SPATIAL WEIGHT MATRICES

Each nonnegative matrix, $W = (w_{ij} : i, j = 1,..,n)$, is a possible *spatial weight matrix* summarizing spatial relations between n spatial units. Here each *spatial weight*, w_{ij}, typically reflects the "spatial influence" of unit j on unit i. Following standard convention, we here exclude "self influence" by assuming that $w_{ii} = 0$ for all $i = 1,..,n$ (so that W has a zero diagonal). The following is a list of spatial weight matrices often used in practice. See also the list on p.261 of [BG].

1. Weights Based on Distance

The following weight matrices are based on the *centroid distances*, d_{ij}, between each pair of spatial units i and j.[1]

1.1 k-Nearest Neighbor Weights. Let centroid distances from each spatial unit i to all units $j \neq i$ be ranked as follows: $d_{ij(1)} \leq d_{ij(2)} \leq \cdots \leq d_{ij(n-1)}$. Then for each $k = 1,..,n-1$, the set $N_k(i) = \{j(1), j(2),.., j(k)\}$ contains the k closest units to i (where for simplicity we ignore ties). For each given k, the *k-nearest neighbor* weight matrix, W, then has spatial weights of the form:

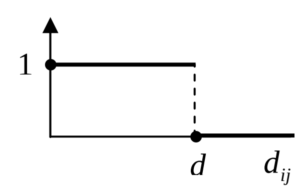
$$\text{(1a)} \qquad \boxed{w_{ij} = \begin{cases} 1 & , \quad j \in N_k(i) \\ 0 & , \quad otherwise \end{cases}} \qquad \text{(Standard form)}$$

Alternatively, one can consider a symmetric version in which positive weights are assigned to all ij pairs for which at least one is among the *k*-nearest neighbors of the other:

$$\text{(1b)} \qquad \boxed{w_{ij} = \begin{cases} 1 & , \quad j \in N_k(i) \text{ or } i \in N_k(j) \\ 0 & , \quad otherwise \end{cases}} \qquad \text{(Symmetric form)}$$

1.2 Radial Distance Weights. If distance itself is an important criterion of spatial influence, and if d denotes a *threshold distance* (or *bandwidth*) beyond which there is no direct spatial influence between spatial units, then the corresponding *radial distance* weight matrix, W, has spatial weights of the form:

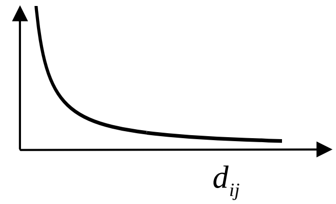
$$\text{(2)} \qquad \boxed{w_{ij} = \begin{cases} 1 & , \quad 0 \leq d_{ij} \leq d \\ 0 & , \quad d_{ij} > d \end{cases}}$$



[1] Such distances may of course be between other relevant representative points for each spatial unit, such as the capital (or largest city) of each state.

1.3 Power Distance Weights. Note that in radial distance weights there is assumed to be no diminishing effect in distance up to threshold *d*. If there are believed to be diminishing effects, then one standard approach is to assume that weights are a *negative power function* of distance of the form

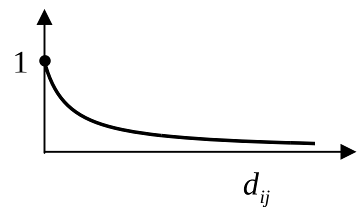
(3) $$\boxed{w_{ij} = d_{ij}^{-\alpha}}$$



where α is any positive exponent, typically $\alpha = 1$ or $\alpha = 2$.

1.4 Exponential Distance Weights. An alternative to negative power functions are *negative exponential functions* of distance of the form:

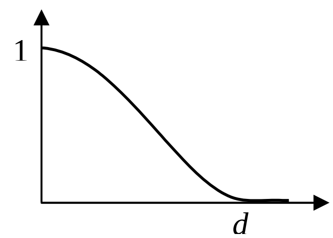
(4) $$\boxed{w_{ij} = \exp(-\alpha\, d_{ij})}$$



where α is any positive exponent.

1.5 Double-Power Distance Weights. A somewhat more flexible family incorporates finite bandwidths with "bell shaped" taper functions. If d again denotes the maximum radius of influence (*bandwidth*) then the class of *double-power* distance weights is defined for each positive integer *k* by

(5) $$\boxed{w_{ij} = \begin{cases} \left[1-\left(d_{ij}/d\right)^k\right]^k & , \; 0 \le d_{ij} \le d \\ 0 & , \; d_{ij} > d \end{cases}}$$



where typical values of k are 2, 3 and 4. Note that w_{ij} falls continuously to zero as d_{ij} approaches d, and is defined to be zero beyond d. The graph shows the case of a *quadratic distance function* with $k = 2$ (see also [BG]. P.85).

2. Weights Based on Boundaries

The advantage of the distance weights above is that distances are easily computed. But in many cases the *boundaries* shared between spatial units play in important role in determining degree of "spatial influence".

2.1 Spatial Contiguity Weights. The simplest of these weights simply indicate whether spatial units share a boundary or not. If the set of boundary points of unit i is denoted by $bnd(i)$ then the so-called *queen contiguity weights* are defined by

(6) $$w_{ij} = \begin{cases} 1 & , \ bnd(i) \cap bnd(j) \neq \varnothing \\ 0 & , \ bnd(i) \cap bnd(j) = \varnothing \end{cases}$$

However, this allows the possibility that spatial units share only a single boundary point (such as a shared corner point on a grid of spatial units). Hence a stronger condition is to require that some *positive* portion of their boundary be shared. If l_{ij} denotes the length of shared boundary, $bnd(i) \cap bnd(j)$, between i and j, then these so-called *rook contiguity weights* are defined by

(7) $$w_{ij} = \begin{cases} 1 & , \ l_{ij} > 0 \\ 0 & , \ l_{ij} = 0 \end{cases}$$

2.2 Shared-Boundary Weights. As a sharper form of comparison, note that if l_i defines the total boundary length of $bnd(i)$ that is shared with other spatial units, i.e., $\Sigma_{j \neq i} l_{ij}$, then fraction of this length shared with any particular unit j is given by l_{ij} / l_i. These fractions themselves yield a potentially relevant set of *shared boundary weights*, defined by

(8) $$w_{ij} = \frac{l_{ij}}{l_i} = \frac{l_{ij}}{\sum_{k \neq i} l_{ik}}$$

3. Combined Distance-Boundary Weights

Finally, it should be evident that in many situations spatial influence may exhibit aspects of *both* distance and boundary relations. One classical example of this is given in the original study of spatial autocorrelation by Cliff and Ord (1969). In analyzing the Eire blood-group data, they found that the best weighting scheme for capturing spatial autocorrelation effects was given by the following combination of *power-distance* and *boundary-shares*,

(9) $$w_{ij} = \frac{l_{ij} d_{ij}^{-\alpha}}{\sum_{k \neq i} l_{ik} d_{ik}^{-\alpha}}$$

with simple inverse-distance, $\alpha = 1$.

3. Normalizations of Spatial Weights

In most cases it is convenient to normalize spatial weights to remove dependence on extraneous scale factors (such as the particular units of distance employed in exponential and power weights). Here there are two standard approaches:

3.1 Row Normalized Weights. Recall that the i^{th} row of W contains all spatial weights influencing spatial unit i, namely $(w_{ij} : j \neq i)$ [recall that $w_{ii} = 0$]. So if the (nonnegative) weights in each row are normalized to have unit sum, i.e.,

$$\boxed{\sum\nolimits_{j=1}^{n} w_{ij} = 1 \ , \ i = 1,..,n} \tag{10}$$

then this produces what called the *row normalization* of *W*. Note that each row-normalized weight, w_{ij}, can then be interpreted as the *fraction* of all spatial influence on unit i attributable to unit j. The appeal of this interpretation has led to the current wide-spread use of row-normalized weight matrices. In fact, many of the spatial weight definitions above are often implicitly defined to be row normalized. The most obvious example is that of shared boundary weights in (8), which by definition are seen to be row normalized. [Also the combined example in (9) was defined by Cliff and Ord (1969) to be in row-normalized form.] Another simple example is provided by the standard form of *k-nearest neighbor weights* in (1a) above, which are often defined using weights $1/k$ rather than 1 to ensure row normalization. A more interesting example is provided by the *power distance weights*, which have the row-normalized form,

$$\boxed{w_{ij} = \frac{d_{ij}^{-\alpha}}{\sum_{k \neq j} d_{ik}^{-\alpha}}} \tag{11}$$

These normalized weights are seen to be precisely the *Inverse Distance Weighting* (IDW) scheme employed in Spatial Analyst for spatial interpolation. A similar example is provided by *exponential distance weights*, with row-normalized form,

$$\boxed{w_{ij} = \frac{\exp(-\alpha\, d_{ij})}{\sum_{k \neq j} \exp(-\alpha\, d_{ik})}} \tag{12}$$

These weights are also used for spatial interpolation. In addition, it should be noted that these normalized weights are commonly used in spatial interaction modeling, where (10) and (11) are often designated, respectively, as *Newtonian* and *exponential* models of spatial interaction intensities or probabilities.

3.2 Scalar Normalized Weights. In spite of its popularity, row-normalized weighting has it drawbacks. In particular, row normalization alters the internal weighting structure of *W* so that comparisons between rows become somewhat problematic. For example, consider *spatial contiguity weighting* with respect to the simple three-unit example shown on the left below:

(13)

i	j	k

$$W = \begin{pmatrix} 0 & w_{ij} & w_{ik} \\ w_{ji} & 0 & w_{jk} \\ w_{ki} & w_{kj} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

As represented in the contiguity weight matrix, W, on the right, unit j is influenced by both i and k, while units i and k are each influenced only by the single unit j. Hence it might be argued that j is subject to *more* spatial influence than either i or k. But row normalization of W changes this relation, as seen by its row-normalized form, W_{rn}, below:

$$(14)\qquad W_{rn} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Here the "total" influence on each unit is by definition the same, so that unit i now influences j only "half as much" as j influences i. While the exact meaning of "influence" is necessarily vague in most applications, this effect of row-normalization can hardly be considered as neutral.[2]

In view of this limitation, it is natural to consider simple *scalar* normalizations which multiply W by a single number, say $\alpha \cdot W$, which removes any measure-unit effects but preserves relations between all rows of W. For example, if $w_{\max}$ denotes the largest element of W, then the choice,

$$(15)\qquad \boxed{\alpha = \frac{1}{w_{\max}} > 0}$$

provides one such normalization that has the advantage of ensuring that the resulting spatial weights, w_{ij}, are all between 0 and 1, and hence can still be interpreted as *relative influence intensities*.

However, for theoretical reasons, it is more convenient to divide W by the maximum eigenvalue, $\lambda_{\max}$, of W (to be discussed later) and hence to set

$$(16)\qquad \boxed{\alpha = \frac{1}{\lambda_{\max}} > 0}$$

[2] Further discussion of this problem can be found in Kelejian and Prucha (2010).

This *max-eigenvalue normalization* is one of the normalization options available in the MATLAB program, **dist_wts.m**, for constructing distance-based weight matrices (to be discussed in class).

References

Cliff, A. D. and J. K. Ord. (1969). "The problem of Spatial autocorrelation." In *London Papers in Regional Science 1*, *Studies in Regional Science*, 25–55, edited by A. J. Scott, London: Pion.

Kelejian, H.H. and Prucha, I.R. (2010) "Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances", *Journal of Econometrics*, 157: 53-67.