# Diagonalization

# Diagonalization (1)

**Similarity**

**Definition:** Two $n \times n$ matrices $A$ and $B$ are **similar** if there exists an invertible $n \times n$ matrix $C$ such that $A = CBC^{-1}$.

For example, $\begin{bmatrix} -12 & -30 \\ 5 & 13 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ are similar because

$$\underbrace{\begin{bmatrix} -12 & -30 \\ 5 & 13 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}}_{C^{-1}}$$

$\det(C) = 2 - 3 = -1$

$$C = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} \frac{-1}{-1} & \frac{-3}{-1} \\ \frac{-1}{-1} & \frac{-2}{-1} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}_{B} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -12 & -30 \\ 5 & 13 \end{bmatrix}}_{A}$$

# Diagonalization (2)

## Similarity

**Fact:** Let $A = CBC^{-1}$. Then for any $n \geq 1$,

$$A^n = CB^nC^{-1}$$

**Proof**: Observe the pattern

$$\underline{A^2} = AA = \overbrace{(CBC^{-1})}^{A}\overbrace{(CBC^{-1})}^{A} = CB\overbrace{(C^{-1}C)}^{I}BC^{-1} = C\overbrace{BI}^{B}BC^{-1} = \underline{CB^2C^{-1}}$$

$$A^3 = A^2A = \underbrace{(CB^2C^{-1})}_{A^2}\underbrace{(CBC^{-1})}_{A} = CB^2\underbrace{(C^{-1}C)}_{I}BC^{-1} = C\underbrace{B^2I}_{B^2}BC^{-1} = \underline{CB^3C^{-1}}$$

## Diagonalizability

Definition: An $n \times n$ matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix, that is, if there exists an invertible $n \times n$ matrix $C$ and a diagonal matrix $D$ such that

$$A = CDC^{-1}$$

For example, $\begin{bmatrix} -12 & -30 \\ 5 & 13 \end{bmatrix}$ is diagonalizable because

$$\underbrace{\begin{bmatrix} -12 & -30 \\ 5 & 13 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}^{-1}}_{C^{-1}}$$

# Diagonalization (3)

**Powers of diagonalizable matrices**

Multiplying diagonal matrices

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}\begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} = \begin{bmatrix} x_1 y_1 & 0 & 0 \\ 0 & x_2 y_2 & 0 \\ 0 & 0 & x_3 y_3 \end{bmatrix}$$

Powers of a diagonal matrix

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}^n = \begin{bmatrix} x_1^n & 0 & 0 \\ 0 & x_2^n & 0 \\ 0 & 0 & x_3^n \end{bmatrix}$$

Powers of a diagonalizable matrix

$$A = C\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}C^{-1} \quad \Longrightarrow \quad A^n = C\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}^n C^{-1} = C\begin{bmatrix} x_1^n & 0 & 0 \\ 0 & x_2^n & 0 \\ 0 & 0 & x_3^n \end{bmatrix}C^{-1}$$

# Diagonalization (4)

**Powers of diagonalizable matrices**

# Example 1

$$C^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Let $\underset{A}{A = \begin{bmatrix} 4 & -6 \\ 3 & -5 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}} \underset{C^{-1}}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}}$. Find a formula for $A^n$ in which the entries are functions of $n$, where $n$ is any positive integer.

$$A^n = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^n \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-2)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-2)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & (-2)^n \\ 1 & (-2)^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2-(-2)^n & -2+2(-2)^n \\ 1-(-2)^n & -1+2(-2)^n \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2-(-2)^4 & -2+2(-2)^4 \\ 1-(-2)^4 & -1+2(-2)^4 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -14 & 30 \\ -15 & 31 \end{bmatrix}}}$$

# Diagonalization (5)

**When is a matrix diagonalizable?**

**Diagonalization Theorem**: An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

In that case, $A = CDC^{-1}$ where

$$C = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues, in the <u>same order</u>.

**Special Case:** An $n \times n$ with $n$ distinct eigenvalues is diagonalizable.

# Diagonalization (6)

## Example 2

Diagonalize the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

$f(\lambda) = \lambda^2 - Tr(A)\lambda + det(A)$

$Tr(A) = 1 \qquad det(A) = \frac{1}{4} - \frac{9}{4} = -2$

$f(\lambda) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) \Longrightarrow \lambda_1 = -1 \quad , \lambda_2 = 2$

$\lambda_1 = -1 \qquad A + I = \begin{bmatrix} \frac{1}{2}+1 & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2}+1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \xrightarrow{\frac{2}{3}R_1} \begin{bmatrix} 1 & 1 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1}$

$\longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = -y \\ y = y \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \qquad A - 2I = \begin{bmatrix} \frac{1}{2}-2 & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2}-2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow{-\frac{2}{3}R_1} \begin{bmatrix} 1 & -1 \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \xrightarrow{R_2 - \frac{3}{2}R_1}$

$\longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = y \\ y = y \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

# Diagonalization (7)

## Example 2 (continued)

Diagonalize the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

$\lambda_1 = -1 \qquad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 2 \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} = CDC^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

# Diagonalization (8)

Note that diagonalizations are not unique since there are infinitely many eigenvectors for a given eigenvalues, and you can switch the order of the eigenvalues and the corresponding eigenvectors (as long as they are consistent).

## Example 3

Diagonalize the matrix $A = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$f(\lambda) = \det(A - \lambda I) =$

$\det \begin{bmatrix} 4-\lambda & -3 & 0 \\ 2 & -1-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda)\left((4-\lambda)(-1-\lambda)+6\right) =$

$(1-\lambda)(-4-3\lambda+\lambda^2+6) = (1-\lambda)(\lambda^2-3\lambda+2) =$

$\underline{(1-\lambda)}(\lambda-2)\underline{(\lambda-1)} = -(\lambda-1)(\lambda-1)(\lambda-2) = -(\lambda-1)^2(\lambda-2)$

$\lambda = 1, \quad \lambda = 2$

$\underline{\lambda = 1}$ has (algebraic) multiplicity $\underline{2}$.

# Diagonalization (9)

## Example 3 (continued)

Diagonalize the matrix $A = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$\lambda = 1 \qquad A - I = \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow$

$\begin{matrix} x = y \\ y = y \\ z = z \end{matrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$v_1$ and $v_2$ form a basis for 1-eigenspace so they are linearly independent.

# Diagonalization (10)

## Example 3 (continued)

Diagonalize the matrix $A = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$\lambda = 2 \qquad A - 2I = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = 3z \\ y = 2z \\ z = z \end{matrix} \Rightarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$v_1, v_2$ and $v_3$ are linearly independent (eigenspaces don't overlap except at the origin)

$$\begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

# Diagonalization (11)

**Recipe: Diagonalization.** Let $A$ be an $n \times n$ matrix. To diagonalize $A$:

1. Find the eigenvalues of $A$ using the characteristic polynomial.
2. For each eigenvalue $\lambda$ of $A$, compute a basis $\mathcal{B}_\lambda$ for the $\lambda$-eigenspace.
3. If there are fewer than $n$ total vectors in all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.
4. Otherwise, the $n$ vectors $v_1, v_2, \ldots, v_n$ in the eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_i$ is the eigenvalue for $v_i$.

# Diagonalization (12)

## Example 4

Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)^2 \quad \Longrightarrow \quad \lambda = 2, \lambda = 1$$

$$\lambda = 2 \qquad (A - 2I)v = 0 \quad \Longrightarrow \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Longrightarrow$$

$$\Longrightarrow \quad \begin{matrix} x = y \\ y = 0 \\ z \text{ is a free variable} \end{matrix} \quad \Longrightarrow \quad x = y = 0 \quad \text{which is the } z\text{-axis}$$

# Diagonalization (13)

## Example 4 (continued)

Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

We just showed that the 2-eigenspace is the $z$-axis, so a basis would be $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\lambda = 1 \qquad (A - I)v = 0 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{matrix} y = 0 \\ z = 0 \\ x \text{ is a free variable} \end{matrix}$$

The 1-eigenspace is the $x$-axis, so a basis would be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. There are only two vectors total in the bases of the eigenspaces and we need three, so the matrix $A$ is not diagonalizable.

# Diagonalization (14)

**Diagonalizability has nothing to do with invertibility.** Of the following matrices, the first is diagonalizable and invertible, the second is diagonalizable but not invertible, the third is invertible but not diagonalizable, and the fourth is neither invertible nor diagonalizable, as the reader can verify:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$