Experiments with Magnetic Torque and Dynamics

Alex Short

October 27, 2023

Abstract

We observe the effects of a magnetic field on a magnetic moment by exploring different ways that torque effects angular motion. We look at the dynamics of simple harmonic oscillation and precession of a magnetic moment in a magnetic field and try to determine the value of the magnetic moment.

1 Introduction

Magnetic dynamics is a field of interest to many different areas of physics. It is believed that properties like magnetic torque and angular momentum play heavy roles in the dynamics of atoms and charged particles. Before I introduce the experiment, I will build a theoretical foundation for all the dynamics being observed in the lab.

The magnetic moment is a vector $\vec{\mu}$ that interacts with the magnetic field. The equation describing that interaction is

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{1}$$

[1] where $\vec{\tau}$ is the torque vector caused by the $\vec{B}$ field. Let us consider a constant magnetic field oriented in the $\hat{z}$ direction. Suppose the magnetic moment is also oriented in the z direction. The cross product would then be zero, and no torque would occur. However, if we tilt the moment an angle θ away from the z-axis, the cross product does not vanish, and we indeed have some torque.

$$|\vec{\tau}| = \mu B \sin\theta \tag{2}$$

Equation (2) describes the magnitude of the torque, as the direction is easily found by the right-hand-rule. The direction of the torque will always be perpendicular to the plane that θ lies in.

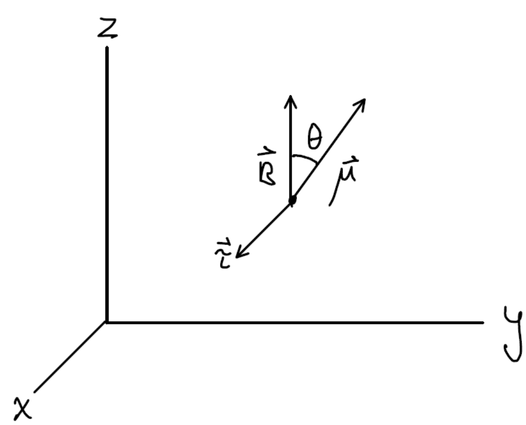

Figure 1: magnetic moment in a magnetic field with torque

Due to this, we can relate the torque to the angular acceleration. The angular acceleration would lie in the plane of θ, so we can say that the torque is related to the acceleration of θ, $\ddot{\theta}$.

$$|\vec{\tau}| = -I\ddot{\theta} \tag{3}$$

Where I is the moment of inertia. Notice the sign. In this case the torque is a *restoring* torque, so we assert that it accelerates in the negative direction. We can then combine equations (2) and (3) to come up with a fully dynamical differential equation that is solvable.

$$I\ddot{\theta} = -\mu B \sin\theta \tag{4}$$

When θ is small, we can make the approximation $\sin\theta \approx \theta$, making equation (4) a fully linear differential equation, which makes it easily solvable. We then solve for θ as a function of time.

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{\mu B}{I}}t\right) \tag{5}$$

The quantities B and I are measurable, and so we can take parts out of this equation to measure. The square root inside the cosine is what we call the angular frequency of the oscillator, the rate at which oscillations occur. By dividing 2π by this frequency, we can get the period T, the time of one oscillation. This is an easily measurable quanitity.

$$\frac{2\pi}{T} = \sqrt{\frac{\mu B}{I}} \tag{6}$$

Squaring this and doing some rearranging provides a linear relationship based on T and B.

$$\frac{1}{T^2} = \frac{\mu}{4\pi^2 I}B \tag{7}$$

By plotting the inverse square of the period for different values of the magnetic field, we can determine the value of the magnetic moment from the slope.

The next dynamical experiment I will discuss is the idea of precession. Precession is the phenomenon in which a spinning object will rotate around different axis than the spinning one. We can prove this using vector algebra. Imagine the magnetic moment, tilted at an angle θ from the z-axis as before, but this time spinning quickly around the axis of $\vec{\mu}$ with angular speed ω_s.

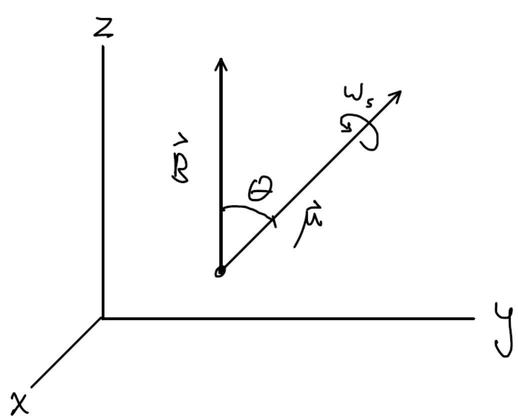

Figure 2: magnetic moment spinning about its own axis in a magnetic field

Again, there is a magnetic field oriented in the z direction, so we find the torque, as in equation (1), to be perpendicular to the plane that θ lies in. The difference in this situation is that initial angular momentum is now a factor. Since the moment is spinning around its axis, there is an initial angular momentum vector $\vec{L}$ oriented in the direction of $\vec{\mu}$.

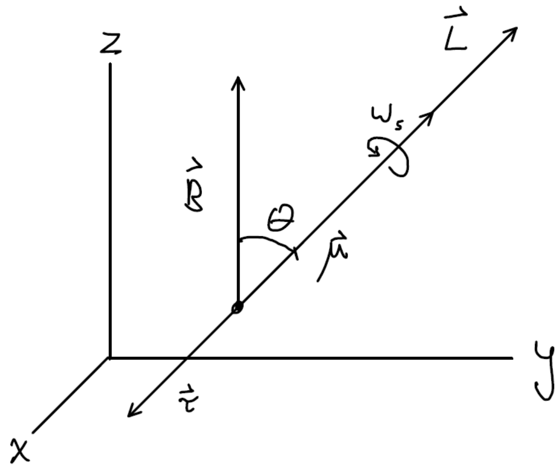

Figure 3: spinning magnetic moment showing torque and angular momentum vectors

The torque is defined to be the time derivative of the angular momentum.

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{8}$$

In the case of simple harmonic motion, the initial angular momentum is zero, meaning any future angular momentum will point in the same direction as the torque. When the moment is spinning around its axis, the same cannot be said.

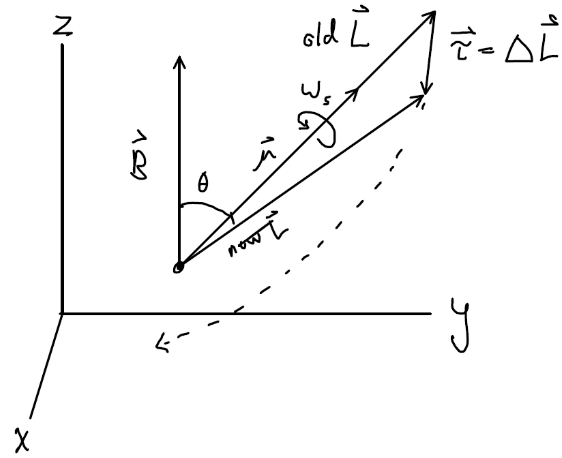

Figure 4: torque causing a change in angular momentum, showing rotation about the z axis

The figure (4) shows how the torque causes a change in angular momentum perpendicular to the current one, essentially rotating the vector around the z-axis. The magnetic moment vector will be aligned with the angular momentum, and thus the magnetic moment also rotates around the z-axis. This is called precession. Mathematically, we can quantify this. Let us call the azimutal angle (angle of μ in the xy-plane) ϕ. This angle measures rotation around the z-axis. We can calculate the torque to be the same as equation (2) in this case. One important measurable quantity is the period, or the amount of time for ϕ to change by 2π. To relate the torque to this angle, we first examine the geometry of the vectors.

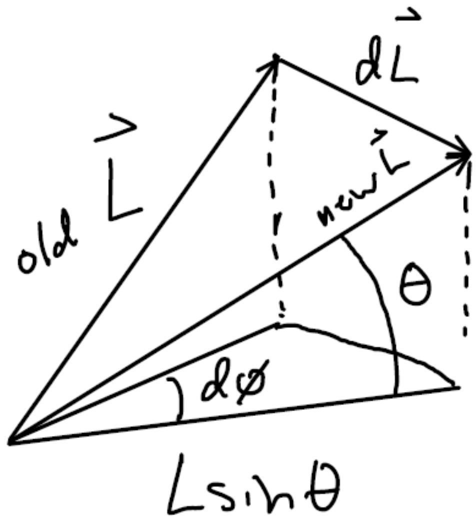

Figure 5: Geometry of the angular momentum vectors in order to find $d\phi$

From the above figure, we see that the change in the angular momentum ($d\vec{L}$) is equal to the arc length along a circle of radius $L\sin\theta$. This is of course just the product of the angle and radius, giving the next equation:

$$dL = L\sin\theta d\phi \tag{9}$$

From the torque equation, we also have that $dL = \tau dt = \mu B \sin\theta dt$. We can then combine these two equations into a relation between ϕ and time.

$$d\phi = \frac{\mu B}{L} dt \tag{10}$$

We can integrate this equation to solve for $\phi(t)$, and also replace L in terms of the angular spin velocity ω_s.

$$\phi(t) = \frac{\mu B}{I\omega_s} t \tag{11}$$

The period is the time at which the moment has completed one full rotation around the z-axis, or in other words it is the time at which $\phi(t) = 2\pi$. Thus we can replace t with the period T and solve.

$$\frac{1}{T} = \frac{\mu B}{2\pi I \omega_s} \tag{12}$$

This is the equation we will be using to determine the value of the magnetic moment when observing precession.

2 Experimental Results

To measure the effects of magnetic torque, I used a macroscopic magnetic moment (shaped like a sphere) observe motion inside a uniform magnetic field. The uniform field was created by a set of Helmholtz coils, measured to have a diameter of 21.25 ± 0.01 cm. Each coil had precisely 131 turns. The equation for determining the value of the magnetic field equidistant between the coils is known:

$$B = \frac{16\mu_0 N I}{\sqrt{125} D} \tag{13}$$

Where I represents the current through the coils, D is the diameter, N is the number of turns, and μ_0 is the permeability of free space [2]. Based on the direction of the current and the use of a field compass, I knew the field was oriented upwards, in the positive z direction. Other quantities measured included the mass (179.3 ± 0.1 g) and radius (2.3 ± 0.1 cm) of the sphere to determine the moment of inertia I_N.

2.1 Simple Harmonic Motion

The first part of the experiment was to measure the period of simple harmonic motion. The moment was displaced by a small angle from its equilibrium position and the oscillations were recorded at 240 frames per second. The current was simultaneously measured and recorded. Using equation (13) I

Current (A) ± 0.001	**Magnetic Field (mT)**	**Frames Per Oscillation** ± 1	**Period (s)** ± 0.004
0.301	0.334 ± 0.001	820	3.417
0.552	0.612 ± 0.001	570	2.375
0.684	0.758 ± 0.001	510	2.125
0.883	0.979 ± 0.001	414	1.725
1.007	1.116 ± 0.001	371	1.546
1.095	1.214 ± 0.001	362	1.508

Table 1: measured data for the simple harmonic motion of a magnetic moment

found the magnetic field, and plotted the relationship in equation (7), with a linear fit.

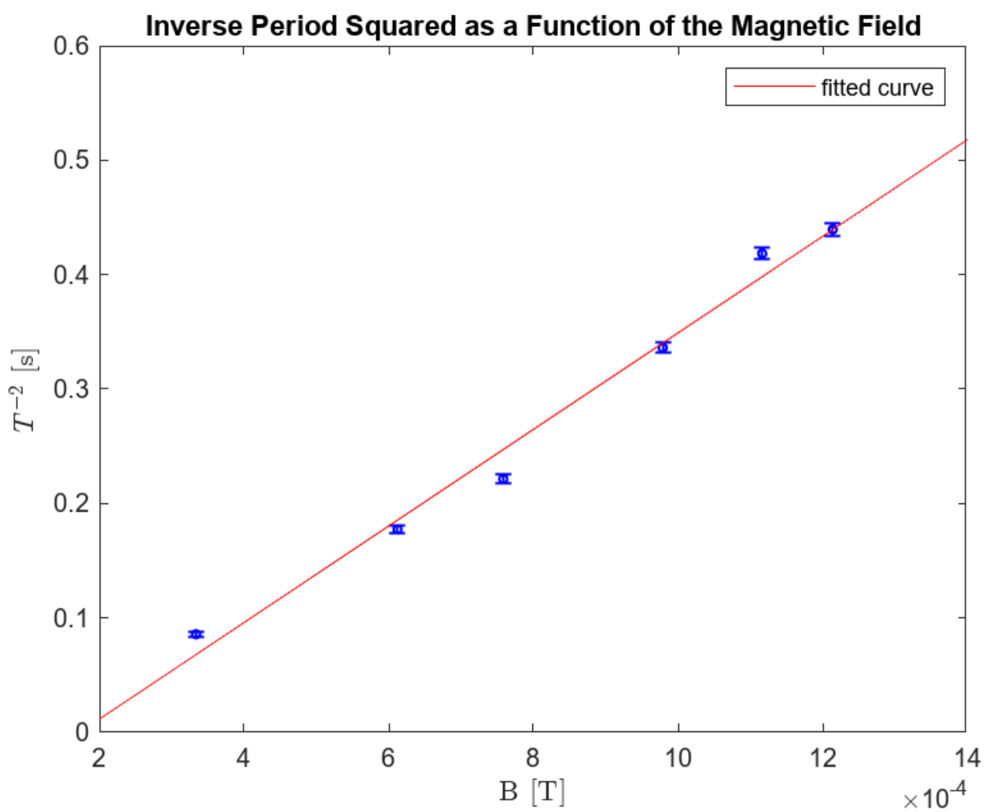

Figure 6: Linear fit of the inverse period squared versus the magnetic field. This relationship is shown in equation (7) and can be used to find the magnetic moment.

The slope of this fit can be used to find μ, the magnetic moment. The slope given by the lineat fit (with 68% confidence intervals) is 422 ± 29. From this, and error propagation [3], I found the value of the magnetic moment to be 0.63 ± 0.5 J/T.

To understand the data better a standard-deviation-of-the-mean analysis was performed on individually calculated magnetic moments for each value of the current. The following table shows these values, as well as the analysis.

Magnetic Moments [J/T]
0.38 ± 0.02
0.43 ± 0.02
0.44 ± 0.02
0.51 ± 0.02
0.56 ± 0.02
0.54 ± 0.02

Mean Average:	0.48
Standard Deviation:	0.07
SDOM:	0.01

Table 2: SDOM analysis of simple harmonic motion

Thus the SDOM-determined value of μ is 0.48 ± 0.01. This is significantly different than the value determined from the linear fit, and this is because the fitted y-intercept was 0.07, which is significant in comparison to the magnitude of the y-axis. The question of which method is more accurate is unclear, as the linear bit gives the best slope for the data, although the line doesn't intersect the origin as theoretically predicted. On the other hand the SDOM analysis is a more accurate description of the purely proportional relationship between the magnetic field and inverse square of the period.

2.2 Precession

The dynamics of precession are more complex than simple harmonic motion, yet we can measure the magnetic moment by observing precession as well. The equation relating the period and magnetic field of precession is given in equation (12). The dependent variale is the inverse of the period, and the independent variable is the magnetic field divide by the spin speed. This was necessary as both the magnetic field and spin speed varied for each measurement. The magnetic field is again found using equation (13).

Current (A) ± 0.001	**Magnetic Field (mT)**	**Spin Speed** ω_s **rad/s**	**Period (s)** ± 0.02
0.301	0.334 ± 0.001	44.36 ± 0.09	30.22
0.449	0.553 ± 0.001	50.5 ± 0.1	15.17
0.698	0.774 ± 0.001	38.45 ± 0.08	15.28
0.896	0.993 ± 0.001	24.94 ± 0.05	15.75
1.054	1.169 ± 0.001	38.96 ± 0.08	12.20

Table 3: measured data for the precession of a magnetic moment

The spin speed ω_s is the angular velocity of the magnetic moment about its own axis, and was calculated from a video recording, as was the period. The recording was 60 frames per second, giving an accuracy of ± 1 frame an accuracy of ± 0.02 seconds.

As with the simple harmonic motion, I performed both an SDOM analysis and a linear fit on the data.

Magnetic Moments [J/T]
0.74 ± 0.02
0.77 ± 0.02
0.78 ± 0.02
0.71 ± 0.02
0.92 ± 0.02

Mean Average:	0.78
Standard Deviation:	0.08
SDOM:	0.02

Table 4: SDOM analysis of precession

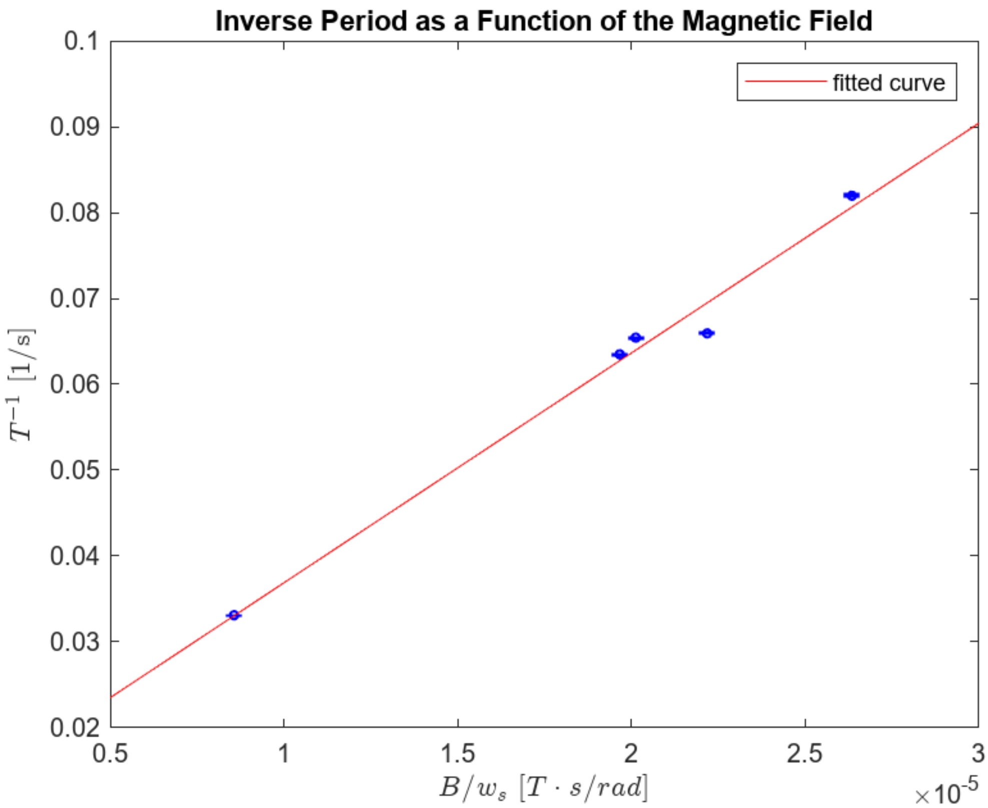

Figure 7: Linear fit of precession data, inverse period versus the magnetic field divided by the angular spin velocity

The value of the magnetic moment determined from the SDOM analysis was $\mu = 0.78 \pm 0.02$. This is still decently different from the values found for the simple harmonic motion data. There are definitely sources of experimental error, which I will discuss in the conclusion. For the linear fit, the determined slope with a 68% confidence interval is 2680 ± 220. Solving for the magnetic moment yields a value of 0.64 ± 0.06 J/T. This is incredibly close to the value determined from the linear fit in the simple harmonic motion section (0.63). That result led me to believe the linear fit may be more accurate than the SDOM for this experiment.

2.3 Further Discussion: Atomic Physics

The macroscopic sphere containing a magnetic moment can be a good analog to an atom. Atoms (and some molecules) have nonzero magnetic moments, and therefore interact with the magnetic field. The results and observations of this experiment lead to the conclusion that an atom such as a hydrogen atom may behave similarly in a uniform magnetic field. A hydrogen atom could potentially exhibit both simple harmonic motion and precession, and the effects may be even more pronounced since the mass and radius are tiny in comparison to the magnetic field. The frequency of oscillations and speed of precession would most likely be much higher. In addition, the quantum mechanics may cause some unforseen differences between the macroscopic and microscopic case. For example, a hydrogen atom may only precess at quantized initial angles.

3 Conclusion

Magnetic moments are one of the most important electromagnetic structures, as they are present in the fundamental building blocks of physics. Atomic spin physics is built on the idea of magnetic moments and their interactions with magnetic fields. This experiment was an attempt to understand those interactions by observing the motion of a macroscopic magnetic moment, which may be analogous to the atomic kind. The equations describing simple harmonic motion and precession were derived and then tested experimentally to try and find the value of the magnetic moment being observed.

With the macroscopic experiment, different sources of error were present. First of all, the small angle approximation adds some error to the linear relationship in equation (7). The angles in the experiment were small, but visible, and therefore the small angle approximation is not exact. In addition, factors such as friction and air resistance may play a factor, although a very small one. Due to the shape of the piece attached to the sphere, gravitation may have affected the motion, and for precession, the moment was never quite spinning around its own axis. The latter would effect the direction of angular momentum, making it slightly off and therefore smaller in the direction I desired. All of these effects led to the calculated values of μ differing greatly between the linear fits and SDOM analyses. The two best values (in my opinion) were the two linear fits, with the two values differing by only 1.5%. Since these values were from different experiments (one from SHM and one from precession) 0.63 or 0.64 is probably the best estimate for the value of the magnetic moment.

References

[1] R. D. Knight, *Physics for scientists and engineers a strategic approach*, 4th ed. (Pearson Education Inc., 2017).

[2] P. J. Mohr, B. N. Taylor, D. B. Newell, and et al., *Crc handbook of chemistry and physics*, edited by W. Haynes, 95th ed. (CRC Press, 6000 Broken Sound Parkway NW, Suite 300, Boca Raton, FL, 2014).

[3] J. R. Taylor, *An introduction to error analysis :* 2nd ed. (University Science Books, Mill Valley, Calif : c1982.).