Applied Spatial Statistics in R, Section 4

Spatial Point Processes

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Outline

Point Pattern Analysis: Aerial Bombardment

- During World War II, Germany launched 1,358 V-2 Rockets at London.
- The V-2's speed and trajectory made it invulnerable to anti-aircraft guns and fighters.
- But its guidance systems were thought to be too primitive to hit specific targets.
- After the strikes began in 1944, bomb damage maps were interpreted by some analysts as showing that impact sites were clustered.
- This evidence appeared to contradict existing intelligence on the V-2 program.
- If the rocket strikes were spatially clustered, the guidance systems must have been more advanced than previously thought.

Point Pattern Analysis: Aerial Bombardment

Figure: Distribution of V-2 Rocket Strikes on Central London, 1944

Point Pattern Analysis: Aerial Bombardment

- R.D. Clarke (1946) decided to apply a statistical test to assess whether any support could be found for the clustering hypothesis.
- He selected an area of 144 km^2 in south London, which he divided into 576 squares of $1/4$ km^2.
- For each square, Clark recorded the total number of observed bomb hits. There were 537 total in the study area.
- He then recorded the number of squares with $k = 1, 2, 3, \ldots$ hits.
- The expected number of squares with k hits was derived from the Poisson distribution $\sum_{k=1}^{n} \frac{e^{-\lambda}\lambda^k}{k!}$, with $\lambda = \frac{537}{576}$ and $n = 576$.

Point Pattern Analysis: Aerial Bombardment

No. of bombs per square	Expected	Observed
1	226.74	229
2	211.39	211
3	98.54	93
4	7.14	7
5+	1.57	1
	$\chi^2 = 1.17, p = 0.88$	

- It is clear from the cross-tabulation that the distribution of V-2 hits conforms quite closely to the Poisson distribution.
- The occurrence of clustering would have been reflected in an excess number of squares with either a high number of bombs or none at all, and fewer squares in the intermediate classes.
- The closeness of fit suggested that V-2 impact sites were random, rather than clustered.

Point Pattern Processes

Point patterns have first- and second- order properties:

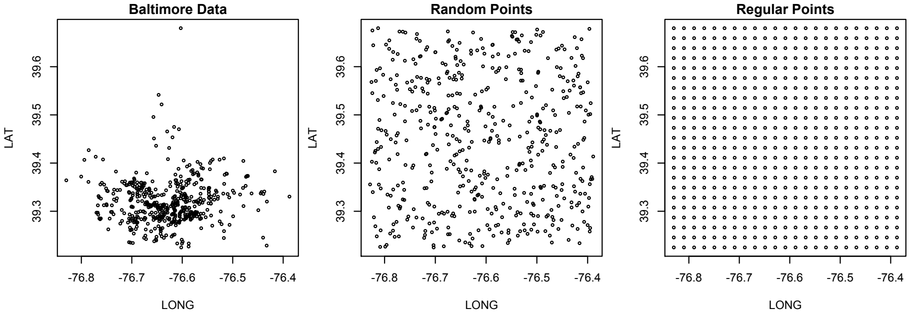
1. First-order properties measure the distribution of events in a study region: intensity and spatial density.
2. Second-order properties measure the tendency of events to appear clustered, independently, or regularly-spaced.

Point Pattern Processes: Complete Spatial Randomness

- The most basic test which can be performed is that of Complete Spatial Randomness (CSR). Under CSR, events are distributed independently and uniformly over a study area.
- A point process which is CSR point process is formally defined as a homogeneous Poisson process (HPP).
  - Under HPP, the location of one point in space does not affect the probabilities of other points' appearing nearby. The intensity of the point process in area A is a constant $\lambda(y) = \lambda > 0, \quad \forall y \in A$.
- A generalization of HPP which allows for non-constant intensity $\lambda(y)$ is called an inhomogeneous Poisson process (IPP).

Point Pattern Processes: Complete Spatial Randomness

- Let's explore conformity to CSR among three point patterns: (1) real data on crime locations in Baltimore, (2) points drawn from uniform distribution over the same study area, (3) regularly-spaced point pattern.



Point Pattern Processes: $\mathcal{G}$ Function

- The $\mathcal{G}$ Function measures the distribution of distances from an arbitrary event to its nearest neighbors.

$$\hat{\mathcal{G}}(r) = \frac{\sum_{i=1}^{n} I_i}{n}$$

$$I_i = \begin{cases} 1 & \text{if } d_i \in \{d_i : d_i \leq r, \forall i\} \\ 0 & \text{otherwise} \end{cases}$$

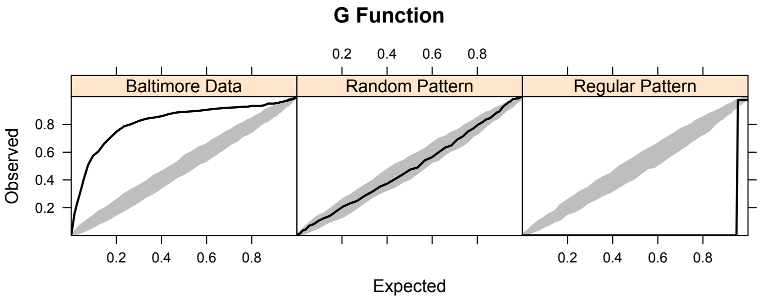
- where $d_i = \min_j\{d_{ij}, \forall j \neq i \in S\}, i = 1, \ldots, n$.
- So, the $\mathcal{G}$ function represents the number of elements in the set of distances up to some threshold r, normalized by the total number of points n in point pattern S.
- Under CSR, the value of the $\mathcal{G}$ function becomes:

$$\mathcal{G}(r) = 1 - e^{\lambda \pi r^2}$$

- where λ is the mean number of events per unit (intensity).

Point Pattern Processes: $\mathcal{G}$ Function

- The comparability of a point process with CSR can be assessed by plotting the empirical function $\hat{\mathcal{G}}(r)$ against the theoretical expectation $\mathcal{G}(r)$.
- For a clustered pattern, observed locations should be closer to each other than expected under CSR. A regular pattern should have higher nearest-neighbor distances than expected under CSR.
- This is shown below for the Baltimore crime locations dataset.



Point Pattern Processes: $\mathcal{F}$ Function

- The $\mathcal{F}$ Function measures the distribution of **all** distances from an arbitrary point k in the plane to the nearest observed event j.

$$\hat{\mathcal{F}}(r) = \frac{\sum_{k=1}^{m} I_k}{m}$$

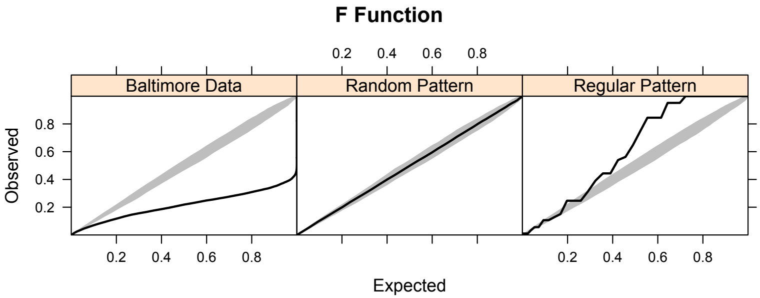
$$I_k = \begin{cases} 1 & \text{if } d_k \in \{d_k : d_k \leq r, \forall k\} \\ 0 & \text{otherwise} \end{cases}$$

- where $d_k = \min_j\{d_{kj}, \forall j \in S\}, k = 1, \ldots, m, \quad j = 1, \ldots, n.$
- Under CSR, the expected value is also

$$\mathcal{F}(r) = 1 - e^{\lambda \pi r^2}$$

Point Pattern Processes: $\mathcal{F}$ Function

- As before, we can plot the empirical function $\hat{\mathcal{F}}(r)$ against its theoretical expectation $\mathcal{F}(r)$.
- For a clustered pattern, observed locations j should be farther away from random points k than expected under CSR. In a regular pattern, random locations should be closer to observed points.
- This is again shown below for the Baltimore crime locations dataset.



Point Pattern Processes: Intensity

- For an HPP point process, intensity is a constant $\lambda(x) = \lambda = \frac{n}{|A|}$, where n is the number of points observed in region A, and $|A|$ is the area of region A.
- For an IPP point process, intensity is non-constant and can be estimated non-parametrically with kernel smoothing (Diggle 1985, Berman and Diggle 1989, Bivand et. al. 2008).

Point Pattern Processes: Kernel Density

- The kernel density estimator is:

$$\hat{\lambda}(x) = \frac{1}{h^2} \sum_{i=1}^{n} \frac{\kappa\left(\frac{||x-x_i||}{h}\right)}{q(||x||)}$$

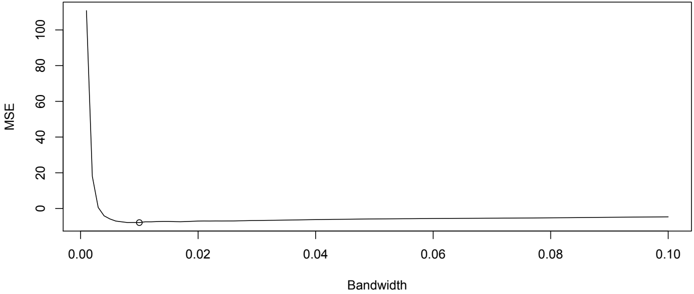
- where $x_i \in \{x_1, \ldots, x_n$ is an observed point, h is the bandwidth, $q(||x||)$ is a border correction to compensate for observations missing due to edge effects, and $\kappa(u)$ is a bivariate and symmetrical kernel function.
- R currently implements a two-dimensional quartic kernel function:

$$\kappa(u) = \begin{cases} \frac{3}{\pi}(1 - ||u||^2)^2 & \text{if } u \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

- where $||u||^2 = u_1^2 + u_2^2$ is the squared norm of point $u = (u_1, u_2)$

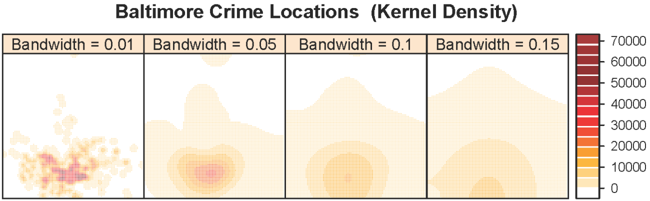
Point Pattern Processes: Kernel Density

- There is no general rule for selecting the <u>bandwidth</u> h, which governs the level of smoothing.
- Small bandwidth $\rightarrow$ spiky map; large bandwidth $\rightarrow$ smooth map.
- Berman and Diggle (1989) propose a criterion based on minimization of mean square error (MSE) of the kernel smoothing estimator.
- The plot below implements this approach for the Baltimore crime dataset. The "optimal" bandwidth here is 0.01.



Point Pattern Processes: Kernel Density

- The plot below shows kernel density estimates for the Baltimore crime locations at different values of the bandwidth h.
- Lighter values indicate greater intensity of the point process.
- Clearly, different bandwidths tell very different stories about the spatial intensity of crime in Baltimore...



Examples in R

Switch to R tutorial script. Section 4.