Linear Transformations Part I

Linear Transformations (1)

Definition: A **linear transformation** is a transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfying the following two properties:

- $T(u+v) = T(u) + T(v)$

- $T(cu) = cT(u)$, for all vectors u, v in $\mathbb{R}^n$ and all scalars c.

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a matrix transformation defined by $T(x) = Ax$ then

$$T(u+v) = A(u+v) = Au + Av = T(u) + T(v) \checkmark$$
$$T(cu) = A(cu) = cAu = cT(u) \checkmark$$

<u>So, a matrix transformation is a linear transformation</u>.

Properties of linear transformations. Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

- $T(0) = 0$ (application of the second defining property with $c = 0$)

- For any vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ and any scalars $c_1, c_2, \ldots, c_k$ we have

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_k T(v_k)$$

(application of the first and second defining properties)

Linear Transformations (2)

Example 1

(a) Let $T: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $T(x) = x - 1$. Is T a linear transformation? No

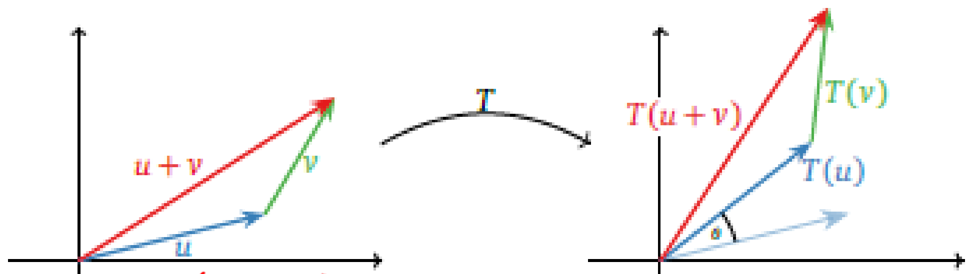
$T(0) = -1 \neq 0$

(b) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T(x) = 2x$. Is T a linear transformation? ✓

$T(u+v) = 2(u+v) = 2u + 2v = T(u) + T(v)$ ✓

$T(cu) = 2(cu) = c(2u) = cT(u)$ ✓

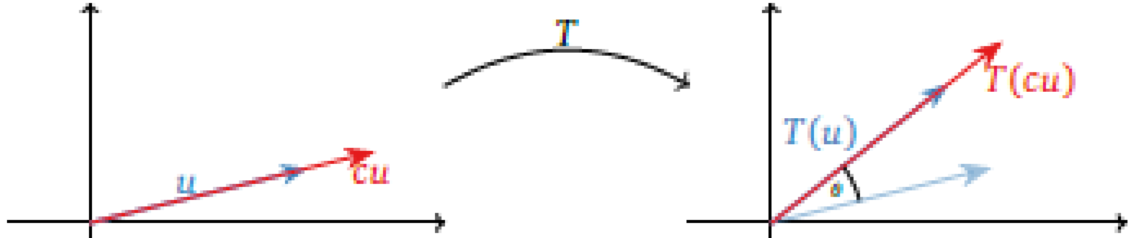
(c) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T(x) =$ the vector x rotated counterclockwise by the angle θ. Is T a linear transformation? Yes.



$T(u+v) \longrightarrow$ add and then rotate

$T(u) + T(v) \longrightarrow$ rotate and then add

$\Big\}$ same result ✓



$T(cu) \longrightarrow$ expand and then rotate

$c \cdot T(u) \longrightarrow$ rotate and then expand

same result. ✓

Linear Transformations (3)

Example 2

Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - y \\ y \\ x \end{bmatrix}$. Is T a linear transformation? Yes

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} 3(x_1 + x_2) - (y_1 + y_2) \\ y_1 + y_2 \\ x_1 + x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} (3x_1 - y_1) + (3x_2 - y_2) \\ y_1 + y_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - y_1 \\ y_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 - y_2 \\ y_2 \\ x_2 \end{bmatrix} =$$

$$= T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \checkmark$$

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} 3cx - cy \\ cy \\ cx \end{bmatrix} = c\begin{bmatrix} 3x - y \\ y \\ x \end{bmatrix} = cT\begin{bmatrix} x \\ y \end{bmatrix} \checkmark$$

Linear Transformations (4)

Example 3

(a) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ y \end{bmatrix}$. Show that T is not linear.

$$T\left(-1\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = T\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$-T\begin{bmatrix} 2 \\ 0 \end{bmatrix} = -\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Not linear.

(b) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ y \end{bmatrix}$. Show that T is not linear.

$$T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\begin{bmatrix} cx \\ cy \end{bmatrix} = \begin{bmatrix} c^2xy \\ cy \end{bmatrix} \neq cT\begin{bmatrix} x \\ y \end{bmatrix}$$

(c) Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + 2 \\ x - 2y \end{bmatrix}$. Show that T is not linear.

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$