More GR

The Metric

the central element of gr is the metric tensor $g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$

it is a map between geometry and coordinates, and we define a line element of the form:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Connection

In curved geometries a derivative becomes more complicated. At each point we can locally define a tangent space in which we do vector analysis.

To account for different basis vectors at different points we define a covariant derivative:

$$\nabla_\mu V^\mu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

Where the term $\Gamma^\nu_{\mu\lambda} V^\lambda$ is a correction to accomt for the curved geometry. The coefficient Γ is called the connection coefficient and it allows us to "connect" neighbors points on a manifold.

Curvature

When parallel transport fails to preserve the original orientation of a vector, we have a curvature!

We can introduce a notion of intrinsic curvature, curvature we cannot see since the motion is constrained to non curved variables.

For example on the surface of a sphere we can only move in the ϕ and θ directions, when the curvature is embedded in the r-dimension.

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The intrinsic curvature can be quantified by the Riemann tensor

$$R^{\sigma}{}_{\rho\mu\nu} V^{\rho} = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^{\lambda}$$

Ie the curvature depends on the commutation relation of the covariant derivative. If the ∇ commutes, then $R = 0$.

Thus we are looking at infinitesimal parallel transport around a closed path.

The Ricci curvature tensor is

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

Contracting this with the metric gives the curvature scalar:

$$R_{\mu\nu} g^{\mu\nu} = R$$

These two are in the Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Tetrad

Normally in GR we use an ordinary coordinate system with a set of orthonormal basis vectors $e_\mu = \partial_\mu$.

A vector is then $V = V^{\mu} \partial_\mu$

A covector is then $W = W_\mu dx^\mu$ where $e^\mu = dx^\mu$

There is an alternative.

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We can model spacetime from the perspective of a local traveller by introducing a new set of orthonormal basis vectors called tetrad

$$\hat{e}_I \equiv \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\}.$$

At each point of the traveler's world line

$$\hat{e}_I = e^{\mu}_{I}\, \partial_\mu$$

The alphabetic letters I are called internal indices and the greek indices are called external indices.

↳ Latin: refers to locally flat tetrad axes
Greek: refers to space-time coordinates.

The cotetrad basis is $\hat{e}^I = e^I_\mu\, dx^\mu$

The tetrad basis is always flat, and adopts a Minkowski metric when defining a line element locally:

$$ds^2 = \eta_{IJ}\, \hat{e}^I \hat{e}^J$$

The tetrad frame is only valid in the local region around each point along a traveler geodesic.

We can set the spacetime interval in the coordinates equal to flat in the tetrad, we find

$$g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{IJ}\, \hat{e}^I \hat{e}^J$$

$$= \eta_{IJ} (e^I_\mu dx^\mu)(e^J_\nu dx^\nu)$$

$$= \eta_{IJ}\, e^I_\mu e^J_\nu (dx^\mu dx^\nu)$$

Thus we have the relation $\boxed{g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}}$

This allows us to transform between coordinates and a local orthonormal basis

The internal or external indices are contracted for tetrad components:

$$e^{\mu}_{I} e^{I}_{\nu} = \delta^{\mu}_{\nu} \quad \text{and} \quad e^{\mu}_{I} e^{J}_{\mu} = \delta^{J}_{I}$$

We can express vectors & 1-forms in this way:

$$V^{\mu} = e^{\mu}_{I} V^{I}$$

$$e^{I}_{\mu} = (e^{\mu}_{I})^{-1}$$

$$W_{I} = e^{\mu}_{I} W_{\mu}$$

e^{I}_{μ} are the cotetrad components & e^{μ}_{I} are the tetrad components.

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Palatini formulation

Spin Connection: A dedicated internal connection $\omega_\mu{}^{IJ}$ which allows us to differentiate tensors with both internal and external indices covariantly.

For example with a mixed tensor $X_{\nu I}$, we have

$$D_\mu X_{\nu I} = \partial_\mu X_{\nu I} + \Gamma_{\mu\nu}^{\alpha} X_{\alpha I} + \omega_{\mu I}{}^{J} X_{\nu J}$$

This generalized derivative has two correction terms:

↳ the metric connection $\Gamma_{\mu\nu}^{\alpha}$ accounts for external corrections

↳ the spin connection $\omega_{\mu I}{}^{J}$ accounts for internal corrections.

To get a better idea of these connections, recall that we demand the covariant derivative to vanish on the metric tensor:

$$D_\mu g_{\alpha\beta} = 0$$

This means the metric connection must be symmetric in the lower indices: $\Gamma_{\beta\gamma}^{\alpha} = \Gamma_{\gamma\beta}^{\alpha}$

The tensor $\Gamma_{\beta\gamma}^{\alpha} = (\Gamma_{\beta\gamma}^{\alpha})^S + (\Gamma_{\gamma\beta}^{\alpha})^A$ is called the torsion tensor

Since the metric connection is symmetric, space-time is torsion free

We can also demand D_μ vanishes on the internal metric η_{IJ}, which results in

$$\omega_{\mu IJ} = -\omega_{\mu JI}$$

Antisymmetric in the internal indices.

We use ∇_μ to represent the torsion free generalized derivative.

With

$$\Gamma_{\mu\nu}{}^{\sigma} = -\frac{1}{2} g^{\sigma\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right)$$

$$\omega_{\mu J}^{I} = -e_J^{\nu} \left(\partial_\mu e_{\nu I} + \Gamma_{\mu\nu}^{\alpha} e_{\alpha I} \right)$$

Palatini Formulation

Palatini Action

We can define a new action equivalent to the Einstein Hilbert Action called the:

Palatini Action: $S_P = \int_M \sqrt{-g}\, e^\rho_K e^\sigma_L F_{\rho\sigma}{}^{KL} d^4x$

$$= \frac{1}{2}\int_M \sqrt{-g}\,(e^\rho_K e^\sigma_L - e^\sigma_K e^\rho_L) F_{\rho\sigma}{}^{KL} d^4x$$

$$= \frac{1}{2}\int_M \sqrt{-g}\,(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\sigma_\rho \delta^\rho_\beta) e^\alpha_K e^\beta_L F_{\rho\sigma}{}^{KL} d^4x$$

There is an identity:

$$(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\sigma_\rho \delta^\rho_\beta) = -\frac{1}{2}\tilde{\varepsilon}^{\mu\nu\rho\sigma}\tilde{\varepsilon}_{\mu\nu\alpha\beta}$$

Placing this in the action gives:

$$S_P = -\frac{1}{4}\int_M \sqrt{-g}\,(\tilde{\varepsilon}^{\mu\nu\rho\sigma}\tilde{\varepsilon}_{\mu\nu\alpha\beta}) e^\alpha_K e^\beta_L F_{\rho\sigma}{}^{KL} d^4x$$

$$= -\frac{1}{4}\int_M \sqrt{-g}\,(\tilde{\varepsilon}^{\mu\nu\rho\sigma}\tilde{\varepsilon}_{\mu\nu KL}) F_{\rho\sigma}{}^{KL} d^4x$$

$$= -\frac{1}{4}\int_M \sqrt{-g}\,(\tilde{\varepsilon}^{\mu\nu\rho\sigma}\tilde{\varepsilon}_{IJKL}\, e^I_\mu e^J_\nu) F_{\rho\sigma}{}^{KL} d^4x$$

There is another identity:

$$\varepsilon_{IJKL} = \sqrt{-g}\,\tilde{\varepsilon}_{IJKL}$$

Thus the action reduces to

$$S_p = -\frac{1}{4}\int_M \tilde{\varepsilon}^{\mu\nu\rho\sigma} \varepsilon_{IJKL}\, e^K_\mu e^2_\nu F_{\rho\sigma}{}^{KL}\, d^4x$$

We can identify our internal curvature tensor F with the torsion free Riemann tensor as:

$$F_{cdK}{}^{L} := \partial_c \omega_{dK}{}^{L} - \partial_d \omega_{cK}{}^{L} + [\omega_c, \omega_d]_K{}^{L} \equiv R_{cdK}{}^{L}$$

Note two spacetime indices have been replaced with internal indices.

We can place R into the palatini action.

$$S_p = -\frac{1}{4}\int_M \tilde{\varepsilon}^{\mu\nu\rho\sigma} \varepsilon_{IJKL}\, e^I_\mu e^J_\nu R_{\rho\sigma}{}^{KL}\, d^4x$$

$$= -\frac{1}{4}\int_M \tilde{\varepsilon}^{\mu\nu\rho\sigma} \varepsilon_{IJKL}\, e^I_\mu e^J_\nu e^K_\alpha e^L_\beta R_{\rho\sigma}{}^{\alpha\beta}\, d^4x$$

$$= -\frac{1}{4}\int_M \tilde{\varepsilon}^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}\, d^4x$$

$$= -\frac{1}{4}\int_M \sqrt{-g}\, \tilde{\varepsilon}^{\mu\nu\rho\sigma} \tilde{\varepsilon}_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta}\, d^4x$$

$$= -\frac{1}{4}\int_M \sqrt{-g}\,(-2)(\delta^\rho_\alpha \delta^\sigma_\beta - \delta^\sigma_\alpha \delta^\rho_\beta) R_{\rho\sigma}{}^{\alpha\beta}\, d^4x$$

$$= \frac{1}{2}\int_M \sqrt{-g}\,(R_{\alpha\beta}{}^{\alpha\beta} - R_{\beta\alpha}{}^{\alpha\beta})\, d^4x$$

Recalling the antisymmetry of R, we write

$$S_p = \int_M \sqrt{-g}\, R_{\alpha\beta}{}^{\alpha\beta}\, d^4x = \boxed{\int_M \sqrt{-g}\, R\, d^4x}$$

Palatini Formulation

This is the Einstein-Hilbert Action. Thus $S_P \equiv S_{EH}$ under the torsion-free condition.

3+1 Decomposition

We introduce new fields called the triad

$$E^{\nu}_{I} = q^{\nu}_{\mu} e^{\mu}_{I} \quad \text{where} \quad q_{ab} = g_{ab} \text{ (spatial metric)}$$

We also require the machinery of extrinsic curvature. We have defined the Lie derivative previously, and now:

Extrinsic Curvature: $\frac{1}{2} \mathcal{L}_n q_{\mu\nu}$

The Lie derivative of the spatial metric along the time-like normal vector n^{μ}.

Or: $\quad K_{\mu\nu} = q^{\alpha}_{\mu} q^{\beta}_{\nu} \nabla_{\alpha} n_{\beta}$

If we contract the extrinsic curvature with the time-like normal:

$$\begin{aligned} n^{\mu} K_{\mu\nu} &= n^{\mu} q^{\alpha}_{\mu} q^{\beta}_{\nu} \nabla_{\alpha} n_{\beta} = n^{\mu}(\delta^{\alpha}_{\mu} - n_{\mu} n^{\alpha}) q^{\beta}_{\nu} \nabla_{\alpha} n_{\beta} \\ &= n^{\mu} q^{\beta}_{\nu} (\nabla_{\mu} n_{\beta} + n_{\mu} n^{\alpha} \nabla_{\alpha} n_{\beta}) \\ &= q^{\beta}_{\mu} (n^{\mu} \nabla_{\mu} n_{\beta} + n^{\mu} n_{\mu} n^{\alpha} \nabla_{\alpha} n_{\beta}) \\ &= q^{\beta}_{\mu} (n^{\mu} \nabla_{\mu} n_{\beta} - n^{\alpha} \nabla_{\alpha} n_{\beta}) \\ &= 0 \end{aligned}$$

K is a purely spatial quantity.

Back to the 3+1, the triad is purely spatial with respect to its spatial indices.

↳ $E^0_I = 0$

We rewrite the integrand of the palatini action:

$$e^\mu_I e^\nu_J F_{\mu\nu}{}^{IJ} = e^\mu_I e^\nu_J \delta^\alpha_\mu \delta^\beta_\nu F_{\alpha\beta}{}^{IJ}$$

$$= e^\mu_I e^\nu_J (q^\alpha_\mu - n^\alpha n_\mu)(q^\beta_\nu - n^\beta n_\nu) F_{\alpha\beta}{}^{IJ}$$

$$= e^\mu_I e^\nu_J (q^\alpha_\mu q^\beta_\nu - q^\alpha_\mu n^\beta n_\nu - q^\beta_\nu n^\alpha n_\mu + n^\alpha n_\mu n^\beta n_\nu) F_{\alpha\beta}{}^{IJ}$$

$$E^\alpha_I \quad E^\beta_J$$

$$= E^\alpha_I E^\beta_J F_{\alpha\beta}{}^{IJ} - 2 e^\mu_I e^\nu_J q^\alpha_\mu n^\beta n_\nu F_{\alpha\beta}{}^{IJ}$$

$$= E^\alpha_I E^\beta_J F_{\alpha\beta}{}^{IJ} - 2 E^\alpha_I n^\beta n_J F_{\alpha\beta}{}^{IJ}$$

We can rewrite the time-like unit vector as

$$n^\beta = \frac{1}{N}(t^\beta - N^\beta)$$

Then

$$e^\mu_I e^\nu_J F_{\mu\nu}{}^{IJ} = E^\alpha_I E^\beta_J F_{\alpha\beta}{}^{IJ} - 2E^\alpha_I \frac{1}{N}(t^\beta - N^\beta) n_J F_{\alpha\beta}{}^{IJ}$$

$$= E^\alpha_I E^\beta_J F_{\alpha\beta}{}^{IJ} - \frac{2}{N}\left(E^\alpha_I n_J t^\beta F_{\alpha\beta}{}^{IJ} - E^\alpha_I n_J N^\beta F_{\alpha\beta}{}^{IJ}\right)$$

Then, recalling the relation

$$\mathcal{L}_t \omega_\alpha{}^{IJ} = t^\beta F_{\beta\alpha}{}^{IJ} + D_\alpha(t^\beta \omega_\beta{}^{IJ})$$

We can rewrite it to:

$$t^\beta F_{\alpha\beta}{}^{IJ} = -\dot{\omega}_\alpha{}^{IJ} + D_\alpha(t^\beta \omega_\beta{}^{IJ})$$

Note $\mathcal{L}_t$ is the lie derivative with respect to the time-like vector field.

Substituting gives:

$$e^{\mu}_{I} e^{\nu}_{J} F_{\mu\nu}^{IJ} = E^{\alpha}_{I} E^{\beta}_{J} F_{\alpha\beta}^{IJ} - \frac{2}{N} n_I E^{\alpha}_{J} \dot{\omega}_{\alpha}^{IJ} + \frac{2}{N} n_I E^{\alpha}_{J} D_{\alpha}(t^{\beta}\omega_{\beta}^{IJ}) + \frac{2}{N} n_I N^{\alpha} E^{\beta}_{J} F_{\alpha\beta}^{IJ}$$

We can place this back in the full action, along with the relation $\sqrt{-g} = N\sqrt{q}$.

We introduce the densitized triad:

$$\tilde{E}^{\alpha}_{I} \equiv \sqrt{q}\, E^{\alpha}_{I}$$

We can utilize the antisymmetry of the internal indices in order to write:

$$S_P = \int_M d^4x \; \mathrm{Tr}\Big(- \tilde{E}^{\alpha} \dot{\omega}_{\alpha} - 2\underset{\sim}{N} \tilde{E}^{\alpha} \tilde{E}^{\beta} F_{\alpha\beta} + N^{\alpha} \tilde{E}^{\beta} F_{\alpha\beta} - (t\cdot\omega)(D_{\alpha}\tilde{E}^{\alpha}) \Big)$$

The configuration variables are then:

$$\omega_{\alpha}^{IJ},\ \tilde{E}^{\alpha}_{IJ},\ \underset{\sim}{N},\ N^{\alpha},\ (t\cdot\omega)^{IJ}$$

The first two are the canonically conjugate pair and the rest are multipliers generating constraints:

$$S = \mathrm{Tr}(\tilde{E}^{\alpha}\tilde{E}^{\beta} F_{\alpha\beta}) \approx 0$$

$$V_{\alpha} = \mathrm{Tr}(\tilde{E}^{\beta} F_{\alpha\beta}) \approx 0$$

$$G_{IJ} = D_{\alpha} \tilde{E}^{\alpha}_{IJ} \approx 0$$

They form a closed algebra under Poisson Brackets

However, there are some extra constraints

$$\phi^{\alpha\beta} = \varepsilon^{IJKL}\tilde{E}^{\alpha}_{IJ}\tilde{E}^{\beta}_{KL} \approx 0 \quad \text{(primary)}$$

$$\chi^{\alpha\beta} = \varepsilon^{IJKL}(D_\gamma \tilde{E}^{\alpha}_{IJ})[\tilde{E}^{\beta}, \tilde{E}^{\gamma}]_{KL} + (\alpha \leftrightarrow \beta) = 0$$

(secondary constraint)

The original constraints (S, V_α, G_{IJ}) all generate weakly vanishing poisson brackets with each other

Thus (S, V_α, G_{IJ}) are first class, and

$(\phi^{\alpha\beta}, \chi^{\alpha\beta})$ are second class

We can eliminate the second class constraints by solving them and imposing them on the phase space variables, and we arrive at a reduced phase space of variables:

$(\tilde{E}^{a}_{i}, K^{i}_{a})$. The first class constraints are reduced to:

$$S' = (\tilde{E}^{b}_{i}\tilde{E}^{a}_{j} - \tilde{E}^{a}_{i}\tilde{E}^{b}_{j})K^{i}_{a}K^{j}_{b}/\sqrt{q} - \sqrt{q}\,R \approx 0$$

$$V'_\alpha = D_\beta(K^{i}_{a}\tilde{E}^{b}_{i} - K^{i}_{c}\tilde{E}^{c}_{i}\delta^{b}_{a}) \approx 0$$

$$G'_i = \varepsilon_{ijk}K^{j}_{a}\tilde{E}^{ak}$$

Where R is the spatial scalar curvature