

Dynamic Privacy Choices

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Abstract

I study a dynamic model of consumer privacy and platform data collection. Consumers choose how actively to use a platform in each period. The activity generates data on consumers, which benefits the platform but hurts consumers. In the long-run, consumers lose privacy and obtain a low payoff, but choose a high activity level. To induce this outcome, the platform adopts a privacy policy that gradually expands the scope of data collection. Platform competition is less likely when consumers are obtaining lower payoffs from an incumbent. Regulating data collection can lower consumer and total welfare in the long-run.

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1 Introduction

Online platforms, such as Amazon, Facebook, Google, and Uber, analyze the activities of consumers and collect a large amount of data. While this data collection may improve their services and benefit consumers, it raises concerns among consumers and policymakers (Cr  mer et al., 2019; Furman et al., 2019; Morton et al., 2019).

As an example, consider a consumer (she) and a social media platform (it). The consumer browses posts and news on the platform, and derives some utility. The platform analyzes her profile and browsing behavior, and collects data such as her race, location, and political preferences. If the consumer uses the platform actively, it can collect more data and earn higher revenue, by, say, improving targeted advertising. The consumer faces a trade-off: On the one hand, she enjoys reading posts and connecting with friends on the platform. On the other hand, the consumer may value her privacy or be concerned about the risk of data leakage, identity theft, and price or non-price discrimination based on collected data.¹ This is the “privacy cost” for the consumer of using the platform. If the consumer anticipates a high privacy cost, then she uses the platform less actively. The platform can influence this trade-off through its privacy policy. For example, it may commit to not use cookies to track consumers, as Facebook did in 2004.²

I formalize this example in a dynamic model. In each period, a consumer chooses her activity level. The activity generates a signal about her time-invariant type X , and a higher activity level generates a more informative signal. The platform’s payoff in each period is increasing in the amount of information it has about X . The consumer enjoys a benefit from the service but incurs a privacy cost, which is increasing in the amount of collected information. The platform designs a privacy policy, which specifies the amount of noise to add to the signal in each period. The amount of noise reflects the privacy level of the platform. A higher privacy level lowers the consumer’s marginal privacy cost and increases her activity level.

The key economic mechanism is that, if the consumer has less privacy, then she has even less incentive to protect her privacy *at the margin*. For example, if Google or Amazon already knows a lot about a consumer, she might not care about letting Google maps track her location or

¹Such concerns are highlighted by, for example, the *Cambridge Analytica* scandal.

²In 2004, Facebook’s privacy policy stated that “we do not and will not use cookies to collect private information from any user.” <https://web.archive.org/web/20050107221705/http://www.thefacebook.com/policy.php>

putting Alexa in the living room. In an extreme case, if the platform knows everything, then the marginal privacy cost is zero, because the consumer's activity no longer affects what information the platform has. Thus, data collection may lower consumer welfare, but increases their marginal incentive to use the platform.

In equilibrium, the consumer chooses higher activity levels but receives lower payoffs over time. In the long-run, the consumer loses privacy and incurs a high privacy cost that can more than offset the benefit from the service; however, the consumer acts as if there is no privacy cost. This occurs even if the consumer values privacy arbitrarily highly. This is because the platform designs its privacy policy to exploit her incentive: It offers high privacy levels in early periods to encourage consumer activity, but offers low privacy levels in the long-run to accelerate data collection.

Platform competition is unlikely to mitigate the problem, because data collected by an incumbent becomes an entry barrier. If there are multiple platforms, the consumer chooses which platform to use partly based on her marginal privacy cost. If an incumbent (e.g. Google) has a lot of data on the consumer, it reduces her marginal cost of using the incumbent. This makes switching to a new player (e.g. DuckDuckGo) less likely. The result offers a novel antitrust implication: Market entry is less likely when consumers are suffering from privacy loss and obtaining lower welfare from an incumbent.

I explore various privacy regulations. For example, I show that mandating the platform to adopt a stricter privacy policy may lower consumer welfare and efficiency in the long-run. In contrast, the "right to be forgotten," which enables the consumer to erase past information, may enhance consumer welfare and induce market entry.

The paper has implications on consumer privacy and data collection by digital platforms. First, the consumer's long-run behavior seems consistent with the so-called privacy paradox: Consumers express concerns about data collection, but many of them actively share data with third parties ([Acquisti et al., 2016](#)). Second, the platform's equilibrium strategy rationalizes how online platforms, such as Facebook, have relaxed its privacy policy over time. Third, the analysis on competition poses a challenge to regulators as well as firms that offer privacy-friendly alternatives to dominant platforms. Finally, the paper offers dynamic implications of privacy regulations that policymakers might consider.

This paper contributes to the literature on markets for data and the economics of privacy. First,

the paper is closely related to [Acemoglu et al. \(2019\)](#), [Bergemann et al. \(2019\)](#), and [Choi et al. \(2019\)](#). They consider static models in which a platform collects consumer data in exchange for monetary compensation. In these papers, if some consumers sell their data, the platform can also learn about other consumers. This data externality lowers the marginal incentive of each consumer to protect privacy, leading to low equilibrium compensation. The key economic force of this paper is analogous to theirs: A consumer’s data provision today lowers her incentive to protect privacy in the future. However, there are three important differences. First, the dynamic model enables me to study new issues, such as market entry, the value of commitment to privacy policies, the right to be forgotten, data retention, and the evolution of a platform’s privacy policy. Second, economic applications are different. The previous works consider a setting in which a consumer has data at the outset, and a platform collect data in exchange for monetary compensation. In contrast, I assume that data arise as a byproduct of online activity from which the consumer derive benefits. This formulation directly applies to digital platforms that offer free services.

This paper also relates to the literature that studies competition and data in a dynamic model. [Hagi and Wright \(2020\)](#) study “data-enabled learning” whereby firms can improve their products and services through learning from the data they obtain from their customers. [Prüfer and Schottmüller \(2017\)](#) assume that the cost of investing in quality is decreasing in the firm’s past sales, and greater investment in quality leads to higher demand in the current period. In contrast to this literature, I assume that data collection lowers consumer welfare. This assumption relates my paper to a consumer privacy problem. The more important difference is whether a consumer regards collected data as sunk. For example, in [Hagi and Wright \(2020\)](#), regardless of how much data a firm has, a consumer’s outside option from not purchasing a product is zero. In my paper, once a platform collects data, the consumer incurs the associated privacy cost even when she does not use the platform. This assumption is important for the key mechanism where data collection lowers a consumer’s payoff but increases her marginal incentive to use the platform.

[Bonatti and Cisternas \(Forthcoming\)](#) study consumer privacy in a continuous-time dynamic model. They consider a long-lived consumer with short lived sellers. Sellers can learn about consumer preferences based on scores that aggregate purchase histories, and sellers use information for price discrimination. In contrast, I consider long-lived platforms and abstract away from how platforms use consumer information.

Fainmesser et al. (2019) study the optimal design of a platform to store data and invest in information security. They consider a platform that cares about both the activity levels of consumers and the amount of data it can extract. They study how different objectives lead to different platform designs. I adopt simpler preferences for consumers and platforms but consider a dynamic environment.

The rest of the paper is organized as follows. Section 2 presents a model of a monopoly platform. Section 3 presents the long-run equilibrium outcome and characterizes the equilibrium privacy policy of the platform. Section 4 considers platform competition and shows that the data collected by an incumbent works as a barrier to entry. Section 5 studies the incentive of the consumer or platforms to erase past information.. Section 6 considers extensions. In particular, I consider a forward-looking consumer and general payoff functions.

2 Model

Time is discrete and infinite, indexed by $t = 1, 2, \dots$. There are a consumer (she) and a platform (it). The consumer's type X is drawn from a normal distribution $\mathcal{N}(0, \sigma_0^2)$. X is realized before $t = 1$ and fixed over time. The consumer does not observe X .³ The platform does not observe X , but receives signals about X .

In each period t , the consumer publicly chooses an *activity level* $a_t \geq 0$. Then, the platform privately observes a signal $s_t = X + \varepsilon_t + z_t$, where $\varepsilon_t \sim \mathcal{N}\left(0, \frac{1}{a_t}\right)$ and $z_t \sim \mathcal{N}(0, \gamma_t)$. A greater a_t reduces the variance of ε_t and makes s_t more informative about X , whereas a greater γ_t increases the variance of z_t and makes s_t less informative about X .⁴ $\gamma_t \geq 0$ is the *privacy level* of the platform in period t . All the random variables, $X, \varepsilon_1, z_1, \varepsilon_2, z_2, \dots$, are mutually independent.

The payoffs are as follows. Suppose that the consumer has chosen activity levels $\mathbf{a}_t = (a_1, \dots, a_t)$ up to period t . Then, at the end of period t , the platform receives a payoff of $\sigma_0^2 - \sigma_t^2(\mathbf{a}_t) \geq 0$. $\sigma_t^2(\mathbf{a}_t)$ is the variance of the conditional distribution of X given \mathbf{a}_t , derived from Bayes' rule on and off equilibrium paths.⁵ A lower $\sigma_t^2(\mathbf{a}_t)$ means that the platform has a more accurate estimate

³If the consumer observes X , then there could be an equilibrium in which the consumer's action becomes a signal of X . However, all the results of this paper hold with respect to a pooling equilibrium, in which the consumer of all types chooses the same activity level.

⁴If $a_t = 0$, then I define s_t as a pure noise, which is independent of X .

⁵The equivalent formulation is that the platform observes (a_t, s_t) , chooses $b_t \in \mathbb{R}$, and obtains a payoff of

of X , or in other words, the consumer has less privacy. $\sigma_t^2(\mathbf{a}_t)$ is decreasing in a_s for any $s \leq t$, and independent of realized signals s_1, \dots, s_t . Where it does not cause confusion, I write $\sigma_t^2(\mathbf{a}_t)$ as σ_t^2 . The platform discounts future payoffs with a discount factor $\delta_P \in (0, 1)$.

The consumer's payoff in period t is $U(\mathbf{a}_t) := u(a_t) - v[\sigma_0^2 - \sigma_t^2(\mathbf{a}_t)]$. The first term $u(a_t)$ is the gross benefit of using the platform. $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly concave, continuously differentiable, maximized at $a^* \in (0, \infty)$, and satisfies $u(0) = 0$. The second term $v[\sigma_0^2 - \sigma_t^2(\mathbf{a}_t)]$ is the *privacy cost*, which captures the negative impact of data collection on the consumer. $v \geq 0$ is an exogenous parameter that captures the value of privacy. The consumer is myopic and chooses a_t to maximize $U(a_1, \dots, a_t)$ in each period t . I later consider a forward-looking consumer. The payoffs are normalized so that if $a_t = 0$ for all t , then the platform and the consumer obtain zero payoffs in all periods. The primitives, σ_0^2 , $u(\cdot)$, and v , are commonly known to the consumer and the platform.

The timing of the game is as follows. Before $t = 1$, the platform commits to a *privacy policy* $\gamma = (\gamma_1, \gamma_2, \dots) \in \mathbb{R}_+^\infty$. After observing γ , the consumer chooses an activity level in each period. I take $(\sigma_t^2(\cdot))_{t \in \mathbb{N}}$ as a primitive, and consider pure-strategy subgame perfect equilibrium in which the consumer chooses the highest activity level among all the optimal activity levels. I will later show that the equilibrium remains the same even if the platform has a weaker commitment power.

2.1 Discussion of Modeling Assumptions

As I have discussed in the introduction, the model captures the situation in which a consumer uses a platform and derives utility from its service, and the platform analyzes her activity and collects data. The platform earns higher revenue from a greater amount of data, but data collection reduces the consumer's welfare. Here, I provide further discussion on modeling assumptions.

Data Generation A higher activity level a_t mechanically generates more information about X . This assumption is suitable for consumer information such as their preferences, health, and friends' network. For example, a consumer may not have a verifiable piece of information about her political preferences before joining a platform. However, as the consumer spends time on browsing and "liking" news, the platform can infer her political preferences. The model captures this by assuming that the informativeness of s_t is increasing in a_t . The model excludes a consumer who $-\mathbb{E}[(X - b_t)^2]$. Writing the payoffs in terms of $\sigma_t^2(\mathbf{a}_t)$ greatly simplifies exposition.

strategically manipulates browsing history or friends' network to influence the platform's learning, which seems to be less relevant. The current formulation distinguishes this paper from the recent literature on markets for data, which typically assumes that consumers are exogenously endowed with data (Acemoglu et al., 2019; Bergemann et al., 2019; Choi et al., 2019; Ichihashi, 2019).

It is crucial that the consumer cannot delete past information (σ_t^2 is weakly decreasing over time). In other words, a privacy cost from past data collection is a sunk cost. This is motivated by "data persistence," where it can be difficult for a consumer to delete collected data completely (Tucker, 2018). Section 5 considers an extension in which the consumer or a platform can erase past information.

Consumer The consumer incurs a higher privacy cost and obtains a lower payoff if the platform has more information. The privacy cost captures monetary or non-monetary reasons for which a consumer wants a platform to have less information. For instance, a consumer may have intrinsic preferences for privacy, which is empirically supported by Kummer and Schulte (2019), Lin (2019), and Tang (2019).⁶ For another instance, a consumer may consider the risk of data breach, identity theft, and price or non-price discrimination by platforms and third parties who have access to consumer data.

I acknowledge that consumer preferences on data collection is more complicated than what is assumed. For example, a consumer may prefer data collection if it leads to better services. Also, the benefit $u(a)$ of using a platform may depend on the amount of data collected. However, I adopt $u(a_t) - v[\sigma_0^2 - \sigma_t^2]$ for two reasons. First, the above extensions seem to push the equilibrium to more data collection. Because the equilibrium involves full privacy loss (i.e. Proposition 2) even if the consumer wants the platform to have no information, I conjecture that the main insight is robust to those extensions. Second, the current payoff specification is a tractable benchmark to capture misalignment between a platform's incentive to gather data and a consumer's incentive to protect privacy.

Platform I comment on the platform's strategy and payoffs. First, this paper considers a platform that either chooses a privacy policy at the outset (as in the baseline model) or sets a privacy

⁶Lin (2019) considers an experiment that empirically distinguishes intrinsic and instrumental preferences for privacy.

level at the beginning of each period. They are not fully general. For example, I could consider a mechanism that maps each history of activity levels $(a_s)_{s=1}^t$ to a privacy level γ_t in period t . However, such a mechanism does not affect the long-run equilibrium outcome, and it seems impractical and prohibitively costly to communicate to consumers.

Second, the platform cannot use monetary transfer. This is consistent with online platforms such as Facebook and Google. If the platform could commit to a mechanism mapping \mathbf{a}_t to monetary transfer in each period t , then the platform may commit to compensate the consumer in exchange for a higher activity level. However, I conjecture that such compensation disappears in the long-run, because the consumer chooses the highest activity level and the platform learns full information even without transfer.

Finally, the platform's payoff function can be more general than $\sigma_0^2 - \sigma_t^2$. For example, I can adopt any strictly decreasing function of σ_t^2 as the platform's per-period profit. All the results and the proof continue to hold without modification.

3 Monopoly Platform

I begin with studying the consumer's behavior, taking the platform's strategy as given. Then, I present the long-run equilibrium outcome. After that, I characterize the platform's equilibrium privacy policy.

3.1 Consumer Behavior

Bayes' rule tells⁷

$$\sigma_t^2(\mathbf{a}_t) = \frac{1}{\frac{1}{\sigma_{t-1}^2(\mathbf{a}_{t-1})} + \frac{1}{\frac{1}{a_t} + \gamma_t}}. \quad (1)$$

⁷This follows from the fact that if $x|\mu \sim N(\mu, \sigma^2)$ and $\mu \sim N(\mu_0, \sigma_0^2)$, then $\mu|x \sim N\left(\frac{\frac{\sigma_0^2}{\sigma^2 + \sigma_0^2}x + \frac{\sigma^2}{\sigma^2 + \sigma_0^2}\mu_0, \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1}\right)$.

Thus, in each period t , the (myopic) consumer chooses a_t to maximize

$$\begin{aligned} & u(a_t) - v [\sigma_0^2 - \sigma_t^2(a_t)] \\ &= u(a_t) - v \left[\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2(a_{t-1})} + \frac{1}{a_t + \gamma_t}} \right], \end{aligned}$$

taking $\sigma_{t-1}^2(a_{t-1})$ and γ_t as given. Define the privacy cost function as $C(a, \gamma, \sigma^2) = v \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{a + \gamma}} \right)$. The following lemma summarizes the key properties of the privacy cost C and the marginal privacy cost $\frac{\partial C}{\partial a}$.

Lemma 1. *The following holds.*

1. $C(a, \gamma, \sigma^2)$ is decreasing in γ and σ^2 , and increasing in a .
2. $\frac{\partial C}{\partial a}(a, \gamma, \sigma^2)$ is decreasing in γ and increasing in σ^2 .

Proof. Point 1 follows from [equation \(1\)](#). Point 2 follows from

$$\frac{\partial C}{\partial a} = v \cdot \frac{\frac{\frac{1}{a^2}}{\left(\frac{1}{a} + \gamma\right)^2}}{\left(\frac{1}{\sigma^2} + \frac{1}{a + \gamma}\right)^2} = \frac{v}{\left(\frac{1}{\sigma^2} (1 + \gamma a) + a\right)^2}. \quad (2)$$

□

The marginal cost $\frac{\partial C}{\partial a}$ of using the platform is increasing in σ^2 . Thus, if the consumer has lower privacy, then she has lower marginal incentive to protect privacy. In particular, if the consumer has no privacy ($\sigma^2 \rightarrow 0$), then the marginal cost is zero, because the activity level has no impact on the platform's information.

The rest of [Lemma 1](#) is intuitive. The privacy cost C is increasing in the activity level a and the amount of information the platform has collected, $\frac{1}{\sigma^2}$. A higher privacy level γ makes the signal less informative and lowers the privacy cost. A higher γ also reduces the marginal cost, because adding noise makes the platform's learning less sensitive to the consumer activity.

Let $a^*(\gamma, \sigma^2)$ denote the optimal activity level given a privacy level γ in the current period and the conditional variance σ^2 from the previous period:

$$a^*(\gamma, \sigma^2) := \max \left\{ \arg \max_{a \geq 0} u(a) - v \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a} + \gamma}} \right) \right\}. \quad (3)$$

Lemma 1 implies that the consumer chooses a higher activity level when she has less privacy or the platform sets a higher privacy level (see [Appendix A](#) for the proof). Recall $a^* = \arg \max_{a \geq 0} u(a)$.

Lemma 2. $a^*(\gamma, \sigma^2)$ is increasing in γ and decreasing in σ^2 . For any (γ, σ^2) , $\lim_{\hat{\sigma}^2 \rightarrow 0} a^*(\gamma, \hat{\sigma}^2) = \lim_{\hat{\gamma} \rightarrow \infty} a^*(\hat{\gamma}, \sigma^2) = a^*$.

The next result presents the long-run outcome when the platform adopts a stationary privacy policy (see [Appendix B](#) for the proof).

Proposition 1. Suppose that a privacy level γ_t is independent of t . Let $(a_t^*)_{t \in \mathbb{N}}$ denote the equilibrium activity levels of this subgame. There is a cutoff value of privacy $v^*(\gamma) \in (0, \infty)$ such that:

1. If $v < v^*(\gamma)$, then a_t^* increases in t , $\lim_{t \rightarrow \infty} a_t^* = a^*$, and $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$. The consumer's per-period payoff decreases over time.
2. If $v > v^*(\gamma)$, then $a_t^* = 0$ and $\sigma_t^2 = \sigma_0^2$ for all $t \in \mathbb{N}$.

Moreover, $v^*(\gamma)$ is increasing in γ .

The intuition is as follows. If the value of privacy is low, then it is optimal for the consumer to choose a positive activity level $a_1^* > 0$ in $t = 1$. The consumer activity generates some information, which reduces her payoff but increases the marginal net benefit of using the platform. Thus, in $t = 2$, the consumer chooses $a_2^* \geq a_1^*$. Repeating this argument, we can conclude that a_t^* increases over time. This also implies that the platform can perfectly learn the consumer's type as $t \rightarrow \infty$. This is associated with $a_t^* \rightarrow a^*$, because the platform's perfectly learning X implies that the marginal privacy cost goes to zero. Thus, in the long-run, the consumer loses her privacy but behaves as if there is no privacy cost. In contrast, the consumer with a high v does not use the platform in $t = 1$, which also implies that she sets $a_t = 0$ in any period $t \geq 2$.

Proposition 1 has a policy implication. Suppose that a regulator, who cares about consumer privacy, mandates a stricter privacy policy (i.e. $\gamma_t = \bar{\gamma} > \gamma$ as opposed to $\gamma_t = \gamma, \forall t \in \mathbb{N}$). Since $v^*(\cdot)$ is increasing, the regulation increases the cutoff v^* , and expands the set of parameter v 's such that the consumer experiences the long-run privacy loss (Point 1). To see the welfare implication of this, suppose that v satisfies $u(a^*) - v\sigma_0^2 < 0$. For a small γ , the consumer might choose $a_t^* = 0$ and obtain a payoff of zero in all periods. If the regulator enforces a large γ , then the consumer chooses $a_1^* > 0$ in $t = 1$ because it is less costly to use the platform. However, this leads to $(a_t^*, \sigma_t^2) \rightarrow (a^*, 0)$, and thus the consumer's per-period payoff converges to $u(a^*) - v\sigma_0^2 < 0$. Thus, the regulation can increase the consumer's per-period payoffs in the short-run but decrease them in the long-run. If the regulator puts a large weight on the long-run welfare, it might consider the regulation as detrimental.⁸

3.2 Equilibrium

I now turn to the equilibrium of the entire game (see [Appendix C](#) for the proof, which uses the characterization result in [Proposition 3](#)).

Proposition 2. *Take any v , and let $(a_t^*, \gamma_t^*, \sigma_t^2)_{t \in \mathbb{N}}$ denote the equilibrium outcome. Then,*

$$\lim_{t \rightarrow \infty} a_t^* = a^*, \lim_{t \rightarrow \infty} \gamma_t^* = 0, \text{ and } \lim_{t \rightarrow \infty} \sigma_t^2 = 0. \quad (4)$$

Moreover, for any $T \in \mathbb{N}$, there is a \underline{v} such that, for any $v \geq \underline{v}$, the equilibrium privacy policy satisfies $\gamma_t^ > 0$ for all $t \leq T$.*

Even if the value v of privacy is arbitrarily high, in the long-run, the consumer becomes an active user with no privacy. This is in contrast to [Proposition 1](#), where the consumer with a high v chooses to not use the platform and secures zero payoffs. Whenever v is high, the equilibrium privacy policy is nonstationary: In early periods, the platform sets positive privacy levels. In the long-run, the platform offers a vanishing level of privacy, fully monetizing information generated

⁸The caveat “if the regulator puts a large weight on the long-run welfare” is important. This is because a higher γ increases the consumer's ex ante sum of discounted payoffs from her optimal policy. This is true even if we consider a forward-looking consumer. Thus, we can argue that a higher γ can be undesirable, only if we evaluate the consumer's behavior under an alternative objective that exhibits more patience than the consumer.

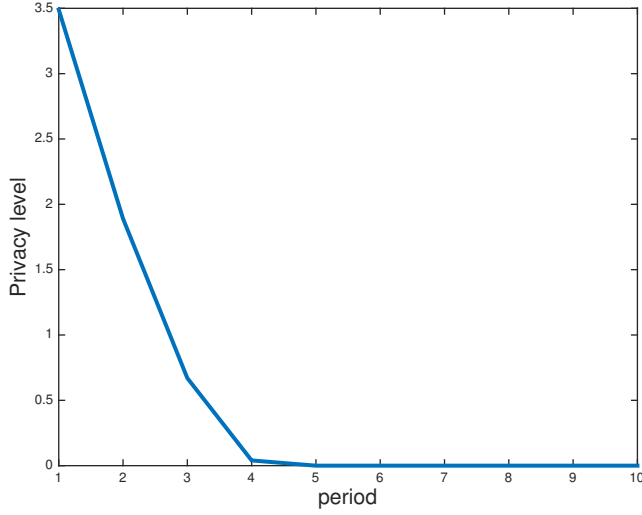


Figure 1(a): Privacy level γ_t

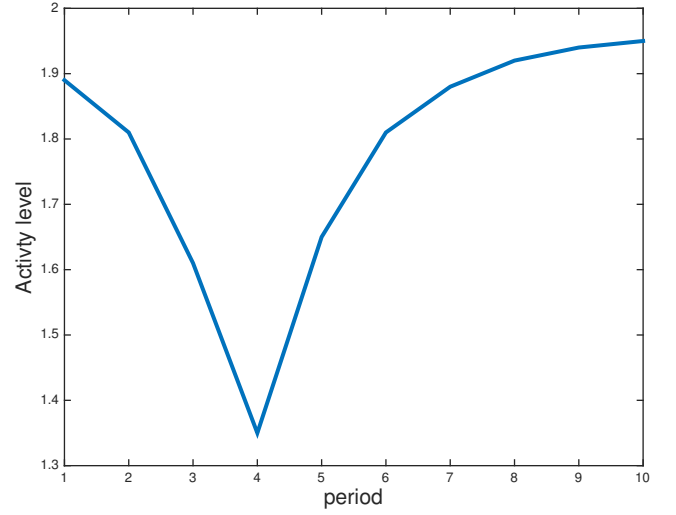


Figure 1(b): Activity level a_t

Figure 1: Equilibrium under $u(a) = 2a - \frac{1}{2}a^2$, $v = 10$, and $\sigma_0^2 = 1$.

by consumer activity. The long-run payoff of the consumer is $u(a^*) - v\sigma_0^2$, which can be arbitrarily low if we consider a large v .

The intuition is as follows. In early periods, consumer activity has a large impact on what the platform knows about her type. Thus, the consumer faces a high marginal privacy cost, which discourages her from raising the activity level. To reduce the marginal cost, the platform sets a high privacy level. Thus, in early periods, the platform offers high privacy and slowly learns the consumer type. After a long period of interaction, the platform accurately knows the consumer type, which implies that she faces a low marginal cost. Then, the platform can lower a privacy level to accelerate learning.

Figure 1 depicts the equilibrium dynamics in a numerical example.⁹ Figure 1(a) shows that the platform offers a decreasing level of privacy, hitting zero in $t = 5$. Figure 1(b) shows that the equilibrium activity level first decreases but eventually approaches $a^* = 2$. This nonmonotonicity contrasts with the case of a constant privacy policy in Proposition 1.¹⁰

Proposition 2 has several implications. First, it gives a potential economic explanation of the so-

⁹I compute the equilibrium strategy profile using Proposition 3.

¹⁰While I have not managed to prove that the nonmonotonicity always occurs, a numerical exercise suggests that, for a wide range of parameters such that the equilibrium privacy level is strictly decreasing in early periods, the activity level also decreases.

called privacy paradox: Consumers seem to casually share their data with online platforms, despite their demand for privacy and concerns about data collection.¹¹ One may view this puzzle as the long-run equilibrium outcome, where the consumer faces a high privacy cost and zero marginal cost. In contrast to the explanation based on information externalities, the long-run outcome in [Proposition 2](#) occurs without a priori market imperfection. Indeed, a similar outcome arises even if the consumer is forward-looking ([Proposition 8](#)).

The key for the above result is that data collection hurts the consumer but increases her incentive to use the platform in the future. This complementarity between present and future consumption is what [Becker and Murphy \(1988\)](#) call (harmful) addiction. One crucial difference from their work is that the platform strategically influences the consumer's choice through its privacy policy. The platform's equilibrium strategy ensures that the consumer is trapped into the "addiction problem" even if she values privacy arbitrarily highly *ex ante*.

At least at an anecdotal level, the equilibrium strategy of the platform is also consistent with how the privacy policies of some online platforms have evolved. In 2004, Facebook's privacy policy stated that they would not use cookies to collect consumer information. However, in 2020, it states that they use cookies to track consumers on and possibly off the website.¹² [Srinivasan \(2019\)](#) describes how Facebook has acquired dominance in the social media market as follows:

"When Facebook entered the market, the consumer's privacy was paramount. The company prioritized privacy, as did its users—many of whom chose the platform over others due to Facebook's avowed commitment to preserving their privacy. Today, however, accepting Facebook's policies in order to use its service means accepting broad-scale commercial surveillance."

Also, [Fainmesser et al. \(2019\)](#) describe how the business models of online platforms have changed from the initial phase where they expand a user base to the mature phase where they monetize collected information. [Proposition 2](#) rationalizes the described pattern as a platform's optimal policy given the decreasing marginal incentive of consumers to protect their privacy.

¹¹[Acquisti et al. \(2016\)](#) contains an insightful review of research on the privacy paradox. Recent empirical work, for example, includes [Athey et al. \(2017\)](#).

¹²In 2020, Facebook's privacy policy states that "we use cookies if you have a Facebook account, use the Facebook Products, including our website and apps, or visit other websites and apps that use the Facebook Products (including the Like button or other Facebook Technologies)." <https://www.facebook.com/policies/cookies>

3.3 Characterizing Equilibrium Privacy Policy

The following result characterizes the platform's optimal privacy policy. Recall that $a^*(\gamma, \sigma^2)$ is the activity level a_t chosen by the consumer given $\gamma_t = \gamma$ and $\sigma_{t-1}^2 = \sigma^2$.

Proposition 3. *The equilibrium privacy policy $(\gamma_1^*, \gamma_2^*, \dots)$ is recursively defined as follows.*

$$\gamma_t^* \in \arg \min_{\gamma \geq 0} \frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)} + \gamma, \quad (5)$$

$$\hat{\sigma}_0^2 = \sigma_0^2, \quad (6)$$

$$\hat{\sigma}_t^2 = \frac{1}{\frac{1}{\hat{\sigma}_{t-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_t^*, \hat{\sigma}_{t-1}^2)} + \gamma_t^*}}, \forall t \in \mathbb{N}. \quad (7)$$

For any privacy policy, the induced conditional variances $(\sigma_t^2)_{t \in \mathbb{N}}$ satisfies $\hat{\sigma}_t^2 \leq \sigma_t^2$ for all $t \in \mathbb{N}$.

Proof. Take any privacy policy $(\gamma_t)_{t \in \mathbb{N}}$, and let $(\sigma_t^2)_{t \in \mathbb{N}}$ denote the sequence of the conditional variances induced by the consumer's best responses. I show $\hat{\sigma}_t^2 \leq \sigma_t^2$ for all $t \in \mathbb{N}$. The inequality holds with equality for $t = 0$. Take any $\tau \in \mathbb{N}$. Suppose that $\hat{\sigma}_t^2 \leq \sigma_t^2$ for $t = 0, \dots, \tau - 1$. Then, it holds

$$\sigma_\tau^2 = \frac{1}{\frac{1}{\sigma_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau, \sigma_{\tau-1}^2)} + \gamma_\tau}} \geq \frac{1}{\frac{1}{\hat{\sigma}_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau, \hat{\sigma}_{\tau-1}^2)} + \gamma_\tau}} \geq \frac{1}{\frac{1}{\hat{\sigma}_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau^*, \hat{\sigma}_{\tau-1}^2)} + \gamma_\tau^*}} = \hat{\sigma}_\tau^2.$$

The first inequality follows from the inductive hypothesis and $a^*(\gamma, \cdot)$ being decreasing. The second inequality follows from (5). $\hat{\sigma}_t^2 \leq \sigma_t^2, \forall t \in \mathbb{N}$ implies that the strategy described by (5), (6), and (7) is optimal, because it gives the platform a weakly greater profit than any other privacy policy in all periods. \square

The minimand of (5), $\frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)} + \gamma$, is the variance of the noise term $\varepsilon_t + z_t$ in $s_t = X + \varepsilon_t + z_t$ given the consumer's best response. Thus, the minimization problem (5) implies that, in equilibrium, the platform chooses γ_t to *statically* maximize the informativeness of the signal in each period. This captures the platform's trade-off. On the one hand, a higher privacy level γ leads to a higher activity level, which leads to a lower variance $\frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)}$ of ε_t . On the other hand, given any activity level, a higher γ lowers the informativeness of the signal. This cost is captured by the second term γ , a variance of z_t . The platform chooses γ_t^* by resolving this trade-off. As the

platform solves (5) in each period, the conditional variance evolves according to (7) with the initial condition (6).

The platform chooses its strategy to maximize the sum of discounted payoffs. However, the optimization in (5) does not include future payoffs. This is an implication of the consumer's decreasing marginal incentive to protect privacy: It is optimal for the platform to statically maximize the amount of information, because it also reduces the consumer's marginal costs and increases her activity levels in the future.

Proposition 3 implies that the platform does not require as strong a commitment power as currently assumed.¹³ To state the result formally, I say that the platform has the *short-run commitment power* if it chooses a privacy level γ_t at the beginning of each period t (before the consumer chooses a_t) without committing to future privacy levels.

Corollary 1. *Let (\mathbf{a}^*, γ^*) denote the equilibrium outcome of the baseline model, in which the platform can commit to any privacy policy. The same outcome (\mathbf{a}^*, γ^*) can arise in an equilibrium of the game in which the platform only has the short-run commitment power.*

Proof. Consider the short-run commitment power. Consider the following strategy profile: For each node with the conditional variance σ^2 , the platform sets $\gamma \in \arg \min_{\gamma \geq 0} \frac{1}{a^*(\gamma, \sigma^2)} + \gamma$. The consumer acts according to $a^*(\cdot, \cdot)$ at any node. By construction, (\mathbf{a}^*, γ^*) arises on the path of play. If the platform deviates at any node, it weakly increases the conditional variances in all periods (Proposition 3). Thus, the platform has no profitable deviation. \square

The result shows that the long-run commitment has no value relative to the short-run commitment. In contrast, the platform could be strictly worse if it has no commitment power: If the platform sets γ_t after observing a_t , then in any equilibrium, the platform sets $\gamma_t = 0$ whenever $a_t > 0$. Anticipating this, the consumer chooses a weakly lower activity level than under the short-run or long-run commitment power. In practice, the short-run commitment seems a reasonable assumption, because a platform could be sanctioned for the outright violation of its privacy policy.

¹³The proof of the following result also reveals that the equilibrium outcome is independent of the platform's discount factor or time horizon.

4 Platform Competition

I now explore the effect of competition among platforms. There are two platforms, an incumbent (I) and an entrant (E). I is in the market from the beginning of $t = 1$. In period $t^* \geq 2$, E enters the market. t^* is exogenous and known to I at the outset, and thus there is no issue of strategic entry.

Before the entry ($t < t^*$), the consumer chooses her activity level $a_t^I \geq 0$ for I . After the entry ($t \geq t^*$), the consumer chooses $(a_t^I, a_t^E) \in \mathbb{R}_+^2$, where a_t^E is the activity level for E . I assume single-homing: The consumer can choose (a_t^I, a_t^E) if and only if $\min(a_t^I, a_t^E) = 0$. This is natural if two platforms offer substitutable services such as search engines.

I consider two games that differ in the timing of moves. One is *competition with long-run commitment*: I publicly commits to $(\gamma_1^I, \gamma_2^I, \dots)$ at the beginning of $t = 1$, and E publicly commits to $(\gamma_{t^*}^E, \gamma_{t^*+1}^E, \dots)$ at the beginning of period t^* . The other is *competition with short-run commitment*: In each period, each platform chooses a privacy level, after which the consumer chooses an activity level. In period $t \geq t^*$, I and E set privacy levels γ_t^I and γ_t^E simultaneously.

As in the baseline model, activity on platform $k \in \{I, E\}$ generates a signal $s_t^k = X + \varepsilon_t^k + z_t^k$ with $\varepsilon_t^k \sim N\left(0, \frac{1}{a_t^k}\right)$ and $z_t^k \sim N(0, \gamma_t^k)$. Each platform k privately observes s_t^k . Thus, there is no information spillover between platforms. All the noise terms (ε_t^k, z_t^k) are independent across (t, k) .

The payoff of platform $k \in \{I, E\}$ in period t is $\sigma_0^2 - \sigma_{t,k}^2$, where $\sigma_{t,k}^2$ is the conditional variance of X given activity levels a_1^k, \dots, a_t^k on platform k . The consumer's payoff in period t is given by

$$u(a_t^I) - v(\sigma_0^2 - \sigma_{t,I}^2) + \mathbf{1}_{\{t \geq t^*\}} \cdot [u(a_t^E) - v(\sigma_0^2 - \sigma_{t,E}^2)], \quad (8)$$

where $\mathbf{1}_{\{t \geq t^*\}}$ is the indicator function that equals 1 or 0 if $t \geq t^*$ or $t < t^*$, respectively. Payoff (8) implies that even if the consumer switches to E in some period and never uses I from then on, she continues to incur a privacy cost $v(\sigma_0^2 - \sigma_{t,I}^2)$ based on the information collected by I (and vice versa). For example, suppose that I collects and shares consumer data with an online retailer in period t . Then, consumers may face the risk of price discrimination on the retailer's website in the future, even after they migrate to E . I will later consider how a regulation such as the “right to be forgotten” alters this assumption and changes the equilibrium.

To ensure the existence of an equilibrium, I impose an upper bound on the feasible privacy

levels. In practice, the upper bound might reflect the minimum amount of data that a platform has to collect to offer services, or the maximum privacy level that a platform can credibly enforce.

Assumption 1. There is a $\bar{\gamma} \in (0, +\infty)$ satisfying $a^*(\bar{\gamma}, \sigma_0^2) > 0$ such that each platform can choose a privacy level of at most $\bar{\gamma}$.

$a^*(\bar{\gamma}, \sigma_0^2) > 0$ implies that if a platform chooses $\bar{\gamma}$, then the consumer chooses $a_t^I > 0$ or $a^E > 0$. This restriction is necessary for a non-trivial equilibrium in which the consumer uses a platform.

4.1 Equilibrium under Competition

The following result shows that in an equilibrium, the consumer never switches to E , and the long-run outcome equals the one under monopoly. If the entry is sufficiently late, then the equilibrium outcome exactly equals the one under monopoly.

Proposition 4. *Regardless of the commitment assumption:*

1. *There is an equilibrium in which $a_t^E = 0$ for all $t \in \mathbb{N}$, $\lim_{t \rightarrow \infty} a_t^I = a^*$, $\lim_{t \rightarrow \infty} \sigma_{I,t}^2 = 0$, and $\lim_{t \rightarrow \infty} \gamma_t^I = 0$.*
2. *There is a $\underline{t} \geq 2$ such that for any $t^* \geq \underline{t}$, I 's privacy levels $(\gamma_t^I)_{t \in \mathbb{N}}$ in the above equilibrium equal the monopoly strategy.*
3. *Switching never occurs: There is no equilibrium in which $a_t^E > 0$ and $a_t^I = 0$ for all $t \geq t^*$.*

The intuition is as follows. Before the entry, I chooses privacy levels that make it optimal for the consumer to choose positive activity levels. Suppose that, upon entry, E chooses the highest privacy level $\bar{\gamma}$. Since the privacy cost from collected data is sunk, the consumer chooses which platform to use based on the marginal (or more precisely, incremental) cost. Because the incumbent has collected data, the consumer faces a lower marginal cost of using I . Thus, if I also chooses $\bar{\gamma}$, then the consumer strictly prefers to use I . This ensures that the consumer only uses I even after the entry. However, the equilibrium privacy level of I may not be $\bar{\gamma}$: I chooses a privacy level to maximize the amount of information generated subject to the constraint that the consumer prefers

I. As time goes by, the constraint of attracting the consumer is relaxed, because the consumer's marginal cost of using *I* goes to zero. This ensures that *I* can offer a vanishing level privacy over time.

The threat of future entry has no impact on *I*'s strategy: Before the entry, *I* chooses the same privacy levels as monopoly, regardless of commitment assumption. This is because maximizing the amount of information makes consumer switching least likely to occur.

Point 3 implies that, for switching to occur, *E* needs some advantage in terms of quality of service or privacy level. For the next result, I say that *E successfully enters the market* if there is an equilibrium in which the consumer switches to the entrant upon entry, i.e. $a_t^E > 0$ and $a_t^I = 0$ for all $t \geq t^*$.

Proposition 5. *Suppose that the gross benefit of *E*'s service is given by $u^E(\cdot) = u(\cdot) + \Delta$. There is a $\Delta^* > 0$ such that for any $\Delta \geq \Delta^*$, there is an equilibrium in which *E successfully enters the market*. The minimum Δ^* with this property is increasing in t^* .*

4.2 Antitrust Implication of Propositions 4 and 5

Why do some online platforms, such as Google and Facebook, have dominant positions? One answer could be that consumers love them: They offer high quality services free of charge, and consumers tend to use the same platform because of network effect. In other words, we may be seeing market concentration because other firms cannot offer services of comparable quality. Indeed, [Furman et al. \(2019\)](#) states that:

“Data can act as a barrier to entry in digital markets. A data-rich incumbent is able to cement its position by *improving its service and making it more targeted for users*, as well as making more money by better targeting its advertising” (italicized by the author).

My results contribute to this discussion with the following argument: We do not see competition because consumers are suffering from a lack of privacy and obtaining low payoffs from dominant incumbents. This seemingly counterintuitive claim is rationalized by consumers' decreasing marginal incentive to protect privacy. If consumers perceive collected data as sunk, they choose

whether to switch based on marginal costs. Then, dominant platforms holding a large amount of data can reduce consumer welfare but make switching less likely to occur. Both the above quote and this idea indicate that data can be an entry barrier. However, the entry barrier in my model is associated with lower consumer welfare.

The insight is relevant to, for example, the market for search engines: The incumbent is Google, and the entrant is a “privacy-friendly” alternative such as DuckDuckGo. The result suggests that it might be difficult for DuckDuckGo to penetrate the market simply by offering better privacy protection. If consumers have already lost privacy against Google, their marginal privacy cost of using Google is nearly zero. Thus, DuckDuckGo may need to offer better privacy protection *and* comparable quality of a search engine.

The result also suggests a potential remedy. If consumers can erase collected data upon switching, then competition is more likely to occur. This is because erasing past information increases the marginal cost of using the incumbent relative to the entrant. In the next section, I formally analyze the impact of this “right to be forgotten.”

5 Erasing Past Information

So far, the amount of information a platform has on the consumer is weakly increasing over time. I now consider the incentive of the consumer or a platform to erase past information.

5.1 The Right to be Forgotten

I consider the right to be forgotten, whereby the consumer can request a platform to erase past information. Below, I describe the model of competition, but a similar description applies to the case of monopoly.

In each period, the consumer makes two decisions. First, the consumer chooses whether to erase past information of each platform in the market. Second, the consumer chooses (a_t^I, a_t^E) . If she erases information of platform $k \in \{I, E\}$ in period t , then the conditional variance for platform k at the beginning of t becomes σ_0^2 . At the end of the period, the consumer still incurs a privacy cost based on information generated in period t . It is costless for the consumer to erase information.

For example, if the consumer erases information of both platforms in period t and uses platform E , then her payoff is

$$u(a_t^E) - v [\sigma_0^2 - \sigma_{1,E}^2(a_t^E)] , \quad (9)$$

where $\sigma_{1,E}^2(a_t^E)$ is the conditional variance for E given one signal based on (a_t^E, γ_t^E) . The privacy cost from E is only based on the signal of period t . Since the consumer has erased information for I and does not use I , she does not incur a privacy cost from I in period t .

In contrast, suppose that the consumer has never erased information. If she uses platform E in period t , then her payoff in period t is

$$u(a_t^E) - v [\sigma_0^2 - \sigma_{t,E}^2(\mathbf{a}_t^E)] - v [\sigma_0^2 - \sigma_{t,I}^2(\mathbf{a}_t^I)] , \quad (10)$$

where $\mathbf{a}_t^E = (a_1^E, \dots, a_t^E)$ and $\mathbf{a}_t^I = (a_1^I, \dots, a_t^I)$. Thus, the consumer incurs a privacy cost from both platforms based on past information. The following result summarizes the impact of the right to be forgotten.

Proposition 6. *If the consumer can costlessly erase past information, then regardless of the commitment assumption, there is the following equilibrium:*

1. *Under monopoly, in all periods, the consumer erases information and the platform sets a privacy level γ_1^* defined in (5).*
2. *Under competition, the consumer erases information in all periods, and both platforms set the highest privacy level $\bar{\gamma}$ in any period after the entry (i.e. in period $t \geq t^*$).*
3. *Suppose that the gross benefit of E 's service is given by $u^E(\cdot) = u(\cdot) + \Delta$. For any $\Delta > 0$, E can successfully enter the market.*

The result implies that the right to be forgotten benefits the consumer in three ways. First, it reduces privacy cost. Second, it incentivizes platforms to choose higher privacy levels: Once the consumer erases information, then she incurs a high marginal privacy cost. This ensures that a monopoly platform always offers a period-1 privacy level in any period (Point 1). Under competition, erasing information makes competing platforms homogeneous. This intensifies competition

and incentivizes platforms offer the highest privacy level (Point 2). Finally, erasing past information eliminates the incumbency advantage and makes market entry more likely (Point 3).

5.2 Data Retention Policies

This section considers a platform's choice of data retention policies, which have recently been paid attention by economists and legal scholars ([Chiou and Tucker, 2017](#)). For simplicity, I focus on the short-run commitment: Under monopoly, the platform chooses whether to erase past information and sets a privacy level. Then, the consumer chooses an activity level. Under competition (after the entry), platforms simultaneously choose whether to erase information. After observing this, they simultaneously set privacy levels. Finally, the consumer chooses an activity level for each platform.

Erasing information affects the conditional variances and payoffs in the same way as the consumer erasing information (see the previous subsection). The following result shows that a platform's optimal data retention policy is to not erase information at all.

Proposition 7. *A platform never erases information:*

1. *A monopoly platform never erases information in any period in any equilibrium. The equilibrium outcome equals [Proposition 2](#).*
2. *Under competition, there is an equilibrium in which platforms never erase information in any period. Among the equilibria with this property, [Proposition 4](#) holds.*

If consumer behavior is exogenous, then both a higher privacy level and erasing information benefit the consumer by reducing the amount of collected information. However, if consumer behavior is endogenous, they have different impacts: A high privacy level reduces the consumer's marginal cost and increases her activity level. In contrast, erasing information increases the consumer's marginal privacy cost and decreases her activity level. Thus, platforms might voluntarily offer a high privacy level in some periods, whereas they have no incentive to erase past information.

6 Extensions

6.1 Forward-looking Consumer

Suppose that the consumer is patient with one-period ahead discount factor $\delta_C \in (0, 1)$. As in the baseline model, a monopoly platform can commit to any privacy policy at the outset. The following result shows that the long-run equilibrium outcome remains the same as the case of a myopic consumer (see [Appendix F](#) for the proof, which also proves the existence of an equilibrium).

Proposition 8. *For any v and δ_C , in any equilibrium:*

$$\lim_{t \rightarrow \infty} a_t^* = a^* \text{ and } \lim_{t \rightarrow \infty} \sigma_t^2 = 0. \quad (11)$$

If the platform adopts a nonstationary privacy policy, the consumer's problem is a nonstationary dynamic programming, which is difficult to solve. Thus, I first show that there is a stationary privacy policy that induces (11). Then, I show that the platform's equilibrium strategy also induces the same long-run outcome. The key is that the consumer's dynamic incentive: The consumer's optimal activity levels are increasing in the amount of information the platform collected in the past and the amount of information the platform will collect in the future. This result comes from the supermodularity of the consumer's objective in activity levels and the informativeness of signals.

6.2 General Privacy Cost Function

Suppose that the consumer's per period payoff is $u(a_t) - C(\sigma_t^2)$. Assume $C(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is continuously differentiable. This implies that the cost and the marginal cost (i.e. $C(0)$ and $C'(0)$) are bounded. $C(\cdot)$ needs not be monotone: For example, it can be convex and minimized at some σ^* , which means that the consumer benefits data collection which reduces σ_0^2 to σ^* , but the additional data collection is harmful. The following result shows that the long-run outcome remains the same (see [Appendix G](#)).

Proposition 9. *If the consumer is myopic, in any equilibrium:*

$$\lim_{t \rightarrow \infty} a_t^* = a^* \text{ and } \lim_{t \rightarrow \infty} \sigma_t^2 = 0. \quad (12)$$

7 Conclusion

I consider a dynamic model in which a consumer chooses how actively to use a platform in each period. The activity generates information about the consumer's type, which benefits the platform but hurts the consumer. Thus, the consumer balances the benefit of using the service and the cost of losing privacy. I show that the long-run equilibrium outcome exhibits no privacy and a high activity level. This is driven by the consumer's decreasing marginal incentive to protect privacy. Such an outcome occurs even if the consumer is patient and values privacy arbitrarily highly, because the platform adopts a nonstationary privacy policy that offers high privacy levels in early periods. I show that competition is unlikely to occur in particular when the consumer suffers from low privacy. Allowing the consumer to erase past information mitigates the problem, although platforms themselves have no incentive to do so.

We may view the current model as an information design problem: A designer commits to a mechanism specifying what information to collect, in order to influence the behavior of an agent whose action generates some information. It would be interesting to consider a general design problem with this feature.

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Appendix

A Properties of the Consumer’s Best Response: Proof of Lemma 2

Proof. Define $U(a, \gamma, \sigma^2) := u(a) - v\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{a + \gamma}}\right)$. Lemma 1 implies that $\frac{\partial U}{\partial a}$ is increasing in γ and decreasing in σ^2 . The standard argument of monotone comparative statics implies that $a^*(\gamma, \sigma^2)$ is increasing in γ and decreasing in σ^2 (e.g. Milgrom et al. (1994)).

Suppose to the contrary that $\lim_{\sigma^2 \rightarrow 0} a^*(\gamma, \sigma^2) = a^*$ fails. As $a^*(\gamma, \sigma^2) \leq a^*$ for all (γ, σ^2) , there are $\varepsilon > 0$ and $(\sigma_n^2)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \sigma_n^2 = 0$, $a^*(\gamma, \sigma_n^2) \leq a^* - \varepsilon$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} a^*(\gamma, \sigma_n^2) = a^* - \varepsilon$. Suppose that the consumer chooses a^* instead of $a < a^*$. Then, the

payoff difference is

$$\Delta(a, \sigma^2) := u(a^*) - u(a) - v \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a^*} + \gamma}} \right) + v \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a} + \gamma}} \right). \quad (13)$$

Note that $\lim_{n \rightarrow \infty} \Delta(a^*(\gamma, \sigma_n^2), \sigma_n^2) = u(a^*) - u(a^* - \varepsilon) > 0$. This implies that $\Delta(a^*(\gamma, \sigma_n^2), \sigma_n^2) > 0$ for a large n . Thus, for a large n , the consumer strictly prefers a^* to $a^*(\gamma, \sigma_n^2)$, which is a contradiction. A similar argument implies that $\lim_{\hat{\gamma} \rightarrow \infty} a^*(\hat{\gamma}, \sigma^2) = a^*$. \square

B The Long-run Outcome Under a Stationary Privacy Policy:

Proof of Proposition 1

Proof. To emphasize that the optimal activity level depends on v , I write $a^*(\gamma, \sigma^2)$ as $a^*(v, \gamma, \sigma^2)$.

Define $v^*(\gamma)$ as follows:

$$v^*(\gamma) = \sup \{v \in \mathbb{R} : a^*(v, \gamma, \sigma_0^2) > 0\}. \quad (14)$$

Note that $\frac{\partial U}{\partial a} = u'(a) - v \frac{1}{\left(\frac{1}{\sigma_0^2} (1 + \gamma a) + a\right)^2}$. This implies that

$$\begin{aligned} \frac{\partial U}{\partial a} \Big|_{a=0} &= u'(0) - v \cdot (\sigma_0^2)^2 \\ \frac{\partial U}{\partial a} \Big|_{a=a'} &\leq u'(0) - v \cdot \frac{1}{\left(\frac{1}{\sigma_0^2} (1 + \gamma a^*) + a^*\right)^2}, \forall a' \in [0, a^*]. \end{aligned}$$

The second inequality holds because $u(\cdot)$ and the privacy cost function are increasing and concave. For a sufficiently small v , the right hand side of the first equality is positive. Thus, $v^*(\gamma)$ is well-defined and positive. For a sufficiently large v , the right hand side of the second inequality is negative. Thus, $v^*(\gamma)$ is finite.

Suppose $v < v^*(\gamma)$. By the construction of $v^*(\gamma)$, $a^*(v, \gamma, \sigma_0^2) > 0$. $\sigma_t^2(\mathbf{a}_t^*)$ decreases in t for any sequence of activity levels. Thus, if γ_t is constant across $t \in \mathbb{N}$, then $a^*(v, \gamma_t, \sigma_t^2(\mathbf{a}_t^*))$ is

increasing in t and greater than $a_1^* > 0$ for all t . This implies that $\lim_{t \rightarrow \infty} \sigma_t^2(\mathbf{a}_t^*) = 0$, because

$$0 \leq \sigma_t^2(\mathbf{a}_t^*) \leq \frac{1}{\frac{1}{\sigma_0^2} + \frac{t}{\left(\frac{1}{a_1^*} + \gamma\right)}} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Lemma 2 implies $\lim_{t \rightarrow \infty} a_t^* \rightarrow a^*$. For $v > v^*(\gamma)$, note that $a^*(v, \gamma, \sigma_0^2) = 0$, which implies that $a_t^* = 0$ for all t . Finally, $v^*(\gamma)$ is increasing in γ , because $a^*(v, \gamma, \sigma_0^2)$ is increasing in γ . \square

C Properties of Equilibrium: Proof of Proposition 2

Proof. Let $(a_t^*, \hat{\sigma}_t^2)_{t \in \mathbb{N}}$ denote the equilibrium activity levels and conditional variances. First, I prove $\lim_{t \rightarrow \infty} \hat{\sigma}_t^2 = 0$. By **Lemma 2** and **Proposition 1**, there is a stationary privacy policy $\gamma_t \equiv \gamma$ such that $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$. By **Proposition 3**, $\hat{\sigma}_t^2 \leq \sigma_t^2$ for all $t \in \mathbb{N}$, which implies $\lim_{t \rightarrow \infty} \hat{\sigma}_t^2 = 0$.

To show $\lim_{t \rightarrow \infty} a_t^* = a^*$, suppose to the contrary that there is an $\varepsilon > 0$ such that $a_t^* \leq a^* - \varepsilon$ for infinitely many t 's. Without loss of generality, suppose $a_t^* \leq a^* - \varepsilon$ for all t . Following the proof of **Lemma 2**, we can conclude that, for a large t , the consumer strictly prefers a^* to a_t^* . Indeed, if the consumer chooses a^* instead of a_t^* , $u(\cdot)$ increases by at least $u(a^*) - u(a^* - \varepsilon) > 0$ whereas the increment of privacy cost goes to zero. This is a contradiction.

Next, suppose to the contrary that there is a $\underline{\gamma} > 0$ such that $\gamma_t^* \geq \underline{\gamma}$ for infinitely many t 's. To simplify exposition, suppose $\gamma_t^* \geq \underline{\gamma}$ for all $t \in \mathbb{N}$. Take any $\varepsilon \in (0, \underline{\gamma})$. Then, as $\lim_{t \rightarrow \infty} a_t^* = a^* > 0$, for a sufficiently large t , the minimized value in (5) is weakly greater than $\frac{1}{a^*} - \varepsilon + \underline{\gamma}$. To show a contradiction, let T denote the first period such that $a^*(0, \hat{\sigma}_{T-1}^2) > 0$. Then, $a^*(0, \hat{\sigma}_{t-1}^2) > 0$ for any $t \geq T$. If the platform chooses $\gamma_t = 0$ instead of γ_t^* in period $t \geq T$, then the minimand in (5) equals $\frac{1}{a^*(0, \hat{\sigma}_t^2)}$, which converges to $\frac{1}{a^*} < \frac{1}{a^*} - \varepsilon + \underline{\gamma}$ for a sufficiently large t . This implies that for a sufficiently large t , the platform can strictly increase its payoff in period t by setting $\gamma_t = 0$, which is a contradiction.

To show the final part, I write $\gamma_t^*(v)$ to emphasize the dependence of the equilibrium privacy level on v . Suppose to the contrary that there is a T such that, for any \underline{v} , there is some $v \geq \underline{v}$ such that $\gamma_t^*(v) = 0$ for some $t \leq T$. Then, we can find $v_n \rightarrow \infty$ and $t^* \leq T$ such that $\gamma_{t^*}^*(v_n) = 0$ for all n . However, for a sufficiently large v_n , $a_{t^*}^* = 0$ if $\gamma_{t^*}^*(v_n) = 0$. This follows from the proof of **Proposition 1**, where I show that the consumer with a sufficiently large v chooses $a_1^* = 0$ for a

fixed γ_1 . This contradicts the optimality of γ^* because if the platform sets a sufficiently large γ_{t^*} , then the consumer chooses a positive activity level and the minimand in (5) becomes finite (i.e. $\lim_{\hat{\gamma} \rightarrow \infty} a^*(\hat{\gamma}, \sigma^2) = a^*$). \square

D Equilibrium under Competition: Omitted Proofs in Section 4

D.1 Proof of Proposition 4

Proof. For each $k \in \{I, E\}$, I use $-k$ to mean E or I if $k = I$ and $k = E$, respectively. Suppose that, at the beginning of period $t \geq t^*$, the conditional variance for platform k is $\sigma_{t-1,k}^2$. Let γ_t^k denote the privacy level of platform k in period t . The consumer weakly prefers to use platform k (i.e. $a_t^{-k} = 0$ maximizes her period- t payoff) if

$$\begin{aligned} & \arg \max_{a \geq 0} u(a) - v[\sigma_0^2 - \sigma_{t,k}^2(\gamma_t^k, a | \sigma_{t-1,k}^2)] - v[\sigma_0^2 - \sigma_{t-1,-k}^2] \\ & \geq \arg \max_{a \geq 0} u(a) - v[\sigma_0^2 - \sigma_{t,-k}^2(\gamma_t^{-k}, a | \sigma_{t-1,-k}^2)] - v[\sigma_0^2 - \sigma_{t-1,k}^2], \end{aligned}$$

$\sigma_{t,k}^2(\gamma, a | \sigma_{t-1,k}^2)$ is the conditional variance at the end of period t when platform k chooses γ , the consumer chooses a , and the conditional variance from the previous period is $\sigma_{t-1,k}^2$. Arranging this inequality, we obtain

$$\arg \max_{a \geq 0} u(a) - v[\sigma_{t-1,k}^2 - \sigma_{t,k}^2(\gamma_t^k, a | \sigma_{t-1,k}^2)] \geq \arg \max_{a \geq 0} u(a) - v[\sigma_{t-1,-k}^2 - \sigma_{t,-k}^2(\gamma_t^k, a | \sigma_{t-1,-k}^2)]. \quad (15)$$

The above inequality implies that the consumer prefers to use k if and only if the gross benefit from the service net of the increment of privacy cost is greater for k than $-k$.

First, assume competition without commitment (a similar proof applies to competition with commitment). Consider the following strategy profile. For each period $t < t^*$, I chooses a monopoly privacy level γ_t^* . Take any period $t \geq t^*$. Let $k^* \in \arg \min_{k=I,E} \sigma_{t-1,k}^2$ denote the platform that has the lower conditional variance (if k^* is not unique, then set $k^* = I$). Then, platform $-k^*$ chooses the highest privacy level $\bar{\gamma}$. Platform k^* chooses a privacy level $\gamma_t^{k^*}$ that

solves

$$\begin{aligned}
& \min_{\gamma \in [0, \bar{\gamma}]} \frac{1}{a^*(\gamma, \sigma_{t-1, k^*}^2)} + \gamma \\
\text{s.t. } & \arg \max_{a \geq 0} u(a) - v[\sigma_{t-1, k^*}^2 - \sigma_{t, k^*}^2(\gamma, a | \sigma_{t-1, k^*}^2)] \\
& \geq \arg \max_{a \geq 0} u(a) - v[\sigma_{t-1, -k^*}^2 - \sigma_{t, -k^*}^2(\bar{\gamma}, a | \sigma_{t-1, -k^*}^2)].
\end{aligned} \tag{16}$$

In each period, the consumer myopically chooses a_t^I or (a_t^I, a_t^E) to maximize her per-period payoff. If indifferent, then the consumer uses the platform for which she chose a positive activity level in the most recent period. (If she chose zero activity levels up to period $t - 1$, then she sets $a_t^k = 0$ for one of $k \in \{I, E\}$ with equal probability, and chooses a_t^{-k} to maximize her period- t payoff.)

I show that the above strategy profile is an equilibrium. First, the consumer's behavior is optimal by construction. Second, I verify that platforms have no profitable deviation. Without loss of generality, consider a node in period t in which $I = k^*$ and $E = -k^*$. The strategy of E is optimal: By construction, even if E chooses $\bar{\gamma}$ in all periods $s \geq t$, the consumer uses I in any future periods as long as I and the consumer follow the above strategy.

Suppose now that I chooses a privacy level such that the consumer chooses E in period t . If $\sigma_{t, E}^2 \leq \sigma_{t, I}^2$, then the consumer uses E in any period $s \geq t + 1$. In this case, I 's deviation is not profitable. Otherwise, $\sigma_{t, E}^2 > \sigma_{t, I}^2$ hold. Note that I obtains a lower payoff in period t , because it is not maximizing the informativeness of the signal. Moreover, at any future period s , I faces an optimization problem

$$\begin{aligned}
& \min_{\gamma} \frac{1}{a^*(\gamma, \sigma_{s-1, I}^2)} + \gamma \\
\text{s.t. } & \arg \max_{a \geq 0} u(a) - v[\sigma_{s-1, I}^2 - \sigma_{s, I}^2(\gamma, a | \sigma_{s-1, I}^2)] \\
& \geq \arg \max_{a \geq 0} u(a) - v[\sigma_{s-1, E}^2 - \sigma_{s, E}^2(\bar{\gamma}, a | \sigma_{s-1, E}^2)].
\end{aligned} \tag{17}$$

After deviation, I faces a strictly lower $\sigma_{s-1, E}^2 - \sigma_{s, E}^2(\bar{\gamma}, a | \sigma_{s-1, E}^2) > 0$ because the consumer generated information on E in period t . This means that the set of γ that satisfies the constraint is smaller. Thus, the minimized value in (17) becomes greater for any period $s \geq t + 1$ after deviation. This implies that I 's payoff is weakly lower for any period $s \geq t$ after the deviation. A similar

argument implies that it is not profitable for I to deviate from a monopoly strategy before entry. This is because the deviation lowers I 's payoff before and after entry. In particular, the deviation shrinks the set of γ 's satisfying the constraint in (17) by increasing $\sigma_{s-1,I}^2 - \sigma_{s,I}^2(\gamma, a|\sigma_{s-1,I}^2)$.

$\lim_{t \rightarrow \infty} \sigma_{I,t}^2 = 0$ holds because it holds even if I adopts $\gamma_t = \bar{\gamma}$ for all t , and I chooses each γ_t^I to achieve even lower conditional variances. Given this result, $\lim_{t \rightarrow \infty} a_t^I = a^*$ follows the same proof as monopoly.

Suppose that γ_t^I does not converge to 0. Then, there is a convergent subsequence $\gamma_{t(n)}^I$ such that $\lim_{n \rightarrow \infty} \gamma_{t(n)}^I = \gamma' > 0$. For a sufficiently large n , both $\gamma = 0$ and $\gamma = \gamma_{t(n)}^I$ satisfy the constraint in (17), because $\sigma_{s-1,E}^2 - \sigma_{s,E}^2(\bar{\gamma}, a|\sigma_{s-1,E}^2) = \sigma_0^2 - \sigma_{1,E}^2(\bar{\gamma}, a^*(\bar{\gamma}, \sigma_0^2)|\sigma_0^2) > 0$, but $\lim_{s \rightarrow \infty} \sigma_{s-1,I}^2 - \sigma_{s,I}^2(0, a^*(0, \sigma_{s-1,I}^2)|\sigma_{s-1,I}^2) \leq \lim_{s \rightarrow \infty} \sigma_{s-1,I}^2 = 0$. As $n \rightarrow \infty$, the value of the objective converges to $\frac{1}{a^*}$ and $\frac{1}{a^*} + \gamma'$ for $\gamma = 0$ and $\gamma = \gamma'$, respectively. Thus, for a large n , $\gamma = 0$ achieves a strictly lower value in (17) than $\gamma = \gamma'$. This completes the proof.

For a sufficiently large t^* , $\sigma_{t^*-1,I}^2 \leq \sigma_0^2 - \sigma_{t^*,E}^2(\bar{\gamma}, a^*(\sigma_0^2, \bar{\gamma})|\sigma_0^2)$. Then, for any period $t \geq t^*$, the constraint (17) holds for any γ . This implies that I 's problem is equal to the monopolist's problem after the entry. Combined with the above result, I 's choice equals the monopolist's.

Finally, there is no equilibrium in which $a_t^E > 0$ and $a_t^I = 0$ for all $t \geq t^*$. This is because I can then choose $\bar{\gamma}$ for all periods. Given this, the consumer strictly prefers to use I for any period $t \geq t^* \geq 2$, because the consumer has generated information on I in periods $t < t^*$, which leads to a strictly lower marginal privacy cost. \square

D.2 Successful Entry: Proof of Proposition 5

Proof. Consider the following strategy profile: In any period $t \leq t^*$, I chooses a monopoly strategy. In any period $t \geq t^*$, I chooses $\bar{\gamma}$, whereas E solves

$$\begin{aligned} & \min_{\gamma \in [0, \bar{\gamma}]} \frac{1}{a^*(\gamma, \sigma_{t-1,E}^2)} + \gamma \\ \text{s.t. } & \arg \max_{a \geq 0} u(a) + \Delta - v[\sigma_{t-1,E}^2 - \sigma_{t,E}^2(\gamma, a|\sigma_{t-1,E}^2)] \\ & \geq \arg \max_{a \geq 0} u(a) - v[\sigma_{t-1,I}^2 - \sigma_{t,I}^2(\bar{\gamma}, a|\sigma_{t-1,I}^2)]. \end{aligned} \quad (18)$$

Let Δ^* denote the lowest Δ such that the set of γ 's that satisfy (18) is nonempty given $t = t^*$, $\sigma_{t^*-1,E}^2 = \sigma_0^2$, and the monopoly outcome $\sigma_{t^*-1,I}^2$. Δ^* is well-defined because the set of all γ 's satisfying the constraint is upper hemicontinuous. The rest of the strategy profile is specified analogously to Proposition 4. The same argument as Proposition 4 confirms that this is an equilibrium. Δ^* is increasing in t^* , because a larger t^* decreases $\sigma_{t^*-1,I}^2 - \sigma_{t^*,I}^2(\bar{\gamma}, a | \sigma_{t^*-1,I}^2)$.

Finally, suppose to the contrary that $\Delta < \Delta^*$. If I adopts a monopoly strategy for any $t < t^*$ and chooses $\gamma_t^I = \bar{\gamma}$ in period t^* , then the consumer strictly prefers to use I in t^* . This weakly increases I 's payoff for any period $t < t^*$ and strictly increases I 's payoff in period t^* . This is a contradiction. \square

E Erasing Past Information: Omitted Proofs from Section 5

E.1 The Right to be Forgotten: Proof of Proposition 6

Proof. Consider monopoly with commitment. Since the consumer's action does not affect a privacy policy, it is optimal for the consumer to erase information in all periods. Anticipating this, the platform maximizes the amount of information generated in each period by solving the problem (5) with $t = 1$. If the platform has only a short-run commitment power, then the platform sets γ_t to maximize the amount of information in each period. Because $\frac{1}{a^*(\sigma^2, \gamma)} + \gamma$ is increasing in σ^2 , it is optimal for the consumer to erase information, which leads to a weakly lower amount of information generated. In either case, the platform's problem leads to $\gamma_t = \gamma_1^*$ for all t .

For competition, consider the following strategy profile: Before entry, the consumer erases information in all periods, and chooses the activity level according to $a^*(\cdot, \cdot)$. After entry, the consumer erases information in all periods, and chooses the platform that offers $\gamma_t = \bar{\gamma}$ for all $t \geq t^*$ if there is such a platform (for a node in which both platform have deviated, I assign an equilibrium of that subgame arbitrarily). I sets $\gamma_t^I = \gamma_1^*$ for all $t < t^*$ and $\gamma_t^I = \bar{\gamma}$ for all $t \geq t^*$. E sets $\gamma_t^E = \bar{\gamma}$ for all $t \geq t^*$ upon entry. I can pick any equilibrium in any subgame in which the consumer deviates and chooses to not erase information, because the consumer is worse off relative to no deviation.

Finally, for any $\Delta > 0$, we can construct an equilibrium in which (i) the consumer erases information in all periods and sets $a_t^I = 0$ for any $t \geq t^*$, (ii) I sets $\bar{\gamma}$ in any period $t \geq t^*$, and

(iii) E sets γ_t^E that makes the consumer indifferent between I and E . This is an equilibrium with successful entry. \square

E.2 Data Retention Policies: Proof of Proposition 7

Proof. A monopolists' problem is to solve (5) by choosing a privacy level and whether to erase information. Whenever $\sigma_{t-1}^2 < \sigma_0^2$, erasing information strictly increases the conditional variance, increases the consumer's marginal cost, and shifts $a^*(\cdot, \sigma^2)$ downward. Thus, erasing information strictly lowers the platform's payoff.

In the model of competition, consider the strategy profile in which platforms never erase information on the path of play, and all players behave in the same way as the strategy profile constructed for Proposition 4. The action of each player straightforwardly extends to nodes in which a platform has deleted information, because the relevant state variable in that strategy profile is $(\sigma_{I,t-1}^2, \sigma_{E,t-1}^2)$. If a platform erases information, it lowers the payoff and increases the consumer's cost of using the platform. Thus, it is optimal for each platform to not erase information. \square

F Forward-looking Consumer: Proof of Proposition 8

This appendix consists of three steps. First, I prove the existence of an equilibrium in which the consumer breaks ties and chooses the “greatest” sequence of activity levels. Second, I prove useful properties of the consumer's value function in the dynamic programming. Finally, I use these results to prove Proposition 8.

I prepare notations. $\mathcal{A} := [0, a^*]^{\mathbb{N}}$ denotes the set of all sequences of activity levels between 0 and a^* . It is without loss of generality to exclude an activity level strictly above a^* . Let \mathbf{a} denote a generic element of \mathcal{A} , with the t -th coordinate denoted by a_t . Let Γ denote the set of all sequences of non-negative real numbers. Let γ denote a generic element of Γ , with the t -th coordinate denoted by γ_t . I consider product topology for \mathcal{A} and Γ .

F.1 Existence of an Equilibrium

Take any privacy policy $\gamma \in \Gamma$. The consumer's problem is

$$\max_{\mathbf{a} \in \mathcal{A}} \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\frac{1}{a_s} + \gamma_s}} \right) \right]. \quad (19)$$

For any $\gamma \in \Gamma$, let $\mathcal{A}^*(\gamma) \subset \mathcal{A}$ denote the set of all maximizers in (19).

Lemma 3. $\mathcal{A}^*(\gamma)$ is non-empty, compact, and upper hemicontinuous in product topology.

Proof. First, \mathcal{A} is compact with respect to product topology. Second, the objective function is continuous: To see this, take any sequence of the consumer's strategies $(\mathbf{a}^n)_{n=1}^{\infty}$ such that $\mathbf{a}^n \rightarrow \mathbf{a}^*$. This implies that, for each $t \in \mathbb{N}$, $\lim_{n \rightarrow \infty} a_t^n \rightarrow a_t^*$. The consumer's period- t payoff $U_t(\mathbf{a}, \gamma) := u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\frac{1}{a_s} + \gamma_s}} \right)$ is bounded from above and below by $u(a^*) > 0$ and $-v\sigma_0^2 < 0$, respectively. Define $B := \max(u(a^*), v\sigma_0^2) > 0$. Take any $\varepsilon > 0$, and let T^* satisfy $\frac{\delta_C^{T^*}}{1-\delta_C} B < \frac{\varepsilon}{4}$. Take a sufficiently large n so that, for each $t \leq T^*$, $\delta_C^{t-1} |U_t(\mathbf{a}^n, \gamma) - U_t(\mathbf{a}^*, \gamma)| < \frac{\varepsilon}{2T^*}$. These inequalities imply that

$$\left| \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\mathbf{a}^n, \gamma) - \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\mathbf{a}^*, \gamma) \right| < \varepsilon.$$

Thus, equation (19) is continuous in \mathbf{a} . For each privacy policy γ , let $\mathcal{A}^*(\gamma) \subset \mathcal{A}$ denote the set of all maximizers. Berge maximum theorem implies that $\mathcal{A}^*(\gamma)$ is non-empty, compact, and upper hemicontinuous (Aliprantis and Border, 2006). \square

Next, I prove some properties of the objective function $U(\mathbf{a}, \gamma) := \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\mathbf{a}, \gamma)$. Abusing notation, for any $\mathbf{a}, \mathbf{a}' \in \mathcal{A}$, write $\mathbf{a} \geq \mathbf{a}'$ if and only if $a_t \geq a'_t$ for all $t \in \mathbb{N}$. \geq is a partial order on \mathcal{A} , and (\mathcal{A}, \geq) is a lattice.

Lemma 4. For any γ , $U(\mathbf{a}, \gamma)$ is supermodular in \mathbf{a} .

Proof. Take any γ . Below, I omit γ and write $U(\cdot, \gamma)$ as $U(\cdot)$. Take any $\mathbf{a}, \mathbf{b} \in \mathcal{A}$. For each

$n \in \mathbb{N}$, define $(\mathbf{a} \vee \mathbf{b})^n$ as

$$(\mathbf{a} \vee \mathbf{b})^n = \begin{cases} \max(a_t, b_t) & \text{if } t \leq n, \\ a_t & \text{if } t > n. \end{cases} \quad (20)$$

Similarly, define $(\mathbf{a} \wedge \mathbf{b})^n$ as

$$(\mathbf{a} \wedge \mathbf{b})^n = \begin{cases} \min(a_t, b_t) & \text{if } t \leq n, \\ a_t & \text{if } t > n. \end{cases} \quad (21)$$

Also, define \mathbf{b}^n as

$$\mathbf{b}^n = \begin{cases} b_t & \text{if } t \leq n, \\ a_t & \text{if } t > n. \end{cases} \quad (22)$$

In product topology, $(\mathbf{a} \vee \mathbf{b})^n \rightarrow \mathbf{a} \vee \mathbf{b}$, $(\mathbf{a} \wedge \mathbf{b})^n \rightarrow \mathbf{a} \wedge \mathbf{b}$, and $\mathbf{b}^n \rightarrow \mathbf{b}$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, $U(\mathbf{a})$ is supermodular in the first n activity levels, $(a_1, \dots, a_n) \in \mathbb{R}_+^n$. Thus, $U((\mathbf{a} \vee \mathbf{b})^n) + U((\mathbf{a} \wedge \mathbf{b})^n) \geq U(\mathbf{a}) + U(\mathbf{b}^n)$. Since $U(\cdot)$ is continuous, we can take $n \rightarrow \infty$ and obtain $U(\mathbf{a} \vee \mathbf{b}) + U(\mathbf{a} \wedge \mathbf{b}) \geq U(\mathbf{a}) + U(\mathbf{b})$. \square

Lemma 5. *There is an $\bar{\mathbf{a}} \in \mathcal{A}^*(\gamma)$ such that, for any $\mathbf{a} \in \mathcal{A}^*(\gamma)$, $\bar{\mathbf{a}} \geq \mathbf{a}$.*

Proof. First, Corollary 2 of [Milgrom et al. \(1994\)](#) implies that $\mathcal{A}^*(\gamma)$ is a sublattice of \mathcal{A} . Since $\mathcal{A}^*(\gamma)$ is compact, for each $t \in \mathbb{N}$,

$$\mathcal{A}_t^*(\gamma) := \{a_t \in [0, a^*] : \exists \mathbf{a}_{-t} = (a_s)_{s \in \mathbb{N} \setminus \{t\}} \in \mathcal{A}^*(\gamma) \text{ s.t. } (a_t, \mathbf{a}_{-t}) \in \mathcal{A}^*(\gamma)\} \quad (23)$$

is compact (here, (a_t, \mathbf{a}_{-t}) is a sequence of activity levels such that the consumer takes a_t in period t and acts according to \mathbf{a}_{-t} in other periods). For each $k \in \mathbb{N}$, let \mathbf{a}^k denote an optimal policy such that $\mathbf{a}^k = \max \mathcal{A}_k^*(\gamma)$. Define $\bar{\mathbf{a}}^k := \mathbf{a}^1 \vee \dots \vee \mathbf{a}^k$. Since $\mathcal{A}^*(\gamma)$ is sublattice, for any each $k \in \mathbb{N}$, $\bar{\mathbf{a}}^k$ maximized (19). Also, $\bar{\mathbf{a}}^k \rightarrow \bar{\mathbf{a}}$, where $\bar{a}_t = \max \mathcal{A}_t^*(\gamma)$ for any $k \in \mathbb{N}$. Since $\mathcal{A}^*(\gamma)$ is compact, $\bar{\mathbf{a}} \in \mathcal{A}^*(\gamma)$. By construction, for any $\mathbf{a} \in \mathcal{A}^*(\gamma)$, $\bar{\mathbf{a}} \geq \mathbf{a}$. \square

For each $\gamma \in \Gamma$, let $\bar{\mathbf{a}}(\gamma) := (\bar{a}_t(\gamma))_{t \in \mathbb{N}}$ denote the “greatest” strategy of the consumer defined

in Lemma 5.

Lemma 6. *For each $t \in \mathbb{N}$, $\bar{a}_t(\gamma)$ is upper semicontinuous in γ .*

Proof. By Lemma 3, $\mathcal{A}^*(\gamma)$ is upper hemicontinuous. Thus, the set $\mathcal{A}_t^*(\gamma)$ of all activity levels for period t is upper hemicontinuous in γ . Thus, it is enough to show that for any upper hemicontinuous and compact-valued correspondence $\phi : X \rightrightarrows \mathbb{R}$, $f(x) := \max \phi(x)$ is upper semicontinuous. To show this, take any $x_n \rightarrow x$. For each n , define $y_n = f(x_n)$. Because there is a subsequence $y_{n(k)}$ of y_n that converges to $\limsup y_n$, it holds that $\limsup y_n = \lim y_{n(k)} = \lim f(x_{n(k)}) \leq f(\lim x_{n(k)}) = f(x)$. The inequality holds because ϕ has a closed graph. Connecting the left and right sides, we establish that $f(\cdot)$ is upper semicontinuous. \square

Lemma 7. *There exists an equilibrium.*

Proof. The platform's objective is

$$\sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\frac{1}{\bar{a}_s(\gamma)} + \gamma_s}} \right). \quad (24)$$

To show it is upper semicontinuous, take $\gamma^n \rightarrow \gamma$. Then,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\bar{a}_s(\gamma^n) + \gamma_s^n}} \right) \\
&= \lim_{k \rightarrow \infty} \sup_{n \geq k} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\bar{a}_s(\gamma^n) + \gamma_s^n}} \right) \\
&\leq \lim_{k \rightarrow \infty} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \sup_{n \geq k} \frac{1}{\bar{a}_s(\gamma^n) + \gamma_s^n}} \right) \\
&= \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \lim_{k \rightarrow \infty} \sup_{n \geq k} \frac{1}{\bar{a}_s(\gamma^n) + \gamma_s^n}} \right) \\
&= \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\liminf_{n \rightarrow \infty} \bar{a}_s(\gamma^n) + \gamma_s^n}} \right) \\
&\leq \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\limsup_{n \rightarrow \infty} \bar{a}_s(\gamma^n) + \lim_{k \rightarrow \infty} \inf_{n \geq k} \gamma_s^n}} \right) \\
&\leq \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\bar{a}_s(\gamma) + \gamma_s}} \right)
\end{aligned}$$

The second equality comes from the dominated convergence theorem, and the last inequality uses the upper semicontinuity of $\bar{a}_s(\gamma)$. Thus, given the consumer's optimal behavior, the platform's objective is upper semicontinuous. Since Γ is compact, there is a privacy policy γ^* that maximizes the platform's objective. $(\gamma^*, \bar{a}(\cdot))$ is an equilibrium. \square

F.2 Properties of Consumer Value Function

For each privacy policy $\gamma \in \Gamma$, define

$$V_\gamma(\rho) := \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(\bar{a}_t(\gamma)) - v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \sum_{s=1}^t \frac{1}{\bar{a}_s(\gamma) + \gamma_s}} \right) \right]. \quad (25)$$

$V_\gamma(\rho)$ is the consumer's maximum value of the objective starting from the conditional variance $\sigma^2 = \frac{1}{\rho}$.

Lemma 8. $V_\gamma(\cdot)$ is decreasing and convex. For any $\rho > 0$ and $\Delta > 0$, $\lim_{\rho \rightarrow \infty} V(\rho) - V(\rho + \Delta) = 0$.

Proof. Take any privacy policy γ . Hereafter, I omit γ from subscripts. Consider the “ T -period problem,” in which the consumer’s payoff in any period $s \geq T$ is exogenously set to zero. For any $t \leq T$, let $V_t^T(\rho)$ denote the consumer’s continuation value in the T -period problem starting from period t given $\frac{1}{\sigma_{t-1}^2} = \rho$:

$$V_t^T(\rho) = \max_{a_t, \dots, a_T} \sum_{s=t}^T \delta_C^{s-t} \left(u(a_s) - v \left(\sigma_0^2 - \frac{1}{\rho_{s-1} + \frac{1}{\frac{1}{a_s} + \gamma_s}} \right) \right).$$

Here, $(\rho_t, \dots, \rho_{T-1})$ are recursively defined by Bayes’ rule given (a_t, \dots, a_{T-1}) and $\rho_{t-1} = \rho$. The standard argument of dynamic programming implies that, for each $t = 1, \dots, T$,

$$V_t^T(\rho) = \max_{a \geq 0} u(a) - v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma_t}} \right) + \delta_C V_{t+1}^T \left(\rho + \frac{1}{\frac{1}{a} + \gamma_t} \right), \quad (26)$$

where $V_{T+1}^T(\cdot) \equiv 0$. I use induction to show that $V_1^T(\rho)$ is decreasing and convex. First, V_{T+1}^T is trivially (weakly) decreasing and convex. Suppose that V_{t+1}^T is decreasing and convex. Since $-v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma_t}} \right)$ is decreasing and convex in a , $V_t^T(\cdot)$ is decreasing and convex. Thus, $V_1^T(\cdot)$ is decreasing and convex.

Define $V^\infty(\rho) := \lim_{T \rightarrow \infty} V_1^T(\rho)$. $V^\infty(\rho)$ is also decreasing and convex, because these properties are preserved under pointwise convergence. I show that $V^\infty(\rho)$ is the value function of the original problem, i.e., $V^\infty(\cdot) = V(\cdot)$. Take any ρ , and let $(\bar{a}_1, \bar{a}_2, \dots) \in \mathcal{A}^*(\gamma)$ denote the optimal policy. For any finite T ,

$$V_1^T(\rho) \geq \sum_{s=1}^T \delta_C^{s-1} \left(u(\bar{a}_s) - v \left(\sigma_0^2 - \frac{1}{\rho_{s-1} + \frac{1}{\frac{1}{\bar{a}_s} + \gamma_s}} \right) \right). \quad (27)$$

By taking $t \rightarrow \infty$, we obtain $V^\infty(\rho) \geq V(\rho)$. Suppose to the contrary that $V^\infty(\rho) > V(\rho)$. Then, there is a sufficiently large $T \in \mathbb{N}$ such that $V_1^T(\rho) - \frac{\delta_C^T}{1-\delta_C} v \sigma_0^2 > V(\rho)$. If the consumer in the original infinite horizon problem adopts the T -optimal policy that gives $V_1^T(\rho)$ up to period t , then she can obtain a strictly greater payoff than $V(\rho)$, which is a contradiction. Thus, $V^\infty(\rho) = V(\rho)$.

Suppose that the consumer starting from $\rho + \Delta$ chooses the policy $(\bar{a}_t^\rho)_{t \in \mathbb{N}}$ that is optimal for ρ . Let $(\hat{\rho}_t)_{t=1}^\infty$ denote the induced sequence of the precisions after $\rho + \Delta$, i.e. $\hat{\rho}_t = \rho + \Delta + \sum_{s=1}^t \frac{1}{\frac{1}{a_s} + \gamma_s}$. Note that $\hat{\rho}_t \geq \rho_t$ for all $t \in \mathbb{N}$. Then, it holds that $0 \leq V(\rho) - V(\rho + \Delta) \leq \sum_{t=1}^\infty \delta_C^{t-1} \left(\frac{1}{\rho} - \frac{1}{\rho + \Delta} \right) = \frac{1}{1 - \delta_C} \left(\frac{1}{\rho} - \frac{1}{\rho + \Delta} \right)$. Take any $\varepsilon > 0$. Thus, $\lim_{\rho \rightarrow \infty} V(\rho) - V(\rho + \Delta) = 0$. \square

F.3 Consequences of Previous Lemmas

The following result extends [Proposition 1](#).

Proposition 10. *Take any $\gamma \in \Gamma$ such that $\gamma_t = \gamma$ for all $t \in \mathbb{N}$. Let $(\bar{a}_t)_{t \in \mathbb{N}}$ denote the equilibrium strategy in the subgame following γ . There is a $v^*(\gamma) > 0$ such that the following holds:*

1. *If $v < v^*(\gamma)$, then \bar{a}_t is increasing in t , $\lim_{t \rightarrow \infty} \bar{a}_t = a^*$, and $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$.*
2. *If $v > v^*(\gamma)$, then $\bar{a}_t = 0$ for all $t \in \mathbb{N}$.*

Moreover, $\lim_{\gamma \rightarrow \infty} v^*(\gamma) = \infty$.

Proof. Since γ_t is constant, $V(\cdot)$ satisfies the Bellman equation

$$V(\rho) = \max_{a \geq 0} u(a) - v \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma}} \right) + \delta_C V \left(\rho + \frac{1}{\frac{1}{a} + \gamma} \right). \quad (28)$$

[Lemma 8](#) implies that $V(\cdot)$ is decreasing and convex. Thus, the maximand in (28) has the increasing differences in (a, ρ) . Thus, $\bar{a}(v, \gamma, \rho)$, the greatest maximizer, is increasing in ρ . By the identical argument with the case of the myopic consumer, we obtain $\lim_{t \rightarrow \infty} \sigma_t^2 \rightarrow 0$, or equivalently, $\lim_{t \rightarrow \infty} \rho_t = \infty$ with $\rho := \frac{1}{\sigma_t^2}$. By [Lemma 8](#), for any $\rho > 0$ and $\Delta > 0$, $\lim_{\rho \rightarrow \infty} V(\rho) - V(\rho + \Delta) = 0$. This, combined with $\lim_{t \rightarrow \infty} \rho_t = \infty$, implies $\lim_{t \rightarrow \infty} \bar{a}_t(v, \gamma, \rho) = a^*$.

Define

$$v^*(\gamma) := \sup \{ v \in \mathbb{R} : \bar{a}_1(v, \gamma, \rho) > 0 \}. \quad (29)$$

$v^*(\gamma)$ is increasing because $\bar{a}_1(v, \gamma, \rho)$ is. Suppose to the contrary that there is a sequence $\gamma^n \rightarrow \infty$ such that $v^*(\gamma^n) \leq \bar{v}$ for some $\bar{v} < \infty$. Take the consumer with $v > \bar{v}$. Suppose that the consumer takes $a_t = a^* = \arg \max_{a \geq 0} u(a)$ for all $t \in \mathbb{N}$. As, $\gamma^n \rightarrow \infty$, for each t , the consumer's per-period

payoff in t converges to $u(a^*)$. As the consumer's objective is continuous in per-period payoffs with product topology, the sum of discounted payoffs converges to $\frac{u(a^*)}{1-\delta_C} > 0$. This contradicts that $v > \bar{v}$ should choose $a_t = 0$ for all t . \square

The following result is specific to the forward looking consumer. It states that whenever the change in a privacy policy leads to more informative signals in some periods, the consumer also chooses greater activity levels in other periods.

Lemma 9. *Take $\gamma, \gamma' \in \Gamma$. Define $\mathcal{T} = \{t \in \mathbb{N} : \gamma_t = \gamma'_t\}$. Suppose that $\frac{1}{\bar{a}_t(\gamma)} + \gamma_t \leq \frac{1}{\bar{a}_t(\gamma')} + \gamma'_t$ for all $t \in \mathbb{N} \setminus \mathcal{T}$. Then, $\bar{a}_t(\gamma) \geq \bar{a}_t(\gamma')$ for all $t \in \mathcal{T}$.*

Proof. Let β be any one of γ and γ' . I decompose the consumer's problem (19) into two steps. First, given any $(a_t)_{t \notin \mathcal{T}}$, the consumer chooses $(a_t)_{t \in \mathcal{T}}$ to maximize the following hypothetical objective function:

$$\sum_{t=1}^{\infty} \delta_C^{t-1} \left[\mathbf{1}_{\{t \in \mathcal{T}\}} u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^t \frac{1}{\frac{1}{a_s} + \beta_s}} \right) \right]. \quad (30)$$

Note that the consumer does not receive a benefit of $u(a_t)$ in period $t \notin \mathcal{T}$. This leads to a mapping that maps any $(a_t)_{t \notin \mathcal{T}}$ to the (greatest) optimal choice of $(a_t)_{t \in \mathcal{T}}$. In the second step, the consumer chooses $(a_t)_{t \notin \mathcal{T}}$ to maximize her original objective, taking the mapping $(a_t)_{t \notin \mathcal{T}} \mapsto (a_t)_{t \in \mathcal{T}}$ as given.

For any $t \notin \mathcal{T}$, a_t affects (30) only through $\frac{1}{a_t} + \gamma_t$, because $\mathbf{1}_{\{t \in \mathcal{T}\}} = 0$. Moreover, the same argument as in the proof of Lemma 4 implies that (30) is supermodular in $\left((a_t)_{t \in \mathcal{T}}, -\left(\frac{1}{a_s} + \gamma_s \right) \right)_{s \notin \mathcal{T}}$. This implies that if $\frac{1}{\bar{a}_t(\gamma)} + \gamma_t \leq \frac{1}{\bar{a}_t(\gamma')} + \gamma'_t$ for all $t \in \mathbb{N} \setminus \mathcal{T}$, then $\bar{a}_t(\gamma) \geq \bar{a}_t(\gamma')$ for all $t \in \mathcal{T}$. \square

F.4 $\lim_{t \rightarrow \infty} a_t^* = a^*$ and $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$: Proof of Proposition 8

Proof. Let γ^* denote the equilibrium privacy policy, and let a^* denote the equilibrium activity levels. First, I show $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$. Suppose to the contrary that $\lim_{t \rightarrow \infty} \sigma_t^2 \neq 0$. As σ_t^2 is weakly decreasing, it holds that $\lim_{t \rightarrow \infty} \sigma_t^2 = \hat{\sigma}^2 > 0$ exists. This implies that $a_t^* \rightarrow 0$, which I prove to be a contradiction.

By Proposition 10, there exists a γ^* such that $v^*(\gamma) > v$. That is, if the platform chooses $\gamma_t = \hat{\gamma}$ for all t , then the consumer chooses $\bar{a} > 0$ in $t = 1$. Define $B := \frac{1}{\bar{a}} + \hat{\gamma}$. Consider T^* such that,

for all $t \geq T^*$, $\frac{1}{a_t^*} > B$. Suppose that the platform replaces γ_t^* with $\hat{\gamma}$ for all $t \geq T^*$, and commits to such a new policy ex ante. Take any period $t \geq T^*$. Since the consumer's activity levels after T^* solve the Bellman equation with the “initial state” of $\rho = \frac{1}{\sigma_{T^*}^2} \geq \frac{1}{\sigma_0^2}$, the consumer chooses an activity level weakly greater than $\bar{a} > 0$ after period T^* . Thus, the variance of the noise $\varepsilon_t + z_t$ in the signal s_t is at most $\frac{1}{\bar{a}} + \hat{\gamma} < \frac{1}{a_t^*} + \gamma_t^*$. Thus, this change strictly increases the platform's profit in any period $t \geq T^*$. By Lemma 9, this change also increases the consumer's activity level for any period $t < T^*$. Thus, as a result of the deviation, the platform's payoffs increase in all periods and strictly increase in some periods, which contradicts γ^* being optimal. Thus, in equilibrium, $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$ holds. Finally, $\lim_{t \rightarrow \infty} \sigma_t^2 = 0$ implies $a_t^* \rightarrow a^*$: Otherwise, there is a convergent subsequence $a_{t(n)}^* \rightarrow a' < a^*$, however, the consumer could be strictly better off by choosing a^* , due to Lemma 8. \square

G General Privacy Cost: Proof of Proposition 9

Proof. Consider the consumer's problem in period t . Given the conditional variance σ^2 at the end of period $t - 1$ and the privacy level γ in period t , the consumer chooses a to maximize $U(a, \gamma, \sigma^2) := u(a) - C\left(\frac{1}{\frac{1}{\sigma^2} + \frac{1}{a} + \gamma}\right)$. It holds that

$$\frac{\partial U}{\partial a} = u'(a) + C'\left(\frac{1}{\frac{1}{\sigma^2} + \frac{1}{a} + \gamma}\right) \cdot \frac{1}{\left(\frac{1}{\sigma^2}(1 + \gamma a) + a\right)} \geq u'(a) - B \cdot \frac{1}{\left(\frac{1}{\sigma^2}(1 + \gamma a) + a\right)}, \quad (31)$$

where $B := \sup_{x \in [0, \sigma_0^2]} |C'(x)| < \infty$. If $\lim_{t \rightarrow \infty} \sigma_t^2 > 0$, then $\lim_{t \rightarrow \infty} a_t^* = 0$. This implies that $\lim_{t \rightarrow \infty} \frac{1}{a_t^*} + \gamma_t^* = \infty$. Consider a hypothetical payoff function

$$U_B(a, \gamma, \sigma^2) = u(a) - B \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{a} + \gamma} \right).$$

(31) implies that $\frac{\partial U}{\partial a} \geq \frac{\partial U_B}{\partial a}$. Take any γ' such that $a_B^*(\gamma', \sigma^2) := \max \{\arg \max_{a \geq 0} U_B(a, \gamma', \sigma^2)\} > 0$. Then, for any $\sigma^2 \leq \sigma_0^2$, $a^*(\gamma', \sigma^2) \geq a_B^*(\gamma', \sigma^2) \geq a_B^*(\gamma', \sigma_0^2) > 0$. Take T such that for all $t \geq T$, $\frac{1}{a_t^*} + \gamma_t^* \geq \frac{1}{a_B^*(\gamma', \sigma_0^2)} + \gamma'$. Then, the platform can achieve a lower $\frac{1}{a_t} + \gamma_t$ for any $t \geq T$ by replacing γ_t^* with γ' , which is a contradiction. A similar argument implies that $\lim_{t \rightarrow \infty} a_t^* = a^*$. \square