# Auction Timing and Market Thickness*

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November 1, 2023

**Abstract**

A seller faces a pool of potential bidders that changes over time. She can delay the auction to have a thicker market later on. The seller imposes static distortions (through her choice of reserve prices) and dynamic distortions (through her choice of market thickness). Under a condition on types that generalizes increasing hazard rates, we show that (i) regulating only static distortions can harm efficiency; (ii) when regulating only dynamic distortions, a social planner should reduce market thickness; (iii) if a planner can affect both types of distortions, she should still choose a lower market thickness than the seller, i.e., market thickness is inefficiently high; (iv) the extent of timing disagreement between the seller and a social planner is higher when they both have to use an efficient auction than when they both have to use an optimal one.

Keywords: Auctions, Market Design, Optimal Stopping, Monopoly, Order Statistics.

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*Many thanks to Kyle Bagwell, Gabriel Carroll, Matt Jackson, Fuhito Kojima, Maciej Kotowski, Paul Milgrom, Ilya Segal, Andy Skrzypacz, and Takuo Sugaya for insightful comments. We also thank audiences at Stanford, IIOC, and Michigan. Chaves thanks the Forman Family Fellowship in Economics. Ichihashi is grateful to Yoshida Scholarship Foundation and the Stanford Graduate Fellowship. An earlier version of this paper circulated under the title "When to Hold an Auction?"

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# 1 Introduction

The study of auctions usually begins with a known set of bidders. This leaves out a crucial part of the design problem: the auctioneer is often uncertain about who will show up for the auction (and by when) and must trade off the delay costs from searching for bidders against the benefits of market thickness.

Consider, for instance, the market for distressed corporate assets, in which a substantial portion of sales happen via auction (Boone and Mulherin, 2009). After the firm decides to restructure and sell a division, it cannot take for granted that there will be a given set of bidders. Interested bidders for these assets are hard to find, and even when found, it takes considerable time and effort to motivate them to bid. The process usually involves both hiring an investment bank at significant cost and several months' delay. Boone and Mulherin (2009) note that the process takes more than six months on average, a long and costly wait for a company in distress. These search frictions are especially salient in corporate bankruptcy sales. LoPucki and Doherty (2007) remark that, in their sample,

> On average, eighty prospects are contacted for each sale, and thirty sign confidentiality agreements. But the average number of bidders is only 1.6 per sale. In fifteen of the twenty-six cases for which we have data (58%), there was only one bidder (p.35).

An earlier study had found that only eighteen of fifty-five sales in the sample had multiple bidders (Hotchkiss and Mooradian, 1998).

Delaying these auctions may be necessary to attract bidders, but the delay imposes substantial discounting costs. By delaying, the company risks missing out on a favorable market moment, and it exposes itself to adverse events—e.g., an employee exodus or a loss of key suppliers—that destroy value and make a successful reorganization impossible. During Polaroid's 2002 bankruptcy court case, its management argued for a speedy sale rather than a reorganization because its "revenues [were] falling off" and "[the company] was like a melting ice cube" (LoPucki and Doherty, 2007, p. 54).

Incorporating these delay costs leads to new auction design questions. The firm in our previous vignette must choose not only how to run the auction, but *when* to hold it: how long should the firm search for bidders?[1] Likewise, the relevant market design concerns change. In a static setting, the main efficiency concern is whether the good goes to the highest value bidder. In contrast, the firm in this example can cause distortions through its choice of auction format (specifically, its use of reserve prices) and through its choice of timing. Moreover, the two choices interact: the (static) distortions caused by a reserve price affect the firm's incentives to delay the auction and can induce further (dynamic) distortions.

Motivated by these examples, we study the problem of auction timing. The pool of potential bidders changes stochastically over time. Bidders have symmetric, independent private values for an indivisible good. An auctioneer chooses a timing policy for an auction, and a reserve price to use once the auction is run. We assume the auctioneer uses a second price auction, and we look at timing policies that rule out indicative bidding, i.e., the auctioneer can condition the timing on the number of bidders but not on their values.[2] Considering this class of policies lets us focus squarely on the optimal choice of *market thickness*, and it leads to a clean distinction between static (auction format/reserve price) and dynamic (auction timing) distortions.

We study a few settings that differ in how bidders arrive and potentially depart. In the baseline model (Section 3), we assume that the auctioneer faces a two-period problem in which the number of bidders arriving in the next period is known. In Section 5, we study a continuous-time model in which the pool of potential bidders evolves according to either (i) arbitrary arrival processes or (ii) arrival-departure processes that satisfy some distributional conditions. The latter allows, for example, bidders arriving and departing according to Poisson processes.

[1] Real estate auctions have some similar features—the seller has a joint choice of auction format and auction timing—but we emphasize auctions for corporate auctions because discounting costs, which are crucial to our model, seem more significant in that market: sellers of single family homes may or may not care about delaying an additional week or three to obtain one more bidder.

[2] Therefore, we do not solve the full dynamic mechanism design problem (see, e.g., Gershkov, Moldovanu, and Strack, 2018 or Board and Skrzypacz, 2016), but we can consider a general class of bidder dynamics. We discuss the advantages of our modeling approach in more detail in Section 2. Note that if the auctioneer can commit in advance to a mechanism that will be run upon stopping, then a revenue maximizing auctioneer will use (WLOG) a second price auction with an optimal reserve price, whereas a surplus-maximizing auctioneer will use (WLOG) a second price auction with an efficient reserve price.

We ask two main questions. First, do revenue-maximizing sellers over- or under-value market thickness, i.e., do they wait too long or too little to hold auctions, relative to the social planner? A revenue-maximizing seller will typically use a different reserve price than a social planner, which in turn feeds into the choice of market thickness, so a full answer requires comparing the different auctioneers' choices across reserve prices.

Second, we investigate the equilibrium effects of *partial regulation*: what is the impact of regulatory interventions that only target the static distortion (choice of reserve price) but leave the dynamic distortion (choice of timing and market thickness) unregulated? Conversely, what about policy interventions that directly target the dynamic distortion (i.e., enforce a timing rule or level of market thickness), but leave the static distortion unregulated?

For example, in internet search advertising markets, regulation that affects an advertising platform's ranking algorithm but not the latency used to sell impressions (i.e., how long the webpage waits for bids before displaying search results to the user) would target static distortions while leaving dynamic distortions unregulated.[3] The reverse is also possible. Regulators require stock exchanges to impose temporary trading halts following market upheaval; trade resumes with a "halt auction" (often uniform price double auctions).[4] The longer the trading halt, the more orders (bidders) would accumulate for the eventual halt auction. Regulatory requirements on the length of the trading halt will target dynamic distortions but not static ones.

We identify a property of the distribution of bidder values that governs both the dynamic distortions caused by revenue maximization and the interactions between static and dynamic

[3]When a large advertising platform like Google Adwords sells an impression, it queries different ad exchanges that aggregate bids from advertisers. The longer platform waits for responses, the more bids it will have for a particular impression, but the slower the page will load. This downgrades the internet user's experience and reduces future traffic for the platform. Since advertisers bid for slots, we can think of ranking regulations as an intervention that targets static distortions (i.e., lowering the reserve price for Yelp on a restaurant query or for Expedia on a flight query), but leaves the choice of market thickness (how long to wait for exchange responses) constant.

[4]For instance, see CBOE (2022) for the description of the halt auction for the Chicago Board Options Exchange.

distortions. Let $F$ be the distribution of valuations with density $f$, and define $\xi_F(v) := \frac{f(v)}{1-F(v)}v$. For an auctioneer $a \in \{\text{seller}, \text{planner}\}$ and a reserve price $r \in \{\text{optimal}, \text{efficient}\}$, say $(a, r) \succeq_{delay} (a', r')$ if, for any arrival process for bidders in a large class, $a$ always holds an auction with reserve $r$ later (with more bidders) than $a'$ holds auction with reserve $r'$.[5] Our main result (Theorem 1), informally stated, says that when $\xi_F$ is increasing,

$$\begin{pmatrix}\text{seller,}\\ \text{efficient } r\end{pmatrix} \succeq_{delay} \begin{pmatrix}\text{seller,}\\ \text{optimal } r\end{pmatrix} \succeq_{delay} \begin{pmatrix}\text{planner,}\\ \text{optimal } r\end{pmatrix} \succeq_{delay} \begin{pmatrix}\text{planner,}\\ \text{efficient } r\end{pmatrix}, \tag{1}$$

whereas if $\xi_F$ is decreasing, the string of "inequalities" is reversed.

In particular, for $\xi_F$ increasing, a seller inefficiently over-invests in market thickness, whereas for $\xi_F$ decreasing, she inefficiently under-invests.[6] Regular distributions in the sense of Myerson (1981) can have increasing or decreasing $\xi_F$'s, and therefore result in sellers who over- or under-invest in market thickness. On balance, however, a revenue-maximizing seller inefficiently over-invests in market thickness for many common distributions, including all those with an increasing hazard rate.

The string of inequalities (1) characterizes the conflict between profit maximization and efficiency in the choice of market thickness. We highlight some of the key implications for the leading case of an increasing $\xi_F(\cdot)$.

1. **Partial regulation on static distortions can lower social surplus.** Forcing the seller to use an efficient reserve price can backfire. When forced to use an efficient auction, the seller responds by waiting *more* than when she is free to use an optimal inefficient auction. As shown in Example 1 below, even for standard distributions, the additional discounting costs can overwhelm the surplus gain from a large bidder pool and lead to lower discounted surplus overall.

[5] Here, we use the minimum of the support of $F$, which can be positive, as an efficient reserve price for the second price auction. This uniquely pins down the revenue from the efficient auction.

[6] Below we connect $\xi_F$ to the *price elasticity* of a related Bulow and Roberts (1989) monopoly pricing problem with demand curve $q(p) = 1 - F(p)$. Prior literature in operations refers to $\xi_F$ as a generalized failure rate and explores the implications of an increasing generalized failure rate. We postpone that discussion to the related literature section.

2. **Partial regulation on dynamic distortions should rush the auction.** Given a fixed auction format, the planner should hold the auction sooner than the seller would have held it. Said differently, if the planner is stuck using the seller-preferred inefficient auction, she should realize the lower surplus from this auction sooner, rather than try and make up the surplus loss by attracting additional bidders.

3. **Thickness is inefficiently high.** If the planner can choose the auction format in an unconstrained way, she will hold her planner-preferred auction (i.e., an efficient one) sooner than an unconstrained seller would hold her seller-preferred auction. In that sense, *market thickness is inefficiently high.*[7]

4. **Timing disagreement is greater for efficient auctions.** The wedge between the planner and the seller's timing choices is smaller when they have to use a revenue-optimal auction than when they use an efficient auction.

The following example illustrates the above discussion:

**Example 1.** *Suppose that the auctioneer chooses the auction date over a time horizon* $t \in \{1, 2, 3, 4\}$*. Denoting the number of bidders in period* $t$ *by* $n_t$*, let* $(n_1, n_2, n_3, n_4) = (1, 2, 4, 9)$*. The discount factor is* $\delta = 0.6$*, so at time* $t$*, both revenue and surplus are discounted by* $\delta^{t-1}$*. Bidders' values are drawn iid from an exponential distribution with mean* $1$*. For each* $n \in \{1, 2, 3, 4\}$*, let* $R(n, p)$ *and* $W(n, p)$ *denote the expected revenue and surplus, respectively, from a second price auction with a reserve price of* $p$ *and* $n$ *bidders.*

*Table 1 then presents the discounted expected revenue and surplus from each possible timing choice and reserve price* $p \in \{0, p^*\}$ *(rounding to two decimal places), where* $p^*$ *denotes the optimal reserve price. The optimal auction time for each row is highlighted in gray.*

[7] If the auctioneers are choosing a stopping time adapted to bidder arrivals that happen according to a homogenous Poisson process, one can show that marginal decreases in thickness are socially beneficial. It is easy to show that the optimal stopping time consists of a time-independent cutoff in the number of bidders. In Appendix B, we show that the planner's objective is single peaked in the choice of cutoff. Therefore, unless the planner and seller have the same cutoff, reducing that cutoff on the margin increases efficiency.

Table 1: Expected Discounted Payoffs

| period $t$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\delta^{t-1}W(t,0)$ | 1.00 | 0.90 | 0.75 | 0.61 |
| $\delta^{t-1}W(t,p^*)$ | 0.74 | 0.76 | 0.70 | 0.61 |
| $\delta^{t-1}R(t,p^*)$ | 0.37 | 0.40 | 0.41 | 0.40 |
| $\delta^{t-1}R(t,0)$ | 0 | 0.30 | 0.39 | 0.40 |

*In this example, $\xi_F(v) = v$ is increasing. The optimal timing choices in gray satisfy (1). For a fixed p, surplus is always maximized at an earlier time than was chosen by the seller, and forcing the seller to hold an auction at an (earlier) efficient time increases discounted total surplus (from 0.61 with $t = 4$ to 1.00 with $t = 1$ under $p = 0$, and from 0.70 with $t = 3$ to 0.76 with $t = 2$ under $p = p^*$). In contrast, forcing the seller to use an efficient reserve price but allowing her free choice of timing reduces discounted total surplus: with an optimal reserve price ($p^* = 1$), the seller holds the auction at $t = 3$ and generates discounted total surplus of $\delta^2 W(3, p^*) = 0.70$, whereas with an efficient reserve price ($p = 0$), the seller holds the auction at $t = 4$ and generates surplus of $\delta^3 W(4, 0) = 0.61$.*

Our main focus is the comparison of auction timings across different objectives and auction formats, but the same proof techniques deliver comparative statics with respect to the value distribution. For example, if the bidder's value distribution belongs to a location-scale family, then increasing the location parameter (shifting the distribution right) leads the seller to hold the optimal auction sooner, while increasing the scale parameter ("spreading" the distribution) leads the seller to hold the optimal auction *later*. If values are distributed according to the uniform distribution between $\mu$ and $\mu + \sigma$, changes in $\mu$ and changes in $\sigma$ have opposite effects on timing, even though in both cases an increase in the parameter increases the value distribution in the first-order stochastic dominance.

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the two-period model in which bidders do not depart. Section 4 describes our notion of right-skewness—the extended star-shaped order—for value distributions. This section establishes several comparative statics using that notion, and it studies the inefficiencies that arise from revenue-maximization. Section 5 extends our auction timing results to a continuous-time setting with rich arrival and departure dynamics. Section 6 concludes. All omitted proofs are relegated to the Appendices.

# 2 Related Literature

Wang (1993) appears to be the first paper that studies the endogenous timing of an auction. Wang analyzes a model in which the seller has one unit of an object to sell to buyers who arrive according to a homogenous Poisson process. The seller, who bears a constant flow cost from holding inventory, can decide to sell the object either by posting a price or by holding an auction at a fixed deadline chosen in advance. Wang shows that the seller's choice of timing typically differs from the socially optimal one. In contrast, we study the timing decisions of an auctioneer across auction formats and objectives and how these relate to the shape of a bidder's value distribution. Cong (2020) studies an auction timing problem in which a seller sells real options to a fixed set of long-lived bidders. Among other things, he finds that the seller inefficiently delays the auction. Du and Zhu (2017) study the trade-off between market thickness and discounting in dynamic double auctions. They look at the welfare-maximizing trading frequency, whereas we focus on the comparison between revenue-maximizing and welfare-maximizing market thickness in a simpler setting. Arefeva and Meng (2020) study a two-period auction timing problem in which delaying an auction enables bidders to learn about their outside options and update their values.

Our results also relate to static models on bidder solicitation in Szech (2011) and Fang and Li (2015). They study models in which the auctioneer can solicit more bidders at a cost that is linear in the number of bidders joining the auction. In our paper, the cost to the auctioneer of having more bidders is time discounting. While either formulation can be more appropriate depending on an application, time discounting is more natural in some leading cases. For example, in bankruptcy auctions, a major source of delay costs is that, before the assets are sold, adverse contingencies may arise that destroy the value of the assets. In such a case, delay costs will enter multiplicatively.[8] Our dynamic model of bidder solicitation

[8]With linear costs, the planner and seller always face the same marginal cost from gathering additional

allows for a fuller comparison of auction timing/market thickness decisions—across different auction formats and auctioneer objectives—than was possible in these static solicitation models. Section 4 provides a formal comparison between our results and the aforementioned papers.

This paper fits within a broader literature on how to sell (or allocate) to stochastically arriving agents with private information (Gallien, 2006; Board and Skrzypacz, 2016; Loertscher, Muir, and Taylor, 2016; Gershkov et al., 2018). These papers derive a fully optimal dynamic mechanism for a more restricted class of agents' arrival processes, and they focus especially on when these mechanisms can be implemented in posted prices. In contrast, we focus on the question of how to time the transaction. At the cost of looking at a more restricted class of mechanisms, this modeling approach has several advantages. Aside from the substantive focus it allows—emphasizing issues of market thickness/timing, as well as cleanly distinguishing between static and dynamic distortions—the approach enables us to consider very general arrival dynamics and some departure dynamics,[9] and it delivers monotone comparative statics results. In contrast with this literature, we can also identify conditions under which revenue maximization leads to inefficiently *low* market thickness.[10]

Other papers in economics and operations research have used the star-shaped order in auction models. Moldovanu, Sela, and Shi (2008) study sellers who decide on their discrete supply of a homogeneous good in multi-unit auctions. They use the star-shaped order to provide comparative statics with respect the distribution of values in the monopoly problem. Paul and Gutierrez (2004) and Li (2005) use the star-shaped order to study how the winner's rent in an auction depends on the number of bidders. We use a generalization of the star-

bidders. In our setting, the seller and the planner face different marginal benefits *and* different marginal costs, which complicates the comparison between the two. This follows from discounting: the planner's cost from delaying an efficient auction by $dt$ is (discount rate) $\times\, dt \,\times$ (expected surplus), whereas the seller's cost is (discount rate) $\times\, dt \,\times$ (expected revenue).

[9]The literature has mostly focused on the case without departures of agents. See Pai and Vohra (2013) and Mierendorff (2016) for exceptions.

[10]In a model of a dynamic two-sided market with random arrivals of finite-type traders, Loertscher et al. (2016) find that revenue maximization leads to inefficiently high market-thickness. The models are substantially different, since we consider a one-sided market but have a richer type space and more general bidder dynamics. We see the approaches as complementary.

shaped order to provide new comparative statics on how the growth rate of the revenue (as a function of the number of bidders) depends on the value distribution and the auction format.

The condition that $\xi_F(\cdot)$ is increasing was introduced by Lariviere and Porteus (2001) in the revenue management literature under the name of "increasing generalized failure rate" (IGFR). For example, the condition ensures that the monopoly price is unique. Ziya, Ayhan, and Foley (2004) provide an insightful review of this literature and the relationship between IGFR and other distributional assumptions.[11] In the auction theory literature, Kleinberg and Yuan (2013) and Schweizer and Szech (2019) use the condition to obtain quantitative bounds on equilibrium outcomes. For example, Kleinberg and Yuan (2013) call the condition hyper-regularity and obtain a worst-case bound on the ratio of expected revenue to expected welfare. Our results highlight a new economic implication of IGFR: it governs the *relative growth rate* of revenue and total welfare as a function of the number of bidders.

# 3 Setting

An auctioneer chooses between allocating one indivisible good to any of $n$ bidders in $t = 0$, or allocating it to any of $n + d$ bidders in $t = 1$, where $n, d \in \mathbb{N}$. The auctioneer and bidders share a one-period discount factor $\delta \in (0, 1)$. The auctioneer's value for the object is zero.

All bidders are symmetric and have an independent private value $v_i$ drawn from a cumulative distribution function $F$. If bidder $i$ obtains the good and pays $b$ in period $t$, her discounted payoff is $\delta^t(v_i - b)$ from the perspective of period 0. Assume that (i) $F$ has a finite expectation, (ii) $F$ has a positive and continuous density $f$ over its support $[\underline{v}, \overline{v})$ for some $\underline{v} \geq 0$ and $\overline{v} \in (\underline{v}, \infty]$, and (iii) $F$ is regular, i.e., $v - \frac{1-F(v)}{f(v)}$ is increasing.[12] We use "increasing" to mean "non-decreasing," and "decreasing" to mean "non-increasing." Following Bulow and Roberts (1989), we define type $v$'s marginal revenue as $MR(v) := v - \frac{1-F(v)}{f(v)}$.

The auctioneer runs a second price auction.[13] The auctioneer, denoted by $a \in \{S, P\}$,

[11] Other contributions include Lariviere (2006) and Banciu and Mirchandani (2013).

[12] Assumption (i) implies $\mathbb{E}[\max\{v_1, \ldots, v_n\}] \leq n\mathbb{E}[v_i] < +\infty$. Since the $i$-th order statistic is always lower than the highest order statistic, this inequality implies that all the order statistics have finite moments.

[13] The auctioneer's timing decision remains the same for any two auction formats that give the same revenue and surplus for any given number of bidders. Thus, by the payoff/revenue equivalence theorem, our

can be a *seller* ($a = S$) who maximizes discounted expected revenue, or a *social planner* ($a = P$) who maximizes discounted expected total surplus. The auction format can be either *optimal*, denoted by $r = \mathbf{Opt}$, or *efficient*, denoted by $r = \mathbf{Eff}$. An optimal auction has a reserve price of $p^* := \inf\{p \in [\underline{v}, \overline{v}) : MR(p) \geq 0\}$, which is independent of the number of bidders and thus of the period in which the auction takes place (Myerson, 1981; Riley and Samuelson, 1981). The reserve price of an efficient auction is set to be $\underline{v}$.[14] The auctioneer is constrained to using the same kind of auction in either period.

Let $R(n, p)$ and $W(n, p)$ denote, respectively, the undiscounted expected revenue and expected surplus from a second price auction with $n$ bidders and reserve price $p$. Because bidder $i$'s undiscounted value to the good is $v_i$, she will bid her true value in the auction regardless of when it is held. As a result, we have $R(n, p^*) = \mathbb{E}[\max\{MR(v_1), \dots, MR(v_n), 0\}]$, $R(n, \underline{v}) = \mathbb{E}[\max\{MR(v_1), \dots, MR(v_n)\}]$, and $W(n, \underline{v}) = \mathbb{E}[\max\{v_1, \dots, v_n\}]$. With this notation, we can then characterize the incentives to delay the auction for any pair $(a, r) \in \{S, P\} \times \{\mathbf{Opt}, \mathbf{Eff}\}$. For instance, a seller using an optimal auction $(S, \mathbf{Opt})$ delays if $R(n, p^*) \leq \delta R(n + d, p^*)$, whereas a planner with an efficient auction $(P, \mathbf{Eff})$ delays if $W(n, \underline{v}) \leq \delta W(n + d, \underline{v})$.[15]

Our goal is to compare timing decisions across arbitrary pairs $(a, r) \in \{S, P\} \times \{\mathbf{Opt}, \mathbf{Eff}\}$. For that purpose, we introduce a partial order that ranks which scenario leads to a later auction:

**Definition 1.** Take two auctioneers $a, a' \in \{S, P\}$ and two auction formats $r, r' \in$

results hold identically when the auctioneer uses first-price auctions instead.

[14] While any reserve price below $\underline{v}$ attains efficiency, reserve price $\underline{v}$ enables us to write the expected revenue from an efficient auction as the expected marginal revenue of the winner (since the expected payoff of the lowest type ($\underline{v}$) bidder equals zero).

[15] Note that, in our specification, the planner discounts the value of allocating to bidder $i$ from the perspective of $t = 0$, not from the perspective of the time at which the bidder arrived at the market. If the planner discounted a bidder from his time of arrival, both revenue calculations and $i$'s incentives for reporting $v_i$ truthfully would be unaffected. However, a utility function that discounts from the time of arrival has the unintuitive feature that the planner would privilege "young" bidders at the expense of "old" ones. (We thank Gabriel Carroll for pointing out this interpretation). In addition, when the good itself decays exponentially, the right discount starts from the time of first availability, regardless of when bidders arrive.

$\{\mathbf{Opt}, \mathbf{Eff}\}$. Then write

$$(a, r) \succeq_d (a', r')$$

(where $d$ in $\succeq_d$ is mnemonic for *delay*) if, whenever auctioneer $a'$ weakly prefers to hold an auction $r'$ in $t = 1$, auctioneer $a$ weakly prefers to hold an auction $r$ in $t = 1$. If $(a, r) \succeq_d (a', r')$ and $(a', r') \succeq_d (a, r)$, then write $(a, r) =_d (a', r')$.

**Remark 1** (General Payoff Specification)**.** We can extend the payoff specification in two ways, which are relevant in the context of bankruptcy auctions. First, in addition to time discounting, we can allow the value of the good itself to depreciate over time: Suppose that the value of the good depreciates at rate $1 - \gamma$ for some $\gamma \in (0, 1)$, so that the undiscounted value of the good for bidder $i$ in period $t$ is $\gamma^t v_i$. Bidder $i$'s utility is then written as $\delta^t(\gamma^t v_i - b)$, and in the auction, bidder $i$ will bid her undiscounted value, $\gamma^t v_i$. In the second price auction with reserve price $p$, the undiscounted payoffs of the seller and the planner are $\gamma^t R(n, p)$ and $\gamma^t W(n, p)$, respectively, and their discounted payoffs are $(\delta\gamma)^t R(n, p)$ and $(\delta\gamma)^t W(n, p)$. As a result, the timing choice of the auctioneer coincides with the choice of the auctioneer in the baseline model in which the discount factor is $\delta\gamma$.

Second, we can allow the auctioneer to derive a positive value $r_S > 0$ from keeping the good, so that she loses $r_S$ by allocating the good to a bidder. Introducing a positive $r_S$ changes the allocation in both the efficient and optimal auctions, but the relationship between the value distribution and timing decisions—which we characterize in Theorem 1—extends to this setting. In Appendix A, we prove a general result that allows for arbitrary $r_S \geq 0$.

# 4 Right-Skewness and Comparative Statics

This section establishes our main result—the comparison of timing decisions across auctioneers and formats. To that end, we first show that a social planner holds an efficient auction later when the value distribution has higher "right-skewness" in a certain sense. We then show that the virtual value is more (less) right-skewed than the actual value whenever $\frac{vf(v)}{1-F(v)}$ is increasing (decreasing). These observations lead to a complete ranking of the incentives to delay across different auctioneers and auction formats.

We first define a stochastic order that captures right-skewness of value distributions. Below, given random variables $X$ and $X'$, we use $X \sim X'$ to denote equality of distribution.

**Definition 2** (Extended Star-Shaped Order)**.** Take two random variables $X$ and $Y$. We say that $Y$ is greater in the **extended star-shaped order** than $X$ if there exist increasing functions $g$ and $h$ and a third random variable $Z$ with $X \sim g(Z)$, $Y \sim h(Z)$, such that (i) for any $c > 0$, $(h - cg)(\cdot)$ has at most one strict crossing of 0, and (ii) that crossing, if it occurs, is from below.[16]

If $h$ and $g$ are strictly positive, the order reduces to the star-shaped order in statistics. The star-shaped order generalizes the convex transformation order, which was first introduced by van Zwet (1964) to formalize right-skewness.[17] To understand why Definition 2 captures right-skewness, focus on the case in which $h$ and $g$ are strictly positive. In that case, $h - cg$ crosses 0 once and from below for all $c > 0$ iff $h/g$ is increasing. Whenever $h/g$ is increasing, $h(Z)$ pushes large draws of $Z$ further out into its right tail (relative to small draws) than $g(Z)$ does, and $h(Z)$ also shrinks small draws of $Z$ towards the left tail more than $g(Z)$ does. This comparison is scale-invariant. The additional flexibility in our definition, where $h$ and $g$ can be negative, helps us capture situations where the value to the auctioneer (net of opportunity cost) could be negative, as is the case when $X$ or $Y$ in the definition is a virtual value.

The extended star-shaped order has the following implication for expected order statistics (see Appendix A for the proof):

**Lemma 1.** *Let $\Pi_A(k; n)$ and $\Pi_B(k; n)$ with $k \in \{1, \dots, n\}$ denote the expected $k$-th highest draws from $n$ iid samples drawn according to continuously distributed random variables $X_i^A$ and $X_i^B$ ($i = 1, \dots, n$), respectively. If $X_i^A$ is greater than $X_i^B$ in the extended star-shaped*

[16] For a real-valued function $w(\cdot)$ defined on an interval $I \subset \mathbb{R}$, the strict zero-crossings of $w$ are defined as

$$S^-(w) = \sup\{\text{the number of sign changes of } w(x_1), w(x_2), \dots w(x_m)\}$$

with the supremum taken over all $x_1 < x_2 < \dots < x_m, x_i \in I, w(x_i) \neq 0$ (Karlin, 1968, p.20).

[17] See Barlow and Proschan (1966) for an insightful reference on the star-shaped order and connections to order statistics. They use the quantile functions of $X$ and $Y$ as $g$ and $h$; for our results, it is convenient to keep $g$ and $h$ general.

*order, then for any* $\underline{n}$ *and* $k \le \underline{n}$ *such that* $\Pi_B(k;\underline{n}) > 0$*,* $\Pi_A(k;n)/\Pi_B(k;n)$ *is increasing in* $n \ge \underline{n}$*.*

Lemma 1 implies that, when a social planner faces a value distribution that is greater in the extended star-shaped order, she holds an efficient auction later:

**Lemma 2.** *Consider two planners* $A$ *and* $B$ *facing bidders with valuations* $X_i^A$ *and* $X_i^B$*, respectively. Assume* $X_i^A$ *is greater in the extended star-shaped order than* $X_i^B$*. If planner* $B$ *prefers to hold the efficient auction in* $t = 1$ *as opposed to* $t = 0$*, planner* $A$ *also prefers to do so.*

*Proof.* Suppose that planner $B$ prefers to hold the efficient auction in $t = 1$. Borrowing the notation in Lemma 1, we can write this condition as $\delta\Pi_B(1;n+d) \ge \Pi_B(1;n)$. Lemma 1 implies $\frac{\Pi_A(1;n+d)}{\Pi_A(1;n)} \ge \frac{\Pi_B(1;n+d)}{\Pi_B(1;n)}$. It follows that $\delta\frac{\Pi_A(1;n+d)}{\Pi_A(1;n)} \ge \delta\frac{\Pi_B(1;n+d)}{\Pi_B(1;n)} \ge 1$, and thus $\delta\Pi_A(1;n+d) \ge \Pi_A(1;n)$. Therefore, planner $A$ also delays the efficient auction. □

For intuition, consider a special case of Definition 2 where $g(Z) = Z$, meaning that $X_i^A = h(X_i^B)$ for some function $h$ such that $h(x)/x$ is increasing (so $h$ is itself also increasing). Suppose that both planners currently have $n-1$ bidders.

By transforming the value of each bidder in planner $B$'s problem with function $h$, we obtain planner $A$'s problem. Consequently, we can express the realized total surpluses for planners $A$ and $B$ as $\max_{i=1,\dots,n-1} h\left(X_i^B\right) = h\left(\max_{i=1,\dots,n-1} X_i^B\right)$ and $\max_{i=1,\dots,n-1} X_i^B$, respectively. Suppose that a new bidder joins the auction, resulting in updated realized total surpluses of $h\left(\max_{i=1,\dots,n} X_i^B\right)$ and $\max_{i=1,\dots,n} X_i^B$. Because $h(x)/x$ is increasing, if the new bidder with value $X_n^B$ increases the total surplus for planner $B$ by $\alpha$ percent, then its transformation by $h$—i.e., the total surplus for planner $A$—increases by more than $\alpha$ percent. In other words, the additional $n$-th bidder grows planner $A$'s realized total surplus proportionally faster than planner $B$'s total surplus. As a result, a planner who faces a more right-skewed value distribution has a stronger incentive to wait for more bidders. We phrased this intuition in terms of realized values and total surpluses, but Lemmata 1 and 2 confirm that the comparison is also valid for expected highest draws.

We can write the expected revenue from an auction as the total surplus of an efficient auction once we treat virtual values as bidders' true values. Thus proper modifications of Lemma 2 deliver a complete ranking of delay decisions across auctioneers and reserve prices (recall the notation for auction timing in Definition 1).

**Theorem 1.**

1. *If* $\xi_F(v) = \frac{f(v)}{1-F(v)}v$ *is increasing, then*

$$(S, \mathbf{Eff}) \succeq_d (S, \mathbf{Opt}) \succeq_d (P, \mathbf{Opt}) \succeq_d (P, \mathbf{Eff}).$$

2. *If* $\xi_F(v)$ *is decreasing, then*

$$(S, \mathbf{Eff}) =_d (S, \mathbf{Opt}) \preceq_d (P, \mathbf{Opt}) =_d (P, \mathbf{Eff}).$$

Here, we present the proof of the comparison between $(S, \mathbf{Opt})$ and $(P, \mathbf{Eff})$, relegating the full proof to Appendix A.[18] This comparison captures distortions imposed by the joint choice of auction format and auction timing. Suppose $\xi_F(v)$ is increasing. Then,

$$\frac{\max(MR(v), 0)}{v} = \mathbb{1}_{MR(v)\geq 0} \cdot \left(1 - \frac{1-F(v)}{f(v)v}\right) = \mathbb{1}_{\xi_F(v)\geq 1} \cdot \left(1 - \frac{1}{\xi_F(v)}\right)$$

is increasing in $v$.[19] Thus for any $c > 0$, $\max(MR(v), 0) - cv$ crosses 0 strictly at most once and from below. As a result, if $\xi_F(v)$ is increasing, the truncated virtual value $\max(MR(v_i), 0)$ is greater in the extended star-shaped order than value $v_i$ itself. Lemma 2

[18] We can generalize Theorem 1 to accommodate reserve prices other than the efficient reserve price $\underline{v}$ or the optimal reserve price $p^*$. *Roughly, the more inefficient the auction, the more the planner wants to delay it; if* $\xi_F$ *is increasing, the opposite is true for the seller: the more inefficient the auction, the less the seller wants to delay it.* Formally, let $(S, p)$ and $(P, p)$ denote, respectively, the seller and the planner who run the second price auction with reserve price $p$. If $\xi_F$ is increasing, then for any $p$ and $p'$ such that $\underline{v} \leq p < p' \leq p^*$, we have $(S, p) \succeq_d (S, p') \succeq_d (P, p') \succeq_d (P, p)$, which reduces to part 1 of Theorem 1 when $p = \underline{v}$ and $p' = p^*$. If $\xi_F$ is decreasing, then for any $p$ and $p'$ such that $\underline{v} \leq p < p' \leq \overline{v}$, we have $(P, p') \succeq_d (P, p) \succeq_d (S, p)$, which reduces to part 2 of Theorem 1 when $p = p' = p^* = \underline{v}$. The proof is a straightforward extension of the proof of Theorem 1.

[19] Here, $\mathbb{1}_{MR(v)\geq 0}$ is the indicator function that takes the value 1 or 0 if $MR(v) \geq 0$ or $MR(v) < 0$, respectively. Other indicator functions are defined analogously.

then implies $(S, \mathbf{Opt}) \succeq (P, \mathbf{Eff})$. A symmetric argument implies that if $\xi_F(v)$ is decreasing, we have $(P, \mathbf{Eff}) \succeq (S, \mathbf{Opt})$. Overall, whether $\xi_F(v)$ is increasing or decreasing determines whether the truncated virtual valuation is more or less right skewed than the value itself, which determines whether the revenue from the optimal auction grows proportionally faster or slower than the welfare from an efficient auction.

Part 1 of Theorem 1 further shows that when $\xi_F(\cdot)$ is increasing, the seller wants to delay more than the planner for a fixed reserve price $p \in \{\underline{v}, p^*\}$. Meanwhile, raising the reserve price from $\underline{v}$ to $p^*$ pushes the planner's and seller's timing decisions closer together, leading the planner to delay more and the seller to delay less. One implication is that forcing the seller to use the efficient reserve price may decrease discounted welfare: As illustrated in Example 1, the efficiency loss from greater delay can dominate the efficiency gain from a lower reserve price, causing the discounted total surplus to decrease.[20]

Part 2 of the theorem shows that the planner holds an auction later than the seller, and the auctioneer with a given object holds the optimal and efficient auction at the same timing. This is because the optimal auction coincides with the efficient auction when $\xi_F(\cdot)$ is decreasing. Indeed, decreasing $\xi_F(\cdot)$ renders the virtual valuation always non-negative, so the optimal reserve price $\underline{v}$ leads to the efficient allocation. Thus we have equalities $(S, \mathbf{Opt}) =_d (S, \mathbf{Eff})$ and $(P, \mathbf{Opt}) =_d (P, \mathbf{Eff})$ in auction timing.

Any distribution with a log-concave density has an increasing hazard rate and thus an increasing $\xi_F(v)$.[21] At the same time, a distribution with an increasing virtual value may have increasing or decreasing $\xi_F(v)$. For example, consider a shifted Pareto distribution with

[20] This observation is reminiscent of findings in the mechanism design literature, whereby a mechanism that is optimal when some variable is exogenous could become suboptimal when the same variable becomes endogenous. For example, Persico (2000) shows that the revenue ranking of first and second price auctions in Milgrom and Weber (1982) can reverse when bidders can endogenously acquire information. As another example, Levin and Smith (1994) show that bidders' endogenous entry decisions render a positive reserve price suboptimal from the seller's perspective. We thank an anonymous reviewer for pointing out this connection.

[21] Examples of distributions with increasing $\xi_F(v)$ include uniform distributions and exponential distributions. However, log-concavity is not necessary for increasing $\xi_F(v)$. For example, a log-normal distribution is not log-concave but has an increasing $\xi_F$. See Banciu and Mirchandani (2013) for the list of distributions with increasing $\xi_F$.

distribution function $F(x) = 1 - (x+a)^{-b}$ with support $(1-a, \infty)$ where $a < 1$ and $b > 0$.[22] For any $b \geq 1$, $F$ has an increasing virtual value, and $\xi_F(x) = \frac{bx}{x+a}$ is increasing if $a \geq 0$ and decreasing if $a < 0$. The knife edge case is the (unshifted) Pareto distribution with $a = 0$, which has constant $\xi_F$, i.e., $\xi_F(x) = b$ for all $x$. In this case, the revenue-maximizing timing coincides with the welfare-maximizing timing.

For a demand curve $q(p) = 1 - F(p)$ (Bulow and Roberts, 1989), the function $\xi_F(\cdot)$ (evaluated at $v = p$) is precisely the price elasticty $\frac{d \log q}{d \log p}$ at price $p$. We can therefore understand the role of $\xi_F$ by studying the analogous monopoly pricing problem, in the spirit of Bulow and Roberts (1989):

**Remark 2** (Monopoly Theory Interpretation)**.** Consider the following "market expansion" monopoly problem:

> *A monopolist facing the demand curve $q(p)$ is currently offering a share $Q < 1$ of its total capacity to one market. It has access to a "market creation" technology: it can open a new market and shift capacity away from the current market and into the new one. In the process of shifting capacity, the monopolist loses a share $1-\beta$ of those resources. What distortions will the monopolist impose as it chooses to shift capacity?*

Bulow and Roberts (1989) show that an optimal auction design problem with $n$ symmetric IPV bidders from a value distribution $F$ is equivalent to a third-degree price discrimination problem for a capacity-constrained monopolist catering $n$ markets with demand curves $q(p) = 1 - F(p)$; quantity in the monopoly problem corresponds to (discounted) probability of receiving the good in the auction problem. Hence, due to the $1 - \beta$ multiplicative cost, we can map inefficiently shifting of capacity to a new market in the thought experiment to inefficiently delaying the auction in our model, since the latter reallocates the discounted probability of winning from the current bidding pool to a future one.

Consider the incentives of the planner and the seller to reallocate a small amount of capacity $dq$ from the current market to a new, previously uncatered market. Let $P(q) := F^{-1}(1-q)$ be the inverse demand curve and $\tilde{MR}(q) := \frac{d}{dq}(qP(q))$ the marginal revenue

[22] See Schweizer and Szech (2019) for further discussion on this example.

as a function of quantity. Letting $\tilde{\eta}(q) := \xi_F(P(q))$ denote the price elasticity of demand evaluated at $P(q)$, we can express the relationship between marginal revenue and demand elasticity from classical price theory as $\tilde{MR}(q) = P(q)\left(1 - \frac{1}{\tilde{\eta}(q)}\right)$. To simplify the exposition, focus on $Q$ such that $\tilde{MR}(Q) > 0$ and an increasing $\xi_F$—the argument generalizes easily to other cases.

Figure 1 illustrates this thought experiment: on the left is the original market, and on the right is the new, initially uncatered market. The lightly shaded red areas show gains and losses from shifting capacity for the seller, while the sums of the dark blue and lightly shaded red areas show gains and losses for the planner. If the seller shifts $dq$ to the new market, she

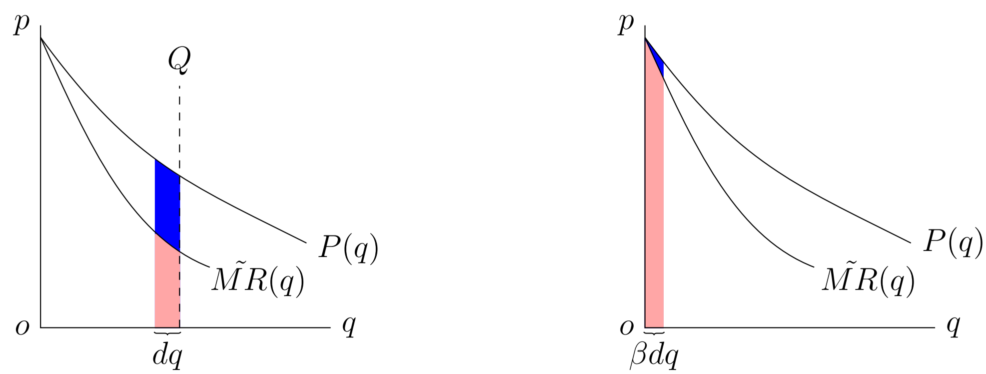


Figure 1: Shifting Consumers to a New Market

earns an $\beta P(0)\left(1 - \frac{1}{\tilde{\eta}(0)}\right) dq$ from those new customers, but she loses $P(Q)\left(1 - \frac{1}{\tilde{\eta}(Q)}\right) dq$ from customers she used to serve in the current market. For the same capacity shift $dq$, the planner earns surplus $\beta P(0)dq$ from the new market, while losing $P(Q)dq$ in the current one. Crucially, whenever the planner wants to move the $dq$ units of capacity, the seller also wants to move that capacity. The planner wants to move the $dq$ units whenever $P(Q) \leq \beta P(0)$. When $\tilde{\eta}(q)$ is decreasing (equivalently, $\xi_F$ is increasing), the above implies

$$P(Q)\left(1 - \frac{1}{\tilde{\eta}(Q)}\right) \leq \beta P(0)\left(1 - \frac{1}{\tilde{\eta}(0)}\right)$$

so the seller also wants to shift capacity into the new market. Hence, if price elasticity is increasing, the seller always has excessive incentives to shift capacity towards new markets. Translated back into the original auction timing problem, the seller inefficiently delays the auction.

**Comparison to Bidder Solicitation Literature** Our setup is closely related to static models of bidder solicitation (e.g., Szech (2011) and Fang and Li (2015)). To highlight our contribution, we formally compare our results to those in Szech (2011). The paper studies an auctioneer who can pay an additive cost $c_n$ to solicit $n$ bidders, where $c_n - c_{n-1}$ is weakly increasing and $c_n$ is strictly increasing in $n \geq 1$ with $c_0 = c_1 = 0$. To compare Szech (2011)'s results to ours, we introduce the following order: Let $(a, r) \succeq_{solicit} (a', r')$ if, for any $\{c_n\}_n$ satisfying the above property, auctioneer $a$ with reserve price $r$ solicits more bidders than auctioneer $a'$ with reserve price $r'$. Also, let $\lambda(v) = \frac{f(v)}{1-F(v)}$ denote the hazard rate. Szech (2011) shows that, for $\lambda(v)$ increasing,[23]

$$(S, \mathbf{Eff}) \succeq_{solicit} \{(S, \mathbf{Opt}), (P, \mathbf{Eff})\}$$

while for $\lambda(v)$ decreasing,[24]

$$(P, \mathbf{Eff}) \succeq_{solicit} (S, \mathbf{Eff}) \succeq_{solicit} (S, \mathbf{Opt}).$$

The above ranking clarifies a difference between the bidder solicitation literature and Theorem 1. The results in that literature do not rank market thickness decisions across auctioneers and reserve prices for the more natural case of increasing hazard rates. In particular, when the value distribution has an increasing hazard rate, the previous studies do not provide a comparison between $(S, \mathbf{Opt})$ and $(P, \mathbf{Eff})$, i.e., the decisions of a seller who maximizes revenue and a planner who maximizes welfare. Using a different cost structure for acquiring bidders, we provide a complete ranking that rests on the monotonicity of $v\lambda(v)$. By doing so, we separate the role of static and dynamic distortions arising from revenue maximization; as a consequence, we can also study the effects of partial interventions that target only one of those two margins.

[23]Proposition 1(ii) and Corollary 2 in Szech (2011) imply $(S, \mathbf{Eff}) \succeq_{solicit} (P, \mathbf{Eff})$ and $(S, \mathbf{Eff}) \succeq_{solicit} (S, \mathbf{Opt})$, respectively.

[24]Proposition 1(i) and Corollary 2 in Szech (2011) imply $(P, \mathbf{Eff}) \succeq_{solicit} (S, \mathbf{Eff})$ and $(S, \mathbf{Eff}) \succeq_{solicit} (S, \mathbf{Opt})$, respectively.

## 4.1 Comparative Statics on Value Distribution

The techniques we developed in the previous section deliver comparative statics with respect to the value distribution for any given auction format. To illustrate this, we describe how the seller's timing decision responds to changes in the valuation distribution. Below, we say that seller $S$ delays an auction more than seller $S'$ if whenever seller $S'$ weakly prefers to hold an auction in $t = 1$, seller $S$ also weakly prefers to hold the auction in $t = 1$. First, the following result considers the seller's timing of an efficient auction:

**Proposition 1.** *Let $S_F$ and $S_H$ denote the seller who faces value distribution $F$ and $H$, respectively. If $F$ is greater in the extended star-shaped order than $H$, then seller $S_F$ delays an efficient auction more than seller $S_H$.*

The proof is almost identical to that of Lemma 2: expected revenue is the expected second highest value, so one applies Lemma 1 with $k = 1$ as opposed to $k = 0$.

Second, focusing on a location-scale family, the following result shows how the seller's timing of the optimal auction responds to change in the location and scale parameters.

**Proposition 2.** *Assume that the value distribution $F$ belongs to a location-scale family, i.e., there exists a cdf $H$ and parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, such that $F(x) = H((x - \mu)/\sigma)$. Let $(\mu, \sigma)$ be any location-scale parameters for which the value distribution is regular. Then, as $\mu$ increases, the seller delays that auction less. If $\mu \geq 0 (\leq 0)$, then, as $\sigma$ increases, the seller delays an optimal auction more (less).*

*Proof.* Let $F(\cdot) = H((\cdot - \mu)/\sigma)$ denote the cdf of values when the location-scale parameters are $(\mu, \sigma)$; $F_\Delta$ with $\Delta > 0$ denote the cdf when parameters are $(\mu + \Delta, \sigma)$; and $F^\beta$ with $\beta > 1$ denote the cdf when parameters are $(\mu, \beta\sigma)$. Likewise define $\mathcal{S}$, $\mathcal{S}_\Delta$, $\mathcal{S}^\beta$ and $MR$, $MR_\Delta, MR^\beta$ as, respectively, the supports and virtual value functions of those distributions.

For any $v \in \mathcal{S}_\Delta$, we have $F_\Delta(v) = F(v - \Delta)$ and $MR_\Delta(v) = MR(v - \Delta) + \Delta$. Letting $F^{(n)}$ denote the distribution of the highest order statistic from $F$, then (with a change of variables $x = v + \Delta$) we can express the revenue from an optimal auction, when values are drawn from $F_\Delta$, as $\int_{x \in \mathcal{S}} \max\{MR(x) + \Delta, 0\} dF^{(n)}(x)$. Via Lemma 1, the comparative statics on $\mu$ now reduce to showing that $w(x) = \max\{MR(x) + \Delta, 0\} - c \max\{MR(x), 0\}$ has at most one strict zero crossing, and if so from above. For $x < MR^{-1}(0)$, $w$ is positive. For $x \geq MR^{-1}(0)$, $w(x) = MR(x)\,[1 + \Delta/MR(x) - c]$, which, by virtue of $MR$ being increasing, changes signs at most once, from positive to negative. This proves the comparative static on $\mu$.

For the analogous result on $\sigma$, we have, for any $v \in \mathcal{S}^\beta$, $F^\beta(v) = F(v/\beta + \mu - \mu/\beta)$ and hence $MR^\beta(v) = v - \beta\frac{1-F((v-\mu)/\beta+\mu)}{f((v-\mu)/\beta+\mu)}$. With a change of variables $x = (v-\mu)/\beta + \mu$, the revenue from an optimal auction with values from $F^\beta$ becomes $\int_{x\in\mathcal{S}} \max\{\beta MR(x) - \mu(\beta - 1), 0\} dF^{(n)}(x)$. Focus on the $\mu \geq 0$ case. By Lemma 1 once again, the result reduces to showing that, for any $c > 0$,

$$w(x) = \max\{\beta MR(x) - \mu(\beta - 1), 0\} - c\max\{MR(x), 0\}$$

has at most one strict zero crossing, and if so from below. For that purpose, notice that, since $\beta > 1$, $\beta MR(x) - \mu(\beta - 1) = 0 \Leftrightarrow MR(x) = (1 - 1/\beta)\mu \geq 0$. Hence $w$ is 0 for $x < MR^{-1}(0)$ and negative for $x \in [MR^{-1}(0), MR^{-1}(\mu(\beta-1)/\beta))$ (recalling that $MR$ is increasing). For $x > MR^{-1}(\mu(\beta-1)/\beta)$, $w(x) = MR(x)[\beta - \mu(\beta-1)/MR(x) - c]$, which, by virtue of $MR$ being increasing and $\mu \geq 0$, changes sign at most once, from negative to positive. The $\mu \leq 0$ case is symmetric. $\square$

The result highlights the usefulness of our techniques for the study of auction timing. Consider the family of uniform distributions $U[\mu, \mu + \sigma]$, for $\mu \geq 0$. One then has a perhaps surprising implication that even though increases in $\mu$ and $\sigma$ both increase values in the first stochastic order, they have exactly opposite effects on the timing decision.

As a second application, consider the case where the seller has an opportunity cost $r_S > 0$ from selling the good. The seller's objective can be re-written in terms of the net value $v - r_S$ supported on $[\underline{v} - r_S, \bar{v} - r_S]$, so that $r_S$ is a location parameter. Proposition 2 then says that increases in $r_S$ lead the seller to delay more for any (regular) value distribution.

**Remark 3.** The proof of Proposition 2 shows that a change in $\mu$ or $\sigma$ affects the value distribution so that the expected revenue grows proportionally faster or slower as a function of the number of bidders. At the same time, the proof does not directly show, for example, that a higher $\mu$ makes the virtual value smaller in the extended star-shaped order. As in Definition 2, the extended star-shaped ordering requires that random variables $MR_\Delta$ and $MR$ be transformations of the same random variable. However, as we increase $\mu$, we change not only the virtual value function (from $MR(\cdot)$ to $MR_\Delta(\cdot)$) but the underlying value distribution (from $F$ to $F_\Delta$). As a result, we cannot simply examine functions $MR_\Delta(\cdot)$ and $MR(\cdot)$ to apply Definition 2. We circumvent this challenge by showing that regardless of $\Delta$ or $\beta$, we can write the auction revenue as the expectation of a transformation of the highest value when values are drawn from $F$.

# 5 General Arrival and Departure Process

We now extend our two-period model to the following more general optimal stopping problem: an auctioneer $a$ with an auction format $r$ observes a stochastic process $\{N_t, t \geq 0\}$ for bidder arrivals and departures, and she must choose a stopping time adapted to $\{N_t, t \geq 0\}$ at which to hold the auction.

Recall that $R(n, p)$ and $W(n, p)$ respectively denote the expected revenue and total surplus from the second price auction with reserve price $p$. Also, $p^*$ is the revenue-optimal reserve price. By iterated expectations, the auction timing problem for $(a, r) = (S, \mathbf{Opt})$ is

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho\tau} R(N_\tau, p^*)\right],$$

and the one for $(P, \mathbf{Eff})$ is

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho\tau} W(N_\tau, \underline{v})\right],$$

where $\rho > 0$ is a discount rate, and $\mathcal{T}$ is the set of stopping times adapted to the filtration $\mathcal{F}^N$ generated by $\{N_t, t \geq 0\}$. The stopping problems for $(S, \mathbf{Eff})$ and $(P, \mathbf{Opt})$ are defined similarly.

In this continuous-time setup, our comparative statics results could be stated in terms of strong set order comparisons on sets of stopping times with a natural lattice structure. However, to simplify the exposition, we follow Quah and Strulovici (2013) and assume that an auctioneer stops the first time she is indifferent between holding the auction and continuing to search for bidders. That is, we focus on the *essential infimum* of the solutions to the relevant optimal stopping problem.[25]

**Definition 3.** Let $\mathcal{T}_{S,\mathbf{Opt}}$ and $\mathcal{T}_{P,\mathbf{Eff}}$ denote the set of the auctioneer's optimal stopping times in

$$\sup_{\tau \in \mathcal{T}} \quad \mathbb{E}\left[e^{-\rho\tau} R(N_\tau, p^*)\right]$$

and

$$\sup_{\tau \in \mathcal{T}} \quad \mathbb{E}\left[e^{-\rho\tau} W(N_\tau, \underline{v})\right],$$

respectively. Define $\mathcal{T}_{S,\mathbf{Eff}}$, and $\mathcal{T}_{P,\mathbf{Opt}}$ analogously. Then, for two auctioneers $a, a' \in \{S, P\}$ and auction formats $r, r' \in \{\mathbf{Opt}, \mathbf{Eff}\}$, write

$$(a, r) \succeq_d^* (a', r')$$

if

$$\operatorname{ess\,inf} \mathcal{T}_{a,r} \geq \operatorname{ess\,inf} \mathcal{T}_{a',r'}$$

with probability 1. If $(a, r) \succeq_d^* (a', r')$ and $(a', r') \succeq_d^* (a, r)$, then write $(a, r) =_d^* (a', r')$.

Below, we extend Theorem 1 to two cases. One is when bidders arrive according to a general arrival process but never depart. The other is when bidders can arrive and depart according to a specific arrival-departure process. We begin with the former.

## 5.1 General Pure-Arrival Process

First, we can extend Theorem 1 whenever bidders never depart, i.e., the process $\{N_t, t \geq 0\}$ is pathwise increasing:

**Theorem 2.** *If $N_t$ is pathwise increasing, all the comparative statics results in Theorem 1 hold with $\succeq_d^*$ in place of $\succeq_d$.*

It suffices to extend Lemma 2 to this setting:

[25] As in Quah and Strulovici (2013), the assumptions on $\{N_t, t \geq 0\}$ and $\mathcal{F}^N$ ensure that the essential infimum of solutions to the stopping problem is itself a solution to the stopping problem.

**Lemma 3.** *Suppose that $N_t$ is pathwise increasing, Consider two planners $A$ and $B$ who face bidders with valuations $X_i^A$ and $X_i^B$, respectively. If $X_i^A$ is greater in the extended star-shaped order than $X_i^B$, then we have $(A, \mathbf{Eff}) \succeq_d^* (B, \mathbf{Eff})$, i.e., the essential infimum of the optimal stopping times for planner $A$ is almost surely larger than that for planner $B$.*

*Proof.* To simplify exposition, let $S_A(n)$ and $S_B(n)$ respectively denote the expected surplus for planner $A$ and $B$ in the efficient auction with $n$ bidders. Let $\tau_A$ and $\tau_B$ be the essential infima over optimal $\mathcal{F}^N$-stopping times for $A$ and $B$'s stopping problems. To derive a contradiction, suppose that $\tau_A < \tau_B$ with a positive probability. The inequalities below are evaluated on this event. Since $\tau_A$ is optimal for $A$,

$$S_A(N_{\tau_A}) \geq \mathbb{E}[e^{-\rho\tau} S_A(N_{\tau_A+\tau})|\mathcal{F}_{\tau_A}]$$

for any feasible stopping time $\tau$ such that $\{\tau = 0\}$ is $\mathcal{F}^N_{\tau_A}$ measurable. In particular, this inequality must hold for $\tau_{B|A}$, B's optimal continuation stopping time starting at $\mathcal{F}^N_{\tau_A}$ (since $\tau_B > \tau_A$, $\{\tau_{B|A} = 0\}$ is $\mathcal{F}_{\tau_A}$-measurable). Therefore,

$$1 \geq \mathbb{E}\Big[e^{-\rho\tau_{B|A}} \frac{S_A(N_{\tau_A+\tau_{B|A}})}{S_A(N_{\tau_A})}\Big|\mathcal{F}^N_{\tau_A}\Big] \geq \mathbb{E}\Big[e^{-\rho\tau_{B|A}} \frac{S_B(N_{\tau_A+\tau_{B|A}})}{S_B(N_{\tau_A})}\Big|\mathcal{F}^N_{\tau_A}\Big],$$

where the second inequality follows from $N_{\tau_A+\tau_{B|A}} \geq N_{\tau_A}$ (since there are no departures) and Lemma 1 applied pathwise inside the expectations. Altogether,

$$S_B(\tau_A) \geq \mathbb{E}[e^{-\rho\tau_{B|A}} S_B(N_{\tau_A+\tau_{B|A}})|\mathcal{F}^N_{\tau_A}],$$

so $\tau_{B|A} = 0$ is an optimal continuation policy for B at $\tau_A$, which contradicts $\tau_B > \tau_A$. □

Lemma 3 enables us to use the same technique as Theorem 1 to establish Theorem 2: We write the stopping payoffs of different auctioneers and auction formats as the expected highest order statistic of random variables that result from certain transformations of bidder values. We then compare the optimal auction timing decisions of two auctioneers by ranking these random variables in the extended star-shaped order. Therefore we obtain Theorem 2.

## 5.2 Stochastic Departures

Without additional structure, our timing comparisons could fail when bidders can depart: When $\frac{f(v)}{1-F(v)}v$ is increasing, the seller gains proportionally more than the planner from adding a bidder, which also means that she loses proportionally more from losing a bidder. Therefore, when bidders can depart, the seller may hold the auction sooner than the planner.

We now impose some structure on the arrival-departure process to extend Theorem 1. We begin with some notation. For each $i \in \mathbb{N}$, let $\theta_i$ denote the time between the arrivals of the $i-1$-th and $i$-th bidders, with $\theta_1$ denoting the arrival time of the first bidder. The arrival time of bidder $i$ is $\alpha_i = \sum_{j=1}^{i} \theta_j$. Let $D_i$ denote bidder $i$'s soujourn time in the bidder pool, so that bidder $i$ is in the pool from time $\alpha_i$ until $\alpha_i + D_i$. Finally, we use $\geq_{FOSD}$ to mean the first-order stochastic dominance ordering. Our assumption on the arrival-departure process $\{N_t, t \geq 0\}$ is as follows.

**Assumption 1** (Semi-Memoryless Structure)**.** There are no bidders at $t = 0$. The inter-arrival times $\theta_i$ are drawn iid from a distribution $G$ satisfying the "New Better than Used" (NBU) property: For any $\theta \sim G$, $t > 0$, we have

$$\theta \geq_{FOSD} (\theta - t)|\{\theta \geq t\}.$$

Sojourn times $D_i$ are drawn iid from an exponential distribution, independently of all values and arrival times. We assume that $\mathbb{E}[e^{-\rho\theta_i}|\theta_i \geq t]$ is well-defined for any $t \geq 0$.

Bidders hence arrive according to a *renewal* process—the arrival time of bidder $i$ gives no information on the additional wait necessary for bidder $i+1$. In the same way, how long bidder $i$ has been in the pool gives no information about the likely time of $i$'s departure. The NBU condition on $G$ generalizes the increasing hazard rate (IHR) property: Under NBU, an auctioneer is more optimistic about arrivals in between arrivals than right after an arrival, while under IHR, the auctioneer gets increasingly optimistic about arrivals the longer it has been since the last arrival. Assumption 1 allows, for example, bidders arriving and departing according to independent Poisson processes.

The informational assumptions are as follows: Bidder $i$ and the auctioneer observe the bidder's arrival time $\alpha_i$. At time $t \geq \alpha_i$, they both know whether $t \leq \alpha_i + D_i$, but neither can observe $D_i$. In other words, both bidder $i$ and the auctioneer know when $i$ arrives and whether he is still available, but neither knows how much longer $i$ might be available for. For example, $D_i$ could represent the sudden arrival of a better outside option for a bidder, or, in the corporate takeover example, it could model the occurrence of an unexpected negative shock to liquidity that prevents the bidder from taking part in the auction. Remark 4 discusses how our results extend to the case in which $i$ can misreport his arrival and availability.

Let $\mathbb{L}_t$ denote the time elapsed since the most recent arrival at calendar time $t$, so that the last arrival occurred at $t - \mathbb{L}_t$. By Assumption 1, the auctioneer only needs to keep track of $\mathbb{L}_t$ to predict future arrival times and can ignore the $D_i$'s since they are irrelevant for predicting departure times.[26] Therefore, to know the future law of motion for $\mathcal{N}$ at $t$, the auctioneer only needs to know $(N_t, \mathbb{L}_t)$, i.e., the number of bidders currently available and the time elapsed since the most recent arrival. Since bidder values are iid and independent of the arrivals process, the auctioneer faces a Markov optimal stopping problem with state variable $(N_t, \mathbb{L}_t)$. We use $(n, l)$ for the generic notation of this state. Our main result for this setting is as follows.

**Theorem 3.** *Under Assumption 1, all the comparative statics results in Theorem 1 hold with $\succeq_d^*$ in place of $\succeq_d$.*

Unlike our previous theorems, Theorem 3 relies on an explicit characterization of the optimal stopping region. Theorem 5 in Appendix B shows that, starting from any state with $n = 0$, the optimal policy is to hold an auction at the first moment in which the number $N_t$ of bidders reaches

$$n^* := \min\left\{n \in \mathbb{N} : \beta(n+1)Q(n+1) \geq Q(n)\right\}. \tag{2}$$

Here, $Q(n)$ is the auctioneer's stopping payoff, such as an (undiscounted) expected surplus or revenue, and $\beta(\cdot)$ is the expected discount rate that the auctioneer has to incur to wait for $n+1$ bidders in the pool starting from state $(n, 0)$ (see the proof of Theorem 5 for the exact definition of $\beta(\cdot)$).

The intuition for the optimal stopping rule is as follows. The NBU property in Assumption 1 ensures that the auctioneer becomes growingly optimistic about the arrival of the next

[26] If we consider a general distribution for $D_i$, then the auctioneer needs to keep track of the time that each current bidder has spent in the market: that time affects the auctioneer's beliefs about how likely the bidder is to leave in the next $dt$ units of time, which in turn would affect the payoffs from stopping in the next $dt$. In this case, the auctioneer has a large and growing state space in her optimal stopping problem, which makes further analysis intractable.

bidder the longer she has been waiting. Thus the auctioneer will never hold an auction while waiting for a new bidder; if the auctioneer holds an auction at time $t$, it coincides with a new bidder's arrival. The auctioneer's problem, therefore, is to determine the number $n$ of bidders to wait for. We show that the auctioneer's discounted revenue from such a threshold policy is concave in $n$, which implies that we can derive the optimal $n$ from a local optimality condition. This condition is given by (3), i.e., $n^*$ is the minimal number of bidders that, once reached, makes waiting for another bidder suboptimal.

Since $n^*$ depends only on the ratios $Q(n+1)/Q(n)$, we can use the same machinery as before, which characterizes the relative growth rates of stopping payoffs.

*Proof of Theorem 3.* Let $S_A(n)$ and $S_B(n)$ denote the expected surplus for planner $A$ and $B$ in the efficient auction with $n$ bidders, respectively. Let $n_A^*$ and $n_B^*$ denote the cutoff in (2) for the corresponding planner and auction. From equation (2), we can see that, starting from an empty pool, planner $A$ waits for a larger number of bidders than planner $B$ if $\frac{S_A(n+1)}{S_A(n)} \geq \frac{S_B(n+1)}{S_B(n)}$ for all $n$, because any $n$ that satisfies the inequality in the infimum defining $n_A^*$ will satisfy the inequality defining $n_B^*$. Hence, if the value distribution faced by planner A is greater in the extended star-shaped order than that faced by $B$, she will stop later, i.e., Lemma 3 holds. We then establish the relevant extended star-shaped order relationships as in the proof of Theorem 1 or 2. □

**Remark 4** (Strategic Arrivals and Departures)**.** So far, we have assumed that the seller observes the arrivals and departures of bidders. This assumption might not always be innocuous: For instance, a sophisticated bidder in a corporate asset auction may pretend that financial backing for a bid takes longer than it actually does, if delaying participation in the auction can disadvantage the seller. Thus when arrivals and availability are privately known to bidders, the feasibility of the optimal timing policy hinges on its form and hence on the dynamics of the bidder pool.[27]

To study this possibility, assume that bidder $i$ privately observes his arrival time $\alpha_i$ and knows at time $t \geq \alpha_i$ whether $\alpha_i + D_i \leq t$. In other words, $D_i$ remains unobservable to

[27]For example, Gershkov, Moldovanu, and Strack (2015) show that an efficient dynamic mechanism with observable arrivals may not be implementable if later arrivals make the seller pessimistic about the time of future arrivals.

anyone, but bidder $i$ privately observes whether or not "his time is up." Bidder $i$ can now delay his arrival until any $\alpha_i' \geq \alpha_i$ such that his time is not up at $\alpha_i'$ and can also pretend to have left at any $t \geq \alpha_i'$ if his time is not up by $t$.[28]

For any arrival-departure process that satisfies Assumption 1, bidders will in fact want to report their arrivals and availability truthfully. The proof adapts an argument by Gershkov et al. (2015) to the case of auction timing. Let $N_t'$ denote the number of bidders observed by the auctioneer at $t$ when bidder $i$ misreports his arrival or departure time. Note that, however $i$ chooses to strategically time his arrival and/or departure, $N_t' \leq N_t$ always holds. In particular, $N_t'$ never hits $n^*$—the threshold number of bidders at which the auctioneer holds an auction—earlier than $N_t$. Thus, bidder $i$ who misreports his availability either (i) misses the opportunity to join the auction or (ii) joins an $n^*$-bidder auction that will be held later than if bidder $i$ had truthfully reported his arrival and departure times. Bidder $i$ does not benefit from the deviation in (i), because the payoff from the auction is non-negative. Nor does he benefit in the case (ii), because he must incur the cost of additional discounting without changing the number of opposing bidders he faces in the auction. Therefore, the auctioneer can implement the optimal timing policy under Assumption 1 even when bidders can "misreport" the timing of their arrival and departure.

# 6 Conclusion

We have characterized the inefficiency consequences of revenue maximization in an auction timing model, identifying a condition on the distribution of values that determines the wedge between a seller and a social planner's relative incentives to trade off delay costs and market thickness. Our model of the auction market is highly stylized. However, it helps separate static and dynamic distortions from revenue maximization, and it identifies unintended consequences of partial interventions that focus on only one of the distortions. As an example, going back to the case of bankruptcy auctions, the legal literature is mostly concerned with the phenomenon of too few bidders (LoPucki and Doherty, 2007). However, to the extent

[28] Similar to the literature on auctions with private budget constraints (Che and Gale, 1998), we assume that agents can only engage in one-sided deviations in reporting their arrival and departure status.

that senior creditors' private benefits from bankruptcy auctions diverge from social benefits (they care about total proceeds more than efficient reorganization), our results suggest that there is also a risk of *excessive* delay and "too many" bidders.

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# Appendix

## A Comparative Statics on Auction Timing: Proofs for Section 4

*Proof of Lemma 1.* The proof slightly generalizes Theorem 3.6 in Barlow and Proschan (1966), which assumes that functions $h$ and $g$ are non-negative. Their proof, however, only relies on the fact that the strict zero crossings of $(h - cg)(\cdot)$ (i.e., $S^-(h - cg)$ in footnote 16) are at most 1 for any $c > 0$, and always from negative to positive.[29] Indeed, reworking the steps in Barlow and Proschan (1966)'s Lemma 3.5, we have from Karlin (1968) (p.155) that the density of the $k$-th highest order statistic out of $n$ independent draws from a density $f$, $f_{k,n}(x)$, is a totally positive kernel of all orders in $n$ and $x$. (Here, $f$ is a density of random variable $Z$ in Definition 2, where $X_i^A \sim h(Z)$ and $X_i^B \sim g(Z)$.) By Theorem 3.1 in Karlin (1968), without the support restrictions in Barlow and Proschan (1966), it follows that for any $c > 0$, $\Pi_A(k; n) - c\Pi_B(k; n) = \int_0^\infty [h(x) - cg(x)] f_{k,n}(x)dx$ has at most one strict zero crossing as a function of $n$ and always from negative to positive. Hence, since $\Pi_A(k; n) - c\Pi_B(k; n) = \Pi_B(k; n) \left( \frac{\Pi_A(k;n)}{\Pi_B(k;n)} - c \right)$ and $\Pi_B(k; n) > 0$, $\frac{\Pi_A(k;n)}{\Pi_B(k;n)}$ must increase in $n$. $\square$

The rest of this section consists of two parts. First, we establish a result that is more general than Theorem 1 and allows the auctioneer to have a value $r_S \geq 0$ to the good. This result accommodates the general payoff specification discussed in Remark 1. Second, we prove that if $r_S = 0$, the result reduces to Theorem 1. The following result generalizes Theorem 1:

**Theorem 4.** *Suppose that the auctioneer has value $r_S \in [0, \bar{v})$ for the good and thus loses the opportunity cost $r_S$ by allocating the good. Then the following hold:*

1. *If $\xi_{F,r_S}(v) = \frac{(v-r_S)f(v)}{1-F(v)}$ is increasing for $v \geq r_S$, then*

$$(S, \mathbf{Eff}) \succeq_d (S, \mathbf{Opt}) \succeq_d (P, \mathbf{Opt}) \succeq_d (P, \mathbf{Eff}). \tag{3}$$

[29] Note the minor typo in their proof of Lemma 3.5. The kernel $K(i, n; x)$ (their notation) is reverse regular order in $n$ and $x$ and totally positive in $i$ and $x$, rather than in $n$ and $x$ as stated in the text. The meaning is clear from the underlying total positivity theory of Karlin.

*2. If* $\xi_{F,r_S}(v) = \frac{(v-r_S)f(v)}{1-F(v)}$ *is decreasing for* $v \geq r_S$*, then*

$$(P, \mathbf{Opt}) \succeq_d (P, \mathbf{Eff}) \succeq_d (S, \mathbf{Eff}) \succeq_d (S, \mathbf{Opt}). \tag{4}$$

*Proof.* We prepare some notation. We can rewrite the different objectives in terms of the net value $\tilde{v} = v - r_S$, which has cdf $F(\tilde{v}+r_S)$ and density $f(\tilde{v}+r_S)$ with support $[\underline{v}-r_S, \bar{v}-r_S]$. The corresponding virtual value is $\tilde{MR}_{r_S}(\tilde{v}) = \tilde{v} - \frac{1-F(\tilde{v}+r_S)}{f(\tilde{v}+r_S)}$. With this notation, the stopping payoffs are

$$\begin{aligned}
\tilde{MR}_{r_S}(\tilde{v})\mathbb{1}_{\tilde{v}\geq 0} \qquad & (S, \mathbf{Eff}) \\
\tilde{MR}_{r_S}(\tilde{v})\mathbb{1}_{\tilde{MR}_{r_S}(\tilde{v})\geq 0} \qquad & (S, \mathbf{Opt}) \\
\tilde{v}\mathbb{1}_{\tilde{v}\geq 0} \qquad & (P, \mathbf{Eff}) \\
\tilde{v}\mathbb{1}_{\tilde{MR}_{r_S}(\tilde{v})\geq 0} \qquad & (P, \mathbf{Opt}).
\end{aligned}$$

For example, the seller who runs an optimal auction, i.e., $(S, \mathbf{Opt})$, allocates the good if and only if the virtual value of the net value is positive, so its undiscounted payoff is written as the expected highest value of $\tilde{MR}_{r_S}(\tilde{v})\mathbb{1}_{\tilde{MR}_{r_S}(\tilde{v})\geq 0}$. Similarly, the efficient auction allocates the good if and only if the net value is positive, so we have $\tilde{MR}_{r_S}(\tilde{v})\mathbb{1}_{\tilde{v}\geq 0}$ for the seller who runs the efficient auction, i.e., $(S, \mathbf{Eff})$.

Let $\tilde{v}^{(1:n)} = \max_{i=1,\ldots,n} \tilde{v}_i$ be the highest value (net of $r_S$) among $n$ bidders. By Lemma 2, we have $(a, r) \succeq_d (a', r')$ if the objectives of $(a, r)$ and $(a', r')$ can be written as $\mathbb{E}[h(\tilde{v}^{(1:n)})]$ and $\mathbb{E}[g(\tilde{v}^{(1:n)})]$, respectively, where functions $h$ and $g$ are increasing and for any $c > 0$, $w(x) = h(x) - cg(x)$ has at most one strict zero crossing, and if then only from below. Indeed, we can set $X_i^A = h(\tilde{v}_i)$ and $X_i^B = g(\tilde{v}_i)$ and use Lemma 2.

To show the first part of the result, suppose that $\xi_{F,r_S}(v) = \frac{(v-r_S)f(v)}{1-F(v)}$ is increasing for $v \geq r_S$. We define functions $h$ and $g$ (and the resulting $w$) as functions of the net value. Below, we write function $w$ for the different comparisons, keeping in mind that the first term is $h$ and the term that follows $c > 0$ is $g$.

$(S, \mathbf{Eff}) \succeq_d (S, \mathbf{Opt})$

$$w(x) = \underbrace{\tilde{MR}_{r_S}(x)\mathbb{1}_{x\geq 0}}_{h(x)} - c\,\underbrace{\tilde{MR}_{r_S}(x)\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0}}_{g(x)}$$

$(S, \mathbf{Opt}) \succeq_d (P, \mathbf{Opt})$

$$w(x) = \tilde{MR}_{r_S}(x)\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0} - cx\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0}$$

$(P, \mathbf{Opt}) \succeq_d (P, \mathbf{Eff})$

$$w(x) = x\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0} - cx\mathbb{1}_{x\geq 0}.$$

We provide details for $(S, \mathbf{Opt}) \succeq_d (P, \mathbf{Opt})$ only, since the remaining ones are verified similarly. In this case, we have $w(x) = 0$ for $x < \tilde{MR}_{r_S}^{-1}(0)$. For $x \geq \tilde{MR}_{r_S}^{-1}(0)$, we have $w(x) = x\left[\frac{\tilde{MR}_{r_S}(x)}{x} - c\right]$. We have $\frac{\tilde{MR}_{r_S}(x)}{x} = 1 - \frac{1-F(x+r_S)}{xf(x+r_S)}$, which we can write as $1 - \frac{1-F(v)}{(v-r_S)f(v)}$ by setting $v = x + r_S$. Because $\frac{(v-r_S)f(v)}{1-F(v)}$ is increasing, so is $w(x)$. Overall, the function $w$ can have at most one strict zero crossing, and always from below. Applying similar arguments to the other functions $w$ defined above, we obtain the first half of the theorem.

To show the second half, suppose that $\xi_{F,r_S}(v) = \frac{(v-r_S)f(v)}{1-F(v)}$ is decreasing for $v \geq r_S$. We define the function $w$ as a function of the net value as follows:

$(P, \mathbf{Opt}) \succeq_d (P, \mathbf{Eff})$

$$w(x) = \underbrace{x\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0}}_{h(x)} - c\underbrace{x\mathbb{1}_{x\geq 0}}_{g(x)}$$

$(P, \mathbf{Eff}) \succeq_d (S, \mathbf{Eff})$

$$w(x) = x\mathbb{1}_{x\geq 0} - c\tilde{MR}_{r_S}(x)\mathbb{1}_{x\geq 0}.$$

$(S, \mathbf{Eff}) \succeq_d (S, \mathbf{Opt})$

$$w(x) = \tilde{MR}_{r_S}(x)\mathbb{1}_{x\geq 0} - c\tilde{MR}_{r_S}(x)\mathbb{1}_{\tilde{MR}_{r_S}(x)\geq 0}$$

Each of these functions has at most one strict zero crossing, and always from below. Therefore we obtain the second part. $\square$

We now show that Theorem 4 reduces to Theorem 1 when $r_S = 0$.

*Proof of Theorem 1.* By setting $r_S = 0$, the first part of Theorem 4 becomes that of Theorem 1. For the second part, we show $(P, \mathbf{Opt}) =_d (P, \mathbf{Eff})$ and $(S, \mathbf{Opt}) =_d (S, \mathbf{Eff})$ when $r_S = 0$. It suffices to show that $MR(v) = v - \frac{1-F(v)}{f(v)} \geq 0$ for all $v \geq \underline{v}$, so that the optimal auction coincides with the efficient auction. Suppose to the contrary that $v - \frac{1-F(v)}{f(v)} = v(1 - \frac{1}{\xi_F(v)}) < 0$ for some $v$. Then, since $v \geq 0$, we have $1 - \frac{1}{\xi_F(v)} < 0$. Because $\xi_F(v)$ is decreasing, we obtain

$$v'\Big(1 - \frac{1}{\xi_F(v')}\Big) < v\Big(1 - \frac{1}{\xi_F(v)}\Big).$$

if $v' > v$. This contradicts the regularity of $F$. □

## B Characterizing the Optimal Timing Policy With Departures

This appendix characterizes the optimal stopping rule described by equation (2). To state the result, let $\tau(n, n+1) := \inf\{t \geq 0 : N_t = n + 1\} - \inf\{t \geq 0 : N_t = n\}$ denote the random time between when the pool first reaches size $n$ and when it first reaches $n+1$. (The possibility of departures makes the distribution of $\tau(n, n + 1)$ different from $G$.) The following result characterizes the optimal stopping rule:

**Theorem 5.** *Suppose that the arrival-departure process $N_t$ satisfies Assumption 1. Let $Q(\cdot)$ be any one of $W(\cdot, \underline{v})$, $W(\cdot, p^*)$, $R(\cdot, \underline{v})$ and $R(\cdot, p^*)$. Let $\mathcal{T}_Q^0$ denote the solution to*

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho\tau} Q(N_\tau)\right].$$

*Let $\beta(n+1) := \mathbb{E}[e^{-\rho\tau(n,n+1)}]$ denote the (unconditional) expected discount factor between the first time the bidder pool reaches size $n$ and the first time it reaches size $n + 1$, and let*

$$n^* := \inf\left\{n \in \mathbb{N} : 1 \geq \beta(n+1)\frac{Q(n+1)}{Q(n)}\right\}. \tag{5}$$

*Then*

$$\operatorname{ess\,inf} \mathcal{T}_Q^0 = \inf\{t : N_t \geq n^*\}.$$

Theorem 5 relies on four lemmas. The first one is a preliminary fact regarding payoffs upon stopping.

**Lemma 4.** *Let $Q(\cdot)$ be any one of $W(\cdot,\underline{v})$, $W(\cdot,p^*)$, $R(\cdot,p^*)$ and $R(\cdot,\underline{v})$. Then, $Q(n)$ is*

1. *non-negative and increasing in $n$;*

2. *concave in $n$, i.e., $Q(n+1)-Q(n)\leq Q(n)-Q(n-1), \forall n\in\mathbb{N}$; and*

3. *their one-step ratios are decreasing, i.e., $\frac{Q(n)}{Q(n-1)}$ is decreasing in $n\in\mathbb{N}$.*[30]

*Proof.* Point 1 holds for $W(n,\underline{v})$ and $W(\cdot,p^*)$ because we can write them, respectively, as $\mathbb{E}[\max_{i=1,\ldots,n} v_i]$ and $\mathbb{E}[\max_{i=1,\ldots,n} v_i \mathbb{1}_{v_i\geq p^*}]$, which are the expected highest draws out of $n$ iid samples. Regularity of $F$ (i.e., increasing $MR(\cdot)$) ensures that $\mathbb{E}(\max\{MR(v^{(n)}),x\})$ is increasing in $n$, and thus $R(n,p)$ is increasing for any $p$. To see that $\mathbb{E}(\max\{MR(v^{(n)}),x\})$ is non-negative, note that $\max\{MR(v^{(n)}),x\}\geq MR(v_1)$ and $E[MR(v_1)]=0$. To show Point 2,[31] take subsets $S,T$ of the set of bidders $B$ with $S\supsetneq T$, and a bidder $i\in B\setminus S$. Then, we claim

$$\max_{j\in S\cup\{i\}} v_j - \max_{j\in S} v_j \leq \max_{j\in T\cup\{i\}} v_j - \max_{j\in T} v_j \tag{6}$$

where $\max_{j\in T} v_j = 0$ if $T=\emptyset$. Indeed, when $v_i = \max_{j\in S\cup\{i\}} v_j$, the comparison becomes $-\max_{j\in S} v_j \leq -\max_{j\in T} v_j$, which holds by the assumption that $T\subset S$; when $v_i<\max_{j\in S\cup\{i\}}$, the left hand side is 0, and the right hand side is non-negative. Taking expectations on both sides of (6) delivers the concavity of $W(n,\underline{v})$.The result extends to $W(\cdot,p^*)$, $R(\cdot,\underline{v})$ and $R(\cdot,p^*)$ by applying the analogous argument to $v_j\mathbb{1}_{v_j\geq p^*}$, $MR(v_j)$, and $MR(v_j)\vee 0$, instead of $v_j$.[32] To show Point 3, note that diminishing returns of $n\mapsto Q(n)$ imply

$$Q(n+1)\left(1-\frac{Q(n)}{Q(n+1)}\right)\leq Q(n)\left(1-\frac{Q(n-1)}{Q(n)}\right) \tag{7}$$

By revenue monotonicity, the terms in the parentheses are non-negative, so for (7) to hold even though $Q(n+1)\geq Q(n)$, $\frac{Q(n)}{Q(n-1)}$ must be decreasing, as required. □

The following lemma presents a useful property of the auctioneer's value function. Let $V(n,l)$ denote the value function at state $(N_t,\mathbb{L}_t)=(n,l)$.

[30] We adopt the convention that $\frac{x}{0}=\infty$.

[31] The proof strategy follows Corollary 2.7. in Dughmi, Roughgarden, and Sundararajan (2009).

[32] We sometimes use the shorthand $a\vee b=\max(a,b)$ and $a\wedge b=\min(a,b)$.

**Lemma 5.** *For any $n$, $l > 0$, $V(n+1,0) \geq V(n,l) \geq V(n,0)$.*

*Proof.* For the first inequality, starting from $(n,l)$, let $q$ be the discounted probability that, under the optimal policy, the state reaches $(n+1,0)$ before the auctioneer holds an auction, and let $q'$ be the discounted probability, under the optimal policy, that the seller holds the auction before the state reaches $(n+1,0)$, i.e., the auction takes place with at most $n$ bidders. Clearly, $q + q' \leq 1$, so $V(n,l) = qV(n+1,0) + q'Q(n) \leq V(n+1,0)$.

Now, define $\theta^x := W - x | \{W \geq x\}$ where $x > 0$. Fix some $l > 0$. Note that by the NBU assumption, $\theta^0 \geq_{FOSD} \theta^l$, so we can always find a probability space such that $\theta^l$ and $\theta^0$ are both defined on that space and $\theta^0 \geq \theta^l$ almost surely.

With that in mind, consider two stochastic processes defined on a common probability space on which $\theta^0 \geq \theta^l$ almost surely. First, define $N_l$ as follows.

- At $t = 0$, initialize $N_l(0) = n$. Also, start $n$ independent exponential departure clocks simultaneously.
- If arrival clock $\theta^l$ "ticks," increase $N_l$ by 1 and draw a new arrival clock $\theta_1 \sim G$ and an exponential departure clock $D_1$.
- For each $k \geq 1$, if arrival clock $\theta_k$ "ticks," increase $N_l$ by 1 and draw a new arrival clock $\theta_{k+1} \sim G$ and an exponential departure clock $D_{k+1}$.
- If any departure clock ticks, decrease $N_l$ by 1.

Second, define $N_0$ in the same way as $N_l$ except that we use $\theta^0$ instead of $\theta^l$ for the first arrival clock. By construction, $N_0$ and $N_l$ are identical until time $\theta^l$.

The values from following the policies that are optimal for $N_l$ and $N_0$ are $V(n,l)$ and $V(n,0)$, respectively, but given the construction of $N_l$ and $N_0$ on a common probability space, we can compare $V(n,l)$ and $V(n,0)$ realization by realization. We refer to the auctioneers who confront $N_l$ and $N_0$ as Auctioneers $l$ and 0, respectively. Likewise, we refer the optimal policies of Auctioneers $l$ and 0 as Policies $l$ and 0, respectively.

Suppose that Auctioneer 0 takes Policy 0, which yields $V(n,0)$. Meanwhile, Auctioneer $l$ uses the following "hybrid" timing policy.

1. As long as no bidders arrive (i.e. $N_l$ and $N_0$ coincide), follow Policy 0.

2. If a new bidder arrives, from that point on adopt Policy $l$.

We show that the hybrid policy, Auctioneer $l$ receives a greater continuation value than $V(n,0)$ for any realizations of the stochastic processes. If Auctioneer $l$ holds an auction at time $t$ while still imitating Auctioneer 0, it must be that $N_l(t) = N_0(t) := \underline{n}$. Hence, both Auctioneer $l$'s payoff from the hybrid policy and Auctioneer 0's payoff from Policy 0 will equal $e^{-rt}Q(\underline{n})$. If instead Auctioneer $l$ switches to Policy $l$ before holding the auction (i.e., at some time $t > \theta^l$), then the time-$\theta^l$ continuation payoff for Auctioneer $l$ is $V(n+1,0)$, while that of Auctioneer 0 (following Policy 0) is $V(n,\theta^l)$. Since $V(n,\theta^l) \leq V(n+1,0)$ was shown above, we conclude that, realization by realization, the hybrid policy gives Auctioneer $l$ at least as much as Policy 0 gives Auctioneer 0, so $V(n,l) \geq V(n,0)$. □

Lemma 5 implies that, when starting from state $(0,0)$, the optimal policy stops the first time the bidder pool reaches some $n^*$. To complete the proof of Theorem 5, it remains to prove the auxiliary results used in the one-step characterization of $n^*$ in (2). Recall that $\beta(j) := \mathbb{E}[e^{-\rho\tau(j-1,j)}]$, where $\tau(j-1,j)$ is the random time elapsed between when the pool size first reaches $j-1$ to when it first reaches $j$. Then we have the following payoff decomposition:

**Lemma 6.** *The auctioneer's expected discounted payoff at time* 0 *from holding the auction upon the* $n$*-th bidder's arrival is given by* $\prod_{j=1}^{n}\beta(j)Q(n)$.

*Proof.* Let $\tau_n = \inf\{t : N_t = n\}$. Then the expected discounted payoff from stopping at the $n$-th bidder's arrival becomes

$$\mathbb{E}[e^{-\rho\tau_n}]Q(n) = \mathbb{E}[e^{-\rho\sum_{j=1}^{n}\tau(j-1,j)}]Q(n). \tag{8}$$

To calculate (8), construct the following "fictitious" process $\mathbf{N}' = \{N'_t, t \geq 0\}$:

- Initialize $N'_0 = 0$.
- Draw arrival clocks sequentially: first draw $\theta_1 \sim G$ at $t = 0$, then when the first clock ticks ($t = \theta_1$) draw $\theta_2 \sim G$ independently at $t = \theta_1$, and so on.
- If an arrival clock $\theta_j$ "ticks":

1. Increase $\mathbf{N}'$ by 1.
2. Remove (ignore thereafter) all remaining departure clocks, and replace them with the same number of new, independent departure clocks.
3. Add an additional exponential departure clock, independent of all arrival clocks.

- If any existing departure clock ticks:

1. Decrease $\mathbf{N}'$ by 1.
2. Remove (ignore thereafter) all remaining departure clocks, and replace them with the same number of new, independent exponential departure clocks.

By the memoryless property of the Poisson clocks, $\mathbf{N}$ and $\mathbf{N}'$ will have the same marginal distributions, so letting $\tau'(j-1,j)$ denote the successive record times for $N'$, $\tau'(j-1,j) \sim \tau(j-1,j)$ and $e^{-\rho\sum_{j=1}^{n}\tau(j-1,j)} \sim e^{-\rho\sum_{j=1}^{n}\tau'(j-1,j)}$. In addition, $N'$'s first-increment times $\{\tau(k-1,k)\}_{k\in\mathbb{N}}$ will be mutually independent, since the increasing part of $\mathbf{N}$ is a renewal, and by the way we construct $\mathbf{N}'$ (replacing all "old" clocks with fresh independent ones at each point of change), all the dependence in between successive records of $\mathbf{N}'$ has been removed. Therefore

$$\mathbb{E}[e^{-\rho\sum_{j=1}^{n}\tau(j-1,j)}] = \mathbb{E}[e^{-\rho\sum_{j=1}^{n}\tau'(j-1,j)}] = \Pi_{j=1}^{n}\mathbb{E}[e^{-\rho\tau'(j-1,j)}] = \Pi_{j=1}^{n}\beta(j),$$

as required. $\square$

Finally, we show that $\beta(\cdot)$ in 5 is decreasing.

**Lemma 7.** *Let $\tau(n-1,n)$ be the time between when the bidder pool first reaches $n-1$ bidders and when it first reaches $n$ bidders. For any $n \in \mathbb{N}$, $\tau(n,n+1) \geq_{FOSD} \tau(n-1,n)$. Thus, $\beta(j) = \mathbb{E}[e^{-\rho\tau(j-1,j)}]$ is decreasing in $j$.*

*Proof.* The proof is by coupling. At $t = 0$, start $n$ independent exponential departure clocks simultaneously. Label one of these clocks "first." Independently of these clocks, draw arrival clocks sequentially: first draw $\theta_1 \sim G$ at $t = 0$, then when the first clock ticks ($t = \theta_1$) draw $\theta_2 \sim G$ independently at $t = \theta_1$, and so on. Add an additional independent departure clock every time a new arrival clock $\theta_j, j \geq 2$ is drawn. Then define two stochastic processes $\mathbf{M} = \{M_t, t \geq 0\}$ and $\mathbf{M}' = \{M'_t, t \geq 0\}$ on this space such that

- Initialize $M_0$ at $n$ and $M'_0$ at $n-1$
- If an arrival clock $\theta_j$ "ticks," increase $\mathbf{M}$ and $\mathbf{M'}$ by 1.
- If any departure clock ticks, decrease $\mathbf{M}$ by 1.
- If any departure clock other than the first one in the original set ticks, decrease $\mathbf{M'}$ by 1.

Note that $M_t = M'_t + 1$ before the first departure clock ticks, and $M_t = M'_t$ thereafter, i.e., $\mathbf{M}$ and $\mathbf{M'}$ eventually "couple."

By the renewals and exponential departures assumptions, $M_t$ is distributed as $N_t$ started from state $(n, 0)$, while $M'_t$ is distributed as $N_t$ started from state $(n-1, 0)$. Therefore, the time $\mathbf{M}$ first crosses $n+1$, denoted $\sigma_{n+1}$, has the same distribution as $\tau(n, n+1)$. Similarly, the time $\mathbf{M'}$ first crosses $n$, denoted $\sigma'_n$, has the same distribution as $\tau(n-1, n)$.

We claim $\sigma_{n+1} \geq \sigma'_n$ almost surely. First, if $\mathbf{M}$ and $\mathbf{M'}$ have not coupled by $\sigma_{n+1} \wedge \sigma'_n$, $M$ will reach $n+1$ at $\sigma'_n$ (since $M_t = M'_t + 1$ before coupling). Hence, $\sigma_{n+1} = \sigma'_n$. Second, suppose that $\mathbf{M}$ and $\mathbf{M'}$ couple before time $\sigma_{n+1} \wedge \sigma'_n$ (so that they meet at some state $k \leq n-1$). Then $\sigma_{n+1} > \sigma'_n$, since $M_t = M'_t$ after coupling, and $\mathbf{M}$ must reach $n$ before reaching $n+1$. Therefore, we conclude that $\tau(n, n+1) \geq_{FOSD} \tau(n-1, n)$ by the usual argument, and the result follows. □

We are now ready to prove Theorem 5.

*Proof of Theorem 5.* Let $\tau^*_{0,0}$ be the essential infimum over optimal stopping times starting from state $(0, 0)$. Suppose, contrary to the theorem, that $\tau^*_{0,0}$ dictates stopping in between arrivals. That is, there exists a state $(n, l)$ with $l > 0$, such that $Q(n) \geq V(n, l)$. Then, using Lemma 5, $Q(n) \geq V(n, l) \geq V(n, 0)$, so if the auctioneer weakly prefers stopping at $(n, l)$, she would also prefer it at $(n, 0)$. In any history starting from state $(0, 0)$, $(n, 0)$ is always reached before $(n, l)$, so $\tau^*_{0,0}$ would have dictated stopping at $(n, 0)$ without ever reaching $(n, l)$, a contradiction. Therefore, if starting from $(0, 0)$ the auctioneer stops at all, she stops at an arrival. Accordingly, we need to check that the auctioneer would indeed stop in finite time almost surely. But this is immediate because $Q(1) > 0$ and the payoff from never stopping is 0. Altogether, we have that the auction will happen the first time $N_t$ reaches an acceptable threshold, i.e., some $n$ such that $Q(n) \geq V(n, 0)$. Let $n^*$ denote the smallest such $n$.

It remains to prove that in fact $n^*$ is given by (2). The optimal stopping policy must be time-consistent, so if the auctioneer stops at $n^*$, then $n^*$ must be the smallest maximizer of the ex ante expected discounted payoff from waiting for exactly $n$ bidders. That is, $n^*$ must be the smallest maximizer of $\delta(n)Q(n)$, where $\delta(n)$ is the expected discount factor from waiting for $n$ bidders. By Lemma 6, our assumptions on bidder dynamics imply that the $n$-step discount (from waiting for $n$ bidders) decomposes into $\delta(n) = \prod_{i=1}^{n} \beta(i)$. The inequality $Q(n) \geq \beta(n+1)Q(n+1)$ is therefore a necessary local condition for optimization of $\delta(n)Q(n)$, i.e., it implies that $\delta(n)Q(n) \geq \delta(n+1)Q(n+1)$. Moreover, a coupling argument (Lemma 7) implies that the successive record times $\tau(n-1, n)$ increase in the first-order stochastic dominance sense as $n$ grows, so $\beta(n) = \mathbb{E}[e^{-\rho\tau(n-1,n)}]$ is decreasing. Together with Point 3 of Lemma 4, since $\beta(\cdot)$ is decreasing, so is $n \mapsto \beta(n+1)\frac{Q(n+1)}{Q(n)}$. It follows that $\prod_{i=1}^{n} \beta(i)Q(n)$ is single-peaked. Therefore the condition in (2) is in fact a sufficient condition for maximizing $\delta(n)Q(n)$, and the optimal policy starting from $(0, 0)$ stops the first time $n^*$ is reached. $\square$