Upcoming Assignments and Assessments

- Homework Assignment #3 is due Friday 2/14

- Quiz #3 will be administered during this week's recitation

Subspaces, Basis and Dimension

Subspaces, Basis and Dimension (1)

Definition:
A **subset** of $\mathbb{R}^n$ is any collection of points in $\mathbb{R}^n$.

Examples of subsets
- The unit circle = $\{(x, y) \text{ in } \mathbb{R}^2 | x^2 + y^2 = 1\}$.
- $\{(0,0)\}$ in $\mathbb{R}^2$
- A plane in $\mathbb{R}^3$ = $\{(x, y, z) \text{ in } \mathbb{R}^3 | x + 3y - z = 0\}$

Definition:
A **subspace** of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ satisfying the following properties:

1. **Non-emptiness**: The zero vector is in V
2. **Closure under addition**: If u and v are in V, then $u + v$ is also in V
3. **Closure under scalar multiplication**: If v is in V and c is in $\mathbb{R}$, then cv is also in V.

Examples of subspaces
- $\mathbb{R}^n$ is a subspace of itself: it contains zero, it is closed under addition and scalar multiplication
- The set $\{0\}$: contains zero, closed under addition and scalar multiplication

Subspaces, Basis and Dimension (2)

Some consequences of the properties of subspaces

- If v is a vector in V, then all scalar multiples of v are in V by the third property. That is, the line through any nonzero vector in V is also contained in V.
- If u, v are in V and c, d are scalars, then cu, dv are also in V by the third property, so, $cu + dv$ is in V by the second property. It follows that $\text{Span}\{u, v\}$ is contained in V.
- Generalizing the above, - if $v_1, v_2, \ldots, v_n$ are all in V then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in V. In other words, *a subspace contains the span of any vectors in it.*

If enough vectors in V are chosen, then eventually their span will fill up V, demonstrating that a subspace is a span.

More examples of subspaces

- A line L through the origin
- A plane P through the origin
- $V = \{\begin{bmatrix} a \\ b \end{bmatrix} \; in \; \mathbb{R}^2 | 5a = 2b\}$ (Let's verify this last one)

Subspaces, Basis and Dimension (3)

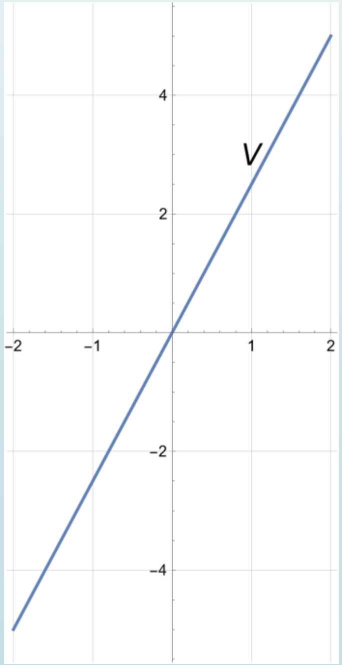
Example 1

Verify that $V = \{\begin{bmatrix} a \\ b \end{bmatrix}\ in\ \mathbb{R}^2 | 5a = 2b\}$ is a subspace.

- V contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $5(0) = 2(0)$.
- Now take two vectors $u = \begin{bmatrix} a \\ b \end{bmatrix}$ where $5a = 2b$ and $v = \begin{bmatrix} c \\ d \end{bmatrix}$ where $5c = 2d$. Note that $5a + 5c = 2b + 2d \Longrightarrow 5(a + c) = 2(b + d)$, so the sum vector $u + v = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$ also satisfies the condition.
- Finally, $5a = 2b \Longrightarrow 5ca = 2cb$ for any scalar c, so, the scalar multiple vector $cu = c\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$ also satisfies the condition.

This subspace is the line through the origin with slope $\frac{5}{2}$

$V = \{\begin{bmatrix} x \\ y \end{bmatrix} in\ \mathbb{R}^2 | 5x = 2y\} \Rightarrow y = \frac{5}{2}x \longrightarrow$



Subspaces, Basis and Dimension (4)

Theorem (Spans are Subspaces and Subspaces are Spans). If $v_1, v_2, \dots, v_p$ are any vectors in $\mathbb{R}^n$, then $\text{Span}\{v_1, v_2, \dots, v_p\}$ is a subspace of $\mathbb{R}^n$. Moreover, any subspace of $\mathbb{R}^n$ can be written as a span of p linearly independent vectors for $p \leq n$.

Proof

Consider the $\text{Span}\{v_1, v_2, \dots, v_p\}$.

1. The zero vector $0 = 0v_1 + 0v_2 + \cdots + 0v_p$ is in the span.
2. If $u = a_1 v_1 + a_2 v_2 + \cdots + a_p v_p$ and $v = b_1 v_1 + b_2 v_2 + \cdots + b_p v_p$ are in $\text{Span}\{v_1, v_2, \dots, v_p\}$, then $u + v = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \cdots + (a_p + b_p)v_p$ is also in $\text{Span}\{v_1, v_2, \dots, v_p\}$.
3. If $v = b_1 v_1 + b_2 v_2 + \cdots + b_p v_p$ is in $\text{Span}\{v_1, v_2, \dots, v_p\}$ and c is a scalar, then $cv = cb_1 v_1 + cb_2 v_2 + \cdots + cb_p v_p$ is also in $\text{Span}\{v_1, v_2, \dots, v_p\}$.

So, $\text{Span}\{v_1, v_2, \dots, v_p\}$ is a subspace of $\mathbb{R}^n$.

Now let V be a subspace of $\mathbb{R}^n$. If V is the zero subspace, then it is the span of the empty set (by definition). So, assume v_1 is a nonzero vector. If $V = \text{Span}\{v_1\}$ we're done. Otherwise, there exists a vector v_2 in V but not in $\text{Span}\{v_1\}$, then $\text{Span}\{v_1, v_2\}$ is in V and the set $\{v_1, v_2\}$ is linearly independent. Again, if $V = \text{Span}\{v_1, v_2\}$, we're done. Otherwise, we keep adding new vectors, one at a time, that are not in the current span but contribute to a larger span. Keep doing this until we can write $V = \text{Span}\{v_1, v_2, \dots, v_p\}$ for some linearly independent set $\{v_1, v_2, \dots, v_p\}$. This process will take at most n steps.

Subspaces, Basis and Dimension (5)

Let $V = \text{Span}\{v_1, v_2, \dots, v_p\}$. We say that V is the subspace **spanned by** or **generated by** the vectors $v_1, v_2, \dots, v_p$. The set $\{v_1, v_2, \dots, v_p\}$ is called a **spanning set** for V.

Definition: Let A be an $m \times n$ matrix.

- The **column space** of A, denoted by $\text{Col}(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A. (*Since it's a span, it must also be a <u>subspace</u>)
- The **null space** of A, denoted by $\text{Nul}(A)$, is the subspace of $\mathbb{R}^n$ consisting of all solutions of the homogeneous equation $Ax = 0$. (It is the solution set of $Ax = 0$)

Now, we must show that $\text{Nul}(A)$ is, indeed, a subspace:

linearity property of the matrix-vector product.

1. The zero vector is in $\text{Nul}(A)$ because $A0 = 0$.
2. Let u and v be in $\text{Nul}(A)$. This means that $Au = 0$ and $Av = 0$. Now, $A(u + v) = Au + Av = 0 + 0 = 0$. It follows that $u + v$ is also in $\text{Nul}(A)$.
3. Let u be in $\text{Nul}(A)$ and let c be a scalar. This means that $Au = 0$. Now $A(cu) = cAu = c0 = 0$ which implies that cu is also in $\text{Nul}(A)$.

Since all three properties are satisfied, we can conclude that $\text{Nul}(A)$ is a subspace,

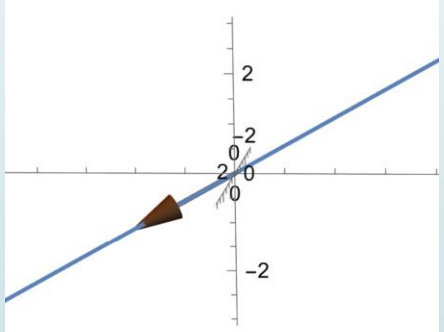
Subspaces, Basis and Dimension (6)

Example 2

Describe the column space and null space of $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix}$

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{array}{c} \checkmark \\ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} -2\checkmark \\ \cancel{\begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}} \end{array} \right\}$$ but notice that the second vector is a scalar multiple of the first, so the two vectors are dependent. We can remove the second vector without changing the span.

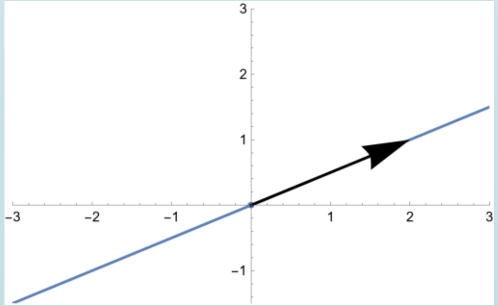
Therefore, $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$. This is a line in $\mathbb{R}^3$.



$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \\ -1 & 2 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -1 & 2 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad x - 2y = 0$$

$$x = 2y \qquad \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$
$$y = y$$

This is a line in $\mathbb{R}^2$.



Subspaces, Basis and Dimension (7)

Example 3

$Ax = 0$

Find a spanning set for the null space of $A = \begin{bmatrix} 2 & 6 & 0 & 4 \\ -1 & 2 & -5 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 6 & 0 & 4 \\ -1 & 2 & -5 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ -1 & 2 & -5 & 3 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 5 & -5 & 5 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$x_1 + 3x_3 - x_4 = 0$

$x_2 - x_3 + x_4 = 0$

$x_1 = -3x_3 + x_4$

$x_2 = x_3 - x_4$

$x_3 = x_3 + 0$

$x_4 = 0 + x_4$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Spanning set $= \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Subspaces, Basis and Dimension (8)

Example 4

Express the subspace $V = \{\begin{bmatrix} a \\ b \end{bmatrix} \ in\ \mathbb{R}^2 | 5a = 2b\}$ as a null space of a matrix.

- Note that V is the solution set of the homogeneous system $5x - 2y = 0$, so let $A = \begin{bmatrix} 5 & -2 \end{bmatrix}$
- The RREF of A is $\begin{bmatrix} 1 & -\frac{2}{5} \end{bmatrix} \implies x - \frac{2}{5}y = 0 \implies x = \frac{2}{5}y$

  $y = y$

  $\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$.
- $V = \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}\right\}$

Subspaces, Basis and Dimension (9)

Definition: Let V be a subspace of $\mathbb{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. the set $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

Note that the above states that if $\{v_1, v_2, \ldots, v_m\}$ is a basis of V, then no proper subset of $\{v_1, v_2, \ldots, v_m\}$ will span V: it is a minimal spanning set. The removal of any vector will shrink the span.

A nonzero subspace has infinitely many different bases, but they all have the same number of vectors.

Definition: Let V be a subspace of $\mathbb{R}^n$. The number of vectors in any basis of V is called the dimension of V, denoted by $\dim V$.

Subspaces, Basis and Dimension (10)

For example, the set $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$:

- Any vector $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} \quad \Rightarrow \quad \mathbb{R}^2 = \operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$
- $x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \Rightarrow \quad x = y = 0 \quad \Rightarrow$ the set is linearly independent

This also shows that $\dim \mathbb{R}^2 = 2$ (not surprisingly)

What do other bases of $\mathbb{R}^2$ look like?

They must contain two vectors, say v_1 and v_2. These vectors must span $\mathbb{R}^2$ and they must be linearly independent.
Now let A be the matrix with columns v_1 and v_2. As a consequence of earlier discussions about spans and linear independence, we have the following:

- $\{v_1, v_2\}$ spans $\mathbb{R}^2$ if and only if A has a pivot position in every row
- $\{v_1, v_2\}$ is linearly independent if and only if A has a pivot position in every column

A 2×2 matrix has a pivot in every row exactly when it has a pivot in every column
It follows that any two noncollinear vectors form a basis of $\mathbb{R}^2$, so, for example, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ will do.

Subspaces, Basis and Dimension (11)

Similarly, it can be shown that $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, e_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ forms a basis for $\mathbb{R}^n$. It follows that $\dim \mathbb{R}^n = n$ (again, not very surprising). The above set of vectors is called the **standard basis** of $\mathbb{R}^n$.

Now, we can generalize the two statements about pivot positions we made earlier.

Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$, and let A be the $n \times n$ matrix with these vectors as its columns.

- $\{v_1, v_2, \ldots, v_n\}$ spans $\mathbb{R}^n$ if and only if A has a pivot position in every row
- $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if and only if A has a pivot position in every column

An $n \times n$ matrix has a pivot position in every row exactly when it has a pivot position in every column, so either of these conditions imply that $v_1, v_2, \ldots, v_n$ form a basis for $\mathbb{R}^n$.

Subspaces, Basis and Dimension (12)

Example 5

Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \mid x - 3y + 2z = 0 \right\}$ and let $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Verify that V is a subspace and show that $\mathcal{B}$ is a basis for V.

V is the solution set of the homogeneous system $x - 3y + 2z = 0 \implies V$ is a span $\implies V$ is a subspace.

- Both vectors are in V:

$$3 - 3(1) + 2(0) = 0$$
$$-2 - 3(0) + 2(1) = 0$$

- Solving the system:

$$x = 3y - 2z \qquad y = y \qquad z = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$, so $\mathcal{B}$ spans V

- $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent since they came from a parametric vector form of the solution set of a homogeneous system.

Subspaces, Basis and Dimension (13)

Theorem: The pivot columns of a matrix A form a basis for $\mathrm{Col}(A)$.

The above theorem is referring to the pivot columns in the *original* matrix, not its reduced row echelon form. Indeed, a matrix and its reduced row echelon form generally have different column spaces. For example, in the matrix A below:

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\-2\\2\end{matrix}} & \boxed{\begin{matrix}2\\-3\\4\end{matrix}} & \begin{matrix}0\\4\\0\end{matrix} & \begin{matrix}-1\\5\\-2\end{matrix} \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟵ pivot columns in RREF

the pivot columns are the first two columns, so a basis for $\mathrm{Col}(A)$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Corollary: The dimension of $\mathrm{Col}(A)$ is the number of pivot positions of A.

Subspaces, Basis and Dimension (14)

Example 6

Find a basis of the subspace $V = \text{Span}\left\{\begin{bmatrix}1\\-2\\2\end{bmatrix}, \begin{bmatrix}2\\-3\\4\end{bmatrix}, \begin{bmatrix}0\\4\\0\end{bmatrix}, \begin{bmatrix}-1\\5\\-2\end{bmatrix}\right\}$

$$\text{Basis} = \left\{ \overset{v_1}{\begin{bmatrix}1\\-2\\2\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}2\\-3\\4\end{bmatrix}} \right\}$$

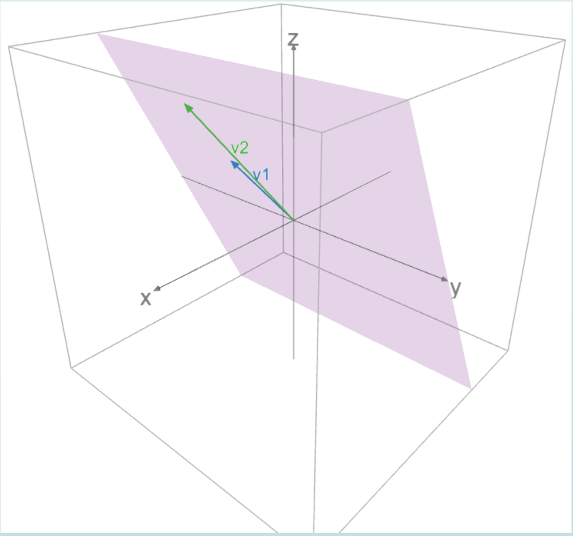
$$A = \begin{bmatrix}1 & 2 & 0 & -1\\ -2 & -3 & 4 & 5\\ 2 & 4 & 0 & -2\end{bmatrix}$$

Find Col(A)

$$\xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & -8 & -7\\ 0 & 1 & 4 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

pivot column

$$\text{Col}(A) = \text{Span}\left\{\begin{bmatrix}1\\-2\\2\end{bmatrix}, \begin{bmatrix}2\\-3\\4\end{bmatrix}\right\}$$



Subspaces, Basis and Dimension (15)

Theorem: The vectors attached to the free variables in the parametric form of the solution set of $Ax = 0$ form a basis of $\mathrm{Nul}(A)$.

Example 7

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix}$. Find a basis for $\mathrm{Nul}(A)$ and determine if the subspace is a line or a plane. If it's a plane, give its equation, if it's a line, give its parametric equations.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \end{bmatrix} \xrightarrow{R_3 - 4R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 + 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -3x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned}$$

a basis

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

let $x_3 = t$

LINE

$$L(t) = (-3t, -2t, t)$$

A basis for a general subspace: when given a subspace written in a different form, in order to compute a basis, it is usually best to re-express it as a column space or a null space of a matrix.

Subspaces, Basis and Dimension (16)

Basis Theorem: Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis of V.
- Any m vectors that span V form a basis for V.

In other words, if we already know that $\dim V = m$ and we're given a set of m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V then we only need to check one of the following conditions:

1. $\mathcal{B}$ is linearly independent, or
2. $\mathcal{B}$ spans V,

in order to conclude that $\mathcal{B}$ is a basis for V.

Example 8

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis of V. Find a different basis for V.

Subspaces, Basis and Dimension (17)

Basis Theorem: Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis of V.
- Any m vectors that span V form a basis for V.

Proof. Suppose that $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a set of linearly independent vectors in V. In order to show that $\mathcal{B}$ is a basis for V, we must prove that $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$. If not, then there exists some vector v_{m+1} in V that is not contained in $\text{Span}\{v_1, v_2, \ldots, v_m\}$. By the increasing span criterion in Section 2.5, the set $\{v_1, v_2, \ldots, v_m, v_{m+1}\}$ is also linearly independent. Continuing in this way, we keep choosing vectors until we eventually do have a linearly independent spanning set: say $V = \text{Span}\{v_1, v_2, \ldots, v_m, \ldots, v_{m+k}\}$. Then $\{v_1, v_2, \ldots, v_{m+k}\}$ is a basis for V, which implies that $\dim(V) = m + k > m$. But we were assuming that V has dimension m, so $\mathcal{B}$ must have already been a basis.

Now suppose that $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ spans V. If $\mathcal{B}$ is not linearly independent, then by this theorem in Section 2.5, we can remove some number of vectors from $\mathcal{B}$ without shrinking its span. After reordering, we can assume that we removed the last k vectors without shrinking the span, and that we cannot remove any more. Now $V = \text{Span}\{v_1, v_2, \ldots, v_{m-k}\}$, and $\{v_1, v_2, \ldots, v_{m-k}\}$ is a basis for V because it is linearly independent. This implies that $\dim V = m-k < m$. But we were assuming that $\dim V = m$, so $\mathcal{B}$ must have already been a basis. $\square$