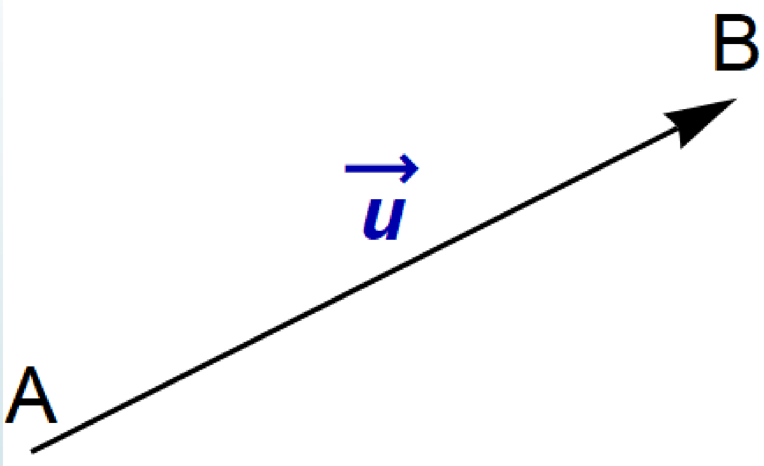
Vectors, Vector Equations and Span Part I

Vectors (1)

A vector is a mathematical object which possesses a ***magnitude*** and a ***direction***. A real number (*scalar*) can be thought of as a one-dimensional vector. A two- or three-dimensional vector can be represented by an arrow, whose direction shows the direction of the vector and whose length represents the magnitude.

“head”, terminal point



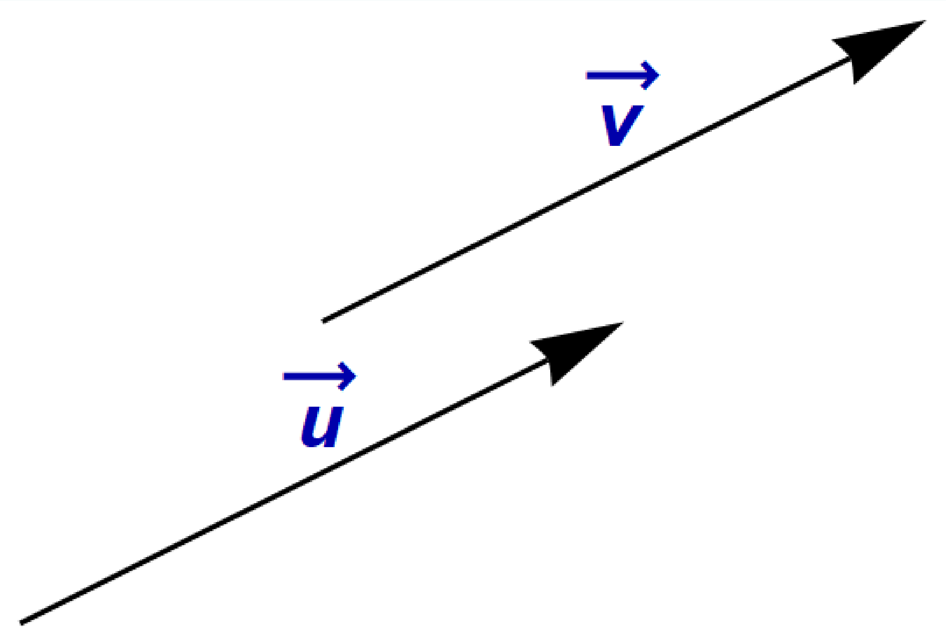
“tail”, initial point

$$\vec{u} = \overrightarrow{AB}$$

The vector whose initial and terminal points coincide has length zero, so we call this the ***zero vector*** and denote it by $\vec{0}$. The zero vector has no natural direction, so we will agree that it can be assigned any direction that is convenient for the problem at hand.

Vectors (2)

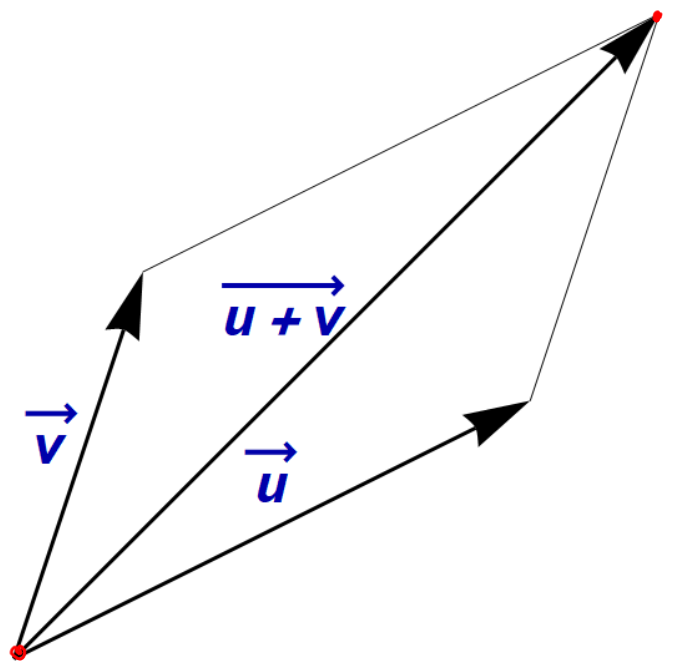
Two vectors are ***equal***, or ***equivalent***, if and only if they have equal magnitudes and the same direction. For example, the two vectors below, $\vec{u}$ and $\vec{v}$ are equal. We can think of them as the same vector positioned differently.

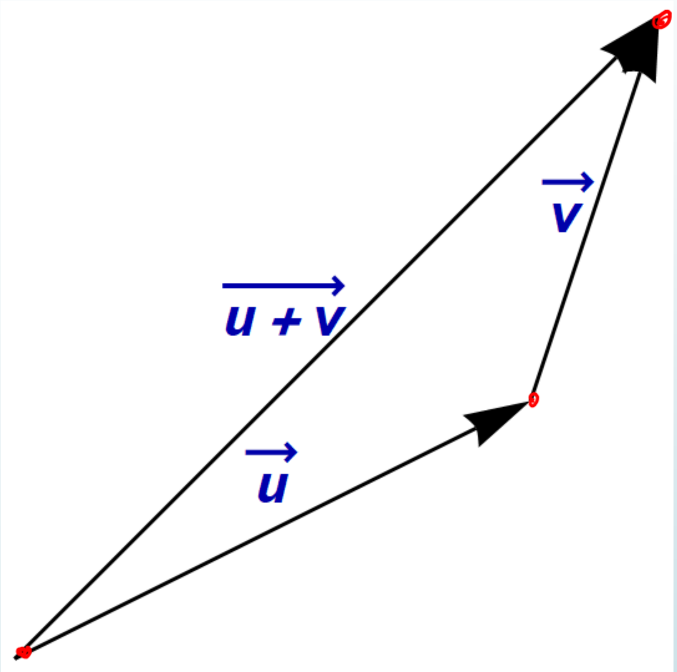


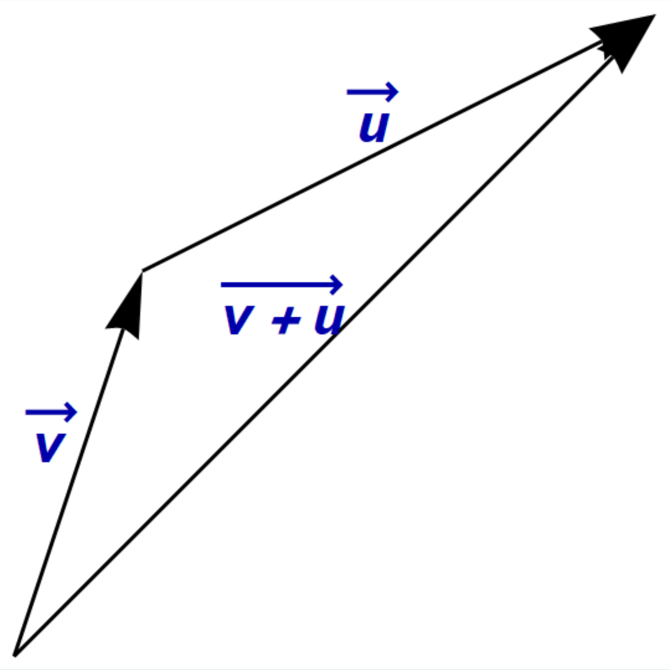
$$\vec{u} = \vec{v}$$

Vectors: Vector Addition (3)

Parallelogram Method **Triangle Method**





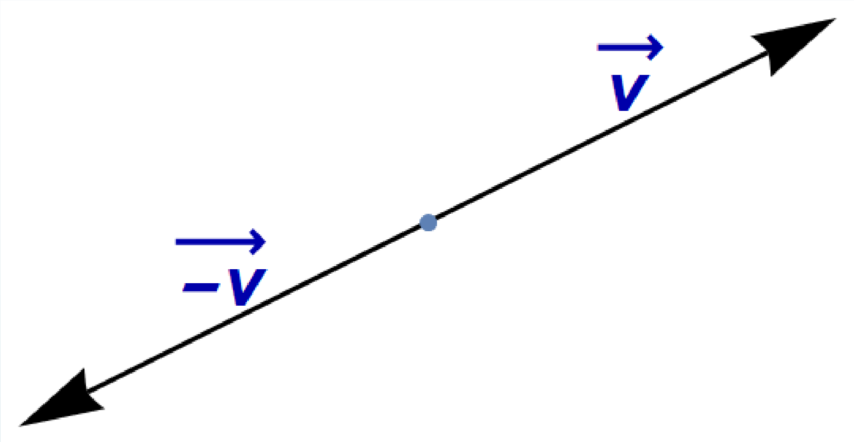


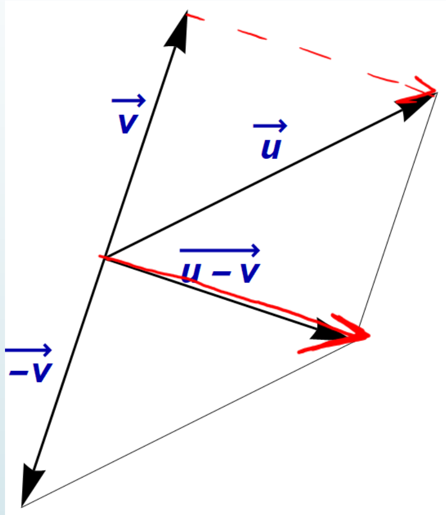
$$\overrightarrow{u+v} = \overrightarrow{v+u}$$

Vectors: Vector Subtraction (4)

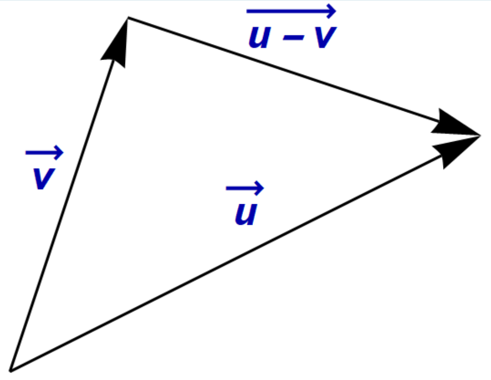
The ***negative*** of a vector $\vec{v}$ is the vector $\overrightarrow{-v}$ whose length is the same as $\vec{v}$ but whose direction is opposite to that of $\vec{v}$

We then define the vector $\overrightarrow{u-v}$ as $\vec{u} + (\overrightarrow{-v})$



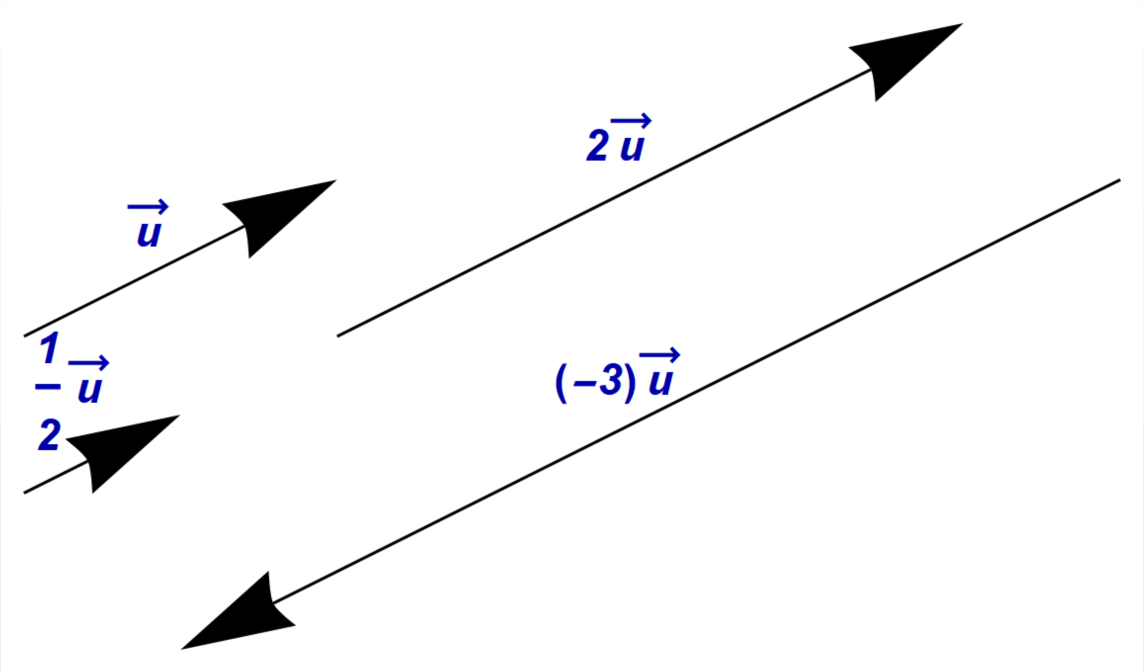


Note that if we move the tail of $\overrightarrow{u-v}$ to the head of $\vec{v}$, the head of $\overrightarrow{u-v}$ will coincide with the head of $\vec{u}$. Thus, we can think of the vector $\overrightarrow{u-v}$ as the vector connecting the head of $\vec{v}$ to the head of $\vec{u}$, in that order.



Vectors: Scalar Multiplication (5)

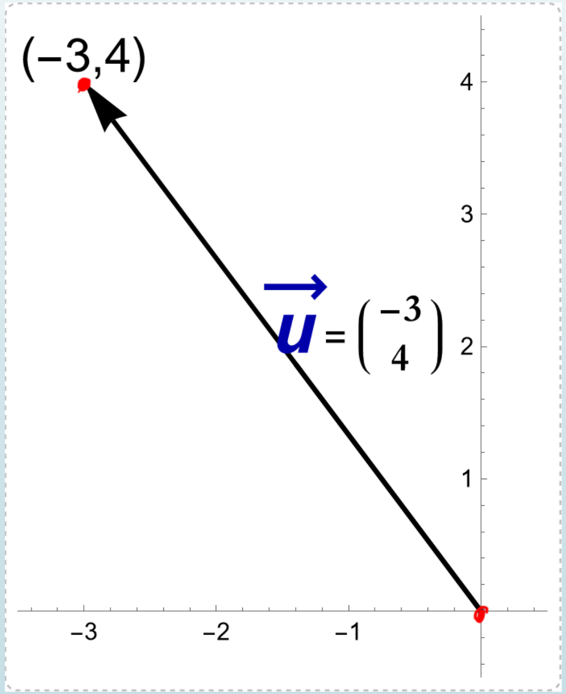
A non-zero scalar quantity k and a vector $\vec{u}$ may be multiplied to obtain a new vector $k\vec{u}$. If k is positive, $k\vec{u}$ will have the same direction as $\vec{u}$, but its length will be k times the length of $\vec{u}$, if k is negative, $k\vec{u}$ will have the opposite direction with length $|k|$ times the length of $\vec{u}$.

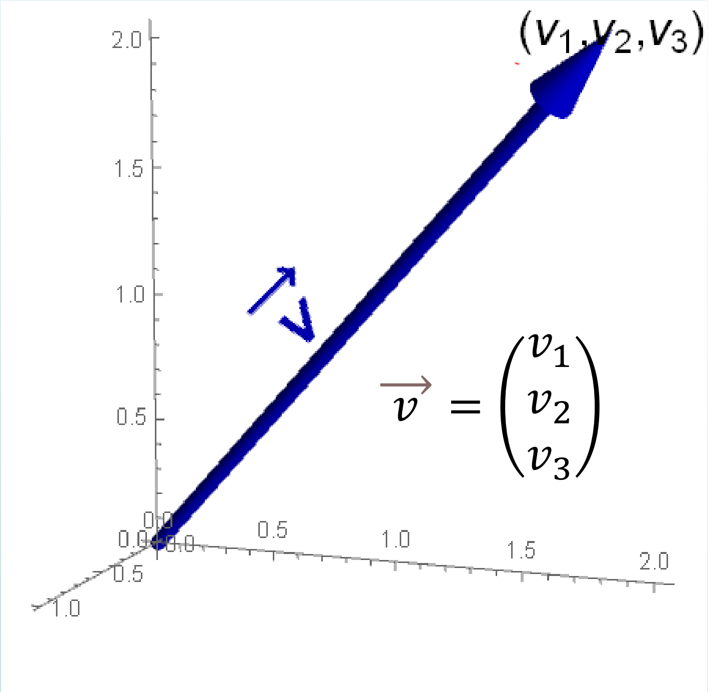


If one vector is a scalar multiple of the other than we say that the two vectors are ***parallel*** (or, ***collinear***).

Vectors: Vector Components (6)

A vector is in ***standard position*** when its tail is at the origin. When a vector is in standard position, the coordinates of the head of the vector are called the ***components*** of the vector. I should point out the importance of mathematical language here; note the use of the word components, and NOT simply coordinates. We will reserve coordinates for points. Components define a unique vector, and every vector has its unique components independent of whether the vector is in standard position or not.

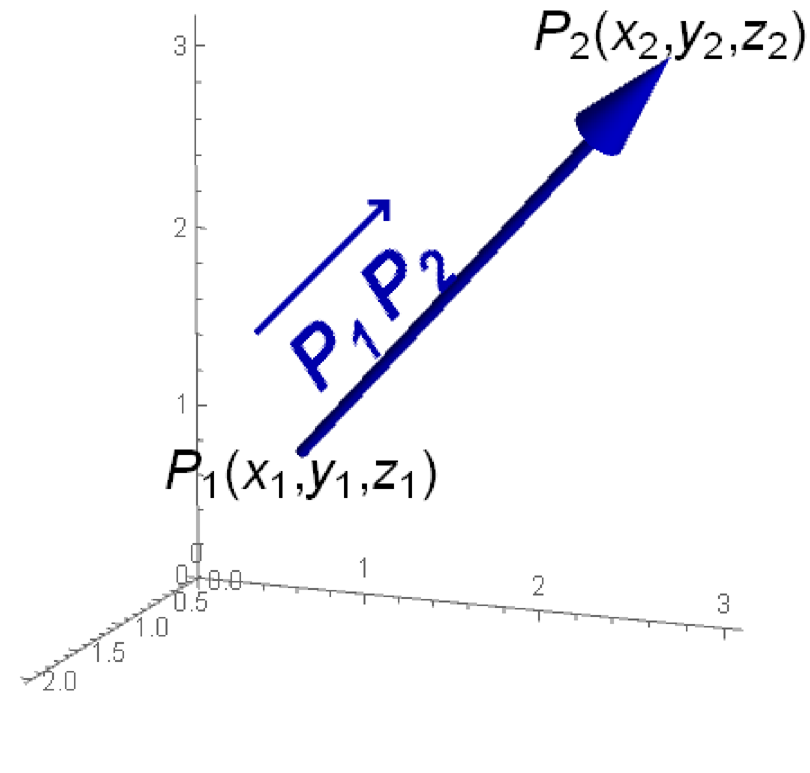




Vectors in standard position.

Vectors: Vector Components (7)

If a vector is not in standard position, then to obtain its components subtract the coordinates of the tail from the coordinates of the head.



$$\overrightarrow{P_1P_2} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

Example 1

Find the components of the vector $\overrightarrow{P_1P_2}$.

(a) $P_1(3,-1), P_2(-2,5)$

$$\overrightarrow{P_1P_2} = \begin{pmatrix} -2-3 \\ 5-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$$

(b) $P_1(1,3,-2), P_2(2,4,1)$

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 2-1 \\ 4-3 \\ 1-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

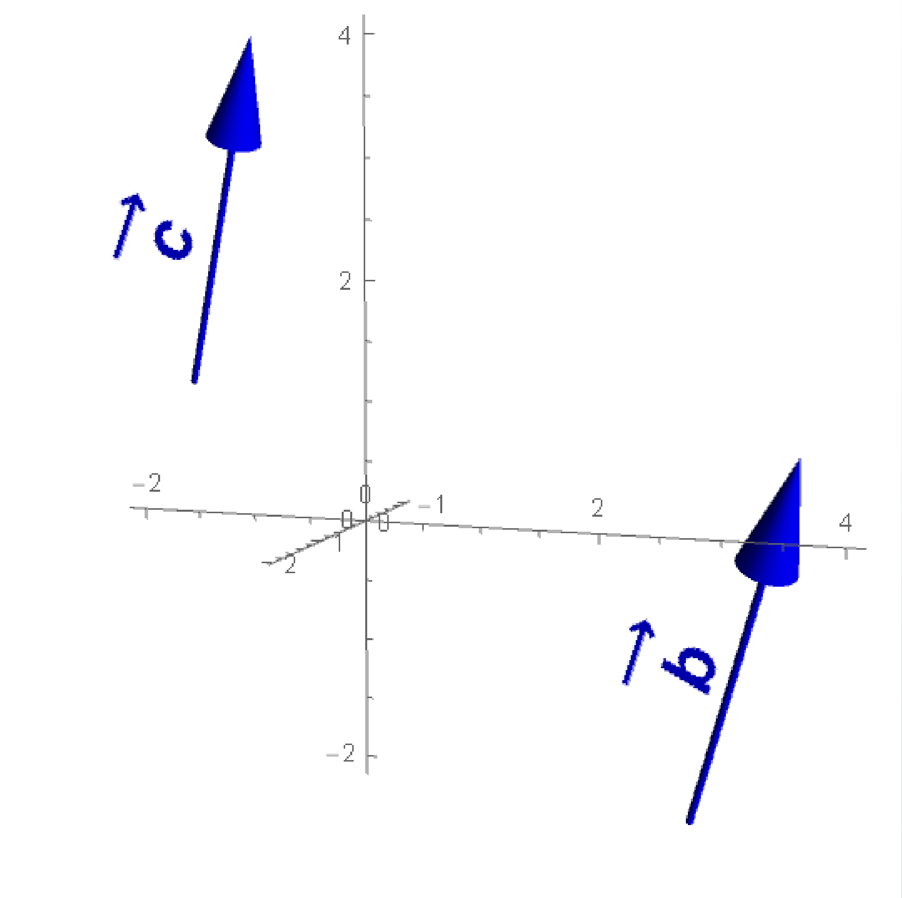
(c) $P_1(-1,-2,1), P_2(0,-1,4)$

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 0-(-1) \\ -1-(-2) \\ 4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Note that the vectors in parts **(b)** and **(c)** are equal.

Vectors: Vector Components (8)

The two equal vectors from parts **(b)** and **(c)** on the previous slide, called $\vec{b}$ and $\vec{c}$, respectively. Note that if both were put in standard position, their heads would coincide and have coordinates $(1,1,3)$.



Vectors: *n*-Space, Addition and Scalar Multiplication (9)

The set of real numbers = the real number line = R^1

The set of ordered pairs of real numbers = R^2

The set of ordered triples of real numbers = R^3

If n is a positive integer, then an ***ordered n-tuple*** is a sequence of n real numbers $(v_1, v_2, v_3, \ldots, v_n)$. The set of all n-tuples is called ***n-space*** and is denoted by R^n.

Define the ***zero vector*** as $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$ be two vectors in R^n and k be any scalar then

$$\overrightarrow{u+v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{pmatrix} \qquad k\vec{u} = \begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \\ \vdots \\ ku_n \end{pmatrix}$$

Vectors: Addition and Scalar Multiplication (10)

Example 2

Let $\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$. Find the components of

(a) $\vec{u} - \vec{v} = \begin{pmatrix} 0 - 1 \\ -1 - 3 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$

(b) $3\vec{u} + 2\vec{w} = \begin{pmatrix} 3 \cdot 0 + 2 \cdot 2 \\ 3(-1) + 2(-4) \\ 3(3) + 2(1) \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ 11 \end{pmatrix}$

(c) $5(\vec{v} - 3\vec{u}) =$ ______

(d) $\vec{u} + \vec{v} - 2\vec{w} =$ ______

Vectors: Vector Operations (11)

Example 3

Let $\vec{u} = (1,1,1)$, $\vec{v} = (0,2,4)$ and $\vec{w} = (-2,6,-6)$. Use properties of vector operations to find the component form of vector $\vec{x}$.

$$2\vec{u} - \vec{v} + \vec{x} = 3\vec{x} + \vec{w}$$

$$2(1,1,1) - (0,2,4) + (x_1, x_2, x_3) = 3(x_1, x_2, x_3) + (-2,6,-6)$$

$$2 - 0 + x_1 = 3x_1 - 2$$

$$-2x_1 = -4 \Rightarrow x_1 = 2$$

$$\cancel{2-2} + x_2 = 3x_2 + 6$$

$$-2x_2 = 6 \Rightarrow x_2 = -3$$

$$2 - 4 + x_3 = 3x_3 - 6$$

$$-2x_3 = -4 \Rightarrow x_3 = 2$$

$$\vec{x} = (2, -3, 2)$$