Matrix Transformations

Matrix Transformations (1)

What is a function?

A **function** is a rule that associates with each element of a set A a unique element in a set B, - a unique output for each input. If f assigns the element y to the element x, we write

$$y = f(x)$$

- x is the independent variable
- y is the dependent variable
- y is the image of x under f, or y is the value of f at x

- The set of all possible inputs is called the **domain** of f (the set A)
- The set B is called the **codomain** of f
- The subset of the codomain that consists of all images of all of the elements in the domain is called the **range** of f

Matrix Transformations (2)

"proper subset of"
$A \subset B$

"subset of"

<u>Single variable functions:</u>

$$f: A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$$

Example: $f(x) = \sin x$
Domain: $\mathbb{R}$
Codomain: $\mathbb{R}$
Range: $[-1, 1]$ $\quad -1 \le y \le 1$

<u>Vector-valued functions of several variables:</u>

$$f(x, y) = (2x - y, \cancel{y}x^2, \cancel{x + y}\sqrt{y})$$

Or we can write

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ x^2 \\ \sqrt{y} \end{bmatrix}$$

Input is a <u>vector</u> in $\mathbb{R}^2$ and output is a <u>vector</u> in $\mathbb{R}^3$

$$f: A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

Evaluate $f\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

Let $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find a vector w in $\mathbb{R}^2$ such that $\cancel{T}f(w) = b$. Is there more than one? $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Find a vector v in $\mathbb{R}^3$ which is not in the range of $\cancel{T}f$ $\quad \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ is not in the range.

Domain: $\{(x, y) \mid y \ge 0\}$
Codomain: $\mathbb{R}^3$
Range: Very hard

Matrix Transformations (3)

Transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$x \longrightarrow T(x)$$

Domain: $\mathbb{R}^n$
Codomain: $\mathbb{R}^m$
Given a vector x in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is the image of x under T
Range: the set of images $\{T(x) | x \in \mathbb{R}^n\}$

Identity transformation $\mathrm{Id}_{\mathrm{R^n}}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$\mathrm{Id}_{\mathrm{R^n}}(x) = x \text{ for all } x \text{ in } \mathbb{R}^n$$

Matrix Transformations (4)

Matrix transformations are defined by matrix-vector products**.**

Let A be an $m \times n$ matrix. The **matrix transformation** associated with A is the transformation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ defined by } \underline{T(x) = Ax}$$

It takes a vector x in $\mathbb{R}^n$ to the vector Ax in $\mathbb{R}^m$.
Another common notation:

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ defined by } T_A(x) = Ax$$

$$x \longrightarrow T_A(x) = Ax$$

Example 1

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}$.

Evaluate $T_A \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \end{bmatrix} - 1\begin{bmatrix} -1 \\ 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Domain of T_A: $\mathbb{R}^3$

Codomain of T_A: $\mathbb{R}^2$

Matrix Transformations (5)

What does the range of a matrix transformation look like?

If A has n columns $v_1, v_2, \ldots, v_n$ and we multiply A by a general vector x, we get

$$T_A(x) = Ax = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

and this is just a linear combination of the vectors $v_1, v_2, \ldots, v_n$. So, the range is the set of all possible linear combinations of the column vectors of A which is the column space of A.

Let A be an $m \times n$ matrix and let $T_A(x)$ be the associated matrix transformation, then

- The <u>domain</u> of T_A is $\mathbb{R}^n$ where n is the number of columns of A
- The <u>codomain</u> of T_A is $\mathbb{R}^m$ where m is the number of rows of A
- The <u>range</u> of T_A is the column space of A

Matrix Transformations (6)

What does the range of a matrix transformation look like?

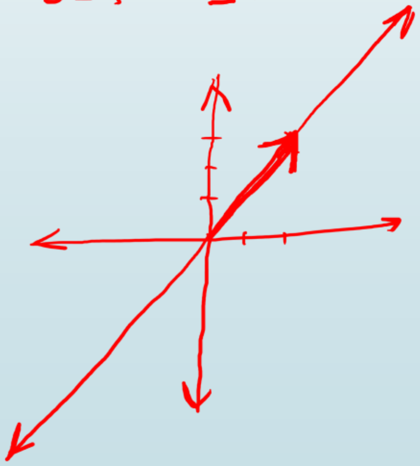
Example 2

Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$. Domain of T_A: $\mathbb{R}^3$ Codomain of T_A: $\mathbb{R}^2$

Describe the range of T_A geometrically.

$$\text{Range} = \text{col}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\} = \underbrace{\text{span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}}_{\text{a basis}} \quad \dim \text{col}(A) = 1$$

A line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$



Matrix Transformations (7)

What does the range of a matrix transformation look like?

Example 3

Let $B = \begin{bmatrix} 1 & 3 \\ -2 & 2 \\ -2 & 1 \end{bmatrix}$. Domain of T_B: $\mathbb{R}^2$ Codomain of T_B: $\mathbb{R}^3$

Describe the range of T_B geometrically.

$$\text{Range} = \text{Col}(B) = \text{span}\underbrace{\left\{ \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}}_{\substack{\text{linearly independent} \\ \Rightarrow \text{ a basis}}} \qquad \dim \text{Col}(B) = 2$$

this is a plane in $\mathbb{R}^3$.

Matrix Transformations (8)

A matrix transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$, when the corresponding matrix is a square $n \times n$ matrix, is called a **matrix operator**, and we say that T <u>operates</u> on $\mathbb{R}^n$

Some examples of matrix operators on $\mathbb{R}^2$. Consider the transformations defined by the following matrices:

$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$ $\quad T_A =$ Reflection over the y-axis

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ Dilation by a factor of 2

$T_A \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

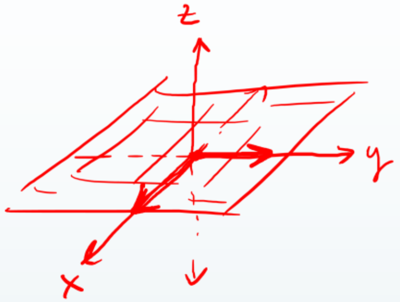
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ $\quad$ Rotation (90 degrees counterclockwise)

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$ $\quad$ Shear

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ Identity $\mathrm{Id}_{\mathbb{R}^2}$

Matrix Transformations (9)

$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \underline{\mathbb{R}^2 \text{ in } \mathbb{R}^3}$



Questions about a matrix transformation

Example 4

$\text{span}\{\ \} = \vec{0}$

$\dim \text{span}\{\ \} = 0$

Let $A = \begin{bmatrix}1 & 1\\0 & 1\\1 & 1\end{bmatrix}$. Domain of T_A: $\mathbb{R}^2$ Codomain of T_A: $\mathbb{R}^3$

- Describe the range of T_A geometrically. $\text{span}\{\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}}_{\text{a basis}}\}$ $\dim \text{Col}(A) = 2$ $\underline{\text{A plane in } \mathbb{R}^3}$

- Evaluate $T_A(u)$ for $u = \begin{bmatrix}1\\-2\end{bmatrix}$ $T_A\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}-2\\-2\\-2\end{bmatrix} = \begin{bmatrix}-1\\-2\\-1\end{bmatrix}$

- Let $b = \begin{bmatrix}7\\5\\7\end{bmatrix}$. Find a vector v in $\mathbb{R}^2$ such that $T_A(v) = b$. Is there more than one? No (No free variables)

  $Av = \begin{bmatrix}7\\5\\7\end{bmatrix}$ $\left[\begin{array}{cc|c}1 & 1 & 7\\0 & 1 & 5\\1 & 1 & 7\end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & 5\\0 & 0 & 0\end{array}\right]$ $v = \begin{bmatrix}2\\5\end{bmatrix}$

- Does there exist a vector w in $\mathbb{R}^3$ such that there is more than one v in $\mathbb{R}^2$ with $T_A(v) = w$?

- Find a vector w in $\mathbb{R}^3$ which is not in the range of T_A. No. $w = \begin{bmatrix}1\\2\\3\end{bmatrix}$ because $\begin{bmatrix}1 & 1\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

  $x + y = 1$
  $y = 2$
  $x + y = 3$
  has no solution