Complex Eigenvalues

Complex Numbers

Complex Numbers (1)

If we try to solve the equation $x^2 + 1 = 0$, we get the following

$$x^2 + 1 = 0 \longrightarrow x^2 = -1 \longrightarrow x = \pm\sqrt{-1}$$

But this is impossible in the real number system since the square of any real number is nonnegative. And, so, this equation has no real solutions.

To make it possible it solve <u>all</u> quadratic equations, we work with an expanded number system called the ***complex number system***.

First define the new number (the imaginary unit): $i = \sqrt{-1}$

A ***complex number*** is an expression of the form $a + bi$ or $a + ib$

where a and b are real numbers and $i^2 = -1$

When convenient we use a single letter , usually z, to denote a complex number and so we write $z = a + bi$ or $z = a + ib$

The ***real part*** of this complex number is a, and is denoted by $\mathrm{Re}(z)$, and the ***imaginary part*** of this complex number is b, denoted by $\mathrm{Im}(z)$. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Complex Numbers (2)

A complex number $z = bi$ whose real part is zero is said to be ***pure imaginary***. A complex number $z = a$ whose imaginary part is zero is a real number, so the real numbers can be viewed as a subset of the complex numbers.

Complex numbers are added, subtracted, and multiplied in accordance with the standard rules of algebra together with the fact that $i^2 = -1$

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = \underline{ac} + \underline{adi} + \underline{bci} - \underline{bd} = \underline{(ac - bd)} + \underline{(ad + bc)}i$$

The term ***complex number system*** refers to the set of complex numbers together with these operations, and is denoted by $\mathbb{C}$, or simply C.

Example 1

Evaluate the following products:

(a) $(3 + 2i)(1 + 4i)$ $= 3 + 12i + 2i + 8i^2 = 3 + 14i - 8 = -5 + 14i$

(b) $(1 + 3i)(2 - i)$ $= 2 - i + 6i - 3i^2 = 2 + 5i + 3 = 5 + 5i$

Complex Numbers (3)

If $z = a + bi$ is a complex number, then the ***complex conjugate*** of z, or simply, the ***conjugate*** of z, is denoted by $\bar{z}$ and is defined by $\bar{z} = a - bi$.

Numerically, $\bar{z}$ is obtained from z by reversing the sign of the imaginary part.

For example:

$z = 3 + 4i$ $\quad\quad \bar{z} = 3 - 4i$

$z = -2 - 10i$ $\quad\quad \bar{z} = -2 + 10i$

$z = i$ $\quad\quad \bar{z} = -i$

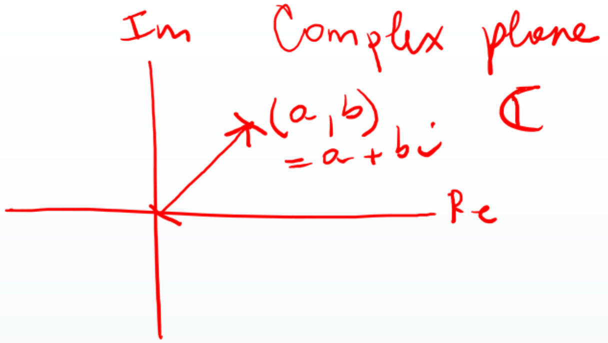
$z = 5$ $\quad\quad \bar{z} = 5$

Note from the last example that $z = \bar{z}$ if and only if z is a real number.

Let’s take a closer look at the product of a complex number and its conjugate

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

Complex Numbers (4)



$$z\bar{z} = a^2 + b^2 \quad \longrightarrow \quad \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

$\sqrt{a^2 + b^2}$ is the **size** of the complex number $z = a + bi$

We call this the ***modulus*** of z and denote it by $|z|$

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$$

Note that if $b = 0$, $z = a$ is a real number and $|z| = \sqrt{a^2} = |a|$ which is simply the absolute value of the real number $z = a$.

Example 2

Find the modulus of the given complex number

(a) $z = 3 - 4i$ $\quad |z| = \sqrt{(3-4i)(3+4i)} = \sqrt{3^2 + 4^2} =$
$= \sqrt{25} = 5$

(b) $z = -12 + 5i$
$|z| = \sqrt{(-12)^2 + 5^2} = \sqrt{169} = 13$

(c) $z = -3i$ $\quad |z| = \sqrt{9} = 3$

(d) $z = -10$ $\quad |z| = 10$

Complex Numbers (5)

If $z \neq 0$, the ***reciprocal*** of z, denoted by $\frac{1}{z}$, is defined by the equation $\quad z\left(\frac{1}{z}\right) = 1$

We can find an explicit formula for $\frac{1}{z}$ by multiplying both sides of the above equation by $\bar{z}$ and using the fact that $z\bar{z} = |z|^2$

$$\bar{z}z\left(\frac{1}{z}\right) = \bar{z} \longrightarrow |z|^2\left(\frac{1}{z}\right) = \bar{z} \longrightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

If $z_2 \neq 0$ then we define the ***quotient*** $\frac{z_1}{z_2}$ to be the product of z_1 and $\frac{1}{z_2}$

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) = z_1\left(\frac{\overline{z_2}}{|z_2|^2}\right) = \frac{z_1\overline{z_2}}{|z_2|^2}$$

For calculation purposes note that $\quad \frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}}$

Example 3

$$\frac{4+3i}{2-i}\cdot\frac{2+i}{2+i} = \frac{(4+3i)(2+i)}{2^2-i^2} = \frac{8+4i+6i+3i^2}{4+1} = \frac{8+10i-3}{5} = \frac{5+10i}{5} =$$

$$= 1+2i$$

Complex Numbers (6)

PROPERTIES OF CONJUGATES

(a) $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

(b) $\overline{z_1 - z_2} = \bar{z_1} - \bar{z_2}$

(c) $\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$

(d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{z_1\bar{z_2}}{z_2\bar{z_2}}\right)} = \overline{\left(\frac{z_1\bar{z_2}}{|z_2|^2}\right)} = \frac{\bar{z_1}\bar{\bar{z_2}}}{|z_2|^2} = \frac{\bar{z_1}z_2}{|z_2|^2} = \frac{\bar{z_1}z_2}{z_2\bar{z_2}} = \frac{\bar{z_1}}{\bar{z_2}}$$

(e) $\bar{\bar{z}} = z$

PROPERTIES OF THE MODULUS

$$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) =$$
$$z_1z_2\bar{z_1}\bar{z_2} = z_1\bar{z_1}z_2\bar{z_2} =$$
$$(z_1\bar{z_1})(z_2\bar{z_2}) = |z_1|^2|z_2|^2$$
$$|z_1z_2|^2 = |z_1|^2|z_2|^2$$

(a) $|\bar{z}| = |z|$

(b) $|z_1z_2| = |z_1||z_2|$

(c) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(d) $|z_1 + z_2| \leq |z_1| + |z_2|$

Complex Eigenvalues

Complex Eigenvalues (1)

Fundamental Theorem of Algebra

A polynomial of degree n has exactly n complex roots, counted with multiplicity.

if $(\lambda-\lambda_0)^n$ is a factor of $f(\lambda)$ then λ_0 has multiplicity n.

Every $n \times n$ matrix has exactly n complex eigenvalues, counted with multiplicity.

Example 1

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$f(\lambda) = \lambda^2 - Tr(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2$$

$a=1$
$b=-2$
$c=2$

$\sqrt{4}\cdot\sqrt{-1}$ 2

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

nonreal complex roots come in conjugate pairs!

$$\lambda_1 = 1 + i \text{ and } \lambda_2 = 1 - i$$

Complex Eigenvalues (2)

Example 1 (continued)

$$\lambda_1 = \underline{1+i} \implies A - \lambda_1 I = A - (1+i)I = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} = \begin{bmatrix} 1-1-i & -1 \\ 1 & 1-1-i \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \xrightarrow{\frac{R_1}{-i}} \begin{bmatrix} 1 & \frac{1}{i} \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & \frac{i}{i^2} \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & \frac{i}{-1} \\ 1 & -i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$x - iy = 0 \implies \begin{matrix} x = iy \\ y = y \end{matrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} i \\ 1 \end{bmatrix} \implies v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = \underline{1-i} \implies A - \lambda_2 I = A - (1-i)I = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1-i & 0 \\ 0 & 1-i \end{bmatrix} = \begin{bmatrix} 1-1+i & -1 \\ 1 & 1-1+i \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \xrightarrow{\frac{R_1}{i}} \begin{bmatrix} 1 & \frac{-1}{i} \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & \frac{-i}{i^2} \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & \frac{-i}{-1} \\ 1 & i \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

Note that $v_2 = \overline{v_1}$

$$x + iy = 0 \implies \begin{matrix} x = -iy \\ y = y \end{matrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -i \\ 1 \end{bmatrix} \implies v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Complex Eigenvalues (3)

Example 2

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 1 & 2 & 2 - \lambda \end{bmatrix} = (2-\lambda)\left(\left(\frac{4}{5} - \lambda\right)\left(\frac{4}{5} - \lambda\right) + \frac{9}{25}\right) =$$

$$= (2-\lambda)\left(\frac{16}{25} - \frac{8}{5}\lambda + \lambda^2 + \frac{9}{25}\right) = (2-\lambda)\left(\lambda^2 - \frac{8}{5}\lambda + 1\right)$$

$a = 1$

$b = -\frac{8}{5}$

$c = 1$

$$\lambda_1 = 2 \qquad \lambda = \frac{\frac{8}{5} \pm \sqrt{\frac{64}{25} - 4}}{2} = \frac{\frac{8}{5} \pm \sqrt{\frac{64}{25} - \frac{100}{25}}}{2} = \frac{\frac{8}{5} \pm \sqrt{-\frac{36}{25}}}{2} = \frac{\frac{8}{5} \pm \left(\frac{6}{5}\right)i}{2} = \frac{4}{5} \pm \frac{3}{5}i$$

$$\lambda_2 = \frac{4+3i}{5} \text{ and } \lambda_3 = \frac{4-3i}{5}$$

Complex Eigenvalues (4)

Example 2 (continued)

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$$\lambda_1 = 2 \quad \Longrightarrow \quad A - 2I = \begin{bmatrix} \frac{4}{5} - 2 & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - 2 & 0 \\ 1 & 2 & 2 - 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5} & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{-\frac{5}{3}R_1} \begin{bmatrix} 1 & 1 & 0 \\ \frac{3}{5} & -\frac{3}{5} & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{5}{3}R_2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{-2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x &= 0 \\ y &= 0 \\ z &= z \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Complex Eigenvalues (5)

Example 2 (continued)

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$$\lambda_2 = \frac{4+3i}{5} \quad \Longrightarrow \quad A - \left(\frac{4+3i}{5}\right) I = \begin{bmatrix} \frac{4}{5} - \frac{4+3i}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \frac{4+3i}{5} & 0 \\ 1 & 2 & 2 - \frac{4+3i}{5} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{-\frac{5}{3}R_1} \begin{bmatrix} i & 1 & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{\frac{5}{3}R_2} \begin{bmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{-iR_1} \begin{bmatrix} 1 & -i & 0 \\ 1 & -i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix}$$

Complex Eigenvalues (6)

Example 2 (continued)

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -i & 0 \\ 1 & -i & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \\ 1 & 2 & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 2+i & \frac{6-3i}{5} \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -i & 0 \\ 0 & 2+i & \frac{6-3i}{5} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\cancel{R_2 \leftrightarrow R_3}}$$

$$\frac{6-3i}{5} \cdot \frac{1}{2+i} = \frac{6-3i}{10+5i} \cdot \frac{(10-5i)}{(10-5i)} = \frac{60-60i+15i^2}{100+25}$$

$$\begin{bmatrix} 1 & -i & 0 \\ 0 & 2+i & \frac{6-3i}{5} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{R_2}{(2+i)}} \begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & \boxed{\frac{9-12i}{25}} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + iR_2} \begin{bmatrix} 1 & 0 & \boxed{\frac{12+9i}{25}} \\ 0 & 1 & \frac{9-12i}{25} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{45-60i}{125} = \qquad = \frac{9-12i}{25}$$

$$x = -\frac{12+9i}{25} z$$

$$y = -\frac{9-12i}{25} z$$

z is a free variable

Letting $z = 25 \quad \Longrightarrow \quad v_2 = \begin{bmatrix} -12-9i \\ -9+12i \\ 25 \end{bmatrix}$

Complex Eigenvalues (7)

Example 2 (continued)

Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$\lambda_3 = \overline{\lambda_2} \quad \Rightarrow \quad v_3 = \overline{v_2}$

$$\lambda_3 = \frac{4-3i}{5} \quad \Longrightarrow \quad v_3 = \begin{bmatrix} -12+9i \\ -9-12i \\ 25 \end{bmatrix}$$

Note that $v_3 = \overline{v_2}$

Complex Eigenvalues (8)

Eigenvalues and eigenvectors come in conjugate pairs

Let A be a matrix with real entries. If λ is a (nonreal) complex eigenvalue with eigenvector v then $\bar{\lambda}$ is a complex eigenvalue with eigenvector $\bar{v}$.

Trick for 2×2 matrices

Let A be a 2×2 matrix and let λ be an eigenvalue of A then

$$A - \lambda I = \begin{bmatrix} z & w \\ * & * \end{bmatrix} \quad \Longrightarrow \quad v = \begin{bmatrix} -w \\ z \end{bmatrix} \text{ is an eigenvector corresponding to } \lambda$$

λ is an eigenvalue $\Longrightarrow$ $A - \lambda I$ is not invertible $\Longrightarrow$ R_2 is a multiple of R_1

$$\begin{bmatrix} z & w \\ * & * \end{bmatrix} = \begin{bmatrix} z & w \\ cz & cw \end{bmatrix}$$

(c is a complex scalar)

$$\begin{bmatrix} z & w \\ cz & cw \end{bmatrix} \xrightarrow{R_2 - cR_1} \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} -w \\ z \end{bmatrix} \text{ is an eigenvector since } \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark$$