# Parallel Transport

To set up the idea of parallel transport, we need to understand curvature on manifolds.

<u>Covariant Derivative</u>: The operator $\nabla$ that performs the functions of the partial derivative, independent of the coordinates.

$\nabla$ is a map from $(k, l)$ tensor fields to $(k, l+1)$ tensor fields that satisfies:

1. Linearity: $\nabla(T+S) = \nabla T + \nabla S$
2. Product Rule: $\nabla(T \otimes S) = (\nabla T) \otimes T + T \otimes (\nabla S)$

↳ thus it can always be written as the partial derivative plus a linear transformation.

Consider a vector $V^\nu$. For each direction $\mu$, the covariant derivative $\nabla_\mu$ is the partial derivative $\partial_\mu$ plus a correction given by a matrix $(\Gamma_\mu)^\rho{}_\sigma$

This matrix is called the <u>connection coefficient</u>: $\Gamma^\nu_{\mu\sigma}$

Thus, $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

We can define the transformation law:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

Thus, the transformative property of $\Gamma^\nu_{\mu\lambda}$:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}$$

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The connection coefficients are <u>not</u> the components of a tensor

We introduce 2 more properties of $\nabla$

3. Commutes with contractions: $\nabla_\mu (T^{\lambda}{}_{\lambda\rho}) = (\nabla T)_\mu{}^{\lambda}{}_{\lambda\rho}$

4. reduces to $\partial$ on scalars: $\nabla_\mu \phi = \partial_\mu \phi$

To take the covariant derivative of an arbitrary tensor:

$$\nabla_\sigma T^{\mu_1,\mu_2,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l}$$

Partial:

$$= \partial_\sigma T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l}$$

Sum for upper indices

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l} + \dots + \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1,\dots,\mu_{k-1},\lambda}{}_{\nu_1,\dots,\nu_l}$$

Subtract for lower indices

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1,\dots,\mu_k}{}_{\lambda,\nu_2,\dots,\nu_l} - \dots - \Gamma^{\lambda}_{\sigma\nu_l} T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_{l-1},\lambda}$$

Now we move on to parallel transport

The machinery of connections is vital to comparing vectors at different points in the manifold.

<u>Parallel Transport</u>: the concept of moving a vector along a path, while keeping it constant.

The result of parallel transport on a curved space is dependent on the <u>path</u>

# Parallel Transport

Let's set up the mathematics.

Imagine a curve $x^\mu(\lambda)$. Now imagine we want to parallel transport a tensor $T$. We need $T$ to <u>not change</u> along the path, so in flat space we say:

$$\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu} = 0$$

In curved space we use the covariant derivative. The covariant derivative along the path is:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$$ replacing the partial from flat space

Thus for parallel transport:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1,\dots,\mu_k}{}_{\nu_1,\dots,\nu_l} = 0$$

This equation is well defined, and for a single index vector:

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

Proof.

$$\frac{dx^\sigma}{d\lambda}\nabla_\sigma V^\mu = \frac{dx^\sigma}{d\lambda}\left[\frac{\partial}{\partial x^\sigma}V^\mu + \Gamma^\mu_{\sigma\rho}V^\rho\right]$$

$$= \frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}V^\rho\cdot\frac{dx^\sigma}{d\lambda}$$

const so commutes

# Parallel Transport

We can find an explicit general solution to parallel transport.

It amounts to finding a matrix $P^{\mu}_{\ \rho}(\lambda, \lambda_0)$ that relates the vector at $\lambda_0$ to its value further down the path:

$$V^{\mu}(\lambda) = P^{\mu}_{\ \rho}(\lambda, \lambda_0) V^{\rho}(\lambda_0)$$

$P^{\mu}_{\ \rho}(\lambda, \lambda_0)$ is called the <u>parallel propogator</u> and it depends on the chosen path $\gamma$.

Define $A^{\mu}_{\ \rho}(\lambda) = -\Gamma^{\mu}_{\sigma\rho} \frac{dx^{\sigma}}{d\lambda}$

Then $\frac{d}{d\lambda} V^{\mu} - A^{\mu}_{\ \rho}(\lambda) V^{\rho} = 0$, thus

$$\frac{d}{d\lambda} V^{\mu} = A^{\mu}_{\ \rho}(\lambda) V^{\rho} = A^{\mu}_{\ \sigma} V^{\sigma}$$

Substitute the $P$ equation into this for:

$$\frac{d}{d\lambda} P^{\mu}_{\ \rho}(\lambda, \lambda_0) \cancel{V^{\rho}(\lambda_0)} = A^{\mu}_{\ \sigma}(\lambda) P^{\sigma}_{\ \rho}(\lambda, \lambda_0) \cancel{V^{\rho}(\lambda_0)}$$

$$\hookrightarrow \frac{d}{d\lambda} P^{\mu}_{\ \rho}(\lambda, \lambda_0) = A^{\mu}_{\ \sigma}(\lambda) P^{\sigma}_{\ \rho}(\lambda, \lambda_0)$$

Now we must solve this equation by integrating.

$$P^{\mu}_{\ \rho}(\lambda, \lambda_0) = \delta^{\mu}_{\rho} + \int_{\lambda_0}^{\lambda} A^{\mu}_{\ \sigma}(\eta) P^{\sigma}_{\ \rho}(\eta, \lambda_0)\, d\eta$$

Now we solve by iteration, plugging the right side into the original equation repeatedly:

The $\delta^{\mu}_{\rho}$ gives the identity matrix when $\lambda = \lambda_0$.

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$$P_{\rho}^{\sigma}(\lambda,\lambda_0)=\delta_{\rho}^{\sigma}+\int_{\lambda_0}^{\lambda}A^{\sigma}_{\sigma}(\eta)P^{\sigma}_{\rho}(\eta,\lambda_0)\,d\eta$$

$$\frac{d}{d\lambda}P^{\mu}_{\rho}(\lambda,\lambda_0)=\delta_{\rho}^{\sigma}A^{\mu}_{\sigma}(\lambda)+A^{\mu}_{\sigma}\int_{\lambda_0}^{\lambda}A^{\sigma}_{\sigma}(\eta)P^{\sigma}_{\rho}(\eta,\lambda_0)\,d\eta$$

$$=A^{\mu}_{\rho}(\lambda)+A^{\mu}_{\sigma}\int_{\lambda_0}^{\lambda}A^{\sigma}_{\sigma}(\eta)P^{\sigma}_{\rho}(\eta,\lambda_0)\,d\eta$$

Then we integrate again:

$$P^{\mu}_{\rho}(\lambda,\lambda_0)=\delta^{\mu}_{\rho}+\int_{\lambda_0}^{\lambda}A^{\mu}_{\rho}(\eta)\,d\eta+\int_{\lambda_0}^{\lambda}A^{\mu}_{\sigma}(\eta)\,d\eta\int_{\lambda_0}^{\eta}A^{\sigma}_{\sigma}(\eta')\,d\eta'$$

$$=\delta^{\mu}_{\rho}+\int_{\lambda_0}^{\lambda}A^{\mu}_{\rho}(\eta)\,d\eta+\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta}A^{\mu}_{\sigma}(\eta)A^{\sigma}_{\sigma}(\eta')P^{\sigma}_{\rho}(\eta',\lambda_0)\,d\eta'\,d\eta$$

Then:

$$P^{\sigma}_{\rho}(\lambda,\lambda_0)=\delta^{\sigma}_{\rho}+\int_{\lambda_0}^{\lambda}A^{\sigma}_{\rho}(\eta)\,d\eta+\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta}A^{\sigma}_{\sigma}(\eta)A^{\sigma}_{\sigma}(\eta')P^{\sigma}_{\rho}(\eta',\lambda_0)\,d\eta'\,d\eta$$

Subbing in:

$$\frac{d}{d\lambda}P^{\mu}_{\rho}(\lambda,\lambda_0)=\delta^{\sigma}_{\rho}A^{\mu}_{\sigma}(\lambda)+A^{\mu}_{\sigma}\int_{\lambda_0}^{\lambda}A^{\sigma}_{\rho}(\eta)\,d\eta$$

$$+A^{\mu}_{\sigma}\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta}A^{\sigma}_{\sigma}(\eta)A^{\sigma}_{\sigma}(\eta')P^{\sigma}_{\rho}(\eta',\lambda_0)\,d\eta'\,d\eta$$

Integrating again:

$$P^{\mu}_{\rho}(\lambda,\lambda_0)=\delta^{\mu}_{\rho}+\int_{\lambda_0}^{\lambda}A^{\mu}_{\rho}(\eta)\,d\eta+\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta}A^{\mu}_{\sigma}(\eta)A^{\sigma}_{\rho}(\eta')\,d\eta'\,d\eta+\dots$$

The pattern continues for infinite integrals.

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The nth term in the series is an integral over an n-dimensional right triangle, or n-simplex.

The n'th order term is then:

$$\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_n}\cdots\int_{\lambda_0}^{\eta_2} A(\eta_n)\cdots A(\eta_1)\,d^n\eta$$

$$=\frac{1}{n!}\int_{\lambda_0}^{\lambda}\cdots\int_{\lambda_0}^{\lambda}\mathcal{P}\left[A(\eta_n)\cdots A(\eta_1)\right]d^n\eta$$

$\mathcal{P}$ is the path ordering symbol. The expression

$$\mathcal{P}\left[A(\eta_n)\cdots A(\eta_1)\right]$$

stands for the product of $n$ matrices $A(\eta_i)$, ordered in such a way that the largest value of $\eta_i$ is on the left and each subsequent $\eta_i$ is less than or equal to the previous one.

Thus we have the solution:

$$P(\lambda,\lambda_0)=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\lambda_0}^{\lambda}\cdots\int_{\lambda_0}^{\lambda}\mathcal{P}\left[A(\eta_n)\cdots A(\eta_1)\right]d^n\eta$$

Which is the taylor expansion for the exponential:

$$P(\lambda,\lambda_0)=\mathcal{P}\exp\left[\int_{\lambda_0}^{\lambda}A(\eta)\,d\eta\right]$$

Or more explicitly:

$$P^{\mu}{}_{\nu}(\lambda,\lambda_0)=\mathcal{P}\exp\left[-\int_{\lambda_0}^{\lambda}\Gamma^{\mu}_{\sigma\nu}\frac{dx^{\sigma}}{d\eta}\,d\eta\right]$$