Upcoming Assignments and Assessments

- Homework Assignment #1 is due Sunday 2/2*
- Quiz #1 will be administered during this week's recitation

*All future HW Assignments will be due on Fridays

Row Reduction and Free Variables Part II

Row Reduction: Pivot Position and Pivot Columns (10)

Definition: A **pivot position** of a matrix is an entry that is a pivot of a row echelon form of that matrix (an entry that eventually becomes a leading 1). A **pivot column** of a matrix is a column that contains a pivot position.

Example 1

Find the pivot positions and pivot columns of the given matrix.

$$\begin{bmatrix} 3 & 2 & 1 & 9 \\ 5 & -1 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & -1 & -1 & 2 \\ 3 & 2 & 1 & 9 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 5R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -6 & -6 & -3 \\ 0 & -1 & -2 & 6 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & -1 & -2 & 6 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{13}{2} \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{13}{2} \end{bmatrix}$$

pivot positions are circled

pivot columns: 1, 2, 3.

$$\xrightarrow[R_2 - R_3]{R_1 - R_3} \begin{bmatrix} 1 & 1 & 0 & \frac{15}{2} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{13}{2} \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -\frac{13}{2} \end{bmatrix}$$

$x_1 = \frac{1}{2}$

$x_2 = 7$

$x_3 = -\frac{13}{2}$

$\left(\frac{1}{2}, 7, -\frac{13}{2}\right)$

all columns except the last one are pivot columns $\implies$ unique solution

Row Reduction: Pivot Position and Pivot Columns (11)

Definition: A **pivot position** of a matrix is an entry that is a pivot of a row echelon form of that matrix (an entry that eventually becomes a leading 1). A **pivot column** of a matrix is a column that contains a pivot position.

Example 2

Solve the linear system: $\begin{aligned} 2x + 10y &= -1 \\ 3x + 15y &= 2 \end{aligned}$ Make a note of the pivot column(s)

$$\begin{bmatrix} 2 & 10 & -1 \\ 3 & 15 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 5 & -\frac{1}{2} \\ 3 & 15 & 2 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{bmatrix} \xrightarrow{\frac{2}{7}R_2}$$

$$\begin{bmatrix} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Pivot colums:
1 and $\underline{\underline{3}}$

$0x + 0y = 1$ No solution

the system is inconsistent.

If the augmented (last) column is pivot column
$\Rightarrow$ No solutions

Row Reduction: Pivot Position and Pivot Columns (12)

Row echelon form of an inconsistent system. An augmented matrix corresponds to an inconsistent system of equations if and only the last column (the augmented column) is a pivot column.

The two cases discussed thus far

When the reduced row echelon form of a matrix has a pivot in every nonaugmented column, then it corresponds to a system with a unique solution

When the reduced row echelon form of a matrix has a pivot in the last column (augmented column), then it corresponds to a system with no solutions

Example 3

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{aligned} x \qquad\quad &= 5 \\ y \quad\; &= -4 \\ z &= 3 \end{aligned}$$

Example 4

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The last line translates to $0 = 4$ which indicates that there are no solutions to this system

Row Reduction: Free Variables (13)

Consider the following reduced row echelon form of a given matrix:

Example: $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{rcl} x_1 \quad + 3x_3 &=& 4 \\ x_2 + 2x_3 &=& -1 \end{array}$

The equation that corresponds to the last row, $0x_1 + 0x_2 + 0x_3 = 0$, can be omitted since it places no restrictions on the unknowns

Since x_1 and x_2 correspond to the leading 1's in the augmented matrix, we call these the ***leading variables***. The remaining variables (in this case x_3) are called ***free variables***. Solving for the leading variables in terms of the free variables gives

$$x_1 = 4 - 3x_3$$
$$x_2 = -1 - 2x_3$$

Treating x_3 as a parameter and assigning to it any arbitrary value t determines the values of the other two variables. So, the solution set is given by the **parametric** equations (in **parametric form**)

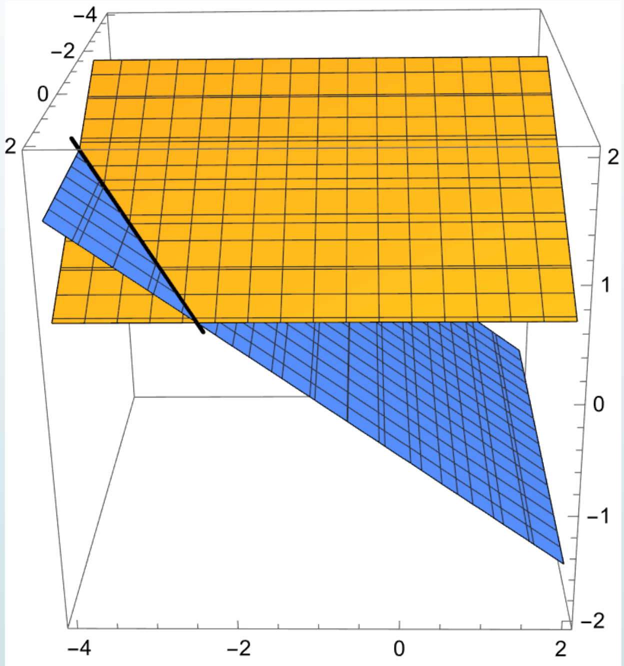
$$x_1 = 4 - 3t, \quad x_2 = -1 - 2t, \; x_3 = t$$

Row Reduction: Free Variables (14)

$$x_1 = 4 - 3x_3$$
$$x_2 = -1 - 2x_3$$

$\longrightarrow$

$$x_1 = 4 - 3t, \quad x_2 = -1 - 2t, \; x_3 = t$$



To obtain a specific solution to the system above, simply replace the free variable with a real number. Or equivalently choose that real number to be the value of the parameter t. So, for example, setting $t = 0$ gives the specific solution $(4, -1, 0)$

Row Reduction: Free Variables and Parametric Form (15)

The **parametric form** of the solution set of a consistent system of linear equations is obtained as follows:

- Write the system as an augmented matrix
- Row reduce to reduced row echelon form
- Write the corresponding (now solved) system of linear equations
- Move all free variables to the right side of the equations

*Free variables may be relabeled if you prefer, to emphasize that they are parameters

Free variables = parameters = *independent* variables
Non-free (leading) variables = *dependent* variables

Row Reduction: Free Variables and Parametric Form (16)

Example 3

Solve the system using Gauss-Jordan reduction

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + x_4 &= 32 \\ 2x_1 - 2x_2 \qquad\quad + 4x_4 &= 16 \\ 3x_1 + x_2 + 2x_3 + 5x_4 &= 38 \\ x_1 - x_2 \qquad\quad + 2x_4 &= 8 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 3 & 2 & 1 & 32 \\ 2 & -2 & 0 & 4 & 16 \\ 3 & 1 & 2 & 5 & 48 \\ 1 & -1 & 0 & 2 & 8 \end{bmatrix}$$

Add -2 times row 1 to row 2;
add -3 times row 1 to row 3;
add -1 times row 1 to row 4

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 32 \\ 0 & -8 & -4 & 2 & -48 \\ 0 & -8 & -4 & 2 & -48 \\ 0 & -4 & -2 & 1 & -24 \end{bmatrix}$$

Multiply row 2 by $-\frac{1}{8}$

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 32 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 6 \\ 0 & -8 & -4 & 2 & -48 \\ 0 & -4 & -2 & 1 & -24 \end{bmatrix}$$

Add 8 times row 2 to row 3;
add 4 times row 2 to row 4

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 32 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Reduction: Free Variables and Parametric Form (17)

Example 3 (continued)

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 32 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -3 times row 2 to row 1

pivot columns

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{7}{4} & 14 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + \quad \frac{1}{2}x_3 + \frac{7}{4}x_4 &= 14 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{4}x_4 &= 6 \end{aligned}$$

The leading variables are x_1 and x_2. The free variables are x_3 and x_4.

$$\begin{aligned} x_1 &= 14 - \frac{1}{2}x_3 - \frac{7}{4}x_4 \\ x_2 &= 6 - \frac{1}{2}x_3 + \frac{1}{4}x_4 \end{aligned}$$

$$\left(14 - \frac{1}{2}x_3 - \frac{7}{4}x_4,\ 6 - \frac{1}{2}x_3 + \frac{1}{4}x_4,\ x_3,\ x_4\right)$$

Letting $x_3 = r$ and $x_4 = s$

$$\begin{aligned} x_1 &= 14 - \frac{1}{2}r - \frac{7}{4}s \\ x_2 &= 6 - \frac{1}{2}r + \frac{1}{4}s \\ x_3 &= r \\ x_4 &= s \end{aligned}$$

Implicit Equations versus the Parametric Form (18)

Solve the following system using row reduction (the solution set defines a line in $\mathbb{R}^3$, - the line of intersection of the planes defined by the initial two equations)

$$\left.\begin{aligned} 2x + 4y - z &= 2 \\ x - 2y + z &= 1 \end{aligned}\right\} \text{Implicit Equations of the line}$$

$$\begin{bmatrix} 2 & 4 & -1 & 2 \\ 1 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 2 & 4 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 8 & -3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{8}R_2}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -\frac{3}{8} & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 1 \\ 0 & 1 & -\frac{3}{8} & 0 \end{bmatrix} \quad \begin{aligned} x + \tfrac{1}{4}z &= 1 \\ y - \tfrac{3}{8}z &= 0 \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= 1 - \tfrac{1}{4}z \\ y &= \tfrac{3}{8}z \end{aligned}}$$

The initial two equations are called **implicit equations** of the resulting line that is the system's solution set

The parametrized solution set is called the **parametric form** of the resulting line

Implicit Equations versus the Parametric Form (19)

Another example (a parametrized plane).

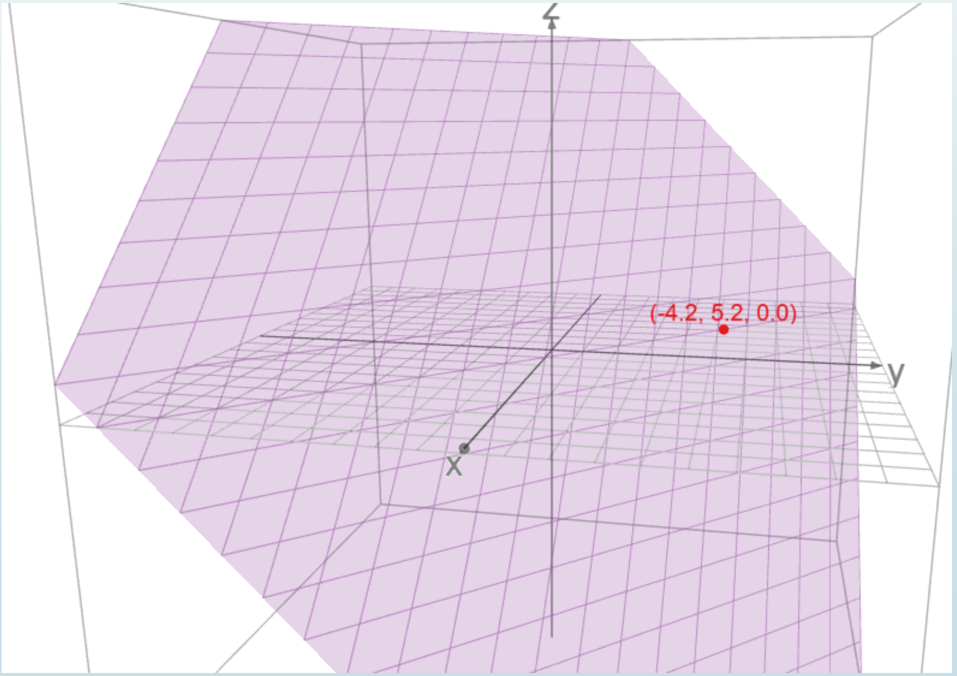
The single equation below defines a plane in $\mathbb{R}^3$ implicitly.

$$x + y + z = 1$$

The free variables are y and z, so the parametric form for the general solution is

$$(x, y, z) = (1 - y - z, y, z)$$

This is referred to as the parametric form of the plane. Note that it has two parameters. So, to find a specific solution we simply need to choose a value for y and a value for z and then calculate the corresponding x-value. That is, any two values of y and z define a point in $\mathbb{R}^3$ and the set of these points make up the plane.



Number of Solutions (20)

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. *The last column is a pivot column.* In this case, the system is *inconsistent*. There are zero solutions, i.e., the solution set is empty. For example, the matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

   comes from a linear system with no solutions.

2. *Every column except the last column is a pivot column.* In this case, the system has a *unique* solution. For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right)$$

   tells us that the unique solution is $(x, y, z) = (a, b, c)$.

3. *The last column is not a pivot column, and some other column is not a pivot column either.* In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). For example, in the system corresponding to the matrix

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right),$$

   any values for x_2 and x_4 yield a solution to the system of equations.

Note that free variables come from the columns without pivots in a matrix in row echelon form.