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Efficient and consistent reliability analysis of soil slope stability using both limit equilibrium analysis and finite element analysis

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ABSTRACT

Limit equilibrium methods (LEMs) and finite element methods (FEMs) of slope stability analysis can be used in computer-based probabilistic simulation approaches (e.g., direct Monte Carlo Simulation (MCS) and Subset Simulation (SS)) to evaluate the slope failure probability (P_f). For a given slope problem, the computational effort for the LEM is generally much less than that required for the FEM, but the FEM tends to give a more realistic prediction of slope failure mechanism and its associated factor of safety. To make use of the advantages of both the LEM (e.g., computationally more efficient) and FEM (e.g., theoretically more realistic and rigorous in terms of slope failure mechanisms), a new probabilistic simulation method is developed in the paper. The proposed approach combines both a simple LEM (i.e., Ordinary Method of Slices considering a limited number of potential slip surfaces) and FEM with the response conditioning method to efficiently calculate P_f of slope stability and to give an estimate of P_f consistent with that obtained from directly performing MCS and SS based on the FEM. It is illustrated through two soil slope examples. Results show that the proposed approach calculates the P_f properly at small probability levels (e.g., $P_f < 0.001$). More importantly, it significantly reduces the number of finite element analyses needed in the calculation, and therefore improves the computational efficiency at small probability levels that are of great interest in slope design practice. In addition, the proposed approach opens up the possibility that makes use of the information obtained using a simple model (e.g., LEM) to guide the reliability analysis based on a relatively sophisticated model (e.g., FEM).

1. Introduction

During the past few decades, several probabilistic simulation methods have been developed to evaluate the reliability (or failure probability, P_f) of slope stability, such as direct Monte Carlo Simulation (direct MCS) (e.g., [1–4]), importance sampling (e.g., [5]), and Subset Simulation (SS) (e.g., [6–8]). These methods involve repeatedly evaluating the safety margin of slope stability using a prescribed deterministic analysis method during the simulation, such as limit equilibrium methods (LEMs) (e.g., [1,5,6,9,10]) and finite element methods (FEMs) (e.g., [2,3,11,12]).

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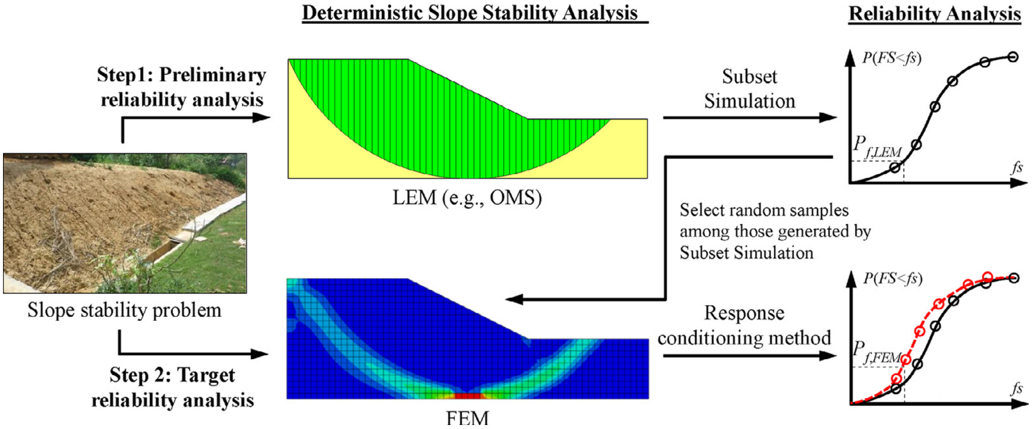


Fig. 1. Slope reliability analysis based on different deterministic analysis models.

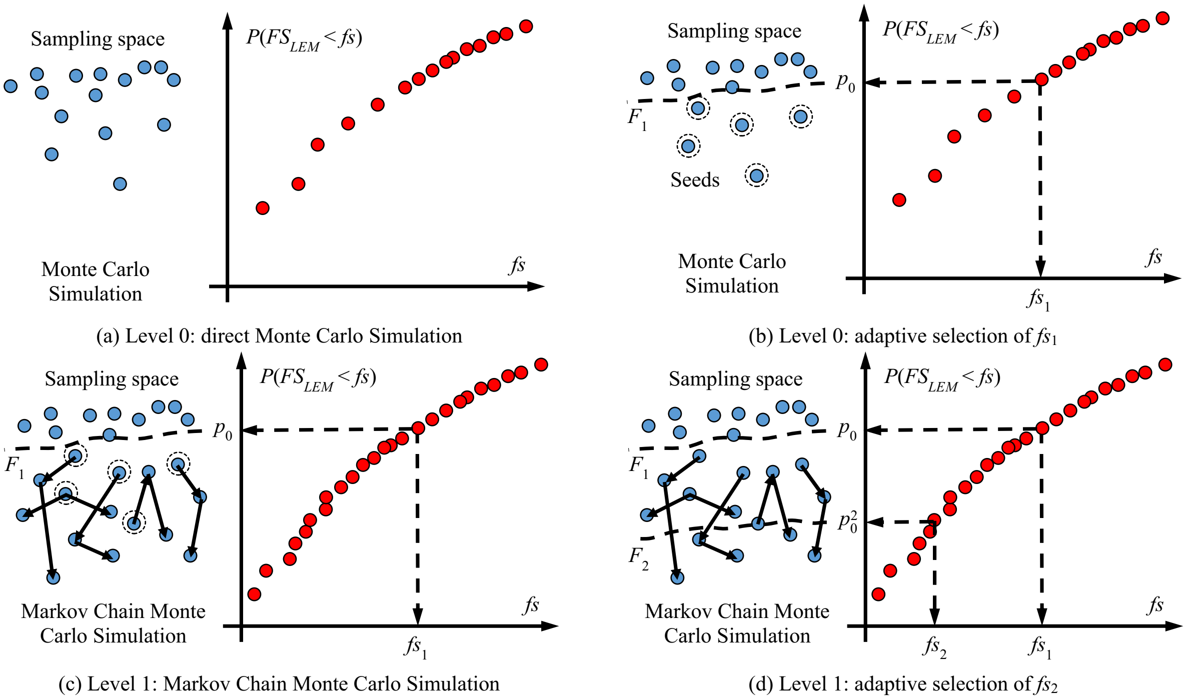
LEMs (e.g., Ordinary Method of Slices (OMS), simplified Bishop's method, and Spencer's method) are widely used in slope engineering practice [13]. Compared with FEMs, LEMs are conceptually simple and require much less computational effort for slope stability analysis, particularly when OMS that has an explicit performance function is applied. However, as pointed out by Griffiths and Lane [14] and Griffiths et al. [2], LEMs need to assume the shape (e.g., circular) and location of slope failure surfaces in the analysis, which are rarely known prior to the analysis, particularly when spatial variability of soil properties is explicitly considered. Inappropriate assumptions on slope failure surfaces in LEMs might lead to negligence of the actual critical slope failure mechanism and, subsequently, result in the estimate of P_f inconsistent with that obtained using more rigorous slope stability analysis methods (e.g., FEMs) in simulation-based reliability analysis. FEMs provide a rigorous and versatile tool for slope stability analysis and alleviate assumptions on slope failure surfaces required in LEMs (e.g., [14,15]). However, FEM-based probabilistic simulation methods (e.g., random finite element method (RFEM)) are sometimes criticized for a lack of computational efficiency and requiring intensive computational power (e.g., [16–18]), particularly at small probability levels (e.g., $P_f < 0.001$). Then, an interesting question arises that how to make use of advantages of both LEMs (e.g., computationally more efficient) and FEMs (e.g., theoretically more realistic and rigorous in terms of the failure mechanism) in reliability analysis of slope stability so as to efficiently obtain consistent reliability estimates. Such a possibility has not been explored in geotechnical literature.

Note that it is not uncommon that there exist different deterministic analysis models/methods (e.g., LEMs and FEMs) for the same geotechnical problem (e.g., slope stability analysis). These methods can be applied in different design stages. For example, at the preliminary design stage, site information (e.g., soil properties) might be too limited to use a relatively sophisticated model (e.g., FEMs) in slope reliability analysis and risk assessment. In such a case, a relatively simple method (e.g., OMS) is an appropriate choice for evaluating slope failure probability and risk. As more site information is collected from site investigation and/or project construction, the understanding of site ground conditions improves, which may allow using a more sophisticated model (e.g., FEM) in slope reliability analysis to make risk-informed decisions. Such different analyses are separately performed at different design stages without interaction. This is somewhat wasted in the sense that the information obtained from the reliability analysis based on the simple model at the preliminary design stage does not inform the later design based on the sophisticated model.

This paper develops a new probabilistic simulation method for slope reliability analysis, which utilizes assets of both LEMs and FEMs in reliability analysis of slope stability to efficiently calculate P_f. The proposed approach provides an estimate of P_f consistent with that obtained by directly performing the finite element analysis of slope stability in probabilistic simulation, but significantly reduces the number of finite element analyses needed in the calculation. As shown in Fig. 1, the proposed approach consists of two major steps: (1) preliminary reliability analysis of slope stability based on SS and a simple LEM (e.g., OMS), which can be finished with practically negligible computational effort; and (2) target reliability analysis of slope stability based on the FEM. The information generated using the LEM in the first step is incorporated into the second step to facilitate the FEM-based slope reliability analysis through "response conditioning method (RCM)" [19]. The paper starts with the introduction of the preliminary reliability analysis of slope stability using SS and OMS, followed by the target reliability analysis based on RCM and FEM. Then, the implementation procedure of the proposed approach is presented and illustrated through two soil slope examples.

2. Preliminary reliability analysis of slope stability using a simple LEM and SS

OMS is one of the simplest LEMs and has an explicit performance function. This section aims to use OMS in SS to generate information on a slope problem concerned and to obtain a preliminary estimate $P_{f,LEM}$ of P_f. Herein, the subscript "*LEM*" indicates that the safety factor of slope stability is evaluated by the LEM during SS. The algorithm of SS and its implementation in reliability analysis of slope stability are provided in the following two subsections, respectively.



(a) Level 0: direct Monte Carlo Simulation
(b) Level 0: adaptive selection of fs_1
(c) Level 1: Markov Chain Monte Carlo Simulation
(d) Level 1: adaptive selection of fs_2

Fig. 2. Schematic diagram of Subset Simulation procedure (modified from Au et al. [6] and Li et al. [40]).

2.1. Subset Simulation algorithm

SS expresses a rare failure event with a small probability as a sequence of intermediate failure events with larger conditional failure probabilities and employs Markov Chain Monte Carlo Simulation (MCMCS) to generate conditional failure samples of these intermediate failure events until the target failure domain is achieved [20–23]. For example, consider the slope stability problem, where the safety factor (FS_{LEM}) of slope stability estimated from the LEM is the critical response. Let $fs_m < \cdots < fs_2 < fs_1$ be an increasing sequence of intermediate threshold values. Then, the intermediate events $\{F_k, k = 1, 2, \ldots, m\}$ are defined as $F_k = \{FS_{LEM} < fs_k\}$. By sequentially conditioning on these intermediate events, the occurrence probability of F_m is written as [8,23]:

$$P(F_m) = P(F_1)\prod_{k=2}^{m} P(F_k|F_{k-1}), \tag{1}$$

where $P(F_1)$ is equal to $P(FS_{LEM} < fs_1)$, and $P(F_k|F_{k-1})$ is equal to $\{P(FS_{LEM} < fs_k|\ FS_{LEM} < fs_{k-1}), k = 2, 3, \ldots, m\}$. In implementations, $fs_1, fs_2, \ldots, fs_m$ are generated adaptively using information from simulated samples so that the sample estimates of $P(F_1)$ and $\{P(F_k|F_{k-1}), k = 2, 3, \ldots, m\}$ always correspond to a specified value of conditional probability p_0 [23].

2.2. Implementation of SS and OMS in slope reliability analysis

As shown in Fig. 2, SS starts with direct MCS, in which N direct MCS samples are generated. The FS_{LEM} values of the N samples are calculated from OMS and are ranked in a descending order. The $N(1 - p_0)$th FS_{LEM} value is chosen as fs_1, and hence, the sample estimate for $P(F_1) = P(FS_{LEM} < fs_1)$ is p_0. Then, the Np_0 samples with $F_1 = \{FS_{LEM} < fs_1\}$ are used as “seeds” for the application of MCMCS to simulate $N(1 - p_0)$ additional conditional samples given $F_1 = \{FS_{LEM} < fs_1\}$, the FS_{LEM} values of which are calculated using OMS during the simulation. Therefore, there are a total of N samples with $F_1 = \{FS_{LEM} < fs_1\}$. These N FS_{LEM} values are ranked again in a descending order, and the $N(1 - p_0)$th FS_{LEM} value is chosen as fs_2, which defines the $F_2 = \{FS_{LEM} < fs_2\}$. The sample estimate for $P(F_2|F_1) = P(FS_{LEM} < fs_2|FS_{LEM} < fs_1)$ is also equal to p_0. Similarly, the procedure described above is repeated $m - 1$ times until the N conditional samples in $F_{m-1} = \{FS_{LEM} < fs_{m-1}\}$ are generated, and their corresponding FS_{LEM} values are calculated and ranked in a descending order. Then, the $N(1 - p_0)$th FS_{LEM} value is chosen as fs_m, and Np_0 samples conditional on $F_m = \{FS_{LEM} < fs_m\}$ are obtained. Finally, a total of m levels of simulations (including one direct MCS level and $m - 1$ levels of MCMCS) are performed in this study, resulting in $mN(1 - p_0) + Np_0$ SS samples.

During SS, the sample space is divided into $m + 1$ mutually exclusive and collectively exhaustive subsets (or bins) Ω_k, $k = 0$, 1, …, m, by intermediate threshold values (i.e., $fs_1, fs_2, \ldots, fs_m$) estimated from OMS, where $\Omega_0 = \{FS_{LEM} \geq fs_1\}$, $\Omega_k = F_k - F_{k+1} = \{fs_{k+1} \leq FS_{LEM} < fs_k\}$ for $k = 1, 2, \ldots, m - 1$, and $\Omega_m = F_m = \{FS_{LEM} < fs_m\}$. According to the Total Probability Theorem [24], the

$P_{f,LEM}$ estimated from SS and OMS is then written as:

$$P_{f,LEM} = P(FS_{LEM} < fs) = \sum_{k=0}^{m} P(F|\Omega_k)P(\Omega_k), \tag{2}$$

where $P(F|\Omega_k)$ = the conditional failure probability given sampling in Ω_k, and it is estimated as the fraction $N_{F,k}/N_k$ of the failure samples with $FS_{LEM} < fs$ in Ω_k, where fs is a prescribed threshold value, $N_{F,k}$ is the number of failure samples in Ω_k, and N_k (i.e., $N(1-p_0)$ for $k = 0, 1, \ldots, m-1$, and Np_0 for $k = m$) is the number of random samples falling into Ω_k; $P(\Omega_k)$ = the occurrence probability of Ω_k, and it is calculated as [23,25]:

$$P(\Omega_k) = \begin{cases} p_0^k(1-p_0), k = 0, 1, \ldots, m-1 \\ p_0^k, k = m \end{cases}. \tag{3}$$

Using the $mN(1-p_0) + Np_0$ samples generated by SS and OMS, the $P_{f,LEM}$ is calculated by Eq. (2). The calculation of $P_{f,LEM}$ can be achieved with practically negligible computational effort because OMS has an explicit performance function and can be efficiently evaluated. Due to assumptions (e.g., a limited number of circular slip surfaces) of slope failure mechanism adopted in OMS, the preliminary reliability analysis, however, only gives an approximate solution (e.g., $P_{f,LEM}$) to the actual failure probability of slope stability. The next section makes use of the information (e.g., Ω_k, $k = 0, 1, \ldots, m$, and random samples in these subsets) obtained from the preliminary reliability analysis to facilitate FEM-based slope reliability analysis so as to efficiently obtain a refined estimate of P_f.

3. Target reliability analysis of slope stability using the FEM and RCM

3.1. Calculation of the target estimate of failure probability using RCM

Since the FEM does not need to make assumptions on the shape and location of slip surfaces, the safety factor (FS_{FEM}) of slope stability obtained from FEM through the shear strength reduction technique (SSRT) (e.g., [14]) reflects the safety margin of slope stability more realistically than FS_{LEM} in terms of failure mechanisms. FS_{FEM} is, hence, considered as the "target" response of slope stability in this study. The estimate ($P_{f,FEM}$) of P_f obtained using the FEM is then referred to as the "target estimate" of P_f in this study. Based on the information generated in the preliminary reliability analysis using SS and OMS, this section strategically performs finite element analyses of slope stability and efficiently calculates the $P_{f,FEM}$ using RCM [19].

In the context of RCM [19], $P_{f,FEM}$ is written as:

$$P_{f,\mathrm{FEM}} = P(F_t = \{FS_{\mathrm{FEM}} < fs\}) = \sum_{i=0}^{M} P(F_t|\mathrm{B}_i)P(\mathrm{B}_i), \tag{4}$$

where B_i, $i = 0, 1, \ldots, M$, are M mutually exclusive and collectively exhaustive subsets from partition of sample space of uncertain parameters involved in slope reliability analysis; $P(F_t|B_i)$ = the conditional probability of the target failure event $F_t = \{FS_{FEM} < fs\}$ given sampling in B_i, where the subscript "t" emphasizes that the occurrence of slope failure is judged using the target response FS_{FEM}; $P(B_i)$ = occurrence probability of B_i. Determination of B_i is an essential step in evaluating $P_{f,FEM}$ by Eq. (4). It is noted that Eq. (4) is an analog of Eq. (2), in which the sample space is stratified into $m+1$ subsets (i.e., Ω_k, $k = 0, 1, \ldots, m$) based on FS_{LEM} values estimated from OMS in the preliminary reliability analysis. It is, therefore, straightforward to use Ω_k, $k = 0, 1, \ldots, m$, as B_i in Eq. (4), i.e., setting $M = m$ and $B_i = \Omega_k$, where $i = k$. Then, Eq. (4) is re-written as:

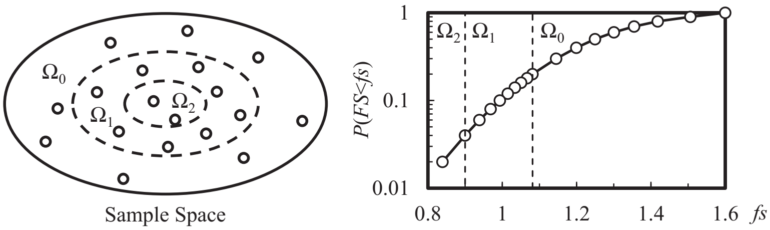
$$P_{f,\mathrm{FEM}} = P(F_t = \{FS_{\mathrm{FEM}} < fs\}) = \sum_{k=0}^{m} P(F_t|\Omega_k)P(\Omega_k), \tag{5}$$

where $P(F_t|\Omega_k)$ = the conditional probability of the target failure event $F_t = \{FS_{FEM} < fs\}$ given sampling in Ω_k. In RCM, $P(F_t|\Omega_k)$ is calculated using a sub-binning strategy [19], in which Ω_k is further divided into N_s sub-bins $\Omega_{k,j}$, $j = 1, 2, \ldots, N_s$. These sub-bins are ranked in a descending order according to FS_{LEM} values estimated from OMS and have the same conditional probability $P(\Omega_{k,j}|\Omega_k)$ given sampling in Ω_k. Then, the N_k random samples in Ω_k fall into different sub-bins $\Omega_{k,j}$, $j = 1, 2, \ldots, N_s$, each of which contains N_k/N_s random samples. Using the Theorem of Total Probability and the fact that $\Omega_{k,j} \subset \Omega_k$, $P(F_t|\Omega_k)$ is written as:

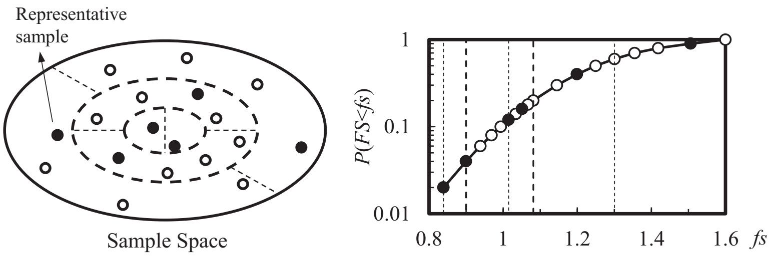
$$P(F_t|\Omega_k) = \sum_{j=1}^{N_s} P(F_t|\Omega_{k,j})P(\Omega_{k,j}|\Omega_k), \tag{6}$$

where $P(\Omega_{k,j}|\Omega_k) = 1/N_s$. Then, one sample is randomly selected from the samples falling into $\Omega_{k,j}$. The selected sample is used as input in the finite element analysis of slope stability to calculate its corresponding FS_{FEM} value (i.e., $FS_{FEM,kj}$) and to judge whether the target failure event (i.e., $F_t = \{FS_{FEM} < fs\}$) occurs in $\Omega_{k,j}$. As a result, a total of N_s random samples and their corresponding FS_{FEM} values are obtained in Ω_k, and $P(F_t|\Omega_k)$ is estimated as [19]:

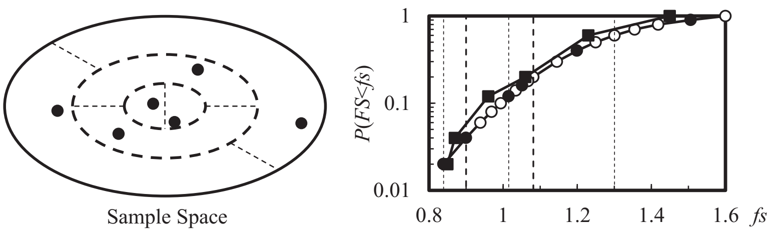
$$P(F_t|\Omega_k) \approx \sum_{j=1}^{N_s} I(F_t|\Omega_{k,j}) \times \frac{1}{N_s}, \tag{7}$$



(a) Subsets obtained during SS



(b) Sub-binning and selection of representative random samples in each subset



(c) Evaluation of $P_{f,FEM}$

Fig. 3. Schematic diagram of response conditioning method ($N = 10$, $p_0 = 0.2$, $m = 2$, $N_s = 2$).

where $I(F_t|\Omega_{k,j})$ = an indicator function. If $FS_{FEM,kj} < fs$, $I(F_t|\Omega_{k,j})$ is equal to 1; otherwise, it is taken as zero. Substituting Eq. (7) into Eq. (5) gives:

$$P_{f,\mathrm{FEM}} \approx \sum_{k=0}^{m} \sum_{j=1}^{N_s} I(F_t|\Omega_{k,j}) \frac{P(\Omega_k)}{N_s}. \tag{8}$$

For illustration, the above procedures are schematically shown in Fig. 3 through a simple example, where $m = 2$, $N = 10$, $p_0 = 0.2$, and $N_s = 2$. Using Eq. (8) to estimate $P_{f,FEM}$ only requires $(m + 1)N_s$ finite element analyses of slope stability. Herein, it is worthwhile to emphasize that the random samples used in the target reliability analysis are selected from those generated by SS in the preliminary reliability analysis, and need not be re-generated. More importantly, it has been proved that the estimate of P_f given by Eq. (8) is asymptotically unbiased by Au [19]. This, herein, means that the $P_{f,FEM}$ estimated from Eq. (8) converges to that obtained from directly performing MCS or SS based on the FEM, and is, therefore, a consistent estimate of P_f.

Note that Ω_k, $k = 0, 1, \ldots, m$, are stratified with respect to FS_{LEM} values estimated using OMS. For a given random sample, the FS_{LEM} value might be different from the FS_{FEM} value because different assumptions are adopted in the two methods. However, it can be intuitively reasoned that FS_{LEM} is correlated to FS_{FEM} to some degree. Such correlation is attributed to the same slope geometry, underground stratification, and soil properties shared by the two methods. The correlation coefficient between FS_{LEM} and FS_{FEM} affects the variability of $P_{f,FEM}$ estimated from Eq. (8) [19], and it can be estimated from the values of FS_{LEM} and FS_{FEM} of $(m + 1)N_s$ selected random samples. This will be illustrated through a slope example later.

3.2. Computational effort

The computational effort of the proposed approach mainly contains two parts: the first part is that used for evaluating $mN(1 - p_0) + Np_0$ FS_{LEM} values by OMS during SS in the preliminary reliability analysis; and the second part is that used for calculating $(m + 1)N_s$ FS_{FEM} values by the FEM. Let ξ denote the ratio of the computational effort used for each evaluation of

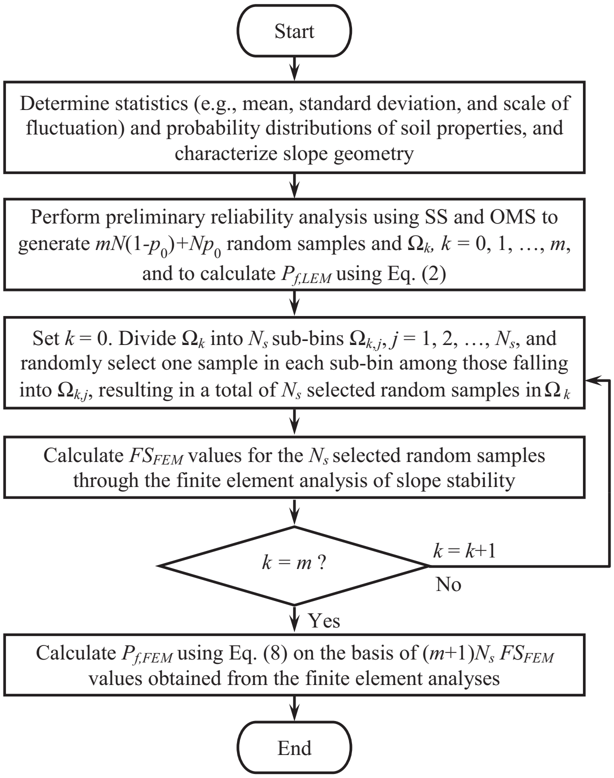


Fig. 4. Implementation procedures of the proposed approach.

FS_{LEM} using the OMS over that used for FS_{FEM}. The total computational effort can be expressed in terms of the equivalent number N_T of the finite element analyses of slope stability:

$$N_T = N_s(m+1) + \xi[mN(1-p_0) + Np_0]. \tag{9}$$

Eq. (9) indicates that N_T relies on p_0, m, ξ, N, and N_s. Previous studies [20,21,23] on SS suggested that $p_0 = 0.1$ is a prudent choice, so it is adopted in this study. Then, the value of m is determined to ensure that ${p_0}^m$ is less than $P_{f,FEM}$ [19]. The ξ is problem-dependent and relies on assumptions adopted in OMS and the finite element model of slope stability analysis. In general, ξ generally decreases as the number of potential slip surfaces considered in OMS decreases and the finite element model becomes more sophisticated (e.g., using a finer finite-element mesh and an advanced constitutive model). For example, if only the deterministic critical slip surface with the minimum FS_{LEM} at mean values of uncertain parameters is considered, evaluation of FS_{LEM} using OMS is finished instantaneously with practically negligible computational effort compared with that used for evaluating FS_{FEM}. In other words, ξ is very small. In such a case, it is suggested using a large value of N (e.g., $N \geq 5000$) in the preliminary reliability analysis because this leads to improvement in computational accuracy in RCM at the expense of minimal increase in computational effort [19]. As ξ is very small, the computational effort required for the proposed approach is mainly attributed to that used for evaluating $(m+1)N_s$ FS_{FEM} values by the FEM, which relies on the choices of N_s. Determination of N_s and its effects on the target estimate (i.e., $P_{f,FEM}$) of failure probability will be further discussed later in this paper.

4. Implementation procedure

Fig. 4 shows schematically implementation procedures of the proposed approach for reliability analysis of slope stability, which consist of six steps and are described as below:

(1) Determine statistics (e.g., mean, standard deviation, and scale of fluctuation) and probability distributions of soil properties, and characterize slope geometry (e.g., slope angle and height).
(2) Perform the preliminary reliability analysis of slope stability using SS and OMS, during which $mN(1-p_0)+Np_0$ random samples are generated and Ω_k, k = 0, 1, …, m, are progressively determined by FS_{LEM} values of these random samples. Then, $P_{f,LEM}$ is calculated using Eq. (2).
(3) Divide Ω_0 into N_s sub-bins $\Omega_{0,j}$, j = 1, 2, …, N_s, and randomly select one sample in each $\Omega_{0,j}$ among those falling into $\Omega_{0,j}$. This leads to a total of N_s selected random samples in Ω_0.
(4) Calculate FS_{FEM} values for the N_s random samples selected in Step (3) by the SSRT in the FEM.

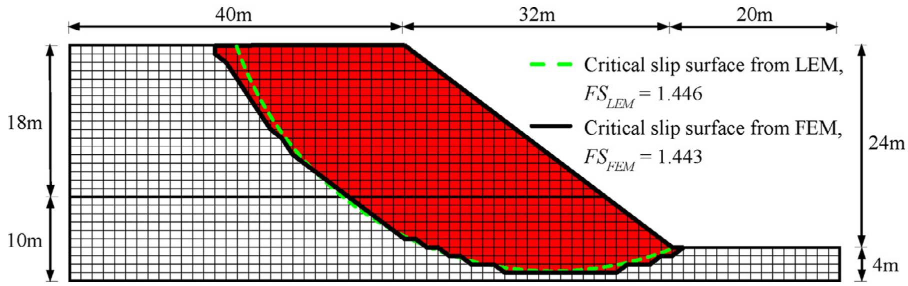


Fig. 5. Two-layered slope example and its corresponding deterministic analysis results.

(5) Repeat Steps (3) and (4) for each Ω_k, $k = 1, 2, \ldots, m$. Finally, $(m+1)N_s$ random samples are selected, and their corresponding FS_{FEM} values are calculated using the FEM.
(6) Calculate $P_{f,FEM}$ using Eq. (8) on the basis of $(m + 1)N_s$ FS_{FEM} values obtained in Steps (3)–(5).

The implementation procedure and algorithm of the proposed approach are somewhat more complicated than those of FEM-based direct MCS. To facilitate the application of the proposed approach, it can be programmed as a user toolbox in computer software, e.g., *MATLAB* [26]. With the toolbox, geotechnical practitioners only need to focus on deterministic slope stability analysis that they are more familiar with, e.g., developing a limit equilibrium model and a finite element model for slope stability analysis in commercial software packages *Slope/W* [27] and *Abaqus* [28], respectively. After these deterministic slope stability analysis models are prepared, the user toolbox repeatedly invokes these models of slope stability for the reliability analysis and returns the failure probability (i.e., $P_{f,LEM}$ and $P_{f,FEM}$) as outputs. Although a thorough understanding of reliability algorithms (e.g., SS and RCM) is always advantageous, this is not a prerequisite for engineers to use the user toolbox, which allows engineers to use the proposed approach without being compromised by the complicated mathematics and, therefore, significantly improves the practicality of the proposed approach. The proposed approach is illustrated through two soil slope examples in the following two sections.

5. Illustrative example I: a two-layered slope

For illustration, this section applies the proposed approach to evaluate the failure probability of a two-layered soil slope. The illustrative slope example is intentionally tailored to explicitly considering the inherent spatial variability of soil properties and having a failure probability less than 10^{-3}, which is of great interest in design practice. As shown in Fig. 5, the slope has a height of 24 m and a slope angle of about 37°, and it is comprised of two soil layers. The undrained shear strength S_{u1} and S_{u2} of the two soil layers are lognormally distributed. Their respective mean values are 80 kPa and 120 kPa, and both of them have a coefficient of variation (COV) of 0.3. Note that these statistics are, herein, defined at the point level. In addition, the unit weights of the two soil layers are taken as 19 kN/m^3. To enable the finite element analysis of slope stability, the information on the Young's modulus and Poisson's ratio are also needed. In the case of no available information on them, they are assumed to be 100 MPa and 0.3 [14], respectively.

As shown in Fig. 5, the respective values of safety factor of the slope example are evaluated as 1.446 and 1.443 at the mean values of soil properties using OMS and FEM. In this study, the finite element analysis of slope stability is performed using an elastic-perfectly plastic constitutive model with a Mohr–Coulomb failure criterion in *Abaqus*. Fig. 5 also shows the critical slip surfaces identified from OMS and FEM by a green dashed line and a bold solid line, respectively. The green dashed line is identified from 16,400 circular potential slip surfaces by explicitly searching the minimum FS_{LEM} value at the mean values of soil properties, and it is referred to as "deterministic critical slip surface" in this example. Moreover, the critical slip surface identified from FEM is determined using the *K*-means clustering method based on the node displacements obtained from the finite element analysis (e.g., [3]).

To incorporate the effects of spatial variability of soil properties into reliability analysis, the inherent spatial variability of S_{u1} and S_{u2} is modeled using two-dimensional (2D) lognormal random fields in this example. The spatial correlation of the logarithm (i.e., ln (S_u)) of undrained shear strength S_u is defined by a 2D single exponential correlation function [29], in which the scale of fluctuation is taken as 24 m and 2.4 m in horizontal and vertical directions, respectively. For each parameter (i.e., S_{u1} and S_{u2}), the random field is discretized using the midpoint method and is simulated using Cholesky decomposition (e.g., [8,29–31]). Herein, it is worthwhile to point out that the lognormal random field of S_u (or equivalently the normal random field of ln(S_u)) are simulated at the element level in this study. This means that the simulated S_u sample in each element represents the locally averaged value of S_u in the element and the local averaging effects over the element are explicitly considered [11]. For this purpose, the locally averaged statistics (including mean, standard deviation, and correlation coefficient) of ln(S_u) over the element are calculated from the point level statistics of ln(S_u), which are functions of point level statistics of S_u. Details of transformation from the point level statistics of ln(S_u) to its locally averaged statistics in a lognormal random field of S_u are referred to Griffiths and Fenton [11] and Vanmarcke [32]. The locally averaged statistics of ln(S_u) are directly used in the midpoint method to simulate the normal

Table 1
Reliability analysis results of the two-layered slope example using the proposed approach for $fs = 1$.

Simulation level k	Ω_k	$P(\Omega_k)$	Preliminary reliability analysis		Target reliability analysis	
			$P(F\|\Omega_k)$	$P_{f,LEM}$	$P(F_t\|\Omega_k)$	$P_{f,FEM}$
0	$1.262 \le FS_{LEM}$	0.9	0/4500		0/50	
1	$1.146 \le FS_{LEM} < 1.262$	0.09	0/4500		0/50	
2	$1.069 \le FS_{LEM} < 1.146$	9×10^{-3}	0/4500		0/50	
3	$1.010 \le FS_{LEM} < 1.069$	9×10^{-4}	0/4500	6.35×10^{-5}	6/50	1.77×10^{-4}
4	$0.966 \le FS_{LEM} < 1.010$	9×10^{-5}	2676/4500		33/50	
5	$FS_{LEM} < 0.966$	1×10^{-5}	500/500		50/50	

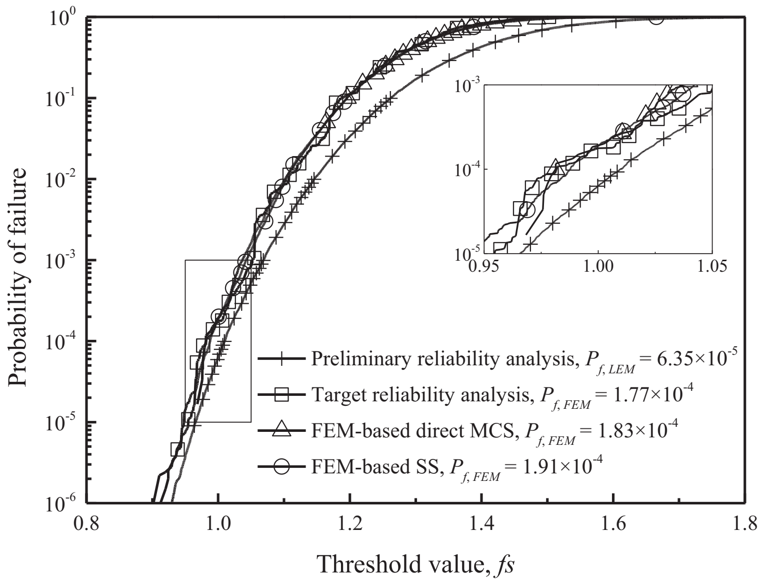


Fig. 6. Cumulative distribution function of safety factor obtained from different methods.

random field of $\ln(S_u)$, which is subsequently converted into the lognormal random field of S_u by taking the exponential. Each realization of the random field is, then, mapped onto the finite-element mesh shown in Fig. 5 to perform slope stability analysis.

5.1. Preliminary estimate of slope failure probability

A SS run with $m = 5$, $N = 5000$, and $p_0 = 0.1$ is performed to preliminarily estimate the slope failure probability. As a result, 23,000 random samples (i.e., realizations of random fields) are generated, and their corresponding FS_{LEM} values are evaluated using OMS. Consider, for example, that only the deterministic critical slip surface (see the green dashed line in Fig. 5) is adopted in OMS during the calculation. For each random sample, the calculation of FS_{LEM} can be finished instantaneously. Therefore, the preliminary reliability analysis can be achieved with practically negligible computational effort. In this example, it only takes about 2 min to evaluate the FS_{LEM} values corresponding to the 23,000 random samples (i.e., about 0.005 s for each evaluation) on a desktop computer with 8 GB RAM and one Intel Core i7 CPU clocked at 3.4 GHz.

Using the 23,000 random samples and their corresponding FS_{LEM} values, the $P_{f,LEM}$ is calculated using Eq. (2) for different fs values. Table 1 summarizes the procedures of calculating $P_{f,LEM}$ for $fs = 1$. During SS, the sample space is divided into six subsets Ω_k, $k = 0, 1, \ldots, 5$, in a descending order of the FS_{LEM} value (see column 2 in Table 1), and conditional samples fall into these six subsets. Based on the conditional samples in the six subsets, $P_{f,LEM}$ for $fs = 1$ is estimated as 6.35×10^{-5} by Eq. (2). $P_{f,LEM}$ varies as the prescribed fs changes. Fig. 6 shows the variation of $P_{f,LEM}$ as a function of f_s (i.e., the cumulative distribution function (CDF) of FS_{LEM}) by a solid line with crosses.

5.2. Target estimate of slope failure probability

Based on the information (i.e., Ω_k, $k = 0, 1, \ldots, 5$, and random samples in these subsets) generated by SS and OMS in the preliminary reliability analysis, the finite element analysis of slope stability is strategically performed to obtain the target estimate (i.e., $P_{f,FEM}$) of failure probability using RCM in this subsection. Consider, for example, dividing Ω_k into 50 sub-bins $\Omega_{k,j}$, $j = 1, 2, \ldots, 50$, i.e., $N_s = 50$. One sample is randomly selected from each sub-bin $\Omega_{k,j}$, resulting in 50 selected random samples in each Ω_k

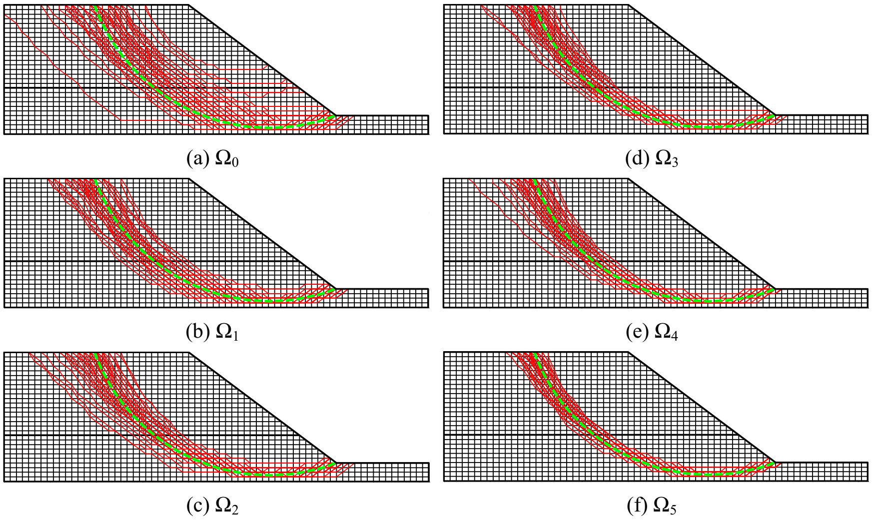


Fig. 7. Comparison of critical slip surfaces identified from the FEM and the deterministic critical slip surface adopted in OMS.

and 300 selected random samples in total. Then, each selected random sample is used as input of the finite element analysis of slope stability in *Abaqus* to calculate its corresponding FS_{FEM}. The computational time required for each finite element analysis of slope stability in this example is about 30 s on the desktop computer with 8 GB RAM and one Intel Core i7 CPU clocked at 3.4 GHz, which is about 6000 times longer than that (i.e., 0.005 s) required for calculating FS_{LEM} by OMS on the same computer in this example. Using Eq. (9), N_T is then calculated as 304. The total computational effort required for the proposed approach is, hence, equivalent to that for performing 304 finite element analyses of slope stability in this example, which takes a total of 2.5 h.

Table 1 also summarizes the procedures of calculating $P_{f,FEM}$ for $fs = 1$. Based on the 300 FS_{FEM} values, a total of 89 samples (including 6, 33, and 50 samples in Ω_3, Ω_4, and Ω_5, respectively) are identified as failure samples. Using Eq. (8), $P_{f,FEM}$ for $fs = 1$ is, then, estimated as 1.77×10^{-4}, which is almost three times greater than $P_{f,LEM}$ (i.e., 6.35×10^{-5}) obtained in the preliminary reliability analysis. Similar observations are also obtained for other fs values, as shown in Fig. 6. It is not surprising to see this since only one slip surface (i.e., the deterministic critical slip surface) is considered in the preliminary reliability analysis, which might be different from the critical slip surface identified from FEM. For example, Fig. 7 (a)–(f) show critical slip surfaces from the FEM for 50 selected random samples in Ω_k, $k = 0, 1, \ldots, 5$, by red solid lines, respectively. For comparison, Fig. 7 also includes the deterministic critical slip surface adopted in OMS by a green dashed line. It is observed that some critical slip surfaces identified from the FEM (i.e., the red solid lines) significantly deviate from the deterministic critical slip surface (i.e., the green dashed line).

From a system reliability analysis point of view, there exist a large number of potential slip surfaces for a soil slope. These potential slip surfaces consist of a series system, and each of them represents a component of the system. Only using the deterministic critical slip surface (i.e., one component of the series system) in slope reliability analysis might lead to underestimation of the slope failure probability because some key failure mechanisms (or system components) are ignored (e.g., [8,33–36]).

5.3. Comparison with the results from FEM-based slope reliability analysis

To validate the results obtained from the proposed approach, a direct MCS run with 60,000 samples and 20 independent SS runs with $m = 4$, $p_0 = 0.1$, and $N = 500$ are performed to calculate the failure probability of the slope example, in which the FEM is directly applied to performing deterministic analysis of slope stability for each random sample. Since the FEM is applied, the resulting failure probability is its target estimate in this study, i.e., $P_{f,FEM} = P(FS_{FEM} < fs)$. In addition, 20 independent runs of the proposed approach are also performed to calculate $P_{f,FEM}$, in which both the LEM (i.e., OMS only considering deterministic critical slip surface) and FEM are applied.

Table 2 summarizes the results obtained from direct MCS and SS based on the FEM (referred to as FEM-based direct MCS and FEM-based SS, respectively) and the proposed approach using both LEM and FEM. The estimate of $P_{f,FEM}$ from FEM-based direct MCS is 1.83×10^{-4}, and the average value of $P_{f,FEM}$ from 20 runs of FEM-based SS is 1.71×10^{-4}, both of which are in good agreement with the mean value (i.e., 1.79×10^{-4}) of $P_{f,FEM}$ obtained from 20 runs of the proposed approach. Moreover, Fig. 6 also plots the CDFs (i.e., target failure probabilities for different fs values) of FS_{FEM} obtained from FEM-based direct MCS, FEM-based SS, and the proposed approach by solid lines with triangles, circles, and squares, respectively. It is shown that, using both a simple LEM (i.e., OMS) and FEM, the proposed approach provides an estimate of failure probability consistent with that obtained by performing direct MCS or SS based on the FEM.

Table 2
Comparison of reliability analysis results of the two-layered slope example.

Method	Number of evaluations		Equivalent number of evaluations, N_T	Failure probability, P_f	COV of P_f	Unit COV	Time (h)
	LEM	FEM					
FEM-based direct MCS	–	60,000[b]	60,000	1.83×10^{-4}	0.30	73.5	500
FEM-based SS	–	1850[b]	1850	1.71×10^{-4}[e]	0.38[e]	16.3	15.4
LEM-based SS[a]	1850[c]	–	463	4.86×10^{-4}[e]	0.31[e]	6.7	3.9
The proposed approach	23,000[d]	300[b]	304	1.79×10^{-4}[e]	0.45[e]	7.8	2.5

[a] 16,400 potential slip surfaces are considered.
[b] Each evaluation takes about 30 s.
[c] Each evaluation takes 7.5 s.
[d] Each evaluation takes 5×10^{-3} s.
[e] Estimated from the results of 20 independent runs.

As shown in Table 2, a total of 60,000 finite element analyses of slope stability are performed in the FEM-based direct MCS, and the corresponding COV of $P_{f,FEM}$ is about 0.3 (i.e., $\sqrt{(1-P_{f,FEM})/N_T P_{f,FEM}}$). It takes about 500 h (about 20.8 days) to finish the 60,000 finite element analyses on the desktop computer used in this study. For each FEM-based SS run, 1850 finite element analyses of slope stability are performed, which takes 15.4 h on average. Based on the 20 independent runs of FEM-based SS, the COV of $P_{f,FEM}$ is estimated as 0.38. In contrast, only 304 equivalent finite element analyses of slope stability are required in each simulation run of the proposed approach. The computational time for each simulation run of the proposed approach is about 2.5 h, which is much less than those (i.e., 500 and 15.4 h, respectively) required for FEM-based direct MCS and FEM-based SS. Based on the 20 independent runs of the proposed approach, the COV of $P_{f,FEM}$ is estimated as 0.45. Note that the COV of estimated $P_{f,FEM}$ relies on the number of samples used in the simulation. To enable a fair comparison, the unit COV defined by Au and Beck [37] and Au [19] is applied as a measure of the computational efficiency in this study, and it is estimated as COV$\sqrt{N_T}$ and accounts for effects of the number (i.e., N_T) of samples used in simulation on the variation of reliability estimates. As shown in column 6 of Table 2, the unit COV values for FEM-based direct MCS, FEM-based SS, and the proposed approach are 73.5, 16.3, and 7.8, respectively. The unit COV for the proposed approach is about 1/9 and 1/2 of that for FEM-based direct MCS and FEM-based SS. This indicates that the proposed approach requires, respectively, about 1/81 and 1/4 of the computational effort used for FEM-based direct MCS and FEM-based SS to achieve the same computational accuracy. The proposed approach significantly reduces the computational effort by incorporating information generated from LEM-based slope reliability analysis into FEM-based slope reliability analysis through RCM. This allows geotechnical practitioners to finish several reliability analyses for different slope designs at small probability levels within hours, which may take a number of weeks using FEM-based direct MCS on the same computer. This is pivotal to applying probabilistic simulation-based reliability analysis methods in slope reliability-based design, in which a trial-and-error procedure is usually involved to find feasible designs satisfying a series of prescribed design requirements (e.g., the target reliability level) and failure probabilities for different designs need to be evaluated.

In addition, it is also noted that the proposed approach provides much more failure samples (i.e., 89 failure samples among 300 selected SS samples, as shown in column 6 of Table 1) than FEM-based direct MCS (i.e., 11 failure samples among 60,000 direct MCS samples). Hence, a large number of failure samples can be obtained from the proposed approach with relative ease, which allows shedding light on relative contributions of different uncertainties to slope failure [38,39] and is useful for evaluating the slope failure consequence (e.g., [40]).

5.4. Comparison with the results from LEM-based slope reliability analysis

For comparison, this subsection performs 20 independent SS runs with $m = 4$, $p_0 = 0.1$, and $N = 500$ to calculate the failure probability of the slope example based on OMS considering 16,400 potential slip surfaces, which are referred to as LEM-based SS runs. As shown in Table 2, 1850 random samples are generated in each LEM-based SS run. For each set of random samples, OMS is applied to performing deterministic analysis of slope stability, and the critical slip surface with the minimum factor of safety is searched from the 16,400 potential slip surfaces. The computational time for each set of random samples is about 7.5 s, which is about 1/4 of that (about 30 s) required for the finite element analysis in this example. Therefore, the computational effort for 1850 evaluations of OMS considering 16,400 potential slip surfaces is equivalent to that for 1850/4 = 463 (i.e. $N_T = 463$) finite element analyses of slope stability.

Based on the 20 LEM-based SS runs performed in this subsection, the COV of estimated failure probability is about 0.31, and the unit COV is then estimated as 6.7, which is close to that (i.e., 7.8) of the proposed approach. However, the mean value of failure probability from the 20 LEM-based SS runs is 4.86×10^{-4}, which obviously deviates from target estimates (i.e., 1.83×10^{-4}, 1.71×10^{-4}, and 1.79×10^{-4}, respectively) obtained from FEM-based direct MCS, FEM-based SS, and the proposed approach. Only using the OMS in slope reliability analysis leads to an inconsistent estimate of slope failure probability in this example though a large number of potential slip surfaces are considered. In contrast, the proposed approach provides consistent reliability estimates with the computational effort similar to that for LEM-based SS.

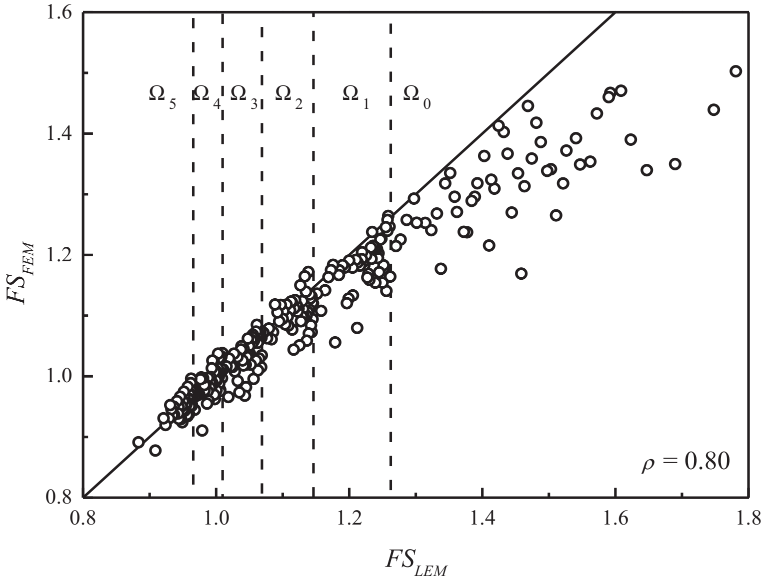


Fig. 8. Correlation between FS_{LEM} and FS_{FEM} in a typical run of the proposed approach.

5.5. Correlation between FS_{LEM} and FS_{FEM}

In this example, 300 random samples are selected in each run of the proposed approach (see Section 5.2). Fig. 8 shows the FS_{LEM} value versus the FS_{FEM} value of the 300 random samples obtained from a typical run of the proposed approach. Using these values of FS_{LEM} and FS_{FEM}, their correlation coefficient ρ is estimated as 0.80. The details of calculating ρ are given in Appendix A. Note that the correlation between FS_{LEM} and FS_{FEM} is relatively strong in this example. On the other hand, if the correlation between FS_{LEM} and FS_{FEM} is relatively weak (i.e., ρ is relatively small), the results obtained from the proposed approach shall be treated with cautions because, in such a circumstance, OMS used in the preliminary reliability analysis provides rare information on the target response (i.e., FS_{FEM}). ρ can be used as an indicator for choosing an appropriate deterministic slope stability analysis model in preliminary reliability analysis.

5.6. Effects of N_s on the target estimate of slope failure probability

The variability of $P_{f,FEM}$ estimated from the proposed approach also relies on the number (i.e., N_s) of random samples selected in each subset Ω_k for the target reliability analysis. To explore the effects of N_s on the estimated $P_{f,FEM}$, a sensitivity study is carried out with different N_s values varying from 5 to 100 in this subsection. For each N_s value, 20 independent simulation runs with $m = 5$, $N = 5000$, and $p_0 = 0.1$ are performed using the proposed approach to evaluate the mean value and COV of $P_{f,FEM}$. Fig. 9 shows the variation of the mean value and COV of $P_{f,FEM}$ as function of N_s by solid lines with squares and circles, respectively. As N_s increases from 5 to 100, the mean value of $P_{f,FEM}$ varies slightly while the COV of $P_{f,FEM}$ decreases significantly. The variability of $P_{f,FEM}$ estimated from the proposed approach is reduced as N_s increases. Such reduction is intuitively reasonable since more information on the target response (i.e., FS_{FEM}) is used in the proposed approach as N_s increases. This is achieved at the expense of more computational effort. Choice of N_s is, therefore, a trade-off between the computational accuracy and efficiency.

6. Illustrative example II: a single-layered slope

The two-layered slope example shown in Fig. 5 is tailored to illustrating the proposed approach in this study, and has not been used by other investigators, making independent validation impossible. For further validation, the section applies the proposed approach to evaluate the failure probability of a relatively simple slope example shown in Fig. 10. The simple slope example was used to illustrate the application of RFEM in evaluating system reliability of slope stability by Huang et al. [12]. As shown in Fig. 10, the simple slope example has a height of 10 m and a slope angle of 26.6°, and it has a single soil layer. The undrained shear strength S_u in the soil layer is modeled by a 2D lognormal random field, in which S_u has as mean value of 30.6 kPa and COV of 0.3. The spatial correlation between $\ln(S_u)$ at different locations is specified by an exponentially decaying isotropic correlation function with a scale of fluctuation of 5 m [12]. In addition, the unit weight, Young's modulus, and Poisson's ratio of the soil layer are taken as 20 kN/m^3, 100 MPa, and 0.3 [12], respectively. Huang et al. [12] performed a FEM-based direct MCS run with 2000 samples to evaluate the failure probability of this slope example, and the resulting $P_{f,FEM}$ value is 0.071, which will be used to validate the results obtained from the proposed approach.

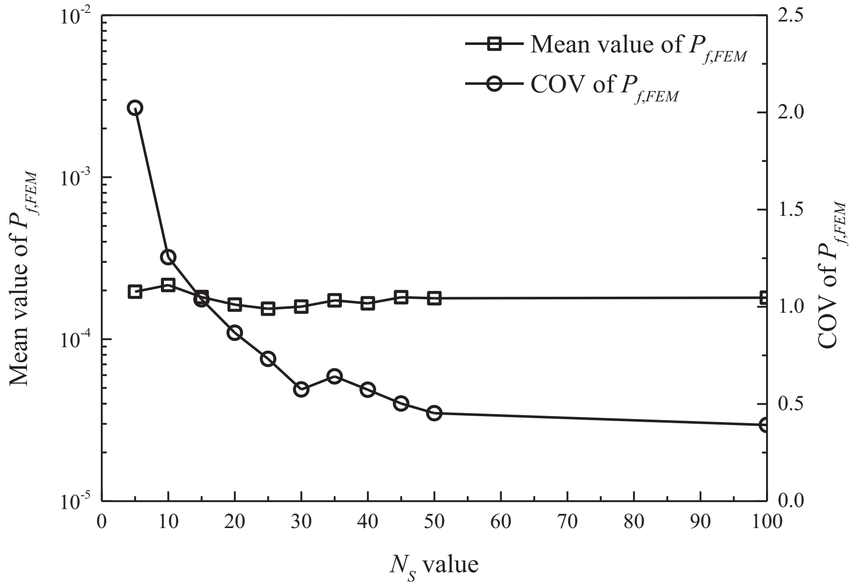


Fig. 9. Effects of N_s on the target estimate $P_{f,FEM}$.

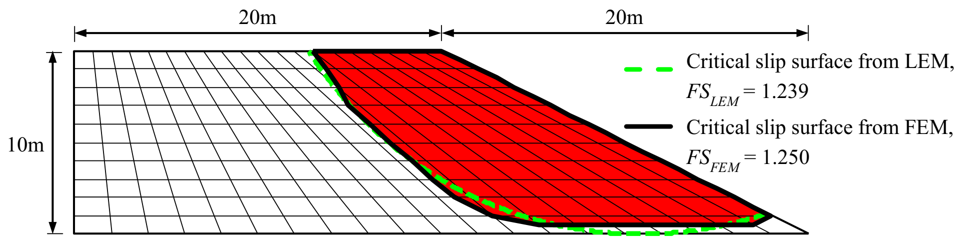


Fig. 10. Single-layered slope example and its corresponding deterministic analysis results.

This section re-evaluates the $P_{f,FEM}$ of the slope example using the proposed approach. Similar to the two-layered slope example, the preliminary reliability analysis is performed using SS and OMS with the deterministic critical slip surface, in which $N = 2000$, $p_0 = 0.1$, and $m = 2$. This step is finished within seconds because the slope example is simple and only one potential slip surface is considered in OMS. Therefore, the computational effort of the preliminary reliability analysis is negligible in this example. The preliminary reliability analysis divides the sampling space into three subsets Ω_0, Ω_1, and Ω_2 with 1800, 1800, and 200 samples, respectively. Then, 20 samples are selected from each subset using the sub-binning strategy, resulting in a total of 60 samples. These 60 selected samples are used as input in finite element analysis of slope stability to estimate $P_{f,FEM}$ in target reliability analysis. To enable a consistent comparison with results reported by Huang et al. [12], the finite element analysis of slope stability in this example is performed using a FEM program *Slope64* developed by Griffiths [41], which is closely based on *Program 6.2* adopted in Huang et al. [12] and Griffiths and Lane [14]. To evaluate the unit COV of $P_{f,FEM}$, 20 repeated runs of the proposed approach are performed to obtain 20 values of $P_{f,FEM}$. This leads to an average $P_{f,FEM}$ value of 0.077, which are favorably comparable with that (0.071) reported by Huang et al. [12]. This further validates the proposed approach. In addition, the unit COV for the proposed approach is about 2.0, which is about 5/9 of that (i.e., $\sqrt{(1-P_{f,FEM})/P_{f,FEM}}=3.6$) for FEM-based direct MCS. This indicates that the proposed approach requires about 3/10 of the computational effort used for FEM-based direct MCS to achieve the same computational accuracy in this example.

7. Summary and conclusion

This paper developed a new probabilistic simulation method that efficiently evaluates the failure probability (P_f) of slope stability using both a simple limit equilibrium method (LEM) (i.e., Ordinary Method of Slices (OMS) considering a limited number of potential slip surfaces) and the finite element method (FEM). It contains two major steps: (1) preliminary reliability analysis of slope stability using an advanced Monte Carlo Simulation (MCS) called “Subset Simulation (SS)” and OMS to generate the information on the slope problem concerned; and (2) target reliability analysis of slope stability based on the FEM to obtain the target estimate (i.e., $P_{f,FEM}$) of P_f. The information generated in the first step is incorporated into the second step to facilitate the FEM-based slope reliability analysis through the response conditioning method (RCM).

The proposed approach was illustrated by two slope examples in spatially variable soils. The results were validated against those obtained from FEM-based direct MCS and FEM-based SS. It was shown that the proposed approach gives the estimate of P_f consistent with those obtained from FEM-based direct MCS and FEM-based SS, and significantly improves the computational efficiency at small probability levels (e.g., $P_f < 10^{-3}$). The slope reliability problem at such small probability levels is of great interest in design practice, but is rarely solved using FEM-based probabilistic simulation methods because intensive computational effort used to be required in terms of the number of finite element analyses. With the proposed approach, the computational effort is reduced by one to two orders of magnitude, making FEM-based probabilistic simulation methods (e.g., RFEM) feasible in slope reliability analysis at small probability levels. It was also observed that the computational effort required for the proposed approach is similar to that for performing SS based on OMS considering a large number of potential slip surfaces (referred to as LEM-based SS) in the two-layered slope example. However, the estimate of P_f obtained from LEM-based SS deviates from the estimates obtained from FEM-based slope reliability analyses in this example. This is mainly attributed to the assumptions (e.g., the circular shape) of potential slip surfaces adopted in OMS, which are alleviated in FEM-based slope reliability analysis.

The proposed approach inherits the assets of both LEM-based slope reliability analysis (e.g., computationally more efficient) and FEM-based slope reliability analysis (e.g., theoretically more realistic and rigorous in terms of slope failure mechanisms). It takes advantage of the information generated using a simple LEM in the preliminary reliability analysis to facilitate the FEM-based slope reliability analysis. This opens up the possibility that makes use of the information obtained at the preliminary design stage of slopes using a simple model (e.g., LEM) to guide the reliability analysis based on a relatively sophisticated model (e.g., FEM) at later design stages.

Acknowledgments

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Appendix A. Calculation of correlation coefficient between FS_{LEM} and FS_{FEM}

Based on the FS_{LEM} and FS_{FEM} values of selected random samples in Ω_k, $k = 0, 1, \ldots, m$, the correlation coefficient ρ between FS_{LEM} and FS_{FEM} is calculated as:

$$\rho = \frac{E(FS_{LEM} \times FS_{FEM}) - E(FS_{LEM})E(FS_{FEM})}{\sigma_{FS_{LEM}}\sigma_{FS_{FEM}}}, \tag{A.1}$$

where $E(FS_{LEM} \times FS_{FEM})$, $E(FS_{LEM})$, and $E(FS_{FEM})$ are expectations of $FS_{LEM} \times FS_{FEM}$, FS_{LEM}, and FS_{FEM}, respectively; $\sigma_{FS_{LEM}}$ and $\sigma_{FS_{FEM}}$ are the standard deviations of FS_{LEM} and FS_{FEM}, respectively. The terms on the right hand side of Eq. (A.1) can be estimated from FS_{LEM} and FS_{FEM} values in Ω_k, $k = 0, 1, \ldots, m$. Since the values of FS_{LEM} and FS_{FEM} in different Ω_k carry different probability weights, the terms required in Eq. (A.1) are calculated as weighted summations:

$$E(FS_{LEM} \times FS_{FEM}) = \sum_{k=0}^{m} E(FS_{LEM} \times FS_{FEM})_k P(\Omega_k), \tag{A.2}$$

$$E(FS_{LEM}) = \sum_{k=0}^{m} E(FS_{LEM})_k P(\Omega_k), \tag{A.3}$$

$$E(FS_{FEM}) = \sum_{k=0}^{m} E(FS_{FEM})_k P(\Omega_k), \tag{A.4}$$

$$\sigma_{FS_{LEM}} = \sqrt{\sum_{k=0}^{m} E(FS_{LEM}^2)_k P(\Omega_k) - E(FS_{LEM})^2}, \tag{A.5}$$

$$\sigma_{FS_{FEM}} = \sqrt{\sum_{k=0}^{m} E(FS_{FEM}^2)_k P(\Omega_k) - E(FS_{FEM})^2}, \tag{A.6}$$

where $E(FS_{LEM} \times FS_{FEM})_k$, $E(FS_{LEM})_k$, $E(FS_{FEM})_k$, $E(FS_{LEM}^2)_k$, and $E(FS_{FEM}^2)_k$ are estimated as the respective averages of $FS_{LEM} \times FS_{FEM}$, FS_{LEM}, FS_{FEM}, FS_{LEM}^2, and FS_{FEM}^2 values corresponding to N_s random samples selected in Ω_k.

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