Dynamic Privacy Choices

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Abstract

I study a dynamic model of consumer privacy and platform data collection. In each period, consumers choose their level of platform activity. Greater activity generates more information about the consumer, thereby increasing platform profits. Although consumers value their privacy, a platform can collect information by committing to gradually lower the level of privacy protection. In the long run, consumers become "addicted" to the platform: They lose privacy and receive low payoffs, but choose high activity levels. If a platform cannot commit to future privacy policies, it may end up offering the highest privacy protection and fail to collect any information.

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1 Introduction

Online platforms, such as Amazon, Facebook, Google, and Uber, analyze user activities and collect a large amount of data. This data collection may improve their services and benefit consumers, but it also raises concerns for consumers and policymakers (Crémer et al., 2019; Furman et al., 2019; Morton et al., 2019).

As an example, consider a consumer (she) and a social media platform (it). The consumer writes posts and reads news on the platform. The platform analyzes her activity and collects data such as her race, location, and political preferences. The platform can then generate revenue—e.g., via improved targeted advertising. The consumer faces a trade-off: On the one hand, she enjoys the services provided by the platform. On the other hand, she may value her privacy, or be concerned about the risk of data leakage, identity theft, and price or non-price discrimination.[1] Such risks are the "privacy costs" of using the platform. If the consumer anticipates a high privacy cost, she may use the platform less actively, or may not join it in the first place. The platform can influence her decision through its privacy policy. For example, Facebook committed to not use (first-party) cookies to track users.[2]

I model such a situation as a dynamic game between a consumer and a platform. In each period, the consumer chooses her level of platform activity. Based on the level of activity, the platform observes a signal about the consumer's time-invariant type. The precision of the signal is increasing in the activity level, but decreasing in the platform's privacy level, which specifies the amount of noise added to the signal. The platform's profit is increasing, but the consumer's payoff is decreasing in the amount of information the platform has collected. As a result, the consumer chooses activity levels that balance the benefits of the service and the privacy costs. Anticipating the consumer's behavior, the platform sets privacy levels.

The main idea is that the consumer has a decreasing marginal privacy cost—i.e., when the consumer has less privacy, she faces a lower marginal privacy cost of using the same platform repeatedly. For example, if Google already knows a lot about a consumer, she might not care about letting Google Maps track her location today. In an extreme case, if the platform knows everything,

[1]Such concerns are highlighted by, for example, the *Cambridge Analytica* scandal.

[2]In 2004, Facebook's privacy policy stated that "we do not and will not use cookies to collect private information from any user." https://web.archive.org/web/20050107221705/http://www.thefacebook.com/policy.php (accessed on July 31, 2020)

the marginal privacy cost is zero, because the consumer's activity on a platform no longer affects what it knows about her.

I examine the dynamic implications of this idea. First, I study the equilibrium dynamics: In early periods, the platform commits to high privacy levels: By committing to not collect too much data, the platform can encourage the consumer, who has not yet lost privacy, to use the service and generate information. As the platform collects more data, the consumer faces lower marginal privacy costs. As a result, in later periods, the platform can reduce a privacy level to speed up data collection. In the long run, the consumer loses privacy and incurs a high privacy cost, but chooses a high activity level. Also, the platform may offer a vanishing privacy level. This equilibrium outcome occurs even if the consumer is patient and values privacy.

Second, I study the role of the platform's commitment. Under a certain condition, the platform can achieve the above equilibrium outcome only with one-period commitment power. In this case, the optimal policy is greedy—i.e., the platform chooses a myopically optimal privacy level in each period. However, with one-period commitment power, there is also an equilibrium in which the platform collects no information: The consumer refuses to provide any positive amount of data, because she anticipates that the platform which fails to collect data today will offer the highest privacy protection in the future. This equilibrium captures the platform's Coasian problem.

The paper mainly analyzes an unregulated monopolist, but I also examine competition and regulation. First, the aforementioned decreasing marginal privacy cost implies that the consumer is more willing to use a platform on which she has less privacy. This consumer's tendency renders competition less effective. Second, ex ante and ex post privacy regulations have different impacts: Mandating that the platform pre-commit to a strict privacy policy may, perversely, lower the privacy and welfare of consumers in the long run. In contrast, enabling the consumer to delete information collected in the past may enhance welfare.

The paper has implications for consumer privacy. First, the consumer's long-run behavior (in the equilibrium with data collection) seems consistent with the so-called privacy paradox: Consumers express concern about data collection, but actively share data with third parties (Acquisti et al., 2016). The platform's equilibrium strategy rationalizes how online platforms, such as Facebook, seem to have relaxed their privacy policies over time. Second, the results clarify the role of commitment and expectation in data collection: Depending on the consumer's expectation about future data collection, a platform that has weak commitment power may be forced to offer the highest privacy protection, or it may collect information even when the consumer highly values their privacy.

The rest of the paper is as follows. Section 2 discusses related literature, and Section 3 presents the model. Section 4 considers the platform with long-run commitment power and presents the equilibrium. Section 5 assumes the platform has one-period commitment. In particular, assuming that the consumer has binary activity level, I characterize the platform-optimal and the consumer-optimal equilibria. Section 6 studies platform competition. Section 7 considers extensions, including the impact of erasing past information.

2 Related Literature

This paper contributes to the literature on the economics of privacy and markets for data. This literature has studied several important questions, such as how to use consumer data to create market segmentation (Ali et al., 2020; Bonatti and Cisternas, 2020; Elliott and Galeotti, 2019; Haghpanah and Siegel, 2019; Loertscher and Marx, 2020; Yang, 2019; Ichihashi, 2020b); how to choose the optimal level of privacy protection and information security (Dwork et al., 2014; Fainmesser et al., 2019; Jullien et al., 2018); how information externalities create inefficiency or influence agents' behavior (Acemoglu et al., 2019; Bergemann et al., 2019; Choi et al., 2019; Easley et al., 2018; Liang and Madsen, 2020; Ichihashi, 2020a); how agents strategically manipulate data (Frankel and Kartik, 2019b,a; Ball, 2020); how consumer data and privacy interact with mechanism design (Brunnermeier et al., 2020; Calzolari and Pavan, 2006; Eilat et al., 2019; Ghosh and Roth, 2011); how to price and sell information (Agarwal et al., 2019; Hörner and Skrzypacz, 2016; Bergemann et al., 2018); and how privacy and data affect competition (Casadesus-Masanell and Hervas-Drane, 2015; De Corniere and Taylor, 2020).

The paper is especially related to Acemoglu et al. (2019); Bergemann et al. (2019); and Choi et al. (2019). They consider static models in which a platform collects data in exchange for money, and the data on some consumers reveal information about others. This "data externality" lowers the cost of consumers to provide their data, which leads to excessive data sharing. In my paper, the consumer's cost of generating information decreases in the stock of data she has already provided and the amount of data the platform will collect in the future. The dynamic model enables me to study new issues, such as the equilibrium dynamics regarding privacy, the role of a platform's commitment, and the impact of erasing past data. My paper also differs from those papers in that data endogenously arise as a byproduct of activities from which consumers derive utility.

This paper also relates to recent work on dynamic competition in digital markets. Hagiu and Wright (2020) study "data-enabled learning," whereby firms can improve their products and services through learning from the data they obtain from their customers. Prufer and Schottmüller (2017) assume that the cost of investing in quality is decreasing in the firm's past sales, and greater investment in quality leads to higher demand in the current period. In contrast to this literature, I assume data collection lowers consumer welfare. Such an assumption enables us to study issues related to consumer privacy. Hagiu and Wright (2020) allow price competition and study rich learning dynamics that incorporate "within-user" and "across-user" learning. In contrast, I focus on within-user learning and the design of a privacy policy, abstracting away from pricing.

How the consumer's incentive changes over time in my model is similar to that of career concern models, which originated with Holmström (1999). In career concern models, a young worker, whose ability has not yet been revealed to the market, works hard to influence the market's belief. In my model, a consumer who has not yet lost privacy uses the platform less actively to generate less information. Over time, the information about the consumer and the worker are revealed, and they have lower incentives to engage in signal jamming. Despite this connection, the two signal jamming activities are different. In career concern models, the market wants the worker to engage in signal jamming, which corresponds to higher effort. Thus, there is a trade-off between learning the worker's ability and motivating high effort (i.e., Hörner and Lambert 2018). In my model, the platform wants the consumer to engage less in signal jamming. Thus, the platform prefers to collect information not only to increase profit today, but also to motivate the consumer to raise activity levels in the future. Many of my results stem from this complementarity between data collection and consumer activity.

3 Model

I study a dynamic game between a consumer (she) and a platform (it). Time is discrete and infinite, indexed by $t \in \mathbb{N}$. The consumer's type X is drawn from a normal distribution $\mathcal{N}(0, \sigma_0^2)$. The type is realized before $t = 1$ and fixed over time. The consumer does not observe X.[3] The platform does not observe X either, but receives signals about it.

In each period $t \in \mathbb{N}$, the consumer chooses an *activity level* a_t from a finite set $A \subset \mathbb{R}_+$ such that $\min A = 0$ and $a_{max} := \max A > 0$. The platform then observes a_t and a signal $s_t = X + \varepsilon_t$, where $\varepsilon_t \sim \mathcal{N}\left(0, \frac{1}{a_t} + \gamma_t\right)$. The consumer does not observe the signal.[4] A higher a_t reduces the variance of ε_t and makes s_t more informative about X. For a fixed a_t, the informativeness of the signal decreases in $\gamma_t \in \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$, which is the *privacy level* of the platform in period t. A higher γ_t implies the platform offers higher privacy protection. If $a_t = 0$ or $\gamma_t = \infty$, signal s_t is totally uninformative. Random variables X and $(\varepsilon_t)_{t\in\mathbb{N}}$ are mutually independent.

The payoffs are as follows. Suppose that the consumer has chosen activity levels $\boldsymbol{a}_t = (a_1, \ldots, a_t) \in A^t$ and the platform has chosen privacy levels $\boldsymbol{\gamma}_t = (\gamma_1, \ldots, \gamma_t) \in \overline{\mathbb{R}}_+^t$ up to period t. At the end of period t, the platform receives a payoff of $\sigma_0^2 - \sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t) \geq 0$, where $\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ is the posterior variance of X given $(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ and Bayes' rule.[5] I take $(\sigma_t^2(\cdot,\cdot))_{t\in\mathbb{N}}$ as a primitive, and analyze the game as that of perfect information. A small $\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ means the platform has an accurate estimate of X, or equivalently, the consumer has low privacy. For any t and $\tau \leq t$, $\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ is decreasing in a_τ, increasing in γ_τ, and independent of s_τ.[6] Where it does not cause confusion, I write $\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ as σ_t^2. The platform discounts future payoffs with a discount factor $\delta_P \in (0, 1)$.

The consumer's flow payoff in period t is $U(\boldsymbol{a}_t, \boldsymbol{\gamma}_t) := u(a_t) - v \cdot [\sigma_0^2 - \sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)]$. The first term $u(a_t)$ is her gross benefit of using the platform, where $u(a)$ is strictly increasing in $a \in A$ and $u(0) = 0$. The second term $v \cdot [\sigma_0^2 - \sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)]$ is a *privacy cost*, which captures the negative

[3] Even if the consumer privately observes X, all results hold with respect to a pooling equilibrium in which consumers of all types choose the same activity level after any history. Unobservable X simplifies exposition without changing the results.

[4] All the results continue to hold even if signals are public, because the payoff of each player does not depend on the realization of a signal.

[5] The equivalent formulation is that the platform observes (a_t, s_t), chooses $b_t \in \mathbb{R}$, and obtains an ex post payoff of $-(X - b_t)^2$, which the platform does not observe. Writing the payoffs in terms of σ_t^2 simplifies exposition. See Acemoglu et al. (2019) for further discussion.

[6] Throughout the paper, "increasing" means "non-decreasing." Similar conventions apply to "decreasing," "higher," "lower," and so on.

impact of data collection on the consumer. The parameter $v \in \mathbb{R}_{++}$ is exogenous and captures her value of privacy. The consumer discounts future payoffs with a discount factor $\delta_C \in [0, 1)$. A special case is a *myopic consumer* (i.e., $\delta_C = 0$), who chooses $a_t \in A$ to maximize $U(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ in each period t. The payoffs of the consumer and the platform are normalized so that if $a_t = 0$ for all t, they obtain zero payoffs in all periods.

The informational assumptions are summarized as follows. The primitives, σ_0^2, A, $u(\cdot)$, and v, are commonly known. The past activity levels and privacy levels are publicly observable. The consumer's type is unobservable, and the signals are observable only to the platform.

I study two games that differ in the timing of moves. One is the game of *long-run commitment.* In this game, before $t = 1$, the platform first commits to a *privacy policy* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots) \in \overline{\mathbb{R}}_+^\infty$. After observing $\boldsymbol{\gamma}$, the consumer chooses an activity level in each period. The other is the game of *one-period (or short-run) commitment.* In this game, the platform and the consumer move sequentially in each period: At the beginning of each period t, the platform sets γ_t. After observing γ_t, the consumer chooses a_t. In this case, the platform can commit to a privacy level only for one period. In either case, the solution concept is subgame perfect equilibrium (SPE) in which the consumer breaks ties in favor of higher activity levels.[7] Hereafter, "equilibrium" refers to an SPE with this restriction.

I do not explicitly model the consumer's decision to join the platform, but we may interpret that the consumer joins in period t if t is the first period in which $a_t > 0$. The results continue to hold even if the consumer incurs a small one-time cost to join.

3.1 Modeling Assumptions

This subsection discusses important modeling assumptions.

Data generation. In practice, consumer data are generated by their activity on a platform, such as browsing content and responding to posts. The model captures such activity by assuming that the precision of a signal is increasing in the activity level. To focus on the consumer's incentives to protect privacy, I abstract away from belief manipulation, such as a consumer strategically

[7] To be precise, for a myopic consumer, an equilibrium refers to a strategy profile such that (i) the consumer chooses a_t to maximize $U(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)$ following every history, breaking ties in favor of higher activity levels, and (ii) the platform, anticipating (i), optimally chooses a privacy policy $\boldsymbol{\gamma}$ before $t = 1$ (under long-run commitment), or chooses γ_t at the beginning of each period t (under one-period commitment).

browsing websites to influence a platform's inference.

Privacy cost function. The privacy cost $v(\sigma_0^2 - \sigma_t^2)$ captures the monetary or nonmonetary reasons why a consumer wants a platform to have less information—e.g., consumers may intrinsically value their privacy, or consider the risk of data breach and price or non-price discrimination by third parties (Kummer and Schulte, 2019; Lin, 2019; Tang, 2019). Section 7 relaxes some of the functional form and informational assumptions.

The privacy cost is sunk. The consumer cannot delete past data. Thus she perceives the privacy cost from past data collection as sunk—i.e., even if $a_t = 0$ for all $t \geq T$, the consumer incurs a cost of $-v(\sigma_0^2 - \sigma_T^2)$ in any $t \geq T$. This assumption reflects the difficulty of deleting data, which is referred to as "data persistence" (Tucker, 2018). For instance, suppose a platform collects sensitive personal information and shares it with third parties. Then the consumer may face a risk of discrimination or malicious targeting even outside of the platform. In another example, if a consumer inadvertently discloses some sensitive information to other users, she may incur a psychological cost because other users know the information. Such a cost is likely to persist even when the consumer is not active on the platform. Because the consumer regards the privacy cost as sunk, she chooses activity levels based on the marginal privacy cost rather than the total privacy cost. Section 7 relaxes this assumption, and studies extensions in which the consumer regards a part of or all of the privacy cost as non-sunk.

Single consumer. I use a model with a single consumer to emphasize that the results do not rely on interactions or externalities between multiple consumers. However, since the consumer's type is Gaussian, one could incorporate multiple consumers with "data externalities," by assuming that consumers' types are correlated (Acemoglu et al., 2019; Bergemann et al., 2019). I conjecture that such an extension does not change the main insight—the externality leads to greater disclosure, but such an economic force already exists in the current model.

4 Equilibrium Under Long-Run Commitment

To examine how the platform designs its privacy policy, I begin by studying the game of long-run commitment. I first present some results under a stationary privacy policy, then study the equilibrium of the entire game.

First, Bayes' rule implies the posterior variance evolves as follows.[8]

$$\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t) = \frac{1}{\frac{1}{\sigma_{t-1}^2(\boldsymbol{a}_{t-1}, \boldsymbol{\gamma}_{t-1})} + \frac{1}{\frac{1}{a_t} + \gamma_t}}. \tag{1}$$

Thus, the consumer's privacy cost in period t is

$$v\left[\sigma_0^2 - \sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)\right] = v\left[\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2(\boldsymbol{a}_{t-1}, \boldsymbol{\gamma}_{t-1})} + \frac{1}{\frac{1}{a_t} + \gamma_t}}\right].$$

Define the privacy cost function as $C(a, \gamma, \sigma^2) := v\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a} + \gamma}}\right)$. The following lemma shows the key properties of privacy cost C and marginal privacy cost $\frac{\partial C}{\partial a}$.

Lemma 1. *The following holds.*

1. $C(a, \gamma, \sigma^2)$ *is decreasing in* γ *and* σ^2*, and increasing in* a*.*
2. $\frac{\partial C}{\partial a}(a, \gamma, \sigma^2)$ *is decreasing in* γ *and increasing in* σ^2*.*

Proof. Point 1 follows from equation (1). Point 2 follows from

$$\frac{\partial C}{\partial a} = v \cdot \frac{\frac{\frac{1}{a^2}}{\left(\frac{1}{a} + \gamma\right)^2}}{\left(\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a} + \gamma}\right)^2} = \frac{v}{\left(\frac{1}{\sigma^2}(1 + \gamma a) + a\right)^2}.$$

□

Lemma 1 implies that if the consumer has less privacy (i.e., σ_t^2 is small), she faces a high privacy cost C but a low marginal privacy cost $\frac{\partial C}{\partial a}$. Intuitively, once a platform has collected a lot of information, the consumer's activity today does not much affect the platform's learning, which leads to a lower marginal privacy cost. As a result, data collection harms the consumer, but incentivizes her to increase an activity level. Another observation is that the marginal privacy cost

[8]If $x|\mu \sim N(\mu, \sigma^2)$ and $\mu \sim N(\mu_0, \sigma_0^2)$, then $\mu|x \sim N\left(\frac{\sigma_0^2}{\sigma^2 + \sigma_0^2}x + \frac{\sigma^2}{\sigma^2 + \sigma_0^2}\mu_0, \left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma^2}\right)^{-1}\right)$.

is decreasing in the level of privacy protection, γ. Thus, the platform can encourage the consumer's activity by committing to add a noise to the signal.

The evolution of posterior variances (1) is written as that of precisions:

$$\frac{1}{\sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)} = \frac{1}{\sigma_{t-1}^2(\boldsymbol{a}_{t-1}, \boldsymbol{\gamma}_{t-1})} + \frac{1}{\frac{1}{a_t} + \gamma_t} = \cdots = \frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{a_s} + \gamma_s}. \tag{2}$$

As a result, if the platform commits to a privacy policy $(\gamma_t)_{t\in\mathbb{N}}$, the consumer solves the following maximization problem:

$$\max_{(a_t)_{t\in\mathbb{N}}\in A^\infty} \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{a_s}+\gamma_s}} \right) \right]. \tag{3}$$

The next result shows the consumer's behavior under a stationary privacy policy (see Appendix B for the proof).

Proposition 1. *Suppose the platform commits to a stationary privacy policy, i.e.,* $\gamma_t = \gamma$, $\forall t \in \mathbb{N}$. *Let* $(a_t^*)_{t\in\mathbb{N}}$ *denote the equilibrium activity levels of this subgame. There is a cutoff value* $v^*(\gamma) \in \mathbb{R}_+$ *such that:*

1. *If* $v < v^*(\gamma)$, *then* a_t^* *increases in* t, $\lim_{t\to\infty} a_t^* = a_{max}$, *and* $\lim_{t\to\infty} \sigma_t^2 = 0$. *The consumer's continuation value decreases over time.*

2. *If* $v > v^*(\gamma)$, *then* $a_t^* = 0$ *and* $\sigma_t^2 = \sigma_0^2$ *for all* $t \in \mathbb{N}$.

The cutoff $v^*(\gamma)$ *is increasing in* γ, *and* $\lim_{\gamma\to\infty} v^*(\gamma) = \infty$.

The intuition is as follows. If the value of privacy is low, the consumer prefers a positive activity level $a_1^* > 0$ in $t = 1$. The consumer activity generates information, which reduces her payoff and the marginal cost of using the platform. As a result, she chooses $a_2^* \geq a_1^*$ in $t = 2$. Repeating this argument, we can conclude that a_t^* increases over time. The platform can then observe the signals to perfectly learn the consumer's type as $t \to \infty$. Perfect learning in $t \to \infty$ implies that the marginal privacy cost goes to zero, and thus $a_t^* \to a_{max}$. To sum up, if v is below the cutoff, the consumer eventually loses her privacy, but acts as if there is no privacy cost. In contrast, the consumer with a high v does not use the platform (Point 2). Finally, $v^*(\gamma)$ is increasing in γ because a higher privacy level reduces the cost of using the platform.

Proposition 1 highlights a perverse effect of privacy regulation: Suppose that a regulator, who cares about consumer privacy, mandates a stricter privacy policy—i.e., $\gamma_t = \gamma$ becomes $\gamma_t = \gamma' > \gamma$ for all $t \in \mathbb{N}$. The result implies that this regulation increases the cutoff from $v^*(\gamma)$ to $v^*(\gamma')$, and expands the range of v's under which the consumer loses privacy (Point 1). To see the welfare implication, suppose $v > \frac{u(a_{max})}{\sigma_0^2}$ holds. For a small γ, the consumer may choose $a_t^* = 0$ and obtain a payoff of zero in all periods. If the regulator enforces a large γ', then the consumer chooses $a_1^* > 0$. However, $a_1^* > 0$ implies $(a_t^*, \sigma_t^2) \to (a_{max}, 0)$, and thus the consumer's per-period payoff converges to $u(a_{max}) - v\sigma_0^2 < 0$. Thus, the regulation can increase the consumer's payoffs in the short run but decrease them in the long run. If the regulator cares about long-run consumer welfare, it may consider a higher γ to be detrimental.[9]

I now present the equilibrium dynamics of the entire game, in which the platform can commit to any (potentially nonstationary) privacy policy, taking the consumer's behavior (3) as given (see Appendix C for the proof; Appendix A proves the existence of an equilibrium).

Theorem 1. *Let $(a_t^*, \gamma_t^*, \sigma_t^2)_{t\in\mathbb{N}}$ denote the outcome of any equilibrium. In the long run, the consumer loses her privacy and chooses the highest activity level: $\lim_{t\to\infty} \sigma_t^2 = 0$ and $\lim_{t\to\infty} a_t^* = a_{max}$. For any $T \in \mathbb{N}$, there is a $\underline{v} \in \mathbb{R}$ such that for any $v \geq \underline{v}$, we have $\gamma_t^* > 0$ for all $t \leq T$. If the consumer is myopic or has a binary activity level (i.e., $|A| = 2$), there is a $T' \in \mathbb{N}$ such that for all $t \geq T'$, $\gamma_t^* = 0$.*

In equilibrium, the platform collects all the information in the long run, regardless of the consumer's value v of privacy and her discount factor δ_C. The result also shows that under certain conditions, the equilibrium privacy policy is nonstationary: The platform commits to positive privacy levels in early periods, but eventually offers a vanishing privacy level.

The intuition is as follows. In early periods, the platform knows little about the consumer, so the consumer's activity has a large impact on what the platform can learn about her type. Thus the consumer faces a high marginal privacy cost, which discourages her from raising the activity

[9]The caveat "if the regulator cares about the long-run consumer welfare" is important, because a higher γ increases the consumer's ex ante sum of discounted payoffs calculated based on δ_C. A higher privacy level is undesirable only if the regulator is more patient than the consumer.

level. The platform then commits to a high level of privacy protection to encourage consumer activity. As a result, in early periods the platform slowly learns about her type. After a long period of interaction, the platform accurately knows the consumer's type, which implies she faces a low marginal cost. The platform then reduces a privacy level to speed up learning.

We may think that when the consumer is patient, the platform benefits from offering a positive privacy protection even in the long run. However, this intuition is inaccurate: The consumer's objective in (3) is supermodular in today's activity level a_t and the precision of future signals $(\frac{1}{(a_{t+s})^{-1}+\gamma_{t+s}})_{s\in\mathbb{N}}$. As a result, the consumer prefers a higher activity level today when she anticipates to lose her privacy in the future.

Theorem 1 illustrates how the decreasing marginal privacy cost and the platform's commitment to underuse data render consumer privacy difficult to sustain. Indeed, if the consumer faced an increasing convex loss for providing data, the platform's learning could stop in the middle. If the platform had no commitment power, the consumer might choose $a_t = 0$, anticipating that the platform will choose $\gamma_t = 0$ ex post.

Figure 1 depicts the equilibrium dynamics for a myopic consumer in a numerical example.[10] Figure 1(a) shows the platform offers a decreasing privacy level, hitting zero in $t = 5$. Figure 1(b) shows that the equilibrium activity level first decreases but eventually approaches $a_{max} = 2$. The non-monotonicity of a^*_t contrasts with the case of a stationary privacy policy.[11]

4.1 Implications of Theorem 1

First, Theorem 1 provides an economic explanation of the so-called *privacy paradox*: Consumers seem to casually share their data with online platforms, despite their concerns about data collection.[12] We may view this puzzle as the long-run equilibrium outcome of this model, in which the consumer faces a high privacy cost and zero marginal cost. The result also points to the difficulty of applying the revealed preference argument to infer a consumer's value of privacy, based on their decision in a single period. Indeed, a static privacy choice may not reveal much about a consumer's

[10] I assume $A = \{0, 0.01, 0.02, \ldots, 2\}$ and compute the equilibrium strategy profile, using Proposition 2.

[11] I have not managed to generally prove the nonmonotonicity of $(a^*_t)_{t\in\mathbb{N}}$. However, a numerical exercise suggests that the nonmonotonicity occurs for a wide range of parameters (v, σ_0^2) with a myopic consumer, when the equilibrium privacy level is strictly decreasing in early periods.

[12] Acquisti et al. (2016) conduct an insightful review of research on the economics of privacy, including the privacy paradox. Recent empirical work includes, for example, Athey et al. (2017).

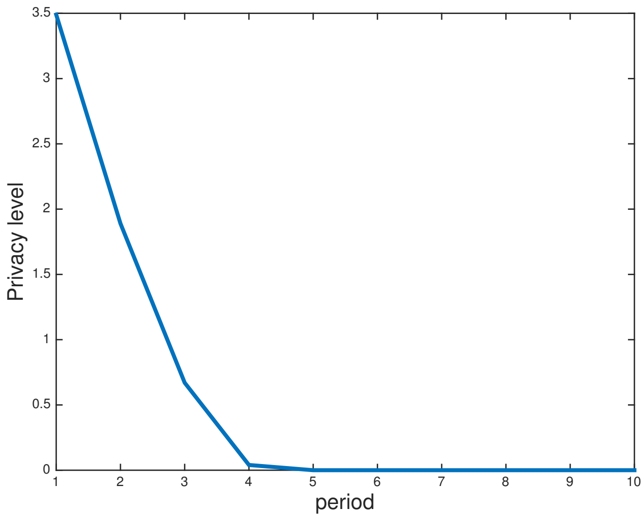

Figure 1(a): Privacy level γ_t

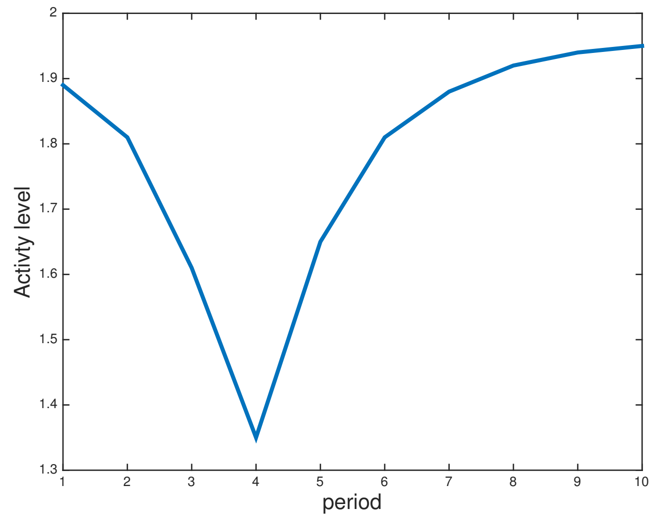

Figure 1(b): Activity level a_t

Figure 1: Equilibrium under $u(a) = 2a - \frac{1}{2}a^2$, $v = 10$, and $\sigma_0^2 = 1$.

preferences for privacy (v), if the choice is made after the consumer has revealed a lot information in the past.

Second, the result establishes a connection between consumer privacy problem and (harmful) rational addiction (Becker and Murphy, 1988). The connection stems from the observation that the consumer's utility is decreasing but marginal utility is increasing in the amount of information collected in the past. One difference with a typical model of rational addiction is that the platform can dynamically adjust the degree of addiction through its privacy policy. As a result, even if consumers are patient and highly value their privacy ex ante, they become "addicted" to the platform.

At an anecdotal level, the equilibrium strategy of the platform seems consistent with how the privacy policies of online platforms have evolved. In 2004, Facebook's privacy policy stated that it would not use (first-party) cookies to collect consumer information. In 2020, the privacy policy states that it uses cookies to track consumers on and possibly off the website.[13] Srinivasan (2019) describes how Facebook has acquired dominance in the social media market:

> When Facebook entered the market, the consumer's privacy was paramount. The com-

[13] In 2020, Facebook's privacy policy states that "we use cookies if you have a Facebook account, use the Facebook Products, including our website and apps, or visit other websites and apps that use the Facebook Products (including the Like button or other Facebook Technologies)." https://www.facebook.com/policies/cookies

> pany prioritized privacy, as did its users—many of whom chose the platform over others due to Facebook's avowed commitment to preserving their privacy. Today, however, accepting Facebook's policies in order to use its service means accepting broad-scale commercial surveillance.

Fainmesser et al. (2019) also describe how online platforms' business models have changed from the initial phase, in which they expand a user base, to the mature phase, in which they monetize the information collected. The equilibrium dynamics in Theorem 1—i.e., positive privacy levels in early periods, but vanishing privacy levels in the long run—rationalize the pattern described, as the platform's best response to consumers' declining incentives to protect their privacy.

4.2 Equilibrium Characterization Under a Myopic Consumer

When the consumer is myopic, we can characterize the equilibrium. To state the next result, let $a^*(\gamma, \sigma^2) \in A$ denote the optimal activity level of a myopic consumer, given a privacy level γ in the current period and the posterior variance σ^2 from the previous period:

$$a^*(\gamma, \sigma^2) := \max \left\{ \arg\max_{a \in A} u(a) - v\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a} + \gamma}}\right) \right\}. \tag{4}$$

Proposition 2. *If the consumer is myopic, the platform adopts a greedy policy that myopically maximizes the precision of the signal in each period. Formally, the equilibrium policy $(\gamma_1^*, \gamma_2^*, \dots)$ is recursively defined as follows:*

$$\gamma_t^* \in \arg\min_{\gamma \geq 0} \frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)} + \gamma, \forall t \in \mathbb{N}, \tag{5}$$

$$\hat{\sigma}_0^2 = \sigma_0^2, \tag{6}$$

$$\hat{\sigma}_t^2 = \frac{1}{\frac{1}{\hat{\sigma}_{t-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_t^*, \hat{\sigma}_{t-1}^2)} + \gamma_t^*}}, \forall t \in \mathbb{N}. \tag{7}$$

Also, given any privacy policy $\boldsymbol{\gamma}$, let $(\sigma_t^2)_{t \in \mathbb{N}}$ denote the posterior variances induced by the consumer's optimal behavior. Then the optimal policy attains uniformly lower posterior variances: $\hat{\sigma}_t^2 \leq \sigma_t^2$ for all $t \in \mathbb{N}$.

Proof. Lemma 1 implies $a^*(\gamma, \sigma^2)$ is increasing in γ and decreasing in σ^2. Take any privacy policy $(\gamma_t)_{t\in\mathbb{N}}$ and let $(\sigma_t^2)_{t\in\mathbb{N}}$ denote the sequence of posterior variances induced by $a^*(\cdot,\cdot)$. I show $\hat{\sigma}_t^2 \le \sigma_t^2$ for all $t \in \mathbb{N}$. The inequality holds with equality for $t = 0$. Take any $\tau \in \mathbb{N}$. Suppose $\hat{\sigma}_t^2 \le \sigma_t^2$ for $t = 0, \dots, \tau - 1$. It holds that

$$\sigma_\tau^2 = \frac{1}{\frac{1}{\sigma_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau, \sigma_{\tau-1}^2)} + \gamma_\tau}} \ge \frac{1}{\frac{1}{\hat{\sigma}_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau, \hat{\sigma}_{\tau-1}^2)} + \gamma_\tau}} \ge \frac{1}{\frac{1}{\hat{\sigma}_{\tau-1}^2} + \frac{1}{\frac{1}{a^*(\gamma_\tau^*, \hat{\sigma}_{\tau-1}^2)} + \gamma_\tau^*}} = \hat{\sigma}_\tau^2.$$

The first inequality follows from the inductive hypothesis and decreasing $a^*(\gamma, \cdot)$. The second inequality follows from (5). We now have $\hat{\sigma}_t^2 \le \sigma_t^2$ for all t, which implies the privacy policy described by (5), (6), and (7) is optimal. □

The objective of the minimization problem (5), $\frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)} + \gamma$, is the variance of the noise ε_t in the signal $s_t = X + \varepsilon_t$, given the consumer's best response. The minimization problem captures the platform's trade-off. On the one hand, a higher privacy level γ leads to a higher activity level, which leads to a lower variance $\frac{1}{a^*(\gamma, \hat{\sigma}_{t-1}^2)}$ of ε_t. On the other hand, given any activity level, a higher γ lowers the informativeness of the signal. This cost is captured by the second term γ. The platform chooses γ_t^* by resolving this trade-off. As the platform solves (5) in each period, the conditional variance evolves according to (7) with the initial condition (6).

The platform chooses its strategy to maximize the sum of discounted profits, but the optimal policy is greedy. The intuition is follows. In principle, the platform chooses (say) γ_1 to maximize the sum of period-1 profit and the continuation value. The period-1 profit is increasing in the precision of the signal in $t = 1$ by construction. If more information is generated in $t = 1$, the consumer faces lower marginal costs and chooses higher activity levels in the future. Thus, the continuation value is also increasing in the precision of the signal in $t = 1$. As a result, the platform can maximize the sum of discounted profits by maximizing the informativeness of signal in $t = 1$. Since the platform's optimal policy is greedy, it is time consistent:

Corollary 1. *Suppose the consumer is myopic, and let $(\boldsymbol{a}^*, \boldsymbol{\gamma}^*)$ denote the equilibrium outcome under the long-run commitment in Proposition 2. The same outcome $(\boldsymbol{a}^*, \boldsymbol{\gamma}^*)$ arises in an equilibrium of the game with one-period commitment.*

Proof. Suppose the platform has one-period commitment power. Consider the following strategy profile: Following every history with posterior variance σ^2, the platform sets $\gamma \in \arg\min_{\gamma \geq 0} \frac{1}{a^*(\gamma,\sigma^2)} + \gamma$. The consumer always acts according to $a^*(\cdot,\cdot)$. By construction, $(\boldsymbol{a}^*, \boldsymbol{\gamma}^*)$ arises on the path of play. Any deviation by the platform increases the posterior variances in all periods (Proposition 2). Thus, it has no profitable deviation. $\square$

We may think the optimality of a greedy policy is specific to a myopic consumer. However, the next section shows that under a certain condition, the equilibrium privacy policy is greedy even if the consumer is patient.

5 Equilibrium Under One-Period commitment

So far, I have assumed a platform has long-run commitment power. I now assume that a platform can commit to a privacy level only for one period. The purpose is to examine how the lack of commitment and consumer expectation could change the platform's ability to collect data. This short-run commitment could be realistic in some contexts. For example, a platform may be sanctioned for the outright violation of its privacy policy, but it may still revise its policy over time. Below, I present the platform-optimal and the consumer-optimal equilibria, assuming a binary activity level. After that, I consider a general set of activity levels.

5.1 Binary Activity Level

To facilitate the analysis, I impose the following assumption.

Assumption 1. The consumer has a binary activity level: $A = \{0, a_{max}\}$.

The following definitions simplify exposition.

Definition 1. A *Markov perfect equilibrium (MPE)* is an equilibrium in which after any history, the platform's choice γ_t depends only on σ^2_{t-1}, and the consumer's choice a_t depends only on $(\sigma^2_{t-1}, \gamma_t)$.

Definition 2. An equilibrium is *platform-best* if it maximizes the platform's ex ante sum of discounted payoffs across all subgame perfect equilibria. We analogously define "*platform-worst*," "*consumer-best*," and "*consumer-worst*."

For example, in a platform-best equilibrium, the platform receives a higher ex ante payoff than in any other equilibrium, which may or may not be an MPE. The following result presents a platform-best equilibrium (see Appendix D for the proof).

Theorem 2. *Under Assumption 1, there is a platform-best and consumer-worst Markov perfect equilibrium, which has the following properties:*

1. *The privacy level γ_t^* is decreasing in t. There is a $T \in \mathbb{N}$ such that for all $t \geq T$, $\gamma_t^* = 0$.*
2. *The consumer loses her privacy in the long run: $\lim_{t\to\infty} \sigma_t^2 \to 0$.*
3. *The platform's strategy is greedy: Given the consumer's strategy, after any history, the platform sets a privacy level γ_t to maximize the informativeness of the signal in period t.*
4. *The privacy levels $(\gamma_t^*)_{t\in\mathbb{N}}$ coincide with an optimal policy under long-run commitment.*

Points 1 and 2 extend the intuition in Theorem 1: The platform offers high privacy levels early to incentivize the consumer to generate information. As the platform collects more information, her incentive to protect privacy declines; correspondingly, the platform sets a decreasing privacy level, which hits zero in a finite period. Lemma 1 alone does not imply that the consumer faces a lower cost of choosing a_{max} when she has less privacy, because her continuation value is endogenous. However, in this equilibrium, the consumer's continuation value $V(1/\sigma_t^2)$, as a function of the amount of information collected, is decreasing and convex in $1/\sigma_t^2$. As a result, the consumer faces a declining marginal loss of generating information.

Point 3 states that the platform adopts a greedy policy, given the consumer's Markov strategy $a(\sigma_{t-1}^2, \gamma)$. The proof also reveals that any deviation by the platform reduces the precision of the signal in any future period. Thus we obtain the same platform-best equilibrium as long as the platform prefers to have more information (see the extension in Section 7). For example, we obtain the same equilibrium even if the platform is myopic.

Point 4 shows that one-period commitment is enough for the platform to attain the best outcome. In this equilibrium, the consumer chooses a_{max} in any period, while obtaining the worst ex ante payoff of zero. As a result, we can strengthen Point 4 as follows: Suppose the platform has the strongest commitment power—i.e., it can commit to any mechanism that determines privacy levels as a function of past and future outcomes. Even in that case, the platform's equilibrium payoff cannot be strictly greater than the best equilibrium payoff with one-period commitment.

Overall Theorem 2 indicates that the lack of long-run commitment and the consumer's patience do not necessarily prevent the platform from collecting information. At the same time, the result does not imply the uniqueness of the equilibrium. The following result shows an MPE in which the platform fails to collect any information (see Appendix E for the proof).

Theorem 3. *Suppose Assumption 1 and $\delta_C \geq \frac{1}{2}$ hold. There is a (v-dependent) $\bar{\rho} > 0$ such that if $\sigma_0^2 \geq \frac{1}{\bar{\rho}}$, there is a consumer-best and platform-worst Markov perfect equilibrium in which the platform sets $\gamma_t = \infty$ and the consumer chooses $a_t = a_{max}$ in all periods.*

In this equilibrium the platform offers full privacy, because whenever it deviates to a finite privacy level, the consumer chooses $a = 0$. The consumer is willing to do so, because she correctly anticipates that the initial privacy loss, no matter how small, will lead to the full privacy loss and impose her a high privacy cost in the future. Formally, after any off-path event in which the platform collects some information (i.e., $\sigma_t^2 < \sigma_0^2$), the platform-best equilibrium in Theorem 2 is played. A simple punishment—i.e., the platform's deviation induces $a_t = 0$ forever— does not work: As Lemma 10 in Appendix F shows, the platform can always set a large but finite γ_t to render such a punishment suboptimal for the consumer.

We may also interpret Theorem 3 as the plaform's Coasian commitment problem: The platform in period t competes with its future self that offers the best privacy protection in any period $s \geq t + 1$. Relative to such a situation, the platform benefits from committing to degrade future privacy protection. The literature shows that a firm's lack of commitment may decrease the equilibrium privacy protection and harm consumers (e.g., Dosis and Sand-Zantman 2019 and De Corniere and Taylor 2020). The above result shows that the opposite could occur in a dynamic setting.

5.2 General Set of Activity Levels

This subsection presents results for a general finite set A of activity levels. First, if the initial uncertainty about the consumer' type is small, the equilibrium is unique. Intuitively, if σ_0^2 is small, the marginal privacy cost is so small that the consumer prefers a_{max}, regardless of the current and future privacy protection.

Proposition 3. *There is a $B > 0$ such that if $\sigma_0^2 \leq B$, then any equilibrium has $(\gamma_t, a_t) = (0, a_{max})$ for all $t \in \mathbb{N}$, and $\lim_{t\to\infty} \sigma_t^2 = 0$.*

Proof. Let a' denote the second highest activity level in A. Take any B that satisfies $u(a_{max}) - u(a') - \frac{v}{1-\delta}B > 0$. In any period, if the consumer chooses $a_t = a_{max}$ instead of $a_t \in A \setminus \{a_{max}\}$, her gross payoff increases by at least $u(a_{max}) - u(a') > 0$. Also, the increment of privacy cost is at most $\frac{v}{1-\delta}B$. Thus, if $\sigma_0^2 \leq B$, the consumer always chooses a_{max}, and the platform chooses $\gamma_t = 0$ to maximize its payoff. $\square$

The general characterization of the platform-best equilibrium is beyond the scope of the paper. However, I can construct an equilibrium that has similar properties to Theorem 2, under the following technical assumption:

Assumption 2. The platform chooses a privacy level from a finite set $\Gamma \subset \overline{\mathbb{R}}_+$ that contains some finite $\bar{\gamma} > \frac{v(\sigma_0^2)^2}{(1-\delta_C)u(a_{max})} - \frac{1}{\sigma_0^2} - \frac{1}{a_{max}}$.

This assumption is consistent with $\min \Gamma = 0$ and $\max \Gamma = \infty$. The previous result implies that in any subgame in which $\sigma_t^2 \leq B$, there is a unique equilibrium. I can then use the "backward induction" with respect to σ_t^2 to construct an MPE, starting from any σ_0^2 (see Appendix F for the proof).

Proposition 4. *Under Assumption 2, for any σ_0^2, there is a Markov perfect equilibrium with the following properties: (i) $\lim\limits_{t\to\infty} \sigma_t^2 = 0$, and (ii) there is a $T \in \mathbb{N}$ such that for all $t \geq T$, $(\gamma_t^*, a_t^*) = (0, a_{max})$.*

6 Platform Competition with a Myopic Consumer

This section examines the implication of a decreasing marginal privacy cost on platform competition. There is an incumbent (I), an entrant (E), and a myopic consumer. I is in the market from the beginning of $t = 1$. In period $t^* \geq 2$, E enters the market. The entry period t^* is exogenous, deterministic, and commonly known. Let γ_t^k denote the privacy level of platform k in t.

Before the entry ($t < t^*$), the consumer chooses an activity level $a_t^I \in A$ for I. After the entry ($t \geq t^*$), the consumer chooses $(a_t^I, a_t^E) \in A^2$, where a_t^E is the activity level for E. The consumer can choose $(a_t^I, a_t^E) \in A^2$ if and only if $\min(a_t^I, a_t^E) = 0$. This restriction captures single-homing, which is natural if platforms offer similar services.

Since the result does not depend on the commitment regime, I examine competition with short-run commitment (Appendix G examines the case of long-run commitment). At the beginning of each period, a platform chooses a privacy level, after which the consumer chooses an activity level. In particular, I and E simultaneously set privacy levels γ_t^I and γ_t^E in each period $t \geq t^*$, without making any commitment to future privacy levels.

As before, platform $k \in \{I, E\}$ receives a signal $s_t^k = X + \varepsilon_t^k$ with $\varepsilon_t^k \sim \mathcal{N}\left(0, \frac{1}{a_t^k} + \gamma_t^k\right)$ in period t. Each platform k privately observes s_t^k, and all of the noise terms $(\varepsilon_t^k)_{k,t}$ are independent across $(k,t) \in \{I, E\} \times \mathbb{N}$. The payoff of platform $k \in \{I, E\}$ in period t is $\sigma_0^2 - \sigma_{t,k}^2$, where $\sigma_{t,k}^2$ is the posterior variance of the consumer's type, given activity levels and privacy levels. The consumer's payoff in period t is

$$u(a_t^I) - v\left(\sigma_0^2 - \sigma_{t,I}^2\right) + \mathbf{1}_{\{t \geq t^*\}} \cdot \left[u(a_t^E) - v\left(\sigma_0^2 - \sigma_{t,E}^2\right)\right], \tag{8}$$

where $\mathbf{1}_{\{t \geq t^*\}}$ is the indicator function that equals 1 or 0 if $t \geq t^*$ or $t < t^*$, respectively. Payoff (8) implies that even if the consumer switches to (say) E and never uses I from some period on, she continues to incur a privacy cost based on the information collected by I in the past (see the discussion in Section 3; Section 7 relaxes this assumption).

To obtain a non-trivial result, I impose an upper bound on the feasible privacy levels. The bound might capture information necessary for a platform to offer services, or the maximum privacy protection a platform can credibly enforce. In particular, the assumption prevents E from offering $\gamma_t^E = \infty$ to poach the consumer.

Assumption 3. There is a $\bar{\gamma} \in \mathbb{R}_+$ satisfying $a^*\left(\bar{\gamma}, \sigma_0^2\right) > 0$ (in (4)) such that I and E can choose a privacy level of at most $\bar{\gamma}$.

6.1 Equilibrium Under Competition

I present an equilibrium in which the long-run outcome equals the monopoly outcome. Also, if the entry is sufficiently late, any equilibrium coincides with monopoly (see Appendix G for the proof).

Proposition 5. *Under Assumption 3, the following holds.*

1. *There is an equilibrium in which $a_t^E = 0$ for all $t \in \mathbb{N}$, $\lim_{t\to\infty} a_t^I = a_{max}$, $\lim_{t\to\infty} \sigma_{I,t}^2 = 0$, and $\lim_{t\to\infty} \gamma_t^I = 0$.*

2. *There is a $\underline{t} \geq 2$ such that if the entry time t^* is greater than $\underline{t}$, any equilibrium outcomes for the consumer and the incumbent, $(a_t^I, \gamma_t^I)_{t\in\mathbb{N}}$, coincide with the monopoly outcomes.*

The intuition is as follows. Suppose that upon entry, the entrant sets the highest privacy level $\bar{\gamma}$. Since the privacy cost from collected data is sunk, the consumer decides which platform to use based on her marginal (or, more precisely, incremental) costs. Because the incumbent has already collected data, the consumer faces a lower marginal cost of using it. Thus if the incumbent also chooses $\bar{\gamma}$, the consumer prefers to use it. However, the equilibrium choice of the incumbent may not be $\bar{\gamma}$: it chooses a privacy level to maximize the precision of the signal, subject to the constraint that the consumer does not switch to the entrant. As time goes by, the constraint is relaxed, because the consumer's marginal cost for the incumbent goes to zero. As a result, it offers a vanishing privacy level over time. Finally, the threat of future entry does not affect the incumbent's strategy: Before the entry, it chooses the same privacy levels as a monopoly, because collecting more information renders consumer switching less likely.

In this model, switching and market entry are less likely when consumers suffer from a lack of privacy and receive low payoffs from the incumbent. This welfare implication contrasts with the idea of "data as an entry barrier," in which dominant platforms use data to improve their services. For example, Furman et al. (2019) state that:

> Data can act as a barrier to entry in digital markets. A data-rich incumbent is able to cement its position by *improving its service and making it more targeted for users*, as well as making more money by better targeting its advertising. (italics added)

As an example, consider search engines: The incumbent is Google, and the entrant is a privacy-preserving alternative of Google, such as DuckDuckGo. If consumers have no privacy on Google, they face negligible marginal privacy costs of using it. Then even if DuckDuckGo is as good a search engine as Google and offers better privacy protection, it may not be able to poach consumers.

7 Extensions

This section examines several extensions. Although we could analyze them under different commitment assumptions and market structures, for simplicity I focus on a monopoly with long-run commitment and a myopic consumer (unless otherwise noted). Omitted proofs are in Appendix H.

7.1 Erasing Past Information

So far, I have assumed the platform indefinitely keeps the data collected in the past. This extension studies the incentive of the consumer or the platform to erase past information.

7.1.1 The Right to be Forgotten

First, I consider the right to be forgotten, whereby the consumer can request a platform to delete past information. At the beginning of each period, the consumer chooses whether to erase past information, then chooses an activity level. If she erases information in period t, the posterior variance at the beginning of t becomes the prior variance σ_0^2. At the end of the period, the consumer still incurs a privacy cost based on information generated in that period. For example, if the consumer erases information in period t, her payoff is $u(a_t) - v\left[\sigma_0^2 - \sigma_1^2(a_t, \gamma_t)\right]$, where $\sigma_1^2(a_t, \gamma_t)$ is the posterior variance given one signal based on (a_t, γ_t). Thus, the privacy cost is only based on the signal of period t. In contrast, if the consumer has never erased information, her payoff in period t is $u(a_t) - v\left[\sigma_0^2 - \sigma_t^2(\boldsymbol{a}_t, \boldsymbol{\gamma}_t)\right]$.

Claim 1. *If the consumer can costlessly erase past information, there is an equilibrium in which the platform commits to a stationary privacy policy $\gamma_t \equiv \gamma_1^*$, where γ_1^* is defined in (5). In this equilibrium, the consumer erases information in every period.*

Once the consumer erases information, she incurs a high marginal privacy cost. Then the platform offers a period-1 privacy level in any period. As a result, the equilibrium involves neither privacy loss nor vanishing privacy protection.

Finally, although I analyzed a monopoly, erasing past information is even more effective when there is competition. If the consumer deletes information, the incumbent and the entrant become homogeneous. As a result, they offer the highest privacy protection to attract consumers.

7.1.2 Data Retention Policies

Does the platform have an incentive to voluntarily erase past data? This question relates to data retention policies, which have recently drawn the attention of economists and legal scholars (Chiou and Tucker, 2017). Here, at the beginning of this game, the platform commits to a privacy policy $(\gamma_t)_{t\in\mathbb{N}}$ and the set $\mathcal{T} \subset \mathbb{N}$ of periods to delete information. The platform erases past information at the beginning of each period $t \in \mathcal{T}$. The platform's erasing information affects the posterior variance and payoffs in the same way as the consumer erasing information (see the previous subsection). As a result, erasing information increases σ_t^2 to σ_0^2, and decreases the myopic consumer's activity level. Thus we obtain the following result.

Claim 2. *In any equilibrium, the platform never erases information: $\mathcal{T} = \emptyset$.*

The result implies that the platform has different incentives to offer ex ante and ex post privacy protections: It may voluntarily offer high privacy levels in early periods, because committing to collect less information encourages the consumer's activity. However, the platform has no incentive to delete past information, because it increases the consumer's marginal cost and decreases her activity level.

7.2 Consumers with Heterogeneous v

The main insight does not depend on whether the platform knows v at the outset. To see this, I extend the model as follows: There is a unit mass of consumers. Each consumer $i \in [0,1]$ has v_i, which is distributed according to a distribution with a finite support $V \subset \mathbb{R}_+$. Let $\alpha_v \in [0,1]$ denote the mass of consumers with $v \in V$. Each consumer i is privately informed of v_i, and the platform knows V and $(\alpha_v)_{v\in V}$.

The game is a natural extension of the baseline model. Before $t = 1$, the monopoly platform chooses a privacy policy $(\gamma_t)_{t\in\mathbb{N}}$, which is common across all consumers. Then each consumer i myopically chooses activity levels $(a_t(i))_{t\in\mathbb{N}}$. The types and signals are independent across consumers.

For each $i \in [0,1]$, let $\sigma_t^2(i)$ denote the posterior variance for consumer i at the end of period t. Then i's payoff is $u(a_t(i)) - v_i[\sigma_0^2 - \sigma_t^2(i)]$, and the platform's payoff is $\int_{i\in[0,1]} \sigma_0^2 - \sigma_t^2(i)di$. If (almost) all consumers who have the same v choose the same activity level, we can write the platform's profit as $\sum_{v \in V} \alpha_v [\sigma_0^2 - \sigma_t^2(v)]$, where $\sigma_t^2(v)$ is the posterior variance of consumers with v. The variance $\sigma_t^2(v)$ is well-defined, because consumers with the same v choose the same sequence of activity levels.

The platform faces a new trade-off: A high privacy level encourages consumers with high v to choose positive activity levels. However, the platform obtains less information from consumers with low v, who would choose high activity levels without privacy protection.[14] This static trade-off also creates a dynamic trade-off: For example, a more myopic platform may set a low privacy level to collect a lot of information only from consumers with low v, whereas a patient platform may set high privacy levels to collect information from all consumers.

However, there is no trade-off for the platform in the long run—i.e., all consumers eventually lose privacy and choose the highest activity levels.

Proposition 6. *Let $(a_t^*(v), \sigma_t^2(v), \gamma_t^*)_{t \in \mathbb{N}, v \in V}$ denote the outcome of any equilibrium. Then,*

$$\forall v \in V, \lim_{t \to \infty} (a_t^*(v), \sigma_t^2(v)) = (a_{max}, 0) \text{ and } \lim_{t \to \infty} \gamma_t^* = 0. \tag{9}$$

To see the intuition, suppose that v is either $L = 0$ or $H > 0$, and the platform sets $\gamma_t = 0$ in early periods to collect information only from L-consumers. During this period, only $\sigma_t^2(L)$ decreases over time. However, once $\sigma_t^2(L)$ gets close to zero, the platform finds it more profitable to increase a privacy level to encourage H-consumers to use the platform. Thus, the platform eventually obtains information from all consumers.

7.3 General Privacy Cost Function

This subsection generalizes consumer preferences in two ways. First, I relax the assumption that the privacy cost is sunk. Second, I relax the assumption that the privacy cost is linear in σ_t^2.

7.3.1 Relaxing "The Privacy Cost is Sunk"

In the baseline model, the consumer incurs a privacy cost of $-v(\sigma_0^2 - \sigma_{t-1}^2)$ even if she chooses $a_t = 0$. Suppose now that the consumer incurs only a fraction $\alpha \in [0, 1)$ of the privacy cost when

[14] A similar trade-off arises in Lefouili and Toh (2019).

$a_t = 0$: If $a_t > 0$, her payoff is $u(a_t) - v\left(\sigma_0^2 - \sigma_t^2\right)$. If $a_t = 0$, it is $-\alpha v\left(\sigma_0^2 - \sigma_{t-1}^2\right)$. The main results under monopoly and competition continue to hold for α close to 1. The following result considers the case of monopoly.

Proposition 7. *Take any $v \in \mathbb{R}_+$. There is an $\alpha^* < 1$ such that for any $\alpha \in [\alpha^*, 1]$, any equilibrium outcome $(a_t^*, \gamma_t^*, \sigma_t^2)_{t\in\mathbb{N}}$ satisfies $\lim_{t\to\infty} a_t^* = a_{max}$, $\lim_{t\to\infty} \gamma_t^* = 0$, and $\lim_{t\to\infty} \sigma_t^2 = 0$.*

The following result considers the case of competition.

Proposition 8. *There is an $\alpha^* < 1$ such that for any $\alpha \in [\alpha^*, 1]$, the result under competition (Proposition 5) holds.*

7.3.2 Relaxing Linearity

Suppose the consumer's per-period payoff is $u(a_t) - C\left(\sigma_t^2\right)$, where $C(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable. In particular, the cost $C(0)$ and the marginal cost $C'(0)$ at no privacy are finite. The cost function can be nonmonotone—e.g., $C(\cdot)$ can be first decreasing and then increasing, which means the consumer prefers some (but not too much) data collection. The following result shows that the long-run outcome remains the same.

Proposition 9. *In any equilibrium, $\lim_{t\to\infty} \sigma_t^2 = 0$. Also, there is a $T \in \mathbb{N}$ such that for all $t \geq T$, $(a_t^*, \gamma_t^*) = (a_{max}, 0)$.*

7.4 Time-Varying Type of the Consumer

The baseline model assumes that the consumer's type X is constant over time. However, we can conceptually extend the model so that her type is some stochastic process $(X_t)_{t\in\mathbb{N}}$. One possibility, which I adopt for a numerical analysis, is as follows: $X_{t+1} = \phi X_t + \zeta_t$ with $\phi \in [0, 1]$, $X_0 \sim \mathcal{N}(0, \sigma_0^2)$, and $\zeta_t \overset{iid}{\sim} \mathcal{N}(0, (1-\phi^2)\sigma_0^2)$. The variance of each ζ_t is normalized so that $Var(X_t) = \sigma_0^2$ for all $t \in \mathbb{N}$. As in the baseline model, given an activity level a_t and a privacy level γ_t in period t, the platform observes a signal $s_t = X_t + \varepsilon_t$ with $\varepsilon_t \sim \mathcal{N}\left(0, \frac{1}{a_t} + \gamma_t\right)$. The posterior variance evolves according to $\sigma_t^2 = \frac{1}{\frac{1}{\phi^2\sigma_{t-1}^2 + (1-\phi^2)\sigma_0^2} + \frac{1}{\frac{1}{a_t} + \gamma_t}}$.

A natural question is how the equilibrium converges to the steady state. However, such an analysis is difficult, partly because the consumer's objective is neither concave nor convex in a_t.

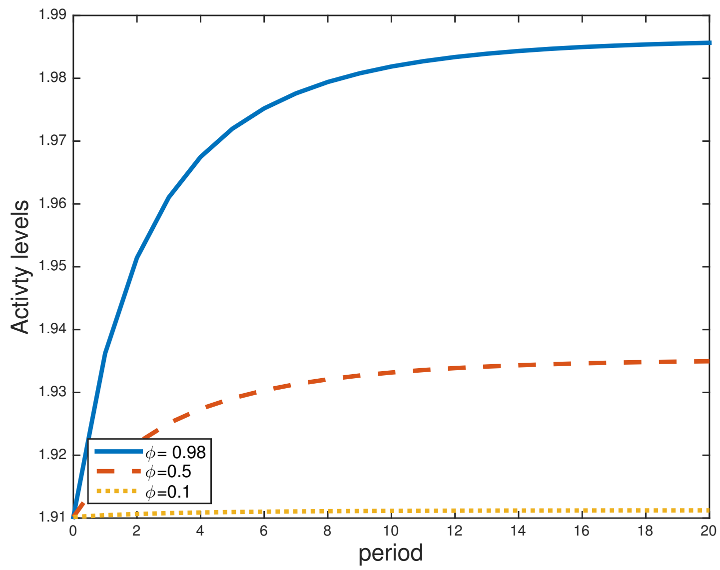

Figure 2: Activity levels $u(a) = 2a - \frac{1}{2}a^2$, $v = 10$, $\sigma_0^2 = 1$, $\phi \in \{0.1, 0.5, 0.98\}$, and $\gamma_t \equiv 4$.

Thus I present a numerical analysis to examine the convergence to the steady state, and how the equilibrium responds to the persistence of the consumer's type. Intuitively, if the type is less persistent (i.e., ϕ is small), a larger amount of new information arrives in each period. Then, she faces a higher marginal cost and chooses a lower activity level. Figure 2 confirms this intuition: Given a stationary privacy policy, the optimal activity levels converge to the steady states, which seem to increase in ϕ.

Figure 3 presents equilibria taking the platform's optimization into account. First, the numerical analysis suggests that the main insight of this paper is not specific to the baseline specification $\phi = 1$. Namely, the platform offers a relatively high privacy level in early periods but later reduces it (Figure 3(a)). While Figure 3 fixes v, a similar numerical exercise shows that the platform is able to obtain a nontrivial amount of information in the steady state even if v is larger.[15] Second, the platform offers a higher privacy level when the consumer's type is less persistent. This observation is consistent with the intuition that the consumer faces a higher privacy cost when her type is less persistent. Finally, the equilibrium activity level is not necessarily decreasing in ϕ. Indeed, the steady-state activity level at $\phi = 0.98$ is higher than the one at $\phi = 0.5$, but lower than the one at $\phi = 0.1$. Thus, the activity level is no longer monotone in ϕ if platform can adopt a nonstationary

[15]For example, if $\phi = 0.5$ and $v = 200$, then in the steady state the platform offers $\gamma_t \approx 90$ and the consumer chooses $a_t = a_{max} = 2$.

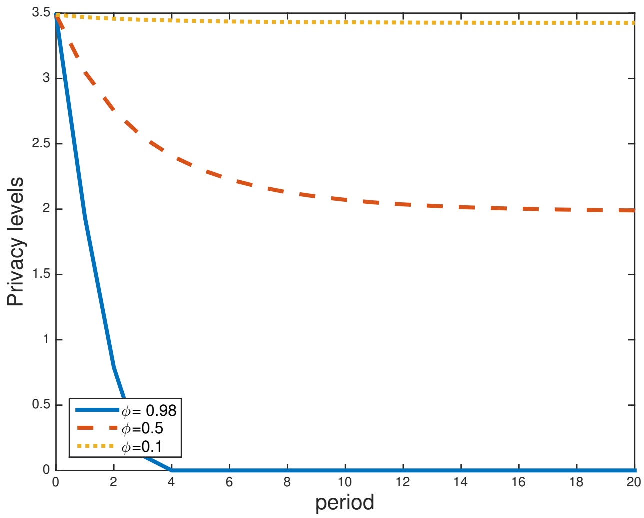

Figure 3(a): Privacy level γ_t

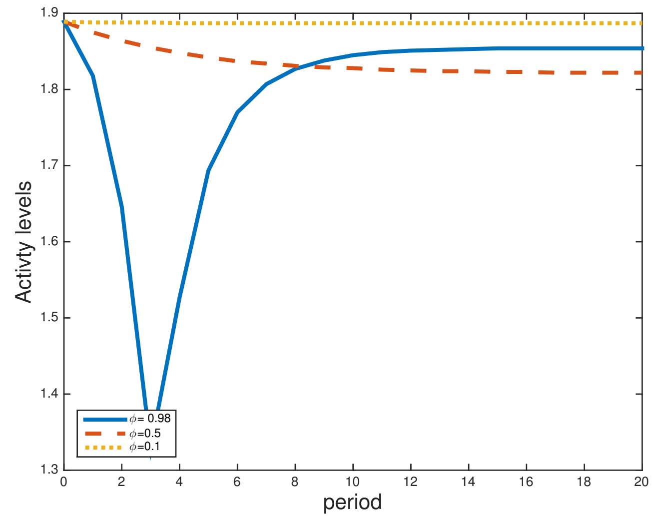

Figure 3(b): Activity level a_t

Figure 3: Activity levels $u(a) = 2a - \frac{1}{2}a^2$, $v = 10$, $\sigma_0^2 = 1$, $\phi \in \{0.1, 0.5, 0.98\}$.

privacy policy.

7.5 General Payoffs of the Platform

This subsection allows the consumer to have any discount factor $\delta_C \in [0, 1)$. First, I relax the functional form assumption of the platform's payoff, maintaining the property that it only depends on information. Second, I consider the platform's payoffs that depend also on activity levels.

7.5.1 General Preferences for Information

All the results of this paper continue to hold if the platform's final payoff from a sequence of posterior variances is $\Pi((\sigma_t^2)_{t\in\mathbb{N}})$, where $\Pi : \mathbb{R}_+^\infty \to \mathbb{R}$ is bounded and coordinate-wise strictly decreasing. This generalization does not change the analysis, because in the equilibrium under monopoly or competition, a deviation by the platform increases σ_t^2 for all $t \in \mathbb{N}$.[16]

One natural specification of $\Pi(\cdot)$ is as follows: Suppose the platform sells information to a sequence of short-lived data buyers. Any information sold in period t is freely replicable later

[16]The platform's deviation may not uniformly increase posterior variances in Theorem 1. However, the proof of this theorem rests on the argument that if the equilibrium fails to meet certain conditions such as $\sigma_t^2 \to 0$, the platform can deviate and uniformly decrease posterior variances. Thus, Theorem 1 continues to hold with the same proof under a general $\Pi(\cdot)$.

and thus has a price of zero in any period $s \geq t+1$. Then, the platform's payoff in period t equals the value of information newly generated in period t—i.e., the platform's ex ante payoff is $\sum_{t=1}^{\infty} \delta_P^{t-1}(\sigma_{t-1}^2 - \sigma_t^2)$. This objective is strictly decreasing in each σ_t^2, because the coefficient of each σ_t^2 is $-\delta_P^{t-1} + \delta_P^t < 0$.

7.5.2 Revenue from Consumer Activity

For some applications, it is natural to assume that the platform's payoffs depend on consumer activity. To capture such a situation, suppose the platform's payoff is $\sum_{t=1}^{\infty} \delta_P^{t-1}\Pi(a_t, \sigma_0^2 - \sigma_t^2)$, where $\Pi(a_t, \sigma_0^2 - \sigma_t^2)$ is strictly increasing in activity a_t and information $\sigma_0^2 - \sigma_t^2$.

If the consumer has a binary activity level, this extension does not change the results, because a higher activity level (i.e., a_{max} as opposed to 0) implies more information. In contrast, for a general set A, this extension creates a new dynamic trade-off for the platform. To see this, suppose the consumer is myopic. If the platform offers a low privacy level, the consumer may choose a low activity level in the current period. However, if a low privacy level leads to greater information collection, the consumer will choose higher activity levels in the future. As a result, the platform now faces a trade-off between offering privacy to induce a high activity level today and collecting more information to induce high activity levels in the future. Although this trade-off may affect some of the results, the main insight continues to hold when the platform is patient.

Proposition 10. *Suppose the platform has long-run commitment power, and fix any $\delta_C \in [0,1)$. For each $\delta_P \in (0,1)$, let $a_\infty(\delta_P)$ and $\sigma_\infty^2(\delta_P)$ denote the long-run activity level and posterior variance in an (arbitrarily chosen) equilibrium. Then $\lim_{\delta_P \to 1}(a_\infty(\delta_P), \sigma_\infty^2(\delta_P)) = (a_{max}, 0)$.*

Proof. Theorem 1 shows that the platform has a strategy $\boldsymbol{\gamma}^*$ that induces $\lim_{t\to\infty}(a_t, \sigma_t^2) = (a_{max}, 0)$. For any $\boldsymbol{\gamma}$ such that $\lim_{t\to\infty}\sigma_t^2 > 0$, a sufficiently patient platform strictly prefers $\boldsymbol{\gamma}^*$ to $\boldsymbol{\gamma}$. Finally, $\sigma_t^2 \to 0$ implies $a_t \to a_{max}$. $\square$

In the short run, the platform may face the above trade-off. However, the platform can always collect full information over time to induce the highest activity level in the long run. Thus a patient platform still chooses to collect information as long as the revenue depends on information.

7.6 Endogenous Quality of Service

So far, the benefit $u(\cdot)$ from the platform's service has been exogenous. Suppose now that before $t = 1$, the platform chooses a quality $q \geq 0$. Given q, the consumer receives a gross benefit of $u_q(a) = q \cdot a$, and the platform receives a payoff of $\sigma_0^2 - \sigma_t^2 - c(q)$ for some strictly increasing $c(\cdot)$. The platform chooses q once, but incurs $c(q)$ in every period. The following result shows that a patient platform does not invest in quality. Recall that $\delta_P \in (0, 1)$ is the platform's discount factor.

Proposition 11. *For any $\delta_P \in (0, 1)$, let $q(\delta_P)$ denote the quality in an (arbitrarily chosen) equilibrium. For any $\delta_C \in [0, 1)$, $\lim_{\delta_P \to 1} q(\delta_P) = 0$. As a result, as $\delta_P \to 1$, the consumer's ex ante sum of discounted payoffs (calculated based on a fixed δ_C) converges to zero.*

The intuition is as follows. The platform may be able to collect information more quickly by choosing a high q. However, the long run outcome—full information collection—is independent of q. As a result, a sufficiently patient platform chooses an arbitrarily low quality. This result holds even if the platform's profit is increasing both in information and activity levels.

8 Conclusion

This paper studies a dynamic model of consumer privacy and platform data collection. The fundamental feature of the model is that data collection today reduces a consumer's marginal loss of giving up privacy in the future. I examine dynamic implications of this idea. First, a monopoly platform is able to collect a lot of information over time by committing to underuse data in early periods. In equilibrium, the consumer eventually loses privacy but keeps choosing a high level of activity. Under a certain condition, the optimal privacy policy is greedy and implementable with a minimal commitment power. Second, if the platform cannot commit to degrade future privacy protection, it may end up offering the highest level of privacy protection in some equilibrium. The result emphasizes the role of consumer expectation in determining a firm's ability to collect data. Finally, a decreasing marginal privacy cost renders competition unhelpful, because a consumer is more likely to stick with a platform on which they have less privacy. I show how a regulation such as the right to be forgotten benefits consumers.

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Appendix

A Existence of an Equilibrium Under Long-Run Commitment

I prove the existence of an equilibrium under long-run commitment with $\delta_C > 0$ (for a myopic consumer, Proposition 2 constructs an equilibrium). I introduce some notations. Let $\mathcal{A} := A^\infty$ denote the set of all sequences of activity levels. Because $A \subset \mathbb{R}_+$ is finite, it is compact, so $\mathcal{A}$ is compact with respect to product topology. Let $\boldsymbol{a}$ denote a generic element of $\mathcal{A}$, with the t-th coordinate denoted by a_t. Let $\boldsymbol{\Gamma} := [0, \infty]^\infty$ denote the set of all privacy policies. Let $\boldsymbol{\gamma}$ denote a generic element of $\boldsymbol{\Gamma}$, with the t-th coordinate denoted by γ_t. I consider the ordered topology for $\overline{\mathbb{R}}_+$ and the product topology for $\boldsymbol{\Gamma}$. Finally, let $U_t(\boldsymbol{a}, \boldsymbol{\gamma})$ denote the consumer's flow payoff in period t, given an outcome $(\boldsymbol{a}, \boldsymbol{\gamma})$. Note that $U_t(\boldsymbol{a}, \boldsymbol{\gamma})$ depends only on $(a_1, \dots, a_t)$ and $(\gamma_1, \dots, \gamma_t)$.

Given any privacy policy $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, the consumer's problem is

$$\max_{\boldsymbol{a} \in \mathcal{A}} \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{a_s} + \gamma_s}} \right) \right]. \tag{10}$$

For any $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, let $\mathcal{A}^*(\boldsymbol{\gamma}) \subset \mathcal{A}$ denote the set of all maximizers of (10).

Lemma 2. *$\mathcal{A}^*(\boldsymbol{\gamma})$ is non-empty, compact, and upper hemicontinuous in $\boldsymbol{\gamma}$.*

Proof. First, $\mathcal{A}$ is compact with respect to product topology. Second, the objective function is continuous: To see this, take any sequence of the consumer's choices $(\boldsymbol{a}^n)_{n=1}^\infty$ such that $\boldsymbol{a}^n \to \boldsymbol{a}^*$. This implies that, for each $t \in \mathbb{N}$, $\lim_{n \to \infty} a_t^n \to a_t^*$. The consumer's period-$t$ payoff $U_t(\boldsymbol{a}, \boldsymbol{\gamma}) := u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{a_s} + \gamma_s}} \right)$ is bounded from above and below by $u(a_{max})$ and $-v\sigma_0^2$, respectively. Define $B := \max(u(a_{max}), v\sigma_0^2) > 0$. Take any $\varepsilon > 0$, and let T^* satisfy $\frac{\delta_C^{T^*}}{1-\delta_C} B < \frac{\varepsilon}{4}$. Take a sufficiently large n so that for each $t \leq T^*$, $\delta_C^{t-1} |U_t(\boldsymbol{a}^n, \boldsymbol{\gamma}) - U_t(\boldsymbol{a}^*, \boldsymbol{\gamma})| < \frac{\varepsilon}{2T^*}$. These inequalities imply that

$$\left| \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\boldsymbol{a}^n, \boldsymbol{\gamma}) - \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\boldsymbol{a}^*, \boldsymbol{\gamma}) \right| < \varepsilon.$$

Thus the objective function in (10) is continuous in $\boldsymbol{a}$. Berge maximum theorem implies that $\mathcal{A}^*(\boldsymbol{\gamma})$ is non-empty, compact, and upper hemicontinuous in $\boldsymbol{\gamma}$. $\square$

Next, I show properties of the consumer's objective $U(\boldsymbol{a}, \boldsymbol{\gamma}) := \sum_{t=1}^{\infty} \delta_C^{t-1} U_t(\boldsymbol{a}, \boldsymbol{\gamma})$. Abusing notation, for any $\boldsymbol{a}, \boldsymbol{a}' \in \mathcal{A}$, write $\boldsymbol{a} \geq \boldsymbol{a}'$ if and only if $a_t \geq a'_t$ for all $t \in \mathbb{N}$. $\geq$ is a partial order on $\mathcal{A}$, and $(\mathcal{A}, \geq)$ is a lattice.

Lemma 3. *For any $\boldsymbol{\gamma}$, $U(\boldsymbol{a}, \boldsymbol{\gamma})$ is supermodular in $\boldsymbol{a}$.*

Proof. Take any $\boldsymbol{\gamma}$. Below, I omit $\boldsymbol{\gamma}$ and write $U(\cdot, \boldsymbol{\gamma})$ as $U(\cdot)$. Take any $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$. For each $n \in \mathbb{N}$, define $(\boldsymbol{a} \vee \boldsymbol{b})^n$ as

$$(\boldsymbol{a} \vee \boldsymbol{b})^n = \begin{cases} \max(a_t, b_t) & \text{if} \quad t \leq n, \\ a_t & \text{if} \quad t > n. \end{cases} \tag{11}$$

Similarly, define $(\boldsymbol{a} \wedge \boldsymbol{b})^n$ as

$$(\boldsymbol{a} \wedge \boldsymbol{b})^n = \begin{cases} \min(a_t, b_t) & \text{if} \quad t \leq n, \\ a_t & \text{if} \quad t > n. \end{cases} \tag{12}$$

Also, define $\boldsymbol{b}^n$ as

$$\boldsymbol{b}^n = \begin{cases} b_t & \text{if} \quad t \leq n, \\ a_t & \text{if} \quad t > n. \end{cases} \tag{13}$$

In product topology, $(\boldsymbol{a} \vee \boldsymbol{b})^n \to \boldsymbol{a} \vee \boldsymbol{b}$, $(\boldsymbol{a} \wedge \boldsymbol{b})^n \to \boldsymbol{a} \wedge \boldsymbol{b}$, and $\boldsymbol{b}^n \to \boldsymbol{b}$ as $n \to \infty$. For each $t \in \mathbb{N}$ and $n \in \mathbb{N}$, $U_t(\boldsymbol{a}, \boldsymbol{\gamma})$ is supermodular in $(a_1, \dots, a_n)$, because the consumer's objective has increasing differences in each pair (a_t, a_s). Thus for each $n \in \mathbb{N}$, $U(\boldsymbol{a})$ is supermodular in the first n activity levels, $(a_1, \dots, a_n) \in \mathbb{R}_+^n$. We then have $U((\boldsymbol{a} \vee \boldsymbol{b})^n) + U((\boldsymbol{a} \wedge \boldsymbol{b})^n) \geq U(\boldsymbol{a}) + U(\boldsymbol{b}^n)$. Because $U(\cdot)$ is continuous, we can take $n \to \infty$ and obtain $U(\boldsymbol{a} \vee \boldsymbol{b})) + U(\boldsymbol{a} \wedge \boldsymbol{b}) \geq U(\boldsymbol{a}) + U(\boldsymbol{b})$. $\square$

The supermodularity implies the consumer has the "greatest" optimal choice.

Lemma 4. *For each $\boldsymbol{\gamma}$, the set $\mathcal{A}^*(\boldsymbol{\gamma})$ of optimal choices is a sublattice of $\mathcal{A}$. There is an $\bar{\boldsymbol{a}} \in \mathcal{A}^*(\boldsymbol{\gamma})$ such that for any $\boldsymbol{a} \in \mathcal{A}^*(\boldsymbol{\gamma})$, $\bar{\boldsymbol{a}} \geq \boldsymbol{a}$.*

Proof. First, Corollary 2 of Milgrom et al. (1994) implies that $\mathcal{A}^*(\boldsymbol{\gamma})$ is a sublattice of $\mathcal{A}$. Let $\mathcal{A}_t^*(\boldsymbol{\gamma})$ denote the projection of $\mathcal{A}^*(\boldsymbol{\gamma})$ on the t-th coordinate, i.e.,

$$\mathcal{A}_t^*(\boldsymbol{\gamma}) := \left\{a_t \in A : \exists \boldsymbol{a}_{-t} = (a_s)_{s\in\mathbb{N}\setminus\{t\}} \in \mathcal{A}^*(\boldsymbol{\gamma}) \text{ s.t. } (a_t, \boldsymbol{a}_{-t}) \in \mathcal{A}^*(\boldsymbol{\gamma})\right\}. \tag{14}$$

Here, $(a_t, \boldsymbol{a}_{-t})$ is a sequence of activity levels such that the consumer takes a_t in period t and acts according to $\boldsymbol{a}_{-t}$ in other periods. For each $k \in \mathbb{N}$, let $\boldsymbol{a}^k$ denote an optimal policy such that the consumer chooses $a_k = \max \mathcal{A}_k^*(\boldsymbol{\gamma})$ in period k. Define $\bar{\boldsymbol{a}}^k := \boldsymbol{a}^1 \vee \cdots \vee \boldsymbol{a}^k$. Because $\mathcal{A}^*(\boldsymbol{\gamma})$ is sublattice, for any $k \in \mathbb{N}$, $\bar{\boldsymbol{a}}^k$ maximizes (10). We also have $\bar{\boldsymbol{a}}^k \to \bar{\boldsymbol{a}}$, where $\bar{a}_t = \max \mathcal{A}_k^*(\boldsymbol{\gamma})$ for any $k \in \mathbb{N}$. Because $\mathcal{A}^*(\boldsymbol{\gamma})$ is compact, $\bar{\boldsymbol{a}} \in \mathcal{A}^*(\boldsymbol{\gamma})$. By construction, for any $\boldsymbol{a} \in \mathcal{A}^*(\boldsymbol{\gamma})$, $\bar{\boldsymbol{a}} \geq \boldsymbol{a}$. $\square$

For each $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, let $\bar{\boldsymbol{a}}(\boldsymbol{\gamma}) := (\bar{a}_t(\boldsymbol{\gamma}))_{t\in\mathbb{N}}$ denote the greatest strategy of the consumer defined in Lemma 4.

Lemma 5. *For each $t \in \mathbb{N}$, $\bar{\boldsymbol{a}}_t(\boldsymbol{\gamma})$ is upper semicontinuous in $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$.*

Proof. By Lemma 2, $\mathcal{A}^*(\boldsymbol{\gamma})$ is upper hemicontinuous, so the set $\mathcal{A}_t^*(\boldsymbol{\gamma})$ of all activity levels that can be chosen in period t is upper hemicontinuous in $\boldsymbol{\gamma}$. Thus, it is enough to show that for any upper hemicontinuous and compact-valued correspondence $\phi : X \twoheadrightarrow \mathbb{R}$, $f(x) := \max \phi(x)$ is upper semicontinuous. To show this, take any $x_n \to x$. For each n, define $y_n = f(x_n)$. Because there is a subsequence $y_{n(k)}$ of y_n that converges to $\limsup y_n$, it holds that $\limsup y_n = \lim y_{n(k)} = \lim f(x_{n(k)}) \leq f(\lim x_{n(k)}) = f(x)$. The inequality holds because ϕ has a closed graph. Connecting the left and right sides, we establish that $f(\cdot)$ is upper semicontinuous. $\square$

Lemma 6. *There exists an equilibrium in the game of long-run commitment power.*

Proof. The platform's objective is

$$\sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma})} + \gamma_s}} \right). \tag{15}$$

To show it is upper semicontinuous, take $\boldsymbol{\gamma}^n \to \boldsymbol{\gamma}$. Then,

$$
\begin{aligned}
&\limsup_{n\to\infty} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma}^n)} + \gamma_s^n}} \right) \\
=& \lim_{k\to\infty} \sup_{n\geq k} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma}^n)} + \gamma_s^n}} \right) \\
\leq& \lim_{k\to\infty} \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \sup_{n\geq k} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma}^n)} + \gamma_s^n}} \right) \\
=& \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \lim_{k\to\infty} \sup_{n\geq k} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma}^n)} + \gamma_s^n}} \right) \\
=& \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\liminf_{n\to\infty} \frac{1}{\bar{a}_s(\boldsymbol{\gamma}^n)} + \gamma_s^n}} \right) \\
\leq& \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{\limsup_{n\to\infty} \bar{a}_s(\boldsymbol{\gamma}^n)} + \lim_{k\to\infty} \inf_{n\geq k} \gamma_s^n}} \right) \\
\leq& \sum_{t=1}^{\infty} \delta_P^{t-1} \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma})} + \gamma}} \right)
\end{aligned}
$$

The second equality comes from the dominated convergence theorem, and the last inequality uses the upper semicontinuity of $\bar{a}_s(\boldsymbol{\gamma})$. Thus, given the consumer's optimal behavior, the platform's objective is upper semicontinuous. Since $\boldsymbol{\Gamma}$ is compact, there is a privacy policy $\boldsymbol{\gamma}^*$ that maximizes the platform's objective. The strategy profile $(\boldsymbol{\gamma}^*, \bar{\boldsymbol{a}}(\cdot))$ is an equilibrium. $\square$

B Consumer Behavior Under a Stationary Privacy Policy: Proof of Proposition 1

This Appendix uses notations introduced at the beginning of Appendix A.

B.1 Properties of Consumer Value Function

First, I prove useful properties of the consumer's value function that hold for any privacy policy. Let $\bar{a}(\boldsymbol{\gamma})$ denote the greatest best response of the consumer constructed in Lemma 4. For each privacy policy $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, define

$$V_{\boldsymbol{\gamma}}(\rho) := \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(\bar{a}_t(\boldsymbol{\gamma})) - v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s(\boldsymbol{\gamma})} + \gamma_s}} \right) \right]. \tag{16}$$

$V_{\boldsymbol{\gamma}}(\rho)$ is the consumer's continuation value, starting from the posterior variance $\sigma^2 = \frac{1}{\rho}$.

Lemma 7. *For any $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}$, $V_{\boldsymbol{\gamma}}(\cdot) : \mathbb{R}_{++} \to \mathbb{R}$ is decreasing and convex. For any $\rho > 0$ and $\Delta > 0$, $\lim_{\rho \to \infty} V_{\boldsymbol{\gamma}}(\rho) - V_{\boldsymbol{\gamma}}(\rho + \Delta) = 0$.*

Proof. Fix any privacy policy $\boldsymbol{\gamma}$. Hereafter, I omit $\boldsymbol{\gamma}$ from subscripts (thus, the consumer value function is $V(\cdot)$). Consider the "T-period problem, " in which the consumer's payoff in any period $s \geq T+1$ is exogenously set to zero. For any $t \leq T$, let $V_t^T(\rho)$ denote the consumer's continuation value in the T-period problem starting from period t given $\frac{1}{\sigma_{t-1}^2} = \rho$:

$$V_t^T(\rho) = \max_{(a_t, \dots, a_T) \in A^{T-t+1}} \sum_{s=t}^{T} \delta_C^{s-t} \left(u(a_s) - v \left(\sigma_0^2 - \frac{1}{\rho_{s-1} + \frac{1}{\frac{1}{a_s} + \gamma_s}} \right) \right).$$

Here, $\rho_{t-1} = \rho$, and $(\rho_t, \dots, \rho_{T-1})$ are recursively defined by Bayes' rule given $(a_t, \dots, a_{T-1})$. The standard argument of dynamic programming implies that for each $t = 1, \dots, T$,

$$V_t^T(\rho) = \max_{a \in A} u(a) - v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma_t}} \right) + \delta_C V_{t+1}^T \left(\rho + \frac{1}{\frac{1}{a} + \gamma_t} \right), \tag{17}$$

where $V_{T+1}^T(\cdot) \equiv 0$. I use induction to show that $V_1^T(\rho)$ is decreasing and convex. First, $V_{T+1}^T \equiv 0$ is trivially decreasing and convex. Suppose V_{t+1}^T is decreasing and convex. Because $-v \cdot \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma_t}} \right)$ has the same property and the upper envelope of decreasing convex functions are decreasing and convex, so does $V_t^T(\cdot)$. This induction argument implies that for each T, $V^T(\rho) = V_1^T(\cdot)$ is decreasing and convex. Also, for any ρ and $\Delta > 0$, $\lim_{\rho \to \infty} V^T(\rho) - V^T(\rho + \Delta) \to 0$.

Define $V^\infty(\rho) := \lim_{T \to \infty} V^T(\rho)$. $V^\infty(\rho)$ is decreasing and convex, because these properties are preserved under pointwise convergence. I show that $V^\infty(\rho)$ is the value function of the original problem, i.e., $V^\infty(\cdot) = V(\cdot)$. Take any ρ, and let $(\bar{a}_1, \bar{a}_2, \dots) \in \mathcal{A}^*(\boldsymbol{\gamma})$ denote the optimal policy.

For any finite T,

$$V^{T}(\rho) \geq \sum_{s=1}^{T} \delta_C^{s-1} \left(u(\bar{a}_s) - v \left(\sigma_0^2 - \frac{1}{\rho_{s-1} + \frac{1}{\frac{1}{\bar{a}_s} + \gamma_s}} \right) \right). \tag{18}$$

By taking $T \to \infty$, we obtain $V^{\infty}(\rho) \geq V(\rho)$. Suppose to the contrary that $V^{\infty}(\rho) > V(\rho)$. Then, there is a sufficiently large $T \in \mathbb{N}$ such that $V^{T}(\rho) - \frac{\delta_C^T}{1-\delta_C} v\sigma_0^2 > V(\rho)$. If the consumer in the original infinite horizon problem adopts the T-optimal policy that gives $V^{T}(\rho)$ up to period t, then she can obtain a strictly greater payoff than $V(\rho)$, which is a contradiction. Thus, $V^{\infty}(\rho) = V(\rho)$.

Finally, I show that for any ρ and $\Delta > 0$, $\lim_{\rho \to \infty} V(\rho) - V(\rho+\Delta) \to 0$. Suppose the consumer starting from $\rho + \Delta$ chooses the policy $(\bar{a}_t^{\rho})_{t \in \mathbb{N}}$ that is optimal for ρ. Let $(\hat{\rho}_t)_{t=1}^{\infty}$ denote the induced sequence of the precisions after $\rho + \Delta$, i.e., $\hat{\rho}_t = \rho + \Delta + \sum_{s=1}^{t} \frac{1}{\frac{1}{\bar{a}_s^{\rho}} + \gamma_s}$. Note that $\hat{\rho}_t \geq \rho_t$ for all $t \in \mathbb{N}$. Then, it holds that $0 \leq V(\rho) - V(\rho+\Delta) \leq \sum_{t=1}^{\infty} \delta_C^{t-1} \left(\frac{1}{\rho} - \frac{1}{\rho+\Delta} \right) = \frac{1}{1-\delta_C} \left(\frac{1}{\rho} - \frac{1}{\rho+\Delta} \right)$. Thus, $\lim_{\rho \to \infty} V(\rho) - V(\rho+\Delta) = 0$. □

B.2 Proof of Proposition 1

Proof. If γ_t is constant across t, the consumer problem is a stationary dynamic programming. Suppose $\gamma_t = \gamma \in \mathbb{R}_+$ for all t. The value function $V(\cdot)$ satisfies the Bellman equation

$$V(\rho) = \max_{a \geq 0} u(a) - v \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma}} \right) + \delta_C V \left(\rho + \frac{1}{\frac{1}{a} + \gamma} \right). \tag{19}$$

Again, I suppress the dependence of $V(\cdot)$ on γ. Lemma 7 implies that $V(\cdot)$ is decreasing and convex. Thus, the maximand in (19) has the increasing differences in (a, ρ). Thus, $\bar{a}(v, \gamma, \rho)$, the greatest maximizer, is increasing in ρ. Note that $\rho_t \leq \rho_{t+1}$, and the inequality is strict if and only if $a_t > 0$. As a result, the consumer's optimal behavior is either (i) $a_t = 0$ for all t, or (ii) $a_1 > 0$ and a_t is increasing in t. Now, define

$$v^*(\gamma) := \sup \{ v \in \mathbb{R} : \bar{a}_1 (v, \gamma, \rho_0) > 0 \}, \quad \text{where} \quad \rho_0 = \frac{1}{\sigma_0^2}. \tag{20}$$

Note that $\bar{a}_1 (v, \gamma, \rho_0) > 0$ if and only if there exists some $\boldsymbol{a} \in \mathcal{A}$ that yields the consumer a positive ex ante payoff. Also, the consumer's payoff from any strategy is decreasing in v and increasing in

γ, whereas her payoff from $a_t = 0$ for all t is independent of (v, γ). As a result, if $\bar{a}_1(v, \gamma, \rho_0) > 0$, then $\bar{a}_1(v', \gamma', \rho_0) > 0$ for any $v' < v$ and $\gamma' > \gamma$. Therefore, the consumer's behavior follows (i) and (ii) above if $v > v^*(\gamma)$ and $v < v^*(\gamma)$, respectively, and $v^*(\gamma)$ is increasing in γ. For any given v, as $\gamma \to \infty$, the consumer's ex ante payoff from (say) $a_t = a_{max} > 0$ for all t becomes positive. Thus, $\lim_{\gamma\to\infty} v^*(\gamma) = \infty$.

If $v < v^*(\gamma)$, then $a_t \geq a_1 > 0$ for all t. Since $\gamma <_\infty$, we obtain $\lim_{t\to\infty} \sigma_t^2 \to 0$, or equivalently, $\lim_{t\to\infty} \rho_t = \infty$ with $\rho_t := \frac{1}{\sigma_t^2}$. By Lemma 7, for any $\rho > 0$ and $\Delta > 0$, $\lim_{\rho\to\infty} V(\rho) - V(\rho+\Delta) = 0$. This, combined with $\lim_{t\to\infty} \rho_t = \infty$, implies $\lim_{t\to\infty} \bar{a}_t(v, \gamma, \rho_t) = a_{max}$. Finally, $v > v^*(\gamma)$ implies $\bar{a}_1 = 0$. This implies $\bar{a}_t = 0$ for all $t \in \mathbb{N}$ because the conditional variance does not change. All of the above arguments hold for a myopic consumer as well, by setting $\delta_C = 0$ in (19). $\square$

C Equilibrium Under Long-Run Commitment: Proof of Theorem 1

C.1 Useful Lemmas

I begin with two useful lemmas. First, suppose the platform changes privacy levels in any period t in some set $\mathcal{T} \subset \mathbb{N}$. If the change affects the consumer behavior and increases the precisions of signals of all periods in $\mathcal{T}$, she chooses higher activity levels in all other periods. Recall that $\bar{a}(\boldsymbol{\gamma}) \in \mathcal{A}$ denote the greatest best response of the consumer constructed in Lemma 4.

Lemma 8. *Take any $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \boldsymbol{\Gamma}$. Define $\mathcal{T} = \{t \in \mathbb{N} : \gamma_t = \gamma_t'\}$. Suppose $\frac{1}{\bar{a}_t(\boldsymbol{\gamma})} + \gamma_t \leq \frac{1}{\bar{a}_t(\boldsymbol{\gamma}')} + \gamma_t'$ for all $t \in \mathbb{N} \setminus \mathcal{T}$. Then, $\bar{a}_t(\boldsymbol{\gamma}) \geq \bar{a}_t(\boldsymbol{\gamma}')$ for all $t \in \mathcal{T}$.*

Proof. Let $\boldsymbol{\beta}$ be any one of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$. I decompose the consumer's problem (10) into two steps. First, given any $(a_t)_{t\notin\mathcal{T}}$, the consumer chooses $(a_t)_{t\in\mathcal{T}}$ to maximize the following hypothetical objective function:

$$\sum_{t=1}^{\infty} \delta_C^{t-1} \left[\mathbf{1}_{\{t\in\mathcal{T}\}} u(a_t) - v \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \sum_{s=1}^{t} \frac{1}{\frac{1}{a_s} + \beta_s}} \right) \right]. \tag{21}$$

The consumer receives a benefit of $u(a_t)$ only in period $t \in \mathcal{T}$. This leads to a mapping that maps any $(a_t)_{t\notin\mathcal{T}}$ to the (greatest) optimal choice of $(a_t)_{t\in\mathcal{T}}$. In the second step, the consumer chooses

$(a_t)_{t\notin\mathcal{T}}$ to maximize her original objective, taking the mapping $(a_t)_{t\notin\mathcal{T}} \mapsto (a_t)_{t\in\mathcal{T}}$ as given.

For any $t \notin \mathcal{T}$, a_t affects (21) only through $\frac{1}{a_t}+\gamma_t$, because $\mathbf{1}_{\{t\in\mathcal{T}\}} = 0$. Also the same argument as in the proof of Lemma 3 implies that (21) is supermodular in $\left((a_t)_{t\in\mathcal{T}}, \left\{\left(\frac{1}{a_s}+\gamma_s\right)^{-1}\right\}_{s\notin\mathcal{T}}\right)$. This implies that if $\frac{1}{\bar{a}_t(\boldsymbol{\gamma})}+\gamma_t \leq \frac{1}{\bar{a}_t(\boldsymbol{\gamma}')}+\gamma_t'$ for all $t \in \mathbb{N}\setminus\mathcal{T}$, then $\bar{a}_t(\boldsymbol{\gamma}) \geq \bar{a}_t(\boldsymbol{\gamma}')$ for all $t \in \mathcal{T}$. □

Next, the platform can commit to a high privacy level to induce a_{max} in any period.

Lemma 9. *There is a $\gamma_{max} < +\infty$ such that if the platform commits to $\gamma_t = \gamma_{max}$, then regardless of the privacy levels in other periods, the consumer chooses $a_t = a_{max}$. Also, there is a $\bar{\sigma}^2$ such that if $\sigma^2_{T-1} \leq \bar{\sigma}^2$, then the consumer chooses $a_t = a_{max}$ for all $t \geq T$ for any $(\gamma_\tau)_{\tau\geq T}$.*

Proof. Let a' denote the second highest activity level in A. Take any $\gamma_{max} < +\infty$ that satisfies

$$u(a_{max}) - u(a') - \frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\frac{1}{a_{max}}+\gamma}}\right) > 0.$$

Suppose the platform commits to a privacy policy such that $\gamma_t = \gamma_{max}$. Take any $(a_t)_{t\in\mathbb{N}} \in \mathcal{A}$ such that $a_t < a_{max}$. Suppose the consumer changes her action in period t from a_t to a_{max}. This change increases her period-t benefit from $u(\cdot)$ by at least $u(a_{max}) - u(a') > 0$. The change also increases the sum of discounted privacy costs (from the perspective of period t) by

$$\begin{aligned}
&\sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2}+\frac{1}{\frac{1}{a_{max}}+\gamma_{max}}+\sum_{\tau=t+1}^{s}\frac{1}{\frac{1}{a_\tau}+\gamma_\tau}}\right) - \sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2}+\frac{1}{\frac{1}{a_t}+\gamma_{max}}+\sum_{\tau=t+1}^{s}\frac{1}{\frac{1}{a_\tau}+\gamma_\tau}}\right)\\
\leq&\sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2}+\frac{1}{\frac{1}{a_{max}}+\gamma_{max}}}\right) - \sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2}+\frac{1}{\frac{1}{a_t}+\gamma_{max}}}\right)\\
\leq&\sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\frac{1}{a_{max}}+\gamma_{max}}}\right) - \sum_{s=t}^{\infty}\delta^{s-t}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\frac{1}{a_t}+\gamma_{max}}}\right)\\
\leq&\frac{1}{1-\delta}\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\frac{1}{a_{max}}+\gamma_{max}}}\right) := D(\gamma_{max}).
\end{aligned}$$

For any given σ_0^2, $\lim_{\gamma_{max}\to\infty} D = 0$. Thus, for a sufficiently large γ_{max}, the consumer's incremental gain from changing a_t to a_{max} is greater than the loss. Also, even for $\gamma_{max} = 0$, $\lim_{\sigma_0^2\to 0} D = 0$. Thus, even if $\gamma_t = 0$, the consumer chooses $a_t = a_{max}$ for a sufficiently small σ_t^2. □

C.2 Proof of Theorem 1

Proof. First, I show $\lim_{t\to\infty}\sigma_t^2 = 0$. Let $\boldsymbol{\gamma}^*$ denote the equilibrium privacy policy, and let $\boldsymbol{a}^*$ denote the equilibrium activity levels. Suppose to the contrary that $\lim_{t\to\infty}\sigma_t^2 \neq 0$. Because σ_t^2 is decreasing, $\lim_{t\to\infty}\sigma_t^2 > 0$ exists. This implies $\frac{1}{a_t^*} + \gamma_t^* \to \infty$. I derive a contradiction.

Let $\gamma_{max} \in [0, +\infty)$ denote the privacy level defined in Lemma 9—i.e., the consumer chooses $a_t = a_{max}$ if $\gamma_t = \gamma_{max}$. If the platform commits to $\gamma_t = \gamma_{max}$, the precision of the signal in period t is $B := \frac{1}{a_{max}} + \gamma_{max}$. Take T^* such that for all $t \geq T^*$, $\frac{1}{a_t^*} + \gamma_t^* > B$. If the platform replaces γ_t^* with γ_{max} for all $t \geq T^*$ and commits to such a new policy ex ante, then the precision of the signal increases from $\frac{1}{\frac{1}{a_t^*}+\gamma_t^*}$ to B^{-1} in any period $t \geq T^*$. Lemma 8 implies that after the policy change, the consumer also chooses a weakly greater a_t for all $t < T^*$. To sum up, the platform can strictly increase its profit by replacing γ_t^* with γ_{max} for all $t \geq T^*$, which is a contradiction. The second part of Lemma 9 then implies that there is some T such that for all $t \geq T$, $a_t^* = a_{max}$.

Next, I write $\gamma_t^*(v)$ to clarify the dependence of the equilibrium privacy level on v. Suppose to the contrary that there is a T such that, for any $\underline{v}$, there is some $v \geq \underline{v}$ such that $\gamma_t^*(v) = 0$ for some $t \leq T$. Then we can find $v_n \to \infty$ and $t^* \leq T$ such that $\gamma_{t^*}^*(v_n) = 0$ for all n. However, for a sufficiently large v_n, $a_{t^*}^* = 0$ if $\gamma_{t^*}^*(v_n) = 0$. The reason is as follows. If the consumer changes her activity level from 0 to some $a > 0$, her gross payoff from $u(\cdot)$ increases by at at most $u(a_{max})$. In contrast, her privacy cost increases by at least

$$v\left(\frac{1}{\frac{1}{\sigma_0^2} + (t^*-1)a_{max}} - \frac{1}{\frac{1}{\sigma_0^2} + (t^*-1)a_{max} + a_{min}}\right) > 0,$$

where a_{min} is the smallest positive activity level in A. This expression is independent of the history and diverges to ∞ as $v \to \infty$. Thus for a large v, the consumer prefers $a = 0$. However, the platform can then commit to a high privacy level for period t^* to induce $a_{t^*} > 0$. By the same argument as the previous paragraph, this change also weakly increases the activity levels in all other periods. This is a contradiction.

Finally, we show that $\gamma_t^* \to 0$ under certain conditions. First, assume the consumer is myopic. Suppose to the contrary that $\gamma_t^* \not\to 0$. Then there is some $\varepsilon > 0$ such that $\gamma_t^* > \varepsilon$ for infinitely many t's. Take T such that $\frac{1}{\frac{1}{\sigma_0^2}+\frac{T-1}{B}} < \bar{\sigma}^2$ and $\gamma_T^* > \varepsilon$. Because the variance of the noise ε_t is at most B in each t, $\sigma^2_{T-1} < \bar{\sigma}^2$ in equilibrium. The second part of Lemma 9 implies that if the platform sets $\gamma_T = 0$, the consumer still chooses a_{max}, which strictly decreases σ^2_T. As a result, the change of γ_T increases the platform payoff in any period $t \geq T$. Also, this change of γ_T does not affect the consumer's choice in $t < T$ because she is myopic. Thus, the platform benefits from changing γ_T from γ^*_T to 0, which is a contradiction. If the consumer has a binary activity level, the result follows from Theorem 2. $\square$

D The Platform-Best Equilibrium: Proof of Theorem 2

Proof. **Step 1: Construction of MPE.** We write the consumer's discount factor δ_C as δ, and use a precision $\rho_t = \frac{1}{\sigma^2_t}$ as a state variable of MPE. Along any path of play, ρ_t is non-decreasing in t. Let $\gamma^*(\rho)$ denote the platform's choice of γ_t given $\rho_{t-1} = \rho$, and let $a^*(\rho, \gamma)$ denote the consumer's choice of a_t given $(\rho_{t-1}, \gamma_t) = (\rho, \gamma)$. Also, let $V_0(\rho)$ denote the consumer's continuation value when the initial state is ρ and $(\gamma_t, a_t) = (0, a_{max})$ for all $t \in \mathbb{N}$:

$$V_0(\rho) = \sum_{t=1}^{\infty} \delta_C^{t-1} \left[u(a_{max}) - v \cdot \left(\sigma^2_0 - \frac{1}{\rho + t a_{max}} \right) \right]. \tag{22}$$

$V_0(\cdot) : \left[\frac{1}{\sigma^2_0}, \rho\right) \to \mathbb{R}$ is continuous, strictly decreasing, and strictly convex in $\rho \geq 0$.

First, we show that there is a $\rho(0) \in \mathbb{R}_{++}$ such that any strategy that satisfies $\gamma^*(\rho) = 0$ and $a^*(\rho, 0) = a_{max}$ for any $\rho \geq \rho(0)$ is an MPE in the game that starts from any $\rho_0 \geq \rho(0)$. Given such $a^*(\cdot, \cdot)$ and the initial state $\rho \geq \rho(0)$, the platform prefers $\gamma = 0$, because the subsequent outcome is $(\gamma_t, a_t) = (0, a_{max})$ for all $t \in \mathbb{N}$, which maximizes the platform's payoff. Given the platform's strategy and the initial state $\rho \geq \rho(0)$, the consumer does not strictly benefit from a one-shot deviation (to $a = 0$) if and only if

$$\begin{aligned} & u(a_{max}) - v \left(\sigma^2_0 - \frac{1}{\rho + a_{max}} \right) + \delta V_0 \left(\rho + a_{max} \right) \geq -v \left(\sigma^2_0 - \frac{1}{\rho} \right) + \delta V_0(\rho) \\ \iff & u(a_{max}) + \frac{v}{\rho + a_{max}} - \frac{v}{\rho} + \delta \left[V_0 \left(\rho + a_{max} \right) - V_0(\rho) \right] \geq 0. \end{aligned}$$

Both $\frac{v}{\rho + \bar{a}} - \frac{v}{\rho}$ and $V_0 \left(\rho + \bar{a} \right) - V_0(\rho)$ are continuous and strictly increasing in ρ, and converge to 0 as $\rho \to \infty$. Since $u(a_{max}) > 0$, there is a unique $\rho(0) < \infty$ such that the inequality holds if and only if $\rho \geq \rho(0)$.

We have constructed an MPE with $\gamma_t^* = 0$ and $a_t^* = a_{max}$ for any initial state $\rho \geq \rho(0)$. Next, we construct an MPE for any initial state $\rho \in [\rho(1), \rho(0)]$, where $\rho(1) < \rho(0)$. Define

$$V(\rho, \gamma) := u(a_{max}) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}\right)$$

$V(\rho, \gamma)$ is the consumer's continuation value when (i) the initial state is ρ, (ii) the platform sets γ and the consumer chooses a_{max} in the first period, and (iii) from the next period on, they always choose $(\gamma_t, a_t) = (0, a_{max})$. Consider the inequality

$$V(\rho, \gamma) \geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta \cdot V(\rho, \gamma),$$

which is written as

$$\begin{aligned} & u(a_{max}) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}\right) \\ \geq & - v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta \cdot \left[u(a_{max}) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}\right)\right], \end{aligned} \tag{23}$$

or equivalently,

$$\begin{aligned} & (1-\delta)u(a_{max}) + \frac{v}{\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}} - \frac{v}{\rho} \\ & + \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}\right) + \delta\left[v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}}\right) - \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}} + \gamma}\right)\right] \geq 0. \end{aligned} \tag{24}$$

We show several properties of the left-hand side of (24). First, both sides of (23) are continuous in γ, and the left-hand side increases more than the right-hand side if γ increases (because of discounting). Thus, the left-hand side of (24) is continuous and strictly increasing in γ. It is also continuous and strictly increasing in ρ. In particular,

$$\begin{aligned}
&V_0\left(\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}\right)+\left[v\left(\sigma_0^2-\frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}}\right)-\delta V_0\left(\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}\right)\right]\\
=&K+v\sum_{t=1}^{\infty}\delta^{t-1}\frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}+ta_{max}}-v\sum_{t=1}^{\infty}\delta^{t-1}\frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}+(t-1)a_{max}}\\
=&K+v\sum_{t=1}^{\infty}\delta^{t-1}\left[\frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}+ta_{max}}-\frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}+(t-1)a_{max}}\right]
\end{aligned}$$

is strictly increasing in ρ, where K is a term that does not depend on ρ.

Because (23) holds with equality at $(\rho,\gamma)=(\rho(0),\gamma)$, it holds with strictly inequality at $\rho=\rho(0)$ for any $\gamma>0$. Then for any ρ that is smaller than but sufficiently close to $\rho(0)$, we can find a unique $\gamma(\rho)>0$ that satisfies (23) with equality. The left-hand side of (24) is increasing in ρ and γ. Thus, if $\gamma(\rho)$ exists for some $\rho<\rho(0)$, $\gamma(\rho')$ exists for any $\rho'\in[\rho,\rho(0))$. If ρ is such that no γ satisfies (23), then define $\gamma(\rho)=\infty$. Then $\gamma(\rho)$ is decreasing in $\rho\le\rho(0)$.

Note that $\rho+\frac{1}{\frac{1}{a_{max}}+\gamma(\rho)}$ is strictly increasing in ρ. Thus for a ρ smaller than but close to $\rho(0)$, we obtain $\rho+\frac{1}{\frac{1}{a_{max}}+\gamma(\rho)}\ge\rho(0)$. As a result, $\rho(1)=\min\left\{\rho\in[\frac{1}{\sigma_0^2},\infty):\rho+\frac{1}{\frac{1}{a_{max}}+\gamma(\rho)}\ge\rho(0)\right\}$ is well-defined. If $\rho(1)>\frac{1}{\sigma_0^2}$, we have $\rho(1)+\frac{1}{\frac{1}{a_{max}}+\gamma(\rho(1))}=\rho(0)$.

We now construct an MPE starting from any $\rho_0\in[\rho(1),\rho(0)]$. Consider the following strategy profile: For any $\rho\in[\rho(1),\rho(0)]$, the platform sets $\gamma(\rho)$ that solves (23) with equality. The consumer chooses a_{max} if $\gamma\ge\gamma(\rho)$ and $\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}\ge\rho(0)$. If $\gamma<\gamma(\rho)$, she chooses $a=0$. If $\rho+\frac{1}{\frac{1}{a_{max}}+\gamma}<\rho(0)$, she chooses some optimal activity level, taking the continuation value $V(\rho,\gamma(\rho))$ as given. Once the state reaches $\rho\ge\rho(0)$, the MPE for $\rho\ge\rho(0)$ is played—i.e., the platform sets $\gamma=0$ and the consumer chooses a_{max} after any history. This strategy profile is an MPE: First, by construction, the consumer has no profitable one-shot deviation after any history. Second, the platform does not benefit from any one-shot deviation: If it increases γ, the deviation decreases the precision in the current and any future periods compared to without deviation. If it decreases γ, the consumer chooses $a=0$ and the deviation decreases the precision in the current and any future periods, compared to without deviation.

For $\rho\in[\rho(1),\rho(0)]$, the consumer's continuation value in the above MPE is $V(\rho):=-\frac{v}{1-\delta}\left(\sigma_0^2-\frac{1}{\rho}\right)$.

We now construct an MPE starting from any $\rho \in [\rho(2), \rho(1)]$, where $\rho(2) < \rho(1)$. Define

$$V_2(\rho, \gamma) := u(a_{max}) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}}+\gamma}}\right) - \frac{\delta}{1-\delta} \cdot v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}}+\gamma}}\right).$$

Consider the inequality

$$V_2(\rho, \gamma) \geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta V_2(\rho, \gamma), \tag{25}$$

or equivalently,

$$u(a_{max}) - \frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}}+\gamma}}\right) \geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta\left[u(a_{max}) - \frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a_{max}}+\gamma}}\right)\right]$$
$$\iff (1-\delta)u(a_{max}) + \frac{v}{\rho + \frac{1}{\frac{1}{a_{max}}+\gamma}} - \frac{v}{\rho} \geq 0.$$

The left-hand side is continuous and strictly increasing in γ, and it is positive for $\gamma = \infty$. It is also continuous and strictly increasing in ρ. As a result, for each $\rho < \rho(1)$, we can find a unique $\gamma(\rho) > 0$ such that (25) holds with equality. By construction, $\gamma(\rho)$ is decreasing. Define $\rho(2) = \min\left\{\rho \in [\frac{1}{\sigma_0^2}, \infty) : \rho + \frac{1}{\frac{1}{a_{max}}+\gamma(\rho)} \geq \rho(1)\right\}$. If $\rho(2) > 0$, then $\rho(2) + \frac{1}{\frac{1}{a_{max}}+\gamma(\rho(2))} = \rho(1)$.

We can then construct a Markov perfect equilibrium for any initial state in $[\rho(2), \rho(1)]$. For any $\rho \in [\rho(2), \rho(1)]$, the platform sets $\gamma(\rho)$ that solves (25) with equality. The consumer chooses a_{max} if $\gamma \geq \gamma(\rho)$ and $\rho + \frac{1}{\frac{1}{a_{max}}+\gamma} \geq \rho(1)$. If $\gamma < \gamma(\rho)$, she chooses $a = 0$. If $\rho + \frac{1}{\frac{1}{a_{max}}+\gamma} < \rho(1)$, she chooses some optimal activity level, taking the continuation value $V_2(\rho, \gamma(\rho))$ as given. Once the state reaches $\rho \geq \rho(1)$, the MPE for $\rho \geq \rho(1)$ is played. We can show that this is an MPE by the same argument as the case of $\rho \in [\rho(1), \rho(0)]$.

Given the initial state $\rho \in [\rho(2), \rho(1)]$, the consumer's continuation value is $V(\rho) = -\frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho}\right)$, which is the same as before. As a result, we can use the incentive constraint (25) to recursively construct a sequence $\rho(3), \rho(4), \ldots$ and an MPE for any k and $\rho \in [\rho(k), \rho(k-1)]$. The smallest ρ we consider is $\rho_0 = \frac{1}{\sigma_0^2}$. Thus, $\frac{1}{\frac{1}{a_{max}}+\gamma(\rho)} \geq \frac{1}{\frac{1}{a_{max}}+\gamma(\rho_0)}$ for any $\rho \geq \rho_0$. As a result, $\rho(k) - \rho(k+1) \geq \frac{1}{\frac{1}{a_{max}}+\gamma(\rho_0)} > 0$ for any $\rho \geq \rho_0$, whenever $\rho(k) > \rho_0$. Thus there is a smallest finite $K^* \in \mathbb{N}$ such that $\rho(K^*) < \rho_0$. Redefine $\rho(K^*)$ as ρ_0. We now have an MPE starting from any $\rho \geq \rho_0$.

Step 2: Platform-best and consumer-worst. The above equilibrium is consumer-worst: If $\rho_0 \geq \rho(0)$, the platform sets $\gamma_t = 0$ for all t, which is the worst choice for the consumer. If $\rho_0 < \rho(0)$, the consumer is indifferent between following the equilibrium strategy and choosing $a_t = 0$ for all t. In either case, there is no equilibrium that gives the consumer a strictly lower payoff. To show the equilibrium is platform-best, let Π denote its ex ante payoff. If there is another equilibrium in which the platform obtains $\Pi' > \Pi$, the consumer's payoff is at most $\frac{u(a_{max})}{1-\delta} - v\Pi' < \frac{u(a_{max})}{1-\delta} - v\Pi = 0$. This contradicts that the consumer's ex ante payoff in any equilibrium is nonnegative.

Step 3: Other properties of the equilibrium. We show that $\gamma(\rho)$ is decreasing in ρ. First, $\gamma(\rho)$ is decreasing on $\rho \leq \rho(1)$, because $\gamma(\rho)$ is determined by the binding (25). Second, $\gamma(\rho)$ is decreasing on $[\rho(1), \rho(0)]$, because it is determined by the binding (23). Third, $\gamma(\rho) = 0$ for all $\rho \geq \rho(0)$. These observations, combined with the continuity of $\gamma(\rho)$, imply $\gamma(\rho)$ is decreasing. From period t to $t+1$, the state increases by $\rho_{t+1} - \rho_t = \frac{1}{\frac{1}{a_{max}} + \gamma(\rho_t)} \geq \frac{1}{\frac{1}{a_{max}} + \gamma(\rho_0)}$. Thus ρ_t is strictly increasing in t and diverges to $+\infty$ (or equivalently, $\sigma_t^2 \to 0$ in equilibrium). As a result, γ_t is strictly decreasing in equilibrium and hits zero in period T, which is the smallest T with $\rho_T \geq \rho(0)$. We now have Points 1 and 2. Also, I constructed the above MPE so that for any σ_t^2, the platform chooses the lowest γ_t that induces a_{max}. Such behavior is equivalent to a greedy policy. Thus, Point 3 holds.

Next, we show that the platform with long-run commitment achieves the same outcome as the above MPE. Let $(\gamma_t^*)_{t \in \mathbb{N}}$ denote the (on-path) equilibrium privacy levels in the above MPE. If the platform commits to $(\gamma_t^*)_{t \in \mathbb{N}}$ ex ante, the consumer chooses a_{max} in all periods. To see this, consider the consumer's (single-agent) decision problem facing $(\gamma_t^*)_{t \in \mathbb{N}}$ under long-run commitment. Suppose the consumer deviates in period t and chooses $a_t = 0$. In the game of one-period commitment, the consumer faces γ_s^* in period $s+1$ for each $s \geq t$. In the game of long-run commitment, the consumer faces γ_{s+1}^* in period $s \geq t$. Thus, the consumer's payoff from any one-shot deviation is lower under long-run commitment than under one-period commitment. As a result, the consumer prefers $a_t = a_{max}$ for all t under long-run commitment.

Also, the platform with long-run commitment cannot be better off by committing to a different privacy policy. Indeed, if the platform has short-run commitment, the consumer obtains a payoff of zero and chooses a_{max} in all periods. If the platform is strictly better off by pre-committing to a different policy, then the consumer has to be strictly worse off, which is a contradiction because the consumer cannot have a negative payoff. This completes Point 4. $\square$

E The Consumer-Best Equilibrium: Proof of Theorem 3

Proof. Let $\rho_0 = \frac{1}{\sigma_0^2}$. Define the strategy profile as follows: Let $\gamma(\rho_0) = \infty$, and for any $\rho > \rho_0$, let $\gamma(\rho) = \gamma^*(\rho)$, which is the platform's strategy in the MPE in Theorem 2. Let $a(\rho_0, \infty) = a_{max}$, and $a(\rho_0, \gamma) = 0$ for any $\gamma < \infty$. For any $\rho > \rho_0$, let $a(\rho, \gamma)$ be the consumer's strategy in the MPE in Theorem 2.

Given the above strategy profile, suppose the platform deviates and offers $\gamma < +\infty$ at $\rho = \rho_0$. If the consumer chooses $a = 0$, her future continuation value is $\frac{1}{1-\delta}u(a_{max})$. Thus, if she chooses $a = a_{max}$, her future continuation value weakly decreases. As a result, a necessary condition for the consumer to choose a_{max} following the platform's deviation is that she obtains a nonnegative payoff in the current period:

$$u(a_{max}) - v\left(\frac{1}{\rho_0} - \frac{1}{\rho_0 + \frac{1}{\frac{1}{a_{max}}+\gamma}}\right) = u(a_{max}) - v\frac{\frac{1}{\frac{1}{a_{max}}+\gamma}}{\rho_0\left(\rho_0 + \frac{1}{\frac{1}{a_{max}}+\gamma}\right)} \geq 0. \tag{26}$$

Let $\hat{\gamma}(\rho_0)$ denote the minimum γ that satisfies this constraint. $\hat{\gamma}(\rho_0)$ is decreasing in ρ_0, positive for a small ρ_0, and $\lim_{\rho_0\to 0}\hat{\gamma}(\rho_0) = \infty$.

Take any $\bar{\rho}$ such that $\bar{\rho} + \frac{1}{\frac{1}{a_{max}}+\hat{\gamma}(\bar{\rho})} \leq \rho(0)$, where $\rho(0)$ is the cutoff constructed for Theorem 2, above which $(\gamma, a) = (0, a_{max})$ is chosen. For any initial state $\rho_0 \leq \bar{\rho}$, the above strategy profile is an equilibrium. First, it is an equilibrium at any (off-path) state $\rho > \rho_0$ by construction. At $\rho = \rho_0$, the consumer has no profitable deviation when the platform offers $\gamma = \infty$, because she can receive the best payoff of $u(a_{max})$ in the current and any future periods. Suppose that the platform deviates and chooses $\gamma_t < \infty$. Suppose to the contrary that the consumer chooses $a = a_{max}$. Then, $\rho_0 + \frac{1}{\frac{1}{a_{max}}+\gamma} \leq \rho(0)$ must hold. Thus her payoff in period t is at most $u(a_{max})$, whereas her continuation value from period $t+1$ is nonpositive (recall that in the platform-best equilibrium, the consumer's continuation payoff starting from $\rho \leq \rho(0)$ is nonpositive). In contrast, if the consumer chooses $a_t = 0$ and follows her strategy thereafter, her payoff is $\frac{\delta}{1-\delta}u(a_{max})$, because she sets $a_t = 0$ in period t and the state remains $(\rho_0, 0)$. Thus, the consumer has a profitable deviation if $\frac{\delta}{1-\delta}u(a_{max}) < u(a_{max})$, which contradicts $\delta \geq 1/2$. $\square$

F A MPE for a General A: Proof of Proposition 4

For simplicity we write δ_C as δ.

Lemma 10. *If the platform sets $\bar{\gamma}$ in Assumption 2 in period t, the consumer strictly prefers (i) $a_t = a_{max}$ and $\forall s \geq t+1, a_s = 0$ to (ii) $a_s = 0, \forall s \geq t$, regardless of the platform's continuation strategy.*

Proof. Define $\rho_t = \frac{1}{\sigma_t^2}$. The consumer prefers (i) to (ii) if and only if

$$u(a_{max}) - \frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho_t + \frac{1}{\frac{1}{a_{max}}+\bar{\gamma}}}\right) \geq -\frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho_t}\right)$$

$$\iff u(a_{max}) - \frac{v}{1-\delta}\left[\frac{1}{\rho_t\left(\rho_t\left(\frac{1}{a_{max}}+\bar{\gamma}\right)+1\right)}\right] \geq 0. \tag{27}$$

The left-hand side of the last inequality is at least

$$H := u(a_{max}) - \frac{v}{1-\delta}\left[\frac{1}{\rho_0\left(\rho_0\left(\frac{1}{a_{max}}+\bar{\gamma}\right)+1\right)}\right].$$

The inequality $H > 0$ is equivalent to the one for $\bar{\gamma}$ in Assumption 2. □

Proof of Proposition 4. Let a^+ denote the smallest positive activity level in A, and let γ^+ denote the highest finite privacy level in Γ. Define $\Delta^* := \frac{1}{\frac{1}{a_+}+\gamma+}$. Proposition 3 implies that there is $\rho(0)$ such that if the initial state is above $\rho(0)$, then $(\gamma_t, a_t) = (0, a_{max})$ for all $t \in \mathbb{N}$ is an MPE. Let $V_0(\rho) : [\rho(0), \infty) \to \mathbb{R}$ and $\Pi_0(\rho) : [\rho(0), \infty) \to \mathbb{R}$ respectively denote the consumer's and the platform's continuation values in such an MPE. $\Pi_0(\cdot)$ is increasing. Also, define $\rho(1) = \rho(0) - \Delta^*$. Finally, let $A_+ = A \setminus \{0\}$ denote the set of all positive activity levels. For any $\rho \in [\rho(1), \rho(0)]$, consider the optimization problem

$$\Pi_1(\rho) = \max_{\gamma\in\Gamma, a(\rho,\gamma)\in A} \sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}} + \Pi_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}\right) \tag{28}$$

$$s.t.\quad a(\rho,\gamma) \in \arg\max_{a\in A_+} u(a) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a}+\gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a}+\gamma}\right), \quad \text{and} \tag{29}$$

$$u(a(\rho,\gamma)) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}\right) \tag{30}$$

$$\geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta\cdot\left[u(a(\rho,\gamma)) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}\right)\right]. \tag{31}$$

Let $\bar{\gamma}$ denote the privacy level in Lemma 10. First, we show that there is $(\gamma, a(\rho,\gamma)) = (\gamma^*, a^*)$ that satisfies the constraints. Take $\gamma^* = \bar{\gamma}$, and let $a(\rho,\gamma^*) = a^*$ denote the solution of (29). Suppose, to the contrary, that (31) fails, i.e., we obtain

$$\begin{aligned} &u(a^*) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a^*}+\gamma^*}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a^*}+\gamma^*}\right) \\ &< -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta\left[u(a^*) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a^*}+\gamma^*}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}\right)\right]. \end{aligned}$$

This inequality implies

$$\begin{aligned} -\frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho}\right) &> u(a^*) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a^*}+\gamma^*}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a^*}+\gamma^*}\right) \\ &\geq u(a_{max}) - v\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma^*}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a_{max}}+\gamma^*}\right) \\ &\geq u(a_{max}) - \frac{v}{1-\delta}\left(\sigma_0^2 - \frac{1}{\rho+\frac{1}{\frac{1}{a_{max}}+\gamma^*}}\right), \end{aligned}$$

which contradicts the definition of $\bar{\gamma}$ in Lemma 10. Let $(\gamma(\rho), a(\rho,\gamma(\rho)))$ denote the solution of the above problem. Let $V_1(\cdot) : [\rho(1),\infty) \to \mathbb{R}$ denote the extension of $V_0(\cdot)$ such that for all

$\rho \in [\rho(1), \rho(0)]$,

$$V_1(\rho) = u(a(\rho, \gamma(\rho)) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma(\rho))}+\gamma(\rho)}}\right) + \delta V_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma(\rho))} + \gamma(\rho)}\right). \tag{32}$$

Let $\Pi_1(\cdot) : [\rho(1), \infty) \to \mathbb{R}$ denote the extension of $\Pi_0(\cdot)$ such that for all $\rho \in [\rho(1), \rho(0)]$,

$$\Pi_1(\rho) = \sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma(\rho))}+\gamma(\rho)}} + \delta \Pi_0\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma(\rho))} + \gamma(\rho)}\right). \tag{33}$$

Now, suppose that we have constructed $(\Pi_{n-1}(\cdot), V_{n-1}(\cdot))$ defined on $[\rho(n-1), \infty)$. Then, consider the problem

$$\Pi_n(\rho) = \max_{\gamma\in\Gamma, a(\rho,\gamma)\in A} \sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}} + \Pi_{n-1}\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)} + \gamma}\right) \tag{34}$$

$$s.t. \quad a(\rho,\gamma) \in \arg\max_{a\in A_+} u(a) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a}+\gamma}}\right) + \delta V_{n-1}\left(\rho + \frac{1}{\frac{1}{a} + \gamma}\right), \quad \text{and} \tag{35}$$

$$u(a(\rho,\gamma)) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}}\right) + \delta V_{n-1}\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)} + \gamma}\right) \tag{36}$$

$$\geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta \cdot \left[u(a(\rho,\gamma)) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}}\right) + \delta V_{n-1}\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)} + \gamma}\right)\right]. \tag{37}$$

By the same argument for $n = 1$, we can find a solution $(\gamma(\rho), a(\rho, \gamma(\rho)))$ for each $\rho \in [\rho(n), \rho(n-1)]$, where $\rho(n) = \rho(n-1) - \Delta^*$. Let n denote the smallest n such that $\rho(n) \leq \frac{1}{\sigma_0^2}$. We now have a functions $\gamma(\rho)$ and $a(\rho,\gamma)$ defined for any $\rho \geq \frac{1}{\sigma_0^2}$.

We use $(\gamma(\rho), a(\rho,\gamma))$ to construct an MPE, $(\gamma^*(\rho), a^*(\rho,\gamma))$. First, set $\gamma^*(\cdot) \equiv \gamma(\cdot)$. Second, to define $a^*(\rho,\gamma)$, take any $\rho \geq \rho_0$ such that $\rho \in [\rho(n), \rho(n-1)]$ and $\gamma < \infty$. Let $a^*(\rho,\gamma) = a(\rho,\gamma)$ if

$$u(a(\rho,\gamma)) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a(\rho,\gamma)}+\gamma}}\right) + \delta V_{n-1}\left(\rho + \frac{1}{\frac{1}{a(\rho,\gamma)} + \gamma}\right) \geq -v\left(\sigma_0^2 - \frac{1}{\rho}\right) + \delta V_n(\rho).$$

If this inequality fails, then $a^*(\rho, \gamma) = 0$. If $\gamma = \infty$, then $a^*(\rho, \gamma) = a_{max}$.

We show that $(\gamma^*(\rho), a^*(\rho, \gamma))$ is an MPE by showing that there is no profitable one-shot deviation. The optimality of $a^*(\rho, \gamma)$ holds by construction. The optimality of $\gamma^*(\rho)$ holds for the following reason. First, it is not optimal for the platform to set γ such that $a(\rho, \gamma) = 0$. Thus, facing $a^*(\rho, \gamma)$, any optimal strategy of the platform induces a positive activity level, i.e., it chooses γ such that $a(\rho, \gamma) \in A_+$. By (35), among such privacy levels, $\gamma(\rho)$ is optimal by (35). Finally, in each period, ρ_t increases by at least $\Delta^* > 0$ defined at the beginning. Once ρ_T exceeds $\rho(0)$, we have $(\gamma_t^*, a_t^*) = (0, a_{max})$ for all $t \geq T$. $\square$

G Equilibrium Under Competition: Proof of Proposition 5

Proof. First, we construct an equilibrium that satisfies Point 1. Suppose that, at the beginning of period $t \geq t^*$, the conditional variance for platform k is $\sigma_{t-1,k}^2$. Let γ_t^k denote the privacy level of platform k in period t. The consumer weakly prefers to use platform k (i.e. $a_t^{-k} = 0$ maximizes her period-t payoff) if

$$\begin{aligned}
&\arg\max_{a \in A} u(a) - v[\sigma_0^2 - \sigma_{t,k}^2(\gamma_t^k, a | \sigma_{t-1,k}^2)] - v[\sigma_0^2 - \sigma_{t-1,-k}^2] \\
\geq &\arg\max_{a \in A} u(a) - v[\sigma_0^2 - \sigma_{t,-k}^2(\gamma_t^{-k}, a | \sigma_{t-1,-k}^2)] - v[\sigma_0^2 - \sigma_{t-1,k}^2],
\end{aligned}$$

where $\sigma_{t,k}^2(\gamma, a | \sigma_{t-1,k}^2)$ is the posterior variance at the end of period t when platform k chooses γ, the consumer chooses a, and the posterior variance from the previous period is $\sigma_{t-1,k}^2$. Arranging this inequality, we obtain

$$\arg\max_{a \in A} u(a) - v[\sigma_{t-1,k}^2 - \sigma_{t,k}^2(\gamma_t^k, a | \sigma_{t-1,k}^2)] \geq \arg\max_{a \in A} u(a) - v[\sigma_{t-1,-k}^2 - \sigma_{t,-k}^2(\gamma_t^{-k}, a | \sigma_{t-1,-k}^2)].$$

This inequality implies that the consumer prefers to use k if and only if the gross benefit from the service minus the incremental privacy cost is greater for k than $-k$.

First, I consider competition with one-period commitment. Consider the following strategy profile. For each period $t < t^*$, I chooses a monopoly privacy level γ_t^*. Take any period $t \geq t^*$. Let $k^* \in \arg\min_{k=I,E} \sigma_{t-1,k}^2$ denote the platform that has the lower posterior variance (if k^* is not unique, we set $k^* = I$). Then platform $-k^*$ chooses the highest privacy level $\bar{\gamma}$. Platform k^*

chooses a privacy level $\gamma_t^{k^*}$ that solves

$$
\begin{aligned}
&\min_{\gamma\in[0,\bar{\gamma}]} \frac{1}{a^*(\gamma, \sigma^2_{t-1,k^*})} + \gamma \\
\text{s.t.} \quad & \arg\max_{a\in A} u(a) - v[\sigma^2_{t-1,k^*} - \sigma^2_{t,k^*}(\gamma, a|\sigma^2_{t-1,k^*})] \\
&\geq \arg\max_{a\in A} u(a) - v[\sigma^2_{t-1,-k^*} - \sigma^2_{t,-k^*}(\bar{\gamma}, a|\sigma^2_{t-1,-k^*})].
\end{aligned} \tag{38}
$$

In each period, the consumer myopically chooses a_t^I (if $t < t^*$) or (a_t^I, a_t^E) (if $t \geq t^*$) to maximize her per-period payoff. If indifferent, she uses the platform for which she chose a positive activity level in the most recent period. (If she chose zero activity levels up to period $t-1$, then she sets $a_t^k = 0$ for one of $k \in \{I, E\}$ with equal probability, and chooses a_t^{-k} to maximize her period-t payoff.)

I show the above strategy profile is an equilibrium. First, the consumer's behavior is optimal by construction. Second, I verify that platforms have no profitable deviation. Without loss of generality, consider a node in period t in which $I = k^*$ and $E = -k^*$. The strategy of E is optimal: Suppose the consumer uses I in period t (i.e. $\sigma^2_{t-1,I} \leq \sigma^2_{t-1,E}$). By construction, even if E chooses $\bar{\gamma}$ in all periods $s \geq t$, the consumer uses I in any future periods as long as I and the consumer follow the above strategy. Thus, E's payoff does not change if E lowers privacy levels. Thus, E has no profitable deviation.

Suppose now that I chooses a privacy level such that the consumer chooses E in period t. If $\sigma^2_{t,E} \leq \sigma^2_{t,I}$, then the consumer uses E in any period $s \geq t+1$. In this case, I's deviation is not profitable. Otherwise, $\sigma^2_{t,E} > \sigma^2_{t,I}$ hold. Note that I obtains a lower payoff in period t, because it is not maximizing the informativeness of the signal. Moreover, at any future period s, I faces an optimization problem

$$
\begin{aligned}
&\min_{\gamma} \frac{1}{a^*(\gamma, \sigma^2_{s-1,I})} + \gamma \\
\text{s.t.} \quad & \arg\max_{a\in A} u(a) - v[\sigma^2_{s-1,I} - \sigma^2_{s,I}(\gamma, a|\sigma^2_{s-1,I})] \\
&\geq \arg\max_{a\in A} u(a) - v[\sigma^2_{s-1,E} - \sigma^2_{s,E}(\bar{\gamma}, a|\sigma^2_{s-1,E})].
\end{aligned} \tag{39}
$$

After deviation, I faces a strictly lower $\sigma^2_{s-1,E} - \sigma^2_{s,E}(\bar{\gamma}, a|\sigma^2_{s-1,E}) > 0$ because the consumer generated information on E in period t. This means the set of γ satisfying the constraint shrinks. Thus, the minimized value in (39) becomes greater for any period $s \geq t+1$ after deviation. This implies that I's payoff is weakly lower for any period $s \geq t$ after the deviation. A similar argument implies that it is not profitable for I to deviate from a monopoly strategy before entry, because the deviation lowers I's payoff before and after entry. In particular, the deviation shrinks the set of γ's satisfying the constraint in (39) by increasing $\sigma^2_{s-1,I} - \sigma^2_{s,I}(\gamma, a|\sigma^2_{s-1,I})$.

On the equilibrium path, $a^E_t = 0$ for all $t \in \mathbb{N}$. $\lim_{t\to\infty} \sigma^2_{I,t} = 0$ holds because it holds even if I adopts $\gamma_t = \bar{\gamma}$ for all t, and I chooses each γ^I_t to achieve even lower posterior variances. Given this result, $\lim_{t\to\infty} a^I_t = a_{max}$ follows the same proof as monopoly.

Suppose γ^I_t does not converge to 0. Then, there is a convergent subsequence $\gamma^I_{t(n)}$ such that $\lim_{n\to\infty} \gamma^I_{t(n)} = \gamma' > 0$. For a sufficiently large n, both $\gamma = 0$ and $\gamma = \gamma^I_{t(n)}$ satisfy the constraint in (39), because $\sigma^2_{s-1,E} - \sigma^2_{s,E}(\bar{\gamma}, a|\sigma^2_{s-1,E}) = \sigma^2_0 - \sigma^2_{1,E}(\bar{\gamma}, a^*(\bar{\gamma}, \sigma^2_0)|\sigma^2_0) > 0$, but $\lim_{s\to\infty} \sigma^2_{s-1,I} - \sigma^2_{s,I}(0, a^*(0, \sigma^2_{s-1,I})|\sigma^2_{s-1,I}) \leq \lim_{s\to\infty} \sigma^2_{s-1,I} = 0$. As $n \to \infty$, the value of the objective converges to $\frac{1}{a_{max}}$ and $\frac{1}{a_{max}} + \gamma'$ for $\gamma = 0$ and $\gamma = \gamma'$, respectively. Thus, for a large n, $\gamma = 0$ achieves a strictly lower value in (39) than $\gamma = \gamma'$. This is a contradiction and thus $\lim_{t\to\infty} \gamma^I_t \to 0$ in the equilibrium.

Next, show Point 2. For a sufficiently large t^*, $\sigma^2_{t^*-1,I} \leq \sigma^2_0 - \sigma^2_{t^*,E}(\bar{\gamma}, a^*(\sigma^2_0, \bar{\gamma})|\sigma^2_0)$. Then, for any period $t \geq t^*$, the constraint (39) holds for any $\gamma \leq \bar{\gamma}$. This implies that in any equilibrium, I's problem is equal to the monopolist's problem, which proves Point 2.

A similar proof applies to competition with long-run commitment. In this game, I commits to $(\gamma^I_1, \gamma^I_2, \dots)$ before $t = 1$, then the consumer (myopically) chooses a^I_t for each $t < t^*$. At the beginning of t^*, E publicly commits to $(\gamma^E_{t^*}, \gamma^E_{t^*+1}, \dots)$, after which the consumer chooses (a^I_t, a^E_t). Here, I consider an equilibrium in which E commits to $\gamma^E_t = \bar{\gamma}\ \forall t \geq t^*$, and I commits to monopoly privacy levels before t^* and sets privacy levels by recursively solving (39) after t^*. $\square$

H Omitted Proofs for Section 7

H.1 Erasing Past Information: Proofs for Section 7.1

Proof of Claim 1. Since the consumer's action does not affect a privacy policy, it is optimal for the consumer to erase information in all periods. Anticipating this, the platform maximizes the amount of information generated in each period, by solving the problem (5) with $t = 1$. Thus the platform sets $\gamma_t = \gamma_1^*$ for all t. $\square$

Proof of Claim 2. The platform's problem is to solve (5) by choosing a privacy level and whether to erase information. Whenever $\sigma_{t-1}^2 < \sigma_0^2$, erasing information strictly increases the posterior variance, increases the consumer's marginal cost, and shifts $a^*(\cdot, \sigma^2)$ downward. Because erasing information strictly lowers the platform's payoff, it chooses $\mathcal{T} = \emptyset$ in equilibrium. $\square$

H.2 Heterogeneous Consumers: Proof of Proposition 6

Proof. Take any equilibrium $(a_t^*(v), \sigma_t^2(v), \gamma_t^*)_{t\in\mathbb{N}, v\in V}$. For each $v \in V$, define $\sigma_\infty^2(v) := \lim_{t\to\infty} \sigma_t^2(v)$. First, suppose, to the contrary, that there is some $v^* \in V$ such that $\sigma_\infty^2(v^*) > 0$. Define

$$\Delta_t := \frac{1}{1-\delta_P} \sum_{v\in V} \alpha_v \left[\sigma_0^2 - \sigma_\infty^2(v)\right] - \frac{1}{1-\delta_P} \sum_{v\in V} \alpha_v \left[\sigma_0^2 - \sigma_{t-1}^2(v)\right]. \tag{40}$$

It holds $\lim_{t\to\infty} \Delta_t = 0$. Now, take any $\gamma_v^* \in \arg\min_\gamma \frac{1}{a^*(v^*,\gamma,\sigma_0^2)} + \gamma$. It holds that for any $\sigma^2 \in [\sigma_\infty^2(v^*), \sigma_0^2]$,

$$\sigma^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a^*(v^*,\gamma_v^*,\sigma^2)} + \gamma_v^*}} \geq \sigma^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a^*(v^*,\gamma_v^*,\sigma_0^2)} + \gamma_v^*}} \geq M := \min_{\sigma^2\in[\sigma_\infty^2(v^*),\sigma_0^2]} \sigma^2 - \frac{1}{\frac{1}{\sigma^2} + \frac{1}{\frac{1}{a^*(v^*,\gamma_v^*,\sigma_0^2)} + \gamma_v^*}} > 0.$$

The first inequality follows from $a^*(v^*, \gamma, \sigma_0^2) \leq a^*(v, \gamma, \sigma^2)$ for $\sigma^2 \leq \sigma_0^2$. The last inequality holds because the minimand is continuous and positive on $[\sigma_\infty^2(v^*), \sigma_0^2]$. For a sufficiently large t, we obtain $\frac{\alpha_v M}{1-\delta_P} > \Delta_t$, or equivalently,

$$\frac{\alpha_v M}{1-\delta_P} + \frac{1}{1-\delta_P} \sum_{v\in V} \alpha_v \left[\sigma_0^2 - \sigma_{t-1}^2(v)\right] > \frac{1}{1-\delta_P} \sum_{v\in V} \alpha_v \left[\sigma_0^2 - \sigma_\infty^2(v)\right].$$

The left hand side is the lower bound of the time-t continuation value that the platform can get by deviating to the privacy level γ_v^* from time t on. The right hand side is the upper bound of the time-t continuation value without deviation. Thus, the platform is strictly better off by committing to a privacy policy that sets γ_v^* from time t on. This is a contradiction. $\lim_{t\to\infty} a_t^*(v) = 0$ and $\lim_{\to\infty} \gamma_t^* = 0$ follow the proof of Theorem 1. $\square$

H.3 General Privacy Cost: Proofs of Propositions 7, 8, and 9

Proof of Proposition 7. Consider any equilibrium. In period t, the consumer chooses a positive activity level if

$$\begin{aligned}
&\max_{a\in A} u(a) - v\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2} + \frac{1}{\frac{1}{a}+\gamma_t^*}}\right) \geq -\alpha v\left(\sigma_0^2 - \sigma_{t-1}^2\right)\\
\iff &\max_{a\in A} u(a) - v\left(\alpha\sigma_{t-1}^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2} + \frac{1}{\frac{1}{a}+\gamma_t^*}}\right) \geq (1-\alpha)v\sigma_0^2.
\end{aligned}$$

Let $\hat{a}_1$ and $\hat{\gamma}_1$ denote the equilibrium activity level and privacy level, respectively, in $t = 1$ of the baseline model (i.e., $\alpha = 1$). Define $y_1 := \frac{1}{\hat{a}} + \hat{\gamma}$ and $f(\alpha, x, y) := \alpha x - \frac{1}{\frac{1}{x}+\frac{1}{y}}$. The function f is strictly convex in x. Thus, on the interval $[0, \sigma_0^2]$, $f(\alpha, \cdot, y)$ is maximized at $x = \sigma_0^2$ if $f(\alpha, \sigma_0^2, y) > f(\sigma, 0, y)$, or equivalently, $\alpha\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{y}} > 0$. Moreover, the left hand side is decreasing in y. Thus, this inequality holds for all $y \leq y_1$ if and only if $\alpha\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{y_1}} > 0$. Let $\alpha^* < 1$ satisfy $\alpha^*\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2}+\frac{1}{\frac{1}{\hat{a}_1}+\hat{\gamma}_1}} > 0$. For any $\alpha \in [\alpha^*, 1]$, we have

$$\begin{aligned}
&u(\hat{a}_1) - v\left(\alpha\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\frac{1}{\hat{a}_1}+\hat{\gamma}_1}}\right) \geq (1-\alpha)v\sigma_0^2\\
\Rightarrow\ &u(\hat{a}_1) - v\left(\alpha\sigma_{t-1}^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2} + \frac{1}{\frac{1}{\hat{a}_1}+\hat{\gamma}_1}}\right) \geq (1-\alpha)v\sigma_0^2\\
\Rightarrow\ &\max_{a\in A} u(a) - v\left(\alpha\sigma_{t-1}^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2} + \frac{1}{\frac{1}{a}+\hat{\gamma}_1}}\right) \geq (1-\alpha)v\sigma_0^2
\end{aligned}$$

The first inequality holds because it is independent of α' and holds for $\alpha' = 1$. The last inequality implies that in any period, if the platform sets $\gamma_t = \hat{\gamma}_1$, then the consumer chooses $a_t > 0$. Also $a_t \geq \hat{a}_1$ holds because $\gamma_t > \hat{\gamma}$ and $\sigma_{t-1}^2 \leq \sigma_0^2$. In equilibrium, the platform sets γ_t to minimize the variance of the noise in s_t subject to the constraint that

$$\max_{a\in A} u(a) - v\left(\alpha\sigma_{t-1}^2 - \frac{1}{\frac{1}{\sigma_{t-1}^2} + \frac{1}{\frac{1}{a}+\gamma_t}}\right) \geq (1-\alpha)v\sigma_0^2.$$

The above argument implies that the variance of the noise in s_t is at most $\frac{1}{\hat{a}_1} + \hat{\gamma} + \varepsilon$, which implies $\sigma_t^2 \to 0$ in equilibrium. By the same proof as Theorem 1, $\sigma_t^2 \to 0$ implies $a_t^* \to a_{max}$ and $\gamma_t^* \to 0$. $\square$

Proof of Proposition 8. I adopt the notations in the proof of Proposition 5. In any period, the consumer weakly prefers to use platform k (i.e., $a_t^k > 0$ and $a_t^{-k} = 0$) if the following two conditions hold:

$$\begin{aligned}
&\arg\max_{a \in A} u(a) - v[\sigma_0^2 - \sigma_{t,k}^2(\gamma_t^k, a|\sigma_{t-1,k}^2)] - \alpha v[\sigma_0^2 - \sigma_{t-1,-k}^2] \\
\geq &\arg\max_{a \in A} u(a) - v[\sigma_0^2 - \sigma_{t,-k}^2(\gamma_t^{-k}, a|\sigma_{t-1,-k}^2)] - \alpha v[\sigma_0^2 - \sigma_{t-1,k}^2],
\end{aligned}$$

and

$$\arg\max_{a \in A} u(a) - v[\sigma_0^2 - \sigma_{t,k}^2(\gamma_t^k, a|\sigma_{t-1,k}^2)] - \alpha v[\sigma_0^2 - \sigma_{t-1,-k}^2] \geq -\alpha v[\sigma_0^2 - \sigma_{t-1,k}^2] - \alpha v[\sigma_0^2 - \sigma_{t-1,-k}^2].$$

These inequalities are respectively equivalent to

$$\arg\max_{a \in A} u(a) - \underbrace{v\left[\alpha\sigma_{t-1,k}^2 - \frac{1}{\frac{1}{\sigma_{t-1,k}^2} + \frac{1}{\frac{1}{a}+\gamma_t^k}}\right]}_{\text{(A)}} \geq \arg\max_{a \in A} u(a) - \underbrace{v\left[\alpha\sigma_{t-1,-k}^2 - \frac{1}{\frac{1}{\sigma_{t-1,-k}^2} + \frac{1}{\frac{1}{a}+\gamma_t^{-k}}}\right]}_{\text{(B)}} \tag{41}$$

and

$$\arg\max_{a \in A} u(a) - \underbrace{v\left[\alpha\sigma_{t-1,k}^2 - \frac{1}{\frac{1}{\sigma_{t-1,k}^2} + \frac{1}{\frac{1}{a}+\gamma_t^k}}\right]}_{\text{(A)}} \geq (1-\alpha)v\sigma_0^2. \tag{42}$$

By the same argument as Proposition 7, there is $\alpha^* < 1$ such that for any $\alpha \geq \alpha^*$, the following holds: For any $\frac{1}{a} + \gamma_t^k \leq \frac{1}{a(\bar{\gamma})} + \bar{\gamma}$, (A) is maximized at $\sigma_{t-1,k}^2 = \sigma_0^2$; for any $\frac{1}{a} + \gamma_t^{-k} \leq \frac{1}{a(\bar{\gamma})} + \bar{\gamma}$, (B) is maximized at $\sigma_{t-1,-k}^2 = \sigma_0^2$. These observations imply the following. First, I can induce $a_t^I > 0$ before the entry, by setting $\gamma_t = \bar{\gamma}$. Second, after I collects some information, if I and E set the same privacy level $\bar{\gamma}$, then the consumer optimally sets $a_t^I > 0 = a_t^E$. We can then apply the proof of Proposition 5 to construct an equilibrium such that (i) E sets $\gamma_t^E = \bar{\gamma}$ for all $t \in \mathbb{N}$, (ii) I sets γ_t^I to minimize the variance of the noise of s_t subject to constraints (41) and (42). The rest of the proof follows the proof of Proposition 5. $\square$

Proof of Proposition 9. Consider the (myopic) consumer's problem in period t. Given the posterior variance σ^2 at the end of period $t-1$ and the privacy level γ in period t, the consumer chooses a to maximize $U(a, \gamma, \sigma^2) := u(a) - C\left(\frac{1}{\frac{1}{\sigma^2}+\frac{1}{\frac{1}{a}+\gamma}}\right)$. It holds that

$$\frac{\partial U}{\partial a} = u'(a) + C'\left(\frac{1}{\frac{1}{\sigma^2}+\frac{1}{\frac{1}{a}+\gamma}}\right) \cdot \frac{1}{\left(\frac{1}{\sigma^2}(1+\gamma a)+a\right)} \geq u'(a) - B \cdot \frac{1}{\left(\frac{1}{\sigma^2}(1+\gamma a)+a\right)}, \tag{43}$$

where $B := \max_{x\in[0,\sigma_0^2]} |C'(x)| < \infty$. If $\lim\limits_{t\to\infty} \sigma_t^2 > 0$, we have $\lim_{t\to\infty} \frac{1}{a_t^*} + \gamma_t^* = \infty$. Consider a hypothetical payoff function

$$U_B(a, \gamma, \sigma^2) := u(a) - B \cdot \left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma^2}+\frac{1}{\frac{1}{a}+\gamma}}\right).$$

Inequality (43) implies $\frac{\partial U}{\partial a} \geq \frac{\partial U_B}{\partial a}$. Take any γ' such that $a_B^*(\gamma', \sigma^2) := \max\left\{\arg\max_{a\in A} U_B(a, \gamma', \sigma_0^2)\right\} > 0$. Then, for any $\sigma^2 \leq \sigma_0^2$, $a^*(\gamma', \sigma^2) \geq a_B^*(\gamma', \sigma^2) \geq a_B^*(\gamma', \sigma_0^2) > 0$. Take T such that for all $t \geq T$, $\frac{1}{a_t^*} + \gamma_t^* \geq \frac{1}{a_B^*(\gamma', \sigma_0^2)} + \gamma'$ The platform can achieve a lower $\frac{1}{a_t} + \gamma_t$ for any $t \geq T$ by replacing γ_t^* with γ', which is a contradiction. A similar argument implies $a_t^* = a_{max}$ and $\gamma_t^* = 0$ for a large but finite t. $\square$

H.4 Endogenous Quality of Service: Proof of Proposition 11

Proof. Given (δ_P, q), Let $\Pi(\delta_P, q)$ denote the platform's ex ante sum of discounted profits. For any $q > 0$, the platform's per-period payoff is at most $\sigma_0^2 - c(q)$. Thus, $(1-\delta_P)\Pi(\delta_P, q) \leq \sigma_0^2 - c(q)$. Suppose to the contrary that there is a sequence $\delta_n \to 1$ such that for some $q' > 0$, $q(\delta_n) \geq q'$ for infinitely many n's (for some selection of equilibria). Without loss of generality, assume $(1-\delta_n)\Pi(\delta_n, q(\delta_n)) \in [0, \sigma_0^2]$ has a limit. Then, $\lim_{n\to\infty}(1-\delta_n)\Pi(\delta_n, q(\delta_n)) \leq \sigma_0^2 - c(q') < \sigma_0^2 - c(q'/2)$. If the platform chooses $q'/2$ and the corresponding optimal policy $\boldsymbol{\gamma}$, then as $\delta_P \to 1$, its average payoff converges to $\sigma_0^2 - c(q'/2)$. Thus, the platform with a large δ_n strictly prefers $q'/2$ to $q(\delta_n)$, which is a contradiction. Thus, $\lim_{\delta_P\to 1} u_{q(\delta_P)}(a_{max}) - v\sigma_0^2 = -v\sigma_0^2$. Also, as the consumer's ex ante payoff is nonnegative but lower than $\frac{u_{q(\delta_P)}(a_{max})}{1-\delta_C}$, it converges to 0 as $\delta_P \to 1$. $\square$