Calculating largest and smallest eigenvalues (in absolute value) and corresponding eigenvectors for large matrices

The Power Method

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Let A be an $n \times n$ diagonalizable matrix with real eigenvalues. Let λ_1 be the eigenvalue of greatest magnitude. Call λ_1 the **dominant** eigenvalue. In many applications, the two most important eigenvalues are the dominant one and the one with smallest magnitude. Because A is diagonalizable there exists a basis of $\mathbb{R}^n$ composed of the eigenvectors of A: $\{\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}, \ldots, \boldsymbol{b_n}\}$. Let $\boldsymbol{b_i}$ be the vector corresponding to λ_i where λ_1 is the dominant eigenvalue, and all others are numbered in accordance with their magnitude in decreasing order, that is, $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \ldots \geq \lambda_n$.
Let w_1 be any nonzero vector in $\mathbb{R}^n$. Then

$$\boldsymbol{w_1} = c_1\, \boldsymbol{b_1} + c_2\, \boldsymbol{b_2} + \ldots + c_n\, \boldsymbol{b_n} \tag{1}$$

for some scalars c_i. Now, let's multiply both sides of (1) by A^s, remembering that $A^s\, \boldsymbol{b_i} = \lambda_i{}^s\, \boldsymbol{b_i}$. So,

$$A^s\, \boldsymbol{w_1} = \lambda_1{}^s\, c_1\, \boldsymbol{b_1} + \lambda_2{}^s\, c_2\, \boldsymbol{b_2} + \lambda_3{}^s\, c_3\, \boldsymbol{b_3} + \ldots + \lambda_n{}^s\, c_n\, \boldsymbol{b_n} \tag{2}$$

Now, factor out $\lambda_1{}^s$

$$A^s\, \boldsymbol{w_1} = \lambda_1{}^s \left(c_1\, \boldsymbol{b_1} + \left(\frac{\lambda_2}{\lambda_1}\right)^s c_2\, \boldsymbol{b_2} + \left(\frac{\lambda_3}{\lambda_1}\right)^s c_3\, \boldsymbol{b_3} + \ldots + \left(\frac{\lambda_n}{\lambda_1}\right)^s \boldsymbol{b_n} \right) \tag{3}$$

When s is large, $\frac{\lambda_i}{\lambda_1}$ are close to zero since $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$. So, as long as $c_1 \neq 0$, $\lambda_1{}^s\, c_1\, \boldsymbol{b_1}$ dominates the right hand side of the equation. So, when s is large enough $A^s\, \boldsymbol{w_1} \approx \lambda_1{}^s\, c_1\, \boldsymbol{b_1}$. This shows that we can approximate an eigenvector of A corresponding to λ_1 by multiplying $\boldsymbol{w_1}$ repeatedly by A. This multiplication may produce large numbers so scaling is used after each multiplication. More precisely, after each multiplication divide the resulting vector by the max of the magnitudes of the components.

Now, if $\boldsymbol{x}$ is an eigenvector corresponding to λ_1, then

$$\frac{A\, \boldsymbol{x} \cdot \boldsymbol{x}}{\boldsymbol{x} \cdot \boldsymbol{x}} = \frac{\lambda_1\, \boldsymbol{x} \cdot \boldsymbol{x}}{\boldsymbol{x} \cdot \boldsymbol{x}} = \lambda_1 \tag{4}$$

As we compute the $\boldsymbol{w_j}$, the quotients $\frac{A\, \boldsymbol{w_j} \cdot \boldsymbol{w_j}}{\boldsymbol{w_j} \cdot \boldsymbol{w_j}}$ should approach λ_1 (these are called Rayleigh quotients).

Summary of the Power Method

The Power Method for Finding the Dominant Eigenvalue λ_1 of A

Step 1 Choose an appropriate vector $\mathbf{w}_1$ in $\mathbb{R}^n$ as first approximation to an eigenvector corresponding to λ_1.

Step 2 Compute $A\mathbf{w}_1$ and the Rayleigh quotient $(A\mathbf{w}_1 \cdot \mathbf{w}_1)/(\mathbf{w}_1 \cdot \mathbf{w}_1)$.

Step 3 Let $\mathbf{w}_2 = (1/d_1)A\mathbf{w}_1$, where d_1 is the maximum of the magnitudes of components of $A\mathbf{w}_1$.

Step 4 Repeat step 2, with all subscripts increased by 1. The Rayleigh quotients should approach λ_1, and the $\mathbf{w}_j$ should approach an eigenvector of A corresponding to λ_1.

The Matrix

$$B = \begin{pmatrix} -3 & 6 & -7 & 2 \\ 13 & -18 & 4 & 6 \\ 21 & 32 & -16 & 9 \\ -4 & 8 & 7 & 11 \end{pmatrix}$$

In[1]:= `Eigen[M_, x_] := (Transpose[M.x].x) / (Transpose[x].x) // N`

$$\text{Eigen}[M\_, x\_] := \frac{(\text{Transpose}[M.x].x)}{\text{Transpose}[x].x} \;//\; N$$

In[2]:= $$\text{next}[M\_, x\_] := \frac{M.x}{\text{Max}[M.x]} \;//\; N$$

```
In[3]:= B = {{-3, 6, -7, 2}, {13, -18, 4, 6}, {21, 32, -16, 9}, {-4, 8, 7, 11}}
Out[3]= {{-3, 6, -7, 2}, {13, -18, 4, 6}, {21, 32, -16, 9}, {-4, 8, 7, 11}}
```

In[4]:= w_1 = {{1}, {1}, {-1}, {1}}

Out[4]= {{1}, {1}, {-1}, {1}}

In[5]:= Eigen[B, w_1]

Out[5]= {{-15.25}}

In[6]:= w_2 = next[B, w_1]

Out[6]= {{0.153846}, {-0.0384615}, {1.}, {0.102564}}

In[7]:= Eigen[B, w_2]

Out[7]= {{-13.2966}}

In[8]:= w_3 = next[B, w_2]

Out[8]= {{-1.02456}, {1.}, {-1.78947}, {0.985965}}

In[9]:= Eigen[B, w_3]

Out[9]= {{-21.2589}}

In[10]:= w_4 = next[B, w_3]

Out[10]= {{0.49119}, {-0.678511}, {1.}, {0.21708}}

In[11]:= Eigen[B, w_4]

Out[11]= {{-26.9769}}

In[12]:= w_5 = next[B, w_4]

Out[12]= {{-0.50669}, {1.}, {-1.06454}, {0.08347}}

In[13]:= **Eigen[B, w_5]**

Out[13]=

{{-32.2882}}

In[14]:= **w_6 = next[B, w_5]**

Out[14]=

{{0.386752}, {-0.724116}, {1.}, {0.0892405}}

In[15]:= **Eigen[B, w_6]**

Out[15]=

{{-30.5133}}

In[16]:= **w_7 = next[B, w_6]**

Out[16]=

{{-0.545484}, {1.}, {-1.33851}, {0.0283977}}

In[17]:= **Eigen[B, w_7]**

Out[17]=

{{-31.087}}

In[18]:= **w_8 = next[B, w_7]**

Out[18]=

{{0.404173}, {-0.717134}, {1.}, {0.026642}}

In[19]:= **Eigen[B, w_8]**

Out[19]=

{{-30.5465}}

In[20]:= **B.w_8**

Out[20]=

{{-12.462}, {22.3225}, {-30.2209}, {-0.060704}}

In[21]:= **-30.5 w_8**

Out[21]=

{{-12.3273}, {21.8726}, {-30.5}, {-0.81258}}

In[22]:= **Eigenvalues[B]**

Out[22]=

{4 -30.8..., 4 16.4..., 4 -5.79... + 11.2... i, 4 -5.79... - 11.2... i}

Another Matrix

$$A = \begin{pmatrix} 1 & -44 & -88 \\ -5 & 55 & 113 \\ 1 & -24 & -48 \end{pmatrix}$$

In[23]:= **A = {{1, -44, -88}, {-5, 55, 113}, {1, -24, -48}}**

Out[23]=

{{1, -44, -88}, {-5, 55, 113}, {1, -24, -48}}

In[24]:=
```
w1 = {{-1}, {1}, {-1}}
```
Out[24]=
```
{{-1}, {1}, {-1}}
```
In[25]:=
```
Eigen[A, w1]
```
Out[25]=
```
{{-39.6667}}
```
In[26]:=
```
w2 = next[A, w1]
```
Out[26]=
```
{{1.}, {-1.23256}, {0.534884}}
```
In[27]:=
```
Eigen[A, w2]
```
Out[27]=
```
{{9.27106}}
```
In[28]:=
```
w3 = next[A, w2]
```
Out[28]=
```
{{1.}, {-1.51282}, {0.60114}}
```
In[29]:=
```
Eigen[A, w3]
```
Out[29]=
```
{{13.8136}}
```
In[30]:=
```
w4 = next[A, w3]
```
Out[30]=
```
{{1.}, {-1.38275}, {0.576452}}
```
In[31]:=
```
Eigen[A, w4]
```
Out[31]=
```
{{11.3651}}
```
In[32]:=
```
w5 = next[A, w4]
```
Out[32]=
```
{{1.}, {-1.43183}, {0.586356}}
```
In[33]:=
```
Eigen[A, w5]
```
Out[33]=
```
{{12.2805}}
```
In[34]:=
```
A.w5
```
Out[34]=
```
{{12.401}, {-17.4921}, {7.21871}}
```
In[35]:=
```
12.2805 w5
```
Out[35]=
```
{{12.2805}, {-17.5835}, {7.20075}}
```

Finding the smallest eigenvalue of A (Find the largest eigenvalue of A^{-1} and take the reciprocal)

In[36]:= **AA = Inverse[A]**

Out[36]=

$$\left\{\left\{-\frac{6}{5}, 0, \frac{11}{5}\right\}, \left\{\frac{127}{60}, -\frac{2}{3}, -\frac{109}{20}\right\}, \left\{-\frac{13}{12}, \frac{1}{3}, \frac{11}{4}\right\}\right\}$$

In[37]:= **Eigen[AA, w_1]**

Out[37]=

{{1.66667}}

In[38]:= **w_2 = next[AA, w_1]**

Out[38]=

{{-0.375}, {1.}, {-0.5}}

In[39]:= **Eigen[AA, w_2]**

Out[39]=

{{1.31311}}

In[40]:= **w_3 = next[AA, w_2]**

Out[40]=

{{-0.514003}, {1.}, {-0.502471}}

In[41]:= **Eigen[AA, w_3]**

Out[41]=

{{0.977146}}

In[42]:= **w_4 = next[AA, w_3]**

Out[42]=

{{-0.496665}, {1.}, {-0.499707}}

In[43]:= **Eigen[AA, w_4]**

Out[43]=

{{1.00691}}

Claim: Largest eigenvalue of A, in absolute value, is λ_1 with $|\lambda_1| \approx 12.2$ and the smallest is λ_3 with $|\lambda_3| \approx 1$.

In[44]:= **Eigenvalues[A]**

Out[44]=

{12, -5, 1}