# Upcoming Assignments and Assessments

- Homework Assignment #5 is due Friday 2/28

Exam will cover sections up to and including 3.4 (matrix multiplication)

- Quizzes #4 and #5 will be administered during this week's recitation

* T is a linear transformation

$\iff$ (i) $T(u+v) = T(u) + T(v)$

(ii) $T(cu) = cT(u)$

# Linear Transformations Part II

* Important fact about linear transformations:

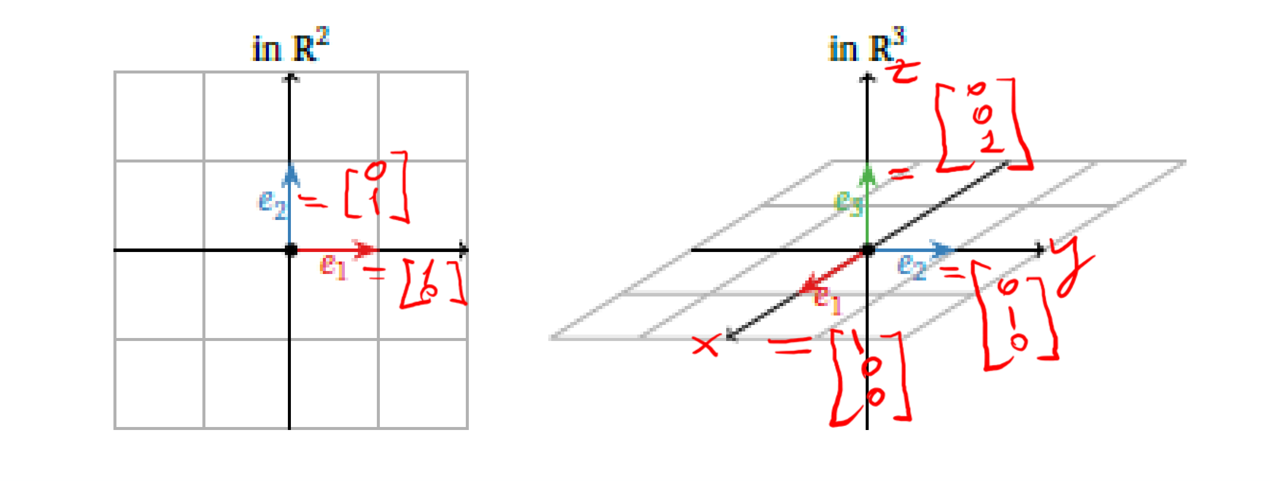
$T(0) = 0$

* All matrix transformations are linear.

# Linear Transformations (5)

The standard coordinate vectors in $\mathbb{R}^n$ are the $n$ vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, e_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$



# Linear Transformations (6)

If $A$ is an $m \times n$ matrix with columns $v_1, v_2 \dots, v_n$ then $Ae_i = v_i$ for each $i = 1,2, \dots, n$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix} e_i = v_i$$

Multiplication by $e_i$ simply selects $A$'s $i^{th}$ column.

For example,

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 0 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

**Definition**: The $n \times n$ identity matrix (columns are standard coordinate vectors),

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

# Linear Transformations (7)

**From linear transformations to matrices**

**Theorem**: Let $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear transformation. Let $A$ be the $m \times n$ matrix

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \dots & T(e_n) \\ | & | & & | \end{bmatrix}$$

Then $T$ is a matrix transformation associated with $A$, that is, $T(x) = Ax$

*Proof*: Suppose $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation. Then

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\left( x \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + y \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + z \overset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right) = T(xe_1 + ye_2 + ze_3) \overset{\text{because } T \text{ is linear}}{=} xT(e_1) + yT(e_2) + zT(e_3)$$

$$= \begin{bmatrix} \overset{v_1}{|} & \overset{v_2}{|} & \overset{v_3}{|} \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$A$ is the **standard matrix** for $T$

**Matrix transformations are the same as linear transformations!**

# Linear Transformations (8)

## Example 4

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T(x) = 3x$. Find the standard matrix $A$ for $T$.

$$T\begin{bmatrix}1\\0\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} \Rightarrow A = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = 3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}5x+y\\y-x\\x\end{bmatrix}$. Find the standard matrix $A$ for $T$.

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}5\\-1\\1\end{bmatrix} \Longrightarrow A = \begin{bmatrix}5 & 1\\-1 & 1\\1 & 0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

# Linear Transformations (9)

$T =$ Reflection over the $x$-axis in $\mathbb{R}^2$

$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} \Longrightarrow A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

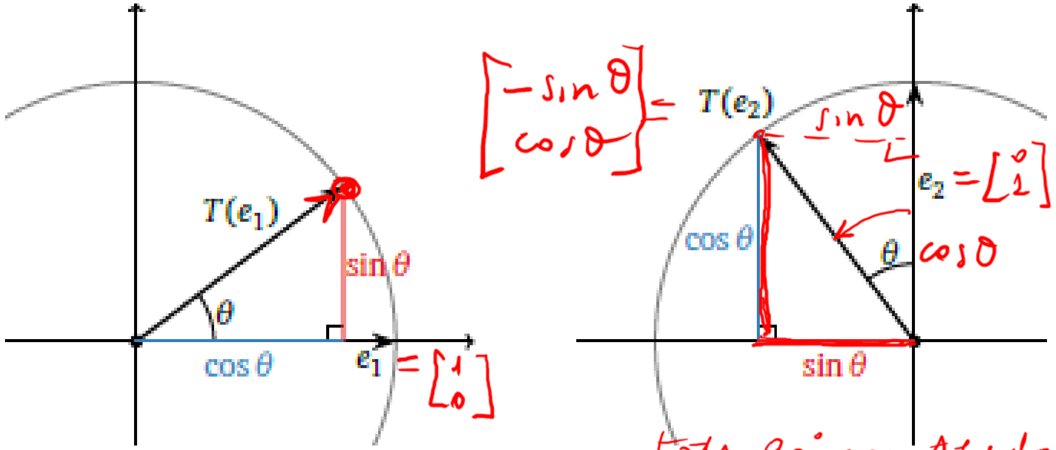
## Example 5

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as follows

$$T(x) = \text{ the vector } x \text{ rotated counterclockwise by the angle } \theta.$$

Find the standard matrix for $T$.

**Solution.** The columns of $A$ are obtained by evaluating $T$ on the standard coordinate vectors $e_1, e_2$. In order to compute the entries of $T(e_1)$ and $T(e_2)$, we have to do some trigonometry.




Rotate $90°$ counterclockwise

$A = \begin{bmatrix}\cos 90° & -\sin 90°\\ \sin 90° & \cos 90°\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

We see from the picture that

$$\left.\begin{array}{l} \checkmark\ T(e_1) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \\ \checkmark\ T(e_2) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\end{array}\right\} \Longrightarrow A = \boxed{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}}$$

## Example 6

Verify that $\mathrm{Id}_{\mathbb{R}^n}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation and that its standard matrix is the identity matrix

$$I_n = \begin{bmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & \cdots & 0 & 1\end{bmatrix}$$

$\checkmark\ \mathrm{Id}_{\mathbb{R}^n}(u+v) = u+v$

$\mathrm{Id}_{\mathbb{R}^n}(u) + \mathrm{Id}_{\mathbb{R}^n}(v) = u+v\ \checkmark$

$\checkmark\ \mathrm{Id}_{\mathbb{R}^n}(cu) = cu = c\cdot \mathrm{Id}_{\mathbb{R}^n}(u)$.