Upcoming Assignments and Assessments

- Homework Assignment #2 is due Friday 2/7
- Quiz #2 will be administered during this week's recitation

*Quiz #1 and solutions will be posted on Brightspace for review. Quiz #1 will not be used in calculation of your final grade.

The Matrix Equation $Ax = b$ and Solution Sets

The Matrix Equation $Ax = b$ (1)

A is a $m \times n$ matrix = m rows and n columns
x and b are vectors, usually not the same size

Matrix-Vector Product

Let A be an $m \times n$ matrix with columns $v_1, v_2, v_3 \dots v_n$:

$$\begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \\ v_1 & v_2 & v_3 & \dots & v_n \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix}$$

Note that here $v_i s$ are vectors in $\mathbb{R}^m$

The product of the matrix A with the vector x is the linear combination

$$Ax = \begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \\ v_1 & v_2 & v_3 & \dots & v_n \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2 + x_3 v_3 + \cdots + x_n v_n$$

In order for Ax to make sense, the number of entries of x has to be the same as the number of columns of A.

The result is a vector in $\mathbb{R}^m$.

The Matrix Equation $Ax = b$ (2)

Example 1

Calculate the product:

(a) $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & -1 & 0 \\ 4 & 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$

Properties of the Matrix-Vector Product. Let A be an $m \times n$ matrix, let u, v be vectors in $\mathbb{R}^n$ and let c be a scalar. Then:

- $A(u + v) = Au + Av$
- $A(cu) = cAu$

The Matrix Equation $Ax = b$ (3)

Definition. A **matrix equation** is an equation of the form $Ax = b$, where A is an $m \times n$ matrix, b is a vector in $\mathbb{R}^m$, and x is a vector whose components $x_1, x_2, x_3, \ldots, x_n$ are unknown.

Two important related questions:

- Given a specific choice of b, what are all of the solutions to $Ax = b$?
- What are all of the choices of b so that $Ax = b$ is consistent?

Matrix and Vector Equations

Matrix Equations and Vector Equations. Let $v_1, v_2, \ldots, v_n$ and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

This is equivalent to the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then $Ax = b$ is equivalent to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b,$$

where $v_1, v_2, \ldots, v_n$ are the columns of A, and $x_1, x_2, \ldots, x_n$ are the entries of x.

The Matrix Equation $\boldsymbol{Ax = b}$ (4)

Example 2

Write the vector equation

$$3v_1 - 5v_2 + v_3 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

as a matrix equation, where v_1, v_2, v_3 are vectors in $\mathbb{R}^3$

$$\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

The Matrix Equation $Ax = b$ (5)

Now we have **four different ways** of expressing a system of linear equations:

(I) As a system of equations

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 7 \\ x_1 \quad - x_2 \quad - 3x_3 &= 5 \end{aligned}$$

(II) As an augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right]$$

(III) As a vector equation $(x_1 v_1 + x_2 v_2 + \cdots x_n v_n = b)$

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

(IV) As a matrix equation $(Ax = b)$

$$\begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

The Matrix Equation $\boldsymbol{Ax = b}$ (6)

Another way to calculate a matrix-vector product – ***row-column multiplication***

Row vector = matrix with one row
First, we must make sure that the size of the row vector (its length) is the same as the size of the column vector (its length). Now let

$$R = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Then the product of the two is defined as $R \cdot C = \begin{bmatrix} r_1 & r_2 & r_3 & \dots & r_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = r_1 c_1 + r_2 c_2 + \cdots + r_n c_n$

which is a scalar.

Example 3

$$\begin{bmatrix} -1 & 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ -2 \\ 1 \end{bmatrix} = (-1)(3) + (2)(5) + (0)(-2) + (4)(1) = -3 + 10 + 0 + 4 = 11$$

The Matrix Equation $Ax = b$ (7)

Another way to calculate a matrix-vector product

Recipe: The row-column rule for matrix-vector multiplication. If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

$$Ax = \begin{pmatrix} -r_1- \\ -r_2- \\ \vdots \\ -r_m- \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}.$$

Example 4

$$\begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} [-1, 2, 0] \cdot \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \\ [-2, 1, 4] \cdot \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -3 + 10 + 0 \\ -6 + 5 - 8 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

The Matrix Equation $\boldsymbol{Ax = b}$ (8)

Let A be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Then

$Ax = b$ has a solution

$A x = b$

$$\iff \text{there exist } x_1, x_2, \ldots, x_n \text{ such that } A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

$$\iff \text{there exist } x_1, x_2, \ldots, x_n \text{ such that } x_1v_1 + x_2v_2 + \cdots + x_nv_n = b$$

vector equation

$$\iff b \text{ is a linear combination of } v_1, v_2, \ldots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

> **Spans and Consistency.** The matrix equation $Ax = b$ has a solution if and only if b is in the span of the columns of A.

This gives an equivalence between an *algebraic* statement ($Ax = b$ is consistent), and a *geometric* statement (b is in the span of the columns of A).

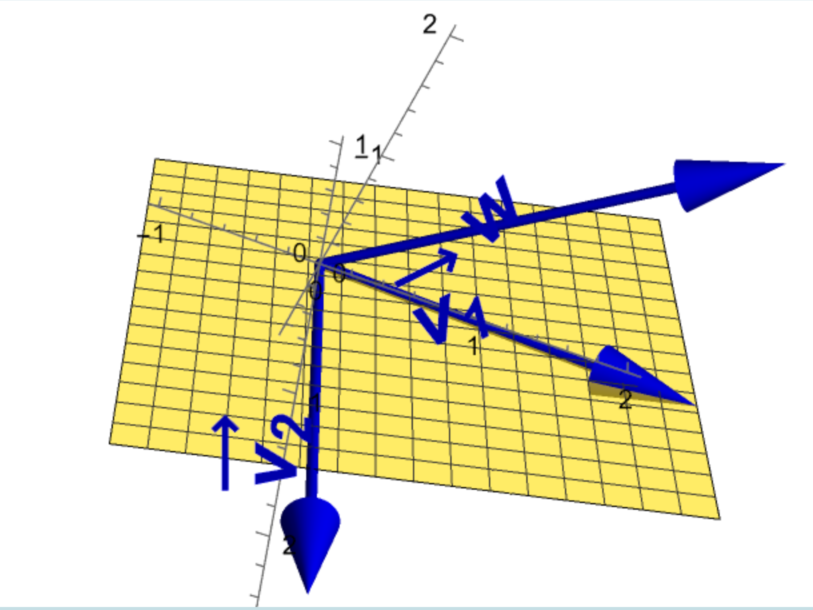
The Matrix Equation $\boldsymbol{Ax = b}$ (9)

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$. Does the equation $Ax = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ have a solution?

The geometric solution:

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and let the target vector be $w = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$.

The equation $Ax = w$ is consistent if and only if w is contained in the span of the columns of A. So, we draw a picture:



Note that w does not lie in the plane spanned by v_1 and v_2, that is, w is not in the span$\{v_1, v_2\}$, which implies that the system is inconsistent.

The Matrix Equation $Ax = b$ (10)

The algebraic solution:

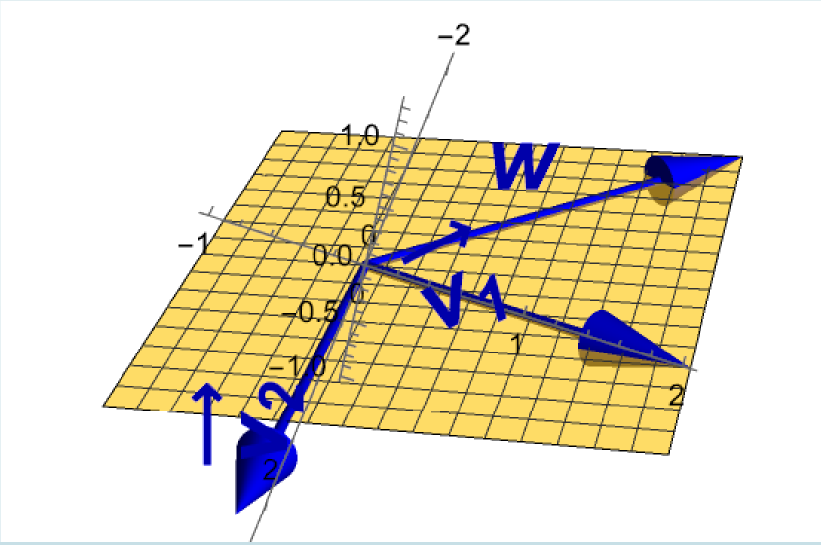
$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & 2 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & 2 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \xrightarrow{\frac{3}{4}R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

REF

No solution

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$, as before. Does the equation $Ax = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ have a solution?

The equation $Ax = w$ is consistent if and only if w is contained in the span of the columns of A. So, we draw a picture:



Note that this time w does lie in the plane spanned by v_1 and v_2, that is, w is in the span$\{v_1, v_2\}$, which implies that the system is consistent.

The Matrix Equation $Ax = b$ (11)

The algebraic solution:

$$\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 6 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3-R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -6 & 6 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{R_3+2R_2}$$

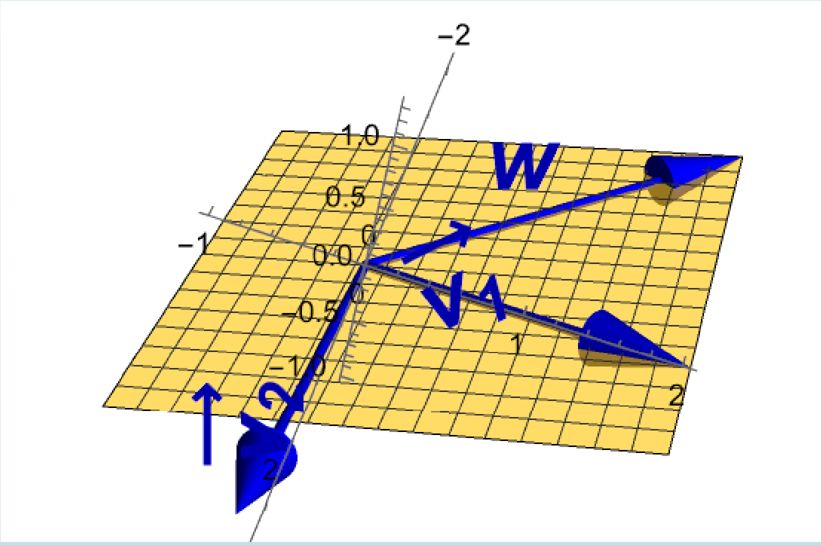
$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The solution is $x_1 = 1$ and $x_2 = -1$

Check: $Ax = b \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \checkmark$$



The Matrix Equation $Ax = b$ (12)

Theorem. *Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:*

1. *$Ax = b$ has a solution for all b in $\mathbf{R}^m$.*
2. *The span of the columns of A is all of $\mathbf{R}^m$.*
3. *A has a pivot position in every row.*

If A does not have a pivot in every row then it will have a row of zeros and when we add the augmented column, if that row of zeros ends with a nonzero entry, then we will have an inconsistent system.

Also, as we will later see, the dimension of the span of the columns is equal to the number of pivots of A. That is, the columns of A span a line if A has one pivot, they span a plane if A has two pivots, etc. The whole space $\mathbb{R}^m$ has dimension m, so this generalizes the fact that the columns of A span $\mathbb{R}^m$ when A has m pivots.

Solution Sets: Homogeneous Systems (13)

Definition. A system of linear equations of the form $Ax = 0$ is called **homogeneous.**
A system of linear equations of the form $Ax = b$ where $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial** solution. Any nonzero solution is called **nontrivial**.

Note that if $Ax = 0$ has a nontrivial solution then it has infinitely many, so
$Ax = 0$ has a nontrivial solution $\Leftrightarrow$ there is a free variable $\Leftrightarrow$ A has a column without a pivot

Example 5 **(No nontrivial solutions)**

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. Solve the homogeneous equation $Ax = 0$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -7 & -8 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & -7 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{7}R_2} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & \frac{8}{7} & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & \frac{8}{7} & 0 \\ 0 & 0 & \frac{3}{7} & 0 \end{bmatrix} \xrightarrow{\frac{7}{3}R_3}$$

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & \frac{8}{7} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 - \frac{8}{7}R_3} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - 4R_3} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The only solution is the trivial one $x = 0$.

Solution Sets: Homogeneous Systems (14)

Parametric Vector Form (homogeneous case)

Let $A = \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Note that A is in reduced row echelon form.

$Ax = 0$ corresponds to the system

$$\begin{aligned} x_1 - 8x_3 - 7x_4 &= 0 \\ x_2 + 4x_3 + 3x_4 &= 0 \end{aligned}$$

Re-expressed in parametric form we get

$$\begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Now, we can turn this into a vector equation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Solution Sets: Homogeneous Systems (15)

Parametric Vector Form (homogeneous case)

The vector equation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

This is called the parametric vector form of the solution set. It states that the solution set is the set of all linear combinations of the vectors $\begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Or equivalently, that the solution set is Span$\{\begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix}\}$

Solution Sets: Homogeneous Systems (16)

Parametric Vector Form (homogeneous case)

Recipe: Parametric vector form (homogeneous case). Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by making the coefficients of x_3, x_6, and x_8 into vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbb{R}^n$, and any scalars x_3, x_6, x_8. This is called the **parametric vector form** of the solution.
In this case, the solution set can be written as $\mathrm{Span}\{v_3, v_6, v_8\}$.

The set of solutions to a homogeneous equation $Ax = 0$ is a span.

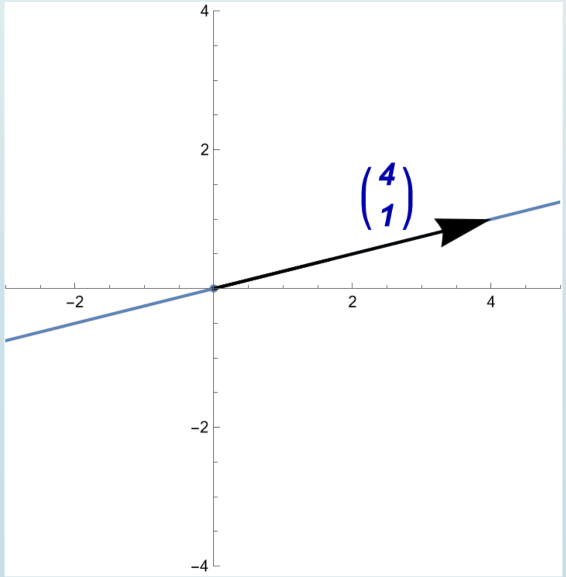
Solution Sets: Homogeneous Systems (17)

Example 6

Compute the parametric vector form of the solution set of $Ax = 0$ where $A = \begin{bmatrix} 1 & -4 \\ 3 & -12 \end{bmatrix}$.

We don't need to construct an augmented matrix since we know that the augmented column will always consist of zeros independent of what row operations we make.

$$\begin{bmatrix} 1 & -4 \\ 3 & -12 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \qquad x_1 - 4x_2 = 0 \qquad \begin{aligned} x_1 &= 4x_2 \\ x_2 &= x_2 \end{aligned} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



Finally, we can write that the solution set is Span$\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\}$.

Line through the origin
containing the vector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Solution Sets: Homogeneous Systems (18)

Example 7

Compute the parametric vector form of the solution set of $Ax = 0$ where $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$.

We don't need to construct an augmented matrix since we know that the augmented column will always consist of zeros independent of what row operations we make.

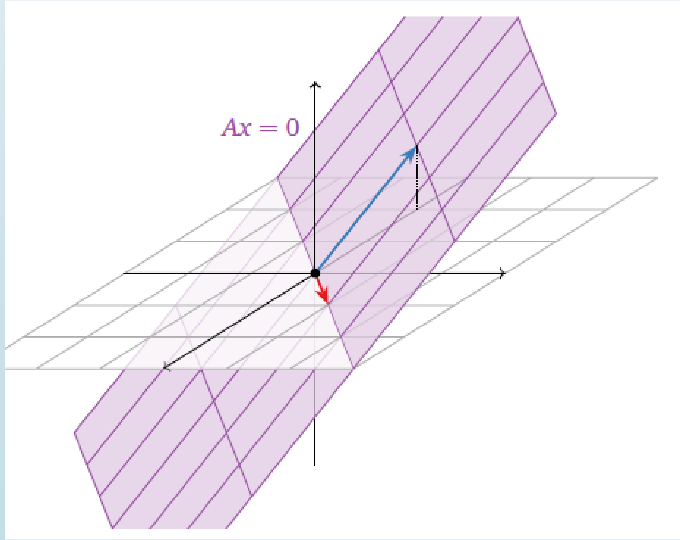
$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_2 + 2x_3 = 0$$

$$\begin{aligned} x_1 &= x_2 - 2x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Finally, we can write that the solution set is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.



Solution Sets: Homogeneous Systems (19)

Dimension of the Solution Set

Dimension of the solution set. The above examples show us the following pattern: when there is one free variable in a consistent matrix equation, the solution set is a line, and when there are two free variables, the solution set is a plane, etc. The number of free variables is called the *dimension* of the solution set.

Intuitively, the dimension of a solution set is the number of parameters you need to describe a point in the solution set. For a line only one parameter is needed, and for a plane two parameters are needed. The more parameters needed to describe a point in the solution set the higher its dimension.

Solution Sets: Inhomogeneous Systems (20)

Example 8

What is the solution set of $Ax = b$ where $A = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$?

Construct the augmented matrix and row reduce.

$$\left[\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

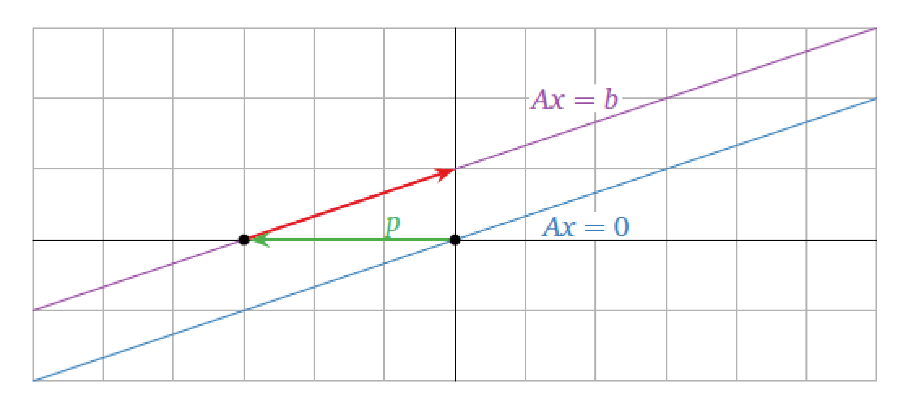
$$x_1 - 3x_2 = -3$$

$$\begin{aligned} x_1 &= 3x_2 - 3 \\ x_2 &= x_2 \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Finally, we can write that the solution set is $\text{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.

The solution set is a *translate* of the line through the origin spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The solution set is a line, but it is NOT a span.



Solution Sets: Inhomogeneous Systems (21)

Example 9

What is the solution set of $Ax = b$ where $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$?

Construct the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x_1 - x_2 + 2x_3 = 1$$

$$\begin{aligned} x_1 &= x_2 - 2x_3 + 1 \\ x_2 &= x_2 \qquad\quad + 0 \\ x_3 &= \qquad\; x_3 + 0 \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Finally, we can write that the solution set is $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

The solution set is a *translate* of the plane through the origin spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ translated by the vector $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The solution set is a plane, but it is NOT a span.

Solution Sets: Inhomogeneous Systems (22)

In the previous example the solution set of $Ax = b$ consisted of all vectors of the form $x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. In this case, $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution (simply set $x_2 = 0$ and $x_3 = 0$).

In Example 7, where A was the same, the solution set of the homogeneous system $Ax = 0$ consisted of all vectors of the form $x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Note that the parametric vector form of the solution set of $Ax = b$ was exactly the same as the parametric vector form of the solution set of $Ax = 0$ plus a particular solution.

Key Observation. If $Ax = b$ is consistent, the set of solutions to is obtained by taking one particular solution p of $Ax = b$, and adding all solutions of $Ax = 0$.

In particular, if $Ax = b$ is consistent, the solution set is a *translate of a span*.