TABLE II
STEP RESPONSE EVALUATION OF ADAPTIVE SAMPLING SCHEMES AGAINST PERIODIC SAMPLING

Response Error E / Sampling Scheme		Sample Numbers 30	20	15
Periodic Sampling $T_i = T$	T (sec)	0.04	0.06	0.08
	E	0.05962	0.09704	0.14314
$T_i = \frac{0.1}{\alpha_1 \dot{e}_i^2 + 1}$	α_1	2.5	0.05	0.008
	E	0.03594	0.05560	0.09876
$T_i = \frac{0.1}{\alpha_2 \lvert\dot{e}_i\rvert + 1}$	α_2	1.4	0.275	0.05
	E	0.03424	0.0550	0.11105
$T_i = \frac{C}{\sqrt{\lvert\dot{e}_i\rvert}}$	C	0.03	0.08	0.15
	E	0.03756	0.05443	0.09125
$T_i = 0.1 - K\lvert\dot{e}_i\rvert$	K	0.065	0.01	0.003
	E	0.03584	0.05587	0.1162

The plant is $G(s) = (10(s + 10))/s^2$ and $T_{\max} = 0.1$ s, $T_{\min} = 0.02$ s. The response error E is $E = \int_0^{1.2} |y^*(t) - y(t)|\, dt$, where $y^*(t)$ is the response of the continuous closed-loop system. The response time considered is 1.2 s.

This same plant has been investigated many times in the past. The bounds for T_i are $T_{\min} = 0.02$ s and $T_{\max} = 0.1$ s. Unit step input is applied. The responses for different control laws over a period of 1.2 s are recorded in Table II, in which the output error E is computed as

$$E = \int_0^{1.2} |y^*(t) - y(t)|\, dt \tag{17}$$

where $y^*(t)$ is the step response of the continuous closed-loop system. Note that the function E is arbitrarily selected [8].

The results in Table II show that, 1) the adjustable coefficient in a given control law directly affects the sample numbers and the output accuracy and 2) all four adaptive systems are more efficient and their efficiency improvements are nearly equal. It is noted that this later observation is highly dependent on the plant, the type of input as well as the error criterion E(17) [8].

X. CONCLUSION

It has been shown that the problem of adaptive sampling control law design can now be analytically solved by way of minimizing an objective function. The generalized objective function is capable of reproducing all the major existing control laws as well as new ones. Hence, the proposed method can be regarded as a unified analytical approach to control law design. Experimental results have demonstrated that adaptive sampling can achieve sampling efficiency.

It is important to point out that the minimization of a cost function over each sampling period as proposed in this paper, results in a design procedure which is independent of the plant and the input. Therefore a more meaningful design approach would be to use an objective function covering the entire interval of operation. This is unquestionably a more difficult problem to solve. New results in this direction would be highly desirable.

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Reduced-Order Observers for Linear Discrete-Time Systems

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***Abstract*—Luenberger's observer is considered as an alternate to the Kalman filter for obtaining state estimates in linear discrete-time stochastic systems. An interesting new solution to the problem of constructing optimal and suboptimal reduced-order observers is presented. The solution contains as special cases both Kalman's optimal filter and the optimal minimal-order observer of Leondes and Novak. Also, the Tse and Athans observer is obtained as a special case of the reduced-order observer solution.**

I. PROBLEM STATEMENT

We consider observable linear discrete-time systems of the form

$$x_{i+1} = A_i x_i + B_i u_i + w_i \tag{1}$$

$$y_i = H_i x_i + v_i. \tag{2}$$

The state of the system at time "i" is given by the n-dimensional vector x_i. u_i is a p-dimensional control input and y_i is an m-dimensional measurement vector. The measurement matrix H_i is assumed to be of rank "m" at each instant "i" in the interval of interest. w_i and v_i are independent white-noise sequences with known statistics

$$E(w_i) = 0,\ E(v_i) = 0, \qquad \text{for all } i \tag{3}$$

$$E(w_i w_j') = Q_i \delta_{ij},\ E(v_i v_j') = R_i \delta_{ij} \tag{4}$$

$$E(w_i v_j') = 0, \qquad \text{for all } i,j \tag{5}$$

where δ_{ij} is the Kronecker delta. The initial state x_0 is an independent random vector with known statistics

$$E(x_0) = \bar{x}_0,\ E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = \Sigma_{0/0}. \tag{6}$$

In the typical state estimation problem it is desired to obtain an estimate $\hat{x}_{i/i}$ of the state vector x_i along with its corresponding error covariance $\Sigma_{i/i}$. If it is desired that the estimate be optimal in the mean-square sense (which implies that the estimate $\hat{x}_{i/i}$ minimizes the quantity $E\{\|\hat{x}_{i/i} - x_i\|^2\}$) then the solution to the *unconstrained* estimation problem is the well-known Kalman filter [5]. Clearly, the Kalman filter is a Luenberger observer [6] of dimension "n" and among the class of all linear observers the Kalman filter provides the best possible performance in the mean-square sense.

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In [3], Leondes and Novak have considered the design of minimal-order observer-estimators for systems of the form (1),(2). In [3] the observer was *constrained* to be of order "$n - m$" and a unique minimal-order observer was constructed which yielded estimates $\hat{x}_i$ of the state vector x_i such that the mean-square estimation error $E\{||\hat{x}_i - x_i||^2\}$ is minimized. Thus, the minimal-order observer of [3] provides the best possible performance in the mean-square sense among the class of all "$n - m$" dimensional observers. It is reasonable to expect that as the number of dynamically filtered observer output variables is increased from the minimum required "$n - m$," the resulting estimation error is correspondingly decreased. Hence, the idea of considering reduced-order observer-estimators is appealing. By definition a reduced-order observer has dynamic order less than the Kalman filter but greater than the minimal-order observer.

The work reported in this paper considers the design of reduced-order observers for discrete systems of the form (1),(2). In this paper it is assumed that all "m" components of the measurement vector are contaminated with measurement noise; however, it is further assumed that "m_1" components of the measurement vector are of "good quality" relative to the remaining "$m - m_1$" components. These "m_1" accurate measurements are utilized to construct a reduced-order observer of dimension "$n - m_1$." When $m_1 = m$ the solution reduces to the minimal-order observer of [3] and when $m_1 = 0$ the solution reduces to the Kalman filter. Finally, if the m_1 accurate measurement components are actually perfect (noise-free), the solution is equivalent to the Tse and Athans reduced-order observer [8].

II. Definition of the Discrete Observer for Stochastic Systems

In this section the notion of an observer for discrete-time stochastic systems is introduced and a precise mathematical description of the observer is given. To begin, we consider discrete-time dynamical systems (observers) of the form

$$z_{i+1} = F_i z_i + G_i u_i + D_i y_i \tag{7}$$

where the observer state vector z_i satisfies the relation

$$z_i = T_i x_i + \varepsilon_i. \tag{8}$$

Thus, for stochastic systems the observer is a linear system whose output vector, z_i, is an estimate of the quantity $T_i x_i$ and ε_i is the error in the estimate of $T_i x_i$. Note that if T_i is an $r \times n$ linear transformation then the observer is of dimension "r" and the matrices F_i, G_i, and D_i are of the order $r \times r$, $r \times p$, and $r \times m$, respectively.

The corresponding Luenberger necessary conditions which must be satisfied at each instant "i" are given as follows (see [1],[2]).

$$T_{i+1} A_i = F_i T_i + D_i H_i \tag{9}$$

$$T_{i+1} B_i = G_i \tag{10}$$

where the observer error, ε_i, satisfies the equation

$$\varepsilon_{i+1} = F_i \varepsilon_i + D_i v_i - T_{i+1} w_i. \tag{11}$$

From (7) it is seen that the observer provides "r" filtered variables which are linearly related to the state vector x_i according to the relation (8). In this paper it is assumed that an estimate of the entire state vector x_i is desired, hence, the estimate $\hat{x}_i$ of the state x_i is obtained by combining these filtered variables, z_i, with the given measurements y_i, in the following linear fashion:

$$\hat{x}_i = P_i z_i + V_i y_i \tag{12}$$

where the weighting matrices P_i and V_i are $n \times r$ and $n \times m$, respectively. Since it is desired that the estimate $\hat{x}_i$ be unbiased, using (2), (8), and (12) one obtains the additional necessary condition

$$P_i T_i + V_i H_i = I_n. \tag{13}$$

But since the rank of the identity matrix I_n is "n," it is clear that the observer dimension "r" must be greater than or equal to $n - m$ if (13) is to be satisfied. Therefore, for the reduced-order observer considered in this paper the observer output, z_i, is taken to be an $(n - m_1)$ dimensional vector where $0 < m_1 < m$. The solutions $m_1 = m$ (minimal-order observer) and $m_1 = 0$ (Kalman filter) are obtained as special cases of the more general solution to be developed.

Finally, the observer design to be presented is based on Huddle's equations[1] which are

$$F_i = T_{i+1} A_i P_i \tag{14}$$

$$D_i = T_{i+1} A_i V_i \tag{15}$$

where P_i, T_i, and V_i are chosen to satisfy the constraint $P_i T_i + V_i H_i = I_n$. As indicated in (14) and (15) the matrices F_i and D_i are not selected directly but are chosen as functions of the matrices P_i, T_i, and V_i. The idea then is to obtain matrices P_i, T_i, and V_i (which satisfy the constraint $P_i T_i + V_i H_i = I_n$) containing arbitrary gain elements which are chosen to minimize the mean-square estimation error at each instant "i."

III. Construction of the Reduced-Order Observer

The construction of a reduced-order observer proceeds as follows. We begin by partitioning the measurement vector into the form

$$\begin{bmatrix} y_i^{(1)} \\ y_i^{(2)} \end{bmatrix} = H_i x_i + \begin{bmatrix} v_i^{(1)} \\ v_i^{(2)} \end{bmatrix} \tag{16}$$

where $y_i^{(1)}$, an "m_1" vector, represents the relatively noise-free measurements and $y_i^{(2)}$, an "$m - m_1$" vector, represents the remaining rather noisy measurements. Separation of the various measurement components into the corresponding subvectors $y_i^{(1)}$ and $y_i^{(2)}$ is, of course, at the discretion of the designer. In the special case where the "m_1" measurement components are actually perfect measurements it is clear how to form the partitions $y_i^{(1)}$ and $y_i^{(2)}$. However, in general we do not assume $v_i^{(1)} = 0$. It is sufficient at this point to assume that an optimum selection for $y_i^{(1)}$ and $y_i^{(2)}$ has been made by the designer. Upon transforming the given system characterized by (1) and (2) to the desired "observer canonical coordinate system" one obtains the equivalent state space representation

$$\begin{bmatrix} q_{i+1}^{(1)} \\ \hline q_{i+1}^{(2)} \end{bmatrix} = \left[\begin{array}{c|c} \bar{A}_{11}^i & \bar{A}_{12}^i \\ \hline \bar{A}_{21}^i & \bar{A}_{22}^i \end{array}\right] \begin{bmatrix} q_i^{(1)} \\ \hline q_i^{(2)} \end{bmatrix} + \begin{bmatrix} \bar{w}_i^{(1)} \\ \hline \bar{w}_i^{(2)} \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} y_i^{(1)} \\ \hline y_i^{(2)} \end{bmatrix} = \left[\begin{array}{c|c|c} I_{m_1} & 0 & \\ \hline 0 & I_{m-m_1} & 0 \end{array}\right] \begin{bmatrix} q_i^{(1)} \\ \hline q_i^{(2)} \end{bmatrix} + \begin{bmatrix} v_i^{(1)} \\ \hline v_i^{(2)} \end{bmatrix}. \tag{18}$$

This equivalent state-space representation is obtained by taking $q_i = M_i x_i$ where M_i is the invertible linear transformation defined by

$$M_i = \begin{bmatrix} H_i \\ \hline \text{--------} \\ \text{--------} \\ \text{--------} \end{bmatrix} \tag{19}$$

and the lower partition of M_i consists of any row vectors which make M_i nonsingular. Since the measurement matrix H_i is assumed to be full rank at each "i" in the interval of interest, the linear transformation defined in (19) always exists and therefore it can be assumed without loss of generality that the original system (1) and (2) is already in the desired canonical form. Thus, the state equations which define our system model are taken to be

[1] To verify that Huddle's equations satisfy the Luenberger conditions one merely substitutes (14) and (15) into (9) and uses the fact that $P_i T_i + V_i H_i = I$ (see [1],[2]).

$$\begin{bmatrix} x_{i+1}^{(1)} \\ \hline x_{i+1}^{(2)} \end{bmatrix} = \left[\begin{array}{c|c} A_{11}^i & A_{12}^i \\ \hline A_{21}^i & A_{22}^i \end{array}\right] \begin{bmatrix} x_i^{(1)} \\ \hline x_i^{(2)} \end{bmatrix} + \begin{bmatrix} w_i^{(1)} \\ \hline w_i^{(2)} \end{bmatrix} \tag{20}$$

$$y_i^{(1)} = x_i^{(1)} + v_i^{(1)} \tag{21}$$

$$y_i^{(2)} = H_i^{(2)} x_i^{(2)} + v_i^{(2)} \tag{22}$$

where

$$H_i^{(2)} \triangleq [I_{m-m_1} | 0]. \tag{23}$$

Note in (20)–(23) that $x_i^{(1)}$ is an "m_1" vector, $x_i^{(2)}$ is an "$n - m_1$" vector, and the A_i matrix is partitioned appropriately.

The estimate $\hat{x}_i$ of the state vector x_i is obtained as follows:

$$\begin{bmatrix} \hat{x}_i^{(1)} \\ \hline \hat{x}_i^{(2)} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ \hline I_{n-m_1} \end{bmatrix}}_{P_i} z_i + \underbrace{\left[\begin{array}{c|c} I_{m_1} & 0 \\ \hline K_i^{(1)} & K_i^{(2)} \end{array}\right]}_{V_i} \begin{bmatrix} y_i^{(1)} \\ \hline y_i^{(2)} \end{bmatrix}. \tag{24}$$

We remark here that the estimate given in (24) is a generalization of the minimal-order observer solution of [3]. As in the development of [3], the $(n - m_1)$ by m_1 gain matrix $K_i^{(1)}$ and the $(n - m_1)$ by $(m - m_1)$ gain matrix $K_i^{(2)}$ are chosen to minimize the mean-square estimation error. Since it is required that the relation $P_i T_i + V_i H_i = I_n$ be satisfied, where P_i is n by $(n - m_1)$, T_i is $(n - m_1)$ by n, V_i is n by m, and H_i is m by n, the observer transformation T_i is found to be

$$T_i = [-K_i^{(1)} | I_{n-m_1} - K_i^{(2)} H_i^{(2)}]. \tag{25}$$

From (12) it is easily shown that the estimation error at time "$i + 1$" is given by the expression

$$e_{i+1} \triangleq [P_{i+1} | V_{i+1}] \begin{bmatrix} \varepsilon_{i+1} \\ \hline v_{i+1} \end{bmatrix} = [\underbrace{\begin{array}{c} 0 \\ \hline I_{n-m_1} \end{array}}_{P_{i+1}} | \underbrace{\begin{array}{c|c} I_{m_1} & 0 \\ \hline K_{i+1}^{(1)} & K_{i+1}^{(2)} \end{array}}_{V_{i+1}}] \begin{bmatrix} \varepsilon_{i+1} \\ \hline v_{i+1}^{(1)} \\ v_{i+1}^{(2)} \end{bmatrix}. \tag{26}$$

Evaluating the trace of the estimation error covariance gives

$$\begin{aligned} \text{trace } \overline{e_{i+1} e_{i+1}'} &= \text{trace } \overline{v_{i+1}^{(1)} v_{i+1}^{(1)\prime}} \\ &\quad + \text{trace } \overline{(\varepsilon_{i+1} + K_{i+1}^{(1)} v_{i+1}^{(1)} + K_{i+1}^{(2)} v_{i+1}^{(2)})} \\ &\quad \overline{\cdot (\varepsilon_{i+1} + K_{i+1}^{(1)} v_{i+1}^{(1)} + K_{i+1}^{(2)} v_{i+1}^{(2)})'}. \end{aligned} \tag{27}$$

Substituting the solution $F_i = T_{i+1} A_i P_i$ and $D_i = T_{i+1} A_i V_i$ into the observer error equation (11) one obtains the observer error covariance given as (28).

$$\overline{\varepsilon_{i+1} \varepsilon_{i+1}'} = T_{i+1}(A_i P_i \overline{\varepsilon_i \varepsilon_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i) T_{i+1}'. \tag{28}$$

It is useful to partition (28) as follows:

$$\overline{\varepsilon_{i+1} \varepsilon_{i+1}'} = T_{i+1} \left[\begin{array}{c|c} \Omega_{11}^i & \Omega_{12}^i \\ \hline \Omega_{21}^i & \Omega_{22}^i \end{array}\right] T_{i+1}' \tag{29}$$

where Ω_{11}^i is m_1 by m_1, Ω_{22}^i is $(n - m_1)$ by $(n - m_1)$, and $\Omega_{21}^i = \Omega_{12}^i$ is $(n - m_1)$ by m_1. Equation (29) plays a fundamental role in the observer design to be developed.

Finally, evaluating (27) using (29) yields the result

$$\begin{aligned} \text{trace } \overline{e_{i+1} e_{i+1}'} &= \text{trace } R_{i+1}^{11} + \text{trace } K_{i+1}^{(1)} R_{i+1}^{11} K_{i+1}^{(1)\prime} \\ &\quad + 2 \text{ trace } K_{i+1}^{(1)} R_{i+1}^{11} + \text{trace } K_{i+1}^{(2)} R_{i+1}^{22} K_{i+1}^{(2)\prime} \\ &\quad + \text{trace } K_{i+1}^{(1)} \Omega_{11}^i K_{i+1}^{(1)\prime} - 2 \text{ trace } \Omega_{21}^i K_{i+1}^{(1)\prime} \\ &\quad + \text{trace } \Omega_{22}^i + \text{trace } K_{i+1}^{(2)} H_{i+1}^{(2)} \Omega_{22}^i H_{i+1}^{(2)\prime} K_{i+1}^{(2)\prime} \\ &\quad + 2 \text{ trace } K_{i+1}^{(1)} \Omega_{12}^i H_{i+1}^{(2)\prime} K_{i+1}^{(2)\prime} \\ &\quad - 2 \text{ trace } \Omega_{22}^i H_{i+1}^{(2)\prime} K_{i+1}^{(2)\prime}. \end{aligned} \tag{30}$$

In obtaining (30) the measurement noise covariance matrix is partitioned appropriately as follows:

$$R_{i+1} \triangleq \left[\begin{array}{c|c} R_{i+1}^{11} & R_{i+1}^{12} \\ \hline R_{i+1}^{21} & R_{i+1}^{22} \end{array}\right] \tag{31}$$

where R_{i+1}^{11} is m_1 by m_1, R_{i+1}^{22} is $(m - m_1)$ by $(m - m_1)$ and $R_{i+1}^{21} = R_{i+1}^{12\prime}$ is $(m - m_1)$ by m_1.

Minimizing the trace $\overline{e_{i+1} e_{i+1}'}$ is straightforward; taking gradients of (30) with respect to the gain matrices $K_{i+1}^{(1)}$ and $K_{i+1}^{(2)}$ yields the necessary and sufficient conditions for a minimum which are the following pair of coupled matrix equations:

$$\begin{aligned} &K_{i+1}^{(1)}(\Omega_{11}^i + R_{i+1}^{11}) + K_{i+1}^{(2)}(H_{i+1}^{(2)} \Omega_{21}^i + R_{i+1}^{21}) - \Omega_{21}^i = 0 \\ &K_{i+1}^{(2)}(H_{i+1}^{(2)} \Omega_{22}^i H_{i+1}^{(2)\prime} + R_{i+1}^{22}) + K_{i+1}^{(1)}(\Omega_{12}^i H_{i+1}^{(2)\prime} \\ &\qquad + R_{i+1}^{12}) - \Omega_{22}^i H_{i+1}^{(2)\prime} = 0. \end{aligned} \tag{32}$$

Finally, the minimizing gain matrices $K_{i+1}^{(1)}$ and $K_{i+1}^{(2)}$ are obtained as follows:

$$[K_{i+1}^{(1)} | K_{i+1}^{(2)}] = [\Omega_{21}^i | \Omega_{22}^i H_{i+1}^{(2)}] \tag{33}$$

$$\cdot \left[\begin{array}{c|c} \Omega_{11}^i + R_{i+1}^{11} & R_{i+1}^{12} + \Omega_{12}^i H_{i+1}^{(2)\prime} \\ \hline H_{i+1}^{(2)} \Omega_{21}^i + R_{i+1}^{21} & H_{i+1}^{(2)} \Omega_{22}^i H_{i+1}^{(2)\prime} + R_{i+1}^{22} \end{array}\right]^{-1}.$$

Note that to guarantee a unique solution of equations (32) it is sufficient to assume the covariances $R_{i+1}^{22} > 0$ and $\overline{w_i^{(1)} w_i^{(1)\prime}} > 0$. Thus, when the measurements $y_i^{(1)}$ are noise-free ($R_{i+1}^{11} = 0$) or only partially noisy ($R_{i+1}^{11} \geq 0$) the partitions $\Omega_{11}^i + R_{i+1}^{11} > 0$ and $H_{i+1}^{(2)} \Omega_{22}^i H_{i+1}^{(2)\prime} + R_{i+1}^{22} > 0$ and the matrix inverse in (33) exists (see [11]).

Initialization of the observer proceeds as follows. Let $z_1 = T_1 \bar{x}_1$ be the observer initial condition, where $\bar{x}_1 = A_0 \bar{x}_0$ is the "expected value" of the state vector x_1. Since $\varepsilon_1 = z_1 - T_1 x_1$, then

$$\overline{\varepsilon_1 \varepsilon_1'} = T_1 \overline{(x_1 - \bar{x}_1)(x_1 - \bar{x}_1)'} T_1'. \tag{34}$$

But $x_1 - \bar{x}_1 = A_0(x_0 - \bar{x}_0) + w_0$; hence, (34) becomes

$$\overline{\varepsilon_1 \varepsilon_1'} = T_1(A_0 \Sigma_{0/0} A_0' + Q_0) T_1'. \tag{35}$$

To initialize the observer, define the covariance matrix Ω_0 to be

$$\Omega_0 = A_0 \Sigma_{0/0} A_0' + Q_0 \tag{36}$$

and evaluate the initial observer gain matrix given in (33).

IV. Limiting Cases of the Reduced-Order Observer Solution

It is of interest to consider the behavior of the reduced-order observer solution for several limiting cases. First we evaluate the observer solution when $m_1 = m$ and $m_1 = 0$. Next, the special case where the noise vector $v_i^{(1)} \equiv 0$ is investigated.

A. Minimal-Order Observer, $m_1 = m$

The special case $m_1 = m$ corresponds to the situation where the entire m-dimensional measurement vector is assumed by the designer to be of sufficient accuracy so that filtering of these states is unnecessary (that is, $\hat{x}_i^{(1)} = y_i$). This solution corresponds to a minimal-order observer and the dynamical portion of the estimator is of dimension "$n - m$." Consider the behavior of the reduced-order observer solution given by (24)–(33). Taking $m_1 = m$ results in an observer of dimension $z_i = n - m$ and

$$K_{i+1}^{(1)} = K_{i+1} = \Omega_{21}^i(\Omega_{11}^i + R_{i+1})^{-1}, \quad K_{i+1}^{(2)} = 0 \tag{37}$$

$$T_i = [-K_i | I_{n-m}], \quad v_i = \begin{bmatrix} I_m \\ \hline K_i \end{bmatrix}, \quad P_i = \begin{bmatrix} 0 \\ \hline I_{n-m} \end{bmatrix}. \tag{38}$$

Comparison of (37) and (38) with the results of [3] shows that the optimal minimal-order observer of Leondes and Novak is merely a special case of the more general low-order observer solution presented in this paper.

B. Kalman Filter, $m_1 = 0$

The special case $m_1 = 0$ corresponds to the situation where none of the components of the measurement vector are assumed by the designer to be of sufficient accuracy and therefore filtered estimates of all "n" states are required. Obviously in this situation the dynamical portion of the observer-estimator has dimension "n." It is easily shown that for $m_1 = 0$ the reduced-order observer solution is equivalent to a Kalman filter. Taking $m_1 = 0$ in (24)–(33) results in an observer of dimension $z_i = n$ and

$$K_i^{(1)} = 0, \quad K_i^{(2)} = K_i(n \times m) \tag{39}$$

$$V_i = K_i, \quad P_i = I_n, \quad T_i = I_n - K_i H_i. \tag{40}$$

Thus the estimate becomes

$$\hat{x}_{i+1} = z_{i+1} + K_{i+1} y_{i+1}. \tag{41}$$

But

$$z_{i+1} = \underbrace{T_{i+1} A_i z_i}_{F_i z_i} + \underbrace{T_{i+1} A_i K_i y_i}_{D_i y_i}$$

$$= (I_n - K_{i+1} H_{i+1}) A_i \underbrace{(z_i + K_i y_i)}_{\hat{x}_i}. \tag{42}$$

Finally, substituting (42) into (41) gives

$$\hat{x}_{i+1} = A_i \hat{x}_i + K_{i+1}(y_{i+1} - H_{i+1} A_i \hat{x}_i). \tag{43}$$

It is clear that the estimate (43) is in the form of a Kalman filter [5] and it remains only to show that the gain matrix K_{i+1} in (43) is identical to Kalman's weighting matrix. However, this equivalence is easily verified by inspection of (33) and using the fact that

$$\Omega_i = A_i \overline{e_i e_i'} A_i' + Q_i \triangleq \Sigma_{i+1/i}. \tag{44}$$

C. Some Perfect Measurements, $v_i^{(1)} = 0$

Recently results have been reported in the literature on the design of reduced-order filters for stochastic systems of the form (1),(2) in the special case when several of the components of the measurement vector are noise-free (i.e., $v_i^{(1)} = 0$) (see Brammer [7], Tse and Athans [8], and Yoshikawa and Kobayashi [9]). It is clear in this case that since some of the states (or linear combinations of the states) are measured perfectly, then it is not necessary to estimate these quantities and thus an optimal estimator of reduced dimension can be constructed. Brammer's approach is based on the orthogonal projection lemma whereas Tse and Athans have applied the matrix minimum principle to obtain the reduced-order optimal filter. It is easy to show that the reduced-order observer developed in this paper is identical to Kalman's optimal filter when the noise term $v_i^{(1)}$ is zero. The proof goes as follows: One simply repeats the derivation of (39)–(43) and then verifies that the gain matrix in (33) is identical to Kalman's gain matrix when $R_{i+1}^{11} = 0$ and $R_{i+1}^{12} = R_{i+1}'^{21} = 0$.

Brammer [7] developed his optimal reduced-order observer solution using the orthogonal projection lemma. His work, apparently overlooked by recent researchers, has an interesting interpretation which we shall present here. Consider the system (1),(2) where the measurements are of the form

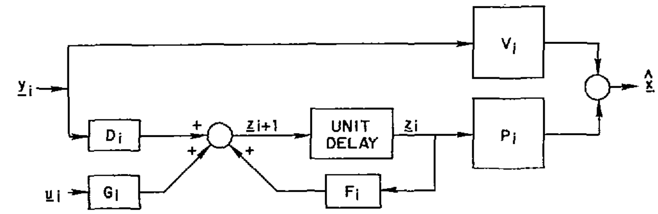
$$\begin{bmatrix} y_i^{(1)} \\ \hline y_i^{(2)} \end{bmatrix} = \begin{bmatrix} \hat{H}_i^{(1)} \\ \hline \hat{H}_i^{(2)} \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \hline v_i^{(2)} \end{bmatrix}. \tag{45}$$

Brammer took as his estimate

$$\hat{x}_i = \hat{P}_i \hat{z}_i + \hat{V}_i^{(1)} y_i^{(1)} \tag{46}$$

where $\hat{T}_i$, the $(n - m_1) \times n$ observer transformation matrix, is chosen such that

$$\begin{bmatrix} \hat{T}_i \\ \hline \hat{H}_i^{(1)} \end{bmatrix}^{-1} = [\hat{P}_i | \hat{V}_i^{(1)}]. \tag{47}$$



OPTIMAL GAIN ALGORITHM

$K_i = [K_i^{(1)} | K_i^{(2)}]$ (EQUATION (33))

$\Omega_i = A_i P_i \overline{\varepsilon_i \varepsilon_i'} P_i' A_i' + A_i V_i R_i V_i' A_i' + Q_i$

$\Omega_0 = A_0 M_0 A_0' + Q_0$

$\overline{\varepsilon_{i+1} \varepsilon_{i+1}'} = T_{i+1} \Omega_i T_{i+1}'$

Fig. 1. Reduced-order observer structure.

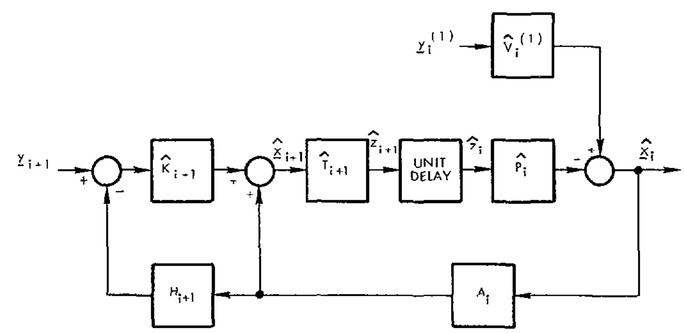


Fig. 2. Brammer's optimal reduced-order observer.

Noting in Brammers' solution (see [7, eq. (13)–(17)]) that his optimal gain matrix is of the form $\hat{L}_{i+1} = \hat{T}_{i+1} \hat{K}_{i+1}$, Brammer's solution can be written

$$\hat{z}_{i+1} = \hat{T}_{i+1}[A_i \hat{x}_i + \hat{K}_{i+1}(y_{i+1} - H_{i+1} A_i \hat{x}_i)]. \tag{48}$$

Brammer's gain matrix is computed as follows:

$$\hat{K}_{i+1} = \Sigma_{i+1/i} H_{i+1}'(H_{i+1} \Sigma_{i+1/i} H_{i+1}' + R_{i+1})^{-1} \tag{49}$$

$$\Sigma_{i+1/i} = A_i \hat{P}_i S_i \hat{P}_i' A_i' + Q_i \tag{50}$$

$$S_{i+1} = \hat{T}_{i+1}(I - \hat{K}_{i+1} H_{i+1}) \Sigma_{i+1/i} \hat{T}_{i+1}'. \tag{51}$$

Obviously, Brammer's optimal reduced-order observer (see Fig. 2) reduces to Kalman's filter when $\hat{T}_i = \hat{P}_i = I_n$ and $\hat{V}_i^{(1)} = 0$. Finally, we remark that the Leondes–Novak reduced-order observer (Fig. 1) may be obtained from Brammer's reduced-order observer (Fig. 2) by noting that for the given state equations (20)–(23) we obtain

$$\hat{H}_i^{(1)} = [I_{m_1} | 0], \quad \hat{T}_i = [0 | I_{n-m_1}] \tag{52}$$

$$\hat{P}_i = \begin{bmatrix} 0 \\ \hline I_{n-m_1} \end{bmatrix}, \quad \hat{V}_i = [\hat{V}_i^{(1)} | 0] = \left[\begin{array}{c|c} I_{m_1} & 0 \\ \hline 0 & \end{array} \right]. \tag{53}$$

Then by taking $T_i = \hat{T}_i - \hat{L}_i \hat{H}_i$, $P_i = \hat{P}_i$, and $V_i = \hat{V}_i + \hat{P}_i \hat{L}_i$ one obtains the results (24) and (25). Thus by a simple rearrangement of the block diagram of Fig. 2 one arrives at the observer configured in Fig. 1.

Tse and Athans [8] applied the matrix minimum principle to obtain their optimal reduced-order observer. However, as pointed out by Yoshikawa and Kobayashi [9], to construct the observer utilizing the approach of Tse and Athans one is required to solve a set of $(n - m)\, n + m_2 m$ simultaneous algebraic equations (see [8, eq. (54)–(64)]. Utilizing the compatibility criterion (theorem 4 of Tse and Athans), Yoshikawa and Kobayashi obtained an explicit closed-form solution for one such optimal reduced-order observer. It is easily verified that the Leondes-Novak observer shown in Fig. 1

satisfies the equations obtained by Yoshikawa and Kobayashi (see [9, eq. (13)–(14)] and is therefore an optimal reduced-order observer when $v_i^{(1)} = 0$.

V. Example: Velocity-Aided Tracking Filter

To illustrate the design of a reduced-order observer the following radar tracking problem is considered.

$$\begin{bmatrix} x_{i+1} \\ \dot{x}_{i+1} \\ \ddot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w_i \tag{54}$$

$$y_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \\ \ddot{x}_i \end{bmatrix} + v_i. \tag{55}$$

As indicated in (55) both position and velocity of the target are measured by the radar. It is assumed the position and velocity measurements are independent, thus the measurement noise covariance R_i is of the diagonal form

$$R_i = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_{\dot{x}}^2 \end{bmatrix} \tag{56}$$

where σ_x^2 is the position measurement noise variance and $\sigma_{\dot{x}}^2$ is the velocity measurement noise variance. W_i is assumed to be an independent zero-mean white sequence with variance σ_m^2 (maneuver variance). The position measurements are assumed sufficiently accurate so that the estimate of target position is simply the radar position measurement. However the velocity measurements are rather noisy so that filtered estimates of target velocity and acceleration are desired. Therefore, a two-dimensional reduced-order observer is designed using the relations (20)–(36). The observer and Kalman filter are initialized by taking as the initial state estimate $\bar{x}_0$ the following "least-squares" estimate:

$$\bar{x}_0 = \begin{bmatrix} y_0 \\ \frac{1}{T}\left(\frac{3}{2}y_0 - 2y_{-1} + \frac{1}{2}y_{-2}\right) \\ \frac{1}{T^2}\left(y_0 - 2y_{-1} + y_{-2}\right) \end{bmatrix} \tag{57}$$

where y_{-2}, y_{-1}, and y_0 are, respectively, the first, second, and third radar position measurements. The corresponding covariance equation $\Sigma_{0/0}$ is given by the following:

$$\Sigma_{0/0} = \begin{bmatrix} \sigma_x^2 & \frac{3}{2}\frac{\sigma_x^2}{T} & \frac{\sigma_x^2}{T^2} \\ \frac{3}{2}\frac{\sigma_x^2}{T} & \frac{13}{2}\frac{\sigma_x^2}{T^2} + \frac{T^2\sigma_m^2}{16} & \frac{6\sigma_x^2}{T^3} + \frac{\sigma_m^2 T}{8} \\ \frac{\sigma_x^2}{T^2} & \frac{6\sigma_x^2}{T^3} + \frac{\sigma_m^2 T}{8} & \frac{6\sigma_x^2}{T^4} + \frac{5\sigma_m^2}{4} \end{bmatrix}. \tag{58}$$

Performance of the reduced-order observer and the corresponding Kalman filter is presented in Fig. 3 which shows the mean-square error in the velocity estimate for both filters. It is clear from Fig. 3 that the performance of the reduced-order filter is excellent and is comparable with the Kalman filter. (A similar comparison of the acceleration estimates for both filters also showed excellent performance for the observer.) Note from Fig. 3 that when velocity data is very inaccurate ($\sigma_{\dot{x}}^2 \to \infty$) the reduced-order observer provides estimates identical to the minimal-order observer of [3]. Similarly, when $\sigma_{\dot{x}}^2 \to \infty$ the Kalman filter performance becomes equivalent to the situation where no velocity input is available. Intuitively one expects these results since both estimators weigh in an optimal manner the position and velocity inputs. Thus, when the velocity input is inaccurate both filters rely almost entirely on the position in information thereby producing essentially the same results as when no

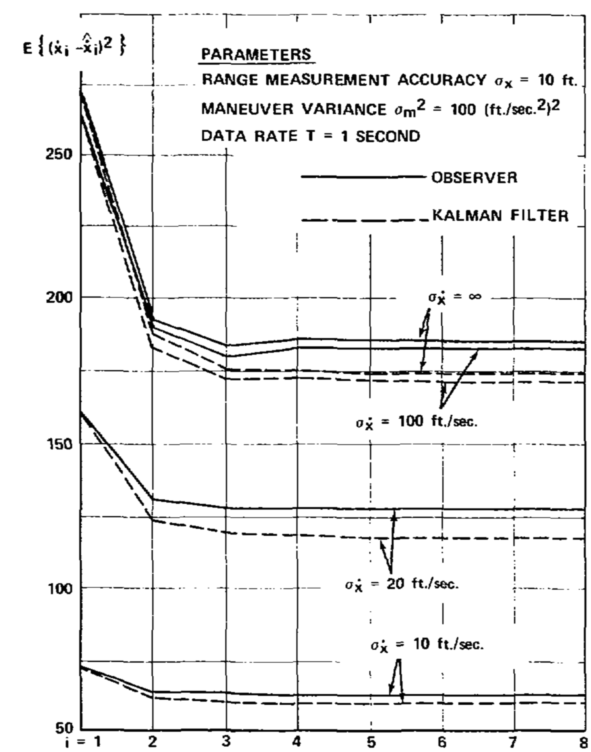


Fig. 3. Mean-square error in velocity.

velocity input is available. When velocity data is accurate, however, it is weighed more heavily by both filters and the resultant accuracies are improved considerably over that obtained in the absence of velocity data.

VI. Conclusions

This paper generalizes and extends the results of Leondes and Novak [3] to the design of optimal and suboptimal reduced-order observers. A unique observer structure is developed. The observer is characterized by a gain matrix which is computed recursively using an algorithm similar to Kalman's gain algorithm. In the special case when some measurements are noise-free ($v_i^{(1)} = 0$) the solution reduces to a Kalman filter [5]. For this case the solution is also shown to be equivalent to Brammer's optimal reduced-order observer [7] and Tse and Athans optimal minimal-order observer [8]. Application of the theory to a simple velocity-aided tracking filter illustrates that the performance obtained may be comparable to that achieved with a Kalman filter.

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