Field Homomorphism Injectivity Proof

Prove that a field homomorphism to the field of complex numbers is injective

Let $f : F \to \mathbb{C}$ be a field homomorphism, where F is any field. To prove that f is injective, we will show that its kernel is trivial.

Step 1. Consider the Kernel of f

The **kernel** of f is defined as:

$$\ker(f) = \{x \in F \mid f(x) = 0\}.$$

A fundamental property of ring homomorphisms is that the kernel is an **ideal** in the domain.

Step 2. Use the Structure of Fields

Since F is a field, it has only two ideals:

1. The trivial ideal $\{0\}$, and
2. The entire field F itself.

Step 3. Eliminate the Possibility $\ker(f) = F$

Assume for contradiction that $\ker(f) = F$. Then, for every $x \in F$, we would have $f(x) = 0$. However, because f is a field homomorphism, it must send the multiplicative identity 1_F of F to the multiplicative identity $1_{\mathbb{C}}$ in $\mathbb{C}$. That is,

$$f(1_F) = 1_{\mathbb{C}}.$$

But $1_{\mathbb{C}} \neq 0$ in the field $\mathbb{C}$. This contradiction shows that $\ker(f)$ cannot equal F.

Step 4. Conclude That $\ker(f) = \{0\}$

Since the only ideals in F are $\{0\}$ and F itself, and we have ruled out F, it must be that:

$$\ker(f) = \{0\}.$$

Step 5. Deduce Injectivity

A homomorphism is injective if and only if its kernel is trivial. Since $\ker(f) = \{0\}$, it follows that f is injective.

Final Conclusion

Thus, any field homomorphism $f : F \to \mathbb{C}$ is injective because its kernel is $\{0\}$.