an arbitrary order.

General Form Of Fuzzy Linear Integral Equations

$$\tilde{\omega}(x) = \tilde{f}(x) + \int_{g(x)}^{h(x)} K(x,t)\tilde{\omega}(t)dt$$

$$(\underline{\omega}(x,r), \overline{\omega}(x,r)) = (\underline{f}(x,r), \overline{f}(x,r)) + \int_{g(x)}^{h(x)} K(x,t)(\underline{\omega}(t,r), \overline{\omega}(t,r))dt$$

An Age-Structured Population

Given

An initial population of known age distribution

Age-dependent birthrate

Age-dependent death rate

Find

Total birthrate

Total population

Age distribution

An Age-Structured Population

Simplified Version

Given

- **An initial population of *newborns***
- **Age-dependent birthrate**
- **Age-*in*dependent death rate**

Find

- **Total birthrate**
- **Total population**

The Founding Mothers

Assume a one-sex population.

Starting population is 1 unit.

Life expectancy is 1 time unit.

$$p_0' = -p_0, \quad p_0(0) = 1$$

Result:

$$p_0(t) = e^{-t}$$

Basic Birthrate Facts

Total Birthrate =

Births to Founding Mothers

+

Births to Native Daughters

$$b(t) = m(t) + d(t)$$

Let $f(t)$ be the number of **births per mother of age** t **per unit time**.

$$m(t) = f(t)\, e^{-t}$$

Births to Native Daughters

Consider daughters of ages x to $x+dx$.

All were born between $t-x-dx$ and $t-x$.

The initial number was $b(t-x)dx$.

The number at time t is $b(t-x)e^{-x}dx$.

The rate of births is $f(x)b(t-x)e^{-x}dx$
$=m(x)b(t-x)dx$.

$$d(t) = \int_0^t m(x)\,b(t-x)\;dx$$

The Renewal Equation

$$b(t) = m(t) + \int_0^t m(x)\, b(t-x)\, dx$$

or

$$b(t) = m(t) + \int_0^t m(t-x)\, b(x)\, dx$$

This is a linear VESK with convolved kernel.
It can be solved by Laplace transform.

Solution of the Renewal Equation

Given the fecundity function f, **define**

$$F(s) = L\left[f(t)\right], \quad r(t) = L^{-1}\left[\frac{F(s)}{1-F(s)}\right]$$

where L is the **Laplace transform. Then**

$$b(t) = e^{-t} r(t)$$

$$p(t) = e^{-t} + e^{-t}\int_0^t r(x)\,dx$$

A Specific Example

Let $f(x) = axe^{-2x}$.

Then

$$b(t) = \frac{\sqrt{a}}{2} e^{(\sqrt{a}-3)t} - \frac{\sqrt{a}}{2} e^{(-\sqrt{a}-3)t}$$

$$p(t) = \frac{\sqrt{a}}{2(\sqrt{a}-2)} e^{(\sqrt{a}-3)t} - \frac{4}{a-4} e^{-t} + \frac{\sqrt{a}}{2(\sqrt{a}\_2)} e^{(-\sqrt{a}-3)t}$$

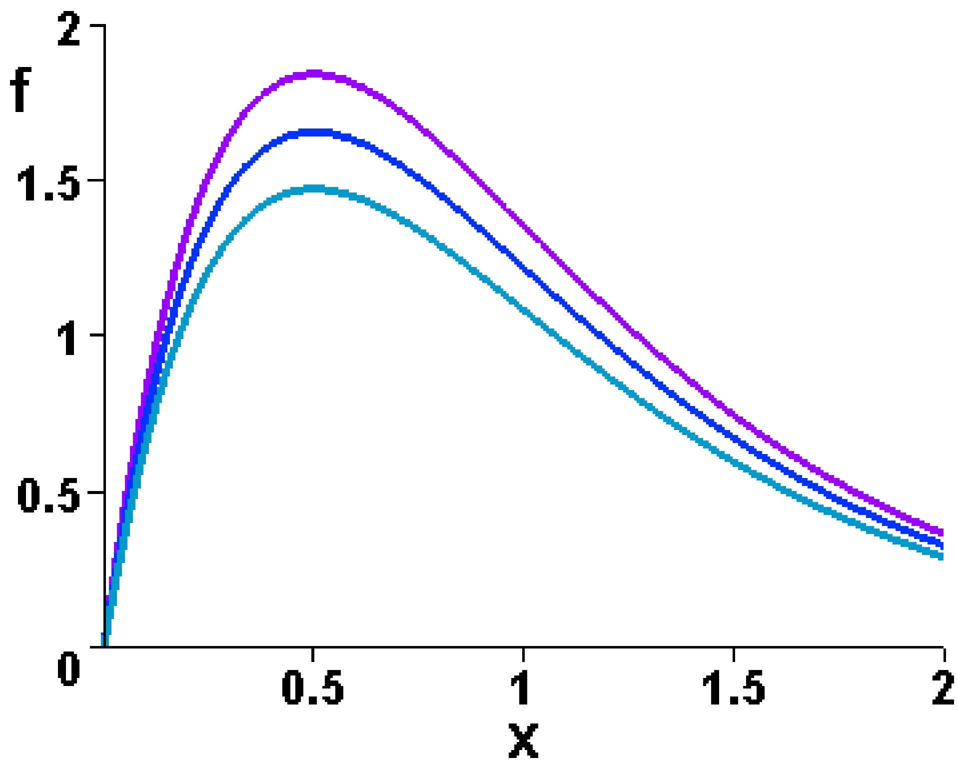
If $a=9$, then $p(t) \to 1.5$.

Exponential growth for $a>9$.

The population dies if $a<9$.

Fecundity

$a = 10$
$a = 9$
$a = 8$

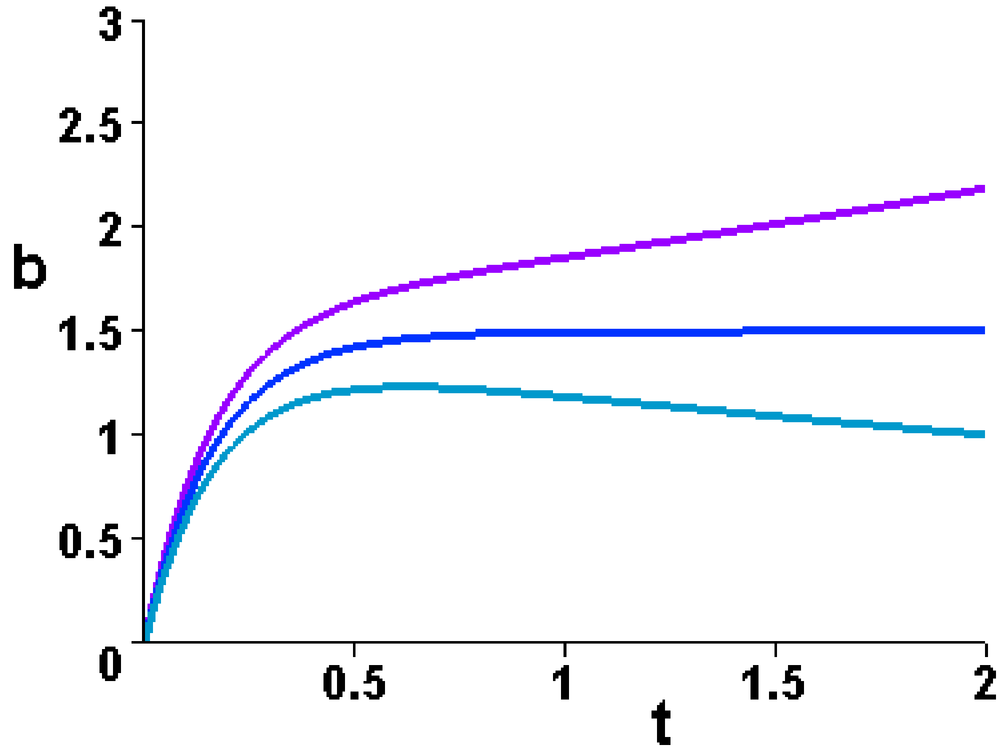


Birth Rate

$a = 10$

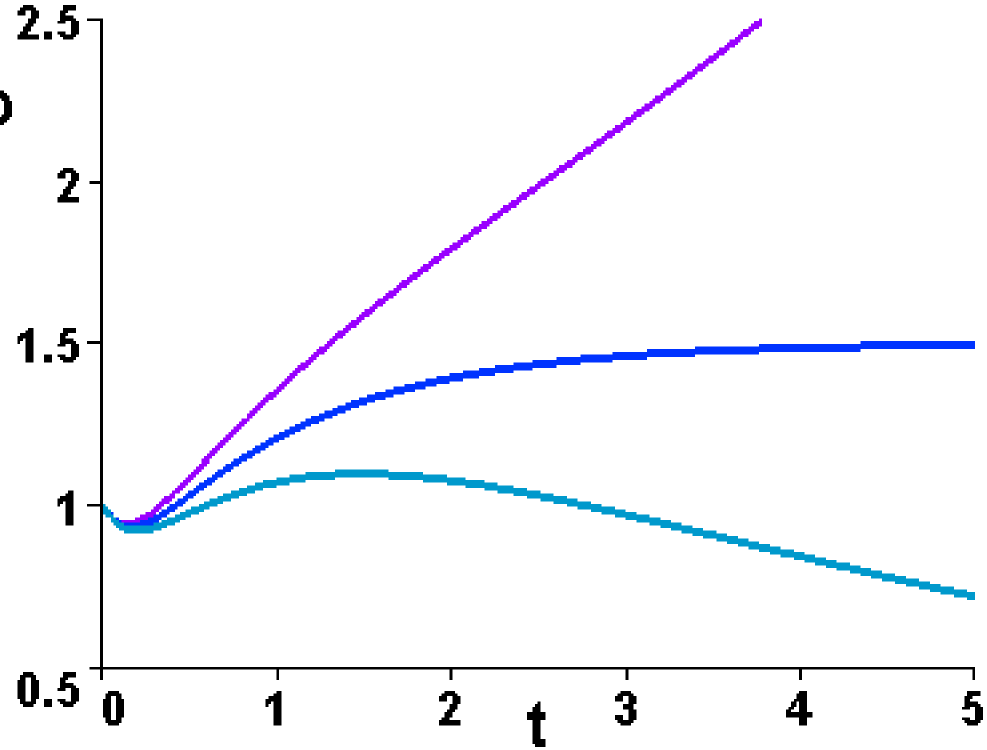
$a = 9$

$a = 8$



Population

$a = 10$
$a = 9$
$a = 8$



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Any
Questions?

Thank you