Linear Independence

Linear Independence (1)

Definition:
A set of vectors $\{v_1, v_2, v_3, \dots, v_k\}$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution $x_1 = x_2 = x_3 = \cdots = x_k = 0$. Otherwise, we say that the set $\{v_1, v_2, v_3, \dots, v_k\}$ is **linearly dependent**.

Example 1

Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$ linearly independent?

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 2 \cdot x_3 = 0$
$x_2 + x_3 = 0$

$x_1 = -2x_3$
$x_2 = -x_3$
$x_3 = x_3$

For example, setting $x_3 = 1$ we get the solution $x_1 = -2$, $x_2 = -1$, $x_3 = 1$

the set is linearly dependent.

Linear Independence (2)

Example 2

Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right\}$ linearly independent?

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 4 & 10 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 4 & 10 \end{bmatrix} \xrightarrow{R_3-4R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{6}R_3} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-3R_3]{R_2-R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} x_1=0 \\ x_2=0 \\ x_3=0 \end{matrix}$$

the set is linearly independent!

Linear Independence (3)

Vectors coming from the parametric vector form of the solution of the equation $Ax = 0$

Example 3

Solve the equation $Ax = 0$ where $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix}$. Express your answer in parametric vector form. Explain why the two vectors in the solution are linearly independent.

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix} \xrightarrow{R_2 + 2R_1} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \underline{x_1 - 3x_2 + 2x_3 = 0}$$

$$x_1 = 3x_2 - 2x_3$$

$$x_2 = x_2 + 0$$

$$x_3 = 0 + x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ 0}}{x_2} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \underset{\substack{\uparrow \\ 0}}{x_3} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vectors coming from the parametric vector form of the solution set of $Ax = 0$ form a linearly independent set.

Linear Independence (4)

Checking Linear Independence

A set of vectors $\{v_1, v_2, v_3, \dots, v_k\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution, if and only if the corresponding matrix equation $Ax = 0$, where

$A = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & \dots & v_k \\ | & | & | & | \end{bmatrix}$, has only the trivial solution, and, if and only if A has a pivot position in every column, that is, every column is a pivot column, which results in no free variables.

Free variables would give rise to a linear dependence relation (as in Example 2)

Wide Matrices

A wide matrix is one with more columns than rows. Note that such a matrix cannot have a pivot in every column, so its columns are automatically <u>linearly dependent</u>. For example, four vectors in $\mathbb{R}^3$ are automatically linearly dependent.

Linear Independence (5)

Some facts about linear independence

1. Two vectors are linearly dependent if and only if they are collinear, that is, one is a scalar multiple of another.
2. Any set containing the zero vector is linearly dependent.
3. If a subset of $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent, then the entire set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent as well.

Proof

1. If $v_1 = cv_2$, then $v_1 - cv_2 = 0$, showing that $\{v_1, v_2\}$ is linearly dependent. In the other direction, if $x_1v_1 + x_2v_2 = 0$ with $x_1 \neq 0$ (let's say), then $v_1 = -\frac{x_2}{x_1}v_2$.
2. For example, if $v_1 = 0$ then $1v_1 + 0v_2 + \cdots + 0v_k = 0$ (this is a linear dependence relation)
3. After reordering, suppose that $\{v_1, v_2, \ldots, v_r\}$ is linearly dependent with $r < k$. This means we can construct a linear dependence relation $x_1v_1 + x_2v_2 + \cdots + x_rv_r = 0$ where at least one x_i is not zero. Now take the coefficients of $v_{r+1}, \ldots, v_k$ to be all zero and add to the left side of the linear dependence relation above. This will still be a linear dependence relation making the entire set $\{v_1, v_2, \ldots, v_k\}$ linearly dependent.

Linear Independence (6)

Some facts about linear independence

Theorem: *A set of vectors* $\{v_1, v_2, \ldots, v_k\}$ *is linearly dependent if and only if one of the vectors is in the span of the other ones. Any such vector can be removed without affecting the span.*

Proof

We'll use an example to illustrate the proof. Suppose v_3 is in Span$\{v_1, v_2, v_4\}$, so we can express v_3 as $v_3 = 2v_1 - 3v_2 + 4v_4$, for example. Subtracting v_3 from both sides we get

$$0 = 2v_1 - 3v_2 - v_3 + 4v_4$$

This is a linear dependence relation ➔ the set is linearly dependent.

Continuing with this case, any linear combination of v_1, v_2, v_3, v_4 is already a linear combination of v_1, v_2, v_4:

$$x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = x_1v_1 + x_2v_2 + x_3(2v_1 - 3v_2 + 4v_4) + x_4v_4 =$$

$(x_1 + 2x_3)v_1 + (x_2 - 3x_3)v_2 + (x_4 + 4x_3)v_4$ ➔ Span$\{v_1, v_2, v_3, v_4\}$is contained in Span$\{v_1, v_2, v_4\}$.

On the other hand, any linear combination of v_1, v_2, v_4 is also a linear combination of v_1, v_2, v_3, v_4 (just let the coefficient of v_3 be zero), so Span$\{v_1, v_2, v_4\}$ is contained in Span$\{v_1, v_2, v_3, v_4\}$, and thus they are equal.

In the other direction, assume a linear dependence relation like $0 = 2v_1 - 3v_2 - v_3 + 4v_4$, then we can solve for any vector v_i. For example, $v_1 = \frac{3}{2}v_2 + \frac{1}{2}v_3 - 2v_4$, making $v_1$in the Span$\{\cancel{v_1}, v_2, v_3, v_4\}$

Linear Independence (7)

Some facts about linear independence

Theorem: *(Increasing Span Criterion) A set of vectors* $\{v_1, v_2, \dots, v_k\}$ *is linearly independent if and only if, for every j<k, the vector* v_j *is not in the* $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$

> If you make a set of vectors by adding one vector at a time, and if the span got bigger every time you added a vector, then your set is linearly independent.

Theorem: Let $v_1, v_2, \dots, v_k$ *be vectors in* $\mathbb{R}^n$, *and consider the matrix* $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_k \\ | & | & & | \end{bmatrix}$. *Then we can delete the columns of A* *<u>without pivot positions (the columns corresponding to the free variables</u>), without changing the* $\text{Span}\{v_1, v_2, \dots, v_k\}$.

The pivot columns are linearly independent, so we cannot delete any more columns without changing the span.

Linear Independence (8)

Some facts about linear independence

Theorem: Let $v_1, v_2, \dots, v_k$ *be vectors in* $\mathbb{R}^n$, *and consider the matrix* $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_k \\ | & | & & | \end{bmatrix}$. *Then we can delete the columns of* A *<u>without pivot positions (the columns corresponding to the free variables</u>), without changing the* $\operatorname{Span}\{v_1, v_2, \dots, v_k\}$.

The pivot columns are linearly independent, so we cannot delete any more columns without changing the span.

Proof

Let's take an example matrix in RREF: $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. It is easy to see that the non-pivot column is in the span of the pivot columns: $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and the pivot columns are linearly independent: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} \Longrightarrow x_1 = x_2 = x_3 = 0.$

Linear Independence (9)

Proof

What if the matrix is not in RREF? Let's look at our RREF example. One solution to the vector equation $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is $x_1 = 2, x_2 = 3, x_3 = -1, x_4 = 0$. What if we start with the matrix $B = \begin{bmatrix} 1 & 7 & 23 & 3 \\ 2 & 4 & 16 & 0 \\ -1 & -2 & -8 & 4 \end{bmatrix}$ where $B \xrightarrow{RREF} A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. So, B and A are row equivalent and, therefore, have the same solution set. So, the solution above also works for B, that is, $2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} 23 \\ 16 \\ -8 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and solving for the third vector we get

$$\begin{bmatrix} 23 \\ 16 \\ -8 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ 4 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

So, $\begin{bmatrix} 23 \\ 16 \\ -8 \end{bmatrix}$ lies in the span of the other three vectors. Furthermore, the other three vectors are linearly independent.

Linear Independence (10)

Example 4

Find the RREF of $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix}$. Then find the smallest subset of column vectors of A whose span is equal to $\text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}\right\}$.

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2+2R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{array}{cccc} \checkmark & \checkmark & * & * \\ \end{array} \begin{bmatrix} 1 & 0 & -8 & -5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}\right\} =$$

$$\text{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}\right\}!$$

Linear Independence (11)

Pivot Columns and Dimension. Let d be the number of pivot columns in the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}.$$

- If $d = 1$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a line.
- If $d = 2$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a plane.
- If $d = 3$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_k\}$ is a 3-space.
- Et cetera.