The Dot Product, Norm and Distance, Orthogonal Vectors and Orthogonal Complements

Vectors: Alternative Notations (1)

Comma-delimited form

$$\vec{u} = (u_1, u_2, u_3, \dots, u_n)$$

Row-matrix form

$$\vec{u} = [u_1 \;\; u_2 \;\; u_3 \;\; \dots \;\; u_n]$$

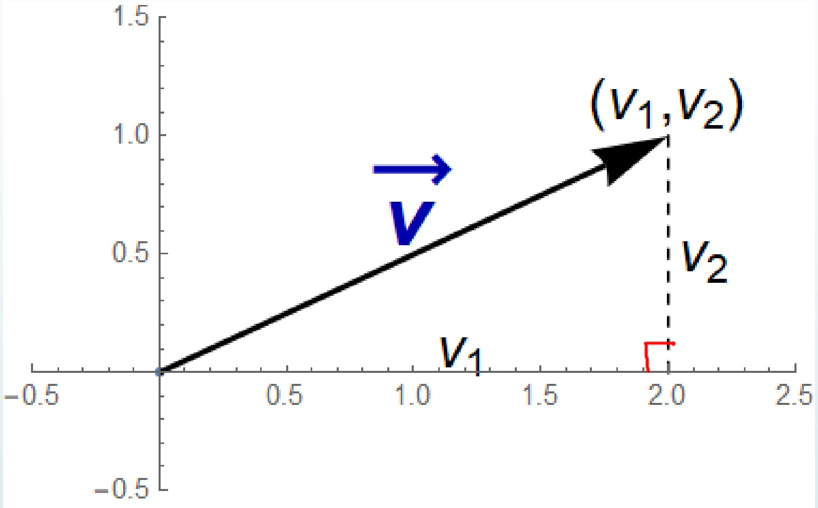
Column-matrix form

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

For most of this lesson, we will use the **comma-delimited form**, although we will come back to the **column-matrix form** at the end. You should be able to work with both depending on the context.

Vectors: Norm (2)

$||\vec{u}||$ = ***norm*** of $\vec{u}$ = ***length*** of $\vec{u}$ = ***magnitude*** of $\vec{u}$



$$||\vec{v}||^2 = {v_1}^2 + {v_2}^2$$

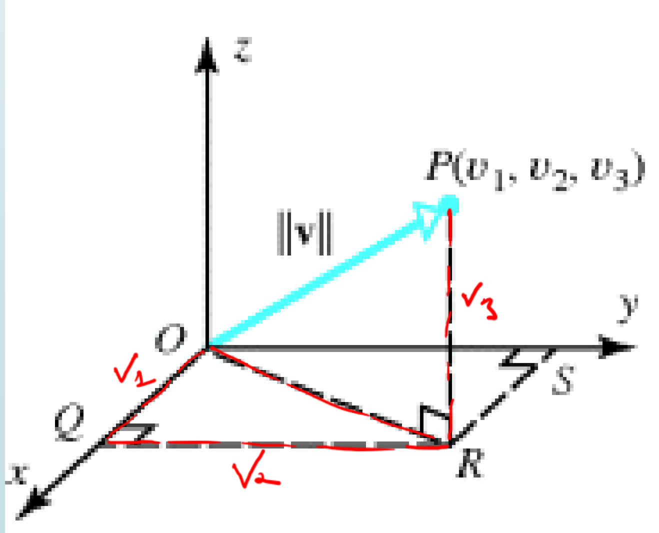
$$||\vec{v}|| = \sqrt{{v_1}^2 + {v_2}^2}$$

$$||\vec{v}||^2 = \underbrace{(OR)^2}+(RP)^2=$$

$$(OQ)^2+(QR)^2+(RP)^2=$$

$${v_1}^2 + {v_2}^2 + {v_3}^2$$

$$||\vec{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$



Vectors: Norm (3)

If $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$ is a vector in R^n , then the ***norm*** of $\vec{v}$ (also called the ***length*** of $\vec{v}$ or the ***magnitude*** of $\vec{v}$) is denoted by $||\vec{v}||$, and is defined by the formula

$$||\vec{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2 + \cdots + {v_n}^2}$$

Example 1

Find the norm of the given vector

(a) $\vec{v} = (-2,1,-3)$

$$\left|\left|\vec{v}\right|\right| = \sqrt{(-2)^2 + 1^2 + (-3)^2} = \sqrt{14}$$

(b) $\vec{w} = (3,-6,0,2)$

$$\left|\left|\vec{w}\right|\right| = \sqrt{3^2 + (-6)^2 + 0^2 + 2^2} = \sqrt{49} = 7$$

Vectors: Norm, Unit Vectors and Direction (4)

Theorem: Let $\vec{v}$ be a vector in R^n and k be any scalar. Then

(a) $||\vec{v}|| \geq 0$

(b) $||\vec{v}|| = 0$ if and only if $\vec{v} = \vec{0}$

(c) $||k\vec{v}|| = |k|\ ||\vec{v}||$

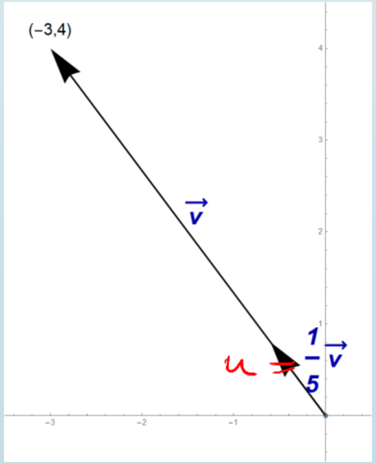
A vector of norm 1 is called a ***unit vector***. Such vectors are useful for specifying a direction when length is not relevant to the problem at hand. We can obtain a unit vector in a desired direction by choosing any *nonzero* vector $\vec{v}$ in that direction and multiplying $\vec{v}$ by the reciprocal of its length.

Example 2

Find the unit vector, call it $\vec{u}$, that has the same direction as the vector $\vec{v} = (-3,4)$.

$||\vec{v}|| = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

$\vec{u} = \frac{1}{5}\vec{v} = \frac{1}{5}(-3,4) = (-\frac{3}{5},\frac{4}{5})$



Vectors: Norm, Unit Vectors and Direction (5)

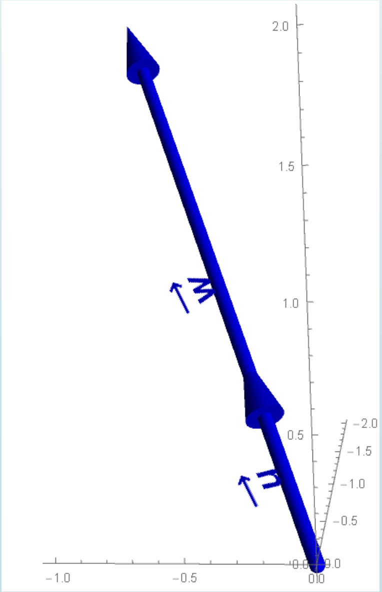
More generally, if $\vec{v}$ is any nonzero vector in R^n, then the unit vector $\vec{u}$ in the direction of $\vec{v}$ is given by

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$$

The process of multiplying a nonzero vector by the reciprocal of its length to obtain a unit vector is called ***normalizing*** $\vec{v}$.

Example 3

Find the unit vector, call it $\vec{u}$, that has the same direction as the vector $\vec{w} = (-2, -1, 2)$.

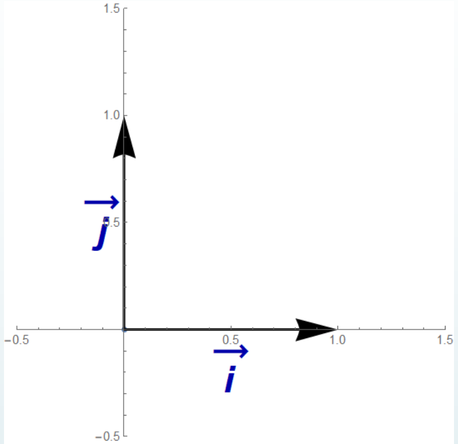


$$||\vec{w}|| = \sqrt{(-2)^2+(-1)^2+2^2} = \sqrt{9} = 3$$

$$\vec{u} = \frac{1}{3}\vec{w} = \frac{1}{3}(-2, -1, 2) = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

Vectors: Standard Unit Vectors (6)

The unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors.*** (standard coordinate vectors)



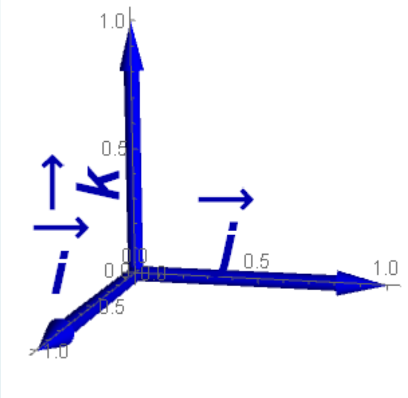
In R^2 they are denoted by

$$\vec{\imath} = (1,0) \text{ and } \vec{\jmath} = (0,1)$$

(e_1) (e_2)

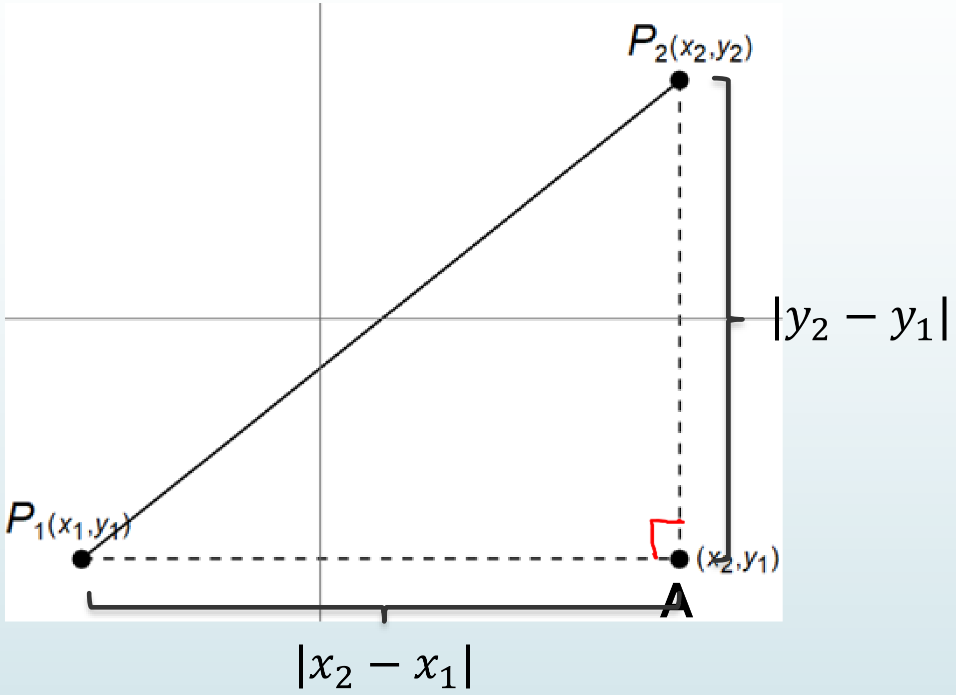
In R^3 they are denoted by

$\vec{\imath} = (1,0,0), \vec{\jmath} = (0,1,0)$ and $\vec{k} = (0,0,1)$ or by e_1, e_2 and e_3



Vectors: Distance in R^n (7)

In R^2



$$|P_1P_2|^2 = |P_2A|^2 + |P_1A|^2$$

$$|P_1P_2|^2 = |y_2 - y_1|^2 + |x_2 - x_1|^2$$

$$|P_1P_2|^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2$$

$$|P_1P_2| = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Using vector notation, the distance d between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

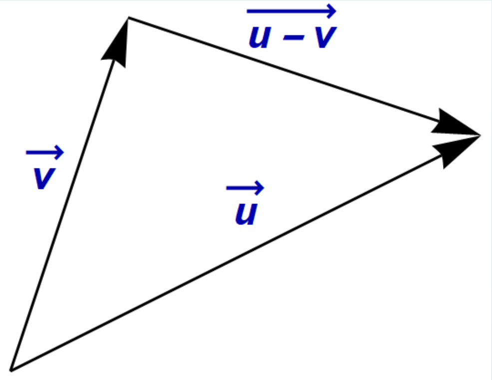
$$d = \left\|\overrightarrow{P_1P_2}\right\| = \|(x_2 - x_1, y_2 - y_1)\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Vectors: Distance in R^n (8)

In R^3, the distance d between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

$$d = \left|\left|\overrightarrow{P_1P_2}\right|\right| = ||(x_2 - x_1, y_2 - y_1, z_2 - z_1)|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Given two vectors $\vec{u} = (u_1, u_2, u_3, \dots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \dots, v_n)$ in R^n, by the distance between $\vec{u}$ and $\vec{v}$, we will mean the <u>distance between their heads </u>(their terminal points) when the vectors are in standard position. (Recall that the vector joining the heads of $\vec{u}$ and $\vec{v}$, when both are in standard position, is the vector $\overrightarrow{u - v}$).



We denote this ***distance*** by $d(\vec{u}, \vec{v})$ and define it to be

$$d(\vec{u}, \vec{v}) = ||\overrightarrow{u - v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

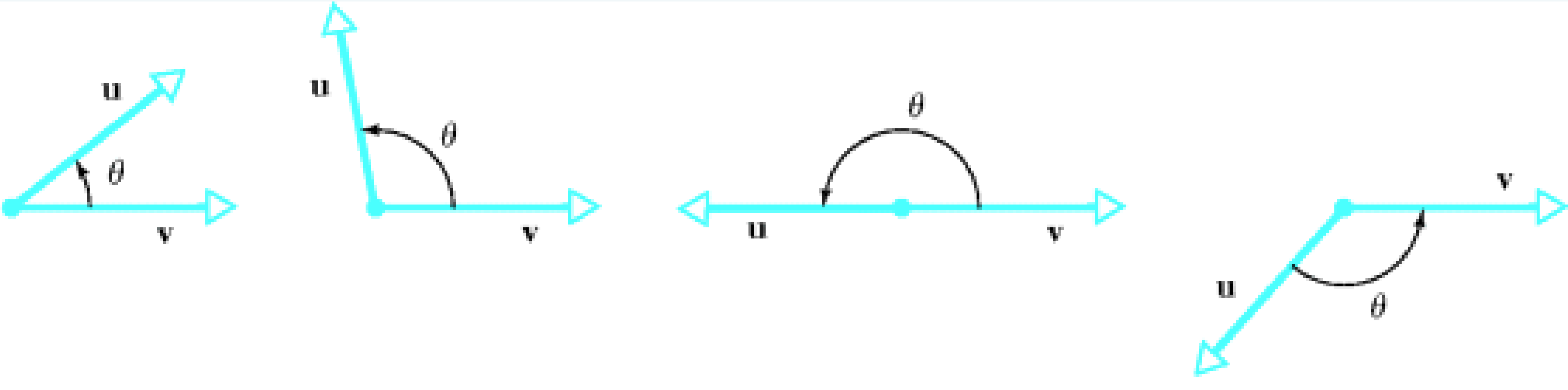
Example 4

Find the distance between $\vec{u} = (2, -1,3,7)$ and $\vec{v} = (1, -1, -1, -1)$

$$d(\vec{u}, \vec{v}) = ||\overrightarrow{u - v}|| = \sqrt{(2 - 1)^2 + (-1 - (-1))^2 + (3 - (-1))^2 + (7 - (-1))^2} = \sqrt{1 + 0 + 16 + 64} = \sqrt{81} = 9$$

Vectors: The Dot Product (9)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in R^2or R^3 that have been positioned so that their initial points coincide. We define the ***angle between*** $\vec{u}$ ***and*** $\vec{v}$ to be the angle θ determined by $\vec{u}$ and $\vec{v}$ that satisfies the inequalities $0 \leq \theta \leq \pi$.



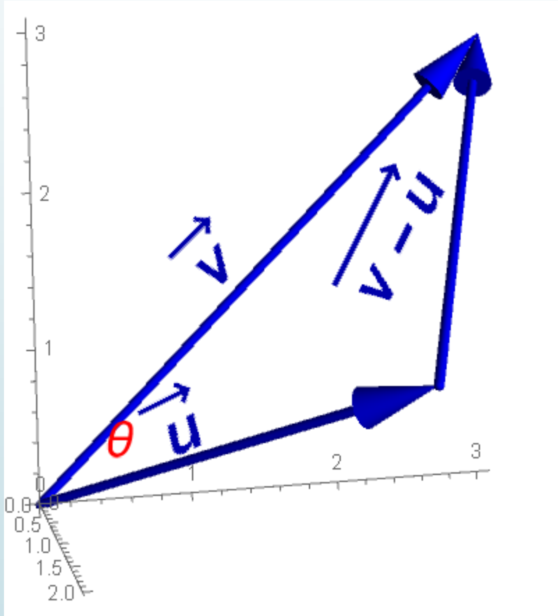
The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$.

Vectors: The Dot Product (10)

The concept of the ***dot product*** of two vectors is closely associated with the idea of the angle between two vectors. So, let's begin by trying to find the angle between two vectors in R^3.

Consider the triangle in R^3 spanned by two vectors $\vec{u}$ and $\vec{v}$, that is, the triangle whose sides are the given vectors and the vector $\overrightarrow{v-u}$ as seen below. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

$c^2 = a^2 + b^2 - 2ab\cos\theta$



By the Law of Cosines, we have $||\overrightarrow{v-u}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||||\vec{v}||\cos\theta$

$$\cos\theta = \frac{||\overrightarrow{v-u}||^2 - ||\vec{u}||^2 - ||\vec{v}||^2}{-2||\vec{u}||||\vec{v}||} =$$

$$= \frac{(v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2-||\vec{u}||^2-||\vec{v}||^2}{-2||\vec{u}||||\vec{v}||}$$

$$= \frac{v_1{}^2 - 2u_1v_1 + \boxed{u_1{}^2} + v_2{}^2 - 2u_2v_2 + \boxed{u_2{}^2} + v_3{}^2 - 2u_3v_3 + \boxed{u_3{}^2} - ||\vec{u}||^2 - ||\vec{v}||^2}{-2||\vec{u}||||\vec{v}||}$$

$$= \frac{(v_1{}^2+v_2{}^2+v_3{}^2)+(u_1{}^2+u_2{}^2+u_3{}^2)-2(u_1v_1+u_2v_2+u_3v_3)-||\vec{u}||^2-||\vec{v}||^2}{-2||\vec{u}||||\vec{v}||}$$

Vectors: The Dot Product (11)

Continuing from the previous slide:

$$\cos\theta = \frac{(v_1{}^2 + v_2{}^2 + v_3{}^2) + (u_1{}^2 + u_2{}^2 + u_3{}^2) - 2(u_1 v_1 + u_2 v_2 + u_3 v_3) - ||\vec{u}||^2 - ||\vec{v}||^2}{-2||\vec{u}||||\vec{v}||} =$$

$$= \frac{\cancel{||\vec{v}||^2} + \cancel{||\vec{u}||^2} - 2(u_1 v_1 + u_2 v_2 + u_3 v_3) \cancel{- ||\vec{u}||^2} \cancel{- ||\vec{v}||^2}}{-2||\vec{u}||||\vec{v}||} = \frac{-2(u_1 v_1 + u_2 v_2 + u_3 v_3)}{-2||\vec{u}||||\vec{v}||} =$$

$$\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{||\vec{u}||||\vec{v}||}$$

Note that the denominator is simply the product of the norms of $\vec{u}$ and $\vec{v}$. The quantity in the numerator is called the ***dot product*** of $\vec{u}$ and $\vec{v}$ and is denoted by " • ". In particular, given any two vectors in R^3, $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, we define the dot product $\vec{u} \cdot \vec{v}$ by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

From the above discussion we obtain the following identities:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} \qquad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||}\right) \qquad \vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$$

Vectors: The Dot Product (12)

Example 5

Find the angle between the vectors $\vec{u} = (2,2,1)$ and $\vec{v} = (0,3,3)$.

$\vec{u} \cdot \vec{v} = 2(0) + 2(3) + 1(3) = 9$ $\quad ||\vec{u}|| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$ $\quad ||\vec{v}|| = \sqrt{0^2 + 3^2 + 3^2} = \sqrt{18}$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}\right) = \cos^{-1}\left(\frac{9}{3\sqrt{18}}\right) = \cos^{-1}\left(\frac{\cancel{9}}{\cancel{3}(\cancel{3}\sqrt{2})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = 45°$$

Example 6

Find the angle between the vectors $\vec{a} = (-1,0,3)$ and $\vec{b} = (0,4,0)$.

$\vec{a} \cdot \vec{b} = 0$ $\quad ||\vec{a}|| = \sqrt{10}$ $\quad \left|\left|\vec{b}\right|\right| = 4$

a and b are perpendicular (orthogonal) $a \perp b$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}\right) = \cos^{-1}\left(\frac{0}{4\sqrt{10}}\right) = \cos^{-1}(0) = \frac{\pi}{2} = 90°$$

Vectors: The Dot Product (13)

Note that from the fact that $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}$ we have the following categorization

$$\vec{u}\cdot\vec{v} > 0 \quad \Longrightarrow \quad \cos\theta > 0 \Longrightarrow \quad \theta \text{ is acute}$$

$$\vec{u}\cdot\vec{v} < 0 \quad \Longrightarrow \quad \cos\theta < 0 \Longrightarrow \quad \theta \text{ is obtuse}$$

$$\vec{u}\cdot\vec{v} = 0 \quad \Longrightarrow \quad \cos\theta = 0 \Longrightarrow \quad \theta = \frac{\pi}{2}$$

The definition of the dot product derived earlier can be generalized to R^n, where $n > 3$.

If $\vec{u} = (u_1, u_2, u_3, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, v_3, \ldots, v_n)$ are vectors in R^n then the ***dot product*** (also called the ***Euclidean inner product***) is denoted $\vec{u}\cdot\vec{v}$ and is defined by

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

Similarly, the formula $\theta = \cos^{-1}(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||})$ is also valid in higher dimensions; as a matter of fact, it may be used as the *definition of the angle* between any two n-dimensional vectors.

Vectors: The Dot Product (14)

Example 7

(a) Find the angle between the vectors $\vec{u} = (1,2,0,2)$ and $\vec{v} = (-3,1,1,5)$ in R^4

$u \cdot v = -3 + 2 + 0 + 10 = 9$ $\quad \|u\| = \sqrt{1+4+0+4} = \sqrt{9} = 3$

$\|v\| = \sqrt{9+1+1+25} = \sqrt{36} = 6$

$\theta = \cos^{-1}\left(\frac{9}{3 \cdot 6}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

(b) Find the angle between the vectors $\vec{a} = (1,2,-2,4,-3)$ and $\vec{b} = (-1,3,2,2,3)$ in R^5

$a \cdot b = -1 + 6 - 4 + 8 - 9 = 0 \implies \theta = \frac{\pi}{2}$

a and b are orthogonal

Vectors: Algebraic Properties of the Dot Product (15)

In the special case where $\vec{u} = \vec{v}$, we obtain the following

$$\vec{v} \cdot \vec{v} = {v_1}^2 + {v_2}^2 + {v_3}^2 + \cdots + {v_n}^2 = ||\vec{v}||^2 \quad \Rightarrow \quad ||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$$

Theorem: If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in R^n, and if k is a scalar, then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(c) $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v}$

(d) $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$

Vectors: Algebraic Properties of the Dot Product (16)

Theorem: If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in R^n, and if k is a scalar, then

(a) $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$

(b) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(c) $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$

(d) $(\vec{u} - \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w}$

(e) $k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$

Vectors: Algebraic Properties of the Dot Product (17)

Example 8

Let $\vec{u}$ and $\vec{v}$ be two vectors in R^n such that $||\vec{u}|| = \sqrt{2}$, $||\vec{v}|| = 5$ and $\vec{u} \cdot \vec{v} = 1$. Use properties of the dot product to find $(2\vec{u} - \vec{v}) \cdot (4\vec{u} + 5\vec{v})$.

$$(2\vec{u} - \vec{v}) \cdot (4\vec{u} + 5\vec{v}) = (2\vec{u}) \cdot (4\vec{u} + 5\vec{v}) - \vec{v} \cdot (4\vec{u} + 5\vec{v}) =$$

$$(2\vec{u}) \cdot (4\vec{u}) + (2\vec{u}) \cdot (5\vec{v}) - \vec{v} \cdot (4\vec{u}) - \vec{v} \cdot (5\vec{v}) =$$

$$8\vec{u} \cdot \vec{u} + 10(\vec{u} \cdot \vec{v}) - 4(\vec{v} \cdot \vec{u}) - 5\vec{v} \cdot \vec{v} =$$

$$8||\vec{u}||^2 + 10(\vec{u} \cdot \vec{v}) - 4(\vec{u} \cdot \vec{v}) - 5||\vec{v}||^2 =$$

$$8||\vec{u}||^2 + 6(\vec{u} \cdot \vec{v}) - 5||\vec{v}||^2 =$$

$$8(2) + 6(1) - 5(25) = 16 + 6 - 125 = -103$$

Vectors: Orthogonal Vectors (18)

As we just saw, given two vectors u and v, u and v are perpendicular, or **orthogonal**, if $\vec{u} \cdot \vec{v} = 0$

Since $0 \cdot v = 0$ for any v, the zero vector is orthogonal to all vectors.

Example 9

Find all vectors orthogonal to $u = (1,1,1)$

$$(x, y, z) \cdot (1,1,1) = x + y + z = 0$$

$$\begin{aligned} x &= -y - z \\ y &= y \\ z &= z \end{aligned} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, the answer is the *plane* given by Span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Vectors: Orthogonal Vectors (19)

As we just saw, given two vectors u and v, u and v are perpendicular, or **orthogonal**, if $\vec{u} \cdot \vec{v} = 0$

Example 10

Find all vectors orthogonal to both $u = (1, -1, 1)$ and $v = (1, 0, 1)$

$$(x, y, z) \cdot (1, -1, 1) = x - y + z = 0$$

$$(x, y, z) \cdot (1, 0, 1) = x + z = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x &= -z \\ y &= 0 \\ z &= z \end{aligned} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, the answer is the *line* given by Span $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Orthogonal Complements

Orthogonal Complements (1)

Definition: Let W be subspace of $\mathbb{R}^n$. Its **orthogonal complement** is the subspace

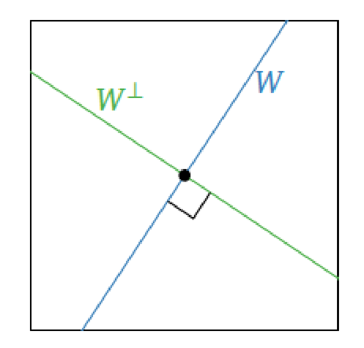
"W perp"

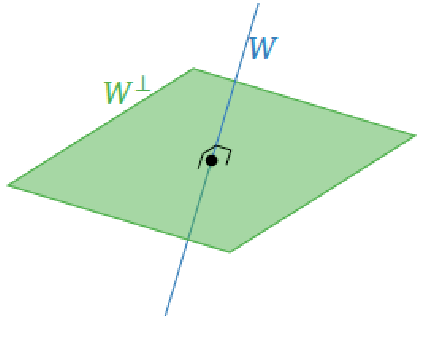
$$W^\perp = \{v \text{ in } \mathbb{R}^n | v \cdot w = 0 \text{ for all } w \text{ in } W\}$$

Some examples:

W $W^\perp$

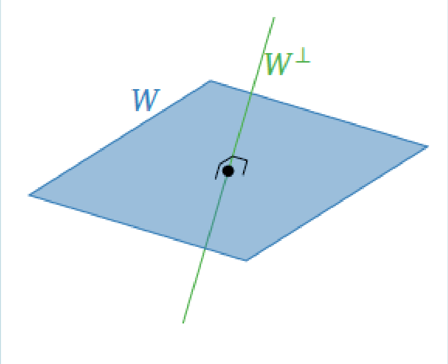
The orthogonal complement of a line in $\mathbb{R}^2$ is the perpendicular line





W $W^\perp$

The orthogonal complement of a line in $\mathbb{R}^3$ is the perpendicular plane



W $W^\perp$

The orthogonal complement of a plane in $\mathbb{R}^3$ is the perpendicular line

Orthogonal Complements (2)

Proposition: Let $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{bmatrix}$ and let $W = \mathrm{Col}(A)$. Then

$$W^\perp = \{\text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m\} = \mathrm{Nul}(A^T)$$

Note that $W = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$. First, we want show that if x is orthogonal to each of the vectors in the spanning set then x is orthogonal to all the vectors in W. So, assume x is perpendicular to each v_i. This means that $x \cdot v_1 = x \cdot v_2 = \cdots = x \cdot v_m = 0$. Any vector v in W has the form $v = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$, so $x \cdot v = x \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_m v_m) = c_1(x \cdot v_1) + c_2(x \cdot v_2) + \cdots + c_m(x \cdot v_m) = c_1(0) + c_2(0) + \cdots + c_m(0) = 0$. Thus, we have shown that x is orthogonal to v, so x is in $W^\perp$.

Now note that $A^T = \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_m^T & - \end{bmatrix}$ and for any x, $A^T x = \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_m^T x \end{bmatrix} = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_m \cdot x \end{bmatrix}$. Therefore, x is orthogonal to each v_i if and only if x is in $\mathrm{Nul}(A^T)$.

Orthogonal Complements (3)

Example 1

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Let $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. Compute $W^{\perp}$.

A plane in $\mathbb{R}^3$

$$W^{\perp} = \text{Nul}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \text{Nul}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

A line in $\mathbb{R}^3$

Orthogonal Complements (4)

Example 2

Let $W = \text{Span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$. Find a system of equations describing the line $W^{\perp}$.

$$\text{Nul}\begin{bmatrix}1 & 1 & -1\\ -1 & 2 & 1\end{bmatrix} = \text{solution set to } \begin{cases} x_1 + x_2 - x_3 = 0 \\ -x_1 + 2x_3 + x_3 = 0 \end{cases}$$

Find a basis for $W^{\perp}$.

$$\begin{bmatrix}1 & 1 & -1\\ -1 & 2 & 1\end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix}1 & 1 & -1\\ 0 & 3 & 0\end{bmatrix} \xrightarrow{\frac{R_2}{3}} \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 0\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$\begin{aligned} x_1 &= x_3 \\ x_2 &= 0 \\ x_3 &= x_3 \end{aligned} \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\0\\1\end{bmatrix} \implies W^{\perp} = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

Orthogonal Complements (5)

Example 3

Find all vectors orthogonal to $v = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$\text{Nul} \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$

$$\begin{aligned} x_1 &= -2x_2 + 3x_3 \\ x_2 &= x_2 \\ x_3 &= \quad\quad x_3 \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

A plane in $\mathbb{R}^3$.

Orthogonal Complements (6)

Facts about orthogonal complements

Let W be a subspace of $\mathbb{R}^n$. Then:

1. $W^\perp$ is also a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim(W) + \dim(W^\perp) = n$

Definition: The **row space** of a matrix A is the span of the rows of A and is denoted by $\mathrm{Row}(A)$.

$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$

$\mathrm{Col}(A^T)^\perp = \mathrm{Nul}(A) \Rightarrow \mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$

Observations and facts:

- $\mathrm{Row}(A) = \mathrm{Col}(A^T)$

Earlier we showed that if A consists of rows $v_1^T, v_2^T, \dots, v_m^T$, then $\mathrm{Row}(A)^\perp = \mathrm{Span}\{v_1, v_2, \dots, v_m\}^\perp = \mathrm{Nul}(A)$

- $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$
- $\mathrm{Row}(A) = \mathrm{Nul}(A)^\perp$
- $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$
- $\mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A)$

Orthogonal Complements (7)

Example 4

Compute the orthogonal complement of the subspace

$$W = \{(x, y, z) \text{ in } \mathbb{R}^3 | 3x + 2y = z\}$$

Re-express W as the solution set of the system $3x + 2y - z = 0$, or, equivalently, the null space of the matrix $A = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$ and we know that $\mathrm{Nul}(A)^{\perp} = \mathrm{Row}(A)$, so $W^{\perp} = \mathrm{Span}\left\{\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right\}$

Recipes: Shortcuts for computing orthogonal complements. For any vectors $v_1, v_2, \ldots, v_m$, we have

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^{\perp} = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$

For any matrix A, we have

$$\mathrm{Row}(A)^{\perp} = \mathrm{Nul}(A) \qquad \mathrm{Nul}(A)^{\perp} = \mathrm{Row}(A)$$
$$\mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A^T) \qquad \mathrm{Nul}(A^T)^{\perp} = \mathrm{Col}(A).$$

Orthogonal Complements (8)

Row rank and column rank

- Row rank = dimension of Row(A)
- Column rank = dimension of Col(A) (Rank)

Theorem: Let A be a matrix. Then the row rank of A is equal to the column rank of A.
(m x n matrix)

Proof: $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n$ (rank theorem)

$\dim \mathrm{Nul}(A)^{\perp} + \dim \mathrm{Nul}(A) = n \implies \dim \mathrm{Col}(A) = \dim \mathrm{Nul}(A)^{\perp}$. But $\mathrm{Nul}(A)^{\perp} = \mathrm{Row}(A)$ which results in $\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A)$.