Diffeomorphisms

Manifold: a topological set, that is mappable to R^n

↳ can define neighborhood & take limits

↳ set up coords to distinguish elements

Diffeomorphism: one-to-one (injective) map that maps a point p in a manifold to a new point $\Phi(p)$, and respects topological notions of proximity

↳ ie differentiability

A diffeomorphism is called an active view coordinate transformation.

Push forward: the operation of mapping a point p in M to a new point $q = \Phi(p)$ is N.

Pull Back: the operation of mapping a function on M back to N:

for $f: \mathbb{R} \to N$, we define

$$\Phi^*(f)(p) \equiv f(\Phi(p)) = f(q).$$

This function acts on M now.

Consider a vector as an operator acting on functions.

↳ Given a vector V^μ, the derivative of a function f along V^μ, is $V^\mu \partial_\mu f$, which is a function.

↳ we can say this is a vector acting on M.

$V^\mu \partial_\mu(\Phi^*(f))$ acts on N, as the vector acts on the pull back of the function.

Diffeomorphisms

The components of V^μ can be found as follows:

$$(\phi V)^\mu \partial_\mu f = V^\mu \partial_\mu (\phi^* f)) = V^\mu \partial_\mu (f \circ \phi)$$

Say the coords of M are x^μ and N are y^μ,

$$V^\mu \frac{\partial (f(\circ \phi)}{\partial x^\mu} = V^\mu \frac{\partial y^r}{\partial x^\mu} \frac{\partial f}{\partial y^r}$$

This assumes the map ϕ is smooth and invertible.

↳ ϕ is called a diffeomorphism

The "effect" on the vector is to multiply its components by the matrix $\partial y^r / \partial x^\mu$.

<u>One-parameter family Φ_t</u>: family of diffeomorphisms such that for $t=0$ $\Phi_0 = I$, and the maps move the points "farther away" for bigger values of t.

The continuous family of diffeomorphisms Φ_t generates a curve on the manifold starting from p.

↳ tangent vector $V^\mu = dx^\mu / dt$

<u>Lie Derivative:</u> A new derivative of vectors and tensors.

For a tensor T and a 1-parameter family of diffeos ϕ_t,

$$L_V T(p) = \lim_{t \to 0} \frac{\phi_t^* (T(\phi_t(p))) - T(p)}{t}$$

↳ push forward p, evaluate tensor, then pull back to p.