Orthogonal Decomposition, Orthogonal Projections and the Method of Least Squares

Orthogonal Decomposition (12)

Back to the Theorem (The $A^T A$ Trick): Let W be a subspace of $\mathbb{R}^n$. Let $v_1, v_2, \ldots, v_m$ be a spanning set for W, and let $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}$. Then for any x in $\mathbb{R}^n$, the matrix equation

$$A^T A v = A^T x \text{ (the unknown vector is } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Proof: Let $x = x_W + x_{W^\perp}$ be the orthogonal decomposition with respect to W. By definition x_W lies in $W = \mathrm{Col}(A)$ and so, there is a vector v in $\mathbb{R}^n$ with $Av = x_W$. Choose any such vector v. We know that $x - x_W = x - Av$ lies in $W^\perp$, which is equal to $\mathrm{Nul}(A^T)$. So, we have $0 = A^T(x - Av) = A^T x - A^T A v$ which implies $A^T A v = A^T x$. This shows that the equation $A^T A v = A^T x$ is consistent. Now, let v be any solution. Reversing the above logic, we see that $x_W = Av$.

Homework #10 is due on Friday 5/2

Quiz #10 will be administered on Friday 5/2

Projects are due Friday 5/9

Orthogonal Projections

Final Exam – 5/12 4:00–5:50 – this room

Orthogonal Projections (13)

Orthogonal projections as transformations

Properties of Orthogonal Projections. Let W be a subspace of $\mathbb{R}^n$ and define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(x) = x_W$. Then:

- T is a linear transformation.
- $T(x) = x$ if and only if x is in W.
- $T(x) = 0$ if and only if x is in $W^\perp$.
- $T \circ T = T$
- The range of T is W.

Proof of the fourth statement:
For any x in $\mathbb{R}^n$ the vector $T(x)$ is in W, so $T \circ T(x) = T\big(T(x)\big) = T(x)$ by the second statement

Orthogonal Projections (14)

Standard Matrices for Projections

Example 6

Let L be the line in $\mathbb{R}^2$ spanned by the vector $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and define $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $T(x) = x_L$.

Compute the standard matrix B for T.

As before, apply T to the standard coordinate vectors to get the columns of B.

$$T(e_1) = (e_1)_L = \frac{u \cdot e_1}{u \cdot u} u = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = (e_2)_L = \frac{u \cdot e_2}{u \cdot u} u = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

Orthogonal Projections (15)

Standard Matrices for Projections

Example 7

Let L be the line in $\mathbb{R}^3$ spanned by the vector $u = \begin{bmatrix}1\\2\\1\end{bmatrix}$, and define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = x_L$.

Compute the standard matrix B for T.

As before, apply T to the standard coordinate vectors to get the columns of B.

$$T(e_1) = (e_1)_L = \frac{u \cdot e_1}{u \cdot u} u = \frac{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix}}{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}} \begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$T(e_3) = (e_3)_L = \frac{u \cdot e_3}{u \cdot u} u = \frac{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}} \begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$T(e_2) = (e_2)_L = \frac{u \cdot e_2}{u \cdot u} u = \frac{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}} \begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{2}{6}\begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$B = \frac{1}{6}\begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}$$

Orthogonal Projections (16)

Recall the corollary below which applies to the case where a basis of W is known.

Corollary: Let A be an $m \times n$ matrix with linearly independent columns and let $W = \mathrm{Col}(A)$. Then the $n \times n$ matrix A^TA is invertible, and for all vectors x in $\mathbb{R}^m$, we have

$$x_W = \underbrace{A(A^TA)^{-1}A^T}_{B}x$$

In this case, we can obtain the standard matrix be computing $A(A^TA)^{-1}A^T$

Example 8

Let $W = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, and define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = x_W$. Compute the standard matrix B for T.

$$A = \begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix} \qquad A^TA = \begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$$

$$B = \frac{1}{3}\overset{A}{\begin{bmatrix}1 & 1\\0 & 1\\-1 & 0\end{bmatrix}}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\overset{A^T}{\begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\end{bmatrix}} = \frac{1}{3}\begin{bmatrix}1 & 1\\-1 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\1 & 1 & 0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 1 & -1\\1 & 2 & 1\\-1 & 1 & 2\end{bmatrix}$$

Orthogonal Projections (17)

Properties of Projection Matrices. Let W be a subspace of $\mathbb{R}^n$ and define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(x) = x_W$ and let B be the standard matrix for T. Then:

- $\mathrm{Col}(B) = W$
- $\mathrm{Nul}(B) = W^{\perp}$.
- $B^2 = B$
- If $W \neq \{0\}$, then 1 is an eigenvalue of B and the 1-eigenspace for B is W
- If $W \neq \mathbb{R}^n$, then 0 is an eigenvalue of B and the 0-eigenspace for B is $W^{\perp}$
- B is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal, where $m = \dim(W)$ (Projection matrices are Diagonalizable)

Example 9

$W^{\perp} = \mathrm{Nul}(A^T)$

Let $W = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$, and define $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x) = x_W$. Diagonalize the standard matrix B for T.

Compute $W^{\perp} = \mathrm{Nul}\begin{bmatrix}1 & 0 & 2\\1 & -1 & 0\end{bmatrix}$ $\quad \begin{bmatrix}1 & 0 & 2\\1 & -1 & 0\end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix}1 & 0 & 2\\0 & -1 & -2\end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\end{bmatrix}$

Orthogonal Projections (17)

Properties of Projection Matrices. Let W be a subspace of $\mathbb{R}^n$ and define $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(x) = x_W$ and let B be the standard matrix for T. Then:

- If $W \neq \{0\}$, then 1 is an eigenvalue of B and the 1-eigenspace for B is W
- If $W \neq \mathbb{R}^n$, then 0 is an eigenvalue of B and the 0-eigenspace for B is $W^\perp$
- B is similar to the diagonal matrix with m ones and $n-m$ zeros on the diagonal, where $m = \dim(W)$

Example 9 (continued)

Compute $W^\perp = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad W^\perp = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$B = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}^{-1}$$

The Method of Least Squares

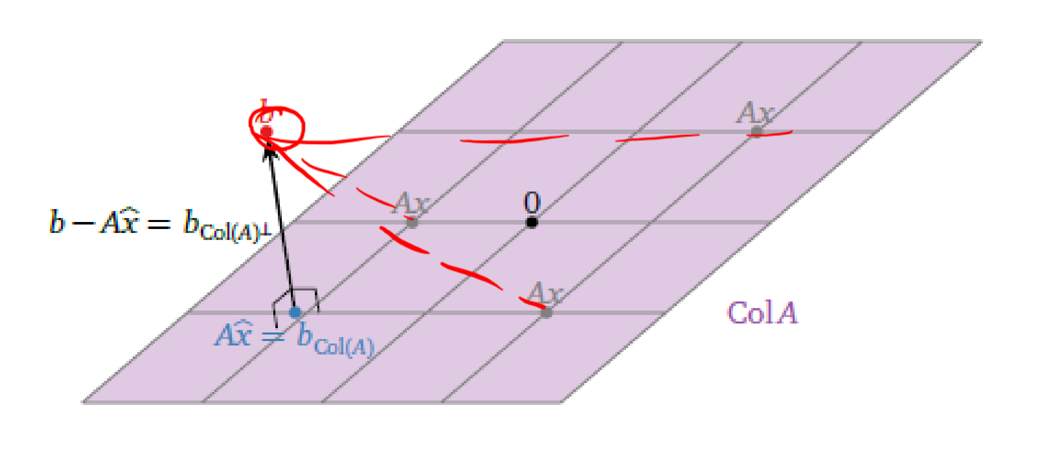
The Method of Least Squares (1)

Definition: Let A be an $m \times n$ matrix and let b be a vector in $\mathbb{R}^m$. A least-squares solution of the matrix equation $Ax = b$ is a vector $\hat{x}$ in $\mathbb{R}^n$such that

$$\text{dist}(b, A\hat{x}) \leq \text{dist}(b, Ax) \quad \text{for any } x \text{ in } \mathbb{R}^n$$

distance

Recall that $\text{dist}(b, Ax) = ||b - Ax||$ is the square root of the sum of the squares of the entries of the vector $b - Ax$. A least-squares solution minimizes the sum of the squares of the differences between the entries of Ax and b.



Here, $Ax = b$ is inconsistent, that is, b is not in $\text{Col}(A)$. $\text{Col}(A)$ is the set of all vectors of the form Ax. So, the closest vector of the form Ax to b is the orthogonal projection of b onto $\text{Col}(A)$, denoted by

$$b_{Col(A)}$$

And $\hat{x}$ is the vector such that $A\hat{x} = b_{Col(A)}$.

A least-squares solution of $Ax = b$ is a solution $\hat{x}$ of the consistent equation $Ax = b_{\text{Col}(A)}$

The Method of Least Squares (2)

To solve the resulting orthogonal projection problem, we use the following.

Theorem: Let A be an $m \times n$ matrix and let b be a vector in $\mathbb{R}^m$. The least-squares solutions of $Ax = b$ are the solutions of the matrix equation

$$\underline{A^T A}x = \underline{A^T b}$$

Recipe: Compute a least-squares solution. Let A be an $m \times n$ matrix and let b be a vector in $\mathbf{R}^n$. Here is a method for computing a least-squares solution of $Ax = b$:

1. Compute the matrix $A^T A$ and the vector $A^T b$.
2. Form the augmented matrix for the matrix equation $A^T Ax = A^T b$, and row reduce.
3. This equation is always consistent, and any solution $\widehat{x}$ is a least-squares solution.

The Method of Least Squares (3)

To solve the resulting orthogonal projection problem, we use the following.

Theorem: Let A be an $m \times n$ matrix and let b be a vector in $\mathbb{R}^m$. The least-squares solutions of $Ax = b$ are the solutions of the matrix equation

$$A^T Ax = A^T b$$

Example 1

Find the least-squares solution of $Ax = b$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. What quantity is being minimized? And what is its value?

$$A^T A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 3 & 0 \\ 3 & 3 & 6 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|c} 5 & 3 & 0 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & 3 & 0 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -10 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 5 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array}\right] \implies \hat{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

The Method of Least Squares (4)

Example 1 (continued)

Find the least-squares solution of $Ax = b$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. What quantity is being minimized? And what is its value?

$$\hat{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

This solution minimizes the distance from Ax to b. That is, $||b - A\hat{x}||$ is this shortest distance. Now, $A\hat{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. So, $\left\| \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\| = \sqrt{1 + 4 + 1} = \sqrt{6}$

The Method of Least Squares (4)

Theorem: Let A be an $m \times n$ matrix and b be a vector in $\mathbb{R}^m$. The following are equivalent:

- $Ax = b$ has a unique least-squares solution
- The columns of A are linearly independent
- $A^T A$ is invertible

In this case, the least-squares solution is $\hat{x} = (A^T A)^{-1} A^T b.$

Example 2

Find the least-squares solution of $Ax = b$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$.

$$A^T A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \qquad (A^T A)^{-1} = \frac{1}{14} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{4} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} =$$

The Method of Least Squares (4)

Theorem: Let A be an $m \times n$ matrix and b be a vector in $\mathbb{R}^m$. The following are equivalent:

- $Ax = b$ has a unique least-squares solution
- The columns of A are linearly independent
- $A^T A$ is invertible

In this case, the least-squares solution is $\hat{x} = (A^T A)^{-1} A^T b$.

Example 2 (continued)

Find the least-squares solution of $Ax = b$ where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix}$.

$$\hat{x} = (A^T A)^{-1} A^T b = \frac{1}{14}\begin{bmatrix} 8 & -4 & -2 \\ 5 & 1 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 70 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{1}{2} \end{bmatrix}$$

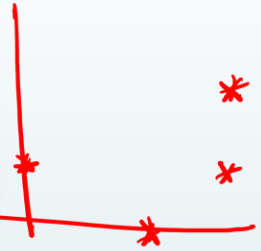
The Method of Least Squares (1)

Example 3

Find the least squares straight line fit to the four points (0,1), (2,0), (3,1) and (3,2).

x y

$$y = Mx + B \qquad \begin{aligned} 1 &= M(0) + 1 \cdot B \\ 0 &= M(2) + 1 \cdot B \\ 1 &= M(3) + 1 \cdot B \\ 2 &= M(3) + 1 \cdot B \end{aligned} \qquad A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$



$$x = \begin{bmatrix} M \\ B \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 8 \\ 8 & 4 \end{bmatrix} \qquad (A^T A)^{-1} = \frac{1}{24} \begin{bmatrix} 4 & -8 \\ -8 & 22 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 & -4 \\ -4 & 11 \end{bmatrix}$$

$$\hat{x} = \underbrace{\frac{1}{12} \begin{bmatrix} 2 & -4 \\ -4 & 11 \end{bmatrix}}_{(A^T A)^{-1}} \underbrace{\begin{bmatrix} 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}}_{b} = \frac{1}{12} \begin{bmatrix} -4 & 0 & 2 & 2 \\ 11 & 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \end{bmatrix} \begin{matrix} = M \\ = B \end{matrix}$$

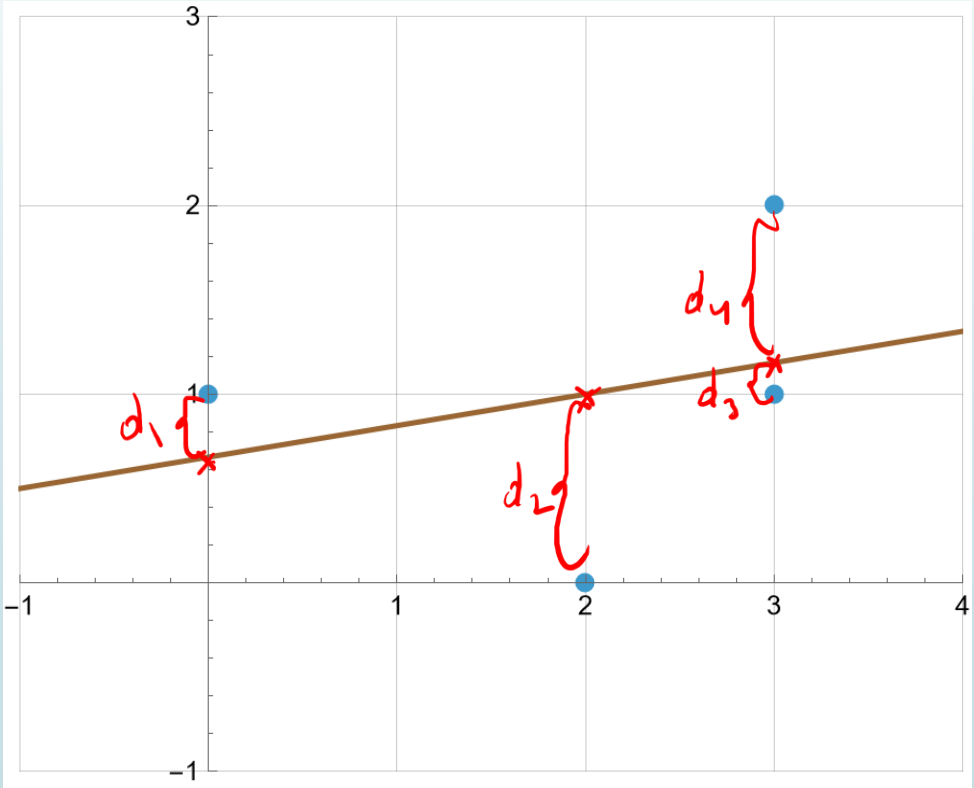
$$y = \frac{1}{6}x + \frac{2}{3}$$

The Method of Least Squares (1)

Example 3

Find the least squares straight line fit to the four points (0,1), (2,0), (3,1) and (3,2).

$$y = \frac{1}{6}x + \frac{2}{3}$$



Note that $A\hat{x}$ is the vector whose entries are the y-coordinates of the graph of the line at the x-values specified by the given data points, and b is the vector whose entries are the y-coordinates of the given data points. The difference $b - A\hat{x}$ is the vector of vertical distances of the graph from the data points.