Determinants: Definition, Properties and Cofactor Expansions Part II

Determinants: Practice (1)

Example 1

Use the definition to calculate det $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 7 & -4 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 7 & -9 \end{bmatrix} \xrightarrow{R_3 - 7R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

$\det = 9$ $\quad$ $\det = -9$ $\quad$ $\det = -9$ $\quad$ $\det = -9$

$$\det = \frac{(-1)^k \text{(diagonal product)}}{\text{(product of scaling factors)}} = (-1)^1(-9) = 9$$

Determinants: Practice (2)

Example 2

Use the definition to calculate $\det \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 6 & 8 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 6 & 8 \end{bmatrix} \xrightarrow[2R_2]{2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 8 \end{bmatrix}$$

$$\xrightarrow{R_3 - 6R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\det = \frac{(-1)^2 (1 \cdot 1 \cdot 8)}{2 \cdot 2} = \boxed{2}$$

Determinants by Cofactor Expansion Continued

Determinants by Cofactor Expansion (1)

If A is a square matrix, then the **minor** of the entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} is called the **cofactor** of entry a_{ij}.

Example 1

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$.

To find M_{11}, we delete the first row and the first column of A and calculate the determinant of the remaining submatrix.

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{3} \\ \cancel{-3} & 2 & 1 \\ \cancel{2} & 4 & -4 \end{bmatrix}$$

$$M_{11} = \det\begin{bmatrix} 2 & 1 \\ 4 & -4 \end{bmatrix} = -12$$

And the corresponding cofactor is

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2(-12) = -12$$

Determinants by Cofactor Expansion (2)

Example 1 Let $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$. **(a)** Find M_{23} and C_{23}. **(b)** Find M_{22} and C_{22}.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$ $\quad M_{23} = \det\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = 4 \quad C_{23} = (-1)^{2+3}M_{23} = (-1)^5(4) = -4$

(b) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$ $\quad M_{22} = \det\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = -10 \quad C_{22} = (-1)^{2+2}M_{22} = (-1)^4(-10) = -10$

Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either + or − in accordance with the following "checkerboard" pattern

$$\begin{bmatrix} + & \boxed{-} & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

For example, $C_{11} = M_{11}$, $C_{12} = -M_{12}$, $C_{13} = M_{13}$
Thus, it is never really necessary to calculate $(-1)^{i+j}$ to calculate C_{ij} —you can simply compute the minor M_{ij} and then adjust the sign in accordance with the checkerboard pattern

Determinants by Cofactor Expansion (3)

$a_{11}a_{22} - a_{21}a_{12}$

Let's see how the determinant of a 2×2 matrix can be expressed in terms of cofactors. The checkerboard pattern is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the cofactors are $C_{11} = a_{22}$, $C_{12} = -a_{21}$, $C_{21} = -a_{12}$, $C_{22} = a_{11}$,

and the determinant may be expressed in the following four ways:

$\det(A) = \boxed{a_{11}C_{11} + a_{12}C_{12}} = a_{11}a_{22} - a_{12}a_{21}$ - <u>cofactor expansion along the first row</u>

$\det(A) = \boxed{a_{21}C_{21} + a_{22}C_{22}} = -a_{21}a_{12} + a_{22}a_{11}$ - <u>cofactor expansion along the second row</u>

$\det(A) = \boxed{a_{11}C_{11} + a_{21}C_{21}} = a_{11}a_{22} - a_{21}a_{12}$ - <u>cofactor expansion along the first column</u>

$\det(A) = \boxed{a_{12}C_{12} + a_{22}C_{22}} = -a_{12}a_{21} + a_{22}a_{11}$ - <u>cofactor expansion along the second column</u>

Each of above four equations is called a **cofactor expansion** of $\det(A)$. In each cofactor expansion the entries and cofactors all come from the same row or same column of A. For example, in the first equation the entries and cofactors all come from the first row of A, in the second they all come from the second row of A, in the third they all come from the first column of A, and in the fourth they all come from the second column of A.

Determinants by Cofactor Expansion (4)

Theorem: If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is the **determinant** of A, and the sums themselves are called **cofactor expansions** of $\det(A)$. That is,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}$$

(Cofactor expansion along the i^{th} row)

and

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj}$$

(Cofactor expansion along the j^{th} column)

Determinants by Cofactor Expansion (5)

Example 2

$$\begin{matrix}+&-&+\\-&+&-\\+&-&+\end{matrix}$$

Find the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & 4 \end{bmatrix}$ by **(a)** cofactor expansion along the first row, **(b)** cofactor expansion along the second column.

(a) $\det(A) = 1\left(\overbrace{\det\begin{bmatrix}2 & 1\\4 & 4\end{bmatrix}}^{4}\right) - 0\left(\det\begin{bmatrix}-3 & 1\\2 & 4\end{bmatrix}\right) + 3\left(\overbrace{\det\begin{bmatrix}-3 & 2\\2 & 4\end{bmatrix}}^{-16}\right) = 1(4) + 3(-16) = -44$

(b) $\det(A) = -0\left(\det\begin{bmatrix}-3 & 1\\2 & 4\end{bmatrix}\right) + 2\left(\overbrace{\det\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}}^{-2}\right) - 4\left(\overbrace{\det\begin{bmatrix}1 & 3\\-3 & 1\end{bmatrix}}^{10}\right) = 2(-2) - 4(10) = -44$

As a rule, the best strategy for cofactor expansion is to expand along a row or column with the most zeros.

Determinants by Cofactor Expansion (6)

Example 3

Find the determinant of the matrix $A = \begin{bmatrix} -2 & 3 & 0 & 4 \\ 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 1 \end{bmatrix}$

To find $\det(A)$ it will be easiest to use cofactor expansion along the third column, since it has the most zeros (be careful with the signs):

$$\det(A) = -2\left(\det\begin{bmatrix} -2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right)$$

To find the 3×3 determinant we will use cofactor expansion along the second row, since it has a zero in it (we could have also chosen to expand along the first column):

$$\det(A) = -2\left(\det\begin{bmatrix} -2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}\right) = -2\left(1\left(\det\begin{bmatrix} -2 & 4 \\ -2 & 1 \end{bmatrix}\right) - 1\left(\det\begin{bmatrix} -2 & 3 \\ -2 & 1 \end{bmatrix}\right)\right) = -2\big(1(6) - 1(4)\big) = -4$$

Determinants by Cofactor Expansion (7)

Example 4

Find the determinant of a general 4×4 lower triangular matrix $T = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$

For each part of the computation, we will use a cofactor expansion along the first row

$$\det(T) = a_{11}\left(\det\begin{bmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{bmatrix}\right) = a_{11}a_{22}\left(\det\begin{bmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{bmatrix}\right) = a_{11}a_{22}a_{33}(\det[a_{44}]) = a_{11}a_{22}a_{33}a_{44}$$

The method shown above can be generalized to obtain the following result.

Theorem: If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is,

$$\det(A) = a_{11}a_{22}a_{33} \ldots a_{nn}$$

Determinants by Cofactor Expansion (8)

A method for computing the determinant of a 3×3 matrix

The formula for the determinant of a 3×3 matrix looks too complicated to memorize outright. Fortunately, there is the following mnemonic device.

Recipe: Computing the Determinant of a 3×3 Matrix. To compute the determinant of a 3×3 matrix, first draw a larger matrix with the first two columns repeated on the right. Then add the products of the downward diagonals together, and subtract the products of the upward diagonals:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boxed{\begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{array}}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} \quad - \quad \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Example 5

Use the above device to find the determinant of $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{bmatrix}$

$$\begin{matrix} 2 & 5 & -3 & 2 & 5 \\ 1 & 3 & -2 & 1 & 3 \\ -1 & 6 & 4 & -1 & 6 \end{matrix} \qquad \begin{matrix} 2 & 5 & -3 & 2 & 5 \\ 1 & 3 & -2 & 1 & 3 \\ -1 & 6 & 4 & -1 & 6 \end{matrix}$$

$$2(3)(4) + 5(-2)(-1) + (-3)(1)(6)$$
$$-(-1)(3)(-3) - 6(-2)(2) - 4(1)(5)$$
$$= 24 + 10 - 18 - 9 + 24 - 20$$
$$= \boxed{}$$

Determinants by Cofactor Expansion (9)

It can be shown that the set of four defining properties of the determinant implies the following three, and that the three below imply the original four. So, this allows us to use the three below as the defining properties of the determinant.

1. **Multilinearity of the determinant:** Let i be a whole number from 1 to n, and fix $n-1$ vectors $v_1, v_2, v_3, \ldots v_{i-1}, v_{i+1}, \ldots v_n$ in $\mathbb{R}^n$. Then the transformation $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$T(x) = \det(v_1, v_2, v_3, \ldots v_{i-1}, x, v_{i+1}, \ldots v_n)$$

is linear.

2. **Two equal rows property**: If A has two identical rows, then $\det(A) = 0$.

3. **The determinant of the identity matrix** is 1.

In turn, one can show that the cofactor definition of the determinant satisfies the three alternative defining properties of the determinant above, implying that these definitions are equivalent.

Determinants by Cofactor Expansion (10)

Summary: methods for computing determinants. We have several ways of computing determinants:

1. *Special formulas for* 2×2 *and* 3×3 *matrices.*

   This is usually the best way to compute the determinant of a small matrix, except for a 3×3 matrix with several zero entries.

2. *Cofactor expansion.*

   This is usually most efficient when there is a row or column with several zero entries, or if the matrix has unknown entries.

3. *Row and column operations.*

   This is generally the fastest when presented with a large matrix which does not have a row or column with a lot of zeros in it.

4. *Any combination of the above.*

   Cofactor expansion is recursive, but one can compute the determinants of the minors using whatever method is most convenient. Or, you can perform row and column operations to clear some entries of a matrix before expanding cofactors.

Remember, *all methods for computing the determinant yield the same number.*

Properties of Determinants and The Inverse (11)

In a cofactor expansion we compute the determinant by multiplying the entries in a row or column by their cofactors and adding the resulting products. It turns out that if one multiplies the entries in any row by the corresponding cofactors from a different row, the sum of these products is always zero.

Let's see how this works in a 3×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Note that if a matrix contains two equal rows, then after a row replacement it will contain a row of zeros, which would make its determinant zero.

Consider the quantity $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$ that is formed by multiplying the entries in the first row by the cofactors of the corresponding entries in the third row and adding the resulting products. To see what the result of this sum is, construct a new matrix A' by replacing the third row of A with another copy of the first row. That is,

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Let C'_{31}, C'_{32} and C'_{33} be the cofactors of the entries in the third row of A'. Since the first two rows of A and A' are the same it follows that $C_{31} = C'_{31}$, $C_{32} = C'_{32}$ and $C_{33} = C'_{33}$.

Properties of Determinants and The Inverse (12)

Now since A' has two identical rows it follows from the equal rows property of the determinant that $\det(A') = 0$. But by cofactor expansion along the third row we get

equal

$$0 = \det(A') = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} = a_{11}C'_{31} + a_{12}C'_{32} + a_{13}C'_{33} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33}$$

which implies that $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$, and this what we were trying to illustrate.

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the **matrix of cofactors from** A. The transpose of this matrix is called the **adjoint** of A and is denoted by $\operatorname{adj}(A)$

Properties of Determinants and The Inverse (13)

Example 6

Given the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$ find $\operatorname{adj}(A)$

$C_{11} = 0 \qquad C_{12} = 1 \qquad C_{13} = 0$

$C_{21} = -2 \qquad C_{22} = 2 \qquad C_{23} = 2$

$C_{31} = -4 \qquad C_{32} = 5 \qquad C_{33} = 6$

So, the matrix of cofactors is $\begin{bmatrix} 0 & 1 & 0 \\ -2 & 2 & 2 \\ -4 & 5 & 6 \end{bmatrix}$ and $\operatorname{adj}(A) = \begin{bmatrix} 0 & -2 & -4 \\ 1 & 2 & 5 \\ 0 & 2 & 6 \end{bmatrix}$

Theorem: If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

Properties of Determinants and The Inverse (14)

Proof: We first show that $A\,\mathrm{adj}(A) = \det(A)\,I$

$$A\,\mathrm{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

The entry in the i^{th} row and j^{th} column of the product $A\,\mathrm{adj}(A)$ is $a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$

If $i = j$, then the above expression is the cofactor expansion of $\det(A)$ along the i^{th} row of A.

(For example, if $i = j = 1$, then the above expression becomes $a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$ which is the cofactor expansion of $\det(A)$ along the first row)

If $i \neq j$, then the a's and the cofactors come from different rows, and from an earlier discussion, the expression above must be equal to zero. It follows that

$$A\,\mathrm{adj}(A) = \begin{bmatrix} \det(A) & 0 & \ldots & 0 \\ 0 & \det(A) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \det(A) \end{bmatrix} = \det(A)\,I$$

Properties of Determinants and The Inverse (15)

Proof (continued):

$$A\,\mathrm{adj}(A) = \det(A)\, I$$

$$\frac{1}{\det(A)}\left(A\,\mathrm{adj}(A)\right) = \frac{1}{\cancel{\det(A)}}\cancel{\det(A)}\, I \qquad (\text{since } A \text{ is invertible } \det(A) \neq 0)$$

$$A\left(\frac{1}{\det(A)}\right)\mathrm{adj}(A) = I$$

$$\underbrace{A^{-1}A}_{I}\left(\frac{1}{\det(A)}\right)\mathrm{adj}(A) = A^{-1}I$$

$$\frac{1}{\det(A)}\mathrm{adj}(A) = A^{-1}$$

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$

Properties of Determinants and The Inverse (16)

Example 7

Given the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix}$ find A^{-1} using the adjoint.

In Example 6 we found that $\operatorname{adj}(A) = \begin{bmatrix} 0 & -2 & -4 \\ 1 & 2 & 5 \\ 0 & 2 & 6 \end{bmatrix}$.

$\det(A) = -2(-3+2) = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 1 & 2 & 5 \\ 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ \frac{1}{2} & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix}$$

Example 8

Determine whether the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$ is invertible, if so use the adjoint method to find the inverse.

matrix of cofactors $= \begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(A) = 12$ $\quad C_{11} = 6 \quad C_{12} = -48 \quad C_{13} = 29 \quad C_{21} = 0 \quad C_{22} = 12$

A is invertible $\quad C_{23} = -6 \quad C_{31} = 0 \quad C_{32} = 0 \quad C_{33} = 2$

Properties of Determinants and The Inverse (17)

Example 8 (continued)

Determine whether the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$ is invertible, if so use the adjoint method to find the inverse.

$\det(A) = 12$ matrix of cofactors $= \begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$ $\operatorname{adj}(A) = \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}$

A is invertible

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Example 9

Determine whether the matrix $A = \begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$ is invertible, if so use the adjoint method to find the inverse.

Note that columns 1 and 3 are proportional; more precisely, (column 3) = 2(column 1) so the matrix columns are not linearly independent which implies that A is not invertible.

Properties of Determinants and Cramer's Rule (18)

Theorem (Cramer's Rule): If $A\vec{x} = \vec{b}$ is a linear system of n equations with n unknowns such that $\det(A) \neq 0$, then the system has a unique solution and this solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)} \qquad \cdots \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained from A by replacing the entries in the j^{th} column of A by the entries in the matrix

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A_j = [v_1 v_2 \cdots v_{j-1}\ \vec{b}\ v_{j+1} \cdots v_n]$$

$$x_j = \frac{\det(A_j)}{\det(A)}$$

Proof: If $\det(A) \neq 0$, then A is invertible and $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

Then, by the adjoint formula for the inverse we have

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\vec{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

Properties of Determinants and Cramer's Rule (19)

Proof (continued):

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj} \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

$$\text{Now let } A_j = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j-1} & b_1 & a_{1j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j-1} & b_2 & a_{2j+1} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nj-1} & b_n & a_{nj+1} & \dots & a_{nn} \end{bmatrix}$$

Since A_j differs from A only in the j^{th} column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the j^{th} column of A. The cofactor expansion of $\det(A_j)$ along the j^{th} column is therefore

$$\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$$

Properties of Determinants and Cramer's Rule (20)

Proof (continued):

So, we have $x_j = \dfrac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)}$ and $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$

$$\Longrightarrow \quad x_j = \frac{\det(A_j)}{\det(A)}$$

Example 10

Use Cramer's Rule to solve the system

$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= 0 \\ 2x_1 \qquad\qquad &= -2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

$b = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -2 & 0 & 3 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 0 & -2 \\ 1 & 1 & 3 \end{bmatrix}$$

$\det(A) = -2(1(1) - 1(3)) = 4$

$\det(A_1) = 2(1(1) - 1(3)) = -4$

$\det(A_2) = -2(-2(1) - 3(0)) + 3(2(3) - (1)(-2)) = 4 + 24 = 28$

$\det(A_3) = -2(0(3) - 1(-2)) - 1(2(3) - (1)(-2)) = -4 - 8 = -12$

Properties of Determinants and Cramer's Rule (21)

Example 10 (continued)

$\det(A) = 4$

$\det(A_1) = -4$

$\det(A_2) = 28$

$\det(A_3) = -12$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-4}{4} = -1$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{28}{4} = 7$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-12}{4} = -3$$

Let's check

$$\begin{aligned} -2x_1 + x_2 + 3x_3 &= 0 \\ 2x_1 \qquad\qquad &= -2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

✓

Example 11

Use Cramer's Rule to solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 3x_3 &= 1 \\ 2x_1 - 6x_2 \qquad &= -4 \\ x_1 + \qquad\quad x_3 &= 2 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & -6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1\big(3(0) - (-6)(3)\big) + 1\big(2(-6) - 2(3)\big) = 18 - 18 = 0$$

Cramer's Rule does not apply