Upcoming Assignments and Assessments

- Homework Assignment #1 is due Sunday 2/2*
- Quiz #1 will be administered during this week's recitation

***All future HW Assignments will be due on Fridays**

Introduction to Systems of Linear Equations

Introduction to Systems of Linear Equations (1)

Definitions/Notation:

$\mathbb{R}$ = the real number line = $(-\infty, \infty)$

$\mathbb{R}^2$= the x-y plane = the set of all points with two coordinates (x, y)

$\mathbb{R}^3$= 3-space = the set of all points with three coordinates (x, y, z)

$\mathbb{R}^n$= n-space = the set of all points with n coordinates

A ***linear equation*** in n variables $x_1, x_2, \dots, x_n$

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n$ and b are constants and the a's are not all zero.

In the special cases where $n = 2$ or $n = 3$, it is common to use variables without subscripts and write linear equations as

$$a_1x + a_2y = b, \qquad (a_1, a_2 \text{ are not both } 0)$$

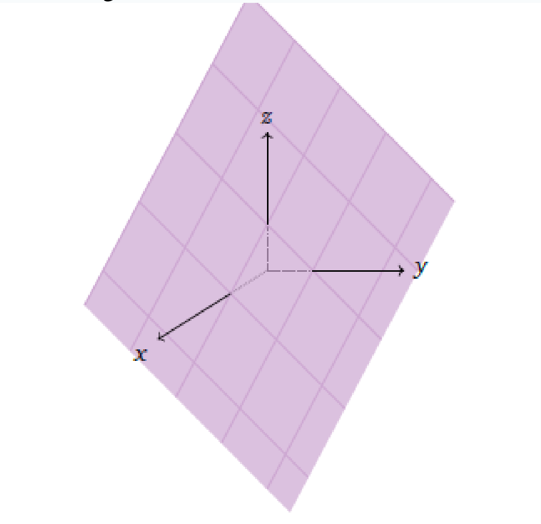
$$a_1x + a_2y + a_3z = b, \qquad (a_1, a_2, a_3 \text{ are not all } 0)$$

In the special cases where $b = 0$, the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ is called a ***homogeneous linear equation*** in variables $x_1, x_2, \dots, x_n$

Introduction to Systems of Linear Equations (2)

One linear equation in two variable. For example, let $x + y = 1$; rewriting this as $y = 1 - x$,we can see that this represents a line with slope -1 and y-intercept at $(0,1)$. For our purposes, we will define a line as a *straight* ray that is *infinite* in both directions.

One linear equation in three variables. For example, $x + y + z = 1$. This equation is an implicit equation that represents a plane in space.



For our purposes, a plane is a flat sheet that is infinite in all directions.

One linear equation in four variables. For example, $x + y + z + w = 1$. This equation defines a 3-plane in 4-space.

In general, a single linear equation in n variables defines an "$(n-1) - plane$" in n-space.

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns.***

Examples:

$$\begin{aligned} 6x + 2y &= 2 \\ 5x - 3y &= 1 \end{aligned} \qquad \begin{aligned} 2x - 3y + 2z &= 1 \\ 2x + 2y - 3z &= 2 \end{aligned}$$

Introduction to Systems of Linear Equations (3)

A general linear system of m equations in the n unknowns $x_1, x_2, \dots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad\;\; \vdots \quad\; &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

a_{ij} is in the i^{th} equation and multiplies x_j

A ***solution*** of a linear system in n unknowns $x_1, x_2, \dots, x_n$ is a sequence of n numbers $s_1, s_2, \dots, s_n$ for which the substitution $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ makes each equation in the system a true statement.

For example, the system $\begin{aligned} 6x + 2y &= 2 \\ 5x - 3y &= 1 \end{aligned}$ has the solution $x = \frac{2}{7}$, $y = \frac{1}{7}$; can be written as $\left(\frac{2}{7}, \frac{1}{7}\right)$

$6\left(\frac{2}{7}\right) + 2\left(\frac{1}{7}\right) \stackrel{?}{=} 2$ ✓

$5\left(\frac{2}{7}\right) - 3\left(\frac{1}{7}\right) \stackrel{?}{=} 1$ ✓

the system $\begin{aligned} 2x - 3y + 2z &= 1 \\ 2x + 2y - 3z &= 2 \end{aligned}$ has the solution $x = 0$, $y = -\frac{7}{5}$, $z = -\frac{8}{5}$; can be written as $\left(0, -\frac{7}{5}, -\frac{8}{5}\right)$

Introduction to Systems of Linear Equations: Linear Systems with Two Unknowns (4)

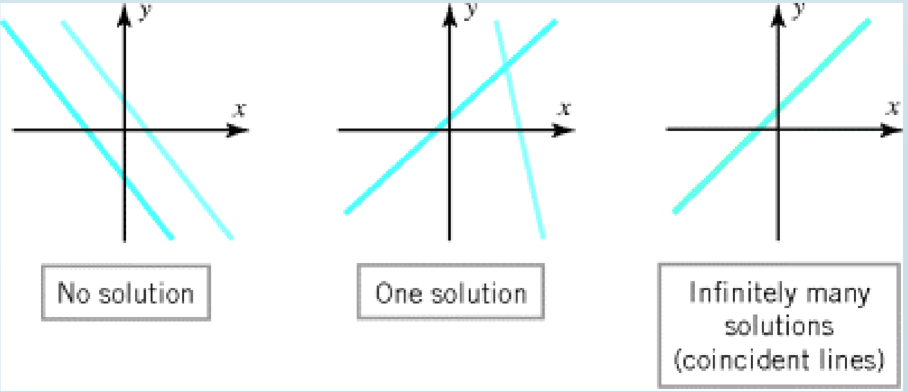
The graphs of the equations in the following system are lines in the xy-plane.

$$a_1x + b_1y = c_1 \quad \text{– line 1}$$
$$a_2x + b_2y = c_2 \quad \text{– line 2}$$

So, each solution (x, y) corresponds to a point of intersection of the two lines and there are three possibilities.

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities.



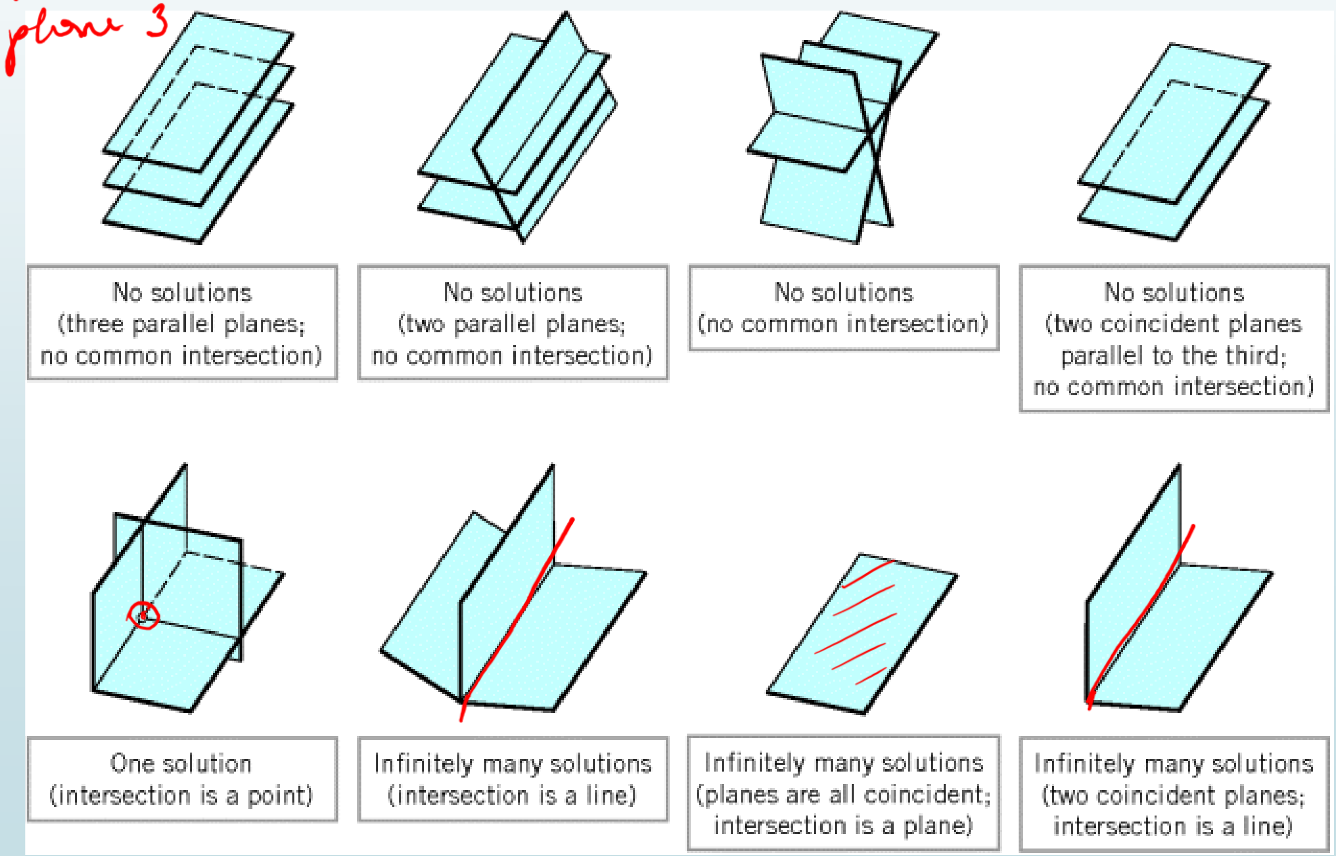
Introduction to Systems of Linear Equations: Linear Systems with Three Unknowns (5)

The graphs of the equations in the following system are planes.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \quad \text{- plane 1} \\ a_2x + b_2y + c_2z &= d_2 \quad \text{- plane 2} \\ a_3x + b_3y + c_3z &= d_3 \quad \text{- plane 3} \end{aligned}$$

The solutions of the system, if any, correspond to points where all three planes intersect, and again there are only three possibilities — no solutions, one solution, or infinitely many solutions

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.



Introduction to Systems of Linear Equations: Three Cases (6)

Example 1

Solve the system $\begin{aligned} x + 3y &= 3 \\ 6x + 2y &= 2 \end{aligned}$

Solution

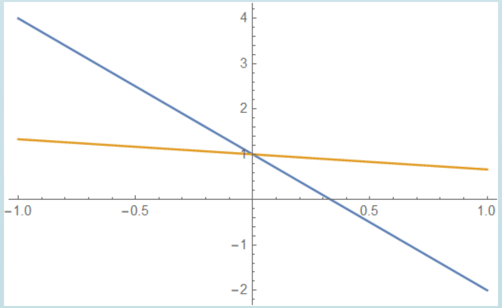
Multiply the first equation by -6 and then add to the second equation to eliminate x

$$\begin{aligned} x + 3y &= 3 \\ 6x + 2y &= 2 \end{aligned} \longrightarrow \begin{aligned} -6x - 18y &= -18 \\ 6x + 2y &= 2 \end{aligned} \longrightarrow -16y = -16 \longrightarrow y = 1$$

Now substitute $y = 1$ into the first equation, obtaining $x + 3(1) = 3 \longrightarrow x = 0$

The system has the unique solution: $x = 0, y = 1$

The two lines intersect at the point (0,1).



Introduction to Systems of Linear Equations: Three Cases (7)

Example 2

Solve the system
$$\begin{aligned} x + 2y &= 5 \\ 2x + 4y &= 3 \end{aligned}$$

Solution

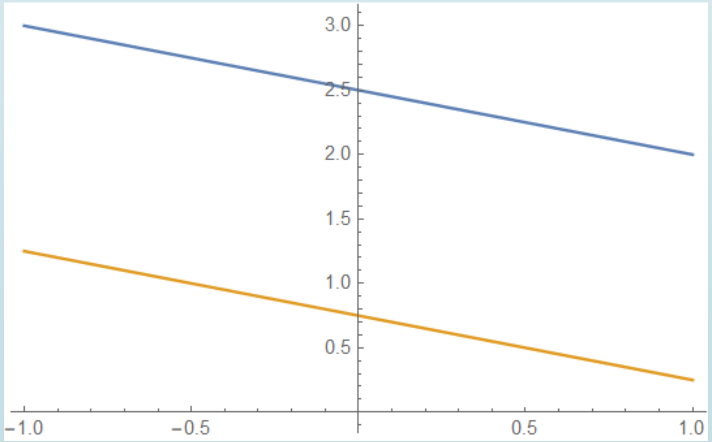
Add -2 times the first equation to the second equation to eliminate x

$$\begin{aligned} x + 2y &= 5 \\ 2x + 4y &= 3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + 2y &= 5 \\ 0 &= -7 \end{aligned}$$

$-2x - 4y = -10$ add

The second equation is contradictory, so the system has no solution

The two lines are parallel.



Introduction to Systems of Linear Equations: Three Cases (8)

Example 3

Solve the system
$$\begin{aligned} 2x - 3y &= 5 \\ 6x - 9y &= 15 \end{aligned}$$

Solution

Add -3 times the first equation to the second equation to eliminate x

$$\begin{aligned} 2x - 3y &= 5 \\ 6x - 9y &= 15 \end{aligned} \longrightarrow \begin{aligned} 2x - 3y &= 5 \\ 0 &= 0 \end{aligned}$$

The second equation can simply be omitted since it doesn't impose any restrictions on x and y

The solutions of the system are those values of x and y that satisfy the single equation $2x - 3y = 5$

One way to describe the solution set is to solve this equation for x in terms of y to obtain $x = \frac{3}{2}y + \frac{5}{2}$ and then assign an arbitrary value t (called a ***parameter***) to y. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \frac{3}{2}t + \frac{5}{2}, \qquad y = t$$

$\left(\frac{3}{2}t + \frac{5}{2}, t\right)$

Introduction to Systems of Linear Equations: Coinciding Planes (9)

Example 4

Solve the system

$$\begin{aligned} x - 2y + z &= 3 \\ 2x - 4y + 2z &= 6 \\ 3x - 6y + 3z &= 9 \end{aligned}$$

Solution

By inspection we can see that the second and third equations are just multiples of the first one. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - 2y + z = 3$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of this equation. We can do this by first solving for x in terms of y and z, then assigning arbitrary values r and s (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 3 + 2r - s, \qquad y = r, \qquad z = s$$

Introduction to Systems of Linear Equations: Parametric Forms of Equations of Linear Spaces (10)

In general, implicit equations of lines, for example,
can be written in parametric form:

$$y = 1 - x \quad \text{Letting } x = t$$

$$\underbrace{(x, y) = (t, 1 - t)}_{\text{parametric form}} \quad \text{for any } t \in \mathbb{R}$$

t is called a parameter because it parametrizes the points on the line.

This can be extended to planes, except now we'll need two parameters. Here's an example. Consider the plane $x + y + z = 1$. The parametric form for this plane is $(x, y, z) = (1 - t - w, t, w)$

Here, the parameters t and w, allow us to use $\mathbb{R}^2$ to label the points on the plane.