

Auction Timing and Market Thickness*

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Abstract

An auctioneer faces a pool of potential bidders that changes over time. She can delay the auction at a cost, in the hopes of having a thicker market later on. We identify a property of the distribution of bidder values—its “price elasticity”—that governs the distortions caused by revenue maximization: a seller inefficiently over-invests in market thickness (delays the auction excessively) if that elasticity is increasing, and under-invests if it is decreasing. We also show that dynamically responding to changes in the bidder pool is essential: committing to delay until an optimal deadline can waste most of the achievable revenue.

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1 Introduction

When an auction house sells a famous painting, it holds the auction in a room on a fixed date, and all bidders have to register with the house before participating. The study of auctions usually begins at this stage, with the assumption that the auctioneer knows the set of bidders in advance. This leaves out a crucial part of the design problem: the auctioneer is often uncertain about who will show up for the auction (and by when) and must trade off the delay costs from searching for bidders against the benefits of market thickness.

Consider, for instance, the market for distressed corporate assets, in which a substantial portion of sales happen via auction (Boone and Mulherin, 2009). After the firm decides to restructure and sell a division, it cannot take for granted that there will be a given set of bidders. Interested bidders for these assets are hard to find, and even when found, it takes considerable time and effort to motivate them to bid. The process usually involves both hiring an investment bank at significant cost and several months' delay. Boone and Mulherin (2009) note that the process takes more than six months on average, a very long and costly wait for a company in distress. While it delays the auction in hopes of having a thicker, more competitive market for its assets, the company risks missing out on a favorable market moment and possibly even entering bankruptcy. Moreover, it risks that promising current bidders may find other opportunities and leave.

Like the auction house selling a painting, the firm selling a division must decide how to allocate the object and determine prices. But the firm faces a new set of trade-offs. It must choose not only how to run the auction, but *when* to hold it: how long should the firm search for bidders?¹ Likewise, the set of market design concerns also changes. In the painting example, the main efficiency concern was whether the painting went to the highest value bidder. Now the selling firm can cause distortions both through its choice of auction format and through its choice of timing. Moreover, these two choices interact: for

¹ Real estate auctions have some similar features—the seller has a joint choice of auction format and auction timing—but we emphasize auctions for corporate auctions because discounting costs, which are crucial to our model, seem more significant in that market: Meanwhile, sellers of single family homes may not care about delaying an additional week or three to obtain one more bidder.

example, the inefficiencies caused by the auction format affect the firm’s incentives to delay the auction.

Motivated by this example, we study the problem of auction timing. The pool of potential bidders changes stochastically over time. Bidders have symmetric, independent private values for an indivisible good. An auctioneer chooses a timing policy and an auction format. We look at timing policies that rule out indicative bidding, i.e., bids are only taken after the auctioneer “stops” and runs the auction format,² but we allow the pool of potential bidders to evolve according to either (i) an arbitrary arrival process, or (ii) an arrival-departure process in a broad class. Considering this class of policies lets us focus squarely on the optimal choice of *market thickness*,³ and it leads to a clean distinction between static (auction format) and dynamic (auction timing) distortions.

First, we ask the following: do revenue-maximizing sellers over- or under-value market thickness, i.e., do they wait too long or too little to hold the auction, relative to the social planner? Notice, first, that there does not seem to be a clear reason to expect one or the other. A seller and a social planner face not only different marginal costs from waiting for an additional bidder—the former suffers discounting costs on the expected revenue from existing bidders, while the latter suffers discounting costs on the expected surplus—but also different marginal benefits, since the seller only cares about the expected marginal revenue as opposed to the expected marginal surplus from that extra bidder. It would seem that, by changing the distributions of values and the law of the arrival process for bidders, one could get these marginal benefit and marginal cost curves to cross any which way, so that

² Therefore, we do not solve the full dynamic mechanism design problem (see, e.g., [Gershkov, Moldovanu, and Strack \(2018\)](#) or [Board and Skrzypacz \(2016\)](#)), but we can consider a very general class of bidder dynamics. We discuss the advantages of our modeling approach in more detail in Section 2.

³ Recently, [Budish, Cramton, and Shim \(2015\)](#) have proposed a frequent batch auction design for securities. To the extent that one rules out having the frequency respond to the actual orders being received, one can interpret our model as a highly stylized model of the optimal choice of frequency for a batch auction. That literature has mostly focused on the endogenous choice of speed by traders, while our arrival-departure process for bidders in most of the analysis is exogenous. The present model is therefore limited in what it can say about the High Frequency Trading example, but the questions we raise have interesting analogues in that market. We discuss these analogues in the conclusion.

the comparison between the planner and the seller’s investment in market thickness would be entirely ambiguous.

Nevertheless, we identify a property of the distribution of bidder values that governs the distortions caused by revenue maximization. Recall [Bulow and Roberts \(1989\)](#)’s monopoly pricing interpretation of auction theory, where an auction problem facing a distribution of values F is equivalent to a third-degree price discrimination problem facing a demand curve $1 - F$. We show that, for all laws of motion for potential bidders in a broad class, a seller inefficiently over-invests in market thickness when that associated demand curve has an increasing price elasticity, and she under-invests for a decreasing elasticity.

Regular distributions in the sense of [Myerson \(1981\)](#) can have increasing or decreasing price elasticities, and therefore result in sellers that over- or under-invest in market thickness. On balance, however, we find that a revenue-maximizing seller inefficiently over-invests in market thickness, unless the distribution of bidders has very fat tails, in which case she will under-invest in market thickness. In particular, price elasticity is increasing for all increasing hazard rate distributions. Meanwhile, we show in [Remark 2](#) that distributions with decreasing elasticity must behave like Pareto or power law distributions in the tails.

We also uncover a subtlety in the inefficiencies caused by revenue maximization. We show that the wedge between a monopolist seller and the social planner is the composition of two distinct effects that work in opposite directions. On the one hand, conditional on using the same auction format, the seller evaluates timing policies according to virtual values as opposed to true values (an *information rent* effect). On the other, the seller wants to use a different selling procedure once she holds the auction, but this feeds back into the decision about when to hold the auction (an *auction format* effect). The auction format effect always pushes the auction earlier, while the information rent effect can push the auction either earlier or later, depending on the distribution of bidder values. Although the two effects can push in different directions, we show that the net effect depends on whether the elasticity of demand is increasing or decreasing, and moreover, the information rent effect always dominates. The following example illustrates how these two effects work:

Example 1. *Suppose that the auctioneer chooses when to hold an auction over a time*

horizon $t \in \{1, 2, 3\}$. One bidder arrives in each period; thus, an auction held in period t has t bidders. The auctioneer discounts the future at a rate $\delta = 0.7$, so at time t both revenue and surplus are discounted by δ^{t-1} . Bidders' values are drawn iid from the uniform distribution on $[0, 1]$. This distribution has an increasing hazard rate, so as we discuss in Remark 2, it has an increasing price elasticity. For each $n \in \{1, 2, 3\}$, let $R(n, 0)$ and $S(n)$ denote the expected revenue and surplus, respectively, from a second price auction with no reserve price and n bidders. Likewise, let $R(n, p^*)$ denote the expected revenue from a second price auction with reserve price $p^* = 1/2$, i.e., a revenue-optimal auction. Table 1 then presents the discounted expected revenue and surplus from each possible timing choice.⁴

t	1	2	3
$\delta^t R(t, 0)$	0	0.23	0.25
$\delta^t S(t)$	0.5	0.47	0.37
$\delta^t R(t, p^*)$	0.25	0.29	0.26

Table 1: Expected Discounted Payoffs - Second Price Auction

The results in Table 1 capture three general features. First, we see that the seller who can use an optimal auction holds the auction inefficiently late ($t = 2$ vs $t = 1$, the efficient choice), since the distribution has an increasing elasticity. Second, the seller who is forced to use a second price auction with no reserve price holds the auction later ($t = 3$) than the planner ($t = 1$)—an information rent effect. Hence, the information rent effect here leads to a later auction. Third, the seller who uses an optimal auction (i.e., chooses an optimal reserve price) holds the auction at $t = 2$, earlier than the seller who is forced to use an efficient (zero reserve price) auction—an auction format effect. Fourth, in this example, the information rent effect pushes the seller's timing later than the planner by two periods, while the auction format effect pushes the seller's timing earlier by one period. Altogether, the information rent effect dominates, so that the net effect of revenue maximization is to inefficiently delay the auction.

⁴The numbers are obtained from the formulae $R(t, 0) = \int_0^1 (2v - 1)tv^{t-1}dv$, $S(t) = \int_0^1 v \cdot tv^{t-1}dv$, and $R(t) = \int_{\frac{1}{2}}^1 (2v - 1)tv^{t-1}dv$.

An implication of these opposing effects is that regulating only the choice of auction format—out of concern over a seller’s inefficient use of market power, say—can *decrease* surplus because of the seller’s endogenous choice of auction timing.

As a consequence of our analysis, we develop a general monotone comparative statics result for the optimal auction timing as a function of the distribution of bidder values. We show that the key condition that determines the relative timing of an auction is how right-skewed the value distribution is, in a sense we define precisely below. If a distribution of types F is more right-skewed than a distribution H , we prove that the social planner holds an auction later under F regardless of how bidders arrive, so long as (a) they never depart, and (b) their arrival times are uninformative about their value for the good being auctioned. When F is more right-skewed than H , then the proportional growth in the highest order statistic from F is faster than that of H , since F has relatively more mass to the right of its support. This gives the social planner a stronger incentive to wait for another bidder under F . Of course, this intuition breaks down when bidders both arrive and depart, since a distribution of types where surplus grows very quickly with new arrivals is also one where surplus *shrinks* very quickly with departures. Nonetheless, our general comparative statics result partially extends to the case with departures: if a distribution of types F is more right-skewed than a distribution H , we prove that the social planner stops later under F so long as inter-arrival times are iid with a “New Better than Used” property (a generalization of the increasing hazard rate property), and bidders’ “lifetimes” in the market are iid exponentially distributed.

The second question we study is whether some simpler heuristics for auction timing that are used in practice can be “approximately” as good as the optimal policy for a broad range of environments (see, for instance, [Hartline and Roughgarden, 2009](#) and [Hartline, 2012](#)). For instance, auctioneers often commit in advance to a fixed date at which to hold the auction (e.g., real estate auctions are commonly run in this fashion). This nonadaptive stopping rule is simpler than changing the date as a function of the arrival and departure of bidders, so it is possible that real-world sellers perceive that fixed dates perform “well enough” across a variety of possible market conditions and are easier to implement. To investigate this

possibility, we verify whether fixed-in-advance deadlines can provide an “ α -approximation” (Hartline, 2012) to the auction timing problem: that is, can the seller always achieve at least a share α of the fully optimal expected discounted revenue in a given environment if she is restricted to using deadlines?⁵ Seen another way, what is the gain to the seller from being able to respond to the history bidder arrivals?

We show that even deadlines that are chosen optimally with full knowledge of the environment can perform arbitrarily poorly in the worst case over different arrival process. Even if the distribution of values is regular in the sense of Myerson (1981), and the arrival process is a renewal (a common tractable choice in applications, e.g., Gershkov, Moldovanu, and Strack (2015) and Zuckerman (1988)), the worst-case ratio between the optimal deadline and the optimal adaptive policy is 0. In contrast, under a strong condition on the arrival time distribution, optimal deadlines do guarantee at least $1/e$ of the best case payoff. The gains from having access the history of bidder arrivals can therefore be very large unless the seller believes there are strong restrictions on the bidder arrival dynamics.

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the environment and formally sets up the auction timing problem. Section 4 introduces the notion of right-skewness for a value distribution. Focusing on the case with no departures, this section establishes several comparative statics using that notion, and it studies the inefficiencies that arise from revenue-maximization. Section 5 extends these results to the case when bidders depart stochastically. Section 6 proves that deterministic timing rules (i.e., deadlines) can perform arbitrarily worse than the fully optimal adaptive rule, and Section 7 concludes. All omitted proofs are relegated to the Appendices.

2 Related Literature

Wang (1993) appears to be the first paper to study the timing of an auction as an endogenous

⁵ Of course fixing a single deadline across environments would perform poorly. Rather, the question is whether by choosing the best deadline for each environment, one can guarantee a given approximation ratio across all environments.

choice by a seller. Wang analyzes a model in which the seller has one unit of an object to sell to buyers who arrive according to a homogenous Poisson process. The seller who bears a constant flow cost from holding inventory can decide to sell the object either by posting a price or by holding an auction at a fixed deadline chosen in advance. Wang shows that the seller’s choice of timing typically differs from the socially optimal one. In contrast, we consider significantly more general arrival and departure dynamics, and we allow the seller much more leeway in choosing when to hold the auction. In our model, the auctioneer can use any stopping time that is adapted to the history of arrivals and departures. The recent paper by [Cong \(2018\)](#) also studies an auction timing problem, but in a very different context (the selling of real options to a fixed set of long-lived bidders). Among other things, he finds that the seller inefficiently delays the auction.

The present paper also relates to a recent literature on bidder solicitation. [Szech \(2011\)](#) and [Fang and Li \(2015\)](#) study a static environment in which an auctioneer must use costly advertising to attract bidders. In contrast to [Wang \(1993\)](#), [Szech \(2011\)](#), or [Fang and Li \(2015\)](#), the solicitation cost in our paper comes from time discounting. This makes the comparison between the seller and the planner harder and requires a different set of arguments: unlike advertising or linear (flow) waiting costs, costs from time discounting are evaluated differently by the seller and the planner. In Remark 3, we give a detailed discussion of the findings in this literature, and how they relate to our paper.

This paper fits within a broader dynamic mechanism design literature on how to allocate objects to stochastically arriving agents ([Gallien, 2006](#); [Gershkov et al., 2018](#); [Board and Skrzypacz, 2016](#); [Loertscher, Muir, and Taylor, 2016](#)). These papers derive a fully optimal dynamic mechanism for a more restricted class of agents’ arrival processes, and they focus especially on when these mechanisms can be implemented in posted prices. In contrast to these papers, we focus on the question of how to time the transaction. At the cost of looking at a more restricted class of mechanisms, this modeling approach has several advantages. Aside from the substantive focus it allows—emphasizing issues of market thickness/timing, as well as cleanly distinguishing between static and dynamic distortions—the approach en-

ables us to consider very general arrival and departure dynamics⁶ and provide monotone comparative statics results. In contrast with this literature, we can also identify conditions under which revenue maximization leads to inefficiently *low* market thickness.⁷ Finally, the dynamic matching literature had considered the robustness of simple rules (e.g., [Akbarpour et al. \(2018\)](#)), but the question seems to be new in dynamic mechanism design with transfers.

The optimal timing of an auction derived in Section 5 has some similarity to the search literature on the optimality of reservation wage policies, as in [Zuckerman \(1984, 1986, 1988\)](#). Zuckerman shows that, in a continuous time job search problem, a reservation wage policy is optimal under general renewal processes satisfying a “New-Better-than-Used” (NBU) property. In our model, a similar logic implies that, when bidders arrive according to an NBU renewal process and have exponential lifetimes, the optimal timing policy is characterized by a cutoff in the number of bidders.

3 Setting

We focus first on the case where bidders never leave the market—the case where bidders both arrive and depart is covered in Section 5. An auctioneer allocates one indivisible good to randomly arriving potential bidders over a time horizon $[0, \infty)$. The bidders arrive according to a counting process $\mathbf{N} = \{N_t, t \geq 0\}$ on some filtered probability space. Let $\mathcal{F}^{\mathbf{N}} = \{\mathcal{F}_t^{\mathbf{N}}\}_{t \geq 0}$ denote the filtration generated by \mathbf{N} . N_t is therefore a random variable representing the number of bidders at time t , with $t \mapsto N_t$ increasing.⁸ We assume that \mathbf{N} is right-continuous with left limits, and that both $\mathcal{F}^{\mathbf{N}}$ and the underlying probability

⁶ One contribution of this paper, relative to the above literature, is our partial characterization of optimal policies for the case with stochastically departing agents. This case is not very much studied. See [Akbarpour, Li, and Oveis Gharan \(2018\)](#), [Pai and Vohra \(2013\)](#), and ? for exceptions.

⁷ In a model of a dynamic two-sided market with random arrivals of finite-type traders, [Loertscher et al. \(2016\)](#) find that revenue maximization leads to inefficiently high market-thickness. The models are substantially different, since we consider a one-sided market but have a richer type space and more general bidder dynamics. We see the approaches as complementary.

⁸ Throughout the paper, we use “increasing” to mean “non-decreasing”. Similarly, by “decreasing” we mean “non-increasing”.

space are appropriately augmented to satisfy the “usual conditions” (Karatzas and Shreve, 1998). Below, when we use the term “counting process,” we implicitly mean an \mathbf{N} , \mathcal{F}^N , and associated filtered probability space satisfying these regularity conditions.

A bidder i is characterized by a value v_i for the good and an arrival time a_i . Each v_i is i ’s private information, and the v_i ’s are drawn iid according to a continuous distribution F independently from $\{N_t, t \geq 0\}$. Throughout the paper, we assume that (i) F has a finite expectation, (ii) F has a positive density f , (iii) $F(0) = 0$, and (iv) F is regular, i.e., $v - \frac{1-F(v)}{f(v)}$ is increasing.⁹

We interpret the arrival time a_i as the time at which i first has demand for (or is able to receive) the good, or as the time i first notices that an auction is taking place. In the corporate assets example, companies engage in months of informal discussion with many possible bidders who may or may not have any intention of actually bidding; a_i would then model the time at which i becomes a “serious” bidder. For most of the analysis, we assume that the auctioneer has access to \mathcal{F}^N (a_i is publicly observed). In Remark 5, we discuss to what extent our results carry over to the case where bidder i can strategically delay his arrival, i.e., report any $\hat{a}_i \geq a_i$. We do not impose specific assumptions regarding when a bidder learns his own value v . Our results hold as long as the bidder knows the realization of v when he makes a bid in the auction.

We assume that the auctioneer can commit to any policy consisting of

1. a timing policy, i.e., a stopping time τ at which to hold an auction, adapted *only* to \mathcal{F}^N , the history of arrival times.
2. an auction format to be used upon stopping, i.e., any mechanism mapping bids into allocations \mathbf{x} and transfers \mathbf{p} for bidders in the market at time τ .

Applying the Revelation Principle to the continuation game after the auctioneer has stopped, we can restrict ourselves to studying truthful direct mechanisms for the auction stage.¹⁰

⁹The assumptions $\mathbb{E}[v_i] < +\infty$ and $F(0) = 0$ imply that all the order statistics of v_i presented below have finite moments. Indeed, $\mathbb{E}[\max\{v_1, \dots, v_n\}] \leq n\mathbb{E}[v_i] < +\infty$. Since the i -th order statistic is always lower than the highest order statistic, this inequality implies that all the order statistics have finite moments.

¹⁰Technically, we only need to assume that the seller can commit to run any static mechanism once she

This restriction on the auctioneer’s space of policies rules out any kind of indicative bidding before the actual auction takes place—so the auctioneer cannot ask for non-binding initial offers from prospective bidders before the final, binding bids are due. This restriction is not without loss of generality.¹¹ Nevertheless, imposing the restriction has a number of advantages, of which we highlight three: (i) it lets us focus cleanly on the question of *timing* and market thickness; (ii) it enables us to consider a very broad range of arrival and departure processes; and (iii) it lets us distinguish static and dynamic distortions and understand the interplay between the two.

Preferences are as follows. If a bidder of value v gets the object and pays p in an auction held at time t , he receives a payoff of $e^{-rt}(v - p)$ where $r > 0$ is the discount factor.¹² If the auctioneer is a *seller*, she receives a payoff of $e^{-rt}p$, the discounted revenue. If on the other hand the auctioneer is a *social planner*, she receives a payoff $e^{-rt}v$, i.e., the discounted surplus.

Given our commitment assumption and our restriction on the class of mechanisms, we can pin down the auction formats that would be used by the seller and the planner. Fix any stopping time τ adapted to the history of arrivals, and any selling mechanism (\mathbf{x}, \mathbf{p}) . Then, the seller’s expected revenue weakly increases if she stops according to τ but instead of using (\mathbf{x}, \mathbf{p}) , she sells the good using an optimal incentive compatible mechanism for the pool of bidders that are present at τ . Therefore, since F is regular, by the Payoff Equivalence Theorem (Myerson, 1981), we can assume without loss of generality that, upon stopping, the seller uses a second price auction with an optimal reserve price $p^* = \inf\{p \in \text{supp} F : \text{stops, but she does not need commitment power with regard to timing. See Remark 4 for details.}$

¹¹ For instance, the fully optimal dynamic mechanism may not lie in this class. Under particular assumptions on the arrival processes $\{N_t, t \geq 0\}$, both Gallien (2006) and Board and Skrzypacz (2016) find optimal dynamic mechanisms that consist of deterministic sequences of posted prices. The optimal direct dynamic mechanism therefore does not decompose into a triple $(\tau, \mathbf{x}, \mathbf{p})$ consisting of a timing policy and a single auction.

¹² One could model i as suffering discounting not from $t = 0$, but starting at a_i . This modification will affect neither revenue calculations nor i ’s incentives for reporting v_i truthfully. However, this utility function has the unintuitive feature that the planner would privilege “young” bidders at the expense of “old” ones. We thank Gabriel Carroll for pointing out this interpretation.

$p - 1/\lambda(p) \geq 0\}$ where $\lambda(p) \equiv \frac{f(p)}{1-F(p)}$ denotes F 's hazard rate.¹³ Repeating the same reasoning for the planner, we can assume that she will use a second price auction with no reserve price. For future reference, let $MR(v) = v - 1/\lambda(v)$ denote v 's virtual value. We sometimes write MR_F and p_F^* to emphasize the dependence of MR and p^* on F .

Definition 1. A seller is *statically constrained* if she is free to choose any feasible (adapted to \mathcal{F}^N) timing policy τ , but she must use the social planner's preferred auction format, i.e., an efficient auction.

Let $R(n, p)$ denote the expected revenue from a second price auction with reserve price p , and $S(n)$ denote the expected surplus from a second price auction with no reserve price. Below, we write $R(n)$ as short-hand for $R(n, p^*)$ whenever it does not cause confusion. Note that $R(n, p^*) = \mathbb{E}[\max\{MR(v_1), \dots, MR(v_n), 0\}]$, $R(n, 0) = \mathbb{E}[\max\{MR(v_1), \dots, MR(v_n)\}]$, and $S(n) = \mathbb{E}[\max\{v_1, \dots, v_n\}]$. Then, by iterated expectations, the unconstrained seller solves

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} R(N_\tau, p^*) \right], \quad (1)$$

the statically constrained seller solves

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} R(N_\tau, 0) \right], \quad (2)$$

whereas the planner solves

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[e^{-r\tau} S(N_\tau) \right] \quad (3)$$

where \mathcal{T} is the set of stopping times adapted to \mathcal{F}^N . In other words, the auction timing problems for both the seller and the social planner reduce to optimal stopping problems with no flow cost and payoffs upon stopping of $R(n, p^*)$ and $S(n)$, respectively.

Our comparative statics results on auction timing can be stated in terms of Strong Set Order comparisons on sets of stopping times with a natural lattice structure. However, to simplify the exposition, we follow [Quah and Strulovici \(2013\)](#) in assuming that an auctioneer

¹³ $\text{supp} F$ denotes the support of F .

stops the first time she is indifferent between holding the auction and continuing to search for bidders. That is, we focus on the *essential infimum* of the solutions to the relevant optimal stopping problem.¹⁴

Definition 2. Consider two auctioneers A and B with preferences over auction times τ given by $\mathbb{E}[e^{-r\tau}S_A(N_\tau)]$ and $\mathbb{E}[e^{-r\tau}S_B(N_\tau)]$. Let \mathcal{T}_A^* and \mathcal{T}_B^* be the solution sets over \mathcal{T} to their respective optimal stopping problems. Say that “A stops later than B” if $\text{ess inf } \mathcal{T}_A^* \geq \text{ess inf } \mathcal{T}_B^*$.

Below we will make repeated use of the following key facts regarding payoffs upon stopping.

Fact 1. Let $Q(\cdot)$ be any one of $S(\cdot)$, $R(\cdot, p^*)$ and $R(\cdot, 0)$. Then, $Q(n)$ is

1. non-negative and increasing in n ;
2. concave in $n \in \mathbb{N}$, i.e., $Q(n+2) - Q(n+1) \leq Q(n+1) - Q(n), \forall n \in \mathbb{N}$;
3. their one-step ratios are decreasing, i.e., $\frac{Q(n+1)}{Q(n)}$ is decreasing in n .

Proof. See Appendix D. □

In words, additional bidders become less valuable the more bidders are already in the market.

4 Right-Skewness and Comparative Statics

We begin by introducing the following stochastic order, which is useful for studying how the optimal timing of an auction varies across different objectives and auction formats.

Definition 3 (Right-skewness). For two random variables X and Y , Y is *more right-skewed* if there exist positive, increasing functions g and h and a third random variable Z such that $X \sim g(Z)$, $Y \sim h(Z)$ and h/g is increasing.

¹⁴ As in Quah and Strulovici (2013), the assumptions on $\{N_t, t \geq 0\}$ and \mathcal{F}^N ensure that the essential infimum of solutions to the stopping problem is itself a solution to the stopping problem.

To understand why the above condition captures higher right-skewness, note that, whenever h/g is increasing, $h(Z)$ pushes large draws of Z further out into its right tail (relative to small draws) than $g(Z)$ does, and $h(Z)$ also shrinks small draws of Z towards the left tail more than $g(Z)$ does. This comparison is scale-invariant, so one should read the comparison “ Y is more right-skewed than X ” as “relatively more of Y ’s mass is on the right of its support,” as opposed to “ Y has more mass on the right than X does”. In the statistics literature, our right-skewness comparison corresponds to $h(Z)$ being larger than $g(Z)$ in the *star-shaped order*.¹⁵ Throughout, we use “more right-skewed” rather than “larger in the star-shaped order” in order to emphasize the underlying intuition.

The following lemma, which connects right-skewness comparisons to growth rates of expected highest draws, drives all of our comparative statics on auction timing:

Lemma 1. *Let $S_A(n-k; n)$ and $S_B(n-k; n)$, $k \in \mathbb{Z}_+$, denote the expected $k+1$ -th highest draws from n iid samples drawn according to the law of random variables X_i^A and X_i^B , respectively. If X_i^A is more right-skewed than X_i^B , then $S_A(n-k; n)/S_B(n-k; n)$ is increasing in n .*

Proof. See Appendix A. □

The intuition for Lemma 1 is easiest to see when $k = 0$ and $h(Z) = Z$ in Definition 3, i.e., $X_i^A = g(X_i^B)$ with $x \mapsto g(x)/x$ increasing. In this case, adding more bidders (i.e., more draws from the distribution) will grow total surplus (i.e., the sample maximum) proportionally more quickly. For instance, if a new draw exceeds the current sample maximum by 10% under X_i^B , the sample maximum under $X_i^A = g(X_i^B)$ grows by more than 10%. This reasoning works realization by realization, but Lemma 1 ensures that this pointwise intuition carries over rigorously to expected highest draws.

For convenience, write $S_m(n)$ as shorthand for $S_m(n; n)$ for $m = A, B$. Using Lemma 1, we can provide unambiguous comparative statics for any two distributions of bidder types that are ordered by their right-skewness.

¹⁵ See Barlow and Proschan (1966) for an insightful reference on the star-shaped order and connections to order statistics. The star-shaped order generalizes the convex transformation order, which was first introduced by Zwet (1964) as a formalization of right-skewness.

Lemma 2. *Let $\{N_t, t \geq 0\}$ be any counting process for bidder arrivals. Consider two social planners A and B facing bidder values drawn according to the laws of X_i^A and X_i^B , respectively. If X_i^A is **more right-skewed** than X_i^B , then social planner A holds the auction later than B .*

Proof. By Lemma 1, since A is facing a more right-skewed distribution, $\frac{S_A(n)}{S_B(n)}$ is increasing, so

$$\frac{S_A(n+1)}{S_A(n)} \geq \frac{S_B(n+1)}{S_B(n)}, \forall n \in \mathbb{N}.$$

Therefore, we have the following inequality:

$$\frac{S_A(n+k)}{S_A(n)} \geq \frac{S_B(n+k)}{S_B(n)}, \forall n \in \mathbb{N}, \forall k \in \mathbb{Z}_+. \quad (4)$$

Let τ_A and τ_B be the essential infima over optimal \mathcal{F}^N -stopping times for A and B , respectively. To derive a contradiction, suppose that $\tau_A < \tau_B$ with a positive probability. The inequalities below are evaluated on this event. Since τ_A is optimal for A ,

$$S_A(N_{\tau_A}) \geq \mathbb{E}[e^{-r\tau} S_A(N_{\tau_A+\tau}) | \mathcal{F}_{\tau_A}]$$

for any feasible stopping time τ such that $\{\tau = 0\}$ is $\mathcal{F}_{\tau_A}^N$ measurable. In particular, this inequality must hold for $\tau_{B|A}$, B 's optimal continuation stopping time starting at $\mathcal{F}_{\tau_A}^N$ (since $\tau_B > \tau_A$, $\{\tau_{B|A} = 0\}$ is \mathcal{F}_{τ_A} -measurable). Therefore,

$$1 \geq \mathbb{E}\left[e^{-r\tau_{B|A}} \frac{S_A(N_{\tau_A+\tau_{B|A}})}{S_A(N_{\tau_A})} \middle| \mathcal{F}_{\tau_A}^N\right] \geq \mathbb{E}\left[e^{-r\tau_{B|A}} \frac{S_B(N_{\tau_A+\tau_{B|A}})}{S_B(N_{\tau_A})} \middle| \mathcal{F}_{\tau_A}^N\right],$$

where the second inequality follows from $N_{\tau_A+\tau_{B|A}} \geq N_{\tau_A}$ (since there are no departures) and (4). Altogether $S_B(\tau_A) \geq \mathbb{E}[e^{-r\tau_{B|A}} S_B(N_{\tau_A+\tau_{B|A}}) | \mathcal{F}_{\tau_A}^N]$, so $\tau_{B|A} = 0$ is an optimal continuation policy for B at τ_A , which contradicts $\tau_B > \tau_A$. \square

These two lemmas deliver our first main result:

Theorem 1. *Let $\{N_t, t \geq 0\}$ be any counting process for bidder arrivals. If $\lambda(v)v = \frac{f(v)}{1-F(v)}v$ is increasing, then the seller holds the auction inefficiently late (later than the planner). If $\lambda(v)v$ is decreasing, the seller holds the auction inefficiently early.*

Proof. Under an optimal auction, the seller has the same expected payoffs as a social planner facing values $X_i \equiv MR(v_i) \vee 0$. By Lemma 2, it therefore suffices to check whether X_i is more or less right-skewed than v_i in the sense of Definition 3. Indeed, since

$$\frac{MR(v) \vee 0}{v} = \max \left\{ 1 - \frac{1}{\lambda(v)v}, 0 \right\},$$

X_i will be more (less) right-skewed than v_i whenever $\lambda(v)v$ is increasing (decreasing). \square

We make two comments on Theorem 1.

Remark 1 (Monopoly Theory Interpretation). The first comment is on how to interpret the quantity $\lambda(v)v$. Recall Bulow and Roberts (1989)’s monopoly pricing interpretation of optimal auctions. They show that an optimal auction design problem with n symmetric IPV bidders, each drawn from a value distribution F , is equivalent to a third-degree price discrimination problem for a capacity-constrained monopolist catering n markets with demand curves $q(p) = 1 - F(p)$. Simple calculus then yields $\frac{d \log q}{d \log p} = \lambda(p)p$. That is, $\lambda(v)v$ is the *elasticity with respect to price* of the associated monopoly pricing problem. Theorem 1 then says that the seller in the original auction problem over-invests in increasing the bidder pool when F ’s elasticity is increasing, and under-invests for decreasing elasticity.

To further unpack the economics behind our monotone price elasticity result, consider the following thought experiment from monopoly theory and third-degree price discrimination:

A monopolist facing a demand curve $q(p)$ is currently offering a share $Q < 1$ of its total capacity to one market. It has access to a “market creation” technology: it can open a new market and shift capacity away from the current market and into the new one. In the process of shifting capacity, the monopolist loses a share $1 - \beta$ of those resources. What distortions will the monopolist impose as it chooses to shift capacity?

Why focus on this particular thought experiment? Quantity in the Bulow and Roberts (1989)’s monopoly problem corresponds to the probability of receiving the good in the auction problem. Therefore, the loss of $1 - \beta$ resources when shifting capacity in the thought experiment corresponds to the way in which waiting for bidders in the auction problem incurs

a multiplicative cost from time discounting: the seller who delays the auction is reducing the probability of allocation in the present and allocating it to the future. Continuing the analogy, we can relate the seller's inefficient incentives to wait for bidders in the optimal auction timing problem to the monopolist's inefficient incentives to *open new markets* in the price discrimination problem.

We introduce some convenient notation. Let $P(q) \equiv F^{-1}(1 - q)$ be the inverse demand curve. Let $\tilde{M}R(q) \equiv \frac{d}{dq}(qP(q))$ denote marginal revenue as a function of quantity. Let $\eta(p) \equiv \frac{d \log q}{d \log p}$ denote the price elasticity of $q(p)$, and let $\tilde{\eta}(q) \equiv \eta(P(q))$ denote the price elasticity of demand evaluated at $P(q)$. Finally, recall the relationship between marginal revenue and demand elasticity from classical price theory, $\tilde{M}R(q) = P(q) \left(1 - \frac{1}{\tilde{\eta}(q)}\right)$.

Consider the incentives of the planner and the seller to reallocate a small amount of capacity dq from the current market to a new, previously uncatered market. To simplify the exposition, focus on Q such that $\tilde{M}R(Q) > 0$. We illustrate this thought experiment in Figure 1. The diagram on the left shows the original market, and the one on the right is the new, initially uncatered market. The lightly shaded red areas show gains and losses from shifting capacity for the seller, while the sums of the dark blue and lightly shaded red areas show gains and losses for the planner. Re-writing $\tilde{M}R(q)$ as $P(q) \left(1 - \frac{1}{\tilde{\eta}(q)}\right)$, we have

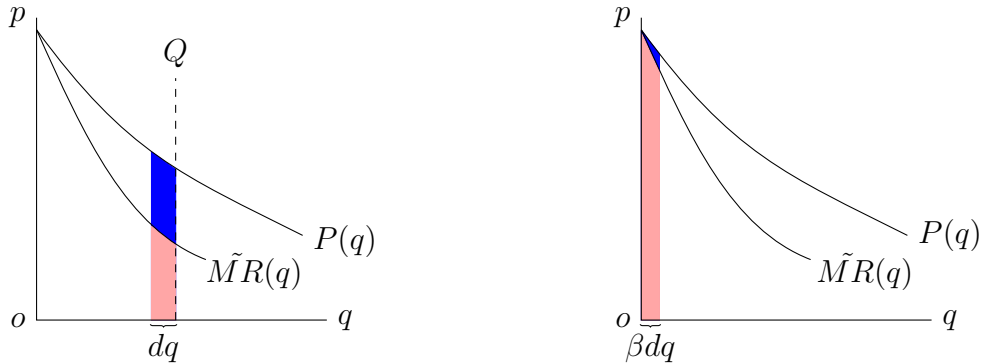


Figure 1: Shifting Consumers to a New Market

that the seller earns revenue $\beta P(0) \left(1 - \frac{1}{\tilde{\eta}(0)}\right) dq$ from the consumers who it now serves in the new market, but she loses revenue $P(Q) \left(1 - \frac{1}{\tilde{\eta}(Q)}\right) dq$ from consumers she used to serve in the current market. Meanwhile, the planner earns surplus $\beta P(0) dq$ from the new market,

while losing $P(Q)dq$ in the current one.

Take first the case where price elasticity of demand η is increasing, so $\tilde{\eta}$ is *decreasing*. We claim that, whenever the planner wants to move the dq units of capacity, the seller also wants to move that capacity. Indeed, if the planner wants to move the dq units, $P(Q) \leq \beta P(0)$. This, together with $\tilde{\eta}(q)$ decreasing, implies that

$$P(Q) \left(1 - \frac{1}{\tilde{\eta}(Q)}\right) \leq \beta P(0) \left(1 - \frac{1}{\tilde{\eta}(0)}\right)$$

so the seller also wants to shift capacity into the new market. Hence, the seller has *excessive* incentives to shift capacity towards new markets. Translated back into the original auction timing problem, we have as in the first part of Theorem 1 that the seller inefficiently delays the auction (over-invests in acquiring bidders) whenever price elasticity η is increasing.

A symmetric argument shows that, whenever $\tilde{\eta}$ is increasing (price elasticity η is decreasing), if the planner does not want to shift the dq consumers, *neither does the seller*, i.e., the seller has insufficient incentives to shift capacity towards new markets. This matches the second part of Theorem 1.

Remark 2 (Inefficient Investments in Market Thickness and Fat Tails). The second comment is about what Theorem 1 means, on balance, for the main question we posed in the introduction: does the seller over- or under-invest in market thickness? Our reading of Theorem 1 is as follows:

A revenue-maximizing seller inefficiently over-invests in market thickness, unless the distribution of bidders has very fat tails, in which case she will under-invest in market thickness.

To see why, note first that distributions with increasing hazard rate must trivially have increasing elasticity, so that the seller will over-invest in market thickness for many standard models.¹⁶

Second, distributions with decreasing elasticity must have fat (power law) tails. This follows from a result by Barlow, Marshall, and Proschan (1963):

¹⁶See Banciau and Mirchandani (2013) for a list of distributions with increasing $\lambda(v)v$.

Fact 2 (From Theorem 6.2 in Barlow et al. (1963)). Let X be a positive random variable drawn from F . Then $\mathbb{E}[X] < \infty$ if $\liminf_{v \rightarrow \infty} \lambda_F(v)v > 1$, and $\mathbb{E}[X] = +\infty$ if $\limsup_{v \rightarrow \infty} \lambda_F(v)v < 1$.

The knife-edge case of constant elasticity, with $\lambda(v) \propto 1/v$, is exactly the Pareto or power law distribution (Lariviere, 2006). Therefore, if F has a decreasing elasticity but a finite first moment, then by Fact 2 F must behave like a Pareto distribution in the tails.¹⁷

4.1 Comparative Statics

With Lemmas 1 and 2, we can characterize how the seller's optimal auction timing responds to changes in the value distribution.

Proposition 1. *Consider two regular value distributions $v^F \sim F$ and $v^H \sim H$, and let $\{N_t, t \geq 0\}$ be any counting process for bidder arrivals.*

1. *If F is more right-skewed than H , then the statically constrained seller holds the auction later when facing F than when facing H .*
2. *If $p_F^* \geq p_H^*$ and $MR_F(x)/MR_H(x)$ is increasing whenever $x \geq p_F^*$, the unconstrained seller holds the auction later when facing F than when facing H .*

The proof of the first point is almost identical to that of Lemma 2, except that it relies on the fact that expected revenue is the expected second highest value and uses Lemma 1 with $k = 1$ as opposed to $k = 0$. The second point follows from a technical extension of Lemma 1 (Lemma 4 in Appendix A) to the case where both h and g can be zero simultaneously (so h/g can be undefined), but the intuition is the same as in Lemma 2: under the above conditions, $MR_F(v_i^F) \vee 0$ is more right-skewed than $MR_H(v_i^H) \vee 0$, so the proportional boost to revenue

¹⁷ There do exist regular distributions with decreasing $\lambda(v)v$ and sufficiently many finite moments, so that the latter part of Theorem 1 is not vacuous. For instance, take $F(v) = 1 - \exp\left(-\int_a^v \left(\frac{1}{x^2} + \frac{2}{v}\right)dx\right)$ on $[a, \infty)$. Then $v - \frac{1-F(v)}{f(v)} = \frac{v+v^2}{1+2v}$ is increasing, while $\lambda(v)v = \frac{1+2v}{v}$ is decreasing. Moreover, $\lim_{v \rightarrow +\infty} \lambda(v)v = 2 > 1$, so Fact 2 implies that F has a finite first moment, and all the relevant expectations are well-defined.

from adding new bidders by waiting is greater under distribution F . Thinking about right-skewness also illuminates why the condition $p_F^* \geq p_H^*$ is necessary. Without this additional assumption, if $MR_H(v_i^H) \vee 0$'s atom at p_H^* is sufficiently far to the right of $MR_F(v_i^F) \vee 0$'s atom at p_F^* , $MR_F(v_i^F) \vee 0$ could fail to be more right-skewed than $MR_H(v_i^H) \vee 0$ even though $MR_F(\cdot)$ pushes large draws further into its right tail than $MR_H(\cdot)$.

4.2 Decomposition of Welfare Effects

Theorem 1 shows that the auction timing chosen by a revenue-seeking seller is typically inefficient. In this subsection, we show that this net welfare loss is the composition of two distinct effects:

Information Rent Effect: While the planner cares about bidder values, the seller cares about values net of information rents that accrue to bidders. Therefore, even if both auctioneers use the same efficient auction, they will face different terminal payoffs, and they will want to hold the auction at different times.

Auction Format Effect: The planner uses an efficient auction upon stopping, which differs from the revenue-optimal auction that the seller chooses upon stopping. This endogenous choice of auction formats creates a further wedge between the timing choices of the seller and the planner.

The auction format effect compares the unconstrained seller's choices to those of the statically constrained seller, while the information rent effect compares the statically constrained seller's choices to those of the social planner.

The following result summarizes how these effects influence the timing of an auction.

Proposition 2. *Let $\{N_t, t \geq 0\}$ be any counting process for bidder arrivals. If $\lambda(v)v$ is increasing,*

- *The statically constrained seller holds the auction later than the planner.*
- *The seller holds the auction later when she is statically constrained.*

If $\lambda(v)v$ is decreasing,

- The statically constrained seller holds the auction earlier than the planner.
- The seller holds the auction at the same time regardless of whether she is statically constrained.

Proof. See Appendix A. □

In sum, (i) the information rent effect always dominates, in the sense that the net effect of revenue maximization on timing always goes in the same direction as the information rent effect; (ii) the auction format effect always leads to weakly thinner markets (earlier auctions); and (iii) the auction format effect is *absent* for markets with decreasing price elasticity.

Proposition 2 has the following implication: *forcing the seller to use a (statically) efficient auction format can decrease the expected discounted welfare*. For instance, in Example 1, the unconstrained seller generates a discounted total surplus of 0.47, while the statically constrained seller generates a discounted total surplus of 0.37.¹⁸ More generally, for any arrival process such that multiple bidders never arrive at the same time, for a sufficiently high discount factor r , the statically constrained seller generates a lower discounted total surplus.¹⁹

Remark 3. Compared to previous *bidder solicitation models* (Szech, 2011; Fang and Li, 2015), we give a more complete characterization of the wedge between revenue and welfare maximization. These papers study static models where the auctioneer can choose the number of bidders to have in the auction. They assume that both the planner and the seller face an additive cost to solicit bidders that is increasing either in the number of bidders (Szech, 2011) or in the probability of a given bidder's attendance (Fang and Li, 2015). As in the current

¹⁸The numbers come from $\delta^2 \int_0^1 v \cdot 3v^2 dv \approx 0.47$ and $\delta \int_0^1 v \cdot 2v dv \approx 0.37$ when $\delta = 0.7$.

¹⁹This claim comes from the following observation. Fix F and $\{N_t, t \geq 0\}$. For a sufficiently large r , the seller with an optimal reserve price stops at the first bidder's arrival, and the statically constrained seller stops at the second bidder's arrival. For a large r , we obtain $\mathbb{E}[e^{-r\tau_2} S(2, 0)] < \mathbb{E}[e^{-r\tau_1} S(1, p^*)]$ where $\tau_n = \inf \{t : N_t = n\}$ for $n = 1, 2$ and $S(n, p)$ is the expected total surplus from an n -bidder second price auction with a reserve price of p .

paper, the auctioneer in these models decides how many bidders to have in the auction, comparing the cost of having one more bidder to the contribution to the revenue.²⁰

Szech (2011) and Fang and Li (2015) sign what we call the pure information rent effect: they show that, when the seller is forced to use an efficient auction format, she over- (under-) advertises relative to the planner whenever λ is increasing (decreasing). The intuition for this result follows from the expression for consumer surplus in an efficient auction, $CS(n) = \mathbb{E}[1/\lambda(v^{(n)})]$, where $v^{(n)}$ denotes the highest draw out of n samples from F . Since the seller and the planner face the same (additive) marginal cost from adding a bidder, the wedge between them depends only on the additional marginal gain, $\Delta S(n)$ vs. $\Delta R(n, 0)$. Using $S(n) = R(n, 0) + CS(n)$, one has that $CS(n+1) - CS(n) = \mathbb{E}[1/\lambda(v^{(n+1)})] - \mathbb{E}[1/\lambda(v^{(n)})]$ is exactly the wedge between the seller's and the planner's net incentives to solicit an additional bidder, starting from a pool size of $n - 1$. Hence, within the class of monotone hazard distributions, λ 's direction of increase governs whether the seller solicits too many or too few bidders for an efficient auction.

However, these papers do not fully sign what we call the net effect, i.e., the total distortions from revenue maximization, net of the choice of auction format. These papers show that, when the seller can both solicit bidders and design the auction optimally, she always under-advertises if the hazard rate λ is decreasing (arguably the less common case), but she may over- or under-advertise for increasing λ . Therefore, the monotone hazard rate condition does not govern distortions from revenue maximization in the same way that our monotone elasticity condition does. In contrast, in our model, the same condition allows us to compare timing/solicitation choices both net and gross of auction format choices, which helps with the economic interpretation of that condition.

Also note that in our setting, not only do the seller and the planner face different marginal benefits from “soliciting” bidders, but they also incur different marginal costs, which in principle complicates the comparison between the two. This follows from discounting. For

²⁰ More precisely, Fang and Li (2015) consider an auctioneer who maximizes a weighted sum of expected revenue and surplus, focusing mainly on a value distribution with a decreasing hazard rate. They also show that the auctioneer prefers to minimize the uncertainty over the number of participating bidders, when the expected number of bidders is fixed.

example, suppose that bidders arrive according to a Poisson process with the constant rate of 1. Then, the planner's cost from delaying the auction by dt is $rS(n)dt$, whereas the statically constrained seller's cost is $rR(n, 0)dt$.

Remark 4. To implement the optimal policy, the seller does not need quite as much commitment power as we assumed. As long as she can commit to sell the object through a one-shot static mechanism once she stops, the seller can implement the optimal policy even without the ability to commit to a timing rule.²¹ To see this, note that the ability to commit to any static mechanism lets the seller implement the optimal static mechanism once she stops, which in turn pins down the expected payoffs from stopping at $N_t = n$. Therefore, the seller faces a reduced-form optimal stopping problem, the solution to which must be time-consistent by the usual dynamic programming reasoning.

5 Departures

In this section, we extend the main result to the case where existing bidders can also leave the market. For instance, a bidder may leave if he finds an outside option better than taking part in the auction.

In general, when bidders can depart, the seller might stop earlier than the planner even if $\frac{f(v)}{1-F(v)}v$ is increasing. Recall that Theorem 1 in the previous section depends on the fact that, when $\frac{f(v)}{1-F(v)}v$ is increasing, the seller gains proportionally more than the planner from having an additional bidder. However, if a bidder can depart, this large gain from one more bidder necessarily implies a large *loss* from having one *fewer* bidder. Depending on the arrival-departure dynamics, the latter effect can dominate the former, so the seller may choose to hold the auction earlier than the planner. We illustrate this possibility in the following example.

Example 2. Suppose that there are two periods $t \in \{1, 2\}$, v_i is uniformly distributed on $[1, 2]$, and there is no discounting. Also, suppose that both the planner and the seller use

²¹Liu, Mierendorff, and Zhong (Forthcoming) study how the lack of commitment to leave an object unsold changes the seller's behavior.

a second price auction with no reserve price. (Note that F here is not only regular but has strictly increasing $\lambda(v)v$, which would guarantee a later auction by the seller in the pure arrivals case.) Define the following arrival and departure process: $N_1 = 2$, $\mathbf{P}(N_2 = 1) = 0.05$, $\mathbf{P}(N_2 = 2) = 0.7$, and $\mathbf{P}(N_2 = 3) = 0.25$. Under this parametrization, we can compute that, approximately, $R(N_1) = 1.333$, $\mathbb{E}[R(N_2)] = 1.308$, $S(N_1) = 1.667$, and $\mathbb{E}[S(N_2)] = 1.679$. Therefore, the planner waits until $t = 2$, while the seller strictly prefers to hold the auction at $t = 1$. In this case, the loss from departure is large enough to deter the seller from delaying the auction, even though she has a chance to hold the auction with three bidders if she waits.

We now formally set up the auction timing problem incorporating bidders' departures. For ease of exposition, we focus on revenue maximization, but the exact same analysis applies to surplus maximization (substituting values for marginal revenues).

To avoid the problem described in Example 2, we impose some structure on the arrival-departure process $\{N_t, t \geq 0\}$. First, we assume that bidders arrive according to a *renewal process*: the time between the arrival of the $k - 1$ -th and k -th bidders, $k = 1, 2, \dots$, is given by a non-negative random variable W_k , where $W_k \stackrel{iid}{\sim} G$. We assume that $\mathbb{E}[e^{-rW_k} | W_k \geq t]$ is well-defined for any $t \geq 0$.

Second, we assume that each bidder might leave after a random time in the pool. We model this stochastic departure by assuming that each bidder i has a sojourn time D_i in the pool, so that if i arrives at a_i , he departs at calendar time $a_i + D_i$. For simplicity, we assume that sojourn times D_i are drawn iid from an exponential distribution independently of the all values and arrival times. We assume that both i and the seller observe

1. a_i ;
2. at time $t \geq a_i$, whether $t \leq a_i + D_i$; but
3. neither can observe D_i .

In other words, both i and the seller know when i arrives and whether he is still available, but neither knows how much longer i might be available for. In Remark 5, we discuss how our results extend to the case where 1. and 2. above are violated, i.e., i can misreport his arrival

and availability. D_i , for instance, could represent the sudden arrival of a better outside option for a bidder, or, in the corporate takeover example, it could model the occurrence of an unexpected negative shock to liquidity that prevents the bidder from taking part in the auction.

Let \mathbb{L}_t denote the time elapsed since the most recent arrival at calendar time t , so that the last arrival occurred at $t - \mathbb{L}_t$. By the renewals assumption, the seller only needs to keep track of \mathbb{L}_t in order to predict future arrival times. Moreover, since the D_i 's are exponential, then by the memoryless property, current bidders' time in the market is irrelevant for predicting their departure times.²² Therefore, to know the future law of motion for \mathcal{N} at t , the seller only needs to know N_t and \mathbb{L}_t .

In addition, recall that the seller's expected terminal payoff from stopping with n bidders is $R(n)$. Then, since bidder values are iid and independent of the arrivals process, the seller faces a Markov optimal stopping problem with state variable (N_t, \mathbb{L}_t) . For future reference, let $V(n, w)$ denote the value function at state $(N_t, \mathbb{L}_t) = (n, w)$. Finally, let $\tau(n, n+1) \equiv \inf \{t \geq 0 : N_t = n+1\} - \inf \{t \geq 0 : N_t = n\}$ denote the random time between when the pool first reaches size n and when it first reaches $n+1$. (The possibility of departures makes the distribution of $\tau(n, n+1)$ different from G .)

The following result describes seller's optimal timing under a condition on the inter-arrival time distribution.

Definition 4. G is “New Better than Used” (NBU) if, for $a \sim G$, $t > 0$, $a \geq_{FOSD} a - t | \{a \geq t\}$.

NBU is a generalization of the increasing hazard rate (IHR) property: under the former, the seller is more optimistic about arrivals in between arrivals than right after an arrival, while under the latter, the seller gets increasingly optimistic about arrivals the longer it has

²² If we consider a general distribution for D_i , then the seller needs to keep track of the time that each current bidder has spent in the market: that time affects her beliefs about how likely the bidder is to leave in the next dt units of time, which in turn would affect the payoffs from stopping in the next dt . In this case, the seller has a large and growing state space in her optimal stopping problem, which makes further analysis intractable.

been since the last one.

Theorem 2. *Suppose that G is NBU and D_i is drawn iid from an exponential distribution. Let $\beta(n+1) = \mathbb{E}[e^{-r\tau(n,n+1)}]$ denote the (unconditional) expected discount factor between the first time the bidder pool reaches size n and the first time it reaches size $n+1$, and let*

$$n^* \equiv \inf \left\{ n \in \mathbb{N} : 1 \geq \beta(n+1) \frac{R(n+1)}{R(n)} \right\}. \quad (5)$$

Then, starting from an empty pool, the seller always holds the auction the first time the pool size reaches n^ .*

By Fact 1, nearly identical arguments hold for the planner.

Without additional structure on G , we have not been able to solve for the optimal timing policy for all initial states (n, w) . However, we can fully characterize the optimal policy for all sufficiently small initial bidder pools, and as we show below, all our comparative statics results from Section 4 extend to this case.²³

Before providing the proof, let us explain the role of the distributional assumptions. First, when G is NBU, continued search is least attractive right after an arrival, since at that point, the seller expects to wait the longest for an additional arrival. This leads the seller to stop only at arrivals, at least for small enough initial pools. Second, the two assumptions of renewal-process arrivals and exponential departures imply that, for each n , the evolution of N_t after first reaching a new record n is independent of all prior history (other than the fact that n was reached). This memorylessness allows us to decompose the waiting time for n bidders into independent one-step increments between successive records, which leads to the one-step characterization of the optimal stopping threshold.

Proof of Theorem 2. Let $\tau_{0,0}^*$ be the essential infimum over optimal stopping times starting from state $(0,0)$. Suppose, contrary to the theorem, that $\tau_{0,0}^*$ dictates stopping in between arrivals. That is, there exists a state (n, w) with $w > 0$, such that $R(n) \geq V(n, w)$. Then,

²³ By the usual dynamic programming arguments, the policy in Theorem 2 must be time-consistent. Hence, starting from any (n, w) that is reachable from $(0,0)$ before stopping, stopping the first time n^* is reached must also be optimal.

using Lemma 5, $R(n) \geq V(n, w) \geq V(n, 0)$, so if the seller weakly prefers stopping at (n, w) , she would also prefer it at $(n, 0)$. In any history starting from state $(0, 0)$, $(n, 0)$ is always reached before (n, w) , so $\tau_{0,0}^*$ would have dictated stopping at $(n, 0)$ without ever reaching (n, w) , a contradiction. Therefore, if starting from $(0, 0)$ the seller stops at all, she stops at an arrival. Accordingly, we need to check that the seller would indeed stop in finite time almost surely. But this is immediate because $R(1) > 0$ and the payoff from never stopping is 0. Altogether, we have that the auction will happen the first time N_t reaches an acceptable threshold, i.e., some n such that $R(n) \geq V(n, 0)$. Let n^* denote the smallest such n .

It remains to prove that in fact n^* is given by (5). The optimal stopping policy must be time-consistent, so if the seller stops at n^* , then n^* must be the smallest maximizer of the ex ante expected discounted revenue from waiting for exactly n bidders. That is, n^* must be the smallest maximizer of $\delta(n)R(n)$, where $\delta(n)$ is the expected discount factor from waiting for n bidders. By Lemma 6 in Appendix B, our assumptions on bidder dynamics imply that the n -step discount (from waiting for n bidders) decomposes into $\delta(n) = \prod_{i=1}^n \beta(i)$. The inequality $R(n) \geq \beta(n+1)R(n+1)$ is therefore a necessary local condition for optimization of $\delta(n)R(n)$, i.e., it implies that $\delta(n)R(n) \geq \delta(n+1)R(n+1)$. Moreover, a coupling argument (Lemma 7 in Appendix B) implies that the successive record times $\tau(n-1, n)$ increase in the first-order stochastic dominance sense as n grows, so $\beta(n) = \mathbb{E}[e^{-r\tau(n-1, n)}]$ is decreasing. Together with Point 3 of Fact 1, since $\beta(\cdot)$ is decreasing, so is $n \mapsto \beta(n+1)\frac{R(n+1)}{R(n)}$. It follows that $\prod_{i=1}^n \beta(i)R(n)$ is single-peaked. Therefore the condition in (5) is in fact a sufficient condition for maximizing $\delta(n)R(n)$, and the optimal policy starting from $(0, 0)$ stops the first time n^* is reached. \square

The one-step characterization of n^* in (5) has an especially tractable structure. The critical threshold function $\beta(n+1)\frac{R(n+1)}{R(n)}$ decouples into a term that depends only on the physical properties of the arrival-departure process ($\beta(n+1)$), and a term that depends only on the proportional gain from having an additional bidder in the pool ($\frac{R(n+1)}{R(n)}$).

This allows us to extend Propositions 1 and 2 to the case with departures:

Theorem 3. *Suppose that G is NBU and D_i is drawn iid from an exponential distribution.*

Then, starting from an empty pool, all the comparative statics results in Theorem 1 and Proposition 2 continue to hold.

Proof. We show that the argument in Lemma 2 applies with this particular structure on $\{N_t, t \geq 0\}$. From the expression for the optimal stopping time in (5), we can see that planner A waits for a larger number of bidders than planner B if $\frac{S_A(n+1)}{S_A(n)} \geq \frac{S_B(n+1)}{S_B(n)}$ for all n , because any n that satisfies the inequality in the infimum defining n_A^* will satisfy the inequality defining n_B^* . Hence, a planner facing this kind of arrival-departure process will stop later when she confronts a more right-skewed value distribution. The same argument as in Theorem 1, Proposition 1, and Proposition 2 then establishes the result. \square

Again, the same result holds for all sufficiently small initial pools.

Remark 5 (Strategic Arrivals and Departures). So far, we have assumed that the seller can observe the arrivals and availability of bidders. However, this assumption might not always be innocuous. For instance, a sophisticated bidder in an auction for corporate assets may pretend that getting financial backing for a bid takes longer than is actually necessary, if it thinks that by delaying it can place the seller in a disadvantageous position. Whether or not the optimal timing policy is implementable when arrivals and availability are privately known will depend on the form of that policy, and therefore indirectly on the law of motion for the bidder pool.²⁴

To study this possibility, assume i privately observes his arrival time a_i , and only he knows at t whether $a_i + D_i \leq t$. (So even though D_i is unobserved, i observes whether or not “his time is up.”) Therefore, i can delay his arrival until any $a'_i \geq a_i$ such that his time is not up at a'_i , and conditional on reporting an arrival at a'_i , he can pretend to have left at any $t \geq a'_i$ if his time is not up by t .²⁵

For the class of arrival-departure processes we have considered in this section, bidders will in fact want to report their arrivals and availability truthfully. The proof adapts an argument

²⁴ For example, Gershkov et al. (2015) show that an efficient dynamic mechanism with observable arrivals may not be implementable if later arrivals make the seller pessimistic about the time of future arrivals.

²⁵ Similar to the literature on auctions with private budget constraints (Che and Gale, 1998), we assume that agents can only engage in one-sided deviations in reporting their arrival and departure status.

by [Gershkov et al. \(2015\)](#) to the case of auction timing. Let N'_t denote the number of bidders observed by the auctioneer at t when bidder i misreports his arrival or departure time. Note that, however i chooses to strategically time his arrival and/or departure, $N'_t \leq N_t$ always holds. In particular, N'_t never hits n^* strictly earlier than N_t . Thus, bidder i either (i) misses the opportunity to participate in the auction and obtains a payoff of zero, or (ii) joins an n^* -bidder auction that will be held later than if i had truthfully reported his arrival and departure times. Bidder i does not benefit from the deviation in (i), because the payoff from the auction is non-negative. Nor does he benefit in the case (ii), because he must incur the cost of additional discounting without changing the number of opposing bidders he faces in the auction. Therefore, the cut-off policy in Theorem 2 is robust to bidders' strategic incentive to "misreport" the timing of their arrival and departure.²⁶

6 Impossibility of α -approximation

In the spirit of a large literature on approximation in mechanism design, one might hope that the seller could do reasonably well by restricting herself to simpler mechanisms.²⁷ In particular, following common practice in, say, real estate auctions, one could conjecture that holding the auction at a given date, fixed in advance, would perform well across a range of environments. In the jargon of approximate mechanism design ([Hartline, 2012](#)), do fixed-in-advance deadlines can provide an " α -approximation" to the auction timing problem? That is, do fixed deadlines always achieve at least a share α of the fully optimal expected discounted revenue, regardless of the distribution of values or arrivals? Alas, the following result shows that the relative loss to the seller from restricting herself to holding the auction at a fixed date (even if the date is set optimally in advance, with full knowledge of the environment) is

²⁶ Here, we show implementability by using specific properties of the stopping region derived from the NBU assumption. However, an almost identical argument shows that bidders will report arrivals truthfully whenever arrivals are given by an arbitrary renewal process, in which case the shape of the stopping region is *a priori* unclear.

²⁷ See, for instance, [Hartline and Roughgarden \(2009\)](#) and [Hartline \(2012\)](#) for approximation results in static auctions.

unbounded. In other words, no α -approximation of optimal revenue is possible: in the worst case over arrival processes, if the seller cannot react to the size of the bidder pool, she will obtain only a vanishing fraction of the revenue generated by an optimal adaptive policy.

The formal analysis proceeds as follows. First, we restrict the class of the auction timing problems. Namely, we consider any revenue-maximization problem such that the bidders arrive according to a renewal process with inter-arrival distribution G , and there are no departures (see Section 5 for the definition of a renewal process). Thus, any auction timing problem we consider is described by a triplet (r, F, G) of discount factor r , value distribution F , the inter-arrival distribution G .

Now, given (r, F, G) , let $OPT(r, F, G)$ be the ex-ante expected revenue from the optimal timing and auction format policy. Also, let $DET(r, F, G)$ be the maximum expected revenue when the seller is restricted to using a deterministic timing policy, i.e., a *deadline*.²⁸ That is, $DET(r, F, G)$ is the maximized value of the problem $\max_{t \in \mathbb{R}_+} e^{-rt} \mathbb{E}[R(N_t)]$. The following result states that, without further restriction on the class of auction timing problems, even the best deterministic stopping rule can waste almost all of the revenue attained by an optimal history-contingent policy.

Theorem 4. *Fix any $r > 0$. Then,*

$$\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \frac{DET(r, F, G)}{OPT(r, F, G)} = 0, \quad (6)$$

where \mathcal{F} is the set of regular distributions, and \mathcal{G} is the set of probability distributions whose supports are subsets of \mathbb{R}_{++} .

The proof, in Appendix C, proceeds by explicitly constructing a sequence of examples that attains the infimum. G is supported on a finite grid of points, and along the sequence, all mass is moving to the right end of the grid. If we take a nearly degenerate value distribution (so that revenue from an auction does not really increase by adding additional bidders beyond the first one), we can then choose grid points so that along the sequence, the optimal deterministic deadline is always to plan on holding the auction at the first time

²⁸ This is the class of policies studied by Wang (1993) in a model with exponential G .

grid point, whereas the fully optimal policy is to wait for the first arrival. Therefore, as we move along the sequence, the best deterministic policy leaves the good unsold with probability approaching one, and achieves a payoff approaching zero. Meanwhile, the optimal history-contingent policy makes sure the good is sold by always waiting for the first bidder, guaranteeing a payoff that is always positive, but far in the future and shrinking along the sequence.

As shown in the proof, along the sequence $\{(F_n, G_n)\}_{n=1}^\infty$, both $DET(r, F_n, G_n)$ and $OPT(r, F_n, G_n)$ converge to zero: that is, by using a fixed (deterministic) deadline, the seller ends up capturing only a vanishing fraction of a vanishing quantity. However, the seller’s loss may not be small in absolute terms because for each fixed n , scaling up v would make $OPT - DET$ arbitrarily large. Given that we can vary the loss in absolute terms by manipulating F , Theorem 4 should not be read as a quantitative statement about the size of losses incurred by deterministic deadlines. Rather, we interpret the result in two ways: unless we restrict the primitives of the timing problem, (i) deterministic deadlines are not an appropriate subclass of stopping times to focus on; and (ii), the gain to the seller from being able to access and react to the history of arrivals can be arbitrarily large.

Our particular construction might give the impression that (optimal) deadlines perform worse if inter-arrival times are more dispersed. This intuition is incorrect, and we can find examples in which a deadline “approximates” the optimal stopping policy even in the limit as G_n becomes arbitrarily dispersed. For example, if we restrict \mathcal{G} to the collection of the uniform distributions $U[0, \bar{T}]$, then given a nearly degenerate F , we obtain $\frac{DET}{OPT} = 1/2$ as $\bar{T} \rightarrow \infty$. Analogously, taking \mathcal{G} to be the collection of all exponential distributions leads to an approximation ratio of e^{-1} . Thus, dispersion is not the only source of our impossibility result. We speculate that, in addition to dispersion, extreme skewness in inter-arrival times is essential for fixed deadlines to perform poorly. We provide some confirmation of this fact in the next subsection: when random discounts have right tails that decrease sufficiently fast (as captured by the increasing hazard rate property), optimal deadlines provide a constant approximation of the best case revenue.

6.1 A Partial Approximation Result

Under strong assumptions on the nature of arrivals and the form of the optimal rule, we can obtain a $1/e$ approximation:

Proposition 3. *Fix any $r > 0$. Let \mathcal{G}^{DIHR+} be the class of inter-arrival time distributions such that (i) for each $n \in \mathbb{N}$, the random discount $e^{-r(T_1+\dots+T_n)}$ until n bidders is IHR, and (ii) there exists $n^* \in \mathbb{N}$ such that the fully optimal auction timing is to wait until n^* -th bidder arrives. Let \mathcal{F} be the set of regular value distributions. Then*

$$\inf_{F \in \mathcal{F}, G \in \mathcal{G}^{DIHR+}} \frac{DET(r, F, G)}{OPT(r, F, G)} \geq \frac{1}{e}. \quad (7)$$

At first glance, condition (ii) might seem like an assumption on the value distribution F . However, by Fact 1, the qualitative properties of terminal payoffs $R(n)$ are essentially identical for any regular F , so in the end the qualitative form of the stopping rule depends on the nature of the arrival process. For example, if G is NBU, then by an argument similar to the one in Theorem 2, the optimal rule stops at some $n^* \in \mathbb{N}$ (the specific value of n^* may depend on parameters like r and F).

The proof, in Appendix C, uses an analogy between our timing problem and the static problem of calculating the worst-case ratio of expected revenue from an optimal posted price to expected efficient surplus in a monopoly pricing problem. Lemma 3.10 in [Dhangwatnotai, Roughgarden, and Yan \(2015\)](#) provides a lower bound of this latter quantity, which we can translate into the current problem.

7 Conclusion

We have studied the question of when to hold an auction in a setting where potential bidders arrive and depart stochastically over time. We show that, for a very general class of arrival and departure processes, a social planner holds an auction later as the distribution of bidder values becomes more right-skewed. As a consequence, we identify a key condition on the

distribution of bidder values that governs the inefficiencies imposed by a revenue-maximizing seller: she over-invests in market thickness (waits for bidders excessively) if the distribution’s price elasticity is everywhere increasing, and she under-invests if that elasticity is everywhere decreasing. We then argue that sellers will over-invest in market thickness unless they face bidders from a distribution with very fat tails.

Moreover, we decompose the welfare losses from revenue maximization into an information rent effect (conditional on using the same selling mechanism, the seller values stopping at each state according to virtual values as opposed to true values) and an auction format effect (the seller wants to use an inefficient selling mechanism upon stopping, but this feeds back into the decision about when to stop). The former leads the seller to hold the auction earlier or later depending on whether the elasticity of demand with respect to price is decreasing or increasing, while the latter always leads the seller to hold the auction earlier. Consequently, we show that even though the two effects can move the optimal timing in opposite directions, whenever we can unambiguously sign the information rent effect, it dominates. Finally, we prove that a fixed deadline, even an optimal one, cannot approximate the performance of the optimal stopping rule without strong restrictions on the arrival process. This suggests that the potential benefit from adopting a history-contingent timing rule may be substantial.

We conclude by highlighting some questions for future research. First, it would be useful to better understand the features of an arrival process that make nonadaptive deadlines perform well compared to the optimal adaptive policy. What exactly are the elements of \mathcal{G}^{DIHR+} , and is there a natural sufficient condition that implies membership in this class?

Second, our discussion of the robustness of simple rules has been somewhat limited, since the rules we consider, while “simple,” still depend on a lot of fine-grained knowledge about the environment. A key question would be whether one can design *prior-free* dynamic mechanisms with good guarantees, i.e., mechanisms whose very description does not depend on the prior distributions of values and arrival times. In a static environment, [Bulow and Klemperer \(1996\)](#) show that a simple second price auction with no reserve price gathers at least as much revenue as the optimal prior-dependent mechanism with one fewer bidder, a result that effectively puts an upper bound on the usefulness of information about the

prior. Can we find guarantees like these for auction timing problems, or more generally for dynamic mechanism design with arriving agents? Obtaining prior-free robustness results in our auction timing framework would be helpful in other stopping problems in economics, such as models of labor search.

Third, while some of the assumptions we make are at odds with key features of the high-frequency market for securities, the design questions we ask have analogues in the design of batch auctions (e.g., [Budish, Cramton, and Shim \(2014\)](#)) that may be of independent interest. Suppose the SEC requires exchanges to use frequent batch auctions but gives exchanges some flexibility in choosing the frequency. Will exchanges choose a clearing frequency that is inefficiently high or inefficiently low, and what features of a security affect which is the case (Theorem 1)? Likewise, should the frequency at which batch auctions are run be fixed, or adapt endogenously to the arrival of orders (Theorem 4)? How does regulation on the type of orders an exchange can accept affect its choice of frequency (Proposition 2)? Altogether, the question of timing in mechanism design appears to be a fruitful direction for research.

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Appendix

A Proofs for Section 4

We shall need the following useful result by [Barlow and Proschan \(1966\)](#)²⁹:

²⁹ In the economics literature, the variation diminishing property of totally positive kernels, on which this Lemma depends, is used, e.g., in [Jewitt \(1987\)](#) and [Bulow and Klemperer \(2009\)](#).

Lemma 3 (From Lemma 3.5 in (Barlow and Proschan, 1966)). *Let $X^{(1:n)} \leq X^{(2:n)} \leq \dots \leq X^{(n:n)}$ be an ordered iid sample from a continuous distribution. Suppose a function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ changes sign k times, and that the expectation of $\gamma(X^{(i:n)})$ exists. Then $\pi(n-i, n) = \mathbb{E}[\gamma(X^{(n-i:n)})]$ changes sign at most k times as a function of n ; if $\pi(n-i, n)$ changes sign in n exactly k times, then the changes occur in the same order as those of $\gamma(x)$.*

We prove a strengthening of Lemma 1, which will come in handy in later arguments:

Lemma 4. *Let $S_A(n-k; n)$ and $S_B(n-k; n)$, $k \in \mathbb{Z}_+$, denote the expected $k+1$ -th highest draws from n iid samples drawn according to the law of continuously distributed random variables X_i^A and X_i^B , respectively. If there exists an iid sequence of random variables $Z_i, i = 1, \dots, n$ and two increasing functions h and g such that*

1. *for any $c \in \mathbb{R}_{++}$, $h - cg$ changes sign at most once and from negative to positive;*
2. *$g \geq 0$; and*
3. *for all $i = 1, \dots, n$, $X_i^A = h(Z_i)$ and $X_i^B = g(Z_i)$,*

then $S_A(n-k; n)/S_B(n-k; n)$ is increasing in n .

Proof. The proof is a slight generalization of Theorem 3.6 in Barlow and Proschan (1966).

From Lemma 3 with $\gamma(z) = h(z) - cg(z)$, we have that

$$\mathbb{E}[h(Z^{(n-k:n)})] - c\mathbb{E}[g(Z^{(n-k:n)})] = S_A(n-k; n) - cS_B(n-k; n) = S_B(n-k; n) \left(\frac{S_A(n-k; n)}{S_B(n-k; n)} - c \right)$$

changes sign at most once and from negative to positive, as a function of n (the first equality uses the fact that h and g are non-decreasing). Because $S_B(n-k; n) > 0$ and this holds for any $c > 0$, we conclude that $S_A(n-k; n)/S_B(n-k; n)$ is increasing in n . \square

To see why this lemma implies Lemma 1, note that for $g > 0$, $\frac{h}{g}$ is increasing if and only if, for any $c > 0$, $h - cg$ changes sign at most once and from negative to positive as the argument increases. The more complicated statement in Lemma 4 is useful for cases where h is sometimes negative or h and g might be zero simultaneously.

Proof of Proposition 2. First, suppose that $\lambda(v)v$ is increasing. Then, $\left(v - \frac{1}{\lambda(v)}\right)v^{-1} = 1 - (\lambda(v)v)^{-1}$ is increasing. Now, applying Lemma 4 $X_i^A = MR(v_i)$ and $X_i^B = v_i$, the statically constrained seller stops later than the planner.

Next, we show that the unconstrained seller stops earlier than the statically constrained seller. For any sample of bidder values v_1, \dots, v_n , let $X_i \equiv MR(v_i)$, and recall that MR is increasing, so $MR(v^{(n:n)}) = X^{(n:n)}$. Also, let \tilde{X} be an additional draw of marginal revenue that is independent from the original sample X_1, \dots, X_n . Define the function $g_p(x) = \mathbb{E}[\max\{\tilde{X}, x, p\}]$, which is non-negative for any $p \in \mathbb{R} \cup \{-\infty\}$. For $n \geq 2$ bidders, we can rewrite expected marginal revenue, censored at p , as

$$\mathbb{E} \left[\mathbb{E} \left[\max\{X^{(n-1:n-1)}, \tilde{X}, p\} \right] \right] = \mathbb{E} \left[\mathbb{E} \left[\max\{X^{(n-1:n-1)}, x, p\} \middle| \tilde{X} = x \right] \right] = \mathbb{E}[g_p(X^{(n-1:n-1)})]. \quad (8)$$

Since g_p is increasing for all $p \in \mathbb{R} \cup \{-\infty\}$, for any $n \geq 2$, the seller who uses an optimal auction has the same payoffs as a social planner facing a distribution of types $Y_i \sim g_0(X_i)$, while the seller who uses a second price auction with no reserve has the same payoffs as a social planner facing a distribution of types $Z_i \sim g_{-\infty}(X_i)$. For any p', p such that $\infty > p' > p \geq -\infty$, $g_{p'}(x)/g_p(x)$ is increasing, so $g_{p'}(X_i)$ is more right-skewed than $g_p(X_i)$, and in particular, for $n \geq 2$ the seller faces a more right-skewed distribution of payoffs under the optimal auction than under a second price auction with no reserve price.³⁰ Therefore, by Lemma 2, the seller stops earlier when she can use an optimal auction.

Next, suppose that $\lambda(v)v$ is decreasing. By a symmetric argument, the planner stops later than the seller if $\lambda(v)v$ is decreasing.

Finally, to prove that the optimal auction coincides with the efficient auction, we show that $v - \frac{1-F(v)}{f(v)} \geq 0$ for any v . Suppose to the contrary that $v - \frac{1-F(v)}{f(v)} = v(1 - \frac{1}{\lambda(v)v}) < 0$ for some v . Then, since $v \geq 0$, $1 - \frac{1}{\lambda(v)v} < 0$ holds. Because $\lambda(v)v$ is decreasing, we obtain

$$v' \left(1 - \frac{1}{\lambda(v')v'}\right) < v \left(1 - \frac{1}{\lambda(v)v}\right).$$

if $v' > v$. This contradicts the regularity of F . □

³⁰ Indeed, $\frac{g_{p'}(x)}{g_p(x)}$ is increasing on $(-\infty, p')$, because only $g_{p'}(x)$ increases in this region. Also, on $(p', +\infty)$, $\frac{g_{p'}(x)}{g_p(x)}$ is equal to 1, so it is trivially increasing.

B Proofs for Section 5

In this section we collect intermediate results necessary in the proof of Theorem 2. The first key step is characterizing how the auctioneer's value function behaves as a function of waiting time:

Lemma 5. *Suppose that inter-arrival times W are NBU, and sojourn times D are iid exponential. Then for any $n, a > 0$, $V(n, a) \geq V(n, 0)$.*

Proof. For any $x \geq 0$, define $W^x := W - x | \{W \geq x\}$. Fix some $a > 0$. First, note that by the NBU assumption, $W^0 \geq_{FOSD} W^a$, so we can always find a probability space such that W^a and W^0 are both defined on that space and $W^0 \geq W^a$ almost surely.

With that in mind, consider two stochastic processes defined on a common probability space on which $W^0 \geq W^a$ almost surely. First, define N_a as follows.

- At $t = 0$, initialize $N_a(0) = n$. Also, start n independent exponential departure clocks simultaneously.
- If arrival clock W^a “ticks,” increase N_a by 1 and draw a new arrival clock $W_1 \sim G$ and an exponential departure clock D_1 .
- For each $k \geq 1$, if arrival clock W_k “ticks,” increase N_a by 1 and draw a new arrival clock $W_{k+1} \sim G$ and an exponential departure clock D_{k+1} .
- If any departure clock ticks, decrease N_a by 1.

Second, define N_0 in the same way as N_a except that we use W^0 instead of W^a for the first arrival clock. Note that the values from following the policies that are optimal for N_a and N_0 are $V(n, a)$ and $V(n, 0)$, respectively, but given the construction of N_a and N_0 on a common probability space, we can compare $V(n, a)$ and $V(n, 0)$ realization by realization.

We prove the result by induction on n . First, as $W^0 \geq W^a$ for sure, we get

$$V(0, a) = \mathbb{E}[e^{-rW^a}]V(1, 0) \geq \mathbb{E}[e^{-rW^0}]V(1, 0) = V(0, 0). \quad (9)$$

Second, apply the induction hypothesis that $V(k, a) \geq V(k, 0)$, $k = 0, \dots, n-1$. We refer to the sellers who confront N_a and N_0 as Sellers a and 0, respectively. Likewise, we refer the

optimal policies of Sellers a and 0 as Policies a and 0, respectively. Suppose that Seller 0 takes Policy 0, which yields $V(n, 0)$. Meanwhile, Seller a uses the following “hybrid” timing policy.

1. As long as no bidders arrive or depart, adopt Policy 0.
2. If some bidder arrives or departs while imitating Seller 0, from that point on adopt Policy a .

We show that this policy gives Seller a a greater revenue than $V(n, 0)$ for any realizations of the stochastic processes. Under the first scenario in this hybrid policy, if Seller a holds an auction (say, at time t) while imitating Seller 0, then both of them obtain a discounted revenue of $e^{-rt}R(n)$ because $W^0 \geq W^a$. Under the second scenario, there are two cases. If Seller a switches to Policy a at t because a bidder has departed, then the time- t continuation payoffs of Seller’s 0 and a are $V(n-1, t)$ and $V(n-1, a+t)$, respectively. By the inductive hypothesis, $V(n-1, a+t) \geq V(n-1, t)$. Finally, if Seller a switches to policy a because a new bidder has arrived at time W^a , then the time- W^a continuation payoff of Seller a is $V(n+1, 0)$, while that of Seller 0 is $V(n, W^a) \leq V(n+1, 0)$.³¹ \square

As shown in the main text, Lemma 5 implies that, when starting from state $(0, 0)$, the optimal policy stops the first time the bidder pool reaches some n^* . To complete the proof of Theorem 2, it remains to prove the auxiliary results used in the one-step characterization of n^* in (5). Recall that $\beta(j) \equiv \mathbb{E}[e^{-r\tau(j-1, j)}]$, where $\tau(j-1, j)$ is the random time elapsed between when the pool size first reaches $j-1$ to when it first reaches j . Then we have the following payoff decomposition:

Lemma 6. *The seller’s expected revenue at time 0 from holding the auction upon the n -th bidder’s arrival is given by $\prod_{j=1}^n \beta(j)R(n)$.*

³¹ $V(n+1, 0) \geq V(n, t)$ holds for any t for the following reason: Starting from (n, t) , let p be the discounted probability that, under the optimal policy, the state reaches $(n+1, 0)$ before the seller holds an auction, and let p' be the discounted probability, under the optimal policy, that the seller holds the auction before the state reaches $(n+1, 0)$, i.e., the auction takes place with at most n bidders. Clearly, $p + p' \leq 1$, so $V(n, t) = pV(n+1, 0) + p'R(n) \leq V(n+1, 0)$.

Proof. Let $\tau_n = \inf\{t : N_t = n\}$. Then the expected revenue from stopping at the n -th bidder's arrival becomes

$$\mathbb{E}[e^{-r\tau_n}]R(n) = \mathbb{E}[e^{-r\sum_{j=1}^n \tau(j-1,j)}]R(n). \quad (10)$$

To calculate (10), construct the following “fictitious” process $\mathbf{N}' = \{N'_t, t \geq 0\}$:

- Initialize $N'_0 = 0$.
- Draw arrival clocks sequentially: first draw $W_1 \sim G$ at $t = 0$, then when the first clock ticks ($t = W_1$) draw $W_2 \sim G$ independently at $t = W_1$, and so on.
- If an arrival clock W_j “ticks”:
 1. Increase \mathbf{N}' by 1.
 2. Remove (ignore thereafter) all remaining departure clocks, and replace them with the same number of new, independent departure clocks.
 3. Add an additional exponential departure clock, independent of all arrival clocks.
- If any existing departure clock ticks:
 1. Decrease \mathbf{N}' by 1.
 2. Remove (ignore thereafter) all remaining departure clocks, and replace them with the same number of new, independent exponential departure clocks.

By the memoryless property of the Poisson clocks, \mathbf{N} and \mathbf{N}' will have the same marginal distributions, so letting $\tau'(j-1, j)$ denote the successive record times for N' , $\tau'(j-1, j) \sim \tau(j-1, j)$ and $e^{-r\sum_{j=1}^n \tau(j-1, j)} \sim e^{-r\sum_{j=1}^n \tau'(j-1, j)}$. In addition, N' 's first-increment times $\{\tau(k-1, k)\}_{k \in \mathbb{N}}$ will be mutually independent, since the increasing part of \mathbf{N} is a renewal, and by the way we construct \mathbf{N}' (replacing all “old” clocks with fresh independent ones at each point of change), all the dependence in between successive records of \mathbf{N}' has been removed. Therefore

$$\mathbb{E}[e^{-r\sum_{j=1}^n \tau(j-1, j)}] = \mathbb{E}[e^{-r\sum_{j=1}^n \tau'(j-1, j)}] = \prod_{j=1}^n \mathbb{E}[e^{-r\tau'(j-1, j)}] = \prod_{j=1}^n \beta(j),$$

as required. □

Finally, $\beta(\cdot)$ is decreasing, which establishes that $\prod_{j=1}^n \beta(j)R(n)$ is single-peaked and yields the one-step characterization of n^* :

Lemma 7. *Let $\tau(n-1, n)$ be the time between when the bidder pool first reaches $n-1$ bidders and when it first reaches n bidders. For any $n \in \mathbb{N}$, $\tau(n, n+1) \geq_{FOSD} \tau(n-1, n)$. Thus, $\beta(j) = \mathbb{E}[e^{-r\tau(j-1, j)}]$ is decreasing in j .*

Proof. The proof is by coupling. At $t = 0$, start n independent exponential departure clocks simultaneously. Label one of these clocks “first.” Independently of these clocks, draw arrival clocks sequentially: first draw $W_1 \sim G$ at $t = 0$, then when the first clock ticks ($t = W_1$) draw $W_2 \sim G$ independently at $t = W_1$, and so on. Add an additional independent departure clock every time a new arrival clock $W_j, j \geq 2$ is drawn. Then define two stochastic processes $\mathbf{M} = \{M_t, t \geq 0\}$ and $\mathbf{M}' = \{M'_t, t \geq 0\}$ on this space such that

- Initialize M_0 at n and M'_0 at $n-1$
- If an arrival clock W_j “ticks,” increase \mathbf{M} and \mathbf{M}' by 1.
- If any departure clock ticks, decrease \mathbf{M} by 1.
- If any departure clock other than the first one in the original set ticks, decrease \mathbf{M}' by 1.

Note that $M_t = M'_t + 1$ before the first departure clock ticks, and $M_t = M'_t$ thereafter, i.e., \mathbf{M} and \mathbf{M}' eventually “couple.”

By the renewals and exponential departures assumptions, M_t is distributed as N_t started from state $(n, 0)$, while M'_t is distributed as N_t started from state $(n-1, 0)$. Therefore, the time \mathbf{M} first crosses $n+1$, denoted σ_{n+1} , has the same distribution as $\tau(n, n+1)$. Similarly, the time \mathbf{M}' first crosses n , denoted σ'_n , has the same distribution as $\tau(n-1, n)$.

We claim $\sigma_{n+1} \geq \sigma'_n$ almost surely. First, if \mathbf{M} and \mathbf{M}' have not coupled by $\sigma_{n+1} \wedge \sigma'_n$, M will reach $n+1$ at σ'_n (since $M_t = M'_t + 1$ before coupling). Hence, $\sigma_{n+1} = \sigma'_n$. Second, suppose that \mathbf{M} and \mathbf{M}' couple before time $\sigma_{n+1} \wedge \sigma'_n$ (so that they meet at some state $k \leq n-1$). Then $\sigma_{n+1} > \sigma'_n$, since $M_t = M'_t$ after coupling, and \mathbf{M} must reach n before

reaching $n + 1$. Therefore, we conclude that $\tau(n, n + 1) \geq_{FOSD} \tau(n - 1, n)$ by the usual argument, and the result follows. \square

C Proofs for Section 6

Proof of Theorem 4. We construct a sequence $\{F_n, G_n\}_{n=1}^\infty$ such that $\frac{DET(r, F_n, G_n)}{OPT(r, F_n, G_n)} \rightarrow 0$. Suppose first that $F_n = F$, where F is a degenerate distribution at 1. The degeneracy of F will be without loss of generality because we can approximate any such F by a sequence of non-degenerate, continuous, and regular distributions, and the same logic used here will apply along that sequence.

Now, consider the following inter-arrival distribution G_n . Fix any positive integer M . Let the support of G_n be $\{t_1, \dots, t_M\}$ for any n . Let p_k denote the probability that G_n puts on t_k for $k = 1, \dots, M$. (We suppress the dependence of p_j on n for notational simplicity.) Then define each p_i inductively as follows.

$$\begin{aligned} p_1 &= \frac{1}{n} \\ p_k &= \sqrt{p_{k-1}} - p_1 - p_2 - \dots - p_{k-1}, \\ p_M &= 1 - p_1 - \dots - p_{M-1}. \end{aligned}$$

There are two remarks. First, for large n , $p_k \geq 0$ for each k , because the first term $\sqrt{p_{k-1}}$ converges to zero strictly slower than the other terms $-p_1 - p_2 - \dots - p_{k-1}$. Second, note that $\sum_{j=1}^k p_j = \sqrt{p_{k-1}}$ holds for each $k < M$.

Now, suppose that the seller, who has to choose a fixed deadline, holds the auction at t_1 (later we show that we can construct t_1, \dots, t_M so that choosing t_1 is indeed optimal). Let $\alpha_k := e^{-rt_k}$ for each $k = 1, \dots, M$. Then it holds that

$$\begin{aligned} \frac{OPT}{DET} &= \frac{\sum_{i=1}^M p_i \alpha_i}{\max_{1 \leq k \leq M} \sum_{i=1}^k p_i \alpha_k} \\ &= \frac{\sum_{i=1}^M p_i \alpha_i}{\alpha_1 p_1} \\ &= 1 + \sum_{j=2}^M \frac{p_j}{p_1} \cdot \frac{\alpha_j}{\alpha_1}. \end{aligned} \tag{11}$$

For $t = t_1$ to be optimal, it is enough that for $j = 1, \dots, M$,

$$p_1 \alpha_1 \geq \alpha_j \sum_{i=1}^j p_i \iff \frac{p_1}{\sum_{i=1}^j p_i} \geq \frac{\alpha_j}{\alpha_1}.$$

Now, since $\frac{p_1}{\sum_{i=1}^j p_i}$ is less than 1 and is strictly decreasing in j , we can choose $0 < t_1 < t_2 < \dots < t_M$ so that $\alpha_k = e^{-rt_k}$ for $k = 1, \dots, M$ makes the above inequalities binding.

Combining this (in)equality and equation (11), we obtain

$$\frac{OPT}{DET} = 1 + \sum_{k=2}^M \frac{p_k}{\sum_{i=1}^k p_i}. \quad (12)$$

Now, we show that as $n \rightarrow +\infty$, the RHS of (12) converges to M . Indeed, each summand in the RHS can be written as

$$\frac{p_k}{\sum_{i=1}^k p_i} = \frac{p_k}{\sqrt{p_{k-1}}} = 1 - \sum_{j=1}^{k-1} \frac{p_j}{\sqrt{p_{k-1}}} \rightarrow 1 \quad \text{as } n \rightarrow +\infty. \quad (13)$$

The last convergence follows from the fact that p_j is of order $n^{-\frac{1}{2j}}$ by construction. Now, (12) and (13) imply that

$$\frac{DET}{OPT} \rightarrow \frac{1}{M} \quad \text{as } n \rightarrow +\infty.$$

That is, for each F , we can choose a sequence of G_n so that $\frac{DET}{OPT}$ converges to $1/M$. Because M is arbitrary, there exists $\{F_n, G_n\}_{n=1}^\infty$ such that $\frac{DET(r, F_n, G_n)}{OPT(r, F_n, G_n)} \rightarrow 0$. (We can construct such a sequence explicitly by a diagonalization argument.) \square

Proof of Proposition 3. Let n^* be the threshold number of bidders characterizing the fully optimal auction timing (this could possibly depend on G and F). Let DET and OPT denote, respectively, the best ex ante payoffs from only using deterministic deadlines and from using any timing policy adapted to arrivals. Consider an auxiliary auction timing problem where a single bidder, whose value is deterministically equal to $R(n^*)$, arrives at the random time $\tilde{T} = T_1 + \dots + T_{n^*}$, and no other bidders arrive before or after. Without loss of generality, assume $R(n^*) = 1$. Let DET_{aux} and OPT_{aux} denote, respectively, the best ex ante payoffs in this auxiliary single-bidder problem from only using deterministic deadlines and from using any feasible policy. If the seller in the auxiliary problem can use any stopping time, she stops at \tilde{T} and posts a price equal to 1. This gives her an ex-ante

payoff of $OPT_{aux} = \mathbb{E}[e^{-r\tilde{T}}]$, which matches OPT from the original many-bidder problem by our assumption on the optimal policy.

Now consider also a static monopoly pricing problem where the buyer's value is given by $v = e^{-r\tilde{T}}$. Given a price $p \in (0, 1]$, the probability of acceptance is $1 - H(p) = \mathbf{P}(v \geq p) = \mathbf{P}(\tilde{T} \leq -r^{-1} \log p) = G(-r^{-1} \log p)$. Note that for any CDF G , $H(x) = 1 - G(-r^{-1} \log x)$ is a well-defined CDF. In this notation, $DET_{aux} = \max_{t \geq 0} e^{-rt} G(t) = \max_{p \geq 0} p(1 - H(p))$, which equals the expected revenue from a posted price p in the static pricing problem, while $OPT_{aux} = \mathbb{E}_G[e^{-r\tilde{T}}] = \mathbb{E}_H[v]$ equals the expected feasible surplus in the static pricing problem. Thus,³² $\frac{DET_{aux}}{OPT_{aux}} = \frac{\max_{p \in [0,1]} p(1-H(p))}{\mathbb{E}_H[v]}$ is the ratio of expected revenue to total feasible surplus in a static pricing problem.

From Lemma 3.10 in [Dhangwatnotai et al. \(2015\)](#) (using $t = 0$ in their notation) we have that for any IHR H , a static pricing problem satisfies

$$\max_{p \geq 0} \{p(1 - H(p))\} \geq \frac{1}{e} \mathbb{E}_{v \sim H}[v]. \quad (14)$$

$G \in \mathcal{G}^{DIHR+}$ by assumption, so $e^{-r\tilde{T}}$ will satisfy IHR. Hence, by (14), $\frac{DET_{aux}}{OPT_{aux}} \geq \frac{1}{e}$.

By the definition of n^* , OPT_{aux} matches OPT in the original problem, while DET_{aux} is a crude lower bound for DET . Indeed, $DET_{aux} = e^{-rt} \mathbf{P}(\tilde{T} \leq t) R(n^*)$ is a lower bound for DET on the event $\{\tilde{T} < t\}$ (since there could be more than n^* bidders at t in that case), and the contribution to DET from the event $\{\tilde{T} < t\}$ is positive. Therefore $DET \geq DET_{aux} \geq \frac{1}{e} OPT_{aux} = \frac{1}{e} OPT$, as required. \square

D Proof of Fact 1

Proof of Fact 1. Point 1 is obvious for $S(n)$, since it equals $\mathbb{E}[v^{(n)}]$, where $v^{(n)}$ denotes the highest draw out of n samples from F . To see that $\mathbb{E}(\max\{MR(v^{(n)}), p\})$ is non-negative, note that $\max\{MR(v^{(n)}), p\} \geq MR(v_1)$ and $E[MR(v_1)] = 0$. To show Point 2, take subsets S, T of the set of bidders B with $S \supseteq T$, and a bidder $i \in B \setminus S$. Then, we claim

$$\max_{j \in S \cup \{i\}} v_j - \max_{j \in S} v_j \leq \max_{j \in T \cup \{i\}} v_j - \max_{j \in T} v_j \quad (15)$$

³² Note that the support of H is a subset of $[0, 1]$ by construction. However, for any H with bounded support, we can always rescale the support of H without changing $\frac{\max_{p \in [0,1]} p(1-H(p))}{\mathbb{E}_H[v]}$.

Indeed, when $v_i = \max_{j \in S \cup \{i\}} v_j$, the comparison becomes $-\max_{j \in S} v_j \leq -\max_{j \in T} v_j$, which holds by the assumption that $T \subset S$; when $v_i < \max_{j \in S \cup \{i\}} v_j$, the left hand side is 0, and the right hand side is non-negative. Taking expectations on both sides of (15) delivers the concavity of $S(n)$.³³ The result extends to $R(n, 0)$ and $R(n, p^*)$ by substituting marginal revenues $MR(v_j)$ and $MR(v_j) \vee 0$, respectively, for v_j .³⁴ To show Point 3, note that diminishing returns of $n \mapsto R(n)$ imply

$$R(n+2) \left(1 - \frac{R(n+1)}{R(n+2)}\right) \leq R(n+1) \left(1 - \frac{R(n)}{R(n+1)}\right) \quad (16)$$

By revenue monotonicity, the terms in the parentheses are non-negative, so for (16) to hold even though $R(n+2) \geq R(n+1)$, $\frac{R(n+1)}{R(n)}$ must be decreasing, as required. An identical argument applies for $R(n, 0)$ and $S(n)$. \square

³³ The proof strategy follows Corollary 2.7. in [Dughmi, Roughgarden, and Sundararajan \(2009\)](#).

³⁴ We sometimes use the shorthand $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$, for notational simplicity.