HW#9 is due today
HW#10 is due next Friday, May 2nd
Quiz#10 will be administered on Friday May 2nd
Final Exam will cover all sections of the text

Orthogonal Decomposition

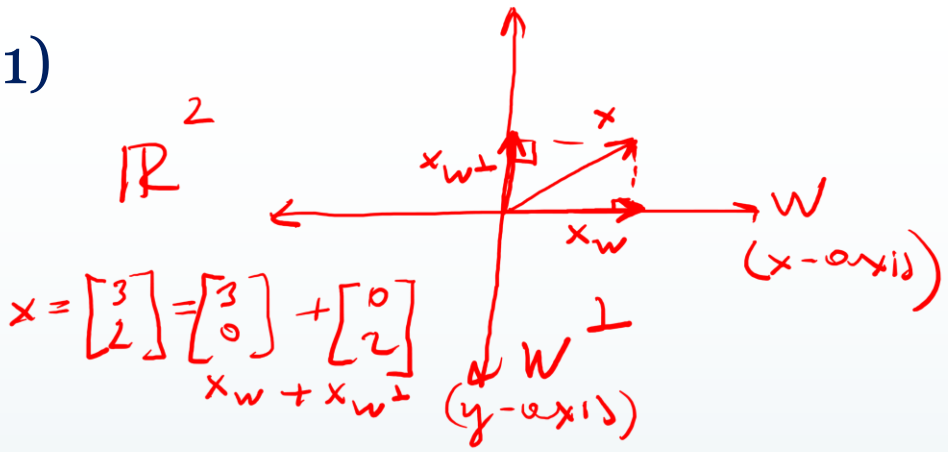
Orthogonal Decomposition (1)

Given a subspace W:

Theorem: Every vector x in $\mathbb{R}^n$ can be written as

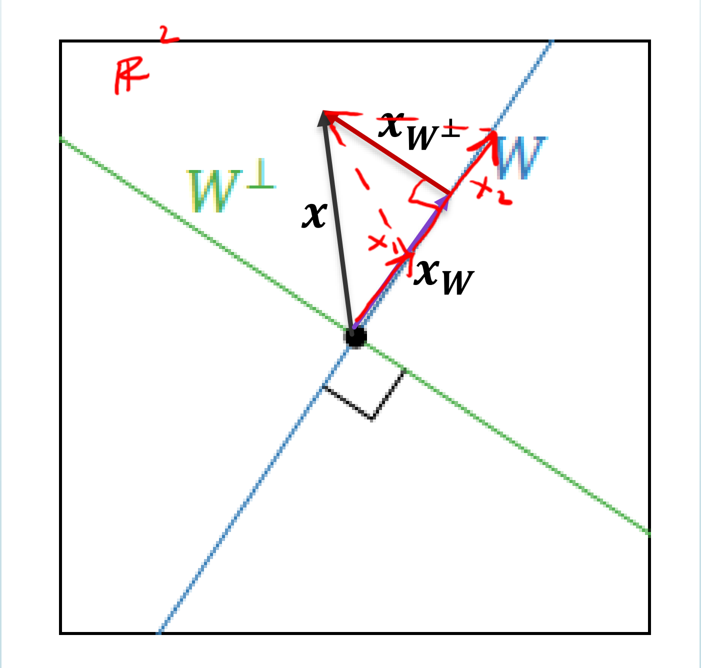
$$x = x_W + x_{W^\perp}$$

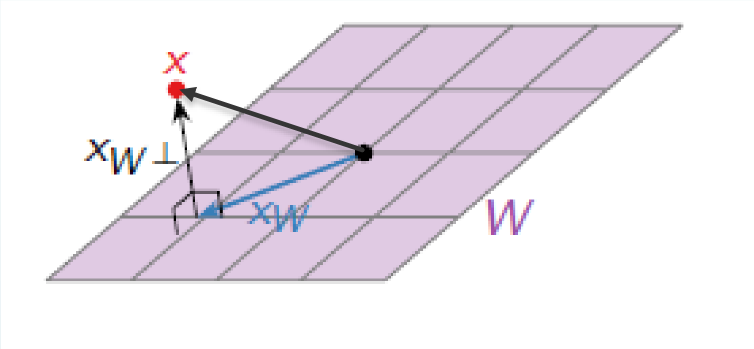
for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.



The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x with respect to W. The vector x_W is called the **orthogonal projection** of x onto W.

Important observation: x_W is the <u>closest</u> vector to x in W.





x_W is the closest vector because the vector $x - x_W$ is orthogonal to W. Note that $x - x_W = x_{W^\perp}$.

The distance from x to W which is the same as the distance from x to x_W is equal to the length of the vector from x_W to x which is exactly the length of $x_{W^\perp}$, $\|x_{W^\perp}\|$.

Orthogonal Decomposition (2)

Theorem: Every vector x in $\mathbb{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

<u>Showing uniqueness</u>: Suppose $x = x_W + x_{W^\perp} = x'_W + x'_{W^\perp}$. Then $\overbrace{x_W - x'_W}^{\text{in } W} = \overbrace{x_{W^\perp} - x'_{W^\perp}}^{\text{in } W^\perp}$. So, the left side is in W, and the right side is in $W^\perp$ which means the left side is orthogonal to the right side but the only vector that is orthogonal to itself is 0. This implies that $x_W - x'_W = 0$ and $x_{W^\perp} - x'_{W^\perp} = 0$ which, in turn, shows that $x_W = x'_W$ and $x_{W^\perp} = x'_{W^\perp}$.

To <u>show existence</u> we will shortly show how to compute orthogonal decomposition.

Closest vector and distance. Let W be a subspace of $\mathbb{R}^n$ and let x be a vector in $\mathbb{R}^n$.

- The orthogonal projection x_W is the closest vector to x in W.
- The distance from x to W is $\|x_{W^\perp}\|$.

Orthogonal Decomposition (3)

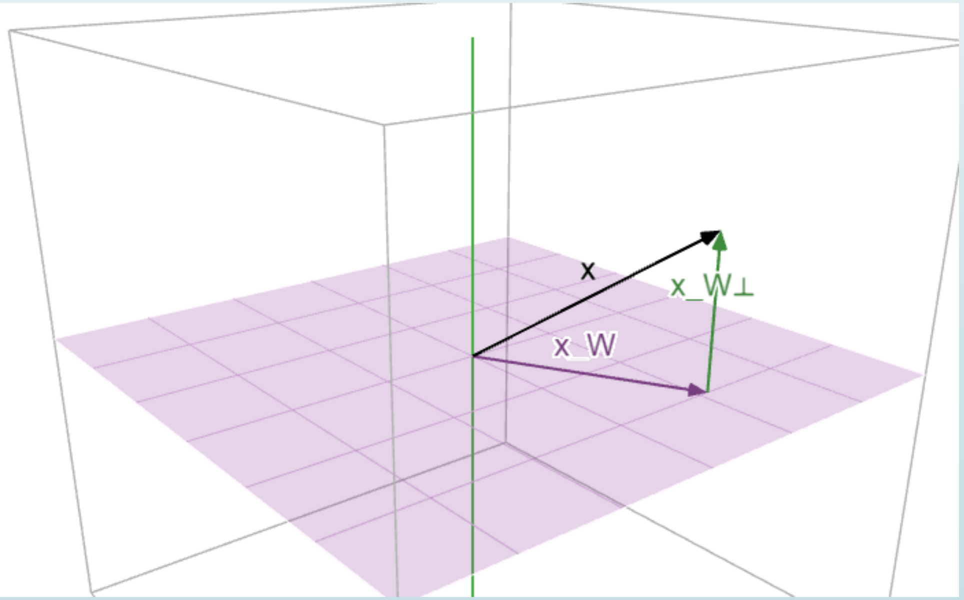
Decomposing a vector in $\mathbb{R}^3$ relative to the *xy*-plane

Let W be the xy −plane. Then what is $W^{\perp}$?

The *z*-axis! Now, let $x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \implies x_W = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $x_{W^{\perp}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

More generally, let $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies x_W = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and $x_{W^{\perp}} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$

The given vector has been decomposed into a “horizontal component” (in the *xy*-plane) and a “vertical component” (on the *z*-axis)



Orthogonal Decomposition (4)

How to compute x_W? And $x_{W^\perp}$?

Theorem (The $A^T A$ Trick): Let W be a subspace of $\mathbb{R}^n$. Let $v_1, v_2, \dots, v_m$ be a spanning set for W, and let $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}$. Then for any x in $\mathbb{R}^n$, the matrix equation

$$A^T A v = A^T x \text{ (the unknown vector is } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Now, to find $x_{W^\perp}$, recall that $x_{W^\perp} = x - x_W$.

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T A v = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

Orthogonal Decomposition (5)

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T Av = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

Example 1

Let $x = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and let $W = \left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ in } \mathbb{R}^3 \,\middle|\, \overbrace{x_1 - x_2 + x_3 = 0}^{\text{A plane in } \mathbb{R}^3}\right\}$. Find x_W and compute the distance from x to W.

We need a basis for $W = \mathrm{Nul}\begin{bmatrix}1 & -1 & 1\end{bmatrix}$. $\begin{bmatrix}1 & -1 & 1\end{bmatrix}$ is already in RREF.

$$\begin{aligned} x_1 &= x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned} \implies \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_2\begin{bmatrix}1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\1\end{bmatrix} \implies W = \mathrm{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\} \implies \text{Let } A = \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix}$$

$$A^T A = \begin{bmatrix}1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix} = \underline{\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}} \qquad A^T x = \begin{bmatrix}1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \underline{\begin{bmatrix}3\\2\end{bmatrix}}$$

Orthogonal Decomposition (6)

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T A v = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

Example 1 (continued)

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad A^T x = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$A^TA \cdot v = A^Tx$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 2 & -1 & 3 \\ -1 & 2 & 2 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & \frac{8}{3} \\ 0 & 1 & \frac{7}{3} \end{array}\right] \implies v = \begin{bmatrix} \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}$$

$A \cdot v$

$$\boxed{x_W = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{8}{3} \\ \frac{7}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}}$$

$$x_{W^\perp} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x} - \underbrace{\begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}}_{x_W} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

distance from x to W

$$\|x_{W^\perp}\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \frac{\sqrt{12}}{3} = \frac{2\sqrt{3}}{3} \approx 1.15$$

Orthogonal Decomposition (7)

Orthogonal Projection Onto a Line

Let $L = \text{Span}\{u\}$ be a line in $\mathbb{R}^n$ and let x be a vector in $\mathbb{R}^n$. Let's compute x_L

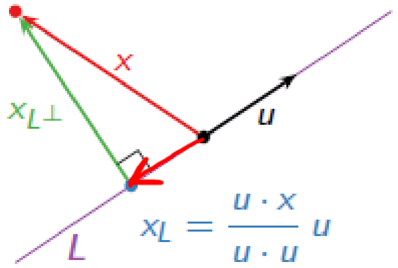
We must solve $u^T uv = u^T x$, where u is an $n \times 1$ matrix. Note that $u^T u = u \cdot u$ and $u^T x = u \cdot x$ so both quantities are scalars. So, $v = \frac{u \cdot x}{u \cdot u} \quad \Longrightarrow \quad x_L = uv = \frac{u \cdot x}{u \cdot u} u$

A v

Projection onto a Line

The projection of x onto a line $L = \text{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u} u \qquad x_{L^\perp} = x - x_L.$$



Orthogonal Decomposition (8)

Projection onto a Line

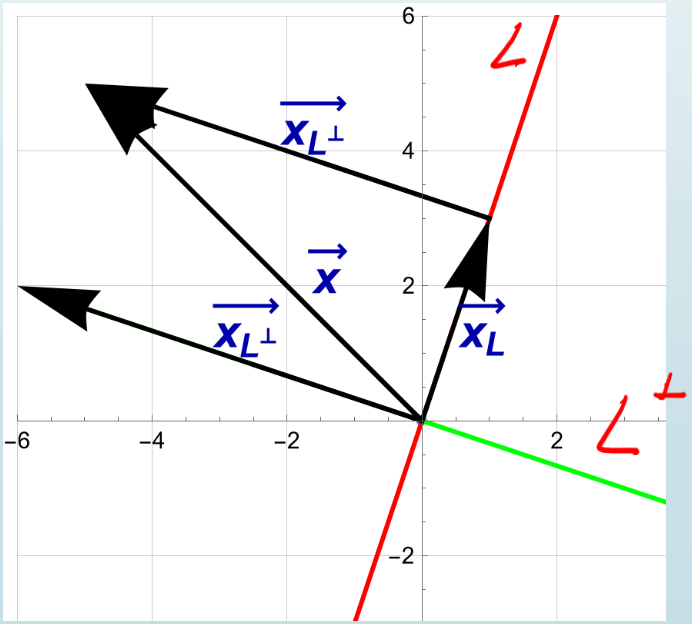
The projection of x onto a line $L = \text{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u} u \qquad x_{L^\perp} = x - x_L.$$

Example 2

Compute the orthogonal projection of $x = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ onto the line L spanned by $u = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and find the distance from x to L.

$$x_L = \frac{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 5 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{10}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad x_{L^\perp} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$



distance from x to L

$$\left|\left|x_{L^\perp}\right|\right| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$

Orthogonal Decomposition (9)

Projection onto a Line

The projection of x onto a line $L = \mathrm{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u}\, u \qquad x_{L^\perp} = x - x_L.$$

Example 3

Let $L = \mathrm{Span}\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ and let $b = \begin{bmatrix}-2\\-3\\-1\end{bmatrix}$. Find b_L and $b_{L^\perp}$.

$$b_L = \frac{\overbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-2\\-3\\-1\end{bmatrix}}^{u\ \cdot\ b}}{\underbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}_{u\ \cdot\ u}}\overset{u}{\begin{bmatrix}-1\\1\\1\end{bmatrix}} = \frac{-2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}$$

$$b_{L^\perp} = \begin{bmatrix}-2\\-3\\-1\end{bmatrix} - \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}-\frac{8}{3}\\-\frac{7}{3}\\-\frac{1}{3}\end{bmatrix}$$

Orthogonal Decomposition (10)

Example 4

Find x_W if $x = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ and $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Steps: Find $A^T A$ and $A^T x$. Then solve $A^T A v = A^T x$ for v. Then use $x_W = Av$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad A^T x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 2 & 11 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & 1 & 10 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & -3 & -12 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad x_W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

How far is x from W?

distance $= \|x_{W^\perp}\|$

$$x_{W^\perp} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \|x_{W^\perp}\| = \sqrt{3}$$

Orthogonal Decomposition (11)

Corollary: Let A be an $m \times n$ matrix with <u>linearly independent</u> columns and let $W = \mathrm{Col}(A)$. Then the $n \times n$ matrix $A^T A$ is invertible, and for all vectors x in $\mathbb{R}^m$, we have

$$x_W = A(A^T A)^{-1} A^T x$$

Example 5

Let $W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find a matrix expression for x_W in terms of x_1, x_2, x_3. Use the above corollary. Use your result to find x_W if $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad A^TA = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad x_W = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 + 2x_2 + x_3 \\ -x_1 + x_2 + 2x_3 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_W = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Orthogonal Decomposition (12)

Orthogonal projections as transformations

Properties of Orthogonal Projections. Let W be a subspace of $\mathbb{R}^n$ and define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(x) = x_W$. Then:

- T is a linear transformation.
- $T(x) = x$ if and only if x is in W.
- $T(x) = 0$ if and only if x is in $W^{\perp}$.
- $T \circ T = T$
- The range of T is W.