Determinants: Definition, Properties and Cofactor Expansions Part I

Determinants: Definition (1)

The three elementary row operations:

- Row replacement ($row + c(another\ row)$)
- Row scaling ($c(row)$)
- Row swap ($row \leftrightarrow another\ row$)

Definition: The ***determinant*** is a function $\det\colon \{square\ matrices\} \longrightarrow \mathbb{R}$ such that

- $\det(A)$ is not affected by row replacements
- Row scaling by c multiplies $\det(A)$ by c
- A row swap multiplies $\det(A)$ by -1
- $\det(I) = 1$

$$\begin{bmatrix}3 & 6\\ 1 & 3\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}1 & 3\\ 3 & 6\end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix}1 & 3\\ 0 & -3\end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix} \xrightarrow{R_1-3R_3} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A \xleftarrow{R_2\leftrightarrow R_1} B \xleftarrow{R_2+3R_1} C \xleftarrow{-3R_2} D \xleftarrow{R_1+3R_3} I$$

$\det(A) = 3 \qquad \det(B) = -3 \qquad \det(C) = -3 \qquad \det(D) = 1 \qquad \det(I) = 1$

Determinants: Properties (2)

What if there is a row of zeros?

If A has a row of zeros, then $\det(A) = 0$

$$\begin{array}{ccc} A & & B \\ \begin{bmatrix} 3 & 2 & 5 \\ 0 & 0 & 0 \\ 1 & 7 & 4 \end{bmatrix} & \xrightarrow{-R_2} & \begin{bmatrix} 3 & 2 & 5 \\ 0 & 0 & 0 \\ 1 & 7 & 4 \end{bmatrix} \end{array}$$

$\det(A) = -\det(B) = -\det(A)$

multiply the zero row by -1 which results in $\underline{\det(A)} = \underline{-\det(A)}$ which means that $\det(A) = 0$

Definition: A matrix is ***upper triangular*** if all its nonzero entries lie on or above the main diagonal, ***lower triangular*** if all its nonzero entries lie on or below the main diagonal. A matrix is ***diagonal*** if it's both upper and lower triangular.

upper triangular

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

diagonal

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

Determinants: Properties (3)

Proposition: If A is upper triangular, lower triang~~le~~ular or diagonal then $\det(A)$ is equal to the product of the entries along its main diagonal.

Proof: Assume A is upper triangular and one of its diagonal entries is zero. All the entries above the nonzero diagonal entries can be cleared using row operations. This will result in a row of zeros, which in turn results in a zero determinant.

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 3 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Now assume A is upper triangular and all the diagonal entries are nonzero. Then A can be reduced to the identity matrix by first scaling by the reciprocals of the diagonal entries and then using row replacements.

$$\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} \xrightarrow{\frac{1}{a}R_1, \frac{1}{b}R_2, \frac{1}{c}R_3} \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row replacements}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det = \underline{\underline{abc}}$ $\qquad$ $\det = 1$ $\qquad$ $\det = 1$

Determinants: Properties (4)

Theorem: Let A be a square matrix. Let's say that after some number of row operations it is changed to a row echelon form B then

$$\det(A) = (-1)^r \frac{\overbrace{(\text{product of the entries on the diagonal of } B)}^{\det(B)}}{(\text{product of scaling factors})}$$

where $r = \#$ of row swaps

Example 1

Calculate $\det\begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix}$

# of row swaps = 1
scaling factors = $\frac{1}{2}$

diagonal product = 20

$$\begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 4 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -8 & 4 \end{bmatrix} \xrightarrow{R_3 + 8R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 20 \end{bmatrix}$$

$$\det\begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & -1 \\ 3 & -2 & 1 \end{bmatrix} = (-1)^1 \left(\frac{20}{\frac{1}{2}}\right) = -40$$

Determinants: Properties (5)

The Existence Theorem: There exists one and only one function from the set of square matrices to the real numbers that satisfies the four defining properties of the determinant.

The determinant is the same independent of how you get to the row echelon form.

Example 2

Calculate $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $a = 0$ then $\det\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = -\det\begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = -bc$

If $a \neq 0$ then $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\frac{1}{a}R_1} = a\det\begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{R_2 - cR_1} = a\det\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - c\left(\frac{b}{a}\right) \end{bmatrix} = a\left(d - \frac{bc}{a}\right) = ad - bc$

In both cases $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

Determinants: Properties (6)

Example 3

Calculate $\det\begin{bmatrix}0 & -7 & -4\\ 2 & 4 & 6\\ 3 & -2 & 1\end{bmatrix} = \det(A)$

$$\begin{bmatrix}0 & -7 & -4\\ 2 & 4 & 6\\ 3 & -2 & 1\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}2 & 4 & 6\\ 0 & -7 & -4\\ 3 & -2 & 1\end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix}1 & 2 & 3\\ 0 & -7 & -4\\ 3 & -2 & 1\end{bmatrix} \xrightarrow{R_3 - 3R_1}$$

$$\begin{bmatrix}1 & 2 & 3\\ 0 & -7 & -4\\ 0 & -8 & -8\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}1 & 2 & 3\\ 0 & -8 & -8\\ 0 & -7 & -4\end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & -7 & -4\end{bmatrix} \xrightarrow{R_3 + 7R_2} \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 0 & 0 & 3\end{bmatrix}$$

# of row swaps $= 2$

scaling factors $= \frac{1}{2}\left(-\frac{1}{8}\right) = -\frac{1}{16}$ $\quad \det(A) = (-1)^2 \dfrac{3}{-\frac{1}{16}} = -48$

diagonal product $= 3$

Determinants: Properties (7)

Example 4

Calculate $\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{bmatrix}$

Determinants: Properties (8)

Example 5

Calculate $\det \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

Determinants, Invertibility and Multiplicativity (9)

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof: If A is invertible its reduced row echelon form is the identity matrix which has determinant equal to 1. If A is not invertible then its reduced row echelon form has a row of zeros in which case the determinant is zero. Row operations do not change whether the determinant is zero.

Theorem: Let A and B be two $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$

Proof: If B is not invertible then AB is also not invertible. It follows that $\det(B) = 0$ and $\det(AB) = 0$ which shows that in this case $\det(AB) = \det(A)\det(B)$ (both sides are 0).

(The hard part) If B is invertible, define a "new" function

$$f(A) = \frac{\det(AB)}{\det(B)}$$

Show that it has the <u>same four defining properties</u> as the determinant which means that it must be the determinant, that is $f(A) = \det(A)$. It follows that

$$\det(A) = \frac{\det(AB)}{\det(B)} \quad \Longrightarrow \quad \det(AB) = \det(A)\det(B)$$

Determinants: More Properties (10)

Corollary 1: If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof: $I = AA^{-1} \Longrightarrow 1 = \det(AA^{-1}) = \det(A)\det(A^{-1}) \quad \Longrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

Corollary 2: $\quad \det(A^n) = \det(A)^n$

Example 6

Calculate $\det(A^8)$ where $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ $\qquad \det(A) = 6 - 4 = 2$

$\det(A^8) = (\det(A))^8 = 2^8 = 256$

Corollary 3: Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices. Then the product $A_1A_2A_3 \cdots A_k$ is invertible if and only if each A_i is invertible.

$$\det(A_1A_2A_3 \cdots A_k) = \det(A_1)\det(A_2)\det(A_3)\cdots\det(A_k)$$

If each A_i is invertible then $\det(A_i) \neq 0$ for all i so RHS above is not equal to zero so neither is the LHS which implies that $A_1A_2A_3 \cdots A_k$ is invertible. Trace backwards to prove the converse.

Determinants: More Properties (11)

Definition: The transpose of A is a matrix obtained by interchanging the rows of A with columns of A, denoted by A^T.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \xrightarrow{\text{transpose}} A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Theorem: Let A be a square matrix then $\det(A^T) = \det(A)$

Theorem: The determinant satisfies the following properties with respect to column operations

- Column replacement does not affect the determinant
- Column swaps multiply the determinant by -1
- Scaling a column by a factor of c multiplies the determinant by c

Theorem: If A has a zero column, then $\det(A) = 0$

Determinants: Summary of Properties (12)

Summary: Magical Properties of the Determinant.

1. There is one and only one function $\det\colon \{n \times n \text{ matrices}\} \to \mathbf{R}$ satisfying the four defining properties.

2. The determinant of an upper-triangular or lower-triangular matrix is the product of the diagonal entries.

3. A square matrix is invertible if and only if $\det(A) \neq 0$; in this case,

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

4. If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

5. For any square matrix A, we have

$$\det(A^T) = \det(A).$$

6. The determinant can be computed by performing row and/or column operations.

Determinants by Cofactor Expansion

Determinants by Cofactor Expansion (1)

Recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant of A, denoted by $\det(A)$ or by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as the quantity $ad - bc$.

Furthermore, A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \begin{bmatrix} \frac{d}{\det(A)} & -\frac{b}{\det(A)} \\ -\frac{c}{\det(A)} & \frac{a}{\det(A)} \end{bmatrix}$

Example 1

Find the determinant of the given matrix and use the formula for the inverse of a 2×2 matrix, to find the inverse, if one exists.

(a) $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ **(b)** $B = \begin{bmatrix} 22 & -3 \\ -4 & 1 \end{bmatrix}$ **(c)** $C = \begin{bmatrix} -15 & 5 \\ 12 & -4 \end{bmatrix}$

Ultimately, we would like to find a similar formula for square matrices of any order n.

Determinants by Cofactor Expansion (2)

To extend the definition of the determinant it is more useful to use subscripted entries, so we'll use the following notation when working with 2×2 matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

We define the determinant of a 1×1 matrix $A = [a_{11}]$ to be $\det(A) = a_{11}$

If A is a square matrix, then the **minor** of the entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} is called the **cofactor** of entry a_{ij}.

Example 2

Let $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & 4 \end{bmatrix}$.

To find M_{11}, we delete the first row and the first column of A and calculate the determinant of the remaining submatrix.

$$\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$$

$$M_{11} = \det \begin{bmatrix} 2 & 1 \\ 4 & -4 \end{bmatrix} = \cancel{4}$$ fix

And the corresponding cofactor is

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2(4) = 4$$

Determinants by Cofactor Expansion (3)

Example 2 Let $A = \begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$. **(a)** Find M_{23} and C_{23}. **(b)** Find M_{22} and C_{22}.

(a) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$ $\quad M_{23} = \det\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = 4 \qquad C_{23} = (-1)^{2+3}M_{23} = (-1)^5(4) = -4$

(b) $\begin{bmatrix} 1 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 4 & -4 \end{bmatrix}$ $\quad M_{22} = \det\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = -10 \qquad C_{22} = (-1)^{2+2}M_{22} = (-1)^4(-10) = -10$

Note that a minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and that the relating sign $(-1)^{i+j}$ is either + or − in accordance with the following "checkerboard" pattern

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

For example, $C_{11} = M_{11}$, $C_{12} = -M_{12}$, $C_{13} = M_{13}$
Thus, it is never really necessary to calculate $(-1)^{i+j}$ to calculate C_{ij} —you can simply compute the minor M_{ij} and then adjust the sign in accordance with the checkerboard pattern