Vectors, Vector Equations and Span Part II

Vectors: Linear Combinations (12)

If $\vec{u}$ is a vector in R^n, then $\vec{u}$ is said to be a ***linear combination*** of vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots \vec{v_r}$ in R^n if it can be expressed in the form

$$\vec{u} = k_1\vec{v_1} + k_2\vec{v_2} + k_3\vec{v_3} + \cdots + k_r\vec{v_r}$$

where $k_1, k_2, k_3, \ldots, k_r$ are scalars.

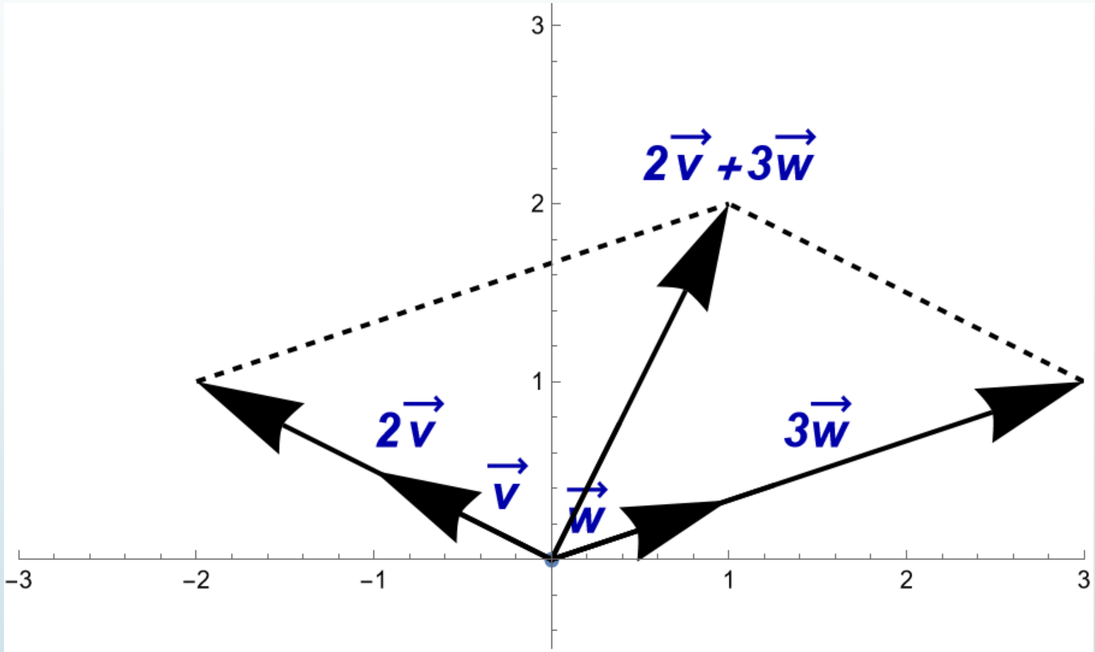
The scalars $k_1, k_2, k_3, \ldots, k_r$ are called the ***coefficients*** of the linear combination.

For example, the vector $\vec{u} = \begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix}$ is a linear combination of the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\vec{v_2} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ because $\vec{u} = 2\vec{v_1} + 3\vec{v_2}$.

$$2\vec{v_1} + 3\vec{v_2} = 2\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -8 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-6 \\ 4+3 \\ -8+3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ -5 \end{pmatrix} = \vec{u}$$

Vectors: Linear Combinations (13)

Geometrically, a linear combination is obtained by stretching / shrinking the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3} \dots \overrightarrow{v_n}$ according to the coefficients, then adding them together using the parallelogram law (or the triangle law)



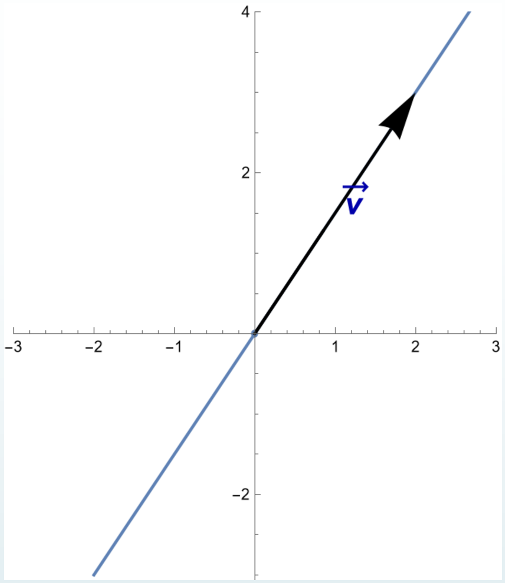
Note that given two distinct nonparallel (noncollinear) nonzero vectors in $\mathbb{R}^2$, say $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, any vector in the plane can be obtained as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ with suitable coefficients.

Vectors: Linear Combinations (14)

A linear combination of a single vector, say, $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is just a scalar multiple of $\vec{v}$.

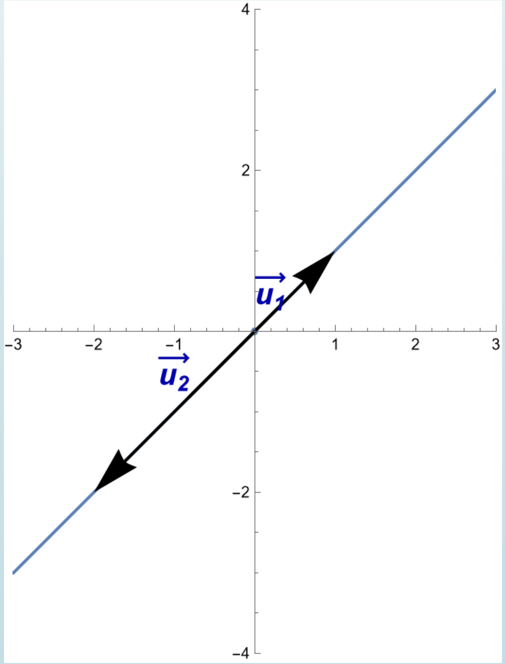
For example, $\frac{1}{2}\vec{v} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$, $-2\vec{v} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$, …

The set of all linear combinations of a single vector creates a *line* that contains $\vec{v}$. Unless $\vec{v} = \vec{0}$.



The set of all linear combinations of two collinear vectors is also line. Let $\overrightarrow{u_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{u_2} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

Then the set of all linear combinations of $\overrightarrow{u_2}$ and $\overrightarrow{u_2}$ creates a line that contains both of these vectors.



Vectors Equations (15)

An equation involving vectors with n coordinates is the same as n equations involving only numbers (not vectors).

For example,

$$x\begin{pmatrix}1\\4\\2\end{pmatrix}+y\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}-5\\1\\5\end{pmatrix} \implies \begin{pmatrix}x\\4x\\2x\end{pmatrix}+\begin{pmatrix}2y\\y\\-y\end{pmatrix}=\begin{pmatrix}-5\\1\\5\end{pmatrix} \implies \begin{pmatrix}x+2y\\4x+y\\2x-y\end{pmatrix}=\begin{pmatrix}-5\\1\\5\end{pmatrix} \implies \begin{matrix}x+2y=-5\\4x+y=1\\2x-y=5\end{matrix}$$

Definition. A **vector equation** is an equation involving a linear combination of vectors with possibly unknown coefficients.

Solving a vector equation is the same as asking if a given vector is a linear combination of the other given vectors which is, in turn, equivalent to solving the corresponding linear system. Let's solve the system:

$$\begin{aligned}x+2y&=-5\\4x+y&=1\\2x-y&=5\end{aligned}$$

$$\left[\begin{array}{cc|c}1&2&-5\\4&1&1\\2&-1&5\end{array}\right]\xrightarrow[R_3-2R_1]{R_2-4R_1}\left[\begin{array}{cc|c}1&2&-5\\0&-7&21\\0&-5&15\end{array}\right]\xrightarrow{-\frac{1}{7}R_2}\left[\begin{array}{cc|c}1&2&-5\\0&1&-3\\0&-5&15\end{array}\right]\xrightarrow{R_3+5R_2}\left[\begin{array}{cc|c}1&2&-5\\0&1&-3\\0&0&0\end{array}\right]$$

$$\xrightarrow{R_1-2R_2}\left[\begin{array}{cc|c}1&0&1\\0&1&-3\\0&0&0\end{array}\right]$$

$x=1$, $y=-3$

Vectors Equations (16)

We should verify that the two numbers we found are indeed the correct coefficient solutions to our original vector equation

$$x\begin{pmatrix}1\\4\\2\end{pmatrix} + y\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}-5\\1\\5\end{pmatrix}$$

$x = 1, y = -3$

$$1\begin{pmatrix}1\\4\\2\end{pmatrix} - 3\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\4\\2\end{pmatrix} - \begin{pmatrix}6\\3\\-3\end{pmatrix} = \begin{pmatrix}-5\\1\\5\end{pmatrix}$$

Recipe: Solving a vector equation. In general, the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = b$$

where $v_1, v_2, \ldots, v_k$, b are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_k$ are unknown scalars, has the same solution set as the linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right)$$

whose columns are the v_i's and the b's.

Vectors Equations (17)

Now we have *three* equivalent ways of thinking about a linear system:

1. As a system of equations:

$$\begin{cases} 2x_1 + 3x_2 - 2x_3 = 7 \\ x_1 - x_2 - 3x_3 = 5 \end{cases}$$

2. As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Spans (18)

Definition. Let $v_1, v_2 \dots v_k$ be vectors in $\mathbb{R}^n$. The **span** of $v_1, v_2 \dots v_k$ is the collection of all linear combinations of $v_1, v_2 \dots v_k$ and is denoted $\text{Span}\{v_1, v_2 \dots v_k\}$**.** That is,

$$\text{Span}\{v_1, v_2 \dots v_k\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_k v_k | x_1, x_2, \dots x_k \text{ in } \mathbb{R}\}$$

Three characterizations of consistency. Now we have three equivalent ways of making the same statement:

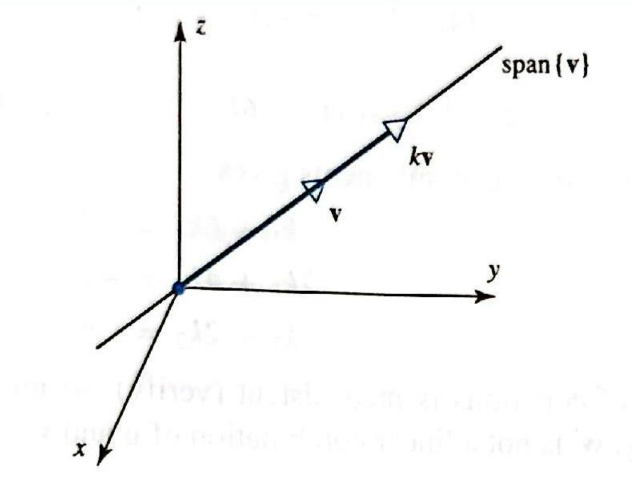
1. A vector b is in the span of $v_1, v_2, \dots, v_k$.
2. The vector equation
$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = b$$
has a solution.
3. The linear system with augmented matrix
$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right)$$
is consistent.

Equivalent means that, for any given list of vectors $v_1, v_2, \dots, v_k$, b, either all three statements are true, or all three statements are false.

Spans (19)

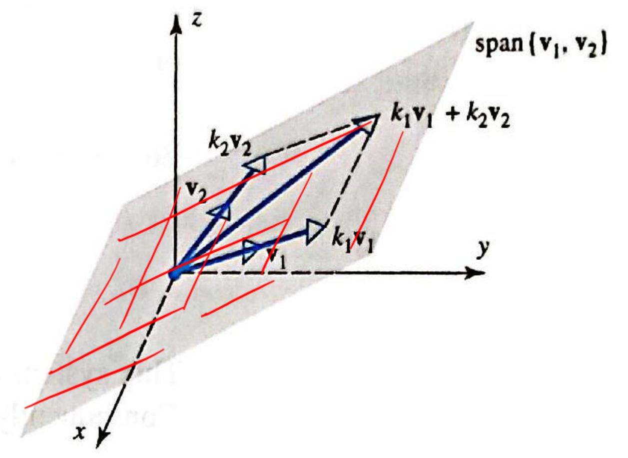
Picturing Spans

If v is a nonzero vector in $\mathbb{R}^3$ in standard position, then span$\{v\}$ is just the line through the origin determined by v.



(a) Span {v} is the line through the origin determined by v.

Let v_1 and v_2 be two nonzero noncollinear vectors in $\mathbb{R}^3$ in standard position. The span$\{v_1, v_2\}$ is the plane through the origin determined by these two vectors



(b) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Spans (20)

An Exercise on Linear Combinations

Consider the vectors $u = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$. Show that $w = \begin{pmatrix} 9 \\ 2 \\ 7 \end{pmatrix}$ is a linear combination of u and v and that $z = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix}$ is not a linear combination of u and v.

$$* \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \xrightarrow{-\frac{1}{8}R_2} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{array}\right] \xrightarrow{R_3 - 8R_2} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right] \qquad \underline{x = -3,\ y = 2}$$

$$* \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right] \xrightarrow{-\frac{1}{8}R_2} \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 8 & 12 \end{array}\right] \xrightarrow{R_3 - 8R_2} \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & 1 & \frac{9}{8} \\ 0 & 0 & 3 \end{array}\right]$$

$0x + 0y = 3 \Rightarrow$
No solution.

z is not in span$\{u, v\}$!