Determinants and Volumes

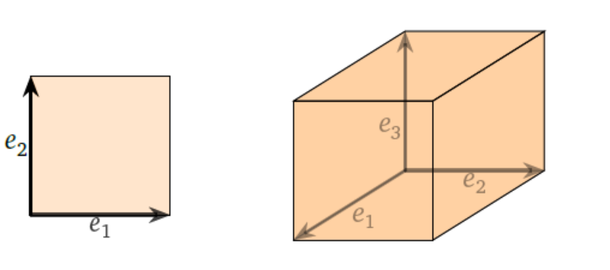
Determinants and Volumes (1)

Definition: The ***parallelepiped*** determined by n vectors $v_1, v_2, \dots, v_n$ in $\mathbb{R}^n$ is the subset

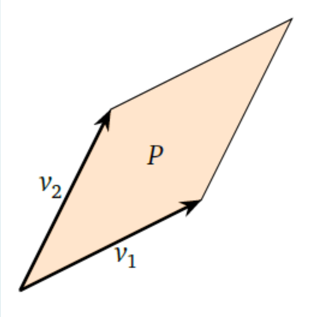
$$P = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n | 0 \le a_1, a_2, \dots, a_n \le 1\}$$

Examples:

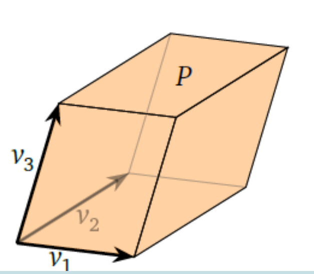
- The parallelepiped determined by the standard coordinate vectors $e_1, e_2, \dots, e_n$ is the unit n-dimensional cube (in $\mathbb{R}^2$, it's the unit square; in $\mathbb{R}^3$, it's the unit cube)



- When $n = 2$, it is a parallelogram



- When $n = 3$, it is a parallelepiped



Determinants and Volumes (2)

When does a parallelepiped have a volume of zero?

When it is *flat*, that is, when it is “squashed into a lower dimension”. This happens when the set $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent which implies that the matrix with rows $v_1, v_2, \ldots, v_n$ is not invertible and has determinant zero.

Theorem:

Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbb{R}^n$, let P be the parallelepiped determined by these vectors, and let A be the matrix with rows $v_1, v_2, \ldots, v_n$. Then the absolute value of the determinant of A is the volume of P:

$$|\det(A)| = \mathrm{vol}(P)$$

Determinants and Volumes (3)

Defining properties of |det|:

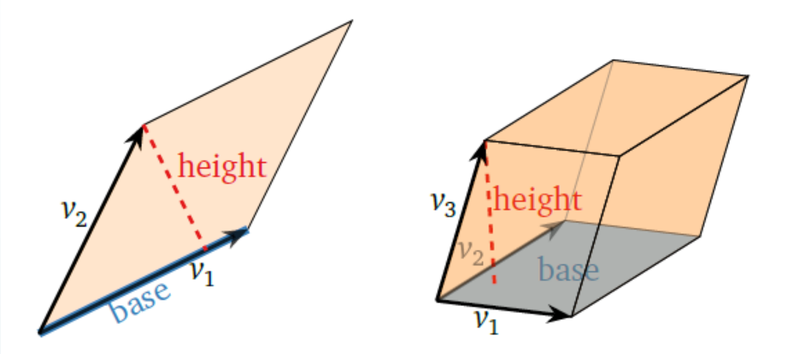
- Row replacements on A do not change $|\det(A)|$
- Row scaling by c multiplies $|\det(A)|$ by $|c|$
- Row swaps do not change $|\det(A)|$
- $|\det(I)| = 1$

Now define vol(A), where A is a square matrix, as the volume of the parallelepiped determined by the rows of A. So, consider vol as a function from the set of square matrices to the real numbers. Need to show that vol satisfies the four properties above.

Note the volume of a parallelepiped is the volume of its base times its height. The base is the parallelepiped determined by $v_1, v_2, \dots, v_{n-1}$, and the height is the perpendicular distance of v_n to the base.

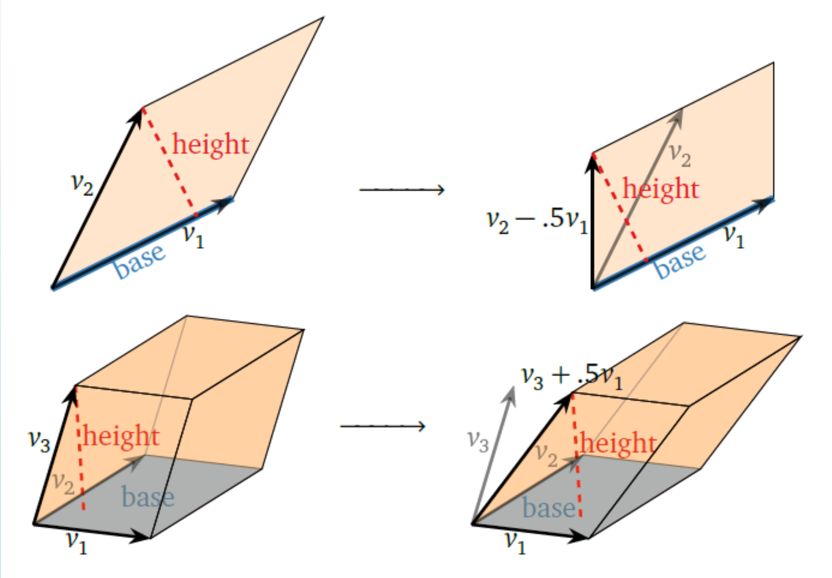
Determinants and Volumes (4)

Base and height



1. Row replacement of the form $R_n = R_n + cR_i$ translates v_n by a multiple of v_i moving v_n in a direction parallel to the base. This does not change the base or the height, so vol(A) is unchanged.

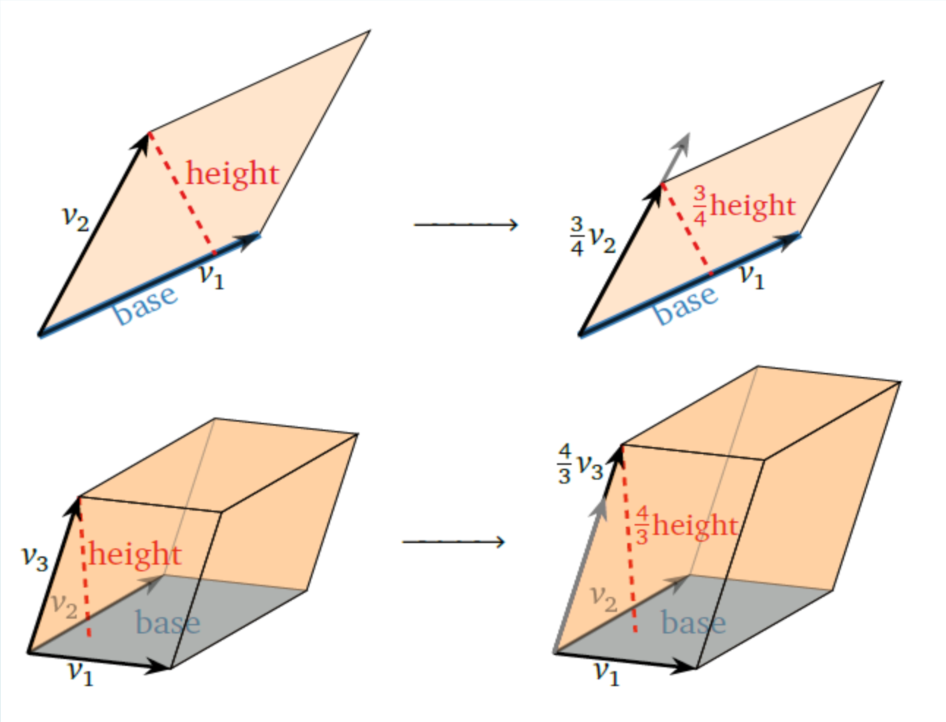
Translating v_n



Determinants and Volumes (5)

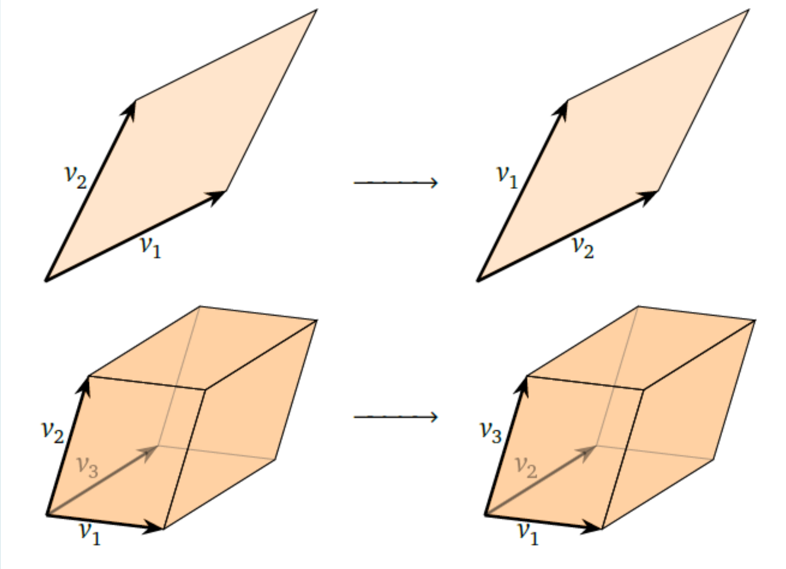
2. Now let's look at a row scaling of the form $R_n = cR_n$ scaling the length of v_n by a factor of $|c|$. This also scales the perpendicular distance of v_n from the base by a factor of $|c|$

Scaling v_n



3. Swapping two rows of A just reorders the determining vectors which has no effect on the parallelepiped. So, vol(A) is unchanged by row swaps.

Reordering vectors



Determinants and Volumes (6)

4. The rows of the identity matrix are the standard coordinate vectors. The associated parallelepiped is the unit cube and its volume is 1. So, we have $\mathrm{vol}(I) = 1$

All four properties satisfied $\Longrightarrow$

$$\mathrm{vol}(P) = \mathrm{vol}(A) = |\det(A)|$$

Example 1

Find the formula for the area of a parallelogram determined by the vectors $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$

$$\text{area} = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |ad - bc|$$

Determinants and Volumes (7)

Example 2

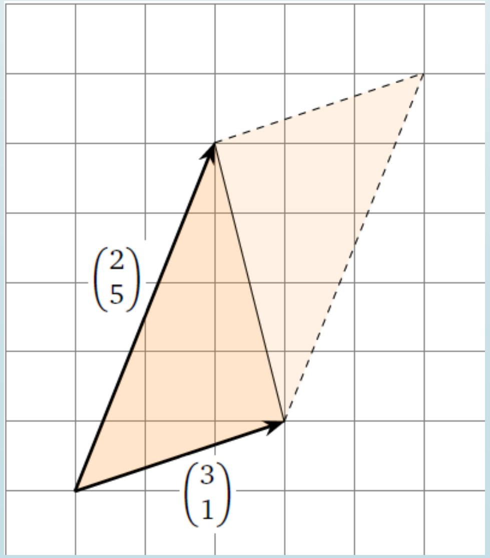
Find the area of a parallelogram determined by the vectors $v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

The corresponding matrix is $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$ and $\left| \det \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \right| = 11$

Example 3

Find the area of the triangle with vertices $(-1, -2), (2, -1), (1,3)$

First, we need to find two determining vectors which can be done by subtracting vector tails from vector heads.



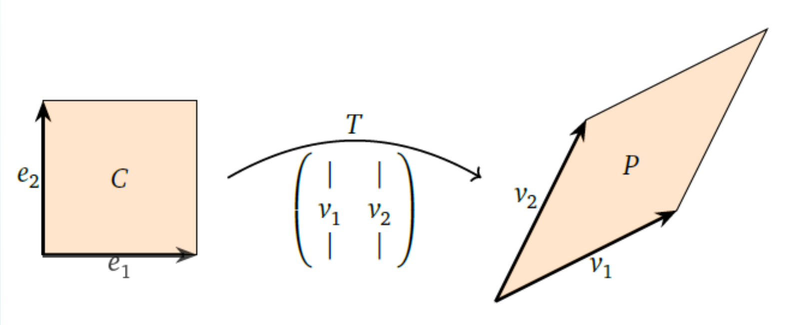
The two vectors we obtain are $v_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

The corresponding matrix is $\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ and $\left| \det \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \right| = |-13| = 13$

Since the area of the parallelogram is 13 the area of the triangle is $\frac{13}{2}$

Determinants and Volumes (8)

Let A be an $n \times n$ matrix with columns $v_1, v_2, \dots, v_n$ and let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the associated matrix transformation. Then $T(e_1) = v_1$, $T(e_2) = v_2$ and so on. So, T takes the unit cube C to the parallelepiped P determined by $v_1, v_2, \dots, v_n$



Note that the area of $P = T(C)$ is just $|\det(A)|$. This implies that T rescaled the volume of the cube by a factor of $|\det(A)|$

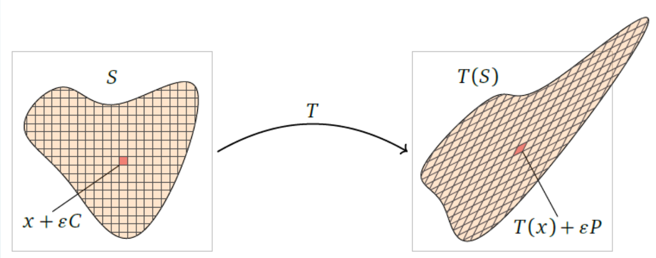
Now let S be any region in $\mathbb{R}^n$, and let $T(S)$ be the image of this region under the transformation T, that is $T(S) = \{T(x) | x \text{ in } S\}$, then

$$\mathrm{vol}\big(T(S)\big) = |\det(A)|\,\mathrm{vol}(S)$$

Determinants and Volumes (9)

Now let S be any region in $\mathbb{R}^n$, and let $T(S)$ be the image of this region under the transformation T, that is $T(S) = \{T(x)|x \text{ in } S\}$, then

$$\text{vol}\big(T(S)\big) = |\det(A)|\,\text{vol}(S)$$



Any region S in $\mathbb{R}^n$ can be approximated by a collection of very small cubes. The image $T(S)$ is then approximated by the image of this collection of cubes, which is a collection of very small parallelepipeds.

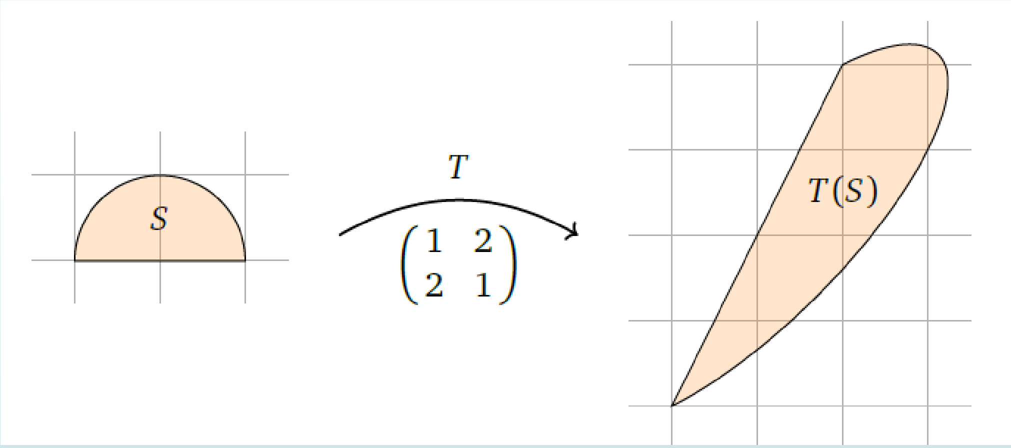
The volume of S is closely approximated by the sum of the volumes of the cubes. The volume of $T(S)$ is closely approximated by the sum of the volumes of the parallelepipeds. The volume of each cube is scaled by $|\det(A)|$. So, the sum of the volumes of the parallelepipeds is $|\det(A)|$ times the sum of the volumes of the cubes.

Determinants and Volumes (10)

Example 4

Let S be a half-circle of radius 1, let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, and define $T_A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T_A(x) = Ax$.

What is the area of $T(S)$?



area of $S = \frac{\pi}{2}$ $\quad |\det(A)| = |-3| = 3$ $\quad \operatorname{vol}\big(T(S)\big) = 3\operatorname{vol}(S) = \frac{3\pi}{2}$

Determinants and Volumes (11)

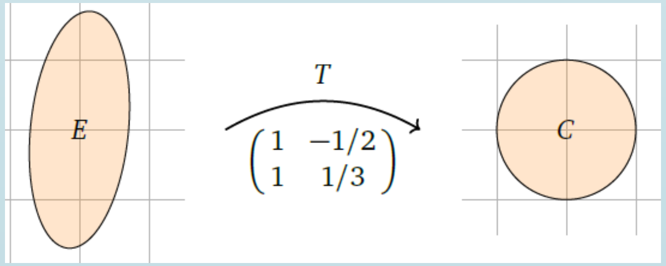
Example 5

Find the area of the interior of the ellipse defined by the equation

$$\left(\frac{2x-y}{2}\right)^2 + \left(\frac{y+3x}{3}\right)^2 = 1$$

This ellipse was obtained from the unit circle $X^2 + Y^2 = 1$ by the linear change of coordinates $X = \frac{2x-y}{2}$ and $Y = \frac{y+3x}{3}$. So, define a linear transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2x-y}{2} \\ \frac{y+3x}{3} \end{bmatrix}$

that takes the ellipse to the circle $\Longrightarrow \quad A = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{3} \end{bmatrix}$ and $|\det(A)| = \left|\frac{1}{3} + \frac{1}{2}\right| = \frac{5}{6}$



$$\pi = \mathrm{vol}(C) = \mathrm{vol}\big(T(E)\big) = |\det(A)|\mathrm{vol}(E) = \frac{5}{6}\mathrm{vol}(E)$$

$$\Longrightarrow \quad \mathrm{vol}(E) = \frac{6\pi}{5}$$