Upcoming Assignments and Assessments

- Homework Assignment #4 is due Friday 2/21

- Quiz #4 will be administered during this week's recitation

Subspaces, Basis and Dimension (16)

Basis Theorem: Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis of V.
- Any m vectors that span V form a basis for V.

In other words, if we already know that $\dim V = m$ and we're given a set of m vectors $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$ in V then we only need to check one of the following conditions:

1. $\mathcal{B}$ is linearly independent, or
2. $\mathcal{B}$ spans V,

in order to conclude that $\mathcal{B}$ is a basis for V.

Example 8

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis of V. Find a different basis for V.

$$V_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$$

and v_1 and v_2 are linearly independent

so, v_1 and v_2 form a basis for V.

The Rank Theorem

The Rank Theorem (1)

Example 1

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Nul}(A)$ and state its dimension.
(b) Find a basis for $\mathrm{Col}(A)$ and state its dimension.

null space = set of solution of $Ax=0$

Column space = set of b's such that $Ax=b$ is consistent = span of the column vectors of A.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ $\begin{matrix} x+z=0 \\ y+z=0 \end{matrix} \Longrightarrow \begin{matrix} x=-z \\ y=-z \\ z=z \end{matrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\ast \quad \ast$

span $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

basis $= \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

dim Nul$(A) = \underline{1}$

(b) pivot columns: 1 and 2

basis of Col$(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

dim col$(A) = \underline{2}$

The Rank Theorem (2)

Example 2

Let $B = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Nul}(B)$ and state its dimension.
(b) Find a basis for $\mathrm{Col}(B)$ and state its dimension.

(a) $$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \xrightarrow{R_3+R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3 + 2x_4$$
$$x_2 = -x_3 - 2x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

basis $= \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim \mathrm{Nul}(B) = \underline{\underline{2}}$.

(b) basis $= \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$

$\dim \mathrm{Col}(B) = \underline{\underline{2}}$

The Rank Theorem (3)

Example 3

Let $C = \begin{bmatrix} 1 & 0 & -3 & 1 \\ -1 & 2 & 1 & -5 \\ -2 & -4 & 10 & 7 \end{bmatrix}$.

(a) How many pivot columns does C have? What is the dimension of $\mathrm{Col}(C)$?

(b) How many free variables does the solution to the matrix equation $Cx = 0$ have? What is the dimension of $\mathrm{Nul}(C)$?

(a) $\begin{bmatrix} 1 & 0 & -3 & 1 \\ -1 & 2 & 1 & -5 \\ -2 & -4 & 10 & 7 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2+R_1} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 2 & -2 & -4 \\ 0 & -4 & 4 & 9 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & 4 & 9 \end{bmatrix}$

$\xrightarrow{R_3+4R_2} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ # pivot columns : 3 dim $\mathrm{Col}(C) = \underline{3}$

(b) # free variables : 1

dim $\mathrm{Nul}(C) = \underline{1}$

The Rank Theorem (4)

Definition:

- The **rank** of a matrix A, denoted by $\operatorname{rank}(A)$, is the dimension of the column space $\operatorname{Col}(A)$.
- The **nullity** of a matrix A, denoted by $\operatorname{nullity}(A)$, is the dimension of the null space $\operatorname{Nul}(A)$.

- $\operatorname{rank}(A) = \dim \operatorname{Col}(A) =$ the number of pivot columns

- $\operatorname{nullity}(A) = \dim \operatorname{Nul}(A) =$ the number of free variables = the number of non-pivot columns

Now, #(pivot columns of A) + #(non-pivot columns of A) = #(columns of A)

Theorem: If A is a matrix with n columns, then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n$$

So, for any consistent system of linear equations,

translate of the null space

(dim of column span) + (dim of solution set) = (number of variables)

The Rank Theorem (5)

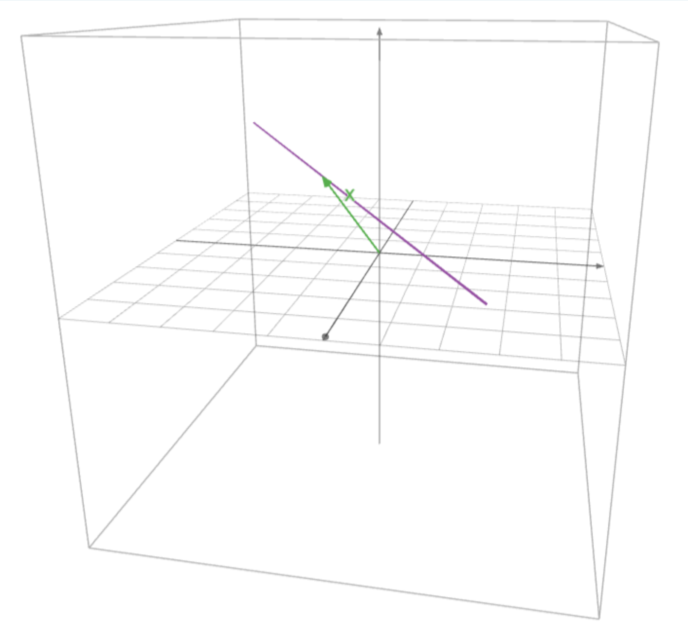
$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 4 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 6 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} \overset{*}{1} & \overset{*}{0} & 8 \\ 0 & 1 & 6 \end{bmatrix}$$

dim col(A) = 2

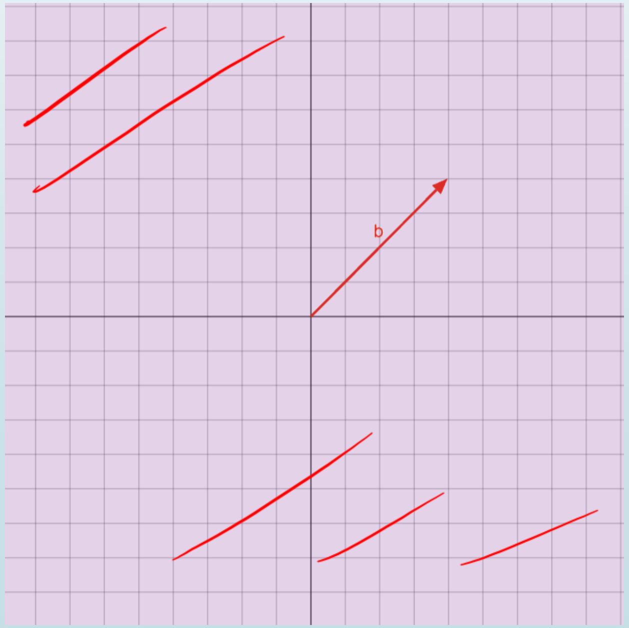
col(A) = $\mathbb{R}^2$

dim Nul(A) = $\underline{\underline{1}}$

The solution set of
$Ax = b$ for a fixed b



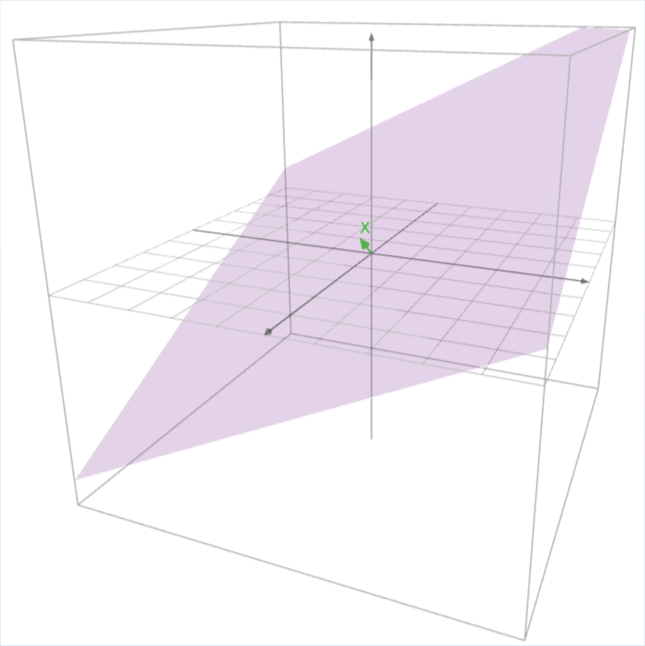
Col(A)
b can be any vector in $\mathbb{R}^2$



The Rank Theorem (6)

$$C = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$$

The solution set of
$Cx = b$ for a fixed b



$\mathrm{Col}(C)$
b must lie on the purple line