# 1

# Sequences and Series

## Learning Objectives:

In this chapter, we will

(a) Give examples of finite and infinite sequences and series using practical examples;

(b) Use the notation $u_n$ to denote the $n^{\text{th}}$ term of a sequence or series;

(c) Know that a sequence can be generated by a formula for the $n^{\text{th}}$ term or a function $y = \mathrm{f}(n)$, where $n$ is a positive integer.

(d) Obtain the $n^{\text{th}}$ term of a sequence using $S_n - S_{n-1}$;

(e) Understand that an infinite sequence or series can converge, and find the limit of the convergent sequence and the sum to infinity when the series is convergent;

(f) Recognise that a sequence is an arithmetic progression (AP) when there is a common difference between consecutive terms;

(g) Recognise that a sequence is a geometric progression (GP) when there is a common ratio between consecutive terms;

(h) Use the formula for the $n^{\text{th}}$ term of an AP or GP;

(i) Use the formula for the sum to $n$ terms of an arithmetic series;

(j) Use the formula for the sum to $n$ terms of a geometric series;

(k) Understand that $r^n \to 0$ as $n \to \infty$ when $|r| < 1$, and use it to deduce the sum to infinity of a geometric series, and the condition for the sum to infinity to exist;

(l) Determine the finite sum or difference of a series made up of arithmetic and geometric series;

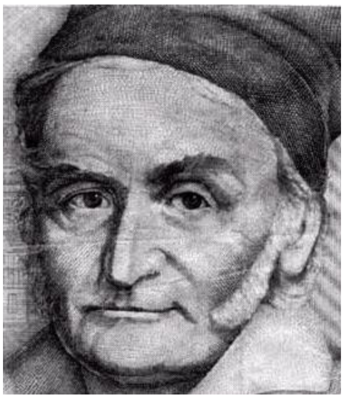
(m) Solve practical problems involving arithmetic and geometric series;

(n) Use $\sum$ notation to express the sum to $n$ terms of a series, e.g. $S_n = \sum_{r=1}^{n} u_r$ ;

(o) Obtain the sum to $n$ terms, $S_n$, for a series given the general term $u_n$ by the method of differences;

## § 1.1 Introduction

Sequences of numbers are often encountered in mathematics. For instance, the numbers 2, 4, 6, 8, 10, … form a sequence of even numbers. Many mathematicians love to explore special sequences and study how to sum them up. One of the greatest mathematicians, Carl Friedrich Gauss was given such a problem one day by his elementary school teacher. His teacher wanted to punish him for being disruptive in class and asked him to add all the whole numbers from 1 to 100. Gauss blurted the answer out almost immediately. His teacher didn't believe Gauss could do it, so he made him show the class how he did it. Gauss simply did $1+100, 2+99, 3+98, ..., 50+51$ and ended up with 50 pairs of 101. Finally, he multiplied $101\times 50$ to get 5050, which is the answer.

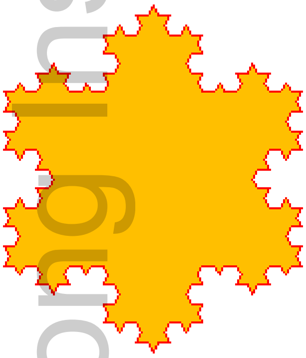


Carl Friedrich Gauss
(1777 – 1855)

There are many real life applications of sequences and series, such as (in the ancient world) in finding the area enclosed by a parabola and a straight line by Archimedes (287 BC – 212 BC),

in working out the value of an annuity (the sum of money to be paid in regular intervals) to a bank for example,


in calculating the area and perimeter of a snowflake,



in investigating the reproduction of a population of honeybees in an ideal environment.

In this chapter, we will study the definitions of special sequences of real numbers and different strategies to sum them including the strategy of summing 1 to 100 that Gauss had "invented" when he was at an age of 9.

## § 1.2 Definitions of Sequences and Series

Consider the following set of numbers

(a) $\{2, 4, 6, 8, 10\}$

(b) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\right\}$

(c) $\{1, 4, 9, 16, 25, ...\}$

In each set, the numbers are listed in a given order and there is a rule for obtaining the next number. For example, (a) is a list of all the even numbers starting from 2 to 10, while (b) is a list of numbers that are reciprocal of the natural numbers.

What is the rule in (c)?

Such sets are called sequences and each member of the set is called a term of the sequence.

A **sequence** is an ordered set in which all the terms are related with each other by a specific rule.

If a sequence consists of a finite number of terms, then it is called a finite sequence.
If a sequence consists of an infinite number of terms, then it is called an infinite sequence. Thus (a) above is a finite sequence while (b) and (c) are infinite sequences.

When the terms of a sequence are added, a series is formed. For example, using (a), $2+4+6+8+10$ is a series.

A **series** is the sum of the terms of a sequence.

If we sum a finite number of terms in a sequence, then we will obtain a finite series. Thus $2+4+6+8+10$ is a finite series. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ is also a finite series even though the terms are taken from (b).
If we sum an infinite number of terms in a sequence (of course the sequence has to be an infinite sequence), then we will obtain an infinite series. Thus $1+4+9+16+25+...$ is an infinite series.

For notation, we call the term in the $n^{\text{th}}$ position of a sequence or a series the $n^{\text{th}}$ term of the sequence or the series. We denote the $n^{\text{th}}$ term by $u_n$ and for the sum of the first $n$ terms of a sequence, we denote it by $S_n$, i.e. $S_n = u_1 + u_2 + u_3 + ... + u_n$.

### Example 1

A sequence $\{u_n\}$ is such that $u_n$ is given by $u_n = n^2$ for $n = 1, 2, 3, \ldots$. and the sum of the first $n$ terms of $u_n$ is denoted by $S_n$.

(i) Write down the first 4 terms of $\{u_n\}$.

(ii) Is $\{u_n\}$ an infinite sequence?

(iii) Find $S_1$, $S_2$, $S_3$ and $S_4$.

**Solution**

(i) $u_1 = 1$, $u_2 = 4$, $u_3 = 9$ and $u_4 = 16$.

(ii) Yes. It is an infinite sequence since there is no final or last number for $n$.

(iii) $S_1 = u_1 = 1$, $S_2 = 1 + 4 = 5$, $S_3 = 1 + 4 + 9 = 14$, $S_4 = S_3 + u_4 = 14 + 16 = 30$ □

### Note

It can be observed from the above example that $u_1 = S_1$ and $S_n = S_{n-1} + u_n$. This is always true and the relationship $S_n = S_{n-1} + u_n$ can be rearranged to $u_n = S_n - S_{n-1}$. The $n^{\text{th}}$ term of a sequence can therefore be obtained by $u_n = S_n - S_{n-1}$ if $S_n$ is given.

> If $u_n$, for $n = 1, 2, 3, \ldots$, denotes the $n^{\text{th}}$ term of a sequence and $S_n$ denotes the sum of the first $n$ terms of a sequence. Then
>
> $u_1 = S_1$ and
>
> $u_n = S_n - S_{n-1}$ (for $n$ starting from 2, i.e. $n \geq 2$).

### Example 2

The sum, $S_n$, of the first $n$ terms of a sequence is given by $S_n = pn + qn^2$. Given also that $S_2 = 8$ and $S_5 = 35$,

(i) find the values of $p$ and $q$,

(ii) deduce an expression, in terms of $n$, for the $n^{\text{th}}$ term of the sequence.

**Solution**

## § 1.3 General Formula of Sequences

There are many ways to define a sequence. For example, we can write a sequence in a list form:

$$\{3, 6, 9, 12, 15, \ldots\}.$$

Another way to define the above sequence is to specify a formula for the general term of the sequence (writing in closed form). Thus we may write

$$u_n = 3n, \ n \in \mathbb{Z}^+.$$

There is another common way of defining a sequence through a recurrence relation. Refer to Appendix A for more details.

**Example 3**

Express the following sequences using a formula for the general term.

(a) $\{1, 2, 3, 4, 5, \ldots\}$
(b) $\{3, 9, 27, 81, 243, \ldots\}$
(c) $\{1, 4, 9, 16, 25, \ldots\}$
(d) $\{-1, 2, -3, 4, \ldots\}$

**Solution**

(a) $\{1, 2, 3, 4, 5, \ldots\}$
General formula: $u_n = n, \ n \in \mathbb{Z}^+$

(b) $\{3, 9, 27, 81, 243, \ldots\}$
General formula: $u_n = 3^n, \ n \in \mathbb{Z}^+$

(c) $\{1, 4, 9, 16, 25, \ldots\}$,
General formula: $u_n = n^2, \ n \in \mathbb{Z}^+$

(d) $\{-1, 2, -3, 4, -5, \ldots\}$
General formula: $u_n = (-1)^n n, \ n \in \mathbb{Z}^+$

□

## § 1.4 Convergence of an Infinite Sequence

The following are two infinite sequences, $\{u_n\}$ and $\{v_n\}$ which have different behaviours when $n \to \infty$.

(a) $u_n = 1 + \dfrac{1}{n^2}, n \in \mathbb{Z}^+$. The terms are 2, $1\frac{1}{4}$, $1\frac{1}{9}$, $1\frac{1}{16}$, $1\frac{1}{25}$, … .
When $n \to \infty$, $u_n \to 1$. Therefore, the terms of the sequence tend towards 1 and we say that the sequence is convergent.

(b) $v_n = n^2, n \in \mathbb{Z}^+$. The terms are 1, 4, 9, 16, 25, … .
When $n \to \infty$, $v_n \to \infty$. Therefore, the terms of the sequence increases to infinity and we say that the sequence is not convergent, i.e. divergent.

Let $u_n$ be the $n^{\text{th}}$ term of a sequence.

If $u_n \to L$ as $n \to \infty$ where $L$ is a real number, the sequence is said to be **convergent**.

$L$ is called the **limit** of the sequence, and it is given by $L = \lim\limits_{n\to\infty} u_n$.

If there is no such $L$, then the sequence is not convergent.
(i.e. the sequence is **divergent**.)

### Example 4

Determine whether the following sequences are convergent. State its limit if the sequence is convergent.

(a) $u_n = \left(\frac{1}{2}\right)^n, n \in \mathbb{Z}^+$ (b) $u_n = (-1)^n, n \in \mathbb{Z}^+$ (c) $u_n = \frac{3^n}{n!}, n \in \mathbb{Z}^+$

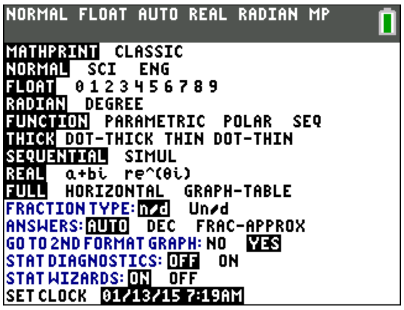
**Solution**

(a) When $n \to \infty$, $u_n = \left(\frac{1}{2}\right)^n \to 0$. The sequence is convergent and its limit is 0.

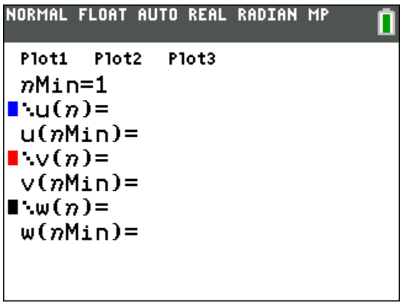
(b) The terms are $-1, 1, -1, 1, \ldots$. The sequence does not converge to a fixed number as the terms alternate between –1 and 1. Therefore, the sequence is not convergent.

(c) Let us investigate the limit of $u_n = \frac{3^n}{n!}$ when $n \to \infty$ using TI-84+.

① Press MODE and ▼ to move the cursor to FUNCTION. Press ▶ to move cursor to SEQ and press ENTER to highlight SEQ.



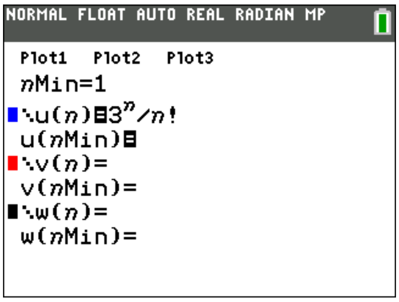

② Press Y=. We need to key in 2 necessary information about the general formula for $u_n$. First, the value of *n*Min refers to the minimum value of $n$ and in this case is 1 which is also the default setting of the G.C..



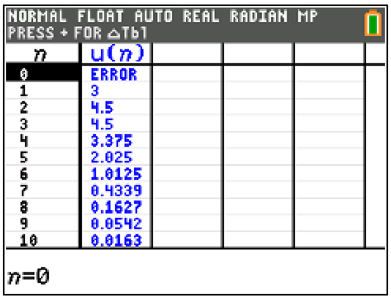
③ Secondly, the general formula can be entered for $u(n)$. In this case, we press

[3] [^] [X,T,Θ,*n*] [►] [÷] [X,T,Θ,*n*] [MATH]
[►] [►] [►] [▼] [▼] [▼] [ENTER]

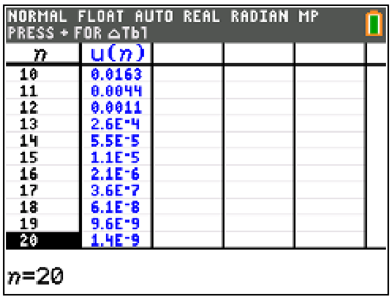
The value of $u(n\text{Min})$ refers to the value of $u_1$ since $n\text{Min} = 1$. This can be computed by the GC using the general formula, so we leave it blank.




④ Press [2nd] [GRAPH] (for Table).




⑤ Press [▼] to scroll down the table to see how the behaviour of the sequence when $n$ increases.




From the G.C., observe that the value of $u_n$ decreases and tends towards 0 as $n$ increases. The sequence is convergent and its limit is 0.

<u>Algebraic Method</u>

We note that $\dfrac{3^n}{n!} > 0$ for $n \in \mathbb{Z}^+$.

Next, we write $\dfrac{3^n}{n!} = \left(\dfrac{3}{1}\times\dfrac{3}{2}\times\dfrac{3}{3}\right)\times\dfrac{3}{4}\times\dfrac{3}{5}\times\ldots\times\dfrac{3}{n} < \dfrac{9}{2}\left(\dfrac{3}{4}\right)^{n-3}$. That is $0 < \dfrac{3^n}{n!} < \dfrac{9}{2}\left(\dfrac{3}{4}\right)^{n-3}$.

Since $\dfrac{9}{2}\left(\dfrac{3}{4}\right)^{n-3} \to 0$ as $n \to \infty$.

Thus, the limit of the sequence is 0, i.e., $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{3^n}{n!} = 0$. Therefore, the sequence is convergent. □

## § 1.5 Convergence of an Infinite Series

The definition of the convergence of a sequence has been introduced and now we shall take a look at the definition of the convergence of a series.

Let $S_n$ be the sum of the first $n$ terms of any sequence.

If $S_n \to L$ as $n \to \infty$ where $L$ is a real number, the series is said to be **convergent**.

The **limit** of the series, $L$ is given by $L = \lim\limits_{n\to\infty} S_n = S_\infty$, where $S_\infty$ is called the sum to infinity of the sequence.

If there is no such $L$, then the series is not convergent, i.e. the series is **divergent**.

**Example 5**

Given that $S_n$ is the sum of the first $n$ terms of a sequence, determine if $S_n$ is convergent. State its sum to infinity, $S_\infty$, if it is convergent.

(a) $S_n = 6 - \dfrac{1}{n+1} - \dfrac{1}{n+2}$,

(b) $S_n = n^2 + 2n$,

(c) $S_n$ is obtained from a *constant sequence*.

**Solution**

We shall continue to look at some special sequences and their series in the next section.

## § 1.6 Arithmetic and Geometric Progression

### Arithmetic Progression (AP)

An arithmetic progression is a sequence of numbers in which each term, other than the first term, is obtained by adding a constant to the preceding term. This constant is called the common difference. For example

(a) 2, 4, 6, 8, 10, …

(b) 7, 4, 1, $-2$, $-5$, …

If the first term of an AP is $a$ and the common difference is $d$, then the arithmetic progression, $\{u_n\}$, can be expressed as

$$u_1 = a,\quad u_2 = a+d,\quad u_3 = a+2d,\ u_4 = a+3d,\ \ldots .$$

The $n^{\text{th}}$ term of an arithmetic progression, $u_n$, is given by

$$u_n = a + (n-1)d .$$

**Note**

1. When $d > 0$, as in (a), the AP is an increasing sequence.
2. When $d < 0$, as in (b) the AP is a decreasing sequence.
3. When $d = 0$, the AP is $\{a, a, a, \ldots\}$. which is called the constant sequence of $a$.

**Example 6**

Given that the first three consecutive terms of an arithmetic progression are $6$, $x$ and $-2$ respectively, find the common difference and hence find

(i) the $5^{\text{th}}$ term of the progression, (ii) an expression for the $n^{\text{th}}$ term.

**Solution**

**Note**

If $a$, $b$ and $c$ are three consecutive terms in an AP, then $b = \dfrac{a+c}{2}$ where $b$ is the **arithmetic mean** of $a$ and $c$. How can we use this idea to find $x$ in Example 6?

**Example 7**

It is given that the 4th and 9th terms of an arithmetic progression are 16 and 39 respectively. Find the common difference and the first term of the progression. Hence find an expression for the $n^{\text{th}}$ term of the progression.

**Solution**

Let $u_n$ be the $n^{\text{th}}$ term of the AP, $a$ be the first term and $d$ be the common difference.

$u_4 = a + 3d = 16$

$u_9 = a + 8d = 39$

Solving both equations simultaneously

$5d = 23 \Rightarrow d = 4.6$

First term, $a = 16 - 3(4.6) = 2.2$

$u_n = 2.2 + (n-1)(4.6) = 4.6n - 2.4$ □

**Sum to *n* terms of an Arithmetic Series**

Recall that when Gauss was asked to sum all the whole numbers from 1 to 100, he simply added the 1 to 100, 2 to 99 and so on to get 50 pairs of 101. We will now use this idea to find the sum to $n$ terms of an arithmetic series.

The arithmetic series is written in its original order, (1) and in its reverse order, (2).

$$S_n = a + (a+d) + (a+2d) + \ldots + [a+(n-2)d] + [a+(n-1)d] \quad \text{--- (1)}$$

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \ldots + (a+2d) + (a+d) + a \quad \text{--- (2)}$$

Summing (1) and (2) gives

$$2S_n = [2a+(n-1)d] + [2a+(n-1)d] + [2a+(n-1)d] + \ldots + [2a+(n-1)d] + [2a+(n-1)d]$$

Since there are $n$ terms of the same expression $[2a+(n-1)d]$,

$$2S_n = n[2a+(n-1)d]$$

$$\Rightarrow S_n = \frac{n}{2}[2a+(n-1)d].$$

The sum to $n$ terms of an arithmetic series, denoted by $S_n$, is given by

$$S_n = \frac{n}{2}[2a+(n-1)d],$$

where $a$ is the first term of the series and $d$ is the common difference.

Alternatively, we can also write $S_n$ as $S_n = \frac{n}{2}(a+l)$,

where $l = a+(n-1)d$ is the last term (i.e. $n^{\text{th}}$ term) of the series.

**Note**

1. For subsequent questions, the result for the sum to $n$ of an arithmetic series can be quoted without proof.
2. When $d = 0$, the sum of the $n$ consecutive terms of the AP is $\underbrace{a + a + a + \ldots + a}_{n \text{ terms}} = na$.
3. The series with each term following the AP is known as the arithmetic series.

**Example 8**

The first term and the third term of an arithmetic series are $-50$ and $-42$. Calculate the value of the first positive term of this series, and the sum of all negative terms.

**Solution**

**Example 9**

The sum of the first forty terms of an arithmetic progression with first term $a$ and common difference $d$ is $S$. The sum of the first forty even-numbered terms is $S - 1440$. Find the value of $d$.

**Solution**

### Test for Arithmetic Progression

To show that the sequence, $u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1},\ \ldots$ is an arithmetic progression, we need to show

$$u_{n+1} - u_n = \text{constant}$$

for all positive integer $n$.

**Note**

1. The constant found is the common difference of the AP.
2. It is **not sufficient** to make use of specific terms such as $u_2 - u_1 = u_3 - u_2 = \text{constant}$ to show that a sequence is an AP. For example, the sequence of odd primes $3, 5, 7, 11, 13, \ldots$ has $u_2 - u_1 = u_3 - u_2$, but it is NOT an AP.

**Example 10**

The sum of the first $n$ terms of a sequence, $S_n$, is given by $S_n = n(n+2)$. Find the $n^{\text{th}}$ term of this sequence and hence, show that the terms of the sequence are in arithmetic progression.

**Solution**

### Geometric Progression (GP)

A geometric progression is a sequence of numbers in which each term, other than the first term, is obtained by multiplying a constant to the preceding term. This constant is called the common ratio. For example

(a) 1, 2, 4, 8, 16, …

(b) 4, $-12$, $36$, $-108$, 324, …

In general, if the first term of a geometric progression is $a$ and the common ratio is $r$, then the geometric progression, $\{u_n\}$, is

$$u_1 = a, \quad u_2 = ar, \quad u_3 = ar^2, \quad u_4 = ar^3, \ \ldots$$

The $n^{\text{th}}$ term of a geometric progression, $u_n$, is given by

$$u_n = ar^{n-1}.$$

**Note**

1. When $r > 0$, as in (a), all the terms in the GP have the same sign.
2. When $r < 0$, as in (b), the terms in the GP alternate in sign.
3. When $r = 1$, the GP is $\{a,\ a, \ldots,\ a\}$, which is called the constant sequence of $a$.

## Example 11

A geometric progression $H$ is such that all its terms are negative. It has first term $-\frac{1}{2}$ and the sixth term is $\frac{9}{4}$ of the fourth term.

Find the $n^{\text{th}}$ term of the geometric progression.

**Solution**

Let $u_n$ be the $n^{\text{th}}$ term of $H$, with first term $a$ and common ratio $r$.

$u_6 = \frac{9}{4}u_4$

$\Rightarrow ar^5 = \frac{9}{4}ar^3$

$\Rightarrow r^2 = \frac{9}{4}$

$\Rightarrow r = \frac{3}{2}$ or $-\frac{3}{2}$ (reject because all the terms are negative)

$$u_n = -\frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$$

$$= -\frac{3^{n-1}}{2^n}$$

□

**Note**

Recall that if $a$, $b$, $c$ are three consecutive terms in an AP then $b$ is the arithmetic mean of $a$ and $c$. We have a similar result for geometric progression:

If $a$, $b$, $c$ are three consecutive terms in a GP, then $b^2 = ac$ where $b$ is the **geometric mean** of $a$ and $c$.

## Sum to $n$ terms of a Geometric Series

Like the arithmetic series, we also have a formula to find the sum to $n$ terms of a geometric series. This formula can be derived easily as follows.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad \text{--- (1)}$$

Multiply both sides by $r$, we obtain

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad \text{--- (2)}$$

(1) – (2) gives

$$\begin{aligned} S_n - rS_n &= a + \cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}} \\ &\quad - \left(\cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{n-1}} + ar^n\right) \\ &= a - ar^n \end{aligned}$$

$$S_n(1-r) = a\left(1-r^n\right) \Rightarrow S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

The sum to $n$ terms of a geometric series, denoted by $S_n$, is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}, \; r \neq 1$$

or

$$S_n = \frac{a\left(r^n-1\right)}{r-1}, \; r \neq 1$$

where $a$ is the first term of the series and $r$ is the common ratio.

**Note**

1. The two formulae above are equivalent: $S_n = \frac{a\left(1-r^n\right)}{1-r} \times \frac{-1}{-1} = \frac{a\left(r^n-1\right)}{r-1}$. The first formula is usually used when the common ratio, $r < 1$ and the second formula is usually used when the common ratio, $r > 1$.
2. The above formula does not apply when $r = 1$. When $r = 1$, the sum of the $n$ consecutive terms of the GP is simply $\underbrace{a + a + a + \ldots + a}_{n \text{ terms}} = na$.
3. The series where each term follows the GP is known as the geometric series.

### Example 12

The first term, second term and last term of a geometric progression are 1024, 512 and 1. Find the sum of the geometric progression.

**Solution**

Let $u_n$ be the $n^{\text{th}}$ term of the GP, with first term $a$ and common ratio $r$.

$$a = 1024 \qquad r = \frac{512}{1024} = \frac{1}{2}$$

$$u_n = ar^{n-1} \Rightarrow 1 = 1024\left(\frac{1}{2}\right)^{n-1} \Rightarrow \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1024} = \left(\frac{1}{2}\right)^{10}$$

$$\therefore\ n-1 = 10 \Rightarrow n = 11$$

$$S_{11} = \frac{a(1-r^n)}{1-r} = \frac{1024\left(1-\left(\frac{1}{2}\right)^{11}\right)}{1-\frac{1}{2}} = 2047$$

□

### Example 13

Find the least number of terms of the geometric series $2+3+\frac{9}{2}+\frac{27}{4}+\ldots$ that must be taken in order to obtain a sum exceeding 48.

**Solution**

### Test for Geometric Progression

To show that the sequence, $u_1,\ u_2,\ \ldots,\ u_n,\ u_{n+1},\ \ldots$ is a geometric progression, we need to show

$$\frac{u_{n+1}}{u_n} = \text{constant}\ .$$

**Note**

1. The constant is the common ratio of the GP.
2. It is **<u>not sufficient</u>** to make use of specific terms such as $\frac{u_2}{u_1} = \frac{u_3}{u_2} = \text{constant}$ to show that a sequence is a GP.

### Example 14

The sum of the first $n$ terms of a series is $-9+\frac{4^n}{3^{n-2}}$. Obtain an expression for the $n^{\text{th}}$ term of the series. Prove that the series follows a geometric progression and state its first term and common ratio.

**Solution**

$$S_n = -9+\frac{4^n}{3^{n-2}}$$

$$\begin{aligned}
u_n &= S_n - S_{n-1} \\
&= \left(-9+\frac{4^n}{3^{n-2}}\right) - \left(-9+\frac{4^{n-1}}{3^{n-3}}\right) \\
&= \frac{4^n}{3^{n-2}} - \frac{4^{n-1}}{3^{n-3}} \\
&= \frac{4^{n-1}}{3^{n-3}}\left(\frac{4}{3}-1\right) \\
&= \frac{4^{n-1}}{3^{n-2}} = 3\left(\frac{4^{n-1}}{3^{n-1}}\right) = 3\left(\frac{4}{3}\right)^{n-1}
\end{aligned}$$

Since $\frac{u_{n+1}}{u_n} = \frac{3\left(\frac{4}{3}\right)^n}{3\left(\frac{4}{3}\right)^{n-1}} = \frac{4}{3} =$ constant, therefore the terms are in GP.

The first term is 3 and the common ratio is $\frac{4}{3}$. □

## Convergence of Geometric Progression and Geometric Series

Consider the following progressions:

(a) $3, 6, 12, 24, \ldots, 3(2)^n$

This is a GP with $r=2$ and when $n \to \infty$, $3(2)^n \to \infty$. Since the terms in the sequence increases to infinity, the GP is not convergent.

(b) $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots, 3\left(\frac{1}{2}\right)^n$

This is a GP with $r=\frac{1}{2}$ and when $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$. Since the terms in the sequence tend towards 0, this GP is convergent.

We will now look at the convergence of series where each of the term follows a geometric progression:

(a) $3+6+12+24+\ldots$

The sum to $n$ terms of the GP, $S_n$, with $r=2$, is $S_n=\frac{3\left(2^n-1\right)}{2-1}=3\left(2^n-1\right)$.

As $n \to \infty$, $2^n \to \infty$. Since $S_n \to \infty$ as $n \to \infty$, the series does not converge.

(b) $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\ldots$

The sum to $n$ terms of the GP, $S_n$, with $r=\frac{1}{2}$, is $S_n=\frac{3\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}=6\left(1-\left(\frac{1}{2}\right)^n\right)$.

As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$. Since $S_n \to 6$ as $n \to \infty$, the series converges.

**Note**

The geometric series converges when $|r|<1$.

Let us consider a geometric series, $S_n=\frac{a\left(1-r^n\right)}{1-r}$.

When $|r|<1$, $r^n \to 0$ as $n \to \infty$.

Thus, $S_n \to \frac{a(1-0)}{1-r}=\frac{a}{1-r}$. Hence the series converges to $\frac{a}{1-r}$.

The sum to infinity of a geometric series, denoted by $S_\infty$, exists if and only if $|r|<1$ and is given by

$$S_\infty = \frac{a}{1-r} \text{ where } |r|<1.$$

**Note**

When $r=1$, the $S_n = \underbrace{a+a+a+\ldots a}_{n \text{ terms}} = na$. As $n \to \infty$, $S_n \to \infty$ provided $a \neq 0$. Therefore the series is not convergent, i.e. $S_\infty$ does not exist.

**Example 15**

Find the set of values of $x$ such that the sum to infinity of the geometric series $2+3x+\frac{9x^2}{2}+\frac{27x^3}{4}+\ldots$ exists.

**Solution**

We will now look at more questions involving arithmetic and geometric progressions. Some of these questions may also include real-life applications.

**Example 16**

The seventh, fourth and third term of an arithmetic progression are the first three consecutive terms of a non-constant geometric progression. Find the common ratio of the geometric progression and show that the geometric progression is convergent. Given that the seventh term of the arithmetic progression is $-2$, find the limit of the geometric series.

**Solution**

**Example 17**

The terms $u_1$, $u_2$, $u_3, \ldots$ form an arithmetic progression with common difference $d$. Another sequence is defined by $v_n = u_{2n} + u_{2n-1}$ for all positive integers $n$.

Show that $\{v_n\}$ is an arithmetic progression.

**Solution**

$$\begin{aligned} v_{n+1} &= u_{2(n+1)} + u_{2(n+1)-1} \\ &= u_{2n+2} + u_{2n+1} \\ &= u_1 + (2n+1)d + u_1 + (2n)d \\ &= 2u_1 + 4nd + d \\ v_n &= u_{2n} + u_{2n-1} \\ &= u_1 + (2n-1)d + u_1 + (2n-2)d \\ &= 2u_1 + 4nd - 3d \end{aligned}$$

Thus, $v_{n+1} - v_n = (2u_1 + 4nd + d) - (2u_1 + 4nd - 3d) = 4d$ (constant)

Since $4d$ is a constant (independent of $n$), therefore $\{v_n\}$ is an AP. □

**Note**

Extension to Example 17:

If the terms $u_1$, $u_2$, $u_3, \ldots$ form a geometric progression with common ratio $r$, and a new sequence is defined by $v_n = u_{2n} \cdot u_{2n-1}$ for all positive integers $n$.

Is $\{v_n\}$ a geometric progression?

**Example 18**

(i) A company is digging for new water source. Using machine $A$, the depth dug in the first day is 110 m. On each subsequent day, the depth dug is $\frac{6}{7}$ of the depth dug on the previous day. How many complete days does it take for the depth dug to exceed 80% of the theoretical maximum total depth?

(ii) Using machine $B$, the depth dug on the first day is 110 m. On each subsequent day, the depth dug is 11 m less than the previous day. Digging continues daily and stops on the day when the depth dug is less than 20 m. What is the total depth when the digging is done?

**Solution**

(ii) Let $U_n$ be the depth dug on the $n^{\text{th}}$ day.

Therefore, $U_n = 110 + (n-1)(-11) = 121 - 11n$

Having a dug death of less than 20 m means $121 - 11n < 20$

$\Rightarrow 101 < 11n \Rightarrow n > 9.18$

Therefore on the 10th day, the depth dug is less than 20 m.

Total depth $= \frac{10}{2}\left[2(110) + 9(-11)\right] = 605$ □

**Example 19**

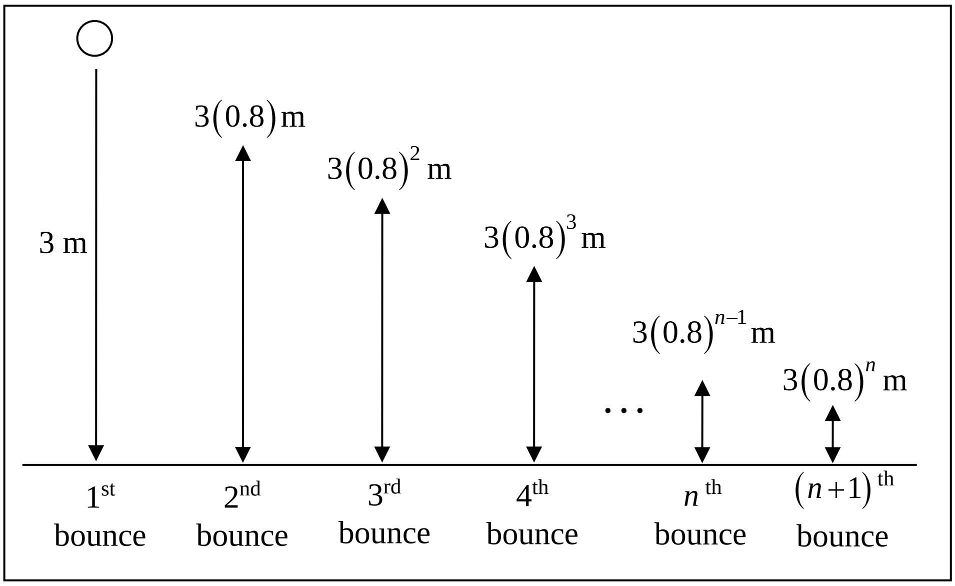
Each time that a tennis ball is released it rebounds to 0.8 of the height from which it fell. It is initially dropped from a point 3 metres from the floor.

(a) Find an expression, in terms of $n$, for the total distance covered by the tennis ball before the $(n+1)^{\text{th}}$ bounce.

(b) Find the total distance covered by the tennis ball before coming to a rest.

**Solution**

(a)




Total distance covered by the tennis ball before the $(n+1)^{\text{th}}$ bounce is

$$=3+2\left[3(0.8)+\ldots+3(0.8)^n\right]$$

$$=3+2(3)\left[0.8+(0.8)^2+\ldots+(0.8)^n\right]$$

$$=3+6\left(\frac{0.8\left[1-(0.8)^n\right]}{1-0.8}\right)=3+24-24(0.8)^n=27-24(0.8)^n$$

Therefore the total distance covered by the tennis ball before the $(n+1)^{\text{th}}$ bounce is $27-24(0.8)^n$.

(b) As $n\to\infty$, $(0.8)^n\to 0$.

Therefore, $27-24(0.8)^n\to 27$. Hence the tennis ball travelled a distance of 27 m before coming to a rest. □

**Example 20**

Karen deposited \$300 into an account at the beginning of one year and then a further \$300 at the beginning of each subsequent year. The interest for the account is added at the end of each year at a fixed rate of 2% of the amount in the account at the beginning of that year. If Karen decides not to withdraw any money out of the account, how much would she have in the account at the end of 15 years? Give your answer to 2 decimal places.

**Solution**

Let $T_n$ be the total amount of money in the account at the end of $n$ years.

$$T_1 = 300 + 300(0.02) = 300(1.02)$$

$$T_2 = \left[300 + 300(1.02)\right](1.02) = 300(1.02) + 300(1.02)^2$$

$$T_3 = \left[300 + 300(1.02) + 300(1.02)^2\right](1.02) = 300(1.02) + 300(1.02)^2 + 300(1.02)^3$$

$$\vdots$$

$$\begin{aligned} T_{15} &= 300(1.02) + 300(1.02)^2 + 300(1.02)^3 + \ldots + 300(1.02)^{15} \\ &= 300\left[(1.02) + (1.02)^2 + (1.02)^3 + \ldots + (1.02)^{15}\right] \\ &= 300 \times \frac{1.02\left(1.02^{15} - 1\right)}{1.02 - 1} \\ &= 15300\left(1.02^{15} - 1\right) \\ &= 5291.786 \end{aligned}$$

That is, Karen would have \$5291.79 at the end of 15 years. □

## Tutorial 1A

1. Determine if the following sequences are increasing, decreasing or neither.

(a) $u_n = -n^2,\ n \in \mathbb{Z}^+$ (b) $u_n = \dfrac{(-1)^n}{n},\ n \in \mathbb{Z}^+$

(c) $u_n = \dfrac{n^2}{n^3 + 2014},\ n \in \mathbb{Z}^+$

2. Determine which of the following sequences, $\{u_n\}$ for $n \in \mathbb{Z}^+$, converge and which diverge. Find the limit of the converging sequence.

(a) $u_n = \dfrac{1}{3} - 2\left(\dfrac{1}{4}\right)^{n-1}$ (b) $u_n = \dfrac{n+1}{n}$

(c) $u_n = \dfrac{e^n}{n}$ (d) $u_n = \sin n\pi$

(e) $u_n = \sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{n}\right)$

3. [A level N82/P1/Q1a, A level J83/P1/Q1a]

(a) The sum of the first nine terms of an arithmetic progression is 75 and the twenty-fifth term is also 75. Find the common difference and the sum of the first hundred terms. [4]

(b) An arithmetic progression has the first term 1000 and common difference −1.4. Calculate the value of the first negative term of the progression and the sum of all the positive terms. [4]

4. [A level N05/P2/Q4 modified]
It is given that $a$, $b$, $c$ are the first three terms of a geometric progression. It is also given that $a$, $c$, $b$ are the first three terms of an arithmetic progression.

(i) Show that $b^2 = ac$ and $c = \dfrac{a+b}{2}$. [2]

(ii) Hence show that $2\left(\dfrac{b}{a}\right)^2 - \left(\dfrac{b}{a}\right) - 1 = 0$. [2]

(iii) Given that the sum to infinity of the geometric progression is $S$, find $S$ in terms of $a$. [4]

5. [A level N88/P1/Q13b]
In a geometric progression, the first term is 12 and the fourth term is $-\dfrac{3}{2}$. Let the sum of the first $n$ terms of the progression be $S_n$ and the sum to infinity of the progression be $S$.
Find the least value of $n$ for which the magnitude of the difference between $S_n$ and $S$ is less than 0.001. [7]

6. [RVHS16/Promo/Q6]

(i) The sum of the first $n$ terms of a series is given by $S_n = \frac{1}{3}n(n+11)$. By finding an expression for the $n^{\text{th}}$ term of the series, show that this is an arithmetic series. Hence, find the common difference of the arithmetic series. [4]

(ii) The first, seventh and $m^{\text{th}}$ term of the series in part (i) are the first three consecutive terms of a geometric series. In either order, find the value of $m$, and determine if the sum to infinity of the geometric series exists. [3]

7. [RI16/Promo/Q9]

(a) The $n$th term of a series is given by $T_n = \mathrm{e}^{2+nx(x+1)}$, where $x$ is a constant.

(i) Show that this series is geometric. [2]

(ii) Find the set of values of $x$ for the sum to infinity to exist. [2]

(b) An arithmetic progression $A$ has $2N$ terms with first term $a$ and fifth term $b$, $N \geq 3$. The sum of all its terms is three times the sum of its first $N$ terms, $S$.

(i) Show that

$$b = \left(\frac{N+9}{N+1}\right)a.$$ [3]

(ii) When $N = 39$, it is known that $S = \frac{1521}{4}$. Find the third term of $A$. [3]

8. [A level 17/P2/Q2]

An arithmetic progression has first term 3. The sum of the first 13 terms of the progression is 156.

(i) Find the common difference. [2]

A geometric progression has first term 3 and common ratio $r$. The sum of the first 13 terms of the progression is 156.

(ii) Show that $r^{13} - 52r + 51 = 0$. Show that the common ratio cannot be 1 even though $r = 1$ is a root of this equation. Find the possible values of the common ratio. [4]

(iii) It is given that the common ratio of the geometric progression is positive, and that the $n$th term of this geometric progression is more than 100 times the $n$th term of the arithmetic progression. Write down an inequality, hence find the smallest possible value of $n$. [3]

9. [A level 19/P1/8]

(a) An arithmetic series has first term $a$ and common difference $2a$, where $a \neq 0$ . A geometric series has first term $a$ and common ratio 2. The $k$th term of the geometric series is equal to the sum of the first 64 terms of the arithmetic series. Find the value of $k$ . [3]

(b) A geometric series has first term $f$ and common ratio $r$, where $f$ , $r \in \mathbb{R}$ and $f \neq 0$. The sum of the first four terms of the series is 0. Find the possible values of $f$ and $r$. Find also, in terms of $f$ , the possible values of the sum of the first $n$ terms of the series. [4]

(c) The first term of an arithmetic series is negative. The sum of the first four terms of the series is 14 and the product of the first four terms is 0. Find the $11^{\text{th}}$ term of the series. [4]

10. [A level 14/P2/Q3]

In a training exercise, athletes run from a starting point $O$ to and from a series of points, $A_1, A_2, A_3, \ldots\ldots$, increasingly far away in a straight line. In the exercise, athletes start at O and run stage 1 from $O$ to $A_1$ and back to $O$, then stage 2 from $O$ to $A_2$ and back to $O$, and so on.

(i)

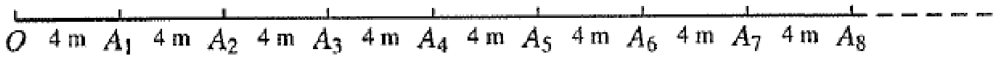



**Fig. 1**

In Version 1 of the exercise, the distances between adjacent points are all 4 m (see Fig. 1).

(a) Find the distance run by an athlete who completes the first 10 stages of Version 1 of the exercise. [2]

(b) Write down an expression for the distance run by an athlete who completes $n$ stages of Version 1. Hence find the least number of stages that the athlete needs to complete to run at least 5 km. [4]

(ii)


**Fig. 2**

In Version 2 of the exercise, the distances between the points are such that $OA_1 = 4$ m , $A_1A_2 = 4$ m , $A_2A_3 = 8$ m and $A_nA_{n+1} = 2A_{n-1}A_n$ (see Fig. 2). Write down an expression for the distance run by an athlete who completes $n$ stages of Version 2.

Hence find the distance from $O$, and the direction of travel, of the athlete after he has run exactly 10 km using Version 2. [5]

11. [A level 18/P1/Q11]

Mr Wong is considering investing money in a savings plan. One plan, $P$, allows him to invest $100 into the account on the first day of every month. At the end of each month the total in the account is increased by $a$%.

(i) It is given that $a = 0.2$

(a) Mr Wong invests $100 on 1 January 2016. Write down how much this $100 is worth at the end of 31 December 2016. [1]

(b) Mr Wong invests $100 on the first day of each of the 12 months of 2016. Find the total amount in the account at the end of 31 December 2016. [3]

(c) Mr Wong continues to invest $100 on the first day of each month. Find the month in which the total in the account will first exceed $3000. Explain whether this occurs on the first or last day of the month. [5]

An alternative plan, $Q$, also allows him to invest $100 on the first day of every month. Each $100 invested earns a fixed bonus of $$b$ at the end of every month for which it has been in the account. This bonus is added to the account. The accumulated bonuses themselves do not earn any further bonus.

(ii) (a) Find in terms of $b$, how much $100 invested on 1 January 2016 will be worth at the end of 31 December 2016. [1]

(b) Mr Wong invests $100 on the first day of each of the 24 months in 2016 and 2017. Find the value of $b$ such that the total value of all investments, including bonuses, is worth $2800 at the end of 31 December 2017. [3]

It is given instead that $a = 1$ for plan $P$.

(iii) Find the value of $b$ for plan Q such that both plans give the same total value in the account at the end of the 60th month. [3]

12 [VJC 2020/BT/Q11]

A pandemic is an outbreak of a disease that occurs over a wide geographic area and affects an exceptionally high proportion of the population.

**(a)** During a particular pandemic, an undergraduate attempted to use a simple geometric progression model to examine the number of infected cases. The model is represented by an equation

$$u_n = ar^{n-1},$$

where $a$ is the number of infected cases in the first week after a city's health authority began tracking the city's infection situation, $u_n$ is the number of infected cases in the $n$th week, and $r$ is a constant representing the rate of infection in the city.

**(i)** For a particular city, $a = 20$ and the city recorded 1000 infected cases in the ninth week. Based on the student's model, find the number of infected cases in the 11th week, and also the week when the total number of infected cases would first exceed 20 000. [6]

**(ii)** In another city, the government started introducing strict movement controls from day X, so that the value of $r$ is lowered to 0.7 for the subsequent weeks. If there were 12 000 infected cases in the first week after day X, use the student's model to find the theoretical maximum total infected cases after day X. [2]

**(b)** On a happier note, at a deserted beach of the second city, it was observed that more baby leatherback sea turtles were being hatched during the pandemic period. A conservationist closely tracking the numbers found that on the first day, second day and third day, there were 24, 27 and 30 turtles hatched respectively, and she assumed that the number hatched on a day would always be three more than the previous day. She wanted to find the number of days, *n*, before a total of 900 turtles would have hatched over the period.

Write down a quadratic inequality for *n* and solve it to find the smallest possible value of *n*. [4]

**Answers**

1. (a) Decreasing (b) Neither (c) Neither

2. (a) Limit is $\frac{1}{3}$ (b) Limit is 1 (c) Divergent
   (d) Limit is 0 (e) Limit is $\frac{\sqrt{2}}{2}$

3. (a) $d = \frac{10}{3}$; 16000 (b) $-1$; 357643

4. (iii) $\frac{2a}{3}$

5. least $n = 13$

6. (i) $u_n = \frac{1}{3}(2n+10)$; $d = \frac{2}{3}$ (ii) $m = 19$

7. (a)(ii) $\{x \in \mathbb{R} : -1 < x < 0\}$ (b)(ii) $\frac{11}{2}$

8. (i) $d = 1.5$ (ii) $r = -1.45, 1.21$ (3 s.f.)
   (iii) Smallest possible $n$ = 42

9. (a) $k = 13$ (b) 0(when $n$ is even) or $f$ (when $n$ is odd)

(c) 63

10. (i)(a) 440 m (i)(b) $4n(n + 1)$; 35 stages

(ii) $8(2^n - 1)$; 1816 m away from $O$, towards $A_{11}$.

11. (i)(a) $102.43 (i)(b) $1215.71

(i)(c) First day of the month in June 2018.

(ii)(a) $100+12b$ (ii)(b) $b=\frac{4}{3}$ (iii) $b=1.23$

12. (a)(i) 2659; 14 (ii) 40 000

(b) $n=19$

## § 1.7 Representation of a Series Using Sigma Notation, $\Sigma$

The sigma notation, $\sum$ provides a compact way to represent the sum of a sequence of numbers. For instance, to express the sum of all positive integers from 1 to 10 inclusive, we may represent it as).

$$2+4+6+8+\ldots+20=\sum_{r=1}^{10}(2r)$$

and it is read as "the sum of $2r$ where $r$ takes integer values from 1 to 10" .

**Note**

1. The smallest and the largest value that $r$ takes is placed below and above the sigma notation respectively.
2. $\sum_{r=1}^{10} r=\sum_{x=1}^{10} x=\sum_{n=1}^{10} n$ (since $r$, $x$ and $n$ are just *dummy variables*)
3. The variable $r$ takes consecutive integer values and it may not necessarily start from 1.
4. $\sum_{r=1}^{\infty} r$ is an infinite series where $r$ takes consecutive integer values from 1 to infinity.

**Example 21**

Write out the following series explicitly.

(a) $\sum_{r=1}^{5} r^2$ , (b) $\sum_{r=2}^{6}(3r-2)(3r+1)$ ,

(c) $\sum_{r=1}^{n} \frac{(-1)^r}{r}$ , (d) $\sum_{r=0}^{n} 2$ .

**Solution**

(a) $\sum_{r=1}^{5} r^2=1^2+2^2+3^2+4^2+5^2$

(b) $\sum_{r=2}^{6}(3r-2)(3r+1)=4(7)+7(10)+10(13)+13(16)+16(19)$

(c) $\sum_{r=1}^{n} \frac{(-1)^r}{r}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\ldots+\frac{(-1)^n}{n}$

(d) $\sum_{r=0}^{n} 2=\overbrace{\underset{\boxed{r=0}}{2}+\underset{\boxed{r=1}}{2}+\underset{\boxed{r=2}}{2}+\ldots+\underset{\boxed{r=n}}{2}}^{n+1 \text{ terms}}$ □

**Note**

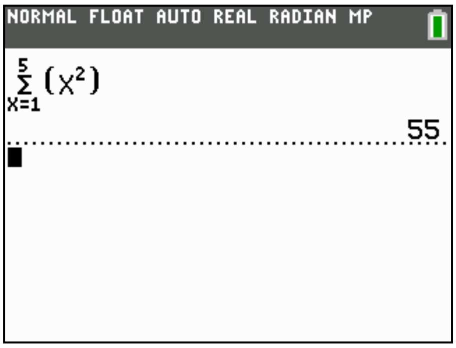
1. TI-84+ can only be used to evaluate Example 21(a) and (b) as the smallest and largest value of $r$ are given numerically.

Evaluating $\sum_{r=1}^{5} r^2$ :

① Press ALPHA WINDOW 2

Key in $\sum_{X=1}^{5}\left(X^2\right)$ and press ENTER

(Press X,T,Θ,$n$ for "X")




2. TI-84+ cannot be used to evaluate Example 21(c) and (d) as the largest value of $r$ for each series is an unknown constant $n$.

In the previous example, we are required to write out the series explicitly. Now, we would look at how a given series can be expressed in sigma notation.

**Example 22**

Express the following series in sigma notation.

(a) $1+3+5+\ldots+(2n-1)$ (b) $3^2+4^2+5^2+\ldots+55^2$

(c) $1-x+x^2-x^3+\ldots$

**Solution**

### Some Useful Results of Sigma Notation

(1) $\sum_{r=1}^{n} a\mathrm{f}(r) = a\sum_{r=1}^{n}\mathrm{f}(r)$ where $a$ is a constant.

E.g. $\sum_{r=1}^{n}(2r) = 2(1)+2(2)+2(3)+\ldots+2(n) = 2(1+2+3+\ldots+n) = 2\sum_{r=1}^{n} r$

(2) $\sum_{r=1}^{n}\left[\mathrm{f}(r)\pm\mathrm{g}(r)\right] = \sum_{r=1}^{n}\mathrm{f}(r)\pm\sum_{r=1}^{n}\mathrm{g}(r)$.

E.g.
$$\begin{aligned}\sum_{r=1}^{n}\left(2^r+3r\right) &= 2^1 + 3(1)\\ &+ 2^2 + 3(2)\\ &+ 2^3 + 3(3)\\ &\quad\vdots\\ &+ 2^n + 3(n)\\ &= \sum_{r=1}^{n}2^r + \sum_{r=1}^{n}3r\end{aligned}$$

(3) $\sum_{r=m}^{n}\mathrm{f}(r) = \sum_{r=1}^{n}\mathrm{f}(r) - \sum_{r=1}^{m-1}\mathrm{f}(r)$.

E.g.
$$\begin{aligned}\sum_{r=3}^{n}r^3 &= 3^3+4^3+5^3+\ldots+n^3\\ &= \underbrace{1^3+2^3+3^3+4^3+5^3\ldots+n^3}_{\sum_{r=1}^{n}r^3} - \underbrace{\left(1^3+2^3\right)}_{\sum_{r=1}^{2}r^3}\\ &= \sum_{r=1}^{n}r^3 - \sum_{r=1}^{2}r^3\end{aligned}$$

(4) $\sum_{r=1}^{n} a = na$ where $a$ is a constant.

In general, $\sum_{r=m}^{n} a = (n-m+1)a$ where $(n-m+1)$ indicates the number of terms of the series; note that from $m$ to $n$ there is $(n-m+1)$ number of terms (for example, from 1 to 10, there is a total of 10 terms, which is obtained by $10-1+1$).

(5) $\sum_{r=m}^{n} r = m+(m+1)+(m+2)+\ldots+n = \frac{1}{2}(n-m+1)(m+n)$ is an arithmetic series.

(6) $\sum_{r=m}^{n} a^r = a^m + a^{(m+1)} + a^{(m+2)} + \ldots + a^n = \frac{a^m\left(1-a^{n-m+1}\right)}{1-a}$ is a geometric series.

**Example 23**

Evaluate the following series.

(a) $\sum_{r=1}^{n} 2r$

(b) $\sum_{m=0}^{n} \left(\frac{1}{2}\right)^m$

(c) $\sum_{r=1}^{2n} \left(5^r + 2\right)$

(d) $\sum_{n=2}^{N} \left(\frac{1}{N-1} - \mathrm{e}^N\right)$

**Solution**

**Example 24**

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, evaluate $\sum_{r=1}^{n}\left[r(r+1)\right]$. Hence find

(i) $\sum_{r=10}^{n}\left[r(r+1)\right]$,

(ii) $\sum_{r=1}^{n}\left[(r+3)(r+4)\right]$.

**Solution**

### Managing Sigma Notation

Consider $\sum_{r=1}^{n} \frac{r^2+3r+1}{(r+2)!} = \frac{3}{2} - \frac{n+3}{(n+2)!}$, find (i) $\sum_{r=m}^{2n} \frac{r^2+3r+1}{(r+2)!}$, (ii) $\sum_{r=1}^{n} \frac{r^2+5r+5}{(r+3)!}$.

**Solution**

## § 1.8 The Method of Differences

This section considers the sum to $n$ terms of a series whose general term can be expressed as a difference of two or more terms.

Suppose the general term $u_r$ can be written as $u_r = \mathrm{f}(r) - \mathrm{f}(r-1)$, then

$$
\begin{aligned}
\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} \left[ \mathrm{f}(r) - \mathrm{f}(r-1) \right] \\
&= \left[ \begin{array}{cccc}
 & \cancel{\mathrm{f}(1)} & - & \mathrm{f}(0) \\
+ & \cancel{\mathrm{f}(2)} & - & \cancel{\mathrm{f}(1)} \\
+ & \cancel{\mathrm{f}(3)} & - & \cancel{\mathrm{f}(2)} \\
\vdots & & & \\
+ & \cancel{\mathrm{f}(n-1)} & - & \cancel{\mathrm{f}(n-2)} \\
+ & \mathrm{f}(n) & - & \cancel{\mathrm{f}(n-1)}
\end{array} \right] \\
&= \mathrm{f}(n) - \mathrm{f}(0)
\end{aligned}
$$

This way of simplifying a finite series whose general term can be expressed as a difference of two or more terms is called the **method of differences**.

**Example 25**

Verify that $\dfrac{1}{(r+1)(r+2)}=\dfrac{1}{r+1}-\dfrac{1}{r+2}$.

(i) By using the method of differences, show that $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\frac{1}{2}-\frac{1}{n+2}$.

(ii) Hence find $\displaystyle\sum_{r=21}^{N}\frac{1}{(r+1)(r+2)}$.

**Solution**

$$\text{RHS}=\frac{1}{r+1}-\frac{1}{r+2}=\frac{(r+2)-(r+1)}{(r+1)(r+2)}=\frac{1}{(r+1)(r+2)}=\text{LHS (verified).}$$

(i) $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\left[\frac{1}{r+1}-\frac{1}{r+2}\right]$

$$=\left[\begin{array}{lll} & \frac{1}{2} & -\cancel{\frac{1}{3}} \\ + & \cancel{\frac{1}{3}} & -\cancel{\frac{1}{4}} \\ + & \cancel{\frac{1}{4}} & -\cancel{\frac{1}{5}} \\ \vdots & & \\ + & \cancel{\frac{1}{n}} & -\cancel{\frac{1}{n+1}} \\ + & \cancel{\frac{1}{n+1}} & -\frac{1}{n+2} \end{array}\right]$$

$$=\frac{1}{2}-\frac{1}{n+2}$$

(ii) $$\sum_{r=21}^{N}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{N}\frac{1}{(r+1)(r+2)}-\sum_{r=1}^{20}\frac{1}{(r+1)(r+2)}$$

$$=\frac{1}{2}-\frac{1}{N+2}-\left[\frac{1}{2}-\frac{1}{20+2}\right]$$

$$=\frac{1}{22}-\frac{1}{N+2}$$

**Example 26**

Express $\dfrac{1}{(2r-1)(2r+1)(2r+3)}$ in partial fractions.

Hence find $\dfrac{1}{1\cdot 3\cdot 5}+\dfrac{1}{3\cdot 5\cdot 7}+\cdots+\dfrac{1}{(2n-1)(2n+1)(2n+3)}$.

Explain why $\displaystyle\sum_{r=1}^{\infty}\frac{1}{(2r-1)(2r+1)(2r+3)}$ is convergent and hence state the sum to infinity.

**Solution**

Let $\dfrac{1}{(2r-1)(2r+1)(2r+3)}=\dfrac{A}{2r-1}+\dfrac{B}{2r+1}+\dfrac{C}{2r+3}$

$\Rightarrow A(2r+1)(2r+3)+B(2r-1)(2r+3)+C(2r-1)(2r+1)=1$.

When $r=\dfrac{1}{2}$, $A=\dfrac{1}{8}$. When $r=-\dfrac{1}{2}$, $B=-\dfrac{1}{4}$. When $r=-\dfrac{3}{2}$, $C=\dfrac{1}{8}$.

Therefore, $\dfrac{1}{(2r-1)(2r+1)(2r+3)}=\dfrac{1}{8(2r-1)}-\dfrac{1}{4(2r+1)}+\dfrac{1}{8(2r+3)}$.

**Example 27**

Given that $\mathrm{f}(r)=(r+1)!$, show that $\mathrm{f}(r)-\mathrm{f}(r-1)=r\cdot r!$.

Hence, find $1\cdot 1!+2\cdot 2!+3\cdot 3!+..+(2N)\cdot(2N)!$.

**Solution**

**Example 28**

By considering $u_n-u_{n-1}$ where $u_n=3n(n+1)(n+2)$ for $n\in\mathbb{Z}^+$, find $\sum_{n=1}^{N} n(n+1)$ in terms of $N$. Hence, determine whether the series $\sum_{n=1}^{\infty} n(n+1)$ is convergent.

**Solution**

$u_n-u_{n-1}=3\left[n(n+1)(n+2)-(n-1)(n)(n+1)\right]$

$=3n(n+1)(n+2-n+1)$

$=9n(n+1)$

$$\sum_{n=1}^{N} n(n+1) = \sum_{n=1}^{N} \frac{1}{9}(u_n - u_{n-1})$$

$$= \frac{1}{9}\left[\begin{array}{rcl} & \cancel{u_1} & - \quad u_0 \\ + & \cancel{u_2} & - \quad \cancel{u_1} \\ & \vdots & \\ + & u_N & - \quad \cancel{u_{N-1}} \end{array}\right]$$

$$= \frac{1}{9}(u_N - u_0) = \frac{1}{3}N(N+1)(N+2)$$

$$\sum_{n=1}^{\infty} n(n+1) = \lim_{N\to\infty} \sum_{n=1}^{N} n(n+1) = \lim_{N\to\infty} \frac{1}{3}N(N+1)(N+2)$$

As $N \to \infty$, $\frac{1}{3}N(N+1)(N+2) \to \infty$. Thus, $\sum_{n=1}^{\infty} n(n+1)$ does not converge. □

**Note**

This is actually the same summation evaluated in Example 24, with some change in symbols.

## Tutorial 1B – Sigma Notation and Method of Differences

1. Evaluate the following series:

   (a) $\sum_{r=0}^{\infty}(-1)^r x^r$, where $-1<x<1$, $x \neq 0$.

   (b) $\sum_{r=1}^{n+1}(\ln r)$

   (c) $0.18+0.0018+0.000018+\ldots$. Give your answer as a fraction in its lowest term.

2. (a) Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, find $\sum_{r=3}^{2n}(2r-1)^2$.

   (b) [A level 06/P1/Q11modified]

   Given $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$, show that $2^3+4^3+6^3+\ldots\ldots+(2n)^3 = 2n^2(n+1)^2$. [3]

   Hence or otherwise find $\sum_{r=1}^{n}(2r-1)^3$, simplifying your answer. [3]

3. (a) [A level 19/P1/Q6]

   **(i)** By writing $\frac{1}{4r^2-1}$ in partial fractions, find an expression for $\sum_{r=1}^{n}\frac{1}{4r^2-1}$. [4]

   **(ii)** Hence find the exact value of $\sum_{r=11}^{\infty}\frac{1}{4r^2-1}$. [2]

   (b) Evaluate $\sum_{r=1}^{N+1}\left[\frac{1}{3(r+1)}+\frac{1}{2r}-\frac{5}{6(r+3)}\right]$.

4. [A level FM9234 N99/P1/Q3a modified]

   Given $u_n = \frac{2^n}{n}$, show that $u_{n+1}-u_n = \frac{2^n(n-1)}{n(n+1)}$ where $n$ is a positive integer. [1]

   Hence, show that $\sum_{n=1}^{N}\frac{2^n(n-1)}{n(n+1)} = \frac{2^{N+1}}{N+1}-2$. [3]

5. Given that $\mathrm{f}(r)=\cos 2r\theta$, write $\mathrm{f}(r)-\mathrm{f}(r+1)$ as a product of two sine functions. Use your result to find the sum of the first $n$ terms of the series

$$\sin 3\theta+\sin 5\theta+\sin 7\theta+\ldots.$$

6. [MJC08/PrelimP2/Q3 modified]

Verify that $\frac{1}{2x}-\frac{1}{(x-1)}+\frac{1}{2(x-2)}=\frac{1}{x(x-1)(x-2)}$. [1]

By using the above result, find $\sum_{n=3}^{N}\frac{1}{n(n-1)(n-2)}$. Hence show that $\sum_{n=1}^{N}\frac{1}{n^3}<\frac{11}{8}$. [6]

7. [SAJC16/C2MidYearP1/Q4b]

(i) Show that $\frac{r^2+3r+1}{(r+2)!}=\frac{1}{r!}-\frac{1}{(r+2)!}$. [1]

(ii) Hence find $\sum_{r=1}^{n}\frac{r^2+3r+1}{(r+2)!}$. [3]

(iii) Using the result in part (b)(ii), show that $\sum_{r=1}^{n}\frac{r^2+5r+5}{(r+3)!}=\frac{2}{3}-\frac{n+4}{(n+3)!}$. [3]

8. [HCI07/C1LectureTest/Q6]

(i) Verify that for any real constant, $m$,

$$\frac{1}{mr+1-m}-\frac{1}{mr+1}=\frac{m}{(mr+1-m)(mr+1)}.$$ [1]

(ii) Hence, show that $\sum_{r=1}^{n}\frac{1}{(mr+1-m)(mr+1)}=\frac{n}{mn+1}$. [3]

(iii) Hence, evaluate

$$\frac{1}{1\cdot 3}+\frac{1}{1\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 7}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 10}+\ldots+\frac{1}{39\cdot 41}+\frac{1}{58\cdot 61}.$$ [4]

9. [NYJC07/PrelimP1/Q8 modified]

Given $\mathrm{f}(r)=\frac{r^2}{2^r}$, show that $\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{(r-1)^2}{2^{r+1}}-\frac{1}{2^r}$. [1]

(i) Hence, find $\sum_{r=1}^{n}\left[\frac{(r-1)^2}{2^{r+1}}-\frac{1}{2^r}\right]$. [3]

(ii) Deduce $\sum_{r=1}^{n}\left[\frac{(r-1)^2}{2^{r+1}}\right]$. [3]

(iii) Show that $\sum_{r=1}^{n}\frac{r^2}{2^r}=6-\frac{n^2+4n+6}{2^n}$. [3]

10. [HCI15/C1BT/Q4, HCI15/Promo/Q8 modified]

A pair of twin brothers, William and John, graduated in 2003 and both of them started their new job in January 2004.

(a) John joined a company that paid him $\$x$ in his first year. In each year, his annual salary increases by an amount equal to 15% of his first year's salary. Show that his total salary after $n$ years is $\frac{nx}{40}(3n+37)$. Hence calculate the least number of years needed for his total salary to exceed 50 times of his first year salary. [4]

(b) William joined a company with a starting monthly pay of \$2000 in January 2004. In January 2005, he received an increment of 50% of his previous monthly pay. In January for each subsequent year, he received an increment of 50% of his previous increment. In other words, in January 2005, his increment was \$1000; in January 2006, his increment was \$500 and so on.
Let $U_n$ denote the pay William received in the $n^{\text{th}}$ year (where 2004 was the $1^{\text{st}}$ year, 2005 was the $2^{\text{nd}}$ year, and so on).

(i) Find $U_1$, $U_2$, $U_3$ and show that $U_n = 48000\left(1-0.5^n\right)$. Hence by considering $\sum_{r=1}^{n} U_r$, find the total pay William received in the first $n$ years. [6]

(ii) William decided to quit if his increment fell below \$10. In which year would he quit the company? [3]

11. [ACJC15/PrelimP1/Q13 modified]

The owner of a newly opened café decided to rent a painting from an artist as part of the decoration of the café. They were discussing how to draft a contract for the terms of the rental.
The artist proposed a rental contract (Version 1), stating that the owner will pay the artist \$15 for the 1st day of rental and for each subsequent day, the daily rental cost will increase by \$0.50.

(i) Express, in terms of $n$, the rental cost for the $n$th day.
On which day of the rental will the owner first have to pay the artist more than \$39 as the daily rental rate? [2]

The owner proposed an alternative contract (Version 2), where the daily rental rate is such that on the $n$th day of the rental, the amount of money, in dollars, the owner has to pay to the artist is given by the function

$$\mathrm{f}(n)=\frac{12000}{4n^2+4n-3}.$$

(ii) Express $\mathrm{f}(n)$ in the form $\dfrac{A}{2n-1}+\dfrac{B}{2n+3}$, where $A$ and $B$ are constants to be determined. [1]

(iii) Hence show that with Version 2 of the contract, the total amount of money the artist will receive at the end of $m$ days of rental is

$$4000-\frac{12000(m+1)}{(2m+1)(2m+3)}.$$ [3]

(iv) The artist accepted Version 2 of the contract, and terminated the contract at the end of $k$ days. Given that the artist received more money in total from Version 2 than if he had chosen Version 1, find the largest possible value of $k$. [3]

(v) Given that the artist accepted Version 2 of the contract, the owner changed his mind and decided to offer the artist \$3,999 to buy his painting. Should the artist accept the offer? Explain your answer clearly. [2]

12. [ACJC JC1 Promo 9758/2019/Q11]

A research team would like to examine the growth of a certain bacteria in a controlled environment. Beginning with a sample amounting to $A_0$ of this bacteria, the researchers noted down the amount of bacteria found in this environment at the end of each day, such that $A_1$ represents the amount present at the end of the first day and $A_2$ for the second day.

**(i)** It was found that the amount of bacteria present at the end of $k^{\text{th}}$ day, denoted by $A_k$, relies on the amount present at the end of the previous day, with the relationship $A_k = A_{k-1} + 360k^2$.

Use this relation to find expressions for $A_1$, $A_2$ and $A_3$, leaving each in terms of $A_0$. Hence, or otherwise, show that the amount of bacteria present at the end of $n^{\text{th}}$ day, $A_n$ is given by $A_n = A_0 + an^3 + bn^2 + cn$, where $a$, $b$ and $c$ are constants to be determined. [You may use the result $\sum_{r=1}^{N} r^2 = \frac{1}{6}N(N+1)(2N+1)$.] [5]

The research team later discovered an antibody to curb the growth of the bacteria. With the introduction of the antibody, the amount of bacteria present (measured in cells per ml) in the new controlled environment, $P(n)$ is given by $P(n) = \sum_{r=1}^{2n}(450 - nr)$, where $n$ is a positive integer that denotes the number of days from which the antibody is administered.

**(ii)** **(a)** Find the number of days after the antibody is administered before it effectively reduces the amount of bacteria present. [3]

**(b)** Calculate the total number of days required for the bacteria to be completely wiped out in the controlled environment. [2]

Based on the chemical composition of the antibody, the team formulated a synthetic medication. During the clinical trial where the synthetic medication is used, the amount of bacteria present (measured in cells per ml) in the experimental controlled environment, $Q(n)$ can be modelled by $Q(n) = 1617 - 20(n-7)^2$, where $n$ is a positive integer that denotes the number of days from which the medication is administered.

**(iii)** Assuming the initial amount of bacteria present in both controlled environments are the same, comment with justification whether the antibody or the synthetic medication is more effective in reducing the amount of bacteria present. [2]

**Answers**

1. (a) $\dfrac{1}{1+x}$ (b) $\ln\left((n+1)!\right)$ (c) $\dfrac{2}{11}$

2. (a) $\dfrac{2n}{3}\left(16n^2-1\right)-10$ (b) $n^2\left(2n^2-1\right)$

3. (a) (i) $\dfrac{1}{2}\left(1-\dfrac{1}{2n+1}\right)$ (ii) $\dfrac{1}{42}$

(b) $\dfrac{43}{36}-\dfrac{1}{2(N+2)}-\dfrac{5}{6(N+3)}-\dfrac{5}{6(N+4)}$

5. $2\sin\theta\sin\left((2r+1)\theta\right)$; $\dfrac{\cos 2\theta-\cos(2n+2)\theta}{2\sin\theta}$

6. $\dfrac{1}{4}-\dfrac{1}{2N(N-1)}$

7. (ii) $\dfrac{3}{2}-\dfrac{n+3}{(n+2)!}$ (iii) $\dfrac{2}{3}-\dfrac{n+4}{(n+3)!}$

8. $\dfrac{2040}{2501}$

9. (i) $\dfrac{1}{2}-\dfrac{(n+1)^2}{2^{n+1}}$ (ii) $\dfrac{3}{2}-\dfrac{(n+1)^2}{2^{n+1}}-\dfrac{1}{2^n}$

10. (a) at least 21 years needed

(b)(i) $U_1=\$24000,\ U_2=\$36000,\ U_3=\$42000$; $48000\left(n-1+0.5^n\right)$

(b)(ii) in year 2012

11. (i) $14.5+0.5n$; on the 50th day (ii) $\dfrac{3000}{2n-1}-\dfrac{3000}{2n+3}$

(iv) largest value of $k$ is 99

12. (i)(a) $A_n=A_0+120n^3+180n^2+60n$

$a=120, b=180, c=60$

(ii)(a) 13 (b) 21

## Supplementary Exercise – Sequences and Series

1. Find the series $\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2 + 7\left(\frac{1}{2}\right)^2 + \cdots + 28\left(\frac{1}{2}\right)^2$.

2. [DHS09/Promo/Q3]
Given that $m$, $-4$, $m+15$ are the fourth, sixth and eighth terms of a geometric progression that has a first term which is positive, find
(i) the common ratio and the first term, [3]
(ii) the sum to infinity. [2]

3. [A level N97/P1/Q15]
A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man and a woman both invest money.

(a) The man decides to invest $\$x$ at the beginning of one year and then a further $\$x$ at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.
(i) How much will there be in the account at the end of 1 year, including the interest? [1]
(ii) Show that, at the end of $n$ years, when the interest for the last year has been added, he will have a total of $\$21\left(1.05^n - 1\right)x$ in his account. [4]
(iii) After how many complete years will he have, for the first time, at least $\$12x$ in his account? [3]

(b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests $\$y$ at the beginning of **each** year. Show that at the end of $n$ years, she will have received a total of $\$\frac{1}{40}n(n+1)y$ in interest. [4]

4. [A level J85/P2/Q1]
A geometric progression has first term 1 and the common ratio $r$ is positive. The sum of the first 5 terms is twice the sum of terms from the 6th to 15th inclusive.
Prove that $r^5 = \frac{1}{2}\left(\sqrt{3} - 1\right)$. [4]

5. [HCI09/Promo/Q8]

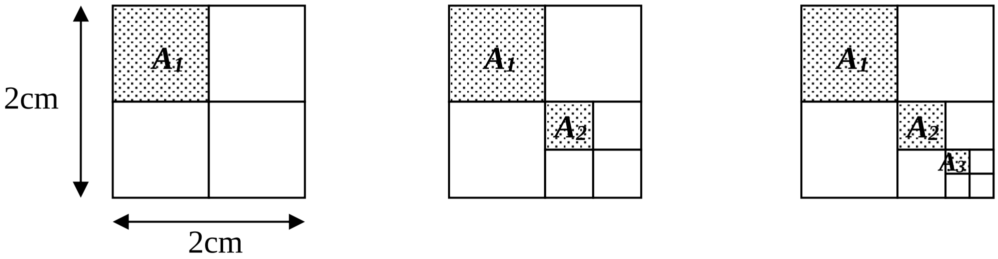



Diagram 1 Diagram 2 Diagram 3

(i) Show that the areas of the shaded squares $A_1, A_2, \dots , A_n$ in the $n^{\text{th}}$ diagram form a geometric progression. [2]

(ii) Show that the total area of the shaded squares in the $n^{\text{th}}$ diagram $S_n$ is $\frac{4}{3}\left(1-\frac{1}{4^n}\right)$. [2]

(iii) Let $S$ be the total shaded area in the $n^{\text{th}}$ diagram as $n \to \infty$. Find the value of $S$. [2]

(iv) Find the least value of $n$ for which the difference between $S_n$ and $S$ is less than 1% of $S$. [3]

6. [IJC09/Promo/Q12]
Adrian has signed up at a driving centre to learn how to drive. His first lesson is 40 minutes long. Each subsequent lesson is 5 minutes longer than the previous lesson, so that the second lesson is 45 minutes long, the third lesson is 50 minutes long, and so on.

(i) Determine the duration of Adrian's $10^{\text{th}}$ lesson. [3]

(ii) The centre requires a student to have attended at least 60 hours of lessons before he is qualified to take the driving test. Find the minimum number of lessons that Adrian has to attend before he can take the test. [5]

7. [JJC09/Promo/Q3]

(a) In a geometric progression, the first term is 2009 and its common ratio is $-\frac{5}{7}$.

(i) Find the least value $n$ such that $|U_n| < \frac{1}{2009}$, where $U_n$ denotes the $n^{\text{th}}$ term of the progression. [3]

(ii) Find, correct to 2 decimal places, the sum of all the negative terms of the progression. [3]

(b) 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5………., $k$ …… is a sequence where the number $k$ appears $k$ times successively ( $k = 1, 2, 3, 4, \dots$ ). Find the $1000^{\text{th}}$ term of the sequence. [4]

8. [NJC09/Promo/Q2]

The $n^{\text{th}}$ term of a series, $T_n$, is given by $T_n = \dfrac{3}{4^{n-2}}$.

(i) Show that the series is a geometric series and that it is convergent. Determine also the sum to infinity. [3]

$T_3$ and $T_4$ form the first and third terms of an arithmetic series respectively. The sum of the first $m$ terms of the arithmetic series is denoted as $S_m$.

(ii) Show that $S_m = \dfrac{m}{64}(57-9m)$ and find the set of possible values of $m$ such that $S_m$ exceeds 1. [4]

9. [RI09/Promo/Q3]

(a) An infinite geometric series has first term $a$ and common ratio $r$. The sum of the first fourteen terms of the series is 127 times the sum to infinity of the remaining terms of the series. Find the two possible values of $r$ in exact form. [3]

(b) From a ribbon, pieces of decreasing lengths are cut. The lengths of the pieces cut follow an arithmetic progression with the 6th piece and the 26th piece cut being of lengths 19 cm and 15 cm respectively.

(i) Find the length of the first piece cut and the common difference of the arithmetic progression. [3]

(ii) Assuming that the ribbon is sufficiently long, find the number of such pieces that can be cut from the ribbon and also the least possible length of the ribbon. [3]

10. [RVHS09/Promo/Q7 modified]

Each year in June approximately 10% of the trees die out and in December, the workers plant 100 new trees. At the end of December 2000 there are 1200 trees in the plantation.

(i) Find the number of living trees at the end of December 2002. [3]

(ii) Consider 2001 as the first year.
Show that the number of living trees at the end of December in the $n^{\text{th}}$ year is given by $200\left(\dfrac{9}{10}\right)^n + 1000$. [3]

(iii) What happens to the number of living trees in the plantation for large $n$? [2]

11. [AJC09/Promo/Q6]

The sum of the first $n$ terms of a series is given by the expression $\left(1-3^{-2n}\right)$.

(i) Find $\mathrm{T}_n$, the $n^{\text{th}}$ term of the series and hence show that the series is a geometric series. [4]

(ii) Find the least value of $k$ such that the sum of the terms from the $k^{\text{th}}$ term is less than $\dfrac{1}{3000}$. [4]

(iii) Express $\displaystyle\sum_{n=1}^{N}\left(\frac{1}{\mathrm{T}_n}\right)$ in terms of $N$. [2]

12. The *Sierpinski sieve*, which is an example of a fractal, is constructed by starting with a solid black equilateral triangle. This triangle is divided into four congruent equilateral triangles, and the middle triangle is removed (see Figure 1). On the next step, each of the three remaining equilateral triangles is divided into four congruent equilateral triangles, and the middle triangle in each of these triangles is removed (see Figure 2). If the process is continued indefinitely, the *Sierpinski sieve* results.

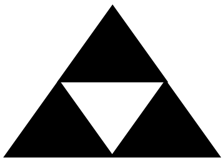


Figure 1

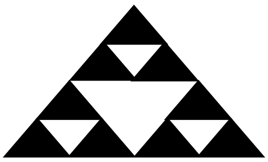


Figure 2

(i) Find $a_k$ that gives the number of triangles removed on the $k^{\text{th}}$ step.

(ii) Calculate the number of triangles removed on the fifteenth step.

(iii) Suppose the initial triangle has an area of 1 unit square. Find $b_k$ that gives the area removed on the $k^{\text{th}}$ step.

(iv) Determine the total area removed after 12 steps.

13. A sequence of numbers is grouped into sets as shown below such that the $r^{\text{th}}$ set contains $r$ terms.

$$\{1\}, \quad \{2, 2^2\}, \quad \{2^3, 2^4, 2^5\}, \quad \{2^6, 2^7, 2^8, 2^9\}, \ \ldots.$$

Find the total number of terms in the first $n$ sets.

(i) Hence find the sum of numbers in the first $n$ sets.

(ii) Deduce, in terms of $n$, the first and the last number in the $n^{\text{th}}$ set.

14. [A level J87/P1/Q19b]

Give that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find an expression, in simplified form, for $\displaystyle\sum_{r=n+1}^{2n}(2r-1)^2$. [6]

15. [A level N99/P1/Q14a]
The $r^{\text{th}}$ term of a series is $3^{r-1}+2r$. Find the sum of the first $n$ terms. [5]

16. [JJC10/PrelimP1/Q2 modified]

(i) Referring to MF26 for the formula of $\binom{n}{r}$, show that

$\binom{k}{2}=\binom{k+1}{3}-\binom{k}{3}$, where $k \in \mathbb{Z}^+$, $k \geq 3$. [2]

(ii) Hence find $\sum_{r=3}^{n}\binom{r}{2}$. [4]

17. [AJC13/C2MidYearP1/Q3]
Show that $(n^2 + 1)(n!) = n\,(n + 1)! - (n - 1)(n!)$ where $n$ is a positive integer. [1]
A series of $n$ terms is given by

$$2(1!) + 5(2!) + 10(3!) + \ldots + (n^2 + 1)(n!) .$$

Find the sum of the series in terms of $n$. [3]
Hence express $65(8!) + 82(9!) + \ldots + (901)(30!)$ in the form $a(b!) - c(d!)$, where $a$, $b$, $c$ and $d$ are constants to be determined. [2]

18. [HCI09/Promo/Q7 modified]
The $n^{\text{th}}$ term of a sequence is given by $u_n = n!(n-1)$, for all positive integers $n$ where $n \geq 2$. Show that

$u_n - u_{n-1} = (n-1)!\left(n^2-2n+2\right)$ and $\displaystyle\sum_{n=2}^{N+1}\left[(n-1)!\left(\frac{n^2}{2}-n+1\right)\right]=\frac{(N+1)!N}{2}$. [6]

19. [NYJC09/Promo/Q6]

(a) Given that $\displaystyle\sum_{r=1}^{n} r^3=\frac{n^2(n+1)^2}{4}$, find $\displaystyle\sum_{r=1}^{n}(r-1)\left(r^2+r+1\right)$ in terms of $n$. [2]

(b) Simplify $\dfrac{1}{r}+\dfrac{3}{r+1}-\dfrac{4}{r+2}$. Hence or otherwise, find $\displaystyle\sum_{r=1}^{n}\frac{5r+2}{r(r+1)(r+2)}$. [4]

20. [SAJC09/Promo/Q9]

Given that $\mathrm{f}(r)=\ln(r)$, where $r$ is a positive integer and $r \geq 2$.

(i) Show that $\mathrm{f}(r-1)-2\mathrm{f}(r)+\mathrm{f}(r+1)=\ln\left(1-\frac{1}{r^2}\right)$. [2]

(ii) Using the method of differences, find the sum of the series $\ln\left(\frac{3}{4}\right)+\ln\left(\frac{8}{9}\right)+\ln\left(\frac{15}{16}\right)+\ln\left(\frac{24}{25}\right)+...+\ln\left[1-\frac{1}{(n+1)^2}\right]$, simplifying your answer. [4]

(iii) Given that the series in part (ii) is convergent, find the sum to infinity of the series. [1]

Deduce that

$$\ln\left(\frac{1}{2}\right)<\ln\left(\frac{3}{4}\right)+\ln\left(\frac{8}{9}\right)+\ln\left(\frac{15}{16}\right)+\ln\left(\frac{24}{25}\right)+...+\ln\left[1-\frac{1}{(n+1)^2}\right]\leq \ln\frac{3}{4},$$

for all $n \in \mathbb{Z}^+$. [2]

21. [HCI10/PrelimP1/Q8]

(i) Express $\frac{4-r}{(r-1)r(r+2)}$ in the form $\frac{A}{r-1}+\frac{B}{r}+\frac{C}{r+2}$. [2]

(ii) Hence find $\sum_{r=2}^{n}\frac{4-r}{(r-1)r(r+2)}$ . [3]

Give a reason why the series is convergent, and state its limit. [2]

(iii) Use your answer to part (ii) to find $\sum_{r=2}^{n}\frac{3-r}{r(r+1)(r+3)}$. [2]

22. [RI10/PrelimP2/Q1]

(i) Given that $\frac{1}{2(n-1)^2}-\frac{1}{2n^2}=\frac{n+a}{n^2(n-1)^2}$, show that $a=-\frac{1}{2}$. [1]

(ii) Given that $S_N=\sum_{n=M}^{N}\frac{2n-1}{2n^2(n-1)^2}$, state the smallest possible value of $M$, where $M \in \mathbb{Z}^+$ and $M \leq N$, such that $S_N$ can be defined. [1]

(iii) If $M=3$, find $S_N$ in terms of $N$. [3]

(iv) Deduce that the sum to infinity of the series

$\frac{1}{(2)(3)^2}+\frac{1}{(3)(4)^2}+\frac{1}{(4)(5)^2}+...$ is less than $\frac{1}{8}$ [3]

23. [AJC13/C1MidYear/Q9]

Let $\mathrm{f}(r)=\dfrac{1}{2+\sin r\theta}$ where $r$ is a positive integer and $0<\theta<\pi$.

(i) Show that $\mathrm{f}(r)-\mathrm{f}(r+2)=\dfrac{2\sin\theta\cos(r+1)\theta}{\left[2+\sin r\theta\right]\left[2+\sin(r+2)\theta\right]}$. [2]

(ii) Hence find, in terms of $n$ and $\theta$, $\displaystyle\sum_{r=1}^{n}\frac{\cos(r+1)\theta}{\left[2+\sin r\theta\right]\left[2+\sin(r+2)\theta\right]}$. [4]

24. [HCI17/PrelimP1/Q11]
A manual hoist is a mechanical device used primarily for raising and lowering heavy loads, with the motive power supplied manually by hand. Three hoists, A, B and C are used to lift a load vertically.

(i) For hoist A, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the load will raise 1.6 cm lesser than the vertical distance covered by the previous pull. Determine the number of pulls needed for the load to achieve maximum total height. Hence find this maximum total height. [4]

(ii) For hoist B, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the vertical distance raised will be 95% of the distance covered by the previous pull. Find the theoretical maximum total height that the load can reach. [2]

(iii) For hoist C, every pull will raise the load by a constant vertical distance of 45 cm. However, after each pull, the load will slip and drop by 2% of the total vertical height the load has reached. Show that just before the 4$^{\text{th}}$ pull, the load would have reached a total vertical height of 130 cm, correct to 3 significant figures.
Hence show that before the $(n+1)^{\text{th}}$ pull, the load would have reached a total vertical height of $X+Y(0.98)^{n+1}$, where $X$ and $Y$ are integers to be determined. [5]

(iv) Explain clearly if hoist C can lift the load up a building of height 25 metres. [2]

25. [HCI17/Promo/Q10]
A mine-sweeping robot is used to sweep mines in a mine-field. The robot is programmed to move through the mine-field in a particular manner. From its starting position denoted as $O$ in the mine field, the robot will move in a straight line covering a distance of 100 metres due east, and then it will turn through an angle of 90°in an anti-clockwise direction. After making the first turn, it will travel in a straight line covering a distance of 80 metres, and then turn through an angle of 90° in an anticlockwise direction. The robot will repeat this process of moving in a straight line and turn through an angle of 90° in an anti-clockwise direction throughout its motion, and the distance covered by the robot between the $n^{\text{th}}$ turn and the $(n+1)^{\text{th}}$ turn is 20% less than the distance covered by the robot between the $(n-1)^{\text{th}}$ turn and the $n^{\text{th}}$ turn, with $n\in\mathbb{Z}^{+}$, as shown in Diagram 3 below.

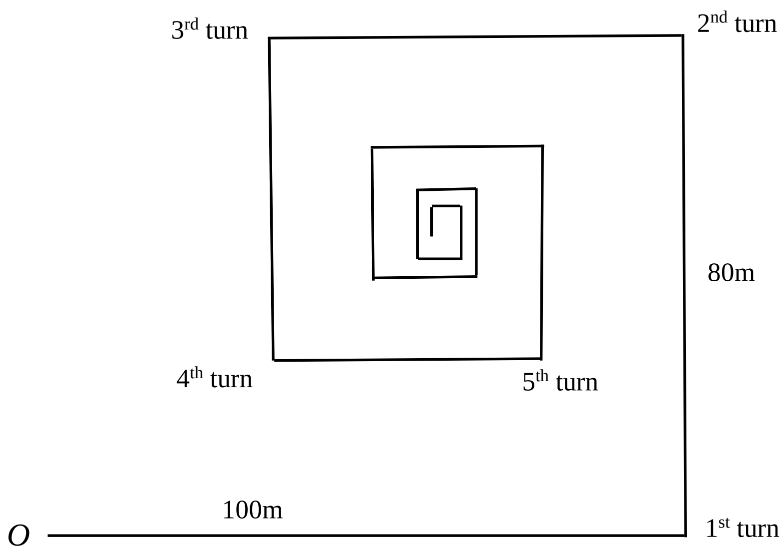



Diagram 3

(i) Show that the distance covered by the robot between its 11th turn and 12th turn is 8.59 metres, correct to 3 significance figures. [2]

(ii) Find the number of turns the robot has made after covering a total distance of 485 metres. [3]

(iii) Determine the coordinates of the theoretical final position that the robot will end up with respect to *O*. [4]

After a change in the robot's setting, the distance covered by the robot between the $n^{\text{th}}$ turn and the $(n+1)^{\text{th}}$ turn is $x$ % less than the distance covered by the robot between the $(n-1)^{\text{th}}$ turn and the $n^{\text{th}}$ turn, $n \in \mathbb{Z}^+$. Given that the initial distance covered by the robot remains as 100 metres, and the robot will cover a total distance of 500 metres just before making its 9th turn, determine the value of $x$. [3]

**Answers**

1. $\dfrac{145}{4}$

2. (i) $r=-\dfrac{1}{2}$, $a=128$ (ii) $\dfrac{256}{3}$

3. (a)(i) \$1.05$x$ (iii) 10

5. (iii) $\dfrac{4}{3}$ (iv) 4

6. (i) 85 (ii) 32

7. (a)(i) 47 (ii) $-2929.79$ (b) 45

8. (i) 16 (ii) $\{2, 3, 4\}$

9. (a) $\dfrac{-1}{\sqrt{2}}$ or $\dfrac{1}{\sqrt{2}}$ (b)(i) $a = 20$, $d = -0.2$ (ii) 100; 1010 cm

10. (i) 1162 (iii) tends to 1000

11. (i) $8(3)^{-2n}$ (ii) 5 (iii) $\dfrac{9}{64}\left(9^N - 1\right)$

12. (i) $3^{k-1}$ (ii) $3^{14}$ (iii) $3^{k-1}\left(\dfrac{1}{4}\right)^k$ (iv) $1-\left(\dfrac{3}{4}\right)^{12}$

13. (i) $\dfrac{n}{2}(n+1)$ (ii) $2^{\frac{n}{2}(n+1)}-1$ (iii) $2^{\frac{n}{2}(n-1)}$; $2^{\frac{n}{2}(n+1)-1}$

14. $\dfrac{n}{3}\left(28n^2 - 1\right)$

15. $\dfrac{3^n - 1}{2} + n^2 + n$

16. (ii) $\dbinom{n+1}{3} - 1$

17. $n(n + 1)!$ ; $30(31)! - 7(8!)$

19. (a) $\dfrac{n^2(n+1)^2}{4} - n$ (b) $\dfrac{3n^2 + 4n}{(n+1)(n+2)}$

20. (ii) $\ln\left(\dfrac{n+2}{2n+2}\right)$ (iii) $\ln\left(\dfrac{1}{2}\right)$

21. (i) $\dfrac{1}{r-1} - \dfrac{2}{r} + \dfrac{1}{r+2}$ (ii) $\dfrac{1}{6} - \dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2}$; $\dfrac{1}{6}$

(iii) $-\dfrac{1}{12} - \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3}$

22. (ii) smallest $M = 2$ (iii) $\dfrac{1}{8} - \dfrac{1}{2N^2}$

23. (ii) $\dfrac{1}{2\sin\theta}\left[\dfrac{1}{2+\sin\theta} + \dfrac{1}{2+\sin 2\theta} - \dfrac{1}{2+\sin(n+1)\theta} - \dfrac{1}{2+\sin(n+2)\theta}\right]$

24. (i) 29 pulls; 655.4 cm (ii) 900 cm (iii) $2205 - 2250(0.98)^{n+1}$

25. (ii) 15 turns (iii) 61.0 m due east and 48.8 m due north of $O$
(iv) $x = 15.7$

## § 1.10 Summary

<table>
<tr><th colspan="3">Sequences and Series</th></tr>
<tr><th></th><th>Sequences, $u_n$</th><th>Series, $S_n$</th></tr>
<tr><td>Definition</td><td>A sequence is an ordered set in which all the terms are related with each other by a specific rule.</td><td>A series is the sum of the terms of a sequence.</td></tr>
<tr><td>Formula</td><td>The rule of a sequence can be defined in 3 ways:<br>(1) Listing, e.g. $\{3, 6, 9, 12, 15, \ldots\}$.<br>(2) Formula for general term, e.g. $u_n = 3n$, $n \in \mathbb{Z}^+$.</td><td>A series can be found by:<br>(1) Recognizing that it is the sum of a special sequence, e.g. of an arithmetic progression, of a geometric progression, of a constant sequence etc.<br>(2) Using the method of differences.</td></tr>
<tr><td>Convergence</td><td>If $u_n \to L$ as $n \to \infty$ where $L$ is a real number, the sequence is said to be convergent.<br>$L$ is called the limit of the sequence, and it is given by $L = \lim\limits_{n\to\infty} u_n$.<br>If there is no such $L$, then the sequence is not convergent, i.e. the sequence is divergent.</td><td>If $S_n \to L$ as $n \to \infty$ where $L$ is a real number, the series is said to be convergent.<br>The limit of the series, $L$ is given by $L = \lim\limits_{n\to\infty} S_n = S_\infty$ where $S_\infty$ is called the sum to infinity.<br>If there is no such $L$, then the series is not convergent, i.e. the series is divergent.</td></tr>
<tr><td>Relationship Between $u_n$ and $S_n$</td><td colspan="2">$u_1 = S_1$ and $u_n = S_n - S_{n-1}$</td></tr>
<tr><td colspan="3">Method of Differences<br>If the general term $u_r$ of a series can be written as $u_r = \mathrm{f}(r) - \mathrm{f}(r-1)$, then

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[\mathrm{f}(r) - \mathrm{f}(r-1)\right] = \left[\begin{array}{cccc} & \cancel{\mathrm{f}(1)} & - & \mathrm{f}(0) \\ + & \cancel{\mathrm{f}(2)} & - & \cancel{\mathrm{f}(1)} \\ + & \cancel{\mathrm{f}(3)} & - & \cancel{\mathrm{f}(2)} \\ \vdots & & & \\ + & \cancel{\mathrm{f}(n-1)} & - & \cancel{\mathrm{f}(n-2)} \\ + & \mathrm{f}(n) & - & \cancel{\mathrm{f}(n-1)} \end{array}\right] = \mathrm{f}(n) - \mathrm{f}(0)$$
</td></tr>
</table>

| Arithmetic and Geometric Progressions | | |
|---|---|---|
| | **Arithmetic Progression** | **Geometric Progression** |
| **Definition** | An arithmetic progression is a sequence of numbers in which each term, other than the first term, is obtained by adding a constant to the preceding term. This constant is called the common difference. | A geometric progression is a sequence of numbers in which each term, other than the first term, is obtained by multiplying a non-zero constant to the preceding term. This constant is called the common ratio. |
| **General Term** | $u_n = a + (n-1)d$ | $u_n = ar^{n-1}$ |
| **Sum to the *n* terms** | $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$ **or** $S_n = \frac{n}{2}\left[a + l\right]$ | $S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1$ **or** $S_n = \frac{a(r^n - 1)}{r-1} \quad r \neq 1$ |
| **Test for A.P / G.P.** | To show that a sequence is in AP, show that the difference $u_{n+1} - u_n$ is a constant (*independent* of $n$). | To show that a sequence is in GP, show that the ratio $\frac{u_{n+1}}{u_n}$ is a constant (*independent* of $n$). |
| **Convergence** | Both the arithmetic progression and arithmetic series do not converge for all non-zero values of $d$. | Both the geometric progression and geometric series converge when $\lvert r \rvert < 1$. The sum to infinity is given by $S_\infty = \frac{a}{1-r}$. |

## Appendix A – Recurrence Relations

Sometimes, it is more convenient to specify the value of the first term, $u_1$ and a formula for $u_{n+1}$ in terms of the preceding terms. Such a definition of a sequence is called the **recurrence definition**, **recurrence relation** or **recurrence formula**. Thus for $\{3, 6, 9, 12, 15, \ldots\}$, we may write

$$u_1 = 3 \text{ and } u_{n+1} = u_n + 3 \text{ for } n \in \mathbb{Z}^+.$$

In Example 3, the sequences can be defined by the following recurrence formulas.

(a) $\{1, 2, 3, 4, 5, \ldots\}$
Recurrence formula: $u_1 = 1$ and $u_{n+1} = u_n + 1$ for $n \in \mathbb{Z}^+$

(b) $\{3, 9, 27, 81, 243, ...\}$
Recurrence formula: $u_1 = 3$ and $u_{n+1} = 3u_n$ for $n \in \mathbb{Z}^+$.

(c) $\{1, 4, 9, 16, 25, ... \}$,
Recurrence formula: $u_1 = 1$ and $u_{n+1} = u_n + 2n + 1$ for $n \in \mathbb{Z}^+$

(d) $\{-1, 2, -3, 4, -5, ...\}$
Recurrence formula: $u_1 = -1$ and $u_{n+1} = (-1)^{n+1}\left(|u_n| + 1\right)$ for $n \in \mathbb{Z}^+$.

The recurrence formulas of AP and GP can be written as

AP: $u_{n+1} = u_n + d$, $u_1 = a$, where $d$ is the common difference,
GP: $u_{n+1} = ru_n$, $u_1 = a$, where $r$ is the common ratio.