Eigenvalues, Eigenvectors and the Characteristic Polynomial

Eigenvalues and Eigenvectors (1)

Definition (for $n \times n$ matrices):

- An **eigenvector** of A is a nonzero vector v in $\mathbb{R}^n$ such that $Av = \lambda v$, for some scalar λ
- An **eigenvalue** of A is a scalar λ such that the equation $Av = \lambda v$ has a nontrivial solution.
- If $Av = \lambda v$ then we say that λ is the eigenvalue for v and v is an eigenvector for λ

❖ Eigenvectors are nonzero by definition; eigenvalues may be zero.

Example 1

Let $A = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}$. Show that $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A and determine its corresponding eigenvalue.

$Av = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4v$; So, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 4$

Eigenvalues and Eigenvectors (2)

Example 2

Let $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$. Show that $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector for A and determine its corresponding eigenvalue.

$$Av = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5v \checkmark \qquad \lambda = 5$$

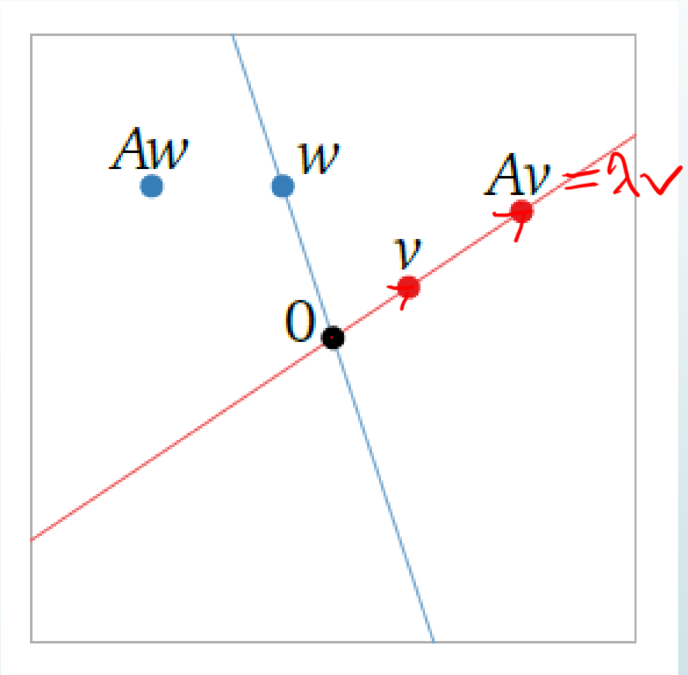
Example 3

Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Show that $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A and determine its corresponding eigenvalue.

$$Av = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0v \quad \text{with } \lambda = 0.$$

Eigenvalues and Eigenvectors (3)

$Av = \lambda v$ means that v and Av are collinear with the origin. That is, Av is a scalar multiple of v and λ is the scaling factor.

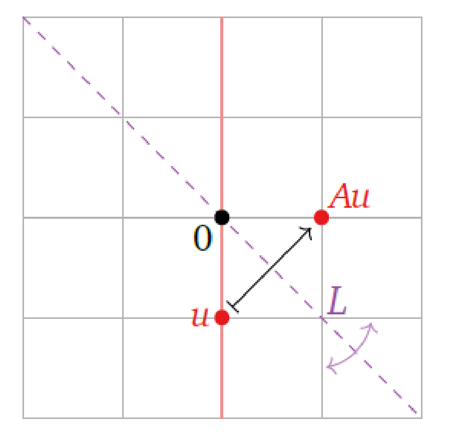


v is an eigenvector of A while w is not.

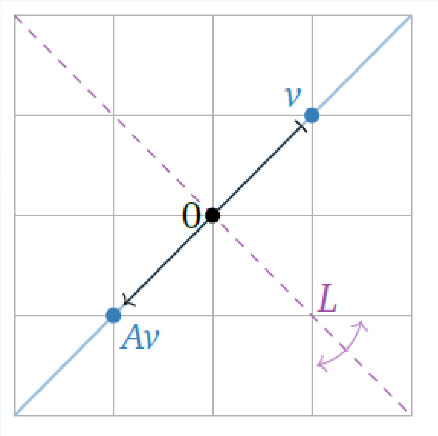
Eigenvalues and Eigenvectors (4)

Recognizing eigenvectors and eigenvalues geometrically

Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation that reflects over the line L defined by $y = -x$ and let A be the standard matrix for T. To find the eigenvectors and eigenvalues of A we examine the picture behind the action of T.



u is not an eigenvector since u and Au are not collinear with the origin

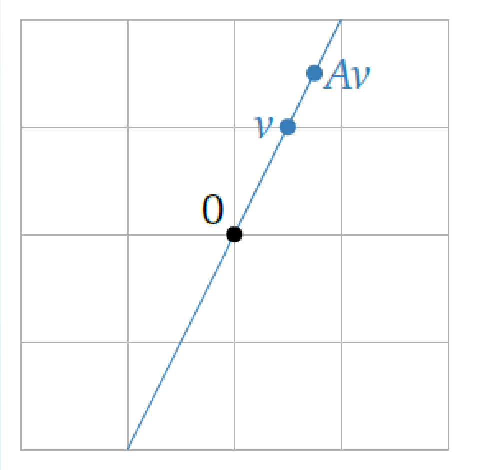


v and Av are collinear with the origin so v is an eigenvector. Av has the same length as v but opposite direction so the corresponding eigenvalue is -1

Eigenvalues and Eigenvectors (5)

Recognizing eigenvectors and eigenvalues geometrically

Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation that dilates by a factor of 1.5 and let A be the matrix for T. Here we have, $T(v) = Av = 1.5v$. So, all nonzero vectors in $\mathbb{R}^2$ are eigenvectors and the eigenvalue is just the scaling factor 1.5.

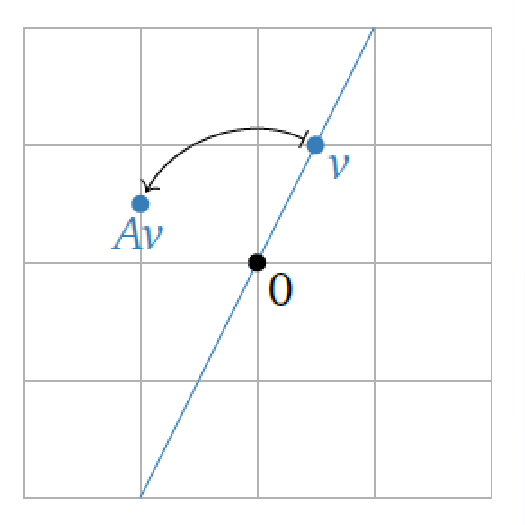


v and Av are collinear with the origin for all nonzero vectors v.

Eigenvalues and Eigenvectors (6)

Recognizing eigenvectors and eigenvalues geometrically

Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation that rotates counterclockwise by 90° and let A be the matrix for T. If v is any nonzero vector, then Av is its image after the rotation, and v and Av cannot be collinear with the origin. The zero vector can't be an eigenvector by definition, so this matrix has no eigenvectors and eigenvalues.



v and Av never lie on the same line that passes through the origin

Eigenvalues and Eigenvectors (7)

Eigenspaces

Given an eigenvalue λ its corresponding eigenvectors are the nonzero solutions of the equation $Av = \lambda v$ which can be rewritten as

$$Av - \lambda v = 0$$
$$Av - \lambda I v = 0$$
$$(A - \lambda I)v = 0$$

So, the eigenvectors are the nontrivial solutions to this homogeneous system.

Definition: Let A be an $n \times n$ matrix and λ be an eigenvalue of A.
The **λ-eigenspace** of A is the solution set of $(A - \lambda I)v = 0$, or equivalently,

$$\mathrm{Nul}(A - \lambda I)$$

Eigenvalues and Eigenvectors (8)

Example 4

For each of the numbers $\lambda = 3, 2, -2$ determine if λ is an eigenvalue for the matrix

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

And if so, compute a basis for the λ-eigenspace.

(Note that λ is an eigenvalue of A if and only if $\text{Nul}(A - \lambda I)$ is nonzero, so we must solve the equation $(A - \lambda I)v = 0$)

(a) $\lambda = 3$ $\quad A - 3I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

$$\begin{aligned} x &= -4y \\ y &= y \end{aligned} \Rightarrow y\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

A basis for the 3-eigenspace is $\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}$

Eigenvalues and Eigenvectors (9)

Example 4 (continued)

For each of the numbers $\lambda = 3, 2, -2$ determine if λ is an eigenvalue for the matrix

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$

And if so, compute a basis for the λ-eigenspace.

(b) $\lambda = 2$

$$A - 2I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -1 & -3 \end{bmatrix}$$

Note that $\det(A - 2I) = -4 \neq 0 \Longrightarrow$ the system has only the trivial solution $\Longleftrightarrow \mathrm{nul}(A-2I) = \{0\}$

$\lambda = 2$ is not an eigenvalue

(c) $\lambda = -2$

$$A + 2I = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1}$$

$$\longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x = y \\ y = y \end{matrix} \Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$ is a basis for the (-2)-eigenspace.

Eigenvalues and Eigenvectors (10)

Example 5

Verify that $\lambda = 1$ and $\lambda = 2$ are eigenvalues of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and find bases for the corresponding eigenspaces.

$$\lambda = 1 \qquad A - I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x &= -2z \\ y &= z \\ z &= z \end{aligned} \implies z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

So, a basis for the 1-eigenspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

Eigenvalues and Eigenvectors (11)

Example 5 (continued)

Verify that $\lambda = 1$ and $\lambda = 2$ are eigenvalues of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and find bases for the corresponding eigenspaces.

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x &= -z \\ y &= y \\ z &= z \end{aligned} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the 2-eigenspace.

(this is a plane in $\mathbb{R}^3$).

Eigenvalues and Eigenvectors (12)

Recipes: Eigenspaces. Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I_n)v = 0$ has a nontrivial solution, if and only if $\mathrm{Nul}(A - \lambda I_n) \neq \{0\}$.
2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\mathrm{Nul}(A - \lambda I_n)$, which can be done by finding the parametric vector form of the solutions of the homogeneous system of equations $(A - \lambda I_n)v = 0$.
3. The dimension of the λ-eigenspace of A is equal to the number of free variables in the system of equations $(A - \lambda I_n)v = 0$, which is the number of columns of $A - \lambda I_n$ without pivots.
4. The eigenvectors with eigenvalue λ are the nonzero vectors in $\mathrm{Nul}(A - \lambda I_n)$, or equivalently, the nontrivial solutions of $(A - \lambda I_n)v = 0$.

Fact: Let A be an $n \times n$ matrix. Then (1) 0 is an eigenvalue of A if and only if A is not invertible and (2) the 0-eigenspace of A is $\mathrm{Nul}(A)$.
(Why?) 0 is an eigenvalue if and only if $Nul(A - 0I) = Nul(A)$ is nonzero which is equivalent to the non-invertibility of A.

Eigenvalues and Eigenvectors (13)

Addenda to the Invertible Matrix Theorem

A is an $n \times n$ matrix and $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the corresponding matrix transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible
2. A has n pivots
3. $\mathrm{Nul}(A) = \{0\}$
4. The columns of A are linearly independent
5. The columns of A span $\mathbb{R}^n$
6. $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$
7. T is invertible
8. T is one-to-one
9. T is onto
10. $\det(A) \neq 0$
11. 0 is not an eigenvalue of A

The Characteristic Polynomial (14)

Definition: Let A be an $n \times n$ matrix.

- The **characteristic polynomial** of A is the function $f(\lambda)$ defined by $f(\lambda) = \det(A - \lambda I)$

Example 6

Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{bmatrix} =$$

$$= 8\left(\frac{1}{4}\right) - \lambda(\lambda^2 - 3) = 2 - \lambda^3 + 3\lambda = -\lambda^3 + 3\lambda + 2$$

The Characteristic Polynomial (15)

Theorem: Let A be an $n \times n$ matrix and let $f(\lambda) = \det(A - \lambda I)$ be its characteristic polynomial. Then a number λ_0 is an eigenvalue of A if and only if $f(\lambda_0) = 0$.

Proof: λ_0 is an eigenvalue of $A \Leftrightarrow Ax = \lambda_0 x$ has a nontrivial solution

$\Leftrightarrow (A - \lambda_0 I)x = 0$ has a nontrivial solution

$\Leftrightarrow A - \lambda_0 I$ is not invertible

$\Leftrightarrow \det(A - \lambda_0 I) = 0$

$\Leftrightarrow f(\lambda_0) = 0$

So, eigenvalues are roots of the characteristic polynomial.

The Characteristic Polynomial (16)

Example 7

Find the eigenvalues and the corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(This is the matrix from **Example 6** whose characteristic polynomial we already know)

$f(\lambda) = -\lambda^3 + 3\lambda + 2$

*You will find it useful to know that one of the eigenvalues is $\lambda = 2$. You should verify this. Then use long division to find the other roots of the characteristic polynomial.

Verify that $\lambda = 2$ is an eigenvalue: $f(2) = -8 + 6 + 2 = 0$ ✓

$$\begin{array}{r|l} & -\lambda^2 - 2\lambda - 1 \\ \hline \lambda - 2 & -\lambda^3 + 0\lambda^2 + 3\lambda + 2 \\ & -\lambda^3 + 2\lambda^2 \\ \hline & -2\lambda^2 + 3\lambda \\ & -2\lambda^2 + 4\lambda \\ \hline & -\lambda + 2 \\ & -\lambda + 2 \\ \hline & 0 \end{array}$$

$-\lambda^2 - 2\lambda - 1 = -(\lambda^2 + 2\lambda + 1) = -(\lambda+1)^2 \Rightarrow \lambda = -1$ is the other eigenvalue.

The Characteristic Polynomial (17)

Example 7 (continued)

Find the eigenvalues and the corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$\lambda = 2$ $\quad A - 2I = \begin{bmatrix} -2 & 6 & 8 \\ \frac{1}{2} & -2 & 0 \\ 0 & \frac{1}{2} & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$x = 16z$
$y = 4z$
$z = z$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$

The 2-eigenspace is Span $\left\{ \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix} \right\}$

(this is a line in $\mathbb{R}^3$).

The Characteristic Polynomial (18)

Example 7 (continued)

Find the eigenvalues and the corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$\lambda = -1$ $\quad A + I = \begin{bmatrix} 1 & 6 & 8 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x &= 4z \\ y &= -2z \\ z &= z \end{aligned} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

the (-1)-eigenspace is $\text{Span}\left\{\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right\}$ (A line in $\mathbb{R}^3$).

The Characteristic Polynomial (19)

Definition: The **trace** of a square matrix A, denoted by $\mathrm{Tr}(A)$, is the sum of the diagonal entries of A

$$\mathrm{Tr}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Theorem: The characteristic polynomial $f(\lambda)$ of an $n \times n$ matrix A is a polynomial of degree n that has the following form

$$f(\lambda) = (-1)^n \lambda^n + \underbrace{(-1)^{n-1}\mathrm{Tr}(A)}\lambda^{n-1} + \cdots + \underbrace{\det(A)}$$

The 2×2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies f(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} =$

$$(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = \underline{\lambda^2 -} \underline{\mathrm{Tr}(A)\lambda} + \underline{\det(A)}$$

The Characteristic Polynomial (20)

Example 8

Find the characteristic polynomial of the matrix below using the preceding theorem and find all eigenvalues

$$A = \begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}$$

$\mathrm{Tr}(A) = 13 + 7 = 20$ $\det(A) = 13(7) - (-4)(-4) = 91 - 16 = 75$

$f(\lambda) = \lambda^2 - 20\lambda + 75$ $f(\lambda) = (\lambda - 5)(\lambda - 15) = 0$ $\lambda = 5$ and $\lambda = 15$

Note that the characteristic polynomial of an $n \times n$ matrix is a polynomial of degree n and therefore has at most n roots which implies that an $n \times n$ matrix has at most n eigenvalues.

The Characteristic Polynomial (21)

Eigenvalues of triangular matrices

Theorem: If A is an upper or lower triangular matrix then its eigenvalues are its diagonal entries

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad \Longrightarrow$$

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$$

The zeros of the above polynomial are exactly a_{11}, a_{22} and a_{33}.

The Characteristic Polynomial (22)

Factoring the characteristic polynomial

Rational Root Theorem: Suppose that A is an $n \times n$ matrix whose characteristic polynomial $f(\lambda)$ has integer coefficients. Then all the rational roots of $f(\lambda)$ are integer divisors of $\det(A)$ (the constant term of the polynomial).

Example 9

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} = (5-\lambda)\big((3-\lambda)(3-\lambda) - 4\big)$$

$$= (5-\lambda)(9 - 6\lambda + \lambda^2 - 4) = (5-\lambda)(5 - 6\lambda + \lambda^2)$$

$$= 25 - 30\lambda + 5\lambda^2 - 5\lambda + 6\lambda^2 - \lambda^3 = -\lambda^3 + 11\lambda^2 - 35\lambda + 25$$

The Characteristic Polynomial (23)

Factoring the characteristic polynomial

Example 9 (continued)

$f(\lambda) = -\lambda^3 + 11\lambda^2 - 35\lambda + 25$

Divisors of 25: $\pm 1, \pm 5, \pm 25$

$f(1) = -1 + 11 - 35 + 25 = 0$ ✓

$\lambda = 1$ is an eigenvalue.

$$
\begin{array}{r|l}
 & -\lambda^2 + 10\lambda - 25 \\
\hline
\lambda - 1 & -\lambda^3 + 11\lambda^2 - 35\lambda + 25 \\
 & -\lambda^3 + \lambda^2 \\
\hline
 & 10\lambda^2 - 35\lambda \\
 & 10\lambda^2 - 10\lambda \\
\hline
 & -25\lambda + 25 \\
 & -25\lambda + 25 \\
\hline
 & 0
\end{array}
$$

$-\lambda^2 + 10\lambda - 25 =$

$-(\lambda^2 - 10\lambda + 25) =$

$= -(\lambda - 5)^2$

$\lambda = 5$ is the other eigenvalue.