Relativistic Electromagnetism

- Maxwell's theory is a vector field theory based on the electric field E^i and magnetic field B^i

Based on 4 equations:

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 4\pi \rho$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Where ρ is charge density and $\vec{J}$ is current density

Levi-Civita symbol: ε^{ijk}, Equal to 1 if i,j,k are an even permutation of 1,2,3, zero if two indices are repeated, and −1 otherwise

↳ useful to express cross product: $(\vec{A} \times \vec{B})^i = \varepsilon^{ijk} A_j B_k$

We can use notation $\partial_i = \partial / \partial x^i$ (∂_0 is then $\partial / \partial t$)

to write the maxwell equations:

① $\varepsilon^{ijk} \partial_j B_k - \partial_0 E^i = 4\pi J^i$ ② $\partial_i E^i = 4\pi J^0$

③ $\varepsilon^{ijk} \partial_j E_k + \partial_0 B^i = 0$ ④ $\partial_i B^i = 0$

Notice we have defined a 4-vector $J^\mu = (\rho, \vec{J})$

Relativistic Electromagnetism

We can now define a field tensor:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

It is antisymmetric, $F_{\mu\nu} = -F_{\nu\mu}$

The components of the fields are all included, for example $E^i = F^{0i}$, $F^{ij} = \varepsilon^{ijk} B_k$

The first 4 maxwell equations can be compressed:

$$\partial_j F^{ij} + \partial_0 F^{i0} = 4\pi J^i$$

$$\partial_i F^{0i} = 4\pi J^0$$

$\hookrightarrow$ or $\boxed{\partial_\mu F^{\nu\mu} = 4\pi J^\nu}$

We can extend the L-C symbol to spacetime, and rewrite the other 4 maxwell equations as

$$\varepsilon^{\sigma\mu\nu\lambda} \partial_\mu F_{\nu\lambda} = 0$$

$\hookrightarrow$ ε is invariant under Lorentz transformations

Maxwell's theory was Lorentz Invariant

Relativistic Electromagnetism

We introduce the idea of potentials:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We can introduce the spacetime potential: $(\phi, \vec{A}) = A^\mu$

Thus, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

All are invariant under L transform, giving easy transforms to moving frames

General Relativity

- inertial mass = gravitational mass ← equiv principle

Distance in curved space-time:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

If $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, it is flat

We cannot mix vectors; a curved spacetime behaves <u>locally</u> at every point like a vector space

↳ vectors at different points are part of different vector spaces

This poses a problem for taking derivatives of vectors

We note that in curvilinear coordinates, the differentials transform linearly:

$$dx^{\mu'} = \Lambda^{\mu'}_{\ \mu}\, dx^\mu \rightarrow \Lambda^{\mu'}_{\ \mu} = \frac{dx^{\mu'}}{dx^\mu}$$

Since we can only think of vectors at a point, we define vectors in curvilinear space as sets of numbers that transform like the coordinate differentials:

↳ $A_{\mu'} = \Lambda_{\mu'}^{\ \mu} A_\mu$ with $\Lambda_{\mu'}^{\ \mu} = \frac{\partial x^\mu}{\partial x^{\mu'}}$

If we assume that Λ is coordinate independent:

$$\partial_{\nu'} A_{\mu'} = \Lambda_{\nu'}^{\ \nu} \Lambda_{\mu'}^{\ \mu}\, \partial_\nu A_\mu$$

↳ the derivative transforms as a <u>tensor</u>

When dealing with curvilinear coordinates, Λ is no longer coordinate independent,

Covariant Derivative : ∇_μ

↳ $\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda$

The objects $\Gamma^\nu_{\mu\lambda}$ are called <u>connections</u>

↳ provided by curved spacetime metric

- is symmetric in lower indices: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$

We can uniquely determine connections from the metric if we demand that $\nabla_\sigma g_{\mu\nu} = 0$.

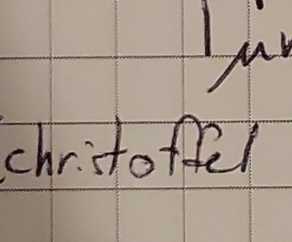
↳ these $g_{\mu\nu}$ are <u>Riemannian geometries</u>

Thus,
$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$
(Christoffel formula)

Curvature

- curvature is not a local concept
- a vector pointing in direction α, when slid around a closed loop in curved space, will point in a different direction. ⟶ called intrinsic curvature

Consider an infinitesimally small loop:



$\longrightarrow (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\lambda$

This gives us an equation:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\lambda = R^\lambda_{\ \rho\mu\nu} V^\rho$$

- R is called the Riemann tensor, or curvature tensor

If the curvature tensor vanishes at all points, the space is flat

<u>Contraction</u>: When one has a tensor with a pair of indices, 1 up and 1 down, one can sum over them and get a tensor with 2 less indices

<u>Ricci Tensor</u>: contraction of the Riemann tensor:

$$R_{\mu\nu} \equiv R^\lambda_{\ \mu\lambda\nu}$$

<u>Curvature Scalar</u>: $R = R_{\mu\nu} g^{\mu\nu}$

Einstein Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

- $T_{\mu\nu}$ is the stress-energy tensor or energy-momentum tensor (constructed by matter fields)

↳ similar in spirit to Newton: $\nabla^2 \phi = 4\pi G \rho$

- G is the Newton constant

The general solution in vacuum with spherical coordinates (symme
is the

Schwartzchild Metric:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)}_{d\Omega^2}$$