Matrix Inverses and the Invertible Matrix Theorem

Matrix Inverses and the Invertible Matrix Theorem (1)

If A is a <u>square</u> matrix, and if a matrix B of the same size can be found such that $AB = I$ and $BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. We write $B = A^{-1}$. If no such matrix B can be found, then A is said to be ***singular.***

Example 1

Show that the matrix $B = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad BA = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

- Matrix invertibility is restricted to <u>square</u> matrices (it can be shown that if A is an $m \times n$ matrix with $m \neq n$ then there is no $n \times m$ matrix B such that $AB = I_m$ and $BA = I_n$).
- We will show that it suffices to check that $AB = I$ <u>or</u> $BA = I$ to conclude that A is invertible.

Matrix Inverses and the Invertible Matrix Theorem (2)

As was stated earlier, not all square matrices are invertible. For example, $\begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$ is not invertible since any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when multiplied by $\begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix}$ will result in $\begin{bmatrix} 4a+b & 0 \\ 4c+d & 0 \end{bmatrix}$ which can never be equal to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

It is important to note that an invertible matrix has exactly one inverse. That is,

Theorem: If B and C are both inverses of the matrix A then $B = C$.

Proof: Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$ which implies that $C = B$.

Matrix Inverses and the Invertible Matrix Theorem (3)

Fact: If A is an invertible matrix, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

Proof: By definition, $A^{-1}A = I$ and $AA^{-1} = I$.

Fact: If A and B are invertible matrices with the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(AA^{-1})B = B^{-1}IB = B^{-1}B = I$$

In general, a product of any number of invertible matrices is invertible and the inverse of the product is the product of the inverses in reverse order.

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Matrix Inverses and the Invertible Matrix Theorem (4)

Theorem: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is (**1**) invertible if and only if $ad - bc \neq 0$, (**2**) in which case the inverse of A is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

So, to obtain the inverse (1) switch a and d, (2) negate b and c, (3) divide all entries by $ad - bc$.

The expression $ad - bc$ is called the ***determinant*** of A and is denoted by $\det(A)$.

Proof: If $\det(A) \neq 0$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You can easily verify this.

Matrix Inverses and the Invertible Matrix Theorem (5)

Theorem: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is (**1**) invertible if and only if $ad - bc \neq 0$, (**2**) in which case the inverse of A is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

Proof (cont'd): Suppose that $\det(A) = 0$ and A is not the zero matrix, then

$$Au = A\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} -ab + ba \\ -bc + ad \end{bmatrix} = \begin{bmatrix} 0 \\ \det(A) \end{bmatrix} = 0$$

$$Av = A\begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad - bc \\ cd - dc \end{bmatrix} = \begin{bmatrix} \det(A) \\ 0 \end{bmatrix} = 0 \quad \text{and}$$

Since the entries of A are not all zero, one of the two vectors above, u or v, is not the zero vector but is in the null space of A. Let's call that vector w. Now, suppose A has an inverse, call it B, then

$$\underline{w} = Iw = (BA)w = B(Aw) = B0 = \underline{0}$$

which gets us to a contradiction, since $w \neq 0$.

Matrix Inverses and the Invertible Matrix Theorem (6)

Example 2

Find the inverse of the given matrix if possible

(a) $A = \begin{bmatrix} -5 & 2 \\ 4 & -2 \end{bmatrix}$ **(b)** $B = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ **(c)** $C = \begin{bmatrix} 4 & -2 \\ -10 & 5 \end{bmatrix}$

$\det(A) = 2$

$A^{-1} = \begin{bmatrix} -\frac{2}{2} & -\frac{2}{2} \\ -\frac{4}{2} & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -\frac{5}{2} \end{bmatrix}$

$\det(B) = 1$

$B^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$\det(C) = 0$

(a) $A^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -\frac{5}{2} \end{bmatrix}$ **(b)** $B^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ **(c)** C is singular since $\det(C) = 4(5) - (-10)(-2) = 0$

Matrix Inverses and the Invertible Matrix Theorem (7)

Solving linear systems using inverses

Let A be an invertible $n \times n$ matrix and let b be a vector in $\mathbb{R}^n$. Then to solve the matrix equation $Ax = b$:

$$Ax = b$$
$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = A^{-1}b$$

Example 3

Use the method just discussed to solve the following matrix equation

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}_{b} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{2} & \frac{-2}{2} \\ \frac{-2}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \longrightarrow$$

$$x = A^{-1} \cdot b$$

$$\det(A) = 2$$

$$\longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + 3 \\ -1 - \frac{9}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{11}{2} \end{bmatrix}$$

Matrix Inverses and the Invertible Matrix Theorem (8)

Example 4

We can now solve a general 2×2 linear system using a matrix equation and the process of inversion

$A \quad x = b \qquad\qquad A^{-1} \cdot A \quad x = A^{-1} \cdot b$

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned} \longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow$$

$$I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{du - bv}{ad - bc} \\ \dfrac{av - cu}{ad - bc} \end{bmatrix} \longrightarrow$$

$$x = \frac{du - bv}{ad - bc} \qquad y = \frac{av - cu}{ad - bc}$$

Matrix Inverses and the Invertible Matrix Theorem (9)

A method for computing A^{-1} for general square matrices

Let A be an $n \times n$ matrix. Create an augmented matrix of the form $(A|I)$. If the reduced row echelon form of $(A|I)$ is $(I|B)$, then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

$A\vec{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $A\vec{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $A\vec{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\left[A \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right] \longrightarrow$ $B = [\vec{x}_1\ \vec{x}_2\ \vec{x}_3]$

Row reducing $(A|I)$ is equivalent to solving n systems of linear equations simultaneously: $A\vec{x}_1 = \vec{e}_1, A\vec{x}_2 = \vec{e}_2, \dots A\vec{x}_n = \vec{e}_n$ where the e_is are the standard coordinate vectors. When reduced to $(I|B)$, the columns of B are the solutions to these systems. Now, recall that the product Be_i simply returns the i^{th} column of B, $\vec{x}_i$, so

$$Be_i = x_i$$
$$\underline{A(Be_i)} = Ax_i = \underline{e_i}$$
$$(AB)e_i = e_i$$

The above shows that the i^{th} column of AB is e_i for all i. It follows that $AB = I$ and, therefore, $B = A^{-1}$.

Matrix Inverses and the Invertible Matrix Theorem (10)

Example 5

Use Gauss-Jordan reduction to find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

$(A \mid I)$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & -1 & -5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -2 & 1 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_1+4R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 4 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R_2-6R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 4 \\ 0 & 1 & 0 & -3 & 5 & -6 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -6 \\ 1 & -1 & 1 \end{bmatrix}$$

Matrix Inverses and the Invertible Matrix Theorem (11)

Example 5

Use Gauss-Jordan reduction to find the inverse of $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

Check

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -6 \\ 1 & -1 & 1 \end{bmatrix}}^{A^{-1}} = \begin{bmatrix} 2-3+2 & -3+5-2 & 4-6+2 \\ 6-6+0 & -9+10+0 & 12-12+0 \\ 4-3-1 & -6+5+1 & 8-6-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 6

Express in matrix form and use the result of Example 5 to solve

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 3x_1 + 2x_2 \phantom{{}+2x_3} &= 1 \\ 2x_1 + x_2 - x_3 &= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{b}$$

$$x = A^{-1}b \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -6 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$x_1 = 1$, $x_2 = -1$ and $x_3 = 1$.

Matrix Inverses and the Invertible Matrix Theorem (12)

Example 7

Use Gauss-Jordan reduction to show that $C = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is not invertible

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ -1 & 2 & 5 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 9 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

The row of zeros indicates that C is not row equivalent to I and is therefore not invertible.

Matrix Inverses and the Invertible Matrix Theorem (13)

Invertible linear transformations

Definition: A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is ***invertible*** if there exists a transformation $U: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T \circ U = \mathrm{Id}_{\mathbb{R}^n}$ and $U \circ T = \mathrm{Id}_{\mathbb{R}^n}$. U is called the ***inverse*** of T and is denoted by T^{-1}.

The above means that $(T \circ U)(x) = T(U(x)) = x$ and $(U \circ T)(x) = U\big(T(x)\big) = x$. T "undoes" the action of U and vice versa.

Examples (Single variable functions):

- $f(x) = 3x$ and $g(x) = \frac{x}{3}$ are inverses of each other since, $(f \circ g) = f\big(g(x)\big) = f\left(\frac{x}{3}\right) = 3\left(\frac{x}{3}\right) = x$

  $(g \circ f) = g\big(f(x)\big) = g(3x) = \frac{3x}{3} = x$
- $f(x) = x^2$ is not an invertible function, because, for example, $f(-2) = 4 = f(2)$. But if f has an inverse g, then $g\big(f(-2)\big) = \underline{-2}$ and $g\big(f(-2)\big) = \underline{g(4)}$ which implies that $g(4) = -2$. At the same time $g\big(f(2)\big) = \underline{2}$ and $g\big(f(2)\big) = \underline{g(4)}$ which implies that $g(4) = 2$, but $g(4)$ is a unique number and cannot be equal 2 and -2 at the same time

Matrix Inverses and the Invertible Matrix Theorem (14)

Invertible linear transformations

Definition: A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is ***invertible*** if there exists a transformation $U: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T \circ U = \mathrm{Id}_{\mathrm{R}^{\mathrm{n}}}$ and $U \circ T = \mathrm{Id}_{\mathrm{R}^{\mathrm{n}}}$. U is called the ***inverse*** of T and is denoted by T^{-1}.

Earlier we examined many transformations $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that are invertible:

- Dilations
- Rotations
- Reflections

Here is a transformation that is not invertible, $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as the projection onto the xy-plane. Every vector on the z-axis projects onto the zero vector. The inverse transformation “would not know” what to send the zero vector to. More precisely, letting U be the potential inverse,

$$\underline{0} = (U \overset{I}{\circ} T)(0) = U\big(T(0)\big) = \underline{U(0)}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (U \overset{I}{\circ} T)\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = U\left(T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \underline{U(0)}$$ but $U(0)$ must be a unique vector.

Matrix Inverses and the Invertible Matrix Theorem (15)

Invertible linear transformations

Proposition:

(1) A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible if and only if it is both one-to-one and onto.

(2) If T is already known to be invertible, then $U: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the inverse of T provided that either $T \circ U = \mathrm{Id}_{\mathbb{R}^n}$ or $U \circ T = \mathrm{Id}_{\mathbb{R}^n}$; it is only necessary to verify one.

Proof:

(1) T is one-to-one and onto $\Leftrightarrow T(x) = b$ has exactly one solution for every b in $\mathbb{R}^n$.

Suppose that T is invertible. Then $T(x) = b$ always has the unique solution $x = T^{-1}(b)$ because

$$T(x) = b$$

$$\underbrace{T^{-1}\big(T}_{I}(x)\big) = T^{-1}(b)$$

$$x = T^{-1}(b)$$

So, T is one-to-one and onto. Now, suppose that T is one-to-one and onto. Let b be a vector in $\mathbb{R}^n$ and let $x = U(b)$ be the unique solution of $T(x) = b$. Then U defines a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any x in $\mathbb{R}^n$, we have $\boldsymbol{U(T(x)) = x}$, because x is the unique solution of the equation $T(x) = b$. For any b in $\mathbb{R}^n$, we have $\boldsymbol{T(U(b)) = b}$ because $x = U(b)$ is the unique solution of $T(x) = b$. We conclude that U is the inverse of T, and T is invertible.

Matrix Inverses and the Invertible Matrix Theorem (16)

Invertible linear transformations

Proposition:
(1) A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible if and only if it is both one-to-one and onto.
(2) If T is already known to be invertible, then $U: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the inverse of T provided that either $T \circ U = \mathrm{Id}_{\mathrm{R^n}}$ or $U \circ T = \mathrm{Id}_{\mathrm{R^n}}$; it is only necessary to verify one.

Proof:
(2) Suppose T is invertible and $T \circ U = \mathrm{Id}_{\mathrm{R^n}}$ we have the following:

$$
\begin{aligned}
T \circ U &= \mathrm{Id}_{\mathrm{R^n}} \\
\underbrace{T^{-1} \circ T}_{} \circ U \circ T &= T^{-1} \circ \mathrm{Id}_{\mathrm{R^n}} \circ T \\
\underline{\mathrm{Id}_{\mathrm{R^n}} \circ U \circ T} &= \underbrace{T^{-1} \circ T}_{} \\
U \circ T &= \mathrm{Id}_{\mathrm{R^n}}
\end{aligned}
$$

We could have started with $U \circ T = \mathrm{Id}_{\mathrm{R^n}}$ and using a similar strategy would end up with $T \circ U = \mathrm{Id}_{\mathrm{R^n}}$

Matrix Inverses and the Invertible Matrix Theorem (17)

Invertible linear transformations

Note:
If a linear transformation is one-to-one and onto then its standard matrix must be a square matrix, so it must be from $\mathbb{R}^n$ to $\mathbb{R}^n$; its domain and codomain must be the same. So, invertibility is restricted to this case.

Theorem:
Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is invertible, in which case T^{-1} is linear with standard matrix A^{-1}.

$U(x+y) = U(x) + U(y)$ and $U(cx) = cU(x)$

Call the inverse of T, U. Check the two defining properties of linearity for the transformation U:

- $\underline{x + y} = T(U(x)) + T(U(y)) = \boxed{T(U(x) + U(y))}$, by linearity of T; now apply U to both sides, $\boldsymbol{U(x+y)} = \underbrace{U(T}_{I}(U(x) + U(y))) = \boldsymbol{U(x) + U(y)}$, using that fact $U \circ T = \mathrm{Id}_{\mathrm{R}^n}$.
- A similar strategy will show that $\boldsymbol{U(cx) = cU(x)}$
- Let B be the standard matrix for U. Then the standard matrix for $T \circ U$ is AB, but $T \circ U = \mathrm{Id}_{\mathrm{R}^n}$ with the standard matrix I. So, $AB = I$, similarly we can show that $BA = I$. Hence A is invertible and $B = A^{-1}$. One can similarly show that invertibility of A implies invertibility of T.

Matrix Inverses and the Invertible Matrix Theorem (18)

Example 8

Invertible linear transformations

Determine whether the matrix operator $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + 2x_2 \\ 5x_1 + 3x_2 \end{bmatrix}$ is invertible. If so, find T^{-1}.

Standard matrix for T: $A = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$ $\det(A) = 2 \Rightarrow$ A is invertible

$\Rightarrow$ T is invertible

$A^{-1} = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{bmatrix}$ $T^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_1 - x_2 \\ -\frac{5}{2}x_1 + 2x_2 \end{bmatrix}$

Example 9

Find the standard matrix A for the $\underbrace{\text{“dilation by a factor of 3” operator in}}_{T}$ $\mathbb{R}^2$. Then find A^{-1} and a formula for T_A^{-1}.

$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ $T_A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_1 \\ \frac{1}{3}x_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det(A) = 9$

Matrix Inverses and the Invertible Matrix Theorem (19)

The invertible matrix theorem

Let A be an $n \times n$ matrix and let transformation $T_A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the associated matrix transformation. Then the following statements are equivalent:

1. A is invertible
2. A has n pivots
3. $\text{Nul}(A) = \{0\}$
4. The columns of A are linearly independent
5. The columns of A span $\mathbb{R}^n$
6. $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$
7. T is invertible
8. T is one-to-one
9. T is onto

Matrix Inverses and the Invertible Matrix Theorem (20)

Other conditions equivalent to matrix invertibility

1. The reduced row echelon form of A is the identity matrix I
2. $Ax = 0$ has no solutions other than the trivial one
3. $\text{nullity}(A) = 0$
4. The columns of A form a basis for $\mathbb{R}^n$
5. $Ax = b$ is consistent for all b in $\mathbb{R}^n$
6. $\text{Col}(A) = \mathbb{R}^n$
7. $\dim \text{Col}(A) = n$
8. $\text{rank}(A) = n$

Corollary: Let A be an $n \times n$ matrix and suppose that there exists an $n \times n$ matrix B such that $AB = I$ or $BA = I$. Then A is invertible and $B = A^{-1}$

Proof: Let $AB = I$. For any b in $\mathbb{R}^n$, $b = Ib = (AB)b = A(Bb)$, which implies that $T_A(Bb) = b$ so b is in the range of T_A which shows that T_A is onto and by the inverse matrix theorem A is invertible. Furthermore,

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B$$

So, $B = A^{-1}$. If $BA = I$, we show that T_A is one-to-one and conclude that A is invertible.