Session 7: Spatial dependence models: estimation and testing

Course on Spatial Econometrics with Applications

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Course Index

- S1: Introduction to spatial econometrics
- S2: Spatial effects, spatial dependence
- S3: Spatial autocorrelation tests
- S4: Exploratory Spatial Data Analysis (ESDA)
- S5: Specification of spatial dependence models
- S6: Spatial regression models: OLS estimation and testing
- PS1: GeoDa: introduction and ESDA
- **S7: Spatial dependence models: estimation and testing**
- S8: Modelling strategies in spatial regression models
- PS2: SpaceStat: confirmatory spatial data analysis
- S9: Specification of spatial heterogeneity models
- S10: Spatial heterogeneity models: estimation and testing
- PS3: Practical exercise and evaluation

. CHASCO, C. (2003), *"Econometría espacial aplicada a la predicción-extrapolación de datos microterritoriales"*. Comunidad de Madrid; pp. 129-140.

Session 7

Overview and Goals

- Limitations of OLS estimation in spatial process models.
- Spatial lag model: estimation and testing.
- Spatial-error model: estimation and testing
- Mixed-regressive-spatial autoregressive model with a spatial autoregressive disturbance (SARAR)

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ANSELIN, L (1988), *"Spatial econometrics: methods and models"*, Kluwer Academic Publishers, pp. 57-59.

7.1. Limitations of OLS estimation in spatial process models

- The spatial dependence shows many similarities to the more familiar time-wise dependence.
- Therefore, the properties of OLS for models with time-lagged dependent variables and/or serial residual autocorrelation do not translate directly to the spatial case.
- This is due to the 2-dimensional and multidirectional nature of dependence in space.
- There are 2 different cases:

7.1.1. OLS in the presence of a spatially-lagged dependent variable: OLS are biased and inconsistent

7.1.2. OLS in the presence of a spatial residual autocorrelation: OLS are inefficient and asymptotically consistent

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7.1. Limitations of OLS estimation in spatial process models

7.1.1. OLS in the presence of a spatially-lagged dependent variable

- In **time-wise models**, OLS estimator remains consistent even when a lagged dependent variable is present, as long as the error term does not show serial correlation (**contemporary independence**). Consequently, even though the estimator is no longer unbiased, it can be used as the basis for asymptotic inference.

- For **spatial autoregressive models**, OLS are unbiased and inconsistent:

E.g.: SAR(1) model $\longrightarrow$ $y = \rho W y + u$

$$\begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{bmatrix} = \rho \begin{bmatrix} 0 & w_{12} & \dots & w_{1N} \\ w_{21} & 0 & \dots & w_{2N} \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ w_{N1} & w_{N2} & \dots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_N \end{bmatrix} = \rho \begin{bmatrix} 0 & w_{12} y_2 & \dots & w_{1N} y_N \\ w_{21} y_1 & 0 & \dots & w_{2N} y_N \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ w_{N1} y_1 & w_{N2} y_2 & \dots & 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_N \end{bmatrix}$$

7.1. Limitations of OLS estimation in spatial process models

7.1.1. OLS in the presence of a spatially-lagged dependent variable (II)

SAR(1) model: $y = \rho W y + u = \rho\, y_L + u$

$$\hat{\rho} = \left[y_L' y_L\right]^{-1} y_L' y = \left[y_L' y_L\right]^{-1} y_L' \left(\rho\, y_L + u\right) = \rho \left[y_L' y_L\right]^{-1}\left[y_L' y_L\right] + \left[y_L' y_L\right]^{-1} y_L' u =$$

$$= \rho + \left[y_L' y_L\right]^{-1} y_L' u$$

1. Biased: $E\left(\hat{\rho}\right) = E\left\{\rho + \left[y_L' y_L\right]^{-1} y_L' u\right\} \neq \rho$

$$E\left[y_L' u\right] = \left\{y = \left(I - \rho\, W\right)^{-1} u\right\} = E\left\{\left[W \cdot \left(I - \rho\, W\right)^{-1} u\right]' u\right\} = \left\{\mathrm{Si}\ \rho = 0\right\} = 0$$

2. Inconsistent: $\underset{N\to\infty}{PLim}\left[y_L' y_L\right]^{-1} y_L' u = \underset{N\to\infty}{PLim}\left[\frac{y_L' y_L}{N}\right]^{-1}\left[\frac{y_L' u}{N}\right] = 0$

$If: \to Q \qquad \boxed{\to 0}$ **Only if ρ = 0**

7.1. Limitations of OLS estimation in spatial process models

Session 7

7.1.2. OLS in the presence of a spatial residual autocorrelation

- Parameter estimates will still be **unbiased, but inefficient**, due to the nondiagonal structure of the disturbance variance matrix.
- The usual properties of OLS and GLS will apply. However, in the spatial case, the multidirectional nature of the spatial dependence will limit the type of FGLS procedures that will lead to consistent estimates.

$$y = X\beta + u \quad ; \quad u = \lambda W u + \varepsilon$$

The multidirectional nature of spatial dependence, FGLS cannot be applied: **no consistent estimators for** ρ, λ **can be obtained from OLS**.

1. Unbiased: $E(b) = E\left\{\beta + [X'X]^{-1} X'u\right\} = \beta$

2. Inefficient: $Cov(b) = E\left[(b-\beta)(b-\beta)'\right] = E\left\{[X'X]^{-1} X'uu'X [X'X]^{-1}\right\} =$

$$= \sigma^2 [X'X]^{-1} X'\boxed{\Omega}X [X'X]^{-1}$$

Ω = non-diagonal

7.2. Spatial lag model: estimation and testing

- Spatial-lag model or mixed regressive spatial autoregressive model or simultaneous spatial autoregressive model

$$y = \rho W y + X\beta + u$$
$$u \approx N\left(0, \sigma^2 I\right)$$

$$y - \rho W y = X\beta + u \;\Rightarrow\; y = (I - \rho W)^{-1} X\beta + (I - \rho W)^{-1} u \;\Rightarrow\; \boxed{y = (I - \rho W)^{-1} X\beta + v}$$

If ρ were known, β could be estimated by OLS

- Spatial-lag model allows to asses:

1. The degree of spatial dependence while controlling for the effect of these other variables.
2. The significance of the other (non-spatial) variables, after the spatial dependence is controlled for.

- Estimation of spatial-lag model:

1. **ML**: based on an underlying normal distribution of the errors.
2. **IV**: more robust.

ANSELIN, L (1988), *"Spatial econometrics: methods and models"*, Kluwer Academic Publishers, pp. 59-65.

Session 7

7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood

- Maximum likelihood estimation of the spatial lag model is based on the assumption of **normal error terms**.

$$L = \Sigma_i \ln(1-\rho\,\omega_i) - \frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln(\sigma^2) - \frac{(y-\rho Wy - X\beta)'(y-\rho Wy - X\beta)}{2\sigma^2}$$

Eigenvalues of W matrix

- *L* is a **nonlinear fuction of the parameters** and must be maximized.
- **β and σ can be expressed in function of** ρ = substitution of these expressions into *L* yields a so-called **concentrated likelihood function**, which only contains ρ:

$$L_C = -\frac{N}{2}\ln\frac{(e_0-\rho\,e_L)'(e_0-\rho\,e_L)}{N} + \Sigma_i \ln(1-\rho\,\omega_i)$$

e_0 and e_L as the residuals in an OLS regression of y on X and Wy on X respectively.

- **ML estimate for ρ is quickly yielded. β and σ = OLS regression of (y- Wy) on X.**

7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood (II)

- Measures of fit:

1) R^2 is not applicable. Instead, **Pseudo R^2** (not comparable with OLS-R^2)

$$\begin{cases} psR^2 = S_{\hat{y}}^2 / S_y^2 \\ Sq.Corr = Corr\left(\hat{y}, y\right)^2 \end{cases}$$

2) Proper measures: **L**

$$L = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - 0,5\frac{e'e}{\sigma^2}$$

AIC: $f\left(N,k\right) = k \cdot \ln\left(N\right)$

SC: $f\left(N,k\right) = 2k$

$$IC = -2L + f\left(k,N\right)$$

7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood (III)

- Hypothesis testing: Inference for estimates of ρ, β-parameters

1) All **statistical inference** for estimates obtained with ML is based on **asymptotic considerations**.

2) In **small data sets**, the asymptotic standard errors for the estimates will tend to be smaller than they should be for the actual sample size used, which will result in a **stronger indication of significance** than may be merited.

3) In practice **one should be cautious** in deciding on the significance of a coefficient when this significance is only marginal.

4) The significance of ρ, β is based on a **standard normal distribution** (a z-value) and not on the Student *t* distribution: $z\text{-}value = b_j / asySE\left(b_j\right)$

7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood (IV)

- Specification diagnostics:

1. In ML approach, a **much narrower range of specification diagnostics** is available
2. **Lagrange Multiplier (LM) tests, Likelihood Ratio (LR) tests**: asymptotic and may lead to inconsistent conclusions in finite samples.

I. Heteroskedasticity:

1. **Breusch-Pagan (BP) test**: based on the residual from the ML estimation, identitical to OLS:
2. **Spatial B-P tests**: includes some adjustments to take into account spatial dependence.

BP = 1/2 the explained sum of squares in a regression of $\left(e_i / \tilde{\sigma}^2_{ML} - 1\right)$ on a constant and the z-variables.

7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood (V)

- Specification diagnostics (II):

II. Spatial lag dependence:

1. **Likelihood Ratio (LR) test on** ρ: $LR = 2\left(L_L - L_0\right)\ ;\ LR \sim \chi_1^2$
2. **Wald test:** $W = z\text{-}value^2 = \left[b_j / asySE\left(b_j\right)\right]^2$
3. **LM-LAG:** → $LM_{LAG} = \dfrac{\left[e'Wy / \tilde{\sigma}_{ML}^2\right]^2}{\dfrac{\left[WXb\right]' MWXb}{\tilde{\sigma}_{ML}^2} + tr\left[W'W + W^2\right]}$
4. Though they are asymptotic, In finite samples: $\boxed{\mathbf{W \geq LR \geq LM\text{-}ERR_L}}$

Changes in this order must be an indication of a **potential specification error**: non-normal error terms, a nonlinear relationship between dependent and explanatory variables, and a poor choice of the variables included in the model and/or of the spatial weights matrix.

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7.2. Spatial lag model: estimation and testing

Session 7

7.2.1. Maximum Likelihood (VI)

- Specification diagnostics (III):

If the spatial lag model you specified is indeed the correct one, then no spatial dependence should remain in the residuals.

III. Spatial-error dependence: LM-ERR in spatial-lag model

A significant result for this test indicates one of two things:

1. If **W for the test is the same as W in the spatial lag model**, the latter must be **misspecified**.

2. **Not all spatial dependence has been eliminated** (or new, spurious patterns of spatial dependence have been created) which casts a serious doubt on the appropriateness of the W specification in the model.

$$LM_{ERR_L} = \frac{\left[\dfrac{e'We}{\hat{\sigma}^2_{ML}}\right]^2}{tr\left(W'W + W^2\right) - tr\left(W'W + W^2\right) A^{-1} \cdot \operatorname{var}(\rho)}$$

$$A^{-1} = \left[I - \rho W\right]^{-1}$$

e = residuals in the ML estimation

var( ) the estimated asymptotic variance for the ρ-estimator

SOLUTIONS: Instead of the current spatial lag model, you could try a higher order spatial autoregressive model, a different W, a completely different model specification (e.g., a spatial error model, mixed autoregressive spatial moving average model).

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7.2. Spatial lag model: estimation and testing

7.2.1. Maximum Likelihood Estimation (VII)

- Spatial autocorrelation: on spatial lag (ρ) and residuals

1. Spatial-lag autocorrelation (ρ)

   - Likelihood Ratio (LR) → $LR = 2\left(L_{lag} - L_{s\tan}\right)$
   - Wald Test → $(t_\rho)^2$
   - Lagrange Multiplier (lag) → In standard model

$W \geq LR \geq LM$

conflicting indications

potential specification errors:

1. Non-normal errors
2. Non-linear relationship
3. Poor choice of variables
4. Poor choice of spatial weight matrix

2. Spatial-error autocorrelation (residuals):

   - Lagrange Multiplier test for spatial error autocorrelation in the spatial lag model

$LMERR_L$ significant: →

1. If $W_{LM} = W_{LAG}$: not all spatial dependence eliminated
2. If $W_{LM} \neq W_{LAG}$: a SARMA model will be better →

1. Higher order W
2. Different W
3. Different model

7.2. Spatial lag model: estimation and testing

7.3.2. Instrumental Variables Estimation

$$y = \rho W y - X\beta + u$$

- It's a robust estimation method since it **doesn't require the assumption of normality in the errors**.
- The presence of the **spatial lag** is similar to the inclusion of **endogenous variables on the RHS in systems of simultaneous equations**. The instrumental variables estimator (IV) or two-stage-least-squares estimator (2SLS) exploits this feature by constructing a proper instrument for the spatial lag.
- The resulting estimate is **consistent, but not necessarily very efficient**. It may be used as the basis for a bootstrap procedure.

7.2. Spatial lag model: estimation and testing

7.3.2. Instrumental Variables Estimation (II)

$$\boxed{y = \rho W y - X\beta + u}$$

Z = [Wy, X]

- The principle of instrumental variables estimation is based on the existence of a set of instruments, **Q, that are strongly correlated with Z but asymptotically uncorrelated with u**.
- Once Q are identified, they are used to construct a proxy for the endogenous variables, which consists of their predicted values in a regression of Wy on Q and Z.
- This **proxy** is then used in a **standard least squares regression**.
- Formally, this process of two-stage-least-squares yields the esitmate:

$$\theta_{IV} = \left[(Z'Q)(Q'Q)^{-1}(Q'Z)\right]^{-1}(Z'Q)(Q'Q)^{-1}Q'y$$

$$\theta_{IV} = \begin{bmatrix} \hat{\rho} \\ \hat{\beta}_1 \\ \dots \\ \hat{\beta}_K \end{bmatrix}_{(K+1,1)}$$

7.2. Spatial lag model: estimation and testing

7.3.2. Instrumental Variables Estimation (III)

- It can be shown that θ_{IV} is **consistent and asymptotically efficient**.
- However, its properties **in finite samples** depend mainly on the choice of Q.
- One potential problem is that ρ_{IV} estimate does not necessarily fall in the acceptable range (-1, +1): values larger than 1 in absolute value may be obtained.
- Typically, this points to **potential problems with the specification** of the model.

- Choice of the instruments:

1. In the standard simultaneous equations framework, Q are the "excluded" exogenous variables.
2. In the spatial lag model, Kelejian and Robinson (1992) have shown that a series of **spatially lagged exogenous variables** for 1st order contiguity matrices are the proper
3. Formally, this results in a matrix Q containing X and WX, where the constant term and other variables that would cause perfect multicollinearity are excluded from WX.

7.2. Spatial lag model: estimation and testing

7.3.2. Instrumental Variables Estimation (IV)

- Measures of fit:

  1) Instead, **Pseudo R^2** (R^2 is not applicable)

$$\begin{cases} psR^2 = S_{\hat{y}}^2 / S_y^2 \\ Sq.Corr = Corr\left(\hat{y}, y\right)^2 \end{cases}$$

  2) Neither of these is comparable with their counterparts in ML spatial-lag model nor with OLS-R^2.

- Hypothesis testing: Inference for estimates of ρ, β-parameters

  1) All **statistical inference** for IV-estimates is based on **asymptotic considerations**.

  2) The significance of ρ, β is based on a **standard normal distribution** (a z-value) and not on the Student *t* distribution:

$$z\text{-}value = b_j / asySE\left(b_j\right)$$

7.2. Spatial lag model: estimation and testing

7.3.3. Bootstrap

$$\boxed{y = \rho W y + X\beta + u}$$

- The bootstrap is a **robust estimator** which exploits the randomness present in artificially created resampled data sets as the basis for statistical inference.
- This leads to alternative parameter estimates, measures of bias and variance, and the construction of pseudo significance levels and confidence intervals.
- In multidimensional space, for the spatial lag model we follow the **approach based on residuals**.

7.2. Spatial lag model: estimation and testing

7.3.3. Bootstrap (II)

- The bootstrap in the spatial lag model is a 2-step procedure:

  1. An **IV estimation** is carried out, which provides an estimate for the error vector(u) in the form of the residuals (e): $e = y - \hat{\rho}_{VI} Wy - X\hat{\beta}_{VI}$

  2. **Pseudo error terms** (e_r) are generated by random resampling (with replacement) from the vector e.

  3. A vector of **pseudo observations on y** may be computed for each set of N such resampled residuals, as: $y_r = (I - \rho_{VI} W)^{-1} (X\beta_{VI} + e_r)$

  4. An estimate for ρ and β in the resampled data set is obtained by means of IV, using Wy_r as the spatial lag. This procedure is repeated a large number of times (R>99), to generate an empirical frequency distribution for the $\hat{\rho}$ and b estimates.

  5. The **bootstrap estimate is the mean** of this empirical frequency distribution.

- Measures of fit: computed as IV. Not comparable.

7.3. Spatial-Error Model

- Spatial-error model or spatial autoregressive error term model

  The spatial error model is a special case of a so called non-spherical error model. The spatial dependence in the error term can take on a number of different forms. In most software, only a spatial autoregressive process for the error term can be estimated.

$$\boxed{\begin{array}{l} y = X\beta + u \\ SAR(1) \rightarrow u = \lambda W u + \varepsilon \end{array}}$$

$$\begin{cases} SAR(1) \rightarrow u = \lambda W u + \varepsilon \\ SMA(1) \rightarrow u = \varepsilon + \theta W \varepsilon \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

- Non-diagonal error variance matrix:

$$E(uu') = \Omega = \sigma^2 \left[(I - \lambda W)'(I - \lambda W) \right]^{-1}$$

If λ were known, the β could be estimated by means of OLS in a model with spatially filtered variables:

$$\boxed{(Y - \lambda W Y)} = \boxed{(X - \lambda W X)}\beta + \varepsilon$$

- OLS are inefficient, but they are still unbiased. **ML is preferred**, though the estimation of the nuisance parameter λ can be biased.

For a known λ, this method is called GLS. In most cases, λ must be estimated jointly with β. In spatial models, FGLS is not applicable.

7.3. Spatial-Error Model

Session 6

7.3.1. Maximum Likelihood Estimation

© ML is based on the **assumption of normal error terms**. It is expressed as:

$$L = \Sigma_i \ln\left(1 - \lambda\, \omega_i\right) - \frac{N}{2}\ln\left(2\pi\right) - \frac{N}{2}\ln\left(\sigma^2\right) - \frac{(y - X\beta)'\left(y - \lambda W\right)'(y - X\beta)}{2\sigma^2}$$

Eigenvalues of W matrix

© Given this assumption, a **likelihood function** can be derived that is a nonlinear function of the parameters and must be maximized.

© As pointed out, the estimates for β and σ can be expressed in function of λ.

© After substituting these expressions into the likelihood, a concentrated likelihood can be found, which is solely a function of the autoregressive parameter:

$$L_C = -\frac{N}{2}\ln\frac{e'e}{N} + \Sigma_i \ln\left(1 - \lambda\, \omega_i\right)$$

e'e = residual sum of squares of the spatially filtered dependent & explanatory variables, which are function of λ.

A ML estimate for λ can be found by means of a search over the acceptable interval ($1/\omega_{min}$, $1/\omega_{max}$).

7.3. Spatial-Error Model

Session 6

7.3.1. Maximum Likelihood Estimation (II)

- Measures of fit:

  1) R^2 is not applicable.

  2) **Pseudo R^2** (not comparable with OLS-R^2)

$$psR^2 = S_{\hat{y}}^2 / S_y^2$$

$$Sq.Corr = Corr\left(\hat{y}, y\right)^2$$

$$R_B^2 = 1 - (e - \lambda We)'(e - \lambda We) / (Y - \iota Y_W)'(I - \lambda W)'(I - \lambda W)(Y - \iota Y_W)$$

$$Y_W = (\iota - \lambda W\iota)'(Y - \lambda WY) / (\iota - \lambda W\iota)'(\iota - \lambda W\iota)$$

ι vector $(N,1)$ of ones.

  3) Proper measures: **L**

$$L = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - 0,5\frac{e'e}{\sigma^2}$$

AIC: $f\left(N,k\right) = 2k$

SC: $f\left(N,k\right) = k \cdot \ln\left(N\right)$

$$IC = -2L + f\left(k,N\right)$$

K does not include λ, what could favor the spatial error over spatial lag model

7.3. Spatial-Error Model

7.3.1. Maximum Likelihood Estimation (III)

- Hypothesis testing: Inference for estimates of λ, β-parameters

  1) All **statistical inference** for estimates obtained with ML is based on **asymptotic considerations**.

  2) In **small data sets**, the asymptotic standard errors for the estimates will tend to be smaller than they should be for the actual sample size used, which will result in a **stronger indication of significance** than may be merited.

  3) In practice **one should be cautious** in deciding on the significance of a coefficient when this significance is only marginal.

  4) The significance of λ, β is based on a **standard normal distribution** (a z-value) and not on the Student *t* distribution: $z\text{-}value = b_j \big/ asySE\left(b_j\right)$

Session 6

7.3. Spatial-Error Model

7.3.1. Maximum Likelihood Estimation (IV)

- Specification diagnostics:

Lagrange Multiplier (LM) tests, Likelihood Ratio (LR) tests: asymptotic and may lead to inconsistent conclusions in finite samples.

I. Heteroskedasticity:

1. **Breusch-Pagan (BP) test**: based on the residual from the ML estimation, identitical to OLS:
2. **Spatial B-P tests**: includes some adjustments to take into account spatial dependence.

BP = 1/2 the explained sum of squares in a regression of on a constant and the z-variables.

7.3. Spatial-Error Model

7.3.1. Maximum Likelihood Estimation (IV)

- Specification diagnostics (II):

II. Spatial-error dependence:

$W_{asy(\lambda)} \geq$ **LR** $\geq$ LM-ERR$_{ols}$

1. Likelihood Ratio (LR) test on λ: $LR = 2\left(L_E - L_0\right)\ ;\ LR \sim \chi_1^2$

2. Common Factor Hypothesis: The spatial error model is equivalent to a spatial lag model referred to as the **common factor or spatial Durbin model**:

$$\left.\begin{aligned} y &= X\beta + u \\ u &= \lambda W u + \varepsilon \end{aligned}\right\} y = X\beta + \lambda W u + \varepsilon \Leftrightarrow y = X\beta + \lambda W\left(y - X\beta\right) + \varepsilon$$

Common factor hypothesis: **H_0: $\lambda.\beta$ = - γ**

$$y = \lambda W y + X\beta + \lambda W X\beta + \varepsilon \quad \text{Constrained}$$

$$y = \lambda W y + X\beta + \lambda W X\gamma + \varepsilon \quad \text{Unconstrained}$$

1. **Wald** test ~ $\chi^2_{k'-1}$: computed from an auxiliary ML estimation of the unconstrained spatial lag model
2. **LR** test ~ $\chi^2_{k'-1}$: LR = 2 ($L_C - L_U$)

7.3. Spatial-Error Model

7.3.1. Maximum Likelihood Estimation (V)

- Specification diagnostics (III):

II. Spatial-error dependence: LM-ERR in spatial error model

If the spatial error model you specified is indeed the correct one, then no spatial dependence should remain in the residuals.

$$LM - ERR_E = \frac{\left[\frac{e'We}{\hat{\sigma}^2_{ML}}\right]^2}{tr\left(W + W'\right)W}$$

e = residuals in the ML estimation

LM-ERR$_E$ $\sim\chi^2_1$

SOLUTIONS: If the null hypothesis were rejected, that might point to a higher order spatial-error model, to a different W or a different model specification (e.g., a mixed-regressive spatial autoregressive cross-regressive model, mixed-regressive-spatial autoregressive model with a spatial autoregressive disturbance).

7.3. Spatial-Error Model

7.3.2. Instrumental Variables estimation

- For the case of non-normal errors, a spatial-error model can be estimated using IV, but applied to the **spatial Durbin** model:

$$\boxed{y = \lambda Wy + X\beta + \lambda WX\beta + \varepsilon}$$

- However, since the WX are already included as explanatory variables in this specification, **additional instruments are needed**, i.e., higher order spatial lags for the exogenous variables.
- This may lead to problems with **multicollinearity.** In any event, the resulting estimates are not likely to be very efficient.
- Nevertheless, an IV approach may provide an indication of the seriousness of the spatial error dependence and how it affects the inference for the other parameters in the model.

. KELEJIAN, HH & IR PRUCHA (1999), "A generalized moments estimator for the autoregressive parameter in a spatial model.". *International Economic Review* 40, 509-533

7.3. Spatial-Error Model

7.3.3. Generalized Method of Moments estimation

- The estimation of λ is consistent, but not efficient.
- It is a nuisance parameter: no inference for it
- It is implemented in SpaceStat.

7.4. Mixed-regressive-spatial autoregressive model with a spatial autoregressive disturbance (SARAR)

7.4.1. Maximum Likelihood estimation

$$y = \rho W_1 y + X\beta + u$$
$$u = \lambda W_2 u + \varepsilon$$

$$\begin{aligned} L &= C - (n/2)ln(\sigma^2) + ln(|A|) + ln(|B|) - (1/2\sigma^2)(e'B'Be) \\ e &= (Ay - X\beta) \\ A &= (I_n - \rho W_1) \\ B &= (I_n - \lambda W_2) \end{aligned} \quad (3.33)$$

We concentrate the function using the following expressions for β and σ^2:

$$\begin{aligned} \beta &= (X'A'AX)^{-1}(X'A'ABy) \\ e &= By - X\beta \\ \sigma^2 &= (e'e)/n \end{aligned} \quad (3.34)$$

Using the expressions in (3.34), we can evaluate the log likelihood for values of ρ and λ. The values of the other parameters β and σ^2 are calculated as a function of the maximum likelihood values of ρ, λ and the sample data in y, X.

Implemented in LeSage's Matlab toolbox

. KELEJIAN, HH & IR PRUCHA (1998), "A generalized spatial two-stage least squares procedure for estimating a spatial autoregressive model with autoregressive disturbances". *Journal of Real Estate Finance and Economics* 17, 99-121

7.4. Mixed-regressive-spatial autoregressive model with a spatial autoregressive disturbance (SARAR)

7.4.2. Generalized Method of Moments estimation

- They give a **feasible generalized spatial two stage least squares** (GS2SLS) estimator for either an error-SAR and a SARAR model.
- The estimation procedure comprises **3 stages**:

1. The model is estimated by 2SLS.
2. They use the resulting 2SLS residuals to estimate ρ and σ^2 using a GM procedure.
3. The estimated ρ is used to perform a Cochrane-Orcutt type transformation to account for the spatial dependence in the residuals.