One-to-one and Onto

One-to-one and Onto (1)

Definition: A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **one-to-one** (*injective*) if, for every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x in $\mathbb{R}^n$.

<u>Equivalent definitions:</u>

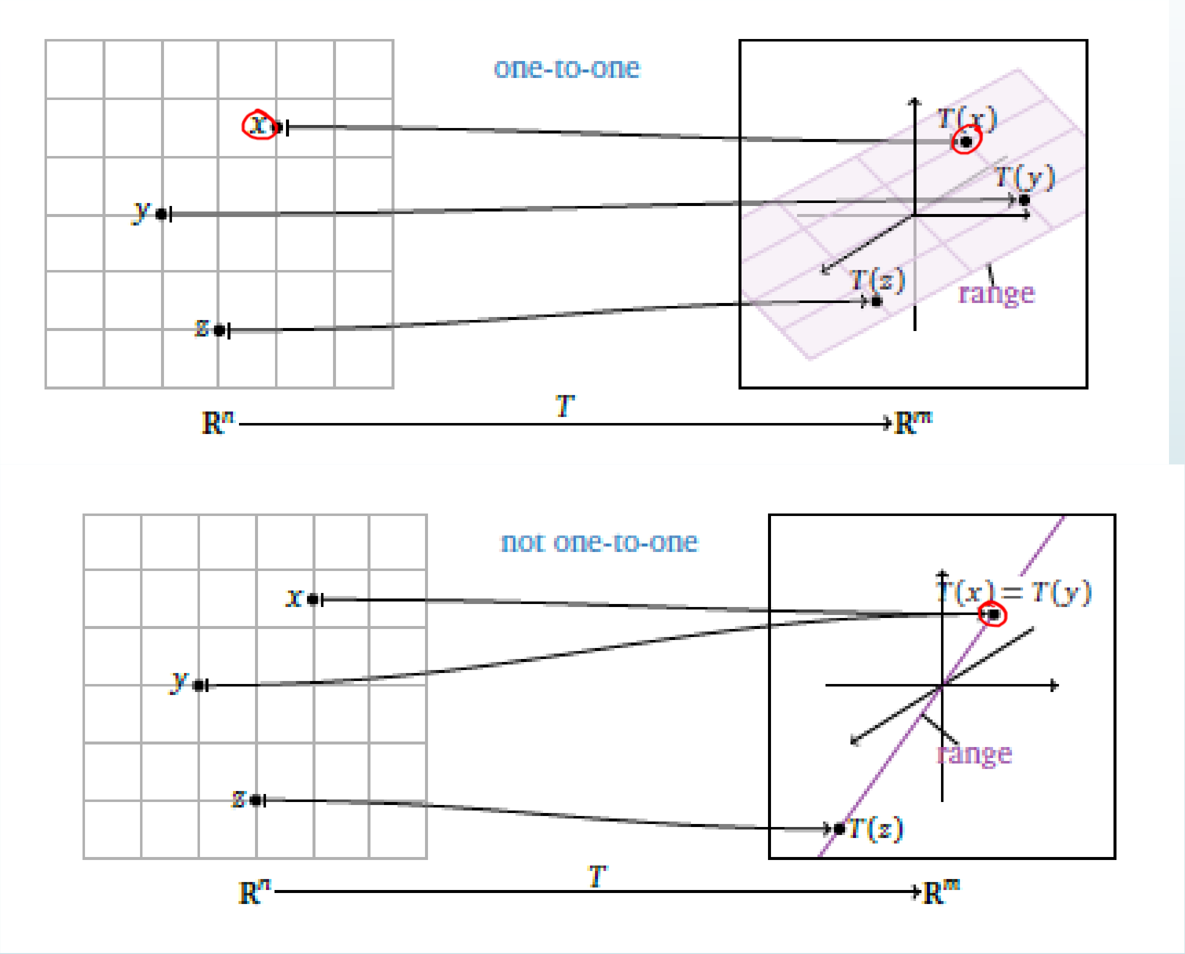
- For every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has one or zero solutions x in $\mathbb{R}^n$
- Different inputs of T have different outputs
- If $T(u) = T(v)$ then $u = v$

T is <u>not</u> one-to-one if (equivalent statements below)

- There exists some vector b in $\mathbb{R}^m$ such that the equation $T(x) = b$ has more than one solution
- There are two different inputs of T with the same output
- There exist vectors u, v such that $u \neq v$ but $T(u) = T(v)$

One-to-one and Onto (2)

Definition: A transformation $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **one-to-one** (*injective*) if, for every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution x in $\mathbb{R}^n$.



One-to-one and Onto (3)

Examples (single variable functions):

- $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \sin x$ is not one-to-one since $\sin(0) = \sin(\pi) = 0$; 0 and π have the same output. The equation $\sin x = 0$ actually has infinitely many solutions: $k\pi$ where k is any integer.

- $g: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = e^x$ is one-to-one since $g(x) = g(y)$ implies that $x = y$;
  $e^x = e^y \implies \cancel{\ln}(\cancel{e}^x) = \cancel{\ln}(\cancel{e}^y) \implies x = y$, or note that the equation $e^x = C$ has one
solution, $\ln C$, if $C > 0$ and no solutions if $C \leq 0$.

- $p: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $p(x) = x^3$ is one-to-one since $x^3 = y^3$ implies that $x = y$ (taking the cube root of both sides)

- $q: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $q(x) = x^3 - x$ is not one-to-one since $q(0) = q(1) = q(-1) = 0$; these are the solutions to the equation $x^3 - x = 0$ $(\implies x(x-1)(x+1) = 0 \implies x = 0,1,-1)$

One-to-one and Onto (4)

Theorem: Let A be an $m \times n$ matrix, and let $T_A = Ax$ be the corresponding matrix transformation. The following statements are equivalent:

1. T is one-to one
2. For every b in $\mathbb{R}^m$ the equation $T(x) = b$ has at most one solution
3. For every b in $\mathbb{R}^m$ the equation $Ax = b$ has a unique solution or is inconsistent
4. $Ax = 0$ has only the trivial solution ✓
5. The columns of A are linearly independent
6. A has a pivot in every column
7. The range of T has dimension n

Proof: (1),(2) and (3) are equivalent by definition; (3) and (4) are equivalent since if $Ax = b$ has a solution it must be a translate of the zero vector which is the solution of $Ax = 0$ which is a single vector (in other words there are no free variables); (4), (5) and (6) are equivalent as a result of the definition of linear independence; (6) and (7) are equivalent because range of T is $\mathrm{Col}(A)$ and $\mathrm{rank}(A)$ is equal to the number of pivot columns.

One-to-one and Onto (5)

Example 1

Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ one-to-one?

$$\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

T_A is one-to-one because there is a pivot in every column

Example 2

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ one-to-one? If not, find two different vectors u, v such that $T_A(u) = T_A(v)$.

Not one-to-one, not all columns are pivot columns.

Solve $Ax = 0$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x = 0$, $y = 0$, z is a free variable.

let $u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow T_A(u) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For example, let $v = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

then $T_A(v) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Verify $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

One-to-one and Onto (6)

Example 3

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ one-to-one? If not, find two different vectors u, v such that $T_A(u) = T_A(v)$.

T_A is not one-to-one because not every column is a pivot column.

$u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow T_A(u) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$x - z = 0$
$y + z = 0$ $\Rightarrow$ $x = z$, $y = -z$

z is a free variable

Let $v = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \Rightarrow T_A(v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example 4

Let $A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 2 & -6 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ one-to-one? If not, find two different vectors u, v such that $T_A(u) = T_A(v)$.

$\begin{bmatrix} 1 & -1 & 3 \\ -2 & 2 & -6 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

T_A is not one-to-one.

$x - y + 3z = 0$ $\quad$ $x = y - 3z$

y is a free variable

z is a free variable

letting $y = 1, z = 1$ $\Leftarrow$

$x = -2$

$v = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow T_A(v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad$ $T_A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

One-to-one and Onto (7)

Some observations

- T_A is <u>not</u> one-to-one $\Leftrightarrow$ Nul(A) is not the zero space $\Leftrightarrow$ nullity$(A) > 0$

- Transformations whose associated matrices are wide ($n > m$) are <u>not</u> one-to-one. (Each column and each row can only contain one pivot, so in order to have a pivot in every column, A must have at least as many rows as columns, so we need $n \leq m$.) Geometric interpretation, - for example, let A be a 2×3 matrix and let $T_A: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the associated transformation. T_A cannot be one-to-one, - $\mathbb{R}^3$ is too big to admit a one-to-one transformation into $\mathbb{R}^2$.

One-to-one and Onto (8)

Definition: A transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **onto** (*surjective*) if, for every vector b in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution x in $\mathbb{R}^n$.

<u>Equivalent definitions:</u>

- Range of T is equal to the codomain of T.
- Every vector in the codomain is the output of some input vector.

T is <u>not</u> onto if (equivalent statements below)

- The range of T is smaller than the codomain of T.
- There exists a vector b in $\mathbb{R}^m$ such that the equation $T(x) = b$ does not have a solution.
- There is a vector in the codomain that is not the output of any input vector.

One-to-one and Onto (9)

Examples (single variable functions):

- $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \sin x$ is not onto since, for example, $\sin x = 2$ has no solution. The range of f is the interval $[-1,1]$ which is smaller than the codomain $\mathbb{R}$.

- $g: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = e^x$ is not onto because the range of g is $(0, \infty)$ which is smaller than the codomain.

- $p: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $p(x) = x^3$ is onto since the equation $x^3 = b$ always has the solution $x = \sqrt[3]{b}$.

- $q: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $q(x) = x^3 - x$ is onto since $x^3 - x = b$ always has a solution, - it is a root of the polynomial $x^3 - x - b$ which is a cubic and must intersect the x-axis at least once guaranteeing at least one real root.

One-to-one and Onto (10)

Theorem: Let A be an $m \times n$ matrix, and let $T_A = Ax$ be the corresponding matrix transformation. The following statements are equivalent:

1. T is onto
2. $T(x) = b$ has at least one solution for every b in $\mathbb{R}^m$
3. $Ax = b$ is consistent for every b in $\mathbb{R}^m$
4. The columns of A span all of $\mathbb{R}^m$
5. A has a pivot in every row
6. The range of T has dimension m

Proof: (1),(2) and (3) are equivalent by definition; (3), (4) and (6) are equivalent since any b for which $Ax = b$ is consistent must be a linear combination of the column vectors of A and the dimension of $\mathrm{Col}(A) = \mathbb{R}^m$ is m. Here's why (5) $\Leftrightarrow$ (3):

If A has a pivot in every row then its RREF potentially looks like $\begin{bmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$ and when augmented with any b it will become $\left[\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{array}\right]$ which is consistent. Conversely, if there is a row of zeros, then $Ax = b$ will be inconsistent for some b.

One-to-one and Onto (11)

Example 5

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ onto?

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A is onto because there is a pivot in every row.

Example 6

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ onto? If not, find a vector b in $\mathbb{R}^3$ such that $T(x) = b$ has no solution.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Not onto.

Let $b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 4 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{array}\right]$$

has no solution.

One-to-one and Onto (12)

Example 7

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$. Is T_A defined by $T_A(x) = Ax$ onto? If not, find a basis for the range of T_A and then find a vector b in $\mathbb{R}^2$ such that $T(x) = b$ has no solution.

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} \overset{*}{1} & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ Not onto.

a basis

range of T_A = col(A) = span$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

Any vector not collinear to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ will do.

Say, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

One-to-one and Onto (13)

Some observations

- Let A be an $m \times n$ matrix. T_A is <u>not</u> onto $\Leftrightarrow$ $\text{Col}(A)$ is a subspace of $\mathbb{R}^m$ whose dimension is less than m, that is, $\text{rank}(A) < m$

- Transformations whose associated matrices are tall ($n < m$) are <u>not</u> onto. (Each column and each row can only contain one pivot, so in order to have a pivot in every row, A must have at least as many columns as rows, so we need $n \geq m$.) Geometric interpretation, - for example, let A be a 3×2 matrix and let $T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the associated transformation. T_A cannot be onto, - $\mathbb{R}^2$ is too small (not enough vectors to fill $\mathbb{R}^3$).

One-to-one and Onto (14)

Some observations

- Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a transformation associated with one of the following matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (T \text{ is a reflection})$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad (T \text{ is a dilation})$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (T \text{ is the identity})$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad (T \text{ is a rotation})$$

In all of the cases above T is both one-to-one and onto. A 2×2 matrix has a pivot in every column if and only if it has a pivot in every row. So, T_A is one-to-one if and only if it is onto.

True for any T_A where A is a square $n \times n$ matrix. Conversely, if $T_A: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is both one-to one and onto then $m = n$.