When to Hold an Auction?*

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Abstract

We study the optimal timing of auctions when bidders can arrive and depart stochastically over time. First, we show that a revenue-maximizing seller holds auctions too late or too early, relative to a surplus-maximizing planner, depending on whenever virtual values (censored at 0) are more or less right-skewed than true values. In particular, revenue-maximizing sellers typically hold auctions inefficiently late. Second, we prove that, by restricting herself to ex ante optimal deadlines, the seller may waste an arbitrarily large fraction of the revenue from the optimal stopping rule. Dynamically responding to changes in the bidder pool is therefore essential.

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1 Introduction

When an auction house sells a famous painting, it holds the auction in a room on a fixed date, and all bidders have to register with the house before participating. Most literature on mechanism design with transfers models situations like these, where the designer knows the set of bidders in advance. This leaves out a crucial part of the design problem: the auctioneer is often uncertain about who will show up for the auction (and by when) and must make a strategic decision about when to hold the auction. Consider, for instance, the market for distressed corporate assets. After the firm decides to restructure and sell a division, it cannot take for granted that there will be a given set of bidders. Interested bidders for these assets are hard to find, and even when found, it takes considerable time and effort to motivate them to bid. The process usually involves both hiring an investment bank at significant cost and several months' delay (Boone and Mulherin (2009) note that the process takes more than six months on average, a very long time for a company in distress). Like the auction house selling a painting, the firm selling a division must decide how to allocate the object and determine prices. But the firm also faces a new set of problems. It must choose not only how to run the auction, but also how long to search for bidders. If the firm does not wait long enough, it may end up with too few bidders and raise little revenue, since the bidders will face less competitive pressure to bid aggressively. On the other hand, by waiting too long in the hopes of receiving more (or better) bidders, it risks that promising current bidders may find other opportunities and leave. Likewise, the things we should worry about as analysts change. In the painting example, the main efficiency concern was whether the painting went to the highest value bidder. Now we need to ask, how long will it take for the bidder to get the division and the firm to get the money? Will the selling firm, who cares about its own time and its profit but not about bidders' time, hold the auction later than would a social planner who wanted to maximize total surplus? We need to consider not only incentives to bid truthfully, but incentives to "arrive truthfully": bidders could, say, delay making offers until they see other bidders enter, in the hopes that the firm will become more desperate to sell and hold an auction with more favorable terms.

Motivated by this example, we study the problem of when to hold an auction in an environment where bidders arrive and depart stochastically over time. First, we ask the following: do profit-maximizing sellers over- or under-value market thickness? Notice, first, that there does not seem to be any a priori reason to expect one or the other. A seller and a social planner face not only different marginal costs from waiting for an additional bidder–the former suffers discounting costs on the expected revenue from existing bidders, while the latter suffers discounting costs on the expected surplus–but also different marginal benefits, since the seller only cares about the expected marginal revenue as opposed to the expected marginal surplus from that extra bidder. It would seem that, by changing the distributions of valuations and the law of the arrival process for bidders, one could get these marginal benefit and marginal cost curves to cross any which way, so that the comparison between the planner and the seller's investment in market thickness would be entirely ambiguous.

In fact, we show that there is a broad sense in which the seller systematically over-values market thickness: for any arrival process for bidders, and for most distributions used in practice, the seller waits for more bidders than the social planner. We identify a property of the distribution–its "elasticity with respect to price," borrowing from Bulow and Roberts (1989)'s monopoly pricing interpretation of auction theory–such that, for *any* arrival process, whenever this elasticity is increasing, the monopolist seller inefficiently delays the auction, whereas when this elasticity is decreasing, she inefficiently accelerates the auction. In particular, for all increasing hazard rate distributions, the seller over-invests in market thickness. Among more general distributions with monotone elasticities, the seller would under-value market thickness only if the valuations have tails that are thicker than Pareto. Since Pareto distributions typically model processes with very thick tails, we see this case as quite unlikely.

We also uncover a key subtlety in the inefficiencies caused by profit maximization. We show that the wedge between a monopolist seller and the social planner is the composition of two distinct effects that work in opposite directions. On the one hand, conditional on using the same auction format, the seller evaluates timing policies according to virtual values as opposed to true values (an *information rent* effect). On the other, the seller wants to use a different selling procedure once she holds the auction, but this feeds back into the decision about when to hold the auction (an *auction format* effect). The auction format effect always pushes the auction earlier, while the information rent effect can push the auction either earlier or later, depending on the distribution of bidder valuations. Although the two effects can push in different directions, we show that the net effect depends on whether the elasiticity of demand is increasing or decreasing, and moreover, the information rent effect always dominates. The following example illustrates how these two effects work:

Example 1. *Suppose that the auctioneer chooses when to hold an auction over a time horizon* $t \in \{1, 2, 3\}$. *One bidder arrives in each period; thus, an auction held in period* t *has* t *bidders. The auctioneer discounts the future at a rate* $\delta = 0.7$, *so at time* t *both revenue and surplus are discounted by* δ^{t-1}. *Bidders' valuations are drawn iid from the uniform distribution on* $[0, 1]$. *For each* $n \in \{1, 2, 3\}$, *let* $R(n, 0)$ *and* $S(n)$ *denote the expected revenue and surplus, respectively, from a second price auction with no reserve price and* n *bidders. Likewise, let* $R(n, p^*)$ *denote the expected revenue from a second price auction with reserve price* $p^* = 1/2$, *i.e., a revenue-optimal auction. Table 1 then presents the discounted expected revenues and surplus from each possible timing choice.*[1]

t	1	2	3
$\delta^t R(t, 0)$	0	0.23	0.25
$\delta^t S(t)$	0.5	0.47	0.37
$\delta^t R(t, p^*)$	0.25	0.29	0.26

Table 1: Expected Discounted Payoffs - Second Price Auction

The results in Table 1 capture three general features. First, we see that the seller who can use an optimal auction holds the auction inefficiently late ($t = 2$ *vs* $t = 1$, *the efficient choice), since the virtual valuation censored below at zero,* $\max\{2v - 1, 0\}$, *is more right-skewed than the actual valuation* v. *(The elasticity for this distribution is* $v(1 - v)^{-1}$, *which is increasing.) Second, the seller who is forced to use an second price auction with no reserve*

[1] The numbers are obtained from the formulae $R(t, 0) = \int_0^1 (2v - 1)tv^{t-1}dv$, $S(t) = \int_0^1 v \cdot tv^{t-1}dv$, and $R(t) = \int_{\frac{1}{2}}^1 (2v - 1)tv^{t-1}dv$.

price holds the auction later ($t = 3$) than the planner ($t = 1$)—an information rent effect. Hence, in Example 1, the information rent effect leads to a later auction. Third, the seller who uses an optimal auction (i.e., chooses an optimal reserve price) in Example 1 holds the auction at $t = 2$, earlier than the seller who is forced to use an efficient (zero reserve price) auction—an auction format effect). Fourth, in this example, the information rent effect pushes the seller's timing later than the planner by two periods, while the auction format effect pushes the seller's timing earlier by one period. Hence, the information rent effect dominates, so that the net effect of revenue maximization is to push the auction later.

An implication of these opposing effects is that regulating only the choice of auction format—out of concern over a seller's inefficient use of market power, say—can *decrease* surplus because of the seller's endogenous choice of auction timing.

As a consequence of our analysis, we develop a general monotone comparative statics result for the optimal auction timing as a function of the distribution of bidder values. We show that the key condition that determines the relative timing of an auction is how right-skewed the valuation distribution is, in a sense we define precisely below. If a distribution of types F is more right-skewed than a distribution H, we prove that the social planner holds an auction later under F regardless of how bidders arrive, so long as they never depart, and their arrival times are uninformative about their value for the good being auctioned. With a very right-skewed valuation distribution, new draws that exceed the sample maximum will tend to do so by larger margins, since they will tend to come from a longer right tail. Adding new bidders will grow total surplus quickly under such a distribution, which gives the social planner a stronger incentive to wait. Of course, this intuition breaks down when bidders both arrive and depart, since a distribution of types where surplus grows very quickly with new arrivals is also one where surplus *shrinks* very quickly with departures. Nonetheless, our general comparative statics result partially extends to the case with departures: if a distribution of types F is more right-skewed than a distribution H, we prove that the social planner stops later under F so long as inter-arrival times are iid with increasing hazard rates, and sojourn times are iid exponential random variables.

The second question we study is whether the simpler heuristics for auction timing that are used in practice can be "approximately" as good as the optimal policy for a broad range of environments (see, for instance, Hartline and Roughgarden, 2009 and Hartline, 2012). For instance, auctioneers often commit in advance to a fixed date at which to hold the auction (e.g., real estate auctions are commonly run in this fashion). This nonadaptive stopping time is simpler than changing the date as a function of the arrival and departure of bidders, so it is possible that real-world sellers perceive that fixed dates perform "well enough" across a variety of possible market conditions and are easier to implement. To investigate this possibility, we verify whether fixed-in-advance deadlines can provide an "α-approximation" (Hartline, 2012) to the auction timing problem: that is, do fixed deadlines always achieve at least a share α of the fully optimal expected discounted revenue, regardless of the distribution of values or arrivals? Seen another way, what is the gain to the seller from being able to respond to the history bidder arrivals?

Surprisingly, we show even deadlines that are chosen optimally in advance with full knowledge of the environment can perform arbitrarily poorly in the worst case over different arrival process. Even if the arrival process is a renewal with "New Better than Used" increments (a common tractable choice in applications, e.g., (Gershkov et al., 2015; Zuckerman, 1988)), and the distribution of values is regular in the sense of Myerson (1981), the worst-case ratio between the optimal deadline and the optimal adaptive policy is 0. In contrast, under a strong condition on the arrival time distribution, optimal deadlines do guarantee at least $1/e$ of the best case payoff. The gains from having access the history of bidder arrivals can therefore be very large unless the seller believes there are strong restrictions on the bidder arrival dynamics.

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the environment and formally sets up the auction timing problem. Section 4 introduces the notion of right-skewness of a valuation distribution. Focusing on the case with no departures, this section establishes several comparative statics using that notion, and it studies the inefficiencies that arise from profit-maximization. Section 5 extends these results to the case when bidders depart stochastically. Section 6 proves that using deterministic timing rules (i.e., deadlines) can perform arbitrarily worse than the fully optimal timing, and Section 7 concludes. All omitted proofs are relegated to the Appendices.

2 Related Literature

Wang (1993) appears to be the first paper to study the timing of an auction as an endogenous choice by a seller. Wang analyzes a model in which the seller has one unit of an object to sell to buyers who arrive according to a homogenous Poisson process. The seller who bears a constant flow cost from holding inventory can decide to sell the object either by posting a price or by holding an auction at a fixed deadline chosen in advance. Wang shows that the seller's choice of timing typically differs from the socially optimal one. In contrast, we consider significantly more general arrival and departure dynamics, and we allow the seller much more leeway in choosing when to hold the auction. In our model, the auctioneer can use any stopping time that is adapted to the history of arrivals and departures. The recent paper by Cong (2015) also studies an auction timing problem, but in a very different context (the selling of real options to a fixed set of long-lived bidders). Among other things, he finds that the seller inefficiently delays the auction.

The present paper also relates to a recent literature on bidder solicitation. Szech (2011) and Fang and Li (2015) study a static environment in which an auctioneer must use costly advertising to attract bidders. In contrast to Wang (1993), Szech (2011), or Fang and Li (2015), the solicitation cost in our paper comes from time discounting. This requires a very different set of arguments from theirs, and it makes the comparison between the seller and planner harder: unlike advertising or linear (flow) waiting costs, costs from time discounting are evaluated differently by the seller and the planner. In Remark 3, we give a detailed discussion of the findings in this literature, and how they relate to ours.

This paper fits within a broader dynamic mechanism design literature on how to allocate objects to stochastically arriving agents (Gallien, 2006; Gershkov, Moldovanu, and Strack, 2014; Board and Skrzypacz, 2016). These papers derive a fully optimal dynamic mechanism for a more restricted class of agents' arrival processes, and they focus especially on when these mechanisms can be implemented in posted prices. In contrast to these papers, we focus on question of the timing of the transaction. Thus we can consider more general arrival and departure dynamics, at the cost of looking at a more restricted class of mechanisms. One contribution of this paper, relative to the above literature, is that we can give full solutions for the case with stochastically departing agents. This case is not very much studied, except for some recent work on dynamic matching market design (e.g., Akbarpour et al. (2016)). Loertscher et al. (2016) consider a similar model to ours, where traders arrive stochastically and the market maker trades off the gains of market thickness against the costs of delay, but where agents' valuations lie in a finite type space. This restriction on the size of the seller's state space lets them consider more intricate market making algorithms that condition trading ("stopping" in our language) on the particular type realizations. Compared to their paper, we have a richer type space, we give general comparative statics results that apply uniformly for any arrival process, and we consider the worst-case robustness of simple rules.

The optimal timing of an auction derived in Section 5 has some similarity to the search literature on the optimality of reservation wage policies, as in Zuckerman (1984, 1986, 1988). He shows that, in a continuous time job search problem, the reservation wage policy is optimal under general renewal processes satisfying New-Better-than-Used (NBU) property. In our model, a similar logic implies that the optimal timing policy is characterized by a cutoff in the number of bidders, whenever bidders arrive according to a renewal process whose inter-arrival times have increasing hazard rates.

3 Setting

A designer allocates one indivisible good to randomly arriving potential bidders over a time horizon $[0, \infty)$. The bidders arrive and depart according to a time-inhomogeneous counting process $\{N(t), t \geq 0\}$, where $N(t)$ is a random variable representing the number of bidders at time t. As it will become clear, we do not need to record the identity of each bidder upon describing the arrival and departure dynamics, because the bidders' valuations are drawn from the same distribution. We say that there are *no departures* if $N(\cdot)$ is path-wise increasing, which will be our main focus in Section 4.[2]

A bidder i is characterized by an arrival time a_i and a value v_i for the good. Each v_i is i's private information, and the v_i's are drawn iid according to a continuous distribution F independently from $N(\cdot)$. Throughout the paper, we assume that (i) F has a finite expectation, (ii) F has a positive density f, (iii) $F(0) = 0$, and (iv) F is regular, i.e., $v - \frac{1-F(v)}{f(v)}$ is increasing.[3]

We can interpret a_i as the time at which i first has demand for (or is able to receive) the good, or as the time i first notices that an auction is taking place. For ease of exposition, we mostly treat a_i as though it is publicly observed, but later we explain how our results extend to the case where bidder i strategically delays his arrival, i.e., can report any $\hat{a}_i \geq a_i$.[4] We do not impose specific assumptions regarding when a bidder learns his own valuation v. Our results hold as long as the bidder knows the realization of v when he makes a bid in the auction.

We assume that the designer can commit to any mechanism, so by a version of the Revelation Principle we can restrict ourselves to studying truthful direct mechanisms.[5] However, we do not consider the full set of possible direct mechanisms; we restrict the analysis to mechanisms of the following form:

1. a timing policy, i.e., a stopping time τ at which to hold an auction, adapted *only* to the history of arrival and departure times.

2. an auction format to be used upon stopping, i.e., a static direct mechanism consisting

[2] Throughout the paper, we use "increasing" to mean "non-decreasing". Similarly, by "decreasing" we mean "non-increasing".

[3] The assumptions $\mathbb{E}[v_i] < +\infty$ and $F(0) = 0$ implies that all the order statistics of v_i presented below have finite moments. Indeed, it holds that $\mathbb{E}[\max\{v_1, \ldots, v_n\}] \leq n\mathbb{E}[v_i] < +\infty$. Because i-th order statistic is always lower than the highest order statistic for sure, this inequality implies that all the order statistics have finite moments.

[4] Similar to the literature on auctions with private budget constraints (Che and Gale, 1998), it will be crucial that agents can only engage in one-sided deviations in their second dimension of private information.

[5] Technically, we only need to assume that the seller can commit to run any static mechanism once she stops, but she does not need commitment power with regard to timing. See Remark 4 for details.

of allocations $\mathbf{x}$ and transfers $\mathbf{p}$ for bidders in the market at τ. This mechanism can be adapted to the entire history observed by the designer.

The economic content of these restrictions is that we rule out any kind of indicative bidding before the actual auction takes place (so the seller cannot, as is often done in corporate takeover auctions, ask for non-binding initial offers from prospective bidders before the final, binding bids are due). We impose this restriction for three reasons: (i) it lets us focus cleanly on the question of *timing*; (ii) it lets us explicitly solve for an optimal policy in an environment where bidders not only arrive but also depart stochastically; and (iii) it lets us consider a much broader range of arrival and departure processes than has previously been done in the literature.

However, the restriction is not without loss of generality. For instance, the fully optimal mechanism will probably not lie in this class. Consider a designer who uses an auction with a binding reserve price. At a minimum, the designer would want to know before stopping at some history of arrivals h_t whether any of the bidders in the pool could even meet the reserve price—otherwise the good would go unsold, and it would have been better to check whether any of the current bidders were willing to submit serious offers.[6]

Preferences are as follows. If a bidder of value v gets the object and pays p in an auction held at time t, he receives a payoff of $e^{-rt}(v-p)$ where $r > 0$ is the discount factor. If the designer is a *seller*, she receives a payoff of $e^{-rt}p$, the discounted revenue. If on the other hand the designer is a *social planner*, she receives a payoff $e^{-rt}v$, i.e., the discounted surplus.

Given the full commitment assumption and our restriction on the class of mechanisms, we can pin down the auction formats that would be used by the seller and the planner under these preferences. Fix any stopping time τ adapted to the history of arrivals, and any selling mechanism $(\mathbf{x}, \mathbf{p})$. Then, the seller's expected profit weakly increases if she stops according to τ but instead of using $(\mathbf{x}, \mathbf{p})$, she sells the good using an optimal incentive compatible

[6] In fact, for the case with no departures, the fully optimal direct mechanism may not consist decompose into a triple $(\tau, \mathbf{x}, \mathbf{p})$ consisting of a timing policy and a single auction. Under particular assumptions on the arrival processes $\{N(t)\}_{t\geq 0}$, both Gallien (2006) and Board and Skrzypacz (2016) find optimal dynamic mechanisms that can be implemented with deterministic sequences of posted prices.

mechanism for the pool of bidders that are present at τ. Therefore, if F is regular, by the Payoff Equivalence Theorem (Myerson, 1981), we can assume without loss of generality that, upon stopping, the seller uses a second price auction with an optimal reserve price $p^* = \inf\{p \in \text{supp}F : p - 1/\lambda(p) \geq 0\}$ where $\lambda(p) \equiv \frac{f(p)}{1-F(p)}$ denotes F's hazard rate.[7] Repeating the same reasoning for the planner, we can assume that she will use a second price auction with no reserve price. For future reference, let $MR(v) = v - 1/\lambda(v)$ denote v's virtual valuation. We sometimes write MR_F and p^*_F to emphasize the dependence of MR and p^* on F.

Definition 1. A seller is *statically constrained* if she is free to choose any feasible (adapted to $\{N(t), t \geq 0\}$) timing policy τ, but she must use the social planner's preferred auction format, i.e., an efficient auction.

Let $R(n, p)$ denote the expected revenue from a second price auction with reserve price p, and $S(n)$ denote the expected surplus from a second price auction with no reserve price. Below, we write $R(n)$ as short-hand for $R(n, p^*)$ whenever it does not cause confusion. Then, by the discussion above, the seller chooses a stopping time τ adapted to the history of arrivals and departures so as to maximize

$$\mathbb{E}\Big[e^{-r\tau} \max\{MR(v_1), \ldots, MR(v_{N(\tau)}), 0\}\Big] = \mathbb{E}\left[e^{-r\tau} R(N(\tau), p^*)\right]$$

whereas the planner chooses τ to maximize

$$\mathbb{E}\Big[e^{-r\tau} \max\{v_1, \ldots v_{N(\tau)}\}\Big] = \mathbb{E}\left[e^{-r\tau} S(N(\tau))\right]$$

where the equalities follow from conditional expectations and the Strong Markov Property. In other words, the auction timing problem for both the seller and the social planner reduce to optimal stopping problems with no flow cost and payoffs upon stopping of $R(n, p^*)$ and $S(n)$, respectively.

Below we will make repeated use of the following key facts regarding payoffs upon stopping.

[7] supp$F$ denotes the support of F.

Fact 1. Let $Q(\cdot)$ be any one of $S(\cdot), R(\cdot, p^*)$ and $R(\cdot, 0)$. Then, $Q(n)$ is

1. non-negative and increasing in n;
2. concave in $n \in \mathbb{N}$, i.e., $Q(n+2) - Q(n+1) \leq Q(n+1) - Q(n), \forall n \in \mathbb{N}$;
3. their one-step ratios are decreasing, i.e., $\frac{Q(n+1)}{Q(n)}$ is decreasing in n.

Proof. See Appendix D. □

Fact 1 captures the intuitive fact that additional bidders become less valuable the more bidders are already in the market.

4 Right-Skewness and Comparative Statics

We begin with introducing the following stochastic order, which is useful for studying how the optimal timing of an auction varies across different objectives and auction formats.

Definition 2 (Right-skewness)**.** For two random variables X and Y, call X be more (less) right-skewed if there are positive, increasing functions g and h and a third random variable Z such that $X \sim g(Z)$, $Y \sim h(Z)$ and $h(z)/g(z)$ is increasing (decreasing).

To understand why the above condition captures higher right-skewness, note that, whenever h/g is increasing, $h(Z)$ pushes large draws of Z further out into its right tail (relative to small draws) than $g(Z)$ does, and $h(Z)$ also shrinks small draws of Z towards the left tail more than $g(Z)$ does. This comparison is scale-invariant, so Y being more right-skewed than X is distinct from Y having a fatter or longer right tail than X, which might be false depending on the relative scales of X and Y.[8] In the statistics literature, our right-skewness comparison corresponds to $h(Z)$ being larger than $g(Z)$ in the *star-shaped order*.[9] Through-

[8] So read this comparison as "relatively more of X's mass is on the right of its support," as opposed to "X has more mass on the right than Y does".

[9] See Barlow and Proschan (1966) for an insightful reference on the star-shaped order and connections to order statistics. The star-shaped order generalizes the convex transformation order, which was first introduced by Zwet (1964) as a formalization of right-skewness.

out, we use "more right-skewed" rather than "larger in the star-shaped order" in order to emphasize the underlying intuition.

The following lemma, which connects right-skewness comparisons to growth rates of expected highest draws, drives all of our comparative statics on auction timing:

Lemma 1. *Let $S_A(n-k;n)$ and $S_B(n-k;n)$, $k \in \mathbb{Z}_+$, denote the expected $k+1$-th highest draws from n iid samples drawn according to the law of random variables X_i^A and X_i^B, respectively. If X_i^A is more right-skewed than X_i^B, then $S_A(n-k;n)/S_B(n-k;n)$ is increasing in n.*

Proof. See Appendix A. □

Remark 1. We can extend both the notion of "more right-skewed" and the conclusions of Lemma 1 to cases where h and g can be zero (possibly simultaneously, so that h/g is not defined). All the arguments below continue to hold so long as $h(z) - cg(z)$ crosses zero only once and from below for any $c > 0$. (This is equivalent to h/g being increasing if $g > 0$ everywhere).

Moreover, the conclusions of Lemma 1 hold even if h (but not g) can be negative. The interpretation of increasing h/g as "more right-skewed" becomes problematic in that case, though, since h pushes low draws of Z far into the left tail, whereas g shrinks them towards zero.

The intuition for Lemma 1 is easiest to see when $k = 0$ and $h(Z) = Z$ in Definition 2, i.e., $X_i^A = g(X_i^B)$ with $g(x)/x$ increasing. In this case, adding more bidders (i.e., more draws from the distribution) will grow total surplus (i.e., the sample maximum) proportionally more quickly. For instance, if a new draw exceeds the current sample maximum by 10% under X_i^B, the sample maximum under $X_i^A = g(X_i^B)$ grows by more than 10%. This reasoning works realization by realization, but Lemma 1 ensures that this pointwise intuition carries over rigorously to expected highest draws.

For convenience, write $S_m(n)$ as shorthand for $S_m(n;n)$ for $m = A, B$. Using Lemma 1, we can provide unambiguous comparative statics for any two distributions of bidder types that are ordered by their right-skewness.

Lemma 2. *Assume that there are no departures and $\{N(t), t \geq 0\}$ is independent of bidder valuations. Consider two social planners A and B facing bidder valuations X_i^A and X_i^B, respectively, where X_i^A is* ***more right-skewed*** *than X_i^B. Then, social planner A holds the auction later than B.*

Proof. By Lemma 1, since A facing a more right-skewed distribution implies that $\frac{S_A(n)}{S_B(n)}$ is increasing, so

$$\frac{S_A(n+1)}{S_A(n)} \geq \frac{S_B(n+1)}{S_B(n)}, \forall n \in \mathbb{N}. \tag{1}$$

Therefore, we have the following inequality:

$$\frac{S_A(n+k)}{S_A(n)} \geq \frac{S_B(n+k)}{S_B(n)}, \forall n \in \mathbb{N}, \forall k \in \mathbb{Z}_+. \tag{2}$$

We now show that, if A stops after a finite history h_t, B also stops after h_t. Take any finite history h_t at which the pool size is n. A stops at h_t only if

$$S_A(n) \geq \mathbb{E}[e^{-r\tau} S_A(N(\tau))|h_t]$$

for any stopping time τ. In particular, this inequality must hold for $\tau = \tau_B$, B's optimal stopping time. Now, this implies that

$$1 \geq \mathbb{E}\Big[e^{-r\tau_B} \frac{S_A(N(\tau_B))}{S_A(n)}\Big| h_t\Big] \geq \mathbb{E}\Big[e^{-r\tau_B} \frac{S_B(N(\tau_B))}{S_B(n)}\Big| h_t\Big],$$

where the second inequality is implied by (2). Then, we obtain $S_B(n) \geq \mathbb{E}[e^{-r\tau_B} S_B(N(\tau_B))|h_t]$. Note that no departures implies $N(\tau_B) \geq n$ after h_t, which enables us to use the inequality (2). □

These two lemmas deliver our main result:

Theorem 1. *Assume that there are no departures and $\{N(t), t \geq 0\}$ is independent of bidder valuations. Then, the seller holds the auction inefficiently late if $\lambda(v)v = \frac{f(v)}{1-F(v)}v$ is increasing, in which case $MR(v) \vee 0$ is more right-skewed than v. Conversely, the seller holds the auction inefficiently early if $\lambda(v)v$ is decreasing, in which case v is more right-skewed than $MR(v) \vee 0$.*

Proof. Under an optimal auction, the seller has the same expected payoffs as a social planner facing a valuation distribution $X_i \equiv MR(v_i) \vee 0$. By Lemma 2, it therefore suffices to check whether X_i is more or less right-skewed than v_i in the sense of Definition 2. Indeed, since

$$\frac{MR(v) \vee 0}{v} = \max\left\{1 - \frac{1}{\lambda(v)v}, 0\right\},$$

X_i will be more (less) right-skewed than v_i whenever $\lambda(v)v$ is increasing (decreasing). $\square$

The statement of Theorem 1 might make it seem that inefficiently early and inefficiently late auctions are equally plausible. On closer inspection, however, the result suggests that sellers *usually* hold auctions inefficiently late. Indeed, a broad range of distributions have increasing $\lambda(v)v$. For instance, it suffices for F to have an increasing hazard rate. In contrast, decreasing $\lambda(v)v$ should be relatively uncommon in applications. To see why, note that F has a constant $\lambda(v)v$ if and only if it is a Pareto distribution (Lariviere, 2006). Thus, for F to have a decreasing $\lambda(v)v$, it must have fatter right tails than the Pareto distribution, which itself usually models processes with very fat tails.[10]

Nevertheless, the latter part of Theorem 1 is not vacuous, in the sense that there do exist regular distributions with decreasing $\lambda(v)v$ and sufficiently many finite moments. For instance, take $F(v) = 1 - \exp\left(-\int_0^v \left(\frac{1}{x^2}+1\right)dx\right)$ on $\mathbb{R}_+$. We can calculate that $v - \frac{1-F(v)}{f(v)} = \frac{v+v^2}{1+2v}$ is increasing, while $\lambda(v)v = \frac{1+2v}{v}$ is decreasing. Moreover, by Theorem 6.2 of Barlow et al. (1963), $\lim_{v\to+\infty} \lambda(v)v = 2 > 1$ implies that F has a finite first moment, so all the relevant expectations are well-defined. By Theorem 1, a seller facing valuation distribution F holds the auction inefficiently *early*. Therefore, our results suggest that the seller typically has an excessive incentive to wait for bidders compared to the planner, unless the distribution of bidder values has extremely thick right tails.

Lemmas 1 and 2 also illuminate how a seller's optimal auction timing responds to changes in the valuation distribution.

Proposition 1. *Consider two regular valuation distributions $v^F \sim F$ and $v^H \sim H$.*

1. *If F is more right-skewed than H, then the statically constrained seller holds the auction later when facing F than when facing H.*

[10] See Banciu and Mirchandani (2013) for a list of distributions with increasing $\lambda(v)v$.

2. *If $p_F^* \geq p_H^*$ and $MR_F(x)/MR_H(x)$ is increasing whenever $x \geq p_F^*$, the unconstrained seller holds the auction later when facing F than when facing H.*

The proof of the first point is almost identical to that of Lemma 2, except that it relies on the fact that expected revenue is the expected second highest value and uses Lemma 1 with $k = 1$ as opposed to $k = 0$. The proof of the second point relies on a technical extension of Lemma 1 to the case where both h and g can be zero simultaneously (so h/g can be undefined), but the intuition is the same as in Lemma 2: under the above conditions, $MR_F(v_i^F) \vee 0$ is more right-skewed than $MR_H(v_i^H) \vee 0$, so the proportional boost to revenue from adding new bidders by waiting is greater under distribution F. Thinking about right-skewness also illuminates why the condition $p_F^* \geq p_H^*$ is necessary. Without this additional assumption, $MR_F(v_i^F) \vee 0$ could fail to be more right-skewed than $MR_H(v_i^H) \vee 0$: if $MR_H(v_i^H) \vee 0$'s atom at p_H^* is sufficiently far to the right of $MR_F(v_i^F) \vee 0$'s atom at p_F^*, $MR_F(v_i^F) \vee 0$ may fail to be more right-skewed, even though $MR_F(\cdot)$ pushes large draws of values further into its right tail than $MR_H(\cdot)$.

4.1 Decomposition of Welfare Effects

Theorem 1 shows that the auction timing chosen by a profit-seeking seller is typically inefficient. In this subsection, we show that this net welfare loss is a composition of the two effects:

Information Rent Effect: While the planner cares about bidder values, the seller cares about values net of information rents that accrue to bidders. Therefore, even if both designers use the same efficient auction, they will face different terminal payoffs, so they will want to hold the auction at different times.

Auction Format Effect: The planner uses an efficient auction upon stopping, which differs from the revenue-optimal auction that the seller chooses upon stopping. This endogenous choice of auction formats creates another wedge between the timing choices of the seller and the planner.

The auction format effect compares the unconstrained seller's choices to those of the statically constrained seller, while the information rent effect compares the statically constrained seller's choices to those of the social planner.

The following result summarizes how these effects influence the timing of an auction. In particular, we show that the information rent effect always dominates, in the sense that the net effect always goes in the same direction as the information rent effect.

Proposition 2. *Assume that there are no departures and* $\{N(t), t \geq 0\}$ *is independent of bidder valuations. If* $\lambda(v)v$ *is increasing,*

- *The statically constrained seller holds the auction later than the planner.*
- *The unconstrained seller holds the auction earlier than when she is statically constrained.*

If $\lambda(v)v$ *is decreasing,*

- *The statically constrained seller holds the auction earlier than the planner.*
- *The unconstrained seller holds the auction later than when she is statically constrained.*

Proof. See Appendix A. □

Without a reserve price, a new bidder contributes to revenue whenever his marginal revenue is higher than the highest marginal revenue among the existing bidders, and it increases revenue by the difference between these two marginal revenues. However, under the optimal reserve price, the new bidder adds to revenue if he has the highest marginal revenue *and* that marginal revenue exceeds zero. Moreover, this new bidder only increases revenue by the difference between his marginal revenue and the maximum of the other bidders' marginal revenues and 0. Thus, a positive reserve price shrinks the set of events on which a new bidder contributes to the revenue, and shrinks the amount by which revenue grows when it does; both forces make the waiting for an additional bidder less valuable.

To understand why the auction format effect is absent for decreasing $\lambda(v)v$, it is helpful to interpret the auction as a monopoly pricing problem as in Bulow and Roberts (1989). As we discuss in detail in Remark 2, $\lambda(p)p$ corresponds to the price elasticity of demand with the demand function $D(p) = 1 - F(p)$, and the virtual valuation $p - \frac{1-F(p)}{f(p)}$ corresponds to the marginal revenue (with respect to quantity) in a certain monopolist problem. Now, if the price elasticity of demand $\lambda(p)p$ is decreasing in price p, then price becomes less and less sensitive to quantity as quantity rises. Thus, if marginal revenue is positive at some $q = D(p)$, then it is positive for any $q' > q$, and a monopolist with this demand curve will make a socially efficient choice of quantity, which corresponds to no reserve price in our model.

Proposition 2 also has an implication for policies that try to restore efficiency: *forcing the seller to use an efficient auction format can decrease the expected discounted welfare*, because the seller who cannot use a reserve price excessively delays the auction. For instance, in Example 1, the seller who can use an optimal reserve generates a discounted total surplus of 0.47, while the seller who is forced to use zero reserve price generates a discounted total surplus of 0.37.[11] More generally, for any arrival process such that multiple bidders never arrive at the same time, for a sufficiently high discount factor r, the seller who is forced use an efficient auction format leads to a lower discounted total surplus.[12]

Remark 2 (Monotone Elasticity of Demand: A Price-Discrimination Explanation)**.** To further illuminate the role of a monotone price elasticity in auction timing, we use Bulow and Roberts (1989)'s monopoly pricing interpretation of optimal auctions. Bulow and Roberts show that the optimal auction design problem with n bidders, each with valuation distribution F_i, is formally equivalent to the third-degree price-discrimination problem of a monopolist catering n markets with demand curves $q_i = 1 - F_i(p_i)$, and a maximum capacity of 1. Continuing their analogy, we can relate the seller's inefficient incentives to wait for bidders in the optimal auction timing problem to the monopolist's inefficient incentives to

[11] The numbers are obtained from $0.47 \approx \delta^2 \int_0^1 v \cdot 3v^2 dv$ and $0.37 \approx \delta \int_0^1 v \cdot 2v dv$ with $\delta = 0.7$.

[12] This claim comes from the following observation. Fix F and $N(\cdot)$. For a sufficiently large r, the seller with an optimal reserve price stops at the first bidder's arrival, and the seller with zero reserve price stops at the second bidder's arrival. For a large r, we obtain $\mathbb{E}[e^{-r\tau_2} S(2,0)] < \mathbb{E}[e^{-r\tau_1} S(1,p^*)]$ where $\tau_n = \inf\{t : N(t) = n\}$ for $n = 1, 2$ and $S(n,p)$ is the expected total surplus from an n-bidder second price auction with a reserve price of p.

open new markets in the price discrimination problem, as follows.

We introduce some convenient notation. Let $P(q) \equiv F^{-1}(1-q)$ be the inverse demand curve. Note that marginal revenue as a function of quantity, $\frac{d}{dq}(qP(q))$, denoted by $\tilde{MR}(q)$, is equal to $MR(P(q))$, the virtual valuation evaluated at $P(q)$. Also, denote the price elasticity of demand, $\frac{d\log q}{d\log p}$, by $\eta(p)$. A simple calculation shows that $\eta(p) = \lambda(p)p$. The price elasticity of demand evaluated at $P(q)$, $\eta(P(q))$, is denoted by $\tilde{\eta}(q)$. Finally, recall the relationship between marginal revenue and demand elasticity from classical price theory, $\tilde{MR}(q) = P(q)\left(1 - \frac{1}{\tilde{\eta}(q)}\right)$.

Suppose that both the planner and the seller are currently offering a share $Q < 1$ of their total capacity to one market. For clarity, we focus on Q such that $\tilde{MR}(Q) > 0$. Now, introduce a "market creation" technology: both the planner and the seller can open a new market into which they can shift capacity from the current market, but in the process they lose a share $1-\beta$ of those resources. (This is analogous to the way in which waiting for bidders in the auction timing problem incurs a multiplicative cost from time discounting).

Consider the incentives of the planner and the seller to reallocate a small amount of capacity dq from the current market to a new, previously uncatered market. We illustrate this thought experiment in Figure 1. The diagram on the left shows the original market, and the one on the right is the new, initially uncatered market. The blue areas show gain and losses from shifting capacity for the seller, while the sums of the blue and red areas show gains and losses for the planner. Re-writing $\tilde{MR}(q)$ as $P(q)\left(1 - \frac{1}{\tilde{\eta}(q)}\right)$, we have that

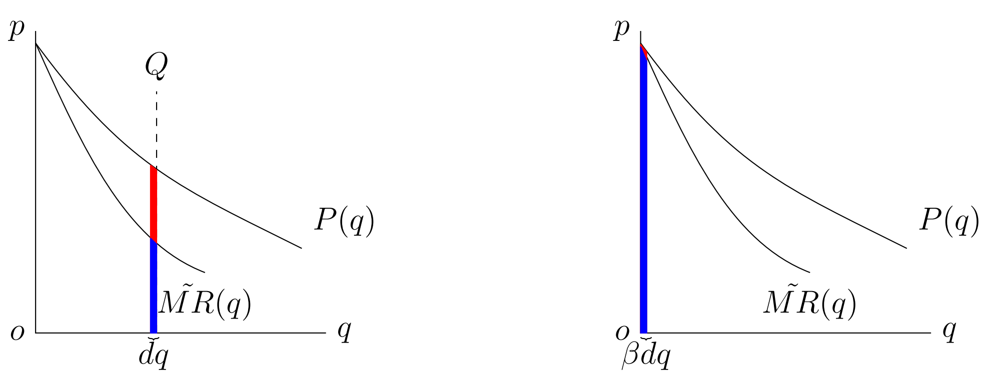

Figure 1: Shifting Consumers to a New Market

the seller earns revenue $\beta P(0)\left(1-\frac{1}{\tilde{\eta}(0)}\right)dq$ from the consumers who it now serves in the new market, but she loses revenue $P(Q)\left(1-\frac{1}{\tilde{\eta}(Q)}\right)dq$ in the current market. Meanwhile, the planner earns surplus $\beta P(0)dq$ from the new market, while losing $P(Q)dq$ in the current one.

Take first the case where price elasticity of demand $\eta(v)$ is increasing, so $\tilde{\eta}(q)$ is *decreasing*. We claim that, whenever the planner wants to move the dq units of capacity, the seller also wants to move that capacity, i.e., the seller has excessive incentives to open new markets. Indeed, if the planner wants to move the dq units, $P(Q) \leq \beta P(0)$. This, together with $\tilde{\eta}(q)$ decreasing, implies that

$$\frac{P(Q)}{P(0)} \leq \beta\left(\frac{1-\frac{1}{\tilde{\eta}(0)}}{1-\frac{1}{\tilde{\eta}(Q)}}\right)$$

so the seller wants to shift capacity into the new market. The same argument shows that, whenever $\tilde{\eta}(q)$ is increasing (price elasticity is decreasing), if the planner does not want to shift the dq consumers, *neither does the seller*. In other words, the seller has insufficient incentives to create new markets.

It remains to clarify why looking at Q's with $\tilde{MR}(Q) > 0$ is sufficient. Whenever $\tilde{MR}(Q) \leq 0$, the seller would always benefit from shifting some consumer mass to a new market, whereas the planner may not, i.e., the seller has excessive incentives to open new markets. One might therefore worry that this contradicts the reasoning given in the decreasing $\eta(v)$ case. However, as shown below in the proof of Proposition 2, for regular F with decreasing $\eta(v)$, $MR(\cdot)$ is always above 0. Thus, for regular F, $\tilde{MR}(Q) \leq 0$ is only ever relevant when $\eta(v)$ is increasing, in which case the fact that the seller faces excessive incentives to open new markets agrees with our discussion in the $\tilde{MR}(Q) > 0$ case.

Remark 3. It is interesting to compare our results with those of *bidder solicitation models*, such as Szech (2011) and Fang and Li (2015). These papers study static models where the auctioneer can choose the number of bidders to have in the auction. They assume that both the planner and the seller face an additive cost to solicit bidders that is increasing and weakly convex either in the number of bidders (Szech, 2011) or in the probability of a given bidder's attendance (Fang and Li, 2015). Similarly to ours, the auctioneer in these models decide how many bidders to have in the auction, comparing the cost of having one more bidder and his contribution to the revenue.

The main difference between our paper and this previous literature is that in our model not only do the seller and the planner face different marginal benefits from "soliciting" bidders, but they also incur different marginal costs. This follows from the fact that solicitation costs in our model are multiplicative. For example, suppose that bidders arrive according to a Poisson process with the constant rate of 1. Then, the planner's cost from delaying the auction by dt is $rS(n)dt$, whereas the seller's cost is $rR(n)dt$. This is in contrast to their papers where the seller and the planner face identical costs at the margin.

Szech (2011) and Fang and Li (2015) sign what we call the pure information rent effect: they show that, when the seller is forced to use an efficient auction format, she over- (under-) advertises relative to the planner whenever λ is increasing (decreasing). The intuition for this result follows from the expression for consumer surplus in an efficient auction, $CS(n) = \mathbb{E}[1/\lambda(v^{(n)})]$, where $v^{(n)}$ denotes the highest draw out of n samples from F. Since the seller and the planner both face the same additive costs, and $S(n) = R(n, 0) + CS(n)$, $CS(n) - CS(n-1)$ is exactly the wedge between the seller's and the planner's net incentives to add a bidder, starting from a pool size of $n - 1$. Hence, monotone λ fully determines whether the seller solicits the bidders more or less than the planner in these models, since it characterizes whether consumer surplus is increasing or decreasing in n.

However, these papers do not fully sign the net effect. They show that, when the seller can both solicit bidders and design the auction optimally, she always under-advertises if the hazard rate λ is decreasing (arguably the less common case), but the seller may over- or under-advertise for increasing λ. Therefore, unlike our right-skewness/monotone elasticity condition, the monotone hazard rate condition does not entirely characterize solicitation decisions, which in turn makes it harder to interpret the economics of monotone λ's. In particular, why does the above intuition about marginal consumer surplus fail once the seller adds a reserve price?[13] In contrast, in our model, the same condition allows us to

[13]It is interesting to note that, in Theorem 3 of Wang (1993), monotone λ *does* determines whether the seller stops later or earlier than the planner even when the seller can use the optimal auction. Wang

compare timing/solicitation choices both net and gross of auction format choices, which makes clear the economic interpretation of that condition.

Remark 4. To implement the optimal policy, the seller does not need to have as much commitment power as we assumed. As long as she can commit to sell the object through a one-shot static mechanism once she stops, the seller can implement the optimal policy without any additional commitment power for the timing rule.[14] This follows from standard dynamic programming arguments. The ability to commit to a static mechanism lets the seller implement the optimal static mechanism once she stops, which in turn pins down the expected payoffs from stopping at $N(t) = n$ (they must equal $R(n)$). Therefore, the seller faces a reduced-form optimal stopping problem, and the solution to that problem must be time-consistent by dynamic programming reasoning. So even if she lacks the ability to commit to a timing policy at the outset, the seller would have the right incentives to carry out the optimal stopping rule.

5 Departures

In this section, we extend the main result to the case where existing bidders can also leave the market. For instance, a bidder may leave if he finds an outside option better than taking part in the auction.

In general, when bidders can depart, the seller might stop earlier than the planner even if $\frac{f(v)}{1-F(v)}v$ is increasing. Recall that Theorem 1 in the previous section depends on the fact that, when $\frac{f(v)}{1-F(v)}v$ is increasing, the seller gains proportionally more than the planner from having an additional bidder. However, if a bidder can depart, this large gain from one more bidder necessarily implies a large *loss* from having one *fewer* bidder. Depending the arrival-

focuses studies a model with homogenous Poisson arrivals and linear waiting costs, and he focuses only on deterministic deadlines. The linear waiting costs mean that his model shares many features with the static solicitation models of Szech (2011) and Fang and Li (2015). He finds that the seller who uses an optimal auction sets an later (earlier) deadline than the planner if F satisfies IHR (DHR).

[14] Liu et al. (2013) studies how the lack of commitment to leave an object unsold changes the seller's behavior.

departure dynamics, the latter effect can dominate the former, so the seller may choose to hold the auction earlier than the planner. We illustrate this possibility in the following example.

Example 2. *Suppose that there are two periods $t \in \{1, 2\}$, v_i is uniformly distributed on $[1, 2]$, and there is no discounting. Also, suppose that both the planner and the seller use a second price auction with no reserve price. (Note that F here is not only regular but has strictly increasing $\lambda(v)v$, which would guarantee a later auction by the seller in the pure arrivals case). Denoting the number of bidders at time t by $N(t)$, define the following arrival and departure process: $N(1) = 2$, $\mathbf{P}(N(2) = 1) = 0.05$, $\mathbf{P}(N(2) = 2) = 0.7$, and $\mathbf{P}(N(2) = 3) = 0.25$. Under this parametrization, we can compute that, approximately, $R(N(1)) = 1.333$, $\mathbb{E}[R(N(2))] = 1.308$, $S(N(1)) = 1.667$, and $\mathbb{E}[S(N(2))] = 1.679$. Therefore, the planner waits until $t = 2$, while the seller strictly prefers to hold the auction at $t = 1$. In this case, the loss from departure is large enough to deter the seller from delaying the auction, even though she has a chance to hold the auction with three bidders if she waits.*

We now formally set up the auction timing problem incorporating bidders' departures. For ease of exposition, we focus on revenue maximization, but the exact same analysis applies to welfare maximization (substituting values for marginal revenues).

To avoid the problem described in Example 2, we impose some structure on the arrival-departure process $N(\cdot)$. First, we assume that bidders arrive according to a *renewal process*: the time between the arrival of the $k-1$-th and k-th bidders, $k = 1, 2, \dots$, is given by a non-negative random variable W_k, where $W_k \overset{iid}{\sim} G$. We assume that $\mathbb{E}[e^{-rW_i} | W_i \geq t]$ is well-defined and finite for any $t \geq 0$.

Second, we assume that each bidder can leave after a random time in the pool. We model this stochastic departure by assuming that each bidder i has a sojourn time D_i in the pool, so that if i arrives at T_i, he departs at calendar time $T_i + D_i$. For simplicity, we assume that sojourn times D_i are drawn iid from an exponential distribution independently of the all valuations and arrival times. We assume that the D_i's are unobservable, either by the bidders or by the seller, so bidder i cannot "report" her departure time. (Later, we discuss how the optimal policy will in fact be robust to bidders' misrepresentation of their departure times.) D_i, for instance, could represent the sudden arrival of a better outside option for a bidder, or, in the corporate takeover example, it could model the occurrence of an unexpected negative shock to liquidity that prevents the bidder from taking part in the auction. Given the assumption that i does not know D_i, our model would not fit the case where the bidder expects the negative liquidity shock in advance.

Given these assumptions on the arrivals and departures, the timing of all future arrivals is independent of the previous arrival history *other than the time of the most recent arrival.*[15] In particular, when D_i follows an exponential distribution, then because of the memoryless property, we do not need to keep track of the time each existing bidder spent in the pool.[16]

Now, let $\mathbb{L}(t)$ denote the time elapsed since the most recent arrival at calendar time t, so that the last arrival occurred at $t - \mathbb{L}(t)$. In addition, recall that the seller's expected terminal payoff from stopping with n bidders is $R(n)$. Then, since bidder values are *iid* and independent of the arrivals process, the seller faces a Markov optimal stopping problem with state variable $(N(t), \mathbb{L}(t))$. For future reference, let $V(n, w)$ denote the value function at state $(N(t), \mathbb{L}(t)) = (n, w)$, and use the short-hand $V(n) \equiv V(n, 0)$. Finally, let $\tau(n, n+1) \equiv \inf\{t \geq 0 : N(t) = n+1\} - \inf\{t \geq 0 : N(t) = n\}$ denote the random time between when the pool first reaches size n and when it first reaches $n+1$. (The possibility of departures makes the distribution of $\tau(n, n+1)$ different from G).

The following result explicitly derives the optimal timing of an auction assuming that the inter-arrival time distribution G has an *increasing hazard rate.* Under this assumption, the optimal rule is characterized by the cutoff with respect to the number of bidders.

Theorem 2. *Suppose that G has an increasing hazard rate (IHR) and D_i is independently*

[15] Suppose there have been n arrivals already at time t, with the most recent one happening at t^*. Then the history of arrivals excluding the most recent one is measurable with respect to $W_1, \ldots W_{n-1}$, whereas the timing of the next arrivals is given by $W_{n+1} + t^* - t, W_{n+1} + W_{n+2} + t^* - t$, and so on.

[16] If we consider a general distribution for D_i, then the seller needs to keep track of time that each current bidder has spent in the market, because it affects her beliefs about how likely that bidder is to leave in the next dt units of time, which in turn would affect the payoffs from stopping in the next dt. In this case, the seller has a large and growing state space in her optimal stopping problem, which makes further analysis intractable.

and identically drawn from an exponential distribution. Then starting from an empty pool at $t = 0$, the seller always holds the auction the first time the pool size reaches a critical threshold

$$n^* \equiv \inf \{n \in \mathbb{N} : R(n) \geq V(n, 0)\} . \tag{3}$$

In fact,

$$n^* = \inf \left\{ n \in \mathbb{N} : 1 \geq \beta(n+1) \frac{R(n+1)}{R(n)} \right\} \tag{4}$$

where $\beta(n+1) = \mathbb{E}[e^{-r\tau(n,n+1)}]$ is the (unconditional) expected discount factor between the first time the pool reaches size n and the first time it reaches size $n+1$.

Before providing the proof, let us give some intuition about the role of the IHR assumption. First, whenever G has an increasing hazard rate, the seller grows more optimistic about the next arrival the longer it has been since the last arrival; accordingly, the value of continued search increases in waiting time ($V(n, w)$ is increasing in w), so stopping search becomes less attractive as w grows. This implies that the seller only stops at the arrivals. In other words, the seller never holds an auction at a departure, or at a point of time where neither arrival nor departure occurred. Second, the two assumptions of renewal-process arrivals and exponential departures implies that, for each n, the evolution of $N(t)$ after first reaching a record n is independent of all prior history (other than the fact that n was reached). This memorylessness of sorts allows us to decompose the waiting time for n bidders into independent one-step increments between successive records, which in turn allows a one-step characterization of the optimal stopping threshold.

Proof of Theorem 2. To show that in any history starting from $(0, 0)$, the seller always stops at an arrival, note that, by Lemma 4 in Appendix B, $V(n, w)$ is non-decreasing in w. Now, suppose to the contrary that the seller is willing to stop in between arrivals. That is, there exists a state (n, w) with $w > 0$, such that $R(n) \geq V(n, w)$. Then, since $w \mapsto V(n, w)$ is non-decreasing, $R(n) \geq V(n, w) \geq V(n, 0)$, and the seller would also want to stop at the state $(n, 0)$. In any history starting from state $(0, 0)$, $(n, 0)$ is always reached before (n, w), so the seller would have stopped at $(n, 0)$ and never reached (n, w), which is a contradiction. Therefore, if starting from $(0, 0)$ the seller stops at all, she stops at an arrival. Accordingly, we need to check that the seller would indeed stop in finite time almost surely. But this is immediate because $R(1) > 0$ and the payoff from never stopping is 0. Altogether, we have that the auction will happen the first time $N(t)$ reaches an acceptable threshold, i.e., some n such that $R(n) \geq V(n, 0)$. Let n^* denote the smallest such n.

It remains to prove that in fact n^* is given by (4). The optimal stopping policy must be time consistent, so if the seller stops at n^*, then n^* must be the smallest maximizer of the ex ante expected discounted revenue from waiting for exactly n bidders. That is, n^* must be the smallest maximizer of $\delta(n)R(n)$, where $\delta(n)$ is the expected discount factor from waiting for n bidders. If $\delta(n)$ decomposes into $\prod_{i=1}^{n} \beta(i)$, then the condition that $R(n) \geq \beta(n+1)R(n+1)$ is simply a necessary local condition for optimization of $\delta(n)R(n)$, i.e., it implies that $\delta(n)R(n) \geq \delta(n+1)R(n+1)$. If we can show that indeed $\delta(n)R(n)$ is single-peaked in n, then this local condition is in fact sufficient, and we are done: $\inf\{n \in \mathbb{N} : R(n) \geq \beta(n+1)R(n+1)\}$ is then precisely the smallest maximizer of ex ante expected revenue. By Lemma 5 in Appendix B, our assumptions on bidder dynamics imply that the n-step discount (from waiting for n bidders) decomposes into $\delta(n) = \prod_{i=1}^{n} \beta(i)$. Moreover, a coupling argument (Lemma 6 in Appendix B) implies that the successive record times $\tau(n-1, n)$ increase in the first-order stochastic dominance as n grows, so $\beta(n) = \mathbb{E}[e^{-r\tau(n-1,n)}]$ is decreasing. Together with Point 3 of Fact 1, we have that $\beta(n+1)\frac{R(n+1)}{R(n)}$ is decreasing. Therefore $\prod_{i=1}^{n} \beta(i)R(n)$ is single-peaked, and we conclude the proof. $\square$

The one-step characterization of n^* in (4) has an especially tractable structure. The critical threshold function $\beta(n+1)\frac{R(n+1)}{R(n)}$ decouples into a term that depends only on the physical properties of the arrival-departure process ($\beta(n+1)$), and a term that depends only on the proportional gain from having an additional bidder in the pool ($\frac{R(n+1)}{R(n)}$).

This allows us to extend Propositions 1 and 2 to the case with departures:

Theorem 3. *Suppose that G has an increasing hazard rate (IHR) and D_i is independently and identically drawn from an exponential distribution. Then,*

- *The unconstrained seller holds the auction earlier than when she is statically constrained.*

- *If $\lambda(v)v$ is increasing (decreasing), both the statically constrained seller and the unconstrained seller hold the auction later (sooner) than the social planner.*

- *For two valuation distributions F and H, if F is more right-skewed, the statically constrained seller hodls the auction later when facing F than when facing H.*

- *For two valuation distributions F and H, if $p_F^* \geq p_H^*$ and $MR_F(x)/MR_H(x)$ is increasing whenever $x \geq p_F^*$, the unconstrained seller holds the auction later when facing F than when facing H.*

Proof. We show that the argument in Lemma 2 applies with this particular structure on $\{N(t), t \geq 0\}$. From the expression for the optimal stopping time in (4), we can see that designer A waits for a larger number of bidders than designer B if $\frac{S_A(n+1)}{S_A(n)} \geq \frac{S_B(n+1)}{S_B(n)}$ for all n, because any n that satisfies the inequality in the infimum for n_A^* will satisfy the inequality defining n_B^*. Hence, the designer facing a more right-skewed type distribution will stop later. The same argument as in Theorem 1, Proposition 1, and Proposition 2 establishes the result. □

Remark 5 (Strategic Arrivals and Departures)**.** So far, we have assumed that the arrival-departure process $N(\cdot)$ is exogenously given. However, this assumption might not always be innocuous. For instance, a sophisticated bidder in the corporate takeover auction may pretend that getting financial backing for a bid takes longer than is actually necessary, if it thinks that by delaying it can place the seller in a disadvantageous bargaining position. In fact, whether or not dynamic mechanisms are manipulable by strategic arrivals or departures has been a major focus of the literature on dynamic mechanism design.[17] For example, if later arrivals make the seller pessimistic about future arrivals, a late arrival by bidder i could lead the seller to "despair" and hold the auction sooner and possibly with fewer bidders than if bidder i had arrived earlier. Knowing this, if i's arrival time is not common knowledge, he could strategically delay his arrival (Gershkov et al., 2015).

Given the optimal auction timing of our model, will bidder i have strategic incentives to arrive later than a_i or depart earlier than $a_i + D_i$? We show that, under the optimal stopping

[17] For example, see Pai and Vohra (2013), Gershkov et al. (2014) and Gershkov et al. (2015).

time identified in Theorem 2, bidders will in fact want to show up (and depart) truthfully.[18] Note that even if bidder i strategically times his arrival and/or departure, $N'(t) \leq N(t)$ always holds. In particular, $N'(t)$ never hits n^* strictly earlier than $N(t)$. Thus, bidder i either (i) misses the opportunity to participate in the auction and obtains a payoff of zero, or (ii) joins an n^*-bidder auction that will be held later than if he would have if he had i truthfully reported his arrival and departure times. Bidder i does not benefit from the deviation in (i), because the payoff from the auction is non-negative. Nor does he benefit in the case (ii), because he must incur the cost of additional discounting without changing the number of opposing bidders he faces in the auction. Therefore, the cut-off policy in Theorem 2 is robust to bidders' strategic incentive to "misreport" the timing of their arrival and departure.

6 Impossibility of α-approximation

In the spirit of a large literature on approximation in mechanism design, one might hope that the seller could do reasonably well by restricting herself to simpler mechanisms.[19] In particular, following common practice in, say, real estate auctions, one could conjecture that holding the auction at a given date, fixed in advance, would perform well across a range of environments. Alas, the following result shows that the relative loss to the seller from restricting herself to holding the auction at a fixed date (even if the date is set optimally in advance, with full knowledge of the environment) is unbounded. In other words, no α-approximation of optimal revenue is possible: in an uncertain environment, the seller may obtain an arbitrarily small fraction of the revenue from the fully optimal policy.

The formal analysis proceeds as follows. First, we restrict the class of the auction timing problems. Namely, we consider any revenue-maximization problem such that the bidders

[18]Following the literature of dynamic mechanism design with strategic arrivals, we assume that even if one bidder strategically arrives later or depart earlier than the actual realizations, it does not affect the (actual) arrival times of other bidders.

[19] See, for instance, Hartline and Roughgarden (2009) and Hartline (2012) for approximation results in static auctions.

arrive according to a renewal process with the inter-arrival distribution G, and there is no departure (see Section 5 for the definition of a renewal process). Thus, any auction timing problem we consider is described by a triplet (r, F, G) of the discount factor r, the valuation distribution F, and the inter-arrival distribution G.

Now, given (r, F, G), let $OPT(r, F, G)$ be the ex-ante expected revenue from the optimal timing and auction format policy. Also, let $DET(r, F, G)$ be the maximum expected revenue when the seller is restricted to use a deterministic auction timing. That is, $DET(r, F, G)$ is the maximized value of the problem $\max_{t\in\mathbb{R}_+} e^{-rt}\mathbb{E}[R(N(t))]$. The following result states that, without further restriction on the class of auction timing problems, focusing only on the deterministic stopping rules can perform arbitrarily worse than the optimal history-contingent policy.

Theorem 4. *Fix any $r > 0$. Then,*

$$\inf_{F\in\mathscr{F},G\in\mathscr{G}} \frac{DET(r, F, G)}{OPT(r, F, G)} = 0, \tag{5}$$

where $\mathscr{F}$ is the set of regular distributions, and $\mathscr{G}$ is the set of probability distributions supported on $\mathbb{R}_{++}$.

The proof, in Appendix C, proceeds by explicitly constructing a sequence of examples that attains the infimum. G is supported on a finite grid of points, and along the sequence, all mass is moving to the right end of the grid. If we take a nearly degenerate valuation distribution (so that profits from an auction do not really increase by adding additional bidders beyond the first one), we can then choose grid points so that along the sequence, the optimal deterministic deadline is always to plan on holding the auction at the first time grid point, whereas the fully optimal policy is to wait for the first arrival. Therefore, as we move along the sequence, the best deterministic policy leaves the good unsold with probability approaching one, and achieves a payoff approaching zero. Meanwhile, the optimal "flexible" policy makes sure the good is sold by always waiting for the first bidder, guaranteeing a payoff that is always positive, but far in the future and shrinking along the sequence.

As we can see in the proof, along the sequence $\{(F_n, G_n)\}_{n=1}^{\infty}$, both $DET(r, F_n, G_n)$ and $OPT(r, F_n, G_n)$ converge to zero: that is, by using a fixed deadline, the seller ends up capturing only a vanishing fraction of a vanishing quantity. However, this does not mean that the seller's loss is necessarily small in absolute terms, because for each fixed n, scaling up v would make $OPT - DET$ arbitrarily large. Given that we can vary the loss in absolute terms by manipulating F, Theorem 4 should not be read as a quantitative statement about the size of losses incurred by deterministic deadlines. Rather, we interpret the result in two ways: unless we restrict the primitives of the timing problem, (i) deterministic deadlines are not an appropriate subclass of stopping times to focus on; and (ii), the gain to the seller from having access and being able to react to the history of arrivals can be arbitrarily large.

Our particular construction might give the impression that deterministic timing performs worse if inter-arrival times are more dispersed. This intuition is incorrect, and we can find examples in which a deadline "approximates" the optimal stopping policy even in the limit as G_n becomes arbitrarily dispersed. For example, if we restrict $\mathscr{G}$ to the collection of the uniform distributions $U[0, \bar{T}]$, then given a nearly degenerate F, we obtain $\frac{DET}{OPT} = 1/2$ as $\bar{T} \to \infty$. Analogously, taking $\mathscr{G}$ to be the collection of all exponential distributions leads to an approximation ratio of e^{-1}. Thus, dispersion is not the only source of our impossibility result. We speculate that, in addition to dispersion, extreme skewness in inter-arrival times is essential for fixed deadlines to perform poorly. We provide some confirmation of this fact in the next subsection: if random discounts have sufficiently thin right tails, then optimal deadlines provide a constant approximation of the best case revenue.

6.1 A Partial Approximation Result

Under strong assumptions on the nature of arrivals and the form of the optimal rule, we can obtain an $1/e$ approximation:

Proposition 3. *Fix any $r > 0$. Let $\mathscr{G}^{DIHR+}$ be the class of inter-arrival time distributions such that (i) for each $n \in \mathbb{N}$, the random discount $e^{-r(T_1+\cdots+T_n)}$ until n bidders is IHR, and (ii) there exists $n^* \in \mathbb{N}$ such that the fully optimal auction timing is to wait until n^*-th bidder arrives. Let $\mathscr{F}$ be the set of regular valuation distributions (not necessarily deterministic).*

Then

$$\inf_{F \in \mathscr{F}, G \in \mathscr{G}^{DIHR+}} \frac{DET(r,F,G)}{OPT(r,F,G)} \geq \frac{1}{e}. \tag{6}$$

At first glance, condition (ii) might seem like an assumption on the valuation distribution F. However, by Fact 1, the qualitative properties of terminal payoffs $R(n)$ are essentially identical for any regular F, so in the end the qualitative form of the stopping rule depends on the nature of the arrival process. For example, if G is "New Better than Used" Zuckerman (1984) ($T - s | T \geq s \geq_{FOSD} T$), a generalization of IHR, then an argument similar to the one in Theorem 2 shows that, for any F, the optimal rule stops at some $n^* \in \mathbb{N}$ (the specific value of n^* may depend on parameters like r and F).

The proof, in Appendix C, uses an analogy between our timing problem and the static problem of calculating the worst-case ratio of expected profit from an optimal posted price to expected welfare in an IPV model. Lemma 3.10 in Dhangwatnotai et al. (2015) provides a lower bound of this latter quantity, which we can translate into the current problem.

7 Conclusion

We have studied the question of when to hold an auction in a setting where bidders arrive and depart stochastically over time. We show that, for a very general class of arrival and departure processes, a social planner holds an auction later as the distribution of bidder values becomes more right-skewed. As a consequence, we also prove that sellers usually hold auctions inefficiently late, since the distribution of marginal revenues (censored at 0) is usually more right-skewed than the distribution of valuations. We identify a key condition on the distribution of bidder values that determines whether censored marginal revenues are more or less right-skewed than valuations: sellers wait too long if the distribution's *price elasticity of demand* is increasing, and too little if it is decreasing. Moreover, we decompose the welfare losses from revenue maximization into an information rent effect (conditional on using the same selling mechanism, the seller values stopping at each state according to virtual values as opposed to true values) and an auction format effect (the seller wants to use an inefficient selling mechanism upon stopping, but this feeds back into the decision about when to stop). The former leads the seller to hold the auction earlier or later depending on whether the elasticity of demand with respect to price is decreasing or increasing, while the latter always leads the seller to hold the auction earlier. Consequently, we show that even though the two effects can move the optimal timing in opposite directions, whenever we can unambiguously sign the information rent effect, it dominates. Finally, we prove that a fixed deadline, even an optimal one, cannot approximate the performance of the optimal stopping rule without strong restrictions on the arrival process. This suggests that the potential benefit from adopting a history-dependent timing rule may be substantial.

We conclude by highlighting some questions for future research. First, it would be useful to better understand what features of an arrival process make simple deadlines perform well compared to the optimal adaptive policy. What exactly are the elements of $\mathscr{G}^{DIHR+}$, and is there a natural sufficient condition that implies membership in this class?

Second, and more importantly, our discussion of the robustness of simple rules has been somewhat limited, since the rules we consider, while "simple", still depend on a lot of fine-grained knowledge about the environment. A key question would be whether one can design *prior-free* dynamic mechanisms with good guarantees, i.e., mechanisms whose very description does not depend on the prior distributions of values and arrival times. In a static environment, Bulow and Klemperer (1996) show that a simple second price auction with no reserve price gathers at least as much revenue as the optimal prior-dependent mechanism with one fewer bidder, a result that effectively puts an upper bound on the usefulness of information about the prior. Can we find guarantees like these for auction timing problems, or more generally for dynamic mechanism design with arriving agents?

Extending these results to dynamic environments is not merely a technical exercise. In fact, prior-free robustness results in our auction timing framework would be helpful in a broad range of stopping problems in economics. Consider, for instance, job search models generalizing the dynamic structure of Mortensen (1986); the optimal stopping rules in these models are often straightforward reservation wage policies (see Zuckerman (1984, 1986, 1988)), but the thresholds depend on fine features of the distributions of wage offers and arrival times between offers that a) would be difficult for agents to verify from limited data, and b) could in fact be violated in general equilibrium. Can workers, say, devise robust search rules that have good guarantees and do not depend on aspects of the environment that might be hard for them to verify? Altogether, the question of timing in mechanism design appears to be a fruitful direction for research.

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Appendix

A Proofs for Section 4

The proof of Lemma 1 depends on a useful result by Barlow and Proschan (1966), which we reproduce here since it is of independent interest and not well known in Economics[20]:

Lemma 3 (From Lemma 3.5 in (Barlow and Proschan, 1966))**.** *Let $X^{(1:n)} \leq X^{(2:n)} \cdots \leq X^{(n:n)}$ be an ordered iid sample from a continuous distribution. Suppose a function $\gamma : \mathbb{R} \to \mathbb{R}$ changes sign k times, and that the expectation of $\gamma(X^{(i:n)})$ exists. Then $\pi(n-i,n) = \mathbb{E}[\gamma(X^{(n-i:n)})]$ changes sign at most k times as a function of n; if $\pi(n-i,n)$ actually does change sign in n exactly k times, then the changes occur in the same order as those of $\gamma(x)$.*

Proof of Lemma 1. The proof is a slight generalization of Theorem 3.6 in Barlow and Proschan (1966). First, note that $\frac{h(z)}{g(z)}$ is increasing if and only if, for any $c > 0$, $h(z) - cg(z)$ changes sign at most once and from negative to positive as z increases. Then, using Lemma 3 with $\gamma(z) = h(z) - cg(z)$, this implies that

$$\mathbb{E}[h(Z^{(n-k:n)})] - c\mathbb{E}[g(Z^{(n-k:n)})] = S_A(n-k;n) - cS_B(n-k;n) = S_B(n-k;n)] \Big(\frac{S_A(n-k;n)}{S_B(n-k;n)} - c \Big)$$

changes sign at most once and from negative to positive, as a function of n (the first equality uses the fact that h and g are non-decreasing). Because $S_B(n-k;n) > 0$ and this holds for any $c > 0$, we conclude that $S_A(n-k;n)/S_B(n-k;n)$ is increasing in n. □

Remark 6. Note that the proof of Lemma 1 goes through unchanged if

1. h is allowed to take negative values, but h/g remains increasing; or

2. h and g are non-negative, but can both equal zero simultaneously (so h/g is undefined on parts of Z's support).

so long as $h(z) - cg(z)$ has the single-crossing property for all $c > 0$.

[20] In the economics literature, the variation diminishing property of totally positive kernels, on which this Lemma depends, is used, e.g., in Jewitt (1987) and Bulow and Klemperer (2009).

Proof of Proposition 2. First, suppose that $\lambda(v)v$ is increasing. Then, $\left(v - \frac{1}{\lambda(v)}\right) v^{-1} = 1 - (\lambda(v)v)^{-1}$ is increasing. Now, applying the generalization of Lemma 1 to (possibly negative) $X_i^A = MR(v_i)$ and $X_i^B = v_i$ by the same argument as Theorem 1, and the seller stops later than the planner if both of them use the efficient auction.

Next, we show that the seller with the optimal auction stops earlier than the seller with the efficient auction. For any sample of bidder values $v_1, \dots, v_n$, let $X_i \equiv MR(v_i)$, and recall that MR is increasing, so $MR(v^{(n:n)}) = X^{(n:n)}$. Also, let $\tilde{X}$ be an additional draw of marginal revenue that is independent from the original sample $X_1, \dots, X_n$. Define the function $g_p(x) = \mathbb{E}[\max\{\tilde{X}, x, p\}]$, which is non-negative for any $p \in \mathbb{R} \cup \{\infty\}$. For $n \geq 2$ bidders, We can re-write expected marginal revenue, censored at p, as

$$\mathbb{E}\left[\mathbb{E}\left[\max\{X^{(n-1:n-1)}, \tilde{X}, p\})\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\max\{X^{(n-1:n-1)}, x, p\} | \tilde{X} = x\right]\right] = \mathbb{E}[g_p(X^{(n-1:n-1)})] \tag{7}$$

Since g_p is increasing for all $p \in \mathbb{R} \cup \{-\infty\}$, for any $n \geq 2$, the seller who uses an optimal auction has the same payoffs as a social planner facing a distribution of types $Y_i \sim g_0(X_i)$, while the seller who uses a second price auction with no reserve has the same payoffs as a social planner facing a distribution of types $Z_i \sim g_{-\infty}(X_i)$. For any p', p such that $\infty > p' > p \geq -\infty$, $g_{p'}(x)/g_p(x)$ is increasing, so $g_{p'}(X_i)$ is more right-skewed than $g_p(X_i)$, and in particular, for $n \geq 2$ the seller faces a more right-skewed distribution of payoffs under the optimal auction than under a second price auction with no reserve price.[21] Therefore, by Lemma 2 the seller stops earlier when she can use an optimal auction.

Next, suppose that $\lambda(v)v$ is decreasing. By the symmetric argument, the planner stops later than the seller if $\lambda(v)v$ is decreasing.

Finally, to prove that the optimal auction coincides with the efficient auction, we show that $v - \frac{1-F(v)}{f(v)} \geq 0$ for any v. Suppose to the contrary that $v - \frac{1-F(v)}{f(v)} = v(1 - \frac{1}{\lambda(v)v}) < 0$ for some v. Then, since $v \geq 0$, $1 - \frac{1}{\lambda(v)v} < 0$ holds. Because $\lambda(v)v$ is decreasing, we obtain

$$v'\left(1 - \frac{1}{\lambda(v')v'}\right) < v\left(1 - \frac{1}{\lambda(v)v}\right).$$

[21] Indeed, $\frac{g_{p'}(x)}{g_p(x)}$ is increasing on $(-\infty, p')$, because only $g_{p'}(x)$ increases in this region. Also, on $(p', +\infty)$, $\frac{g_{p'}(x)}{g_p(x)}$ is equal to 1, so it is trivially increasing.

if $v' > v$. This contradicts the regularity of F. $\square$

B Proofs for Section 5

In this section we collect intermediate results necessary in the proof of Theorem 2. The first key step is establishing a monotonicity of the auctioneer's value function in waiting time:

Lemma 4. *Suppose that inter-arrival times W have an increasing hazard rate, and sojourn times D are iid exponential. Then for any n, $V(n, \cdot)$ is non-decreasing.*

Proof of Lemma 4. For any $\Delta \geq 0$, define $W_\Delta := W - \Delta | \{W \geq \Delta\}$. Take any $a, b \in \mathbb{R}_+$ such that $a > b$. By the IHR assumption, $W_b \geq_{FOSD} W_a$. Assume that W_b and W_a are defined on the same probability space and $W_b \geq W_a$ almost surely.

We consider two stochastic processes defined on the same probability space. First, define N_a as follows.

- At $t = 0$, initialize $N_a(0) = n$. Also, start n independent exponential departure clocks simultaneously.
- If arrival clock W_a "ticks," increase N_a by 1 and draw a new arrival clock $W_1 \sim G$ and an exponential departure clock D_1.
- For each $k \geq 1$, if arrival clock W_k "ticks," increase N_a by 1 and draw a new arrival clock $W_{k+1} \sim G$ and an exponential departure clock D_{k+1}.
- If any departure clock ticks, decrease N_a by 1.

Second, define N_b in the same way as N_a except we use W_b instead of W_a for the first arrival clock. Note that the values from optimal policies given N_a and N_b are $V(n, a)$ and $V(n, b)$, respectively.

We prove the result by induction on n. First, as $W_b \geq W_a$ for sure, we get

$$V(0, a) = \mathbb{E}[e^{-rW_a}]V(1, 0) \geq \mathbb{E}[e^{-rW_b}]V(1, 0) = V(0, b). \tag{8}$$

Second, suppose that $V(k, \cdot)$ is non-decreasing for $k = 0, \ldots, n-1$. We call the sellers who confront N_a and N_b as Sellers a and b, respectively. Correspondingly, we call the optimal policies of Sellers a and b as Policies a and b, respectively. Suppose that Seller b takes Policy b, which yields $V(n,b)$. Consider the following timing policy of Seller a.

1. As long as no bidders arrive or depart, adopt Policy b.

2. Once some bidder arrives or departs before holding an auction (while imitating Seller b), from then on, adopt Policy a.

I show that this policy gives Seller a a greater revenue than $V(n,b)$ for any realizations of the stochastic processes. For Point 1, if Seller a holds an auction (say, at time t) while imitating Seller b, then both of them obtain a discounted revenue of $e^{-rt}R(n)$ because $W_b \geq W_a$. For Point 2, there are two cases. If Seller a switches to Policy a at t because a bidder departs, then time-t continuation payoffs of Seller a and b are $V(n-1,a+t)$ and $V(n-1,b+t)$, respectively. By the inductive hypothesis, $V(n-1,a+t) \geq V(n-1,b+t)$. Finally, if Point 2 holds because a new bidder arrives at time W_a, then time-W_a continuation payoff of Seller a is $V(n+1,0)$ and that of Seller b is $\max\{V(n,b+W_a),V(n+1,0)\} \leq V(n+1,0)$.[22] □

As shown in the main text, Lemma 4 implies that, when starting from state $(0,0)$, the optimal policy stops the first time the bidder pool reaches n^*. To complete the proof of Theorem 2, it remains to prove the auxiliary results used in the one-step characterization of n^* in (4). Recall that $\beta(j) \equiv \mathbb{E}[e^{-r\tau(j-1,j)}]$, where $\tau(j-1,j)$ is the random time elapsed between when the pool size first reaches $j-1$ to when it first reaches j. Then we have the following payoff decomposition:

Lemma 5. *The seller's expected revenue at time* 0 *from holding the auction upon the* n*-th bidder's arrival is given by* $\prod_{j=1}^{n}\beta(j)R(n)$.

Proof. Let $\tau_n = \inf\{t : N(t) = n\}$. Then the expected revenue from stopping at the n-th bidder's arrival becomes

$$\mathbb{E}[e^{-r\tau_n}]R(n) = \mathbb{E}[e^{-r\sum_{j=1}^{n}\tau(j-1,j)}]R(n). \tag{9}$$

[22] $V(n+1,0) \geq V(n,t)$ holds for any t for the following reason: Starting from (n,t), let p be the probability that the state reaches $(n+1,0)$ before the seller holds an auction when she follows the optimal policy. With probability $1-p$, the seller holds an auction with at most n bidders. Taking into account discounting, we get $V(n,t) \leq pV(n+1,0) + (1-p)R(n) \leq V(n+1,0)$.

Let $\{N(t), t \geq 0\}$ denote the true bidder pool process. To calculate (9), construct the following "fictitious" process $\{N'(t), t \geq 0\}$:

1. Whenever N increases,
   - Add another Poisson clock to N', remove all N' existing Poisson clocks and replace them with N' new independent Poisson clocks.
2. Whenever N decreases,
   - Remove all N' existing Poisson clocks from N' and replace them with $N'-1$ new independent Poisson clocks.

By the memoryless property of the Poisson clocks, N and N' will have the same marginal distributions, so letting $\tau'(j-1,j)$ denote the successive record times for N', $\tau'(j-1,j) \sim \tau(j-1,j)$ and $e^{-r\sum_{j=1}^{n}\tau(j-1,j)} \sim e^{-r\sum_{j=1}^{n}\tau'(j-1,j)}$. In addition, N''s first-increment times $\{\tau(k-1,k)\}_{k\in\mathbb{N}}$ will be mutually independent, since the increasing part of N is a renewal, and by the way we construct N' (replacing all "old" clocks with fresh independent ones at each change of N), all the dependence in between successive records of N' has been removed. Therefore

$$\mathbb{E}[e^{-r\sum_{j=1}^{n}\tau(j-1,j)}] = \mathbb{E}[e^{-r\sum_{j=1}^{n}\tau'(j-1,j)}] = \Pi_{j=1}^{n}\mathbb{E}[e^{-r\tau'(j-1,j)}] = \Pi_{j=1}^{n}\beta(j),$$

as required. □

Finally, $\beta(\cdot)$ is decreasing, which estabishes that $\prod_{j=1}^{n}\beta(j)R(n)$ is single-peaked and yields the one-step characterization of n^*:

Lemma 6. *Let $\tau(n-1,n)$ be the time between when the bidder pool first reaches $n-1$ bidders and when it first reaches n bidders. For any $n \in \mathbb{N}$, $\tau(n,n+1) \geq_{FOSD} \tau(n-1,n)$. Thus, $\beta(j) = \mathbb{E}[e^{-r\tau(j-1,j)}]$ is decreasing in j.*

Proof. The proof is by coupling. At $t = 0$, start n independent exponential departure clocks simultaneously. Label one of these clocks "first." Independently of these clocks, draw arrival clocks sequentially: first draw $W_1 \sim G$ at $t = 0$, then when the first clock ticks ($t = W_1$)

draw $W_2 \sim G$ independently at $t = W_1$, and so on. Add an additional independent departure clock every time a new arrival clock $W_j, j \geq 2$ is drawn. Then define two stochastic processes $M(t)$ and $M'(t)$ on this space such that

- At $t = 0$, initialize $M(t)$ at n and $M'(t)$ at $n - 1$
- If an arrival clock W_j "ticks," increment M and M' by 1.
- If any departure clock ticks, decrease M by 1.
- If any departure clock other than the first one in the original set ticks, decrease M' by 1.

Then, by the renewals and exponential departures assumptions, $M(t)$ is distributed as $N(t)$ started from state $(n, 0)$, while $M'(t)$ is distributed as $N(t)$ started from state $(n - 1, 0)$. Therefore, the time M first crosses $n + 1$, denoted σ_{n+1}, has the same marginal distribution as $\tau(n, n + 1)$. Similarly, the time M' first crosses n, denoted σ_n, has the same marginal distribution as $\tau(n - 1, n)$.

We claim $\sigma_{n+1} \geq \sigma_n$ almost surely. First, if M and M' have not coupled at time $\sigma_{n+1} \wedge \sigma_n$, since $M(t) = M'(t) + 1$ until coupling time, M will cross $n + 1$ at time σ_n. Second, suppose M and M' couple before time $\sigma_n \wedge \sigma_{n+1}$ (so they meet at some state $k < n - 1$). Then $\sigma_{n+1} > \sigma_n$, since $M = M'$ after coupling, and M must reach n before reaching $n + 1$. Therefore, we conclude that $\tau(n, n + 1) \geq_{FOSD} \tau(n - 1, n)$ by the usual argument, and the result follows. $\square$

C Proofs for Section 6

Proof of Theorem 4. Suppose that $r_n = r$, $F_n = F$ for any n, where F is a degenerate distribution. The degeneracy of F is without loss of generality, because we can approximate any such F by a sequence of non-degenerate, continuous, and regular distributions, and the same logic used here will apply along that sequence. Throughout the proof, r and F are fixed.

Now, consider the following inter-arrival distribution G_n. Fix any positive integer M. Let the support of G_n be $\{t_1, \ldots, t_M\}$ for any n. Let p_k denote the probability that G_n puts on t_k for $k = 1, \ldots, M$. (We suppress the dependence of p_j on n for notational simplicity.) Then define each p_i inductively as follows.

$$\begin{aligned}
p_1 &= \frac{1}{n} \\
p_k &= \sqrt{p_{k-1}} - p_1 - p_2 - \cdots - p_{k-1}, \\
p_M &= 1 - p_1 - \cdots - p_{M-1}.
\end{aligned}$$

There are two remarks. First, for large n, $p_k \geq 0$ for each k, because the first term $\sqrt{p_{k-1}}$ converges to zero strictly slower than the other terms $-p_1 - p_2 - \cdots - p_{k-1}$. Second, note that $\sum_{j=1}^{k} p_j = \sqrt{p_{k-1}}$ holds for each $k < M$.

Now, suppose that the seller, who has to choose the deterministic timing, holds the auction at t_1 (later we show that we can construct $t_1, \ldots, t_M$ so that choosing t_1 is indeed optimal). Let $\alpha_k := e^{-rt_k}$ for each $k = 1, \ldots, M$. Then it holds that

$$\begin{aligned}
\frac{OPT}{DET} &= \frac{\sum_{i=1}^{M} p_i \alpha_i}{\max_{1 \leq k \leq M} \sum_{i=1}^{k} p_i \alpha_k} \\
&= \frac{\sum_{i=1}^{M} p_i \alpha_i}{\alpha_1 p_1} \\
&= 1 + \sum_{j=2}^{M} \frac{p_j}{p_1} \cdot \frac{\alpha_j}{\alpha_1}. \qquad (10)
\end{aligned}$$

For $t = t_1$ to be optimal, it is enough that for $j = 1, \ldots, M$,

$$p_1 \alpha_1 \geq \alpha_j \sum_{i=1}^{j} p_i \iff \frac{p_1}{\sum_{i=1}^{j} p_i} \geq \frac{\alpha_j}{\alpha_1}.$$

Now, since $\frac{p_1}{\sum_{i=1}^{j} p_i}$ is less than 1 and is strictly decreasing in j, we can choose $0 < t_1 < t_2 \cdots < t_M$ so that $\alpha_k = e^{-rt_k}$ for $k = 1, \ldots, M$ makes the above inequalities binding.

Combining this (in)equality and equation (10), we obtain

$$\frac{OPT}{DET} = 1 + \sum_{k=2}^{M} \frac{p_k}{\sum_{i=1}^{k} p_i}. \qquad (11)$$

Now, we show that as $n \to +\infty$, the RHS of (11) converges to M. Indeed, each summand in the RHS can be written as

$$\frac{p_k}{\sum_{i=1}^k p_i} = \frac{p_k}{\sqrt{p_{k-1}}} = 1 - \sum_{j=1}^{k-1} \frac{p_j}{\sqrt{p_{k-1}}} \to 1 \quad \text{as} \quad n \to +\infty. \tag{12}$$

The last convergence follows from the fact that p_j is is of order $n^{-\frac{1}{2j}}$ by construction. Equality (11) and (12) imply that

$$\frac{DET}{OPT} \to \frac{1}{M} \quad \text{as} \quad n \to +\infty.$$

That is, we can choose a sequence of G_n so that $\frac{DET}{OPT}$ converges to $1/M$. Because M is arbitrary, there exists $\{F_n, G_n\}_{n=1}^{\infty}$ such that $\frac{DET(r,F_n,G_n)}{OPT(r,F_n,G_n)} \to 0$. (We can construct such a sequence explicitly by a diagonalization argument.) $\square$

Proof of Proposition 3. Let n^* be the threshold number of bidders characterizing the fully optimal auction timing (this could possibly depend on G and F). Let DET and OPT denote, respectively, the best ex ante payoffs from using only deadlines from using any timing policy adapted to arrivals. Consider an auxiliary auction timing problem where a single bidder, whose valuation is deterministically equal to $R(n^*)$, arrives at the random time $\tilde{T} = T_1 + \cdots + T_{n^*}$, and no other bidders arrive before or after. Let DET_{aux} and OPT_{aux} denote the best ex ante payoffs from only deadlines and from any policy in this auxiliary single-bidder problem. If the seller in the auxiliary problem can use any stopping time, she stops at $\tilde{T}$ and posts a price equal to $R(n^*)$. This gives her an ex-ante payoff of $OPT_{aux} = \mathbb{E}[e^{-r\tilde{T}}]R(n^*)$, which matches OPT from the original many-bidder problem by our assumption on the optimal policy.

Now consider also a static monopoly pricing problem where with bidder valuation given by $v = e^{-r\tilde{T}}$. Given a price $p \in (0, 1]$, the probability of acceptance is $1 - H(p) = \mathbf{P}(v \geq p) = \mathbf{P}(\tilde{T} \leq -r^{-1} \log p) = G(-r^{-1} \log p)$. Note that for any CDF G, $H(x) = 1 - G(-r^{-1} \log x)$ is a well-defined CDF. In this notation, $DET_{aux} = \max_{t \geq 0} e^{-rt} G(t) = \max_{p \geq 0} p(1 - H(p))$, which equals the expected profits from a posted price p in the static pricing problem, while $OPT_{aux} = \mathbb{E}_G[e^{-r\tilde{T}}] = \mathbb{E}_H[v]$ equals the expected feasible surplus in the static pricing problem. Thus,[23], $\frac{DET_{aux}}{OPT_{aux}} = \frac{\max_{p\in[0,1]} p(1-H(p))}{\mathbb{E}_H[v]}$ is the ratio of expected profits to total welfare in a static pricing problem.

From Lemma 3.10 in Dhangwatnotai et al. (2015) (using $t = 0$ in their notation) we have that for any IHR H, a static pricing problem satisfies

$$\max_{p\geq 0}\{p(1-H(p))\} \geq \frac{1}{e}\mathbb{E}_{v\sim H}[v], \tag{13}$$

. $G \in \mathscr{G}^{DIHR+}$ by assumption, so $e^{-r\tilde{T}}$ will satisfy IHR. Hence, by (13), $\frac{DET_{aux}}{OPT_{aux}} \geq \frac{1}{e}$.

By the definition of n^*, OPT_{aux} matches OPT in the original problem, while DET_{aux} is a crude lower bound for DET. Indeed, $DET_{aux} = e^{-rt}\mathbf{P}(\tilde{T} \leq t)R(n^*)$ is a lower bound for DET on the event $\{\tilde{T} < t\}$ (since there could be more than n^* bidders in that case), and the contribution to DET from the event $\{\tilde{T} < t\}$ is positive. Therefore $DET \geq DET_{aux} \geq \frac{1}{e}OPT_{aux} = \frac{1}{e}OPT$, as required. □

D Proof of Fact 1

Proof of Fact 1. Point 1 is obvious for $S(n)$, since it equals $\mathbb{E}[v^{(n)}]$, where $v^{(n)}$ denotes the highest draw out of n samples from F. To see that $\mathbb{E}(\max\{MR(v^{(n)}), p\})$ is non-negative, note that $\max\{MR(v^{(n)}), p\} \geq MR(v_1)$ and $E[MR(v_1)] = 0$. To show Point 2, take subsets S, T of the set of bidders B with $S \supseteq T$, and a bidder $i \in B \setminus S$. Then, we claim

$$\max_{j\in S\cup\{i\}} v_j - \max_{j\in S} v_j \leq \max_{j\in T\cup\{i\}} v_j - \max_{j\in T} v_j \tag{14}$$

Indeed, when $v_i = \max_{j\in S\cup\{i\}} v_j$, the comparison becomes $-\max_{j\in S} v_j \leq -\max_{j\in T} v_j$, which holds by the assumption that $T \subset S$; when $v_i < \max_{j\in S\cup\{i\}}$, the left hand side is 0, and the right hand side is non-negative. Taking expectations on both sides of (14) delivers the concavity of $S(n)$.[24] The result extends to $R(n, 0)$ and $R(n, p^*)$ in the regular case by

[23] Note that the support of H is a subset of $[0, 1]$ by construction. However, for any H with the bounded support, we can always rescale the support of H without changing $\frac{\max_{p\in[0,1]} p(1-H(p))}{\mathbb{E}_H[v]}$.

[24] The proof strategy follows Corollary 2.7. in Dughmi et al. (2009). Dughmi et al. (2009) give an example of a Vickrey auction with sub-optimal reserve price where expected revenue fails to be sub-modular/concave. Upon inspection, their construction relies on having graphical matroid feasibility constraints. As the argument above shows, expected revenue from a single-unit second price auction is concave in n *for any reserve price*.

substituting (possibly ironed) marginal revenues $MR(v_j)$ or $MR(v_j) \vee 0$, respectively, for v_j. [25] To show Point 3, note that diminishing returns of $n \mapsto R(n)$ imply

$$R(n+2)\left(1 - \frac{R(n+1)}{R(n+2)}\right) \leq R(n+1)\left(1 - \frac{R(n)}{R(n+1)}\right) \tag{15}$$

By revenue monotonicity, the terms in the parentheses are non-negative, so for (15) to hold even though $R(n+2) \geq R(n+1)$, $\frac{R(n+1)}{R(n)}$ must be decreasing, as required. An identical argument applies for $R(n,0)$ and $S(n)$. □

E Implementability under Unobservable Arrival and Departure

Whether or not dynamic mechanisms are manipulable by strategic arrivals or departures has been a major focus of the literature on dynamic mechanism design.[26] We have discussed above why the timing policy we derive is robust to the strategic arrivals, but our arguments used specific features of the optimal policy, which we only knew ex post. In many cases, however, the explicit form of the optimal or efficient policy is unknown (e.g., it can be computed numerically, or only qualitative features are known without further assumptions), but one still needs to determine whether the policy is vulnerable to manipulation by strategic arrivals. This raises the question: are there general features of the environment such that the optimal policy under observable arrivals is always implementable?

We give a partial answer here that nonetheless covers a broad range of problems: in the pure arrivals case, bidders will want to report arrival times truthfully under the designer's optimal stopping time for *any* renewal process $\{N(t)\}_{t\geq 0}$. The proof strategy, based on similar arguments in Gershkov et al. (2015), extends the logic used in Remark 5.

Theorem 5. *Suppose that there are no departures, and the arrival process is a renewal process. Then the optimal auction timing under observable arrivals is implementable under unobservable arrivals.*

Proof of Theorem 5. The proof strategy is similar to Proposition 3 in Gershkov et al. (2014). Without loss of generality, the seller can look at Markov policies, i.e., policies that at each

[25]We sometimes use the shorthand $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$, for notational simplicity.

[26]For example, see Pai and Vohra (2013), Gershkov et al. (2014) and Gershkov et al. (2015).

time T condition only on the number of bidders $N(T)$ and the time since the latest arrival, $\mathbb{L}(T)$. Consider a bidder i with true type $(v_i, a_i) = (v, t)$ who won the auction at time T^* by reporting value v' and arrival time $t' > t$. By discounting, we know that the time T^* at which the auction is held satisfies

$$T^* = \inf\{s : R(N(s)) \geq V(N(s), \mathbb{L}(s))\}$$

where $V(\cdot, \cdot)$ is the value from continued search at the given Markov state. For any value of $N(T^*)$, $\mathbb{L}(T^*)$ is therefore a *pool-size-specific critical waiting period*, such that whenever there are $N(T^*)$ bidders in the pool for a period $\mathbb{L}(T^*)$ with no other arrivals, an auction is triggered.

We claim that bidder i would also win the auction, at a time T^* or sooner and with weakly lower expected payment, by reporting (v', t'') with $t'' \in [t, t')$. Note first that the new report cannot possibly delay the auction past T^*. The new arrival time t' will trigger an auction after a new critical waiting period corresponding to a pool size that might differ from $N(T^*)$. But if the new auction time corresponding to the (possibly new) pool size were later than T^*, the auction would be triggered at T^* once again: since i's new arrival time is at most $t' \leq T - \mathbb{L}(T^*)$, there would be $N(T^*)$ bidders in the pool for a waiting time of $\mathbb{L}(T^*)$. Hence, the new report could either (i) trigger the auction after a new critical waiting period for a strictly lower pool size $n < N(T^*)$ elapses; (ii) trigger an auction after the same critical wait period $\mathbb{L}(T^*)$ for pool size $N(T^*)$ elapses, but starting at a possibly different time; or (iii), not change outcomes. i is strictly better off in case (i), since her expected payments are smaller and the auction happens weakly earlier than before the deviation. Similarly, in case (ii), the auction must be happening strictly earlier than before; expected payments are unchanged, but by discounting, i is strictly better off. □