Final Review

Final Review (1)

Example 1

Suppose that at the present 120,000 people live in a certain city and 30,000 live in its suburbs. The Regional Planning Commission determines that each year 10% of the city population moves to the suburbs and 5% of the suburban population moves to the city.

(a) Set up the transition matrix for this trend.
(b) Assuming that the total population remains constant, how many people will be living in the suburbs a year from now?
(c) In the long run, how many people will be living in the city?

(a)
$$\begin{array}{c|cc} & C & S \\ \hline C & 0.9 & 0.05 \\ S & 0.1 & 0.95 \end{array}$$

(b)
$$\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}\begin{bmatrix} 120,000 \\ 30,000 \end{bmatrix} = \begin{bmatrix} 108,000 + 1,500 \\ 12,000 + 28,500 \end{bmatrix} = \begin{bmatrix} 109,500 \\ 40,500 \end{bmatrix}$$

40,500 will be in the suburbs.

(c) $\lambda = 1$ $\quad A - I = \begin{bmatrix} -0.1 & 0.05 \\ 0.1 & -0.05 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} -0.1 & 0.05 \\ 0 & 0 \end{bmatrix} \xrightarrow{-10R_1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = \frac{1}{2}x_2$, $x_2 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow v = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \frac{1}{2} + 1 = \frac{3}{2} \quad w = \frac{2}{3}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$\frac{1}{3}(150,000) = 50,000$ will be in the city.

Final Review (2)

Example 2

$v = v_W + v_{W^\perp}$

Let $v = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and let $W = \text{Span}\left\{\overset{u}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}\right\}$. (a) Find v_W, the orthogonal projection of v onto W. (b) Compute $v_{W^\perp}$ and verify orthogonality. (c) Find the distance from v to W.

ⓐ $v_W = \dfrac{u \cdot v}{u \cdot u} \cdot u = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{12}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$

ⓑ $v_{W^\perp} = v - v_W = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$

Check $v_W \cdot v_{W^\perp} \overset{?}{=} 0$

$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = 0$ ✓

ⓒ $\text{dist}(v, W) = \|v_{W^\perp}\| = \sqrt{8} = 2\sqrt{2}$

Final Review (3)

Example 3

Let W be subspace spanned by the vectors $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Find $W^{\perp}$, the orthogonal complement of W in terms of its basis. Describe $W^{\perp}$ geometrically.

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad W^{\perp} = Nul(A^T)$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$x_1 = x_3$
$x_2 = -x_3$
$x_3 = x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$W^{\perp} = Span\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

This is a line in $\mathbb{R}^3$.

Final Review (4)

Example 4

$A^TAx = A^Tv \Rightarrow$

$v_W = A(A^TA)^{-1}A^T v$

Let W be the plane defined by the equation $x + 2y + z = 0$ and let $v = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$. (a) Find v_W, the orthogonal projection of v onto W. (b) Find the distance between v and W.

(a) $[1 \ 2 \ 1] \Rightarrow \begin{matrix} x_1 = -2x_2 - x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$W = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(A^TA)^{-1} = \left(\begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

$v_W = \frac{1}{6} \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 18 \\ -12 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

(b)

$v_{W^\perp} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$\|v_{W^\perp}\| = \sqrt{24} = 2\sqrt{6}$

Final Review (5)

Example 5

Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. (a) Find the eigenvalues of A. (b) Find the corresponding eigenspaces.

(c) Diagonalize A, that is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$. Find C^{-1} explicitly.

(a) $\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 =$

$12 - 7\lambda + \lambda^2 - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$

$\lambda_1 = 2, \quad \lambda_2 = 5.$

(b) $\lambda_1 = 2 \quad A - 2I = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = -\frac{1}{2}x_2 \\ x_2 = x_2 \end{matrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

$\text{Span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = 5 \quad A - 5I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $A = \underbrace{\begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{C^{-1}}$

$C^{-1} = -\frac{2}{3} \begin{bmatrix} 1 & -1 \\ -1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Final Review (6)

Example 6

$A^TAx = A^Tb$ and solve for x

Find the least squares best fit line to the following data points:

(1,2), (2,4), (3,5), (4,6)

$y = Mx + B$ $\quad$ $Mx + B = y$

$$\boxed{\begin{aligned} M\cdot 1 + B &= 2 \\ M\cdot 2 + B &= 4 \\ M\cdot 3 + B &= 5 \\ M\cdot 4 + B &= 6 \end{aligned}}$$

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$ $\quad$ $b = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 6 \end{bmatrix}$

$x = \begin{bmatrix} M \\ B \end{bmatrix}$

$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix} \frac{13}{10} \\ 1 \end{bmatrix}$ $\quad$ $\boxed{y = \frac{13}{10}x + 1}$

Final Review (7)

Example 7

Find $\det(A^3B^{-1})$ if $A = \begin{bmatrix} 1 & -3 & 4 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 3 \\ 2 & 2 \end{bmatrix}$

$\det(A) = 1 \cdot \det\begin{bmatrix} 0 & 5 & 0 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} =$

$= 1(-5)\det\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = 1(-5)(2) =$

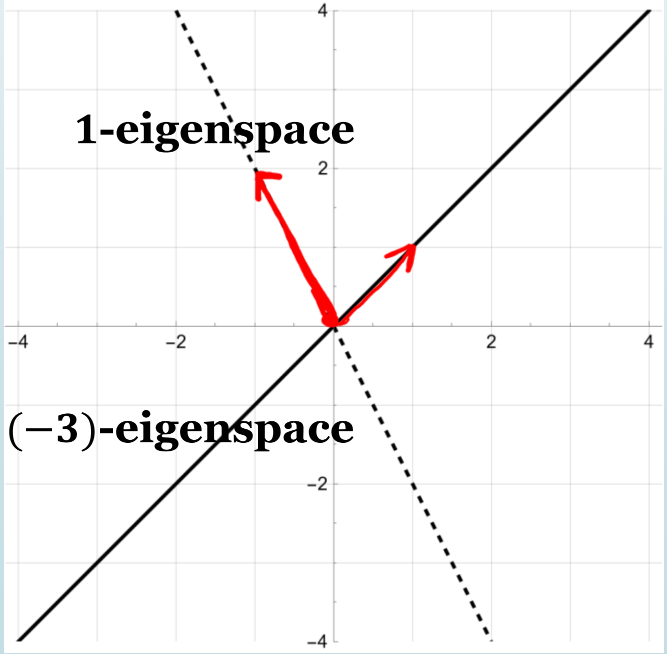
-10

$\det(B) = 10$

$\det(A^3B^{-1}) = \dfrac{(\det(A))^3}{\det(B)} = \dfrac{-1000}{10} = \boxed{-100}$

Example 8

Find the 2×2 matrix A whose eigenspaces are graphed on the right. (The 1-eigenspace is the dashed line, and the (-3)-eigenspace is the solid line)



$A = CDC^{-1}$

$\lambda_1 = 1$ $\quad v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2 = -3$ $\quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$

then compute the product!

Final Review (8)

Recipe 1: Compute the steady state vector. Let A be a positive stochastic matrix. Here is how to compute the steady-state vector of A.

1. Find any eigenvector v of A with eigenvalue 1 by solving $(A - I_n)v = 0$.
2. Divide v by the sum of the entries of v to obtain a vector w whose entries sum to 1.
3. This vector automatically has positive entries. It is the unique steady-state vector.

Recipes: Shortcuts for computing orthogonal complements. For any vectors $v_1, v_2, \ldots, v_m$, we have

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$

For any matrix A, we have

$$\mathrm{Row}(A)^\perp = \mathrm{Nul}(A) \qquad \mathrm{Nul}(A)^\perp = \mathrm{Row}(A)$$
$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T) \qquad \mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A).$$

Recipe: Diagonalization. Let A be an $n \times n$ matrix. To diagonalize A:

1. Find the eigenvalues of A using the characteristic polynomial.
2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.
3. If there are fewer than n total vectors in all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in the eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i.

Recipe: Computing determinants by row reducing. Let A be a square matrix. Suppose that you do some number of row operations on A to obtain a matrix B in row echelon form. Then

$$\det(A) = (-1)^r \cdot \frac{(\text{product of the diagonal entries of } B)}{(\text{product of scaling factors used})},$$

where r is the number of row swaps performed.

Final Review (9)

Invertible Matrix Theorem. *Let A be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the matrix transformation $T(x) = Ax$. The following statements are equivalent:*

1. *A is invertible.*
2. *A has n pivots.*
3. $\mathrm{Nul}(A) = \{0\}$.
4. *The columns of A are linearly independent.*
5. *The columns of A span $\mathbf{R}^n$.*
6. *$Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.*
7. *T is invertible.*
8. *T is one-to-one.*
9. *T is onto.*
10. $\det(A) \neq 0$.
11. *0 is not an eigenvalue of A.*

Recipe: Orthogonal projection onto a line. If $L = \mathrm{Span}\{u\}$ is a line, then

$$x_L = \frac{u \cdot x}{u \cdot u} u \quad \text{and} \quad x_{L^\perp} = x - x_L$$

for any vector x.

Corollary. *Let A be an $m \times n$ matrix with linearly independent columns and let $W = \mathrm{Col}(A)$. Then the $n \times n$ matrix $A^T A$ is invertible, and for all vectors x in $\mathbf{R}^m$, we have*

$$x_W = \underbrace{A(A^T A)^{-1} A^T}_{B} x.$$

standard matrix for orthogonal projections.

Theorem. *Let A be an $m \times n$ matrix and let b be a vector in $\mathbf{R}^m$. The following are equivalent:*

1. *$Ax = b$ has a unique least-squares solution.*
2. *The columns of A are linearly independent.*
3. *$A^T A$ is invertible.*

In this case, the least-squares solution is

$$\widehat{x} = (A^T A)^{-1} A^T b.$$