Row Reduction and Free Variables Part 1

Row Reduction: Augmented Matrices (1)

The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

has as its corresponding ***augmented matrix*** the matrix (rectangular array of numbers)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

For example, the augmented matrix for the system

$$\begin{aligned} x_2 - 3x_3 &= -5 \\ 2x_1 + 3x_2 - x_3 &= 7 \\ 4x_1 + 5x_2 - 2x_3 &= 10 \end{aligned}$$

is

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right]$$

Row Reduction: Elementary Row Operations (2)

The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations***

The process of performing row operations on a matrix does not change the solution set of the corresponding linear system.

Row Reduction: An Example (3)

$$\begin{aligned} x_2 - 3x_3 &= -5 \\ 2x_1 + 3x_2 - x_3 &= 7 \\ 4x_1 + 5x_2 - 2x_3 &= 10 \end{aligned}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{bmatrix}$$

Interchange row 1 and row 2

$$\begin{bmatrix} 2 & 3 & -1 & 7 \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{bmatrix}$$

Multiply row 1 by $\frac{1}{2}$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 4 & 5 & -2 & 10 \end{bmatrix}$$

Add -4 times row 1 to row 3

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & -1 & 0 & -4 \end{bmatrix}$$

Add row 2 to row 3

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

Multiply row 3 by $-\frac{1}{3}$

$$\begin{bmatrix} 1 & \frac{3}{2} & -\frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $\frac{1}{2}$ times row 3 to row 1

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 5 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 3 times row 3 to row 2

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row Reduction: An Example Continued (4)

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{3}{2}$ times row 2 to row 1

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{aligned} x_1 \qquad\qquad &= -1 \\ x_2 \qquad &= 4 \\ x_3 &= 3 \end{aligned}$$

So, the solution to this system is $(-1,4,3)$

Geometrically, this means that the three planes defined by the three equations in this system intersect at the point $(-1,4,3)$

Definition: Two matrices are called **row equivalent** if one can be obtained from the other by performing some number of row operations.

Row Reduction: Reduced Row Echelon Form (5) (RREF)

A matrix is in ***row echelon form*** if :

1. In a row that does not consist entirely of zeros the first nonzero number in the row, often called a **pivot**, is a 1. (Call this a ***leading 1***).
2. Any rows that consist entirely of zeros are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Examples: $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If in addition, (4) each column that contains a leading 1 has zeros everywhere else in that column, then the matrix is in ***reduced row echelon form.***

Examples: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row Reduction: Reduced Row Echelon Form (6)

If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in reduced row echelon form, then the solution set can be obtained either by inspection or by converting certain linear equations to parametric form (the latter will be discussed later).

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{aligned} x_1 &= -1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 1 \end{aligned}$$

The system has a unique solution: $(-1,2,3,1)$

Example:

$$\begin{bmatrix} 1 & -4 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the last row becomes

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

This equation is not satisfied by any values of x_1, x_2, x_3 and x_4. So, the system has no solution and is, therefore, inconsistent.

Row Reduction: The Algorithm (7)

Theorem: Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

Step-by-step procedure:

- Make sure that the first entry in row 1 is not zero, interchange rows if necessary.
- If the first entry in row 1 is a, multiply the row by $\frac{1}{a}$ to introduce a leading 1
- Add appropriate multiples of row 1 to the other rows to obtain zeros below the leading 1

Example 1

$$\begin{bmatrix} 3 & 6 & -6 & 3 \\ 2 & 5 & 1 & 9 \\ 1 & 3 & 4 & 9 \end{bmatrix}$$

Multiply row 1 by $\frac{1}{3}$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 5 & 1 & 9 \\ 1 & 3 & 4 & 9 \end{bmatrix}$$

Add -2 times row 1 to row 2;
add -1 times row 1 to row 3

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 6 & 8 \end{bmatrix}$$

Row Reduction: The Algorithm (8)

Step-by-step procedure (continued):

- Cover up the first row and locate the leftmost column that does not consist entirely of zeros in the resulting submatrix
- In the resulting submatrix, interchange rows, if necessary, to bring a nonzero entry to the top of the column found in the previous step.
- If the top entry in that column is a, multiply the row by $\frac{1}{a}$ to introduce a leading 1
- Add appropriate multiples of the top row to the rows below to obtain zeros below the leading 1

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 6 & 8 \end{bmatrix}$$

Add -1 times row 2 to row 3

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- Cover up the top row again and repeat the last three steps in the new resulting submatrix and continue in this way until the entire matrix is in <u>row echelon</u> form.

Row Reduction: The Algorithm (9)

To get to the reduced row echelon form:

- Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Add -5 times row 3 to row 2;
add 2 times row 3 to row 1

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Add -2 times row 2 to row 1

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{array}{lll} x_1 & & = -1 \\ & x_2 & = 2 \\ & & x_3 = 1 \end{array}$$

The procedure (or algorithm) described for reducing a matrix to reduced row echelon form is called ***Gauss-Jordan reduction.***