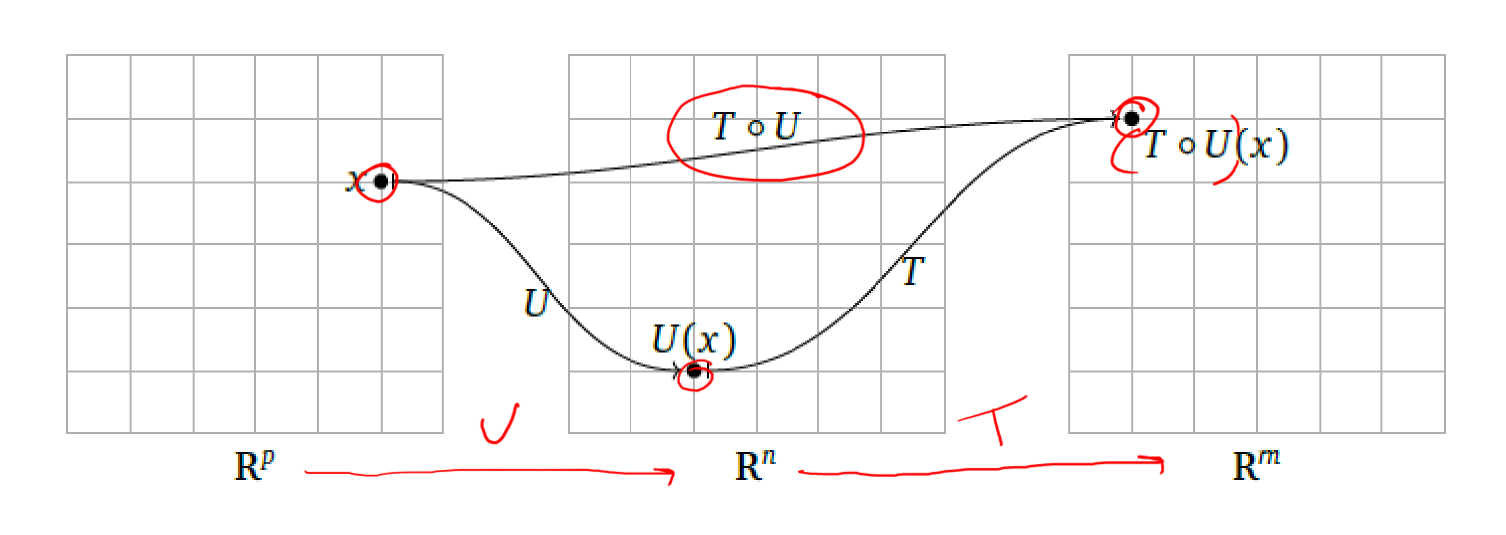
Matrix Multiplication

Matrix Multiplication: Compositions (1)

Definition: Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $U\colon \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be transformations. Their **composition** $T \circ U\colon \mathbb{R}^p \longrightarrow \mathbb{R}^m$ is the transformation defined by

$$(T \circ U)(x) = T(U(x))$$

To evaluate $T \circ U$ of an input vector x, first evaluate $U(x)$, then you take this output vector of U and use it as an input vector of T, so that $(T \circ U)(x) = T(U(x))$. Note this only makes sense when the outputs of U are valid inputs of T, that is, when the range of U is contained in the domain of T.



Matrix Multiplication: Compositions (2)

- In order for $T \circ U$ to be defined the codomain of U must equal to the domain of T
- The domain of $T \circ U$ is the domain of U
- The codomain of $T \circ U$ is the codomain of T

Let S, T,U be transformations and let c be a scalar. Suppose that $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, and that in each of the following identities, the domains and the codomains are compatible when necessary for the composition to be defined. The following properties are easily verified:

$$S \circ (T + U) = S \circ T + S \circ U$$
$$(S + T) \circ U = S \circ U + T \circ U$$
$$c(T \circ U) = (cT) \circ U$$
$$c(T \circ U) = T \circ (cU) \text{ if } T \text{ is linear}$$
$$T \circ \mathrm{Id}_{\mathrm{R^n}} = T \qquad \mathrm{Id}_{\mathrm{R^m}} \circ T = T$$
$$S \circ (T \circ U) = (S \circ T) \circ U \Leftarrow \text{Associative property}$$

Compositions are generally noncommutative, that is $T \circ U \neq U \circ T$
See next example.

Matrix Multiplication: Compositions (3)

Example 1

Let $T, U: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $T(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x$ and $U(x) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x$. Show that the composition of these two transformations is not commutative using the input vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$* \ (T \circ U)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(U\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) =$$

$$T\left(1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$* \ (U \circ T)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = U\left(T\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = U\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = U\left(1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$$

$$= U\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underline{T \circ U \neq U \circ T}$$

Matrix Multiplication: Matrices (4)

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called ***entries.***

The ***size*** of a matrix refers to its number of rows and columns. An $m \times n$ matrix has m rows and n columns.

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ -5 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 4 \\ 5 & 8 & 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 & -3 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 7 \\ 0 \\ 5 \\ -2 \\ 3 \end{bmatrix}$$

$$3 \times 2 \qquad 3 \times 4 \qquad 1 \times 4 \qquad 5 \times 1$$

A matrix with only one column is called a ***column vector***. A matrix with one row is called a ***row vector***.

Here's a general $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{ij} is the entry in row i and column j, sometimes denoted by $(A)_{ij}$

Matrix Multiplication: Matrices (5)

For example, given the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ -5 & 0 \end{bmatrix}$$

$(A)_{11} = 1$, $(A)_{12} = -2$, $(A)_{21} = 2$, $(A)_{22} = 1$, $(A)_{31} = -5$ and $(A)_{32} = 0$

A matrix A with n rows and n columns is called a ***square matrix of order n***,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are said to lie along the ***main diagonal*** of A.

Two matrices are ***equal*** if they have the same size, and their corresponding entries are equal.

Matrix Multiplication (6)

Before defining matrix multiplication let's define ***row-column multiplication***

First, we must make sure that the size of the row (its length) is the same as the size of the column (its length). Now let

$$R = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

Then the product of the two is defined as the inner product or the dot product

$$R \cdot C = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = r_1 c_1 + r_2 c_2 + \cdots + r_n c_n$$

which is a scalar.

Example 2

$$\begin{bmatrix} -1 & 2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ -2 \\ 1 \end{bmatrix} = (-1)(3) + (2)\overset{5}{\cancel{(3)}} + (0)(-2) + (4)(1) = -3 + \overset{10}{\cancel{6}} + 0 + 4 = \overset{11}{\cancel{7}}$$

Matrix Multiplication (7)

Given two matrices A and B, for the ***product matrix*** AB to be defined *the number of columns in matrix A* (the first matrix) must equal to the *number of rows in matrix B* (the second matrix). The product matrix AB will have the same number of rows as A (the first matrix) and the same number of columns as B (the second matrix). So, if A is an $m \times n$ matrix and B is an $n \times k$ matrix then their product AB will be an $m \times k$ matrix**.** The entry in row i and column j of AB will equal to the product of the i^{th} row of A and the j^{th} column of B.

Example 3

Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix}_{2\times 3}$

Since A is a 3×2 matrix and B is a 2×3, their product is defined and will be a 3×3 matrix

$$AB = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

Now to obtain the value of the entry d_{11} we must multiply row 1 of A by column 1 of B:

$$d_{11} = [4 \quad 2] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4 - 4 = 0$$

Matrix Multiplication (8)

Example 3 (continued)

$$AB = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

$$d_{11} = [4 \quad 2] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4 - 4 = 0$$

Now, for the rest of the entries of the product matrix:

$$d_{12} = [4 \quad 2] \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix} = -4 + 10 = 6$$

$$d_{13} = [4 \quad 2] \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 - 2 = -2$$

$$d_{21} = [-1 \quad 3] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 - 6 = -7$$

$$d_{22} = [-1 \quad 3] \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix} = 1 + 15 = 16$$

$$d_{23} = [-1 \quad 3] \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 - 3 = -3$$

$$d_{31} = [2 \quad 1] \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2 - 2 = 0$$

$$d_{32} = [2 \quad 1] \cdot \begin{bmatrix} -1 \\ 5 \end{bmatrix} = -2 + 5 = 3$$

$$d_{33} = [2 \quad 1] \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 - 1 = -1$$

Putting all these together $AB = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -2 \\ -7 & 16 & -3 \\ 0 & 3 & -1 \end{bmatrix}$

Matrix Multiplication (9)

Example 4

Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix}$ Find BA

Since B is a 2×3 matrix and A is a 3×2, the product BA is defined and will be a 2×2 matrix

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+1+0 & 2-3-0 \\ -8-5-2 & -4+15-1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -15 & 10 \end{bmatrix}$$

Note that

~~As was noted earlier~~, matrix multiplication is not commutative.

If A is a square matrix, then we can multiply it by itself; we define its **powers** to be

$$A^2 = AA, A^3 = AAA, \ldots, A^n = \underbrace{AAA \ldots AA}_{n \text{ factors}}$$

Algebraic Properties of Matrices (10)

PROPERTIES OF MATRIX ARITHMETIC

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) $A(BC) = (AB)C$ $\Leftarrow$ Associative

(d) $A(B + C) = AB + AC$

(e) $(B + C)A = BA + CA$

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

Algebraic Properties of Matrices and Inverses: The Identity Matrix (11)

A square matrix with 1's on the main diagonal and zeros everywhere else is called an ***identity matrix.***

Examples: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

An identity matrix is denoted by the letter I or by I_n if knowledge of size is important.

Example 5

Let A be a general 3×2 matrix then

$$AI_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$I_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

In general, if A is any $m \times n$ matrix then $AI_n = A$ and $I_mA = A$

Algebraic Properties of Matrices (12)

A matrix whose entries are all zero is called a ***zero matrix***. It may be denoted simply by 0 or by $0_{m\times n}$ if knowledge of the size is important.

Examples: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$

PROPERTIES OF ZERO MATRICES

If c is a scalar, and if the sizes of the matrices are such that the operations can be performed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) If $cA = 0$ then $c = 0$ or $A = 0$

Algebraic Properties of Matrices (13)

The cancellation law of real arithmetic: If $\cancel{a}b = \cancel{a}c$ and $a \neq 0$, then $b = c$

In general, not true for matrix multiplication.

Example 5

$$A = \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 10 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix} \qquad AC = \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$$

Although $A \neq 0$ and $AB = AC$, $B \neq C$

Algebraic Properties of Matrices (14)

The zero-product property of real arithmetic: If $ab = 0$ then either $a = 0$ or $b = 0$

In general, not true for matrix multiplication.

Example 6

$$A = \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here $AB = 0$ but $A \neq 0$ and $B \neq 0$

Matrix Multiplication: A Different Approach (15)

Alternative approach to matrix products. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Denote the columns of B by $v_1, v_2, \dots, v_p$

$$B = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The product AB is the $m \times p$ matrix with columns $Av_1, Av_2, \dots Av_p$:

$$AB = \begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{bmatrix}$$

$$AB = \begin{bmatrix} A\begin{bmatrix} 0 \\ 1 \end{bmatrix} & A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} =$$

Here, matrix multiplication is defined column by column.

$$= \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Matrix Multiplication and Compositions (16)

Theorem: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $U: \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be linear transformations. Let A and B be their corresponding standard matrices. So, A is an $m \times n$ matrix and B is an $n \times p$ matrix.
The $T \circ U: \mathbb{R}^p \longrightarrow \mathbb{R}^m$ is a linear transformation and its standard matrix is the $m \times p$ matrix AB.

Products and compositions. The matrix of the composition of two linear transformations is the product of the matrices of the transformations.

$T \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ $\quad U \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad T \circ U \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

Proof. First we verify that $T \circ U$ is linear. Let u, v be vectors in R^p. Then

$$\begin{aligned} T \circ U(u+v) &= T(U(u+v)) = T(U(u)+U(v)) \\ &= T(U(u)) + T(U(v)) = T \circ U(u) + T \circ U(v). \end{aligned}$$

If c is a scalar, then

$$(T \circ U)(cv) = T(U(cv)) = T(cU(v)) = cT(U(v)) = c(T \circ U)(v).$$

Matrix Multiplication and Compositions (17)

Theorem: Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $U\colon \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be linear transformations. Let A and B be their corresponding standard matrices. So, A is an $m \times n$ matrix and B is an $n \times p$ matrix.
The $T \circ U\colon \mathbb{R}^p \longrightarrow \mathbb{R}^m$ is a linear transformation and its standard matrix is the $m \times p$ matrix AB.

Now that we know that $T \circ U$ is linear, it makes sense to compute its standard matrix. Let C be the standard matrix of $T \circ U$, so $T(x) = Ax$, $U(x) = Bx$, and $(T \circ U)(x) = Cx$. By this theorem in Section 3.3, the first column of C is Ce_1, and the first column of B is Be_1. We have

$$(T \circ U)(e_1) = T(U(e_1)) = T(Be_1) = A(Be_1).$$

By definition, the first column of the product AB is the product of A with the first column of B, which is Be_1, so

Associative property

$$Ce_1 = (T \circ U)(e_1) = A(Be_1) = (AB)e_1. \implies C = AB$$

It follows that C has the same first column as AB. The same argument as applied to the ith standard coordinate vector e_i shows that C and AB have the same ith column; since they have the same columns, they are the same matrix. $\square$