25/23 Hamiltonian Mechanics

Most systems can be characterized by a Lagrangian $L(q_i, \dot{q}_i)$.

We can define the canonical momenta

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i}$$

We then define the hamiltonian as

$\mathcal{H} = \sum_{i=1}^{N} p_i \dot{q}_i - L$, as a function of only q_i and p_i

Phase space: span p_i, q_i

Configuration Space: span q_i

We can then define motion: $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Poisson Backet: operation between two functions of phase space: f, g

$$\hookrightarrow \{f, g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

On property is: $\{q_i, p_j\} = \delta_{ij}$

for motion:

$$\dot{q}_i = \{q_i, \mathcal{H}\}$$

$$\dot{p}_i = \{p_i, \mathcal{H}\}$$

Hamiltonian Mechanics

Constraints

When we describe a system with more variables than needed we get a relation between these variables, called a constraint.

We can write constraints as $\phi(p_i, q_i) = 0$

↳ these can be called conserved quantities

We can define special trajectories $q(\lambda), p(\lambda)$ in phase space that satisfy

$$\frac{dq_i}{d\lambda} = \{q_i, \phi\} \quad \text{and} \quad \frac{dp_i}{d\lambda} = \{p_i, \phi\}$$

This is a curve where the hamiltonian remains unchanged

Lagrange Multipliers

If you have $2N$ canonical vars and M constraints ϕ_i,

then $H = H_{orig} + \sum_{i=1}^{M} \lambda_i \phi_i$

Where λ_i is a parameter called a Lagrange Multiplier

One sign of constraints is the lack of a time derivative of a var in the lagrangian

Canonical Formulation of Maxwell Theory

We can define the lagrangian density of Maxwell's Theory

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Rearranging:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}\left(2F_{0a}F^{0a} + F_{ab}F^{ab}\right)$$

Where a, b are spatial indices

It may be better to express the lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\lambda\kappa} \eta^{\mu\lambda}\eta^{\nu\kappa}$$

$$\mathcal{L} = -\frac{1}{4}\left(F_{0a}F_{0a}\,\eta^{00}\eta^{aa} + (-F_{0a})(-F_{0a})\,\eta^{00}\eta^{aa} + F_{ab}F_{ab}\,\eta^{aa}\eta^{bb}\right)$$

$$= \frac{1}{2} F_{0a}F_{0a} - \frac{1}{4} F_{ab}F_{ab} = \frac{1}{2}(\partial_0 A_a - \partial_a A_0)^2 - \frac{1}{4}(\partial_a A_b - \partial_b A_a)^2$$

We can now find the canonical momentum π

$$\pi^a(x) = \frac{\delta L}{\delta \dot{A}_a(x)}$$

Where L is the total lagrangian $L = \int d^4x\, \mathcal{L}$

$$\delta L = \int d^4y \Big[(\partial_0 A_a - \partial_a A_0)(\delta \dot{A}_a(y) - \partial_a \delta A_0) - \frac{1}{2}(\partial_a A_b - \partial_b A_a)(\delta \partial_a A_b - \delta \partial_b A_a)\Big]$$

$$\frac{\delta L}{\delta \dot{A}_b(x)} = \int d^4y\, (\dot{A}_a - \partial_a A_0)\,\delta_{ab}\,\delta(x-y)$$

$$= \int d^4y\, F_{0a}\,\delta_{ab}\,\delta(x-y) = F_{0a}(x)$$

Thus, $\pi^a(x) = F_{0a}(x) = \dot{A}_a(x) - \partial_a A_0(x) = -E^a$

We now have the phase space variables A and π, thus we can define some properties:

① $\{A_a(x), A_b(y)\} = 0$

② $\{\pi_a(x), \pi_b(x)\} = 0$

③ $\{A_a(x), \pi_b(y)\} = \delta_{ab}\delta^3(x-y)$

Proof of ③

$$\{A_a(x), \pi_b(y)\} = \frac{\delta A_a(x)}{\delta A_j(z)}\frac{\delta \pi_b(y)}{\delta \pi_j(z)} - \cancel{\frac{\delta A_a(x)}{\delta \pi_j(z)}}^{\,0}\frac{\delta \pi_b(y)}{\delta A_j(z)}$$

$$= \frac{\delta A_a(x)}{\delta A_j(z)}\frac{\delta \pi_b(y)}{\delta \pi_j(z)} = \delta_{aj}\delta^3(x-z)\delta_{bj}\delta^3(y-z)$$

$$= \delta_{ab}\delta(x-y)$$

Now we want to find the hamiltonian, H, or rather the hamiltonian density $\mathcal{H}$.

$$\mathcal{H} = \pi_a(x)\dot{A}^a(x) - \mathcal{L}$$

We use the relation $\dot{A}_a(x) = \pi^a(x) + \partial_a A_0(x)$

$$\mathcal{H} = \pi_a(x)\left(\pi^a(x) + \partial_a A_0(x)\right) - \frac{1}{2}\left(\pi_a^2 - \frac{1}{2}F_{ab}^2\right)$$

$$= \frac{1}{2}\pi_a^2(x) + \frac{1}{2}F_{ab}^2(x) + \pi_a(x)\partial_a A_0(x)$$

We express the full hamiltonian as an integral

$$H = \int d^3x \left[\pi_a^2(x) + \frac{1}{2} F_{ab}^2(x) + \pi_a(x) \partial_a A_0(x) \right]$$

Using integration by parts on the third term, we have

$$\int \pi_a \partial_a A_0 = \cancel{\pi_a A_0} - \int A_0 \partial_a \pi^a$$

The constant vanishes in all important uses, Thus

$$H = \int d^3x \left[\pi_a^2 + \frac{1}{2} F_{ab}^2 + A_0 \partial_a \pi^a \right]$$

We can now start finding equations of motion.

$$\dot{A}_a(x) = \{A_a(x), H\}$$

$$= \int d^3z \left(\frac{\delta A_a(x)}{\delta A_c(z)} \frac{\delta H(y)}{\delta \pi^c(z)} - \underbrace{\frac{\delta A_a(x)}{\delta \pi^c(z)}}_{0} \frac{\delta H(y)}{\delta A_c(z)} \right)$$

$$= \int d^3z \left(\delta_{ac} \delta(x-z) \frac{\delta H_b(y)}{\delta \pi^c(z)} \right)$$

Now,

$$\delta H_b(y) = \int d^3y \left(\pi_b \delta \pi_b + \partial_b A_0 \delta \pi^b \right)$$

Then $\dfrac{\delta H_b(y)}{\delta \pi_c(z)} = \int d^3y \left(\pi_b \delta_{bc} \delta(y-z) + \partial_b A_0 \delta_{bc} \delta(y-z) \right)$

$$= \pi_c + \partial_c A_0$$

Then, $\dot{A}(x)$

$$= \int d^3z \, \delta_{ac} \delta(x-z)(\pi_c(z) + \partial_c A_0(z))$$

$$= \pi_a(x) + \partial_a A_0(x)$$ which already was determined for the canonical momenta:

$$\dot{\pi}_a(x) = \{\pi_a(x), H\}$$

$$= \int d^3z \left(\cancel{\frac{\delta \pi_a(x)}{\delta A_c(z)}}_{0} \frac{\delta H_b(y)}{\delta \pi_c(z)} - \frac{\delta \pi_a(x)}{\delta \pi_c(z)} \frac{\delta H_b(y)}{\delta A_c(z)} \right)$$

$$= -\int d^3z \, \delta_{ac} \delta(x-z) \frac{\delta H_b(y)}{\delta A_c(z)}$$

$$\frac{\delta H_b(y)}{\delta A_c(z)} = \int d^3y \, \frac{1}{2} F_{bc}(\partial_b \delta A_c - \partial_c \delta A_b) \frac{1}{\delta A_c(z)}$$

$$= \int d^3y \, \partial_c F_{bc} \, \delta A_b \frac{1}{\delta A_c(z)}$$

$$= \int d^3y \, \partial_c F_{bc} \, \delta_{bc} \delta^3(y-z) = \partial_b F_{cb}(z)$$

Thus $\dot{\pi}_a(x) = -\int d^3z \, \delta_{ac} \delta(x-z) \partial_b F_{cb}(z)$

$$= -\partial_b F_{ab}(x)$$ which is a form of Maxwell's equations

Poisson Brackets in Field Theories

For two fields F and G defined by canonical variables $\Phi_a(x)$ and $\Pi_b(x)$, we can define the poisson bracket as

$$\{F[\Phi_a(x), \Pi^a(x)], G[\Phi_b(y), \Pi^b(y)]\}$$

$$= \int d^3z \left(\frac{\delta F[\Phi_a(x), \Pi^a(x)]}{\delta \Phi_c(z)} \frac{\delta G[\Phi_b(y), \Pi^b(y)]}{\delta \Pi^c(z)} - \frac{\delta F[\Phi_a(x), \Pi^a(x)]}{\delta \Pi^c(z)} \frac{\delta G[\Phi_b(y), \Pi^b(y)]}{\delta \Phi_c(z)} \right)$$

$$\equiv \int d^3u\, d^3v \left(\frac{\delta F(x)}{\delta \Phi_c(u)} \{\Phi_c(u) \Pi^d(v)\} \frac{\delta G(y)}{\delta \Pi^d(v)} + \frac{\delta F(x)}{\delta \Pi^d(u)} \{\Pi^d(u), \Phi_c(v)\} \frac{\delta G(y)}{\delta \Phi_c(v)} \right)$$

from ③ on page 20,

$$= \int d^3u\, d^3v \left(\frac{\delta F(x)}{\Phi_c(u)} (\delta_c^d \delta^3(u-v)) \frac{\delta G(y)}{\delta \Pi^d(v)} + \frac{\delta F(x)}{\delta \Pi^d(u)} (-\delta_c^d \delta^3(v-u)) \frac{\delta G(y)}{\delta \Phi_c(v)} \right)$$