Dynamic Privacy Choices

Preliminary Draft

Shota Ichihashi*

November 29, 2019

Abstract

I consider a dynamic model of consumer privacy. Consumers choose how much to use a platform in each period. Using the platform generates information about consumers' types, which benefits the platform but hurts consumers. In equilibrium, consumers choose a higher level of usage over time, and in the long-run, they behave as if there is no privacy concern while incurring a high privacy cost. Regulating the platform's data collection can exacerbate the problem. Platform competition improves privacy only if platforms can commit to future privacy policies. The impacts of the right to be forgotten and data retention policies are examined.

*Bank of Canada, 234 Wellington Street West, Ottawa, ON K1A 0G9, Canada. Email: shotaichihashi@gmail.com.
I thank Byung-Cheol Kim and Catherine Tucker for valuable comments.

1 Introduction

Online platforms, such as Facebook, Google, Amazon, and Uber, analyze the activities of consumers and collect a large amount of data. The extensive data collection may improve the quality of services and benefit consumers, however, it also produces concerns about its potential harms, such as price or non-price discrimination, a loss of privacy, and data leakage. Such concerns are highlighted by, for example, the *Cambridge Analytica* scandal.

I study such a situation in the following dynamic model: There are a consumer and a platform. The consumer is characterized by her type X, which is fixed over time and initially unknown to the platform. For example, X could represent political preferences. In each period, the consumer chooses her activity level on the platform (e.g., how much time to spend on browsing news and posts). The activity generates information about X. To focus on harmful data collection, I assume that the platform benefits but the consumer loses if the platform knows more about X. Thus, the consumer chooses her activity level balancing the benefit of using the platform and the cost of losing privacy.

The key economic force is that the marginal privacy cost of using the platform is *decreasing* in the amount of information that the consumer has already provided. To see this, imagine old consumers who have used a social media for 10 years, and new consumers who start to use it today. The platform already knows a lot about old consumers. Therefore, old consumers incur a lower *marginal* loss of privacy than new consumers from a fixed level of activity.

The decreasing marginal privacy cost leads to the following equilibrium dynamics: The consumer's activity level on the platform increases over time and converges to the level that she would choose without any privacy costs. However, the consumer incurs a high privacy cost because the platform accurately learns her type after long periods of interactions. As a result, the long-run welfare of the consumer can be lower than in the absence of the platform. The result holds even if the consumer is forward-looking.

One may think that restricting the platform's data collection may mitigate the problem. However, this is not necessarily the case: Such a regulation may benefit consumers in the short-run but hurt them in the long-run. This is because regulating data collection encourages privacy-sensitive consumers to join the platform, which, in turn, increases the number of consumers who lose their privacy in the long-run.

The dynamic model enables me to study how the platform's optimal privacy policy evolves. I show that the optimal policy offers a high level of privacy (i.e., committing to monetize less information) in early periods but reduces the level of privacy as time goes by. Such a policy resolves the platform's trade-off between encouraging consumer participation and speeding up data collection. I argue that the equilibrium policy is consistent with the evolution of the business models of platforms such as Facebook.

Another potential remedy for consumer privacy problem is platform competition. I show that the effect of competition on privacy depends on whether platforms can commit to future privacy policies. Without commitment power, a platform to which consumers have provided much information can lower its privacy level while keeping their high activity levels.

The dynamic model enables us to examine several key concepts relevant to consumer privacy. First, I consider *the right to be forgotten*, whereby consumers can request platforms to delete past information. Second, I consider the choice of *data retention periods* by a platform. I clarify how these concepts interact with the market structure of platforms and their commitment power.

This paper contributes to the literature on the economics of privacy and markets for data. Acemoglu et al. (2019) and Bergemann et al. (2019) study static models in which platforms collect consumer data in exchange for monetary compensation. The key concept in their work is data externality, where the data of some consumers reveal information about other consumers. These papers show that data externality enables platforms to collect a large amount of data cheaply. I consider a dynamic model with *intertemporal* data externality. Considering a dynamic model is important because several key concepts relevant to consumer privacy problem can be examined only in dynamic environments. In particular, my paper sheds light on consumer myopia, the right to be forgotten, data retention, and platforms' commitment to future privacy policies.[1]

Prufer and Schottmüller (2017) study competition between duopolists that repeatedly choose how much to invest in quality. The key economic force is "data-driven indirect network externality," where the marginal cost of improving the quality is decreasing in the past demand (data). They show that in the long-run, one firm typically dominates the market and firms have low incentives

[1] Tucker (2018) states that "introducing the potential for myopia or hyperbolic discounting into the way we model privacy choices over the creation of data seems therefore an important step."

to invest. There are two major differences between this literature and my paper. First, the literature studies positive externalities, whereas I consider a negative externality. A negative externality is important for studying issues relevant to privacy. Indeed, if externality is only positive, then neither consumers nor regulators would value privacy, and thus platforms do not offer it. Second, while the literature models "data" as a one-dimensional variable, I model data (or information) as a signal correlated with some uncertain state of the world. This formulation enables me to consider a rich space of privacy regulations and policies, such as adding noise to collected data, erasing past information, and time-varying privacy levels.

Bonatti and Cisternas (Forthcoming) study consumer privacy in a continuous-time dynamic model. They consider a long-lived consumer with short lived sellers. Sellers can learn about consumer preferences based on scores that aggregate purchase histories, and sellers use information for price discrimination. In contrast, I consider long-lived platforms and abstract away from how platforms use consumer information.

Fainmesser et al. (2019) study the optimal design of a platform to store data and invest in information security. They consider a platform that cares about both the activity levels of consumers and the amount of data it can extract. They study how different objectives lead to different platform designs. I adopt simpler preferences for consumers and platforms but consider rich dynamics.

In my model, the more consumers partake of the activity on a platform, the more they want to partake in the future. This connects the present paper with the literature of rational addiction such as Becker and Murphy (1988). However, unlike alcohol or cigarettes, the process of "addiction" in my model is influenced by strategic choices of a regulator and a platform regarding data collection and usage. For example, a regulation that mandates a platform to erase old personal data "resets" the addiction problem, which may have no counterpart for usual addictive goods.

The rest of the paper is organized as follows. Section 2 presents a model of a monopoly platform and a myopic consumer. Section 3 presents the dynamics of consumer behavior and characterizes the platform's optimal privacy policy. Section 4 considers platform competition and examines the role of commitment power to future privacy policies. Section 5 studies the incentive of consumers or platforms to erase past data, and how it interacts with the market structure of platforms. Section 6 considers extensions, and in particular, I consider a forward-looking consumer.

2 Model

Time is discrete and infinite, indexed by $t = 1, 2, \ldots$. There is a consumer (she) and a platform (it). The type of the consumer is denoted by X, which has a normal distribution $N(0, \sigma_0^2)$ with $\sigma_0^2 > 0$. X is realized before $t = 1$ and fixed over time. X represents the consumer's persistent characteristics such as political preferences, health information, or financial status. The platform does not observe X but receives signals about X over time, as described below.

In each period t, the consumer chooses her *activity level* $a_t \geq 0$. For example, a_t is how long the consumer spends on a social media platform on each day. The activity level a_t yields the consumer a benefit of $u(a_t)$ in period t. We can interpret $u(a_t)$ as the value of using the platform (e.g., browsing contents) minus the opportunity costs (e.g., sleep). I assume that $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, continuously differentiable, maximized at $a^* \in (0, \infty)$, and $u(0) = 0$.

The consumer's activity on the platform produces information about her type X. For example, the browsing activity of consumers might be indicative of their political preferences. Formally, in period t, if the consumer chooses a_t, then the platform observes a_t itself and a signal $s_t = X + \varepsilon_t + z_t$, where $\varepsilon_t \sim N\left(0, \frac{1}{a_t}\right)$ and $z_t \sim N(0, \gamma_t)$. $\gamma_t \geq 0$ is a part of the platform's strategy and represents the *privacy level* of the platform in period t. All the random variables, X, $\varepsilon_1, z_1, \varepsilon_2, z_2, \ldots$, are mutually independent.

The payoff of each player is as follows. Suppose that the consumer chooses a sequence of activity levels $\mathbf{a}_t = (a_1, \ldots, a_t)$. Then, at the end of period t, the platform receives a payoff of $\sigma_0^2 - \sigma_t^2(\mathbf{a}_t) \geq 0$, where $\sigma_t^2(\mathbf{a}_t)$ is the variance of the conditional distribution of X given $\mathbf{a}_t$. (When it does not cause confusion, I write $\sigma_t^2(\mathbf{a}_t)$ as σ_t^2.) Note that the conditional variance depends on $\mathbf{a}_t$ but not on realized signals $s_1, \ldots, s_t$. The consumer's payoff in period t is given by $U(\mathbf{a}_t) := u(a_t) - v\left[\sigma_0^2 - \sigma_t^2(\mathbf{a}_t)\right]$, where $v > 0$ is the (marginal) *value of privacy*. As to the intertemporal preferences, the platform discounts future payoffs with a discount factor $\delta_P \in (0, 1)$. For now, I consider a myopic consumer, who chooses a_t to maximize $U(a_1, \ldots, a_t)$ in each period t. Later, I consider a forward-looking consumer.

I describe the evolution of conditional variances, which is frequently used in the analysis. Suppose that the consumer has chosen a sequence of activity levels $\mathbf{a}_{t-1} = (a_1, \ldots, a_{t-1})$, which leads to the platform's information $\sigma_{t-1}^2(\mathbf{a}_{t-1})$ at the beginning of period t. If the consumer chooses

a_t in period t, then $\sigma_t^2(\mathbf{a}_t)$ is given by[2]

$$\sigma_t^2(\mathbf{a}_t) = \frac{1}{\frac{1}{\sigma_{t-1}^2(\mathbf{a}_{t-1})} + \frac{1}{\frac{1}{a_t}+\gamma_t}}. \tag{1}$$

To simplify exposition, I sometimes write the platform's information in terms of a precision $\rho_t = \frac{1}{\sigma_t^2}$.

Note that the payoffs are normalized so that if $a_t = 0$ for all t, then the platform and the consumer receive zero payoffs in all periods. If $a_t > 0$, then I say that the consumer *uses* the platform in period t. If $a_1 > 0$, then I say that the consumer *joins* the platform.

The timing of the game is as follows. First, before $t = 1$, the platform commits to a *privacy policy* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots)$. The commitment assumption may seem strong but I will show that the equilibrium is independent of commitment power. After observing $\boldsymbol{\gamma}$, the consumer chooses an activity level in each period. Instead of explicitly modeling beliefs about X, I take $(\sigma_t^2(\cdot))_{t\in\mathbb{N}}$, which is recursively defined by (1), as a primitive of the model. I consider subgame perfect equilibrium in which the consumer breaks ties in favor of greater activity levels in each period.

2.1 Discussion of Modeling Assumptions

Myopic consumer

For now, I assume that the consumer myopically chooses an activity level in each period. We can think of myopia as the consumer's true preferences or behavioral bias. The assumption intends to capture consumers who fail to consider how their activities on a platform today might leak information about them and affect their welfare in the future. However, I will also consider the forward-looking consumer and show that the main insight continues to hold.

Payoffs

I assume that the consumer incurs a loss if the platform learns more about the consumer's type. For example, the loss may reflect the consumer's intrinsic demand for privacy. Alternatively, a platform may monetize consumer data by selling targeted advertising spaces and it may cause

[2]This follows from the fact that if $x|\mu \sim N(\mu, \sigma^2)$ and $\mu \sim N(\mu_0, \sigma_0^2)$, then $\mu|x \sim N\left(\frac{\sigma_0^2}{\sigma^2+\sigma_0^2}x + \frac{\sigma^2}{\sigma^2+\sigma_0^2}\mu_0, \left(\frac{1}{\sigma_0^2}+\frac{1}{\sigma^2}\right)^{-1}\right)$.

privacy concerns. Finally, consumers may be worse off if third parties access consumer information and use it for price or non-price discrimination.

To simplify the analysis, I abstract from several features that can be relevant in practice. For example, platforms may collect data to improve their services. Then, data collection may benefit consumers depending on what kind of data is collected. Also, platforms may earn profits not only from consumers' information but also their attention (a_t). Although I conjecture that some of the main insights can extend to these richer settings, they are out of the scope of the paper.

3 Monopoly Platform

First, I consider the long-run behavior of the consumer when the platform restricts attention to the following simple class of privacy policies:

Definition 1. A privacy policy $\boldsymbol{\gamma} = (\gamma_t)_{t\in\mathbb{N}}$ is a *constant privacy policy* if γ_t is independent of t. If $\gamma_t = \gamma \in [0,\infty)$ for all t, it is called a γ*-constant privacy policy*.

Recall that $a^* = \arg\max_{a\geq 0} u(a)$ is the activity level chosen by the consumer in the absence of privacy costs. Also, a lower σ_t^2 means that the consumer has less privacy. The following result summarizes the long-run dynamics of the activity levels and privacy (see Appendix A for the proof).

Proposition 1. *Take any constant privacy policy. Let $(a_t^*)_{t\in\mathbb{N}}$ denote the equilibrium activity levels. There is a cutoff value of privacy $v^* \in (0,\infty]$ such that the following holds:*

1. *If $v < v^*$, then a_t^* is increasing in t, $\lim_{t\to\infty} a_t^* = a^*$, and $\lim_{t\to\infty} \sigma_t^2(a_1^*,\dots,a_t^*) = 0$.*
2. *If $v > v^*$, then $a_t^* = 0$ and $\sigma_t^2 = \sigma_0^2$ for all $t \in \mathbb{N}$.*

Point 1 says that if the (marginal) value of privacy for a consumer is below a cutoff, then her activity level on the platform increases over time and converges to a^*, which is what the consumer would choose without any privacy costs. However, this is associated with a high privacy cost in the long-run, because the platform eventually learns her type. In contrast, Point 2 says that if the value of privacy for a consumer is above the cutoff, then the consumer is inactive on the platform. In this case, the platform learns nothing about the consumer's type.

The intuition of Proposition 1 is somewhat similar to *rational addiction* in Becker and Murphy (1988). The consumer's activity on the platform today reveals information about her. The information revelation hurts the consumer through a privacy cost, however, it also lowers the *marginal* privacy cost that the consumer will incur by using the platform in the future. This is because the platform learns less from the activity of a given period if the consumer has already provided a lot of information. This implies that the equilibrium activity level is increasing over time conditional on that the consumer uses the platform in $t = 1$. In the long-run, the platform almost perfectly learns the consumer's type. At this point, the consumer's marginal loss of privacy is negligible. As a result, the consumer's activity level converges to the maximum level a^*. Now, recall that the rational addiction model admits a forward-looking consumer. Consistently, Section 6.1 shows qualitatively the same result as Proposition 1 when the consumer is forward-looking.

Proposition 1 gives a potential economic explanation of the observed puzzle: Consumers seem to casually share their information with online platforms despite growing concerns about their data collection. In light of the result, one may view this puzzle as the long-run behavior of consumers with decreasing marginal privacy costs.

The result also provides a policy implication. Suppose that a regulator, who cares about consumer privacy (or the long-run welfare of consumers), mandates the platform to adopt a higher privacy level in all periods. The following result shows that such a regulation can backfire.

Corollary 1. *Take any γ-constant privacy policy, and let $v^*(\gamma)$ denote the cutoff in Proposition 1. $v^*(\gamma)$ is increasing in γ.*

Proof. The result follows from the observation that $\bar{a}\left(v, \frac{1}{\sigma_0^2}, \gamma\right)$ in (16), the consumer's equilibrium activity level in $t = 1$, is increasing in γ. □

If consumer behavior is exogenous, then a stricter privacy policy increases the consumer's per-period payoff in any period and for any v, because it reduces the amount of information the platform can obtain. However, if consumer behavior is endogenous, then a stricter privacy policy may decrease consumer privacy and welfare in the long-run. For example, suppose that v is so high that $u(a^*) - v\sigma_0^2 < 0$ holds. For a small γ, the consumer may choose $a_t = 0$ and obtain a payoff of zero in all periods. However, for a sufficiently large γ, the consumer chooses $a_1 > 0$. Conditional on $a_1 > 0$, a_t converges to a^*, and the consumer's per-period payoff goes to $u(a^*) - v\sigma_0^2 < 0$.

Thus, if we consider consumers with heterogeneous v, then a higher γ leads to a higher fraction of consumers who lose their privacy in the long-run.

Corollary 1 suggests that policymakers could consider the dynamic implication of privacy regulations. Restricting data collection by online platforms may improve consumer privacy in the short-run. However, such a regulation encourages some of privacy sensitive consumers to join the platform. These consumers then become increasingly active users of the platform. As a result, the regulation may increase the number of consumers who incur high privacy costs in the long-run. Note that a higher γ still “benefits” myopic consumers from the ex ante perspective, because they only care about their payoffs in $t = 1$. However, if a regulator cares about the long-run consumer welfare more than consumers themselves do, then the regulator could take into account the potential adverse impact of regulating data collection.

Corollary 1 also illustrates the platform’s trade-off. One the one hand, a stricter privacy policy reduces the amount of information available to the platform for any given activity level. However, it also encourages the consumer to choose a higher activity level. The following result characterizes the platform’s optimal privacy policy, which balances this trade-off (see Appendix B for the proof).

Proposition 2. *Let $\bar{a}(\rho, \gamma)$ denote the consumer’s optimal activity level, where*

$$\bar{a}(\rho, \gamma) := \max \left\{ \arg\max_{a \geq 0} u(a) - v \left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a} + \gamma}} \right) \right\}. \tag{2}$$

The platform’s equilibrium privacy policy $(\gamma_1^*, \gamma_2^*, \dots)$ is recursively defined as follows.

$$\rho_0^* = \frac{1}{\sigma_0^2}, \tag{3}$$

$$\rho_t^* = \rho_{t-1}^* + \frac{1}{\frac{1}{\bar{a}(\rho_{t-1}^*, \gamma_t^*)} + \gamma_t^*}, \tag{4}$$

$$\gamma_t^* \in \arg\min_{\gamma \geq 0} \frac{1}{\bar{a}(\rho_{t-1}^*, \gamma)} + \gamma. \tag{5}$$

The interpretation of each condition is as follows. (2) describes the optimal activity level of the consumer in any period t when she faces the privacy level γ in t and the posterior precision ρ (i.e., the reciprocal of the conditional variance of X). (3) is the initial condition for the precision at the beginning of $t = 1$, and (4) describes the platform’s belief updating in terms of the precision, given the privacy level actually chosen by the platform and the optimal behavior of the consumer.

The minimization problem in (5) captures the key property of the optimal policy: In equilibrium, the platform chooses γ_t to *statically* minimizes the variance of the joint noise term $\varepsilon_t + z_t$ of a signal $s_t = X + \varepsilon_t + z_t$ in each t. This captures the platform's trade-off. On the one hand, a higher privacy level γ leads to a higher activity level. This benefit is captured by a lower variance $\frac{1}{\overline{a}(\rho^*_{t-1}, \gamma)}$ of ε_t. On the other hand, given any activity level, a higher γ reduces the amount of information. This cost is captured by the second term γ, a variance of z_t.

The optimality of static minimization is *not* a direct implication of consumer myopia. Indeed, if the platform's privacy level today can affect the future behavior of a myopic consumer, then the equilibrium privacy policy might depend on whether the platform has commitment power. However, this is not the case in the current model, because setting γ_t to maximize the amount of available information in period t is optimal not only for period t but also for the future. This is because the consumer's activity level in any period is increasing in the amount of information she has already provided.

Proposition 2 implies that the platform's optimal policy does not depend on its discount factor. Moreover, the platform can implement the optimal policy even without commitment power. "Without commitment" means that the platform chooses γ_t at the beginning of each period t before the consumer chooses a_t.[3] To state the next result, for any privacy policy $\boldsymbol{\gamma}$, let $(\sigma_t^2(\boldsymbol{\gamma}))_t$ denote the sequence of conditional variances of X induced by the consumer's optimal behavior. Let $\boldsymbol{\gamma}^*$ denote the platform's equilibrium privacy policy in Proposition 2. See Appendix C for the proof of the following result.

Corollary 2. *For any privacy policy $\boldsymbol{\gamma}$, $\sigma_t^2(\boldsymbol{\gamma}^*) \leq \sigma_t^2(\boldsymbol{\gamma})$ for all $t \in \mathbb{N}$. Thus, The platform's optimal privacy policy is independent of its discount factor δ_P. Moreover, the optimal policy is independent of whether the platform commits to $(\gamma_t)_{t\in\mathbb{N}}$ before $t = 1$ or it chooses γ_t at the beginning of each t.*

The next result summarizes key properties of the equilibrium under the platform's optimal privacy policy: For *any* v , the consumer's activity level converges to a^*, and she loses privacy in

[3]In other words, the platform has commitment power within each period but does not have commitment power for future periods.

the long-run. Moreover, the optimal privacy policy is not constant: The equilibrium privacy level can be positive in early stages but eventually converges to zero (see Appendix D for the proof).

Proposition 3. *Take any v. In equilibrium, the following holds:*

$$\lim_{t\to\infty} a_t^* = a^*, \lim_{t\to\infty} \sigma_t^2 = 0, \textit{ and } \lim_{t\to\infty} \gamma_t^* = 0. \tag{6}$$

The result states that no matter how high is the value v of privacy, the platform can choose privacy levels so that consumers will eventually choose high activity levels and lose their privacy. Thus, if we consider an arbitrarily large v, then the long-run equilibrium payoff of the consumer is unbounded from below.

The platform can induce the participation of consumers with a high v by offering high privacy levels in early periods. As time goes by, the platform obtains a large amount of information about consumers' types, and correspondingly, consumers have a vanishingly small marginal cost of using the platform. From then on, the platform can lower the level of privacy to speed up data collection without deterring consumers' usage of the platform. Figure 1 illustrates this dynamics.[4] In this numerical example, the platform offers a monotonically decreasing level of privacy. The equilibrium activity level first decreases but eventually approaches a^*. This non-monotonicity contrasts with the case of a constant privacy policy (Proposition 1).[5]

The equilibrium privacy policy seems consistent with the observed pattern of privacy policies adopted by some online platforms. For example, Srinivasan (2019) provides the in-depth study of how Facebook has acquired dominance in the social media market, and states that "When Facebook entered the market, the consumer's privacy was paramount. The company prioritized privacy, as did its users–many of whom chose the platform over others due to Facebook's avowed commitment to preserving their privacy. Today, however, accepting Facebook's policies in order to use its service means accepting broad-scale commercial surveillance." Also, Fainmesser et al. (2019) describe how the business model of online platforms has changed from the initial phase where they expand a user base to the mature phase where they monetize collected information. The equilibrium dynamics of Proposition 3 seem to capture such a transition.

[4]For any $T \in \mathbb{N}$, we can compute the equilibrium behavior up to period T using Proposition 2.

[5]While I do not prove this, a numerical exercise suggests that this non-monotonicity holds for a wide range of parameters such that the privacy level is strictly decreasing in early periods.

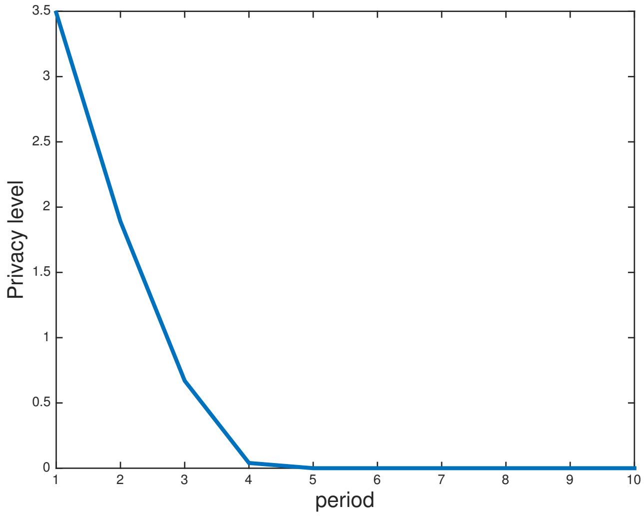

Figure 1(a): Privacy level γ_t

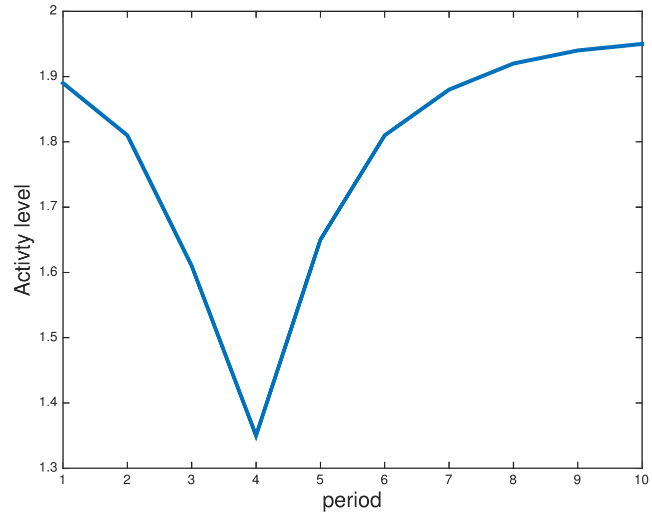

Figure 1(b): Activity level a_t

Figure 1: Equilibrium under $u(a) = 2a - \frac{1}{2}a^2$, $v = 10$, and $\sigma_0^2 = 1$.

4 Platform Competition

I now consider platform competition, where two platforms choose privacy levels to attract a consumer. I show that whether competition improves privacy depends on two things: platforms' commitment power to future privacy policies, and the timing of market entry.

Consider two platforms A and B. For $k = A$ (resp. $k = B$), I use $-k$ to mean B (resp. A). Let γ_t^k denote the privacy level of platform k in period t. In each t, the consumer chooses (a_t^A, a_t^B), where $a_t^k \geq 0$ is her activity level on platform k. I assume single-homing, that is, the consumer can choose (a_t^A, a_t^B) if and only if $\min(a_t^A, a_t^B) = 0$. The consumer's activity on platform k generates a signal $s_t^k = X + \varepsilon_t^k + z_t^k$ with $\varepsilon_t^k \sim N\left(0, \frac{1}{a_t^k}\right)$ and $z_t^k \sim N(0, \gamma_t^k)$. Each platform k observes s_t^k but not s_t^{-k}, and all the noise terms (ε_t^k, z_t^k) are independent across t and k. Thus, the posterior variance $\sigma_{t,k}^2$ of X for platform k in period t is independent of $(a_t^{-k})_{t \in \mathbb{N}}$.

As before, the payoff of platform k in period t is given by $\sigma_0^2 - \sigma_{t,k}^2$. The consumer's payoff in period t is given by

$$\sum_{k=A,B} \left[u(a_t^k) - v\left(\sigma_0^2 - \sigma_{t,k}^2\right) \right]. \tag{7}$$

Note that the consumer incurs a privacy cost $v(\sigma_0^2 - \sigma_{t,k}^2)$ from platform k in period t even when she uses platform $-k$ in that period. This is reasonable when a platform and third parties can store and use data collected in the past, and the use of data can affect consumers outside of platforms. For example, if a platform shares consumer data with an online retailer in period t, then consumers may face the risk of price discrimination on the retailer's website in any period $t' \geq t$.

I consider two modes of competition. One is *competition with commitment*. In this game, before $t = 1$, each platform k simultaneously chooses the entire sequence of its privacy levels $\boldsymbol{\gamma}^k = (\gamma_1^k, \gamma_2^k, \dots)$. The consumer observes $(\boldsymbol{\gamma}^A, \boldsymbol{\gamma}^B)$ and (myopically) chooses activity levels in each period. The other mode is *competition without commitment*. In this game, at the beginning of each period t, platforms simultaneously choose privacy levels of that period, without making any commitment to future privacy levels.

To ensure the existence of an equilibrium, I assume that the maximum privacy level that platforms can choose is bounded from above. The upper bound may reflect the maximum privacy level that a platform can technologically attain while offering valuable services.

Assumption 1. There is a $\bar{\gamma} \in (0, +\infty)$ satisfying $\bar{a}\left(\frac{1}{\sigma_0^2}, \bar{\gamma}\right) > 0$ such that each platform can choose a privacy level of at most $\bar{\gamma}$, where $\bar{a}\left(\frac{1}{\sigma_0^2}, \bar{\gamma}\right)$ is defined by (2).

$\bar{a}\left(\frac{1}{\sigma_0^2}, \bar{\gamma}\right) > 0$ implies that if one platform chooses the highest privacy level $\bar{\gamma}$, then the consumer chooses $a_t^k > 0$ for some $k \in \{A, B\}$. This restriction is necessary for obtaining a non-trivial result: If $\bar{a}\left(\frac{1}{\sigma_0^2}, \bar{\gamma}\right) = 0$, then in any equilibrium, the consumer chooses $(a_t^A, a_t^B) = (0, 0)$ for all $t \in \mathbb{N}$.

The following result shows that whether competition increases privacy depends on platforms' commitment power (see Appendix E for the proof).

Proposition 4. *Under Assumption 1, the following holds.*

1. *Under competition with commitment, there is an equilibrium in which both platforms offer the highest privacy level in all periods, i.e., $\gamma_t^k = \bar{\gamma}$ for all k and t.*

2. *Under competition without commitment, there is no equilibrium in which both platforms choose $\bar{\gamma}$ in all periods. Moreover, there is an equilibrium in which the consumer uses the same platform in all periods, and the platform offers privacy levels converging to zero over time.*

The intuition is as follows. If platforms can commit to future privacy levels, then consumers can decide which platform to use in $t = 1$ based on the entire privacy policy of each platform. Although consumers are myopic and thus they are indifferent between platforms' offering different privacy levels in periods $t > 1$, consumers can still break a tie in favor of a platform that offers the highest privacy levels in all periods. Anticipating this, platforms offer the most stringent privacy policy. In contrast, if platforms cannot commit to future privacy levels, then a platform to which consumers have provided a large amount of information can lower privacy levels and still attract consumers. This eliminates the equilibrium with highest privacy levels.

It is worth to note that qualitatively the same result as Point 2 of Proposition 4 arises even if platforms have a stronger commitment power. To see this, suppose that there is a $\bar{T} \in \mathbb{N}$ such that, for each $n \in \mathbb{Z}_+$, two platforms simultaneously commit to privacy levels $(\gamma^k_{1+n\bar{T}}, \ldots, \gamma^k_{(n+1)\bar{T}})$ at the beginning of period $1 + n\bar{T}$. $\bar{T} = 1$ corresponds to competition without commitment, and $\bar{T} \geq 2$ captures a stronger commitment power. It holds that, for any finite $\bar{T}$, there is an equilibrium such that the consumer sticks to one platform that offers privacy levels converging to zero. Thus, competition with commitment (i.e., $\bar{T} = \infty$), which gives the consumer a higher payoff, seems to be the knife-edge case.

4.1 Market Entry

So far, I have assumed that competing platforms are in the market from the beginning. I now consider the case in which one platform (incumbent) is in the market in $t = 1$ whereas the other platform (entrant) enters the market at some future periods. The analysis reveals the following counterintuitive idea: Entering a market is harder when the incumbent is offering low privacy and consumers are incurring high privacy costs.

Suppose that the market initially consists of only platform A. In period $t^* > 1$, platform B enters the market. For simplicity, assume that t^* is publicly known. To obtain stronger results, I assume that both platforms can commit to privacy policies.

We can formally analyze this model as a version of competition with commitment where the consumer is restricted to choosing $a^B_t = 0$ for all $t < t^*$. The following result shows that if a competitor enters the market late, then the entry has no impact on the market outcome.

Proposition 5. *There is $\underline{t}$ such that for any $t^* \geq \underline{t}$, the following holds: In equilibrium, platform A chooses a monopoly privacy policy and the consumer uses platform A in all periods, even after platform B enters the market.*

Proof. It is sufficient to show that for a large t^*, the platform A chooses a monopoly strategy and the consumer uses A in all periods even if the strategy of B is exogenously given by to offer $\gamma_t^B = \bar{\gamma}$ for all $t \geq t^*$. Note that for any period $t < t^*$, the consumer's activity levels coincide with the monopoly outcome in Proposition 2 because the consumer is myopic. Suppose to the contrary that for any t^*, there is $\tau(t^*) \geq t^*$ such that the consumer uses platform B in period $\tau(t^*)$. Assume that $\tau(t^*)$ is the first period in which the consumer uses B. Then, her activity level in period $\tau(t^*)$ is $a^B_{\tau(t^*)} = \bar{a}(\frac{1}{\sigma_0^2}, \bar{\gamma}) > 0$. However, for a sufficiently large t^*, the increment of the consumer's payoff from choosing $(a^B_{\tau(t^*)}, 0)$ relative to $(0,0)$ exceeds the one from $(0, a^B_{\tau(t^*)})$, which is a contradiction. $\square$

For the next result, suppose that platform A exogenously follows a γ-constant privacy policy such that the consumer uses A in $t = 1$. I say that platform B *can successfully enter the market* if B can choose some $\gamma^B_{t^*}$ in t^* so that the consumer uses B in any period $s \geq t^*$.

Proposition 6. *Let $\gamma^B_{t^*}$ denote the lowest privacy level such that platform B can successfully enter the market. $\gamma^B_{t^*}$ is increasing in t^* and $\lim_{t\to\infty} \gamma^B_{t^*} = +\infty$.*

Proof. The proof follows from the fact that (i) the marginal privacy cost for A is decreasing in $\sigma^2_{t,A}$ and (ii) the marginal privacy cost function for A uniformly converges to zero as $\sigma^2_{t,A} \to 0$. $\square$

Typically, it is easy for an entrant to penetrate the market when an incumbent offers low quality products. Interpreting privacy as quality, we may think that entry is easy when an incumbent offers low privacy and consumers incur high privacy costs. However, the results show the opposite. Indeed, if we consider various t^*'s, then it is exactly when consumers have low privacy that the entry is less likely and a successful entrant offers a high privacy level. The intuition is as follows. If consumers incur high privacy costs, then they incur low marginal privacy costs. Because consumers decide whether to switch based on the marginal costs, lower privacy is associated with the higher barrier to enter the market.

The insight is relevant to the market for search engines. There, platforms A is Google, whereas platform B is DuckDuckGo, which states that "we don't collect or share any of your personal information."[6] We might think that for consumers who value privacy, DuckDuckGo is a good option. However, if consumers choose whether to switch based on marginal privacy cost, then they might stick to Google, which already knows a lot about them. Note that one crucial assumption here is that even if consumers switch to the entrant, they cannot erase the information they provided to the incumbent.

5 Erasing Past Information

The previous sections model privacy policies in terms of the variance γ_t of a noise term z_t. This formulation implies that the platform's information about consumers (i.e., $1/\sigma_t^2$) is non-decreasing over time. This section considers the incentives of the consumer and platforms to *erase past information*. First, I consider the right to be forgotten, under which the consumer can request platforms to erase past information. Second, I study whether competition incentivizes platforms to voluntarily erase past information.

5.1 The Right to be Forgotten

Recent privacy regulations such as the General Data Protection Regulation in the EU try to ensure that consumers have control over their data. The current model partly captures this idea because the consumer can always provide no information by setting $a_t = 0$. This section further assumes that the consumer has "the right to be forgotten." Namely, at the beginning of each period t, the consumer can choose whether to erase all information she provided up to $t - 1$.

Formally, the consumer's choice in each period now consist of two parts (the description below applies to a monopoly platform and competing platforms). First, the consumer chooses whether to erase past information of each platform in the market. If she does, then platform k's information becomes σ_0^2 instead of $\sigma_{t,k}^2(\mathbf{a}_t^k)$. For simplicity, assume that it is costless for the consumer to erase information. Second, the consumer chooses a_t (under monopoly) or (a_t^A, a_t^B) (under competition).

[6] https://duckduckgo.com

For example, suppose that there are two platforms. If the consumer erases information of both platforms in t and uses platform A, then her payoff in t is

$$u(a_t^A) - v\left[\sigma_0^2 - \sigma_{1,A}^2(a_t^A)\right]. \tag{8}$$

The above expression holds because platform B has no information about the consumer ($\sigma_{t,B}^2 = \sigma_0^2$) and platform A's information is based on a signal of only period t.

In contrast, if the consumer has never erased information and uses platform A in period t, then her payoff is

$$u(a_t^A) - v\left[\sigma_0^2 - \sigma_{t,A}^2(\mathbf{a}_t^A)\right] - v\left[\sigma_0^2 - \sigma_{t,B}^2(\mathbf{a}_t^B)\right]. \tag{9}$$

There are two differences. First, if the consumer erases information, then she incurs no privacy costs from a platform that she does not use. Second, the consumer's privacy cost from the platform she uses depends only on the activity level of that period. The following result summarizes the impact of the right to be forgotten.

Proposition 7. *If the consumer can erase past information at no cost, then she does so in all periods. Moreover, the following holds.*

1. *Regardless of commitment power, a monopoly platform chooses a γ_1^*-constant privacy policy, where γ_1^* is defined in* (5).

2. *Regardless of commitment power, competing platforms choose the highest privacy level $\bar{\gamma}$ in all periods in any equilibrium.*

Proof. Given that the consumer erases information at the beginning of each period, a monopoly platform can maximize the amount of available information by setting γ_1^* that solves $\min_{\gamma \geq 0} \frac{1}{\bar{a}(\rho_0, \gamma)} + \gamma$, which implies Point 1. If there are two platforms, then the consumer uses the platform that offers a higher privacy level. This implies Point 2. $\square$

There are two takeaways from the above result. First, the right to be forgotten might indirectly benefit consumers by incentivizing platforms to offer better privacy policies even in the long-run. Once the consumer erases information, then she will incur relatively high marginal privacy costs of using the platform. This ensures that a monopoly platform offers a constant privacy level even in the long-run (Point 1), and competing platforms offer the highest level of privacy. Second, the right to be forgotten and platform competition can be complements. Erasing past information makes two platforms homogeneous, which intensifies their competition in providing privacy.

5.2 Data Retention Policies

This section considers a platform's choice of *data retention policies*, which has recently been paid attention by economists and legal scholars (Chiou and Tucker, 2017). To simplify the analysis, assume that any platform chooses $\gamma_t = 0$ for all $t \in \mathbb{N}$ and $\bar{a}\left(\frac{1}{\sigma_0^2}, 0\right) > 0$. A platform now chooses a *data retention period*. If a platform has commitment power, then it chooses $T \in \mathbb{N} \cup \{\infty\}$ before $t = 1$. T means that the platform commits to erase all information it has collected at the beginning of periods $T, 2T, 3T, \ldots$. For example, $T = 1$ means that the platform erases information at the beginning of every period. If a platform does not have commitment power, then it chooses whether to erase information at the beginning of each t (before the consumer chooses a_t). The result is summarized as follows.

Proposition 8. *The following holds.*

1. *Regardless of commitment power, in any equilibrium, a monopolist never erases past information in any periods.*

2. *Without commitment, in any equilibrium, the consumer uses only one platform, which never erases past information in any periods.*

3. *With commitment, there is an equilibrium in which competing platforms commit to erase past information in every period (i.e., $T_A = T_B = 1$).*

Proof. Point 1 holds because the consumer's activity level in each period is increasing in the amount of information the platform has. Point 2 holds because the consumer uses a platform that has the lowest σ_t^2. Point 3 follows from the same argument as competition without commitment in Proposition 4. □

It is illustrative to compare the choice of privacy levels $(\gamma_1, \gamma_2, \dots)$ and data retention periods. Note that if consumer behavior is exogenous, then these policies reduce the amount of information collected, benefit consumers, and hurt the platform. However, the consumer's endogenous response to these policies are different: Erasing information in period t increases the consumer's marginal privacy cost and decreases a_t. In contrast, a high privacy level γ_t reduces the consumer's marginal cost and increases a_t. Thus, a profit-maximizing platform voluntarily offers a high γ in early periods, however, the platform may not choose finite data retention periods.

6 Extensions

6.1 Forward-Looking Consumer

Suppose now that the consumer is forward-looking, discounting future payoffs by $\delta_C \in [0, 1)$. If the platform commits to a constant privacy policy, then the consumer's problem becomes a simple dynamic programming. The Bellman equation is given by

$$V(\rho) = \max_{a \geq 0} u(a) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a}+\gamma}}\right) + \delta_C V\left(\rho + \frac{1}{\frac{1}{a} + \gamma}\right). \tag{10}$$

Here, I write the state variable in terms of precision ρ instead of variance. The standard argument of dynamic programming implies the following.

Lemma 1. *Let $V^*(\rho)$ denote the consumer's discounted sum of payoffs from the optimal activity levels given ρ. $V^*(\cdot)$ is a unique bounded solution of Equation (10). Moreover, $V^*(\cdot)$ is continuous, strictly decreasing, and strictly convex.*

Proof. To show that Equation (10) has a solution with the desired properties, I first consider the following slightly different functional equation

$$V(\rho) = \max_{a \in [0, a^*]} u(a) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a}+\gamma}}\right) + \delta_C V\left(\rho + \frac{1}{\frac{1}{a} + \gamma}\right). \tag{11}$$

The only difference is that a is restricted to be between 0 and $a^* = \arg\max_{a \geq 0} u(a)$. Let S denote the set of all bounded continuous functions from $\left[\frac{1}{\sigma_0^2}, \infty\right)$ to $\mathbb{R}$, and let T denote the operator defined by the right-hand side of (11). It holds that $T : S \to S$. Indeed, for any $V \in S$, TV is bounded because $-\sigma_0^2 + \delta_C \inf V \le TV(x) \le u(a^*) + \delta_C \sup V$ for all x. TV is continuous by the maximum theorem. T also satisfies Blackwell's sufficient conditions for a contraction. Thus, T has a unique solution $V^* \in S$ satisfying $TV^* = V^*$.

Now, let S' denote the set of all bounded, continuous, weakly decreasing, and weakly convex function from $\left[\frac{1}{\sigma_0^2}, \infty\right)$ to $\mathbb{R}$. It holds that for any $V \in S'$, $TV \in S'$. In particular, TV is weakly convex because the right hand side of (10) is the maximum of convex functions. Also, S' is a compact metric space with sup norm. Thus, the unique fixed point of T belongs to S'. Having established that $V^* \in S'$, I can conclude that V^* also satisfies (10) (i.e., unrestricted $a \ge 0$), because the consumer never chooses $a > a^*$. Also, for any $(a_1, a_2, \dots)$, the induced sequence of precisions $(\rho_1, \rho_2, \dots)$ satisfies $\lim_{t\to\infty} \delta_C^t V^*(\rho_t) = 0$, because $-\frac{\sigma_0^2}{1-\delta_C} \le V^*(x) \le \frac{1}{1-\delta_C} u(a^*)$ for all x. Thus, V^* corresponds to a value function of the original sequential problem. This also establishes the uniqueness of a bounded solution of (10). $\square$

Lemma 2. *Let $V = V^*$ in equation (10). Let $\bar{a}(\rho)$ denote the largest maximizer of the right-hand side. Then, $\bar{a}(\rho)$ is well-defined and increasing in ρ.*

Proof. Convexity of $V^*(\cdot)$ implies that $u(a) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a}+\gamma}}\right) + \delta_C V^*\left(\rho + \frac{1}{\frac{1}{a}+\gamma}\right)$ has increasing differences in (ρ, a). Thus, $\bar{a}(\rho)$ is increasing. $\square$

The following result shows that even if consumers are forward-looking, they experience the long-run loss of privacy associated with high activity levels.

Proposition 9. *Take any constant privacy policy. Let $(a_t^*)_{t\in\mathbb{N}}$ denote the equilibrium activity levels. There is a cutoff value of privacy $v^* > 0$ such that the following holds:*

1. *If $v < v^*$, then a_t^* is increasing in t and $\lim_{t\to\infty} \sigma_t^2(a_1^*, \dots, a_t^*) = 0$.*

2. *If $v > v^*$, then $a_t^* = 0$ and $\sigma_t^2 = \sigma_0^2$ for all $t \in \mathbb{N}$.*

Moreover, v^* is increasing in γ.

Thus, we can interpret the long-run loss of privacy as the optimal choice of rational and forward-looking consumers.

6.2 Exogenous Data Collection

I have shown that the platform offers privacy in early periods to encourage consumer participation. In contrast, the platform often prefers to acquire exogenous information. For example, suppose that before $t = 1$, the platform can choose to acquire an exogenous signal $s_0 = X + \varepsilon_0$ with $\varepsilon_0 \sim N(0, \sigma^2_{\varepsilon_0})$. If the cost of acquiring s_0 is low, then the platform prefers to acquire s_0 because the exogenous signal benefits the platform in two ways: One is the direct effect, that is, s_0 enables the platform to learn about X. The other is the indirect effect, that is, the initial reduction of the conditional variance decreases the consumer's marginal cost of using the platform, and thus the consumer chooses a greater a_t for any $t \geq 1$ compared to the case in which the platform has no access to s_0. Such information acquisition strictly decreases the consumer's payoffs in all periods.

Interestingly, when exogenous information is available, platform competition can hurt consumers. To formalize the idea, let $V(\alpha)$ denote the profit of a *monopoly* platform when it acquires $s_0 = X + \varepsilon_0$ with $\varepsilon_0 \sim N(0, \frac{1}{\alpha})$ and adopts $\gamma_t = 0$ for all $t \in \mathbb{N}$. Suppose that the (potentially forward-looking) consumer chooses $a_1 > 0$ given $\alpha = 0$.

To examine the impact of competition, I consider a version of the game studied in Section 4. There are two platforms $k = A, B$. Each platform k can pay a cost of $c \cdot \alpha^k > 0$ to acquire a signal $s^k_0 = X + \varepsilon^k_0$ with $\varepsilon^k_0 \sim N\left(0, \frac{1}{\alpha^k}\right)$, where $c > 0$ and $\alpha^k \geq 0$. To ensure that there is a pure-strategy equilibrium, I assume sequential move: First, platform A chooses α^A. Second, after observing α^A, platform B chooses α^B. Finally, after observing (α^A, α^B), the consumer chooses $(a^A_t, a^B_t)_{t\in\mathbb{N}}$ subject to single-homing constraint. For simplicity, suppose that both platforms adopt $\gamma^k_t = 0$ for all $k \in \{A, B\}$ and all $t \in \mathbb{N}$. A key observation is that the consumer joins the platform that chooses a *greater* α^k. Intuitively, the exogenous data collection lowers the payoff of the consumer by reducing $(\sigma^2_{k,t})_{t\in\mathbb{N}}$. However, the cost is sunk, and the consumer chooses which platform to use based on marginal costs. Thus, a greater α^k lowers the marginal cost and makes platform k more attractive to the consumer. As a result, competition incentivizes platforms to collect more data from exogenous sources, which lowers consumer welfare:

Proposition 10. *Suppose that $V(\alpha) - c\alpha$ changes its sign once from positive to negative. Then, there is an equilibrium under competition such that the consumer's payoff is lower under competition than under monopoly.*

Proof. Suppose that there are two platforms. Consider an equilibrium where platform A chooses α^* that satisfies $V(\alpha^*) - c\alpha^* = 0$ and platform B chooses $\alpha^B = 0$. If platform A deviates to $\alpha < \alpha^*$, then platform B will also choose $\alpha^B = \alpha$ and the consumer chooses B. If platform A deviates to $\alpha > \alpha^*$, then platform B chooses $\alpha^B = 0$. This is clearly an equilibrium. Moreover, α^* is greater than the monopoly level of data collection α^M. Indeed, if $\alpha^M \geq \alpha^*$, then $V(\alpha^M) - c\alpha^M \leq 0$. However, a monopolist can choose $\alpha = 0$ to earn a positive profit. Thus, competition leads to a lower privacy and a lower payoff of the consumer. □

7 Conclusion

The key idea of this paper is that the more information consumers have revealed in the past, the less they care about their privacy *at the margin*. This decreasing marginal privacy cost implies that consumers become increasingly active users of the platform, and in the long-run, they behave as if there is no privacy cost . Attributing such a behavior to the absence of privacy cost could be misleading, because consumers might incur no privacy cost only at the margin. I show that stricter privacy policies could lower the long-run welfare of consumers. While competition could be an effective remedy, it requires platform's commitment power on future privacy levels. In contrast, enabling consumers to erase past information could benefit consumers.

Appendix

A Proof of Proposition 1

Proof. Define

$$U(v, \rho, \gamma, a) := u(a) - v\left(\sigma_0^2 - \frac{1}{\rho + \frac{1}{\frac{1}{a}+\gamma}}\right) \tag{12}$$

$$= u(a) - v\left(\sigma_0^2 - \frac{\frac{1}{a}+\gamma}{\rho\left(\frac{1}{a}+\gamma\right)+1}\right) \tag{13}$$

$$= u(a) - v\left(\sigma_0^2 - \frac{1+a\gamma}{\rho\left(1+\gamma a\right)+a}\right). \tag{14}$$

Given the conditional variance σ_{t-1}^2 at the beginning of period t, the consumer's problem in t is to choose $a \geq 0$ to maximize $U(v, \rho_t, \gamma, a)$, where $\rho_t := \frac{1}{\sigma_{t-1}^2}$.

$U(v, \rho, \gamma, a)$ is continuous in a. Also, any element of $\max_{a \geq 0} U(v, \rho, \gamma, a)$ is at most $a^* = \arg\max_{a \geq 0} u(a)$ because the privacy cost (i.e., the negative of the second term of $U(v, \rho, \gamma, a)$) is increasing in a. Thus, $A^*(v, \rho, \gamma) = \arg\max_{a \geq 0} U(v, \rho, \gamma, a)$ is non-empty and compact. Define $\bar{a}(v, \rho, \gamma) := \max A^*(v, \rho, \gamma)$. Note that

$$\begin{aligned}
\frac{\partial}{\partial a} U(v, \rho, \gamma, a) &= u'(a) + v \cdot \frac{\frac{-\frac{1}{a^2}}{\left(\frac{1}{a}+\gamma\right)^2}}{\left(\rho + \frac{1}{\frac{1}{a}+\gamma}\right)^2} \\
&= u'(a) + v \cdot \frac{-\frac{1}{a^2}}{\left(\rho\left(\frac{1}{a}+\gamma\right)+1\right)^2} \\
&= u'(a) - v \frac{1}{\left(\rho\left(1+\gamma a\right)+a\right)^2}
\end{aligned} \tag{15}$$

This expression is decreasing in v and increasing in ρ and γ. (It is ambiguous how it depends on a because both the marginal benefit and the marginal cost are decreasing.) Thus, $\bar{a}(v, \rho, \gamma)$ is decreasing in v and increasing in ρ and γ. Define $v^*(\gamma)$ as the cutoff value of privacy:

$$v^*(\gamma) = \inf\left\{v \in \mathbb{R} : \bar{a}\left(v, \frac{1}{\sigma_0^2}, \gamma\right) = 0\right\}. \tag{16}$$

The set in the right hand side is nonempty. Indeed, (15) and the concavity of $u(\cdot)$ imply that for any $a \in [0, a^*]$,

$$\frac{\partial}{\partial a} U(v, \rho, \gamma, a) \leq u'(0) - v \frac{1}{\left(\frac{1}{\sigma_0^2}\left(1+\gamma a^*\right)+a^*\right)^2}.$$

The right hand side of this inequality is negative for a sufficiently large v.

First, I show Point 1. Suppose $v < v^*$. $a_0^* = \bar{a}\left(v, \frac{1}{\sigma_0^2}, \gamma\right) = 0$ would contradict the definition of v^*. Thus, $a_0^* > 0$. Under a constant privacy policy, $\sigma_t^2(a_1, \ldots, a_t)$ is decreasing in t for any $(a_t)_{t \in \mathbb{N}}$. Thus, $\bar{a}(v, \frac{1}{\sigma_t^2(a_1^*, \ldots, a_t^*)}, \gamma)$ is increasing in t and greater than $a_0 > 0$. This implies that $\lim_{t \to \infty} \sigma_t^2(a_1^*, \ldots, a_t^*) = 0$, because

$$0 \leq \sigma_t^2(a_1^*, \ldots, a_t^*) \leq \frac{1}{\frac{1}{\sigma_0^2} + \frac{t}{\left(\frac{1}{a_0^*}+\gamma\right)}} \to 0.$$

This also implies $\lim_{t\to\infty} a_t^* \to a^*$. To show this, suppose to the contrary that $\lim_{t\to\infty} a_t^* = a' < a^*$. Then, $\lim_{t\to\infty} \frac{\partial}{\partial a} U\left(v, \frac{1}{\sigma_t^2}, \gamma, a\right)\Big|_{a=a'} = u'(a') > 0$. Since $u'(a) > 0$ if a is in a small neighbourhood of a', the consumer would choose some $a' > a_t^*$ for a sufficiently large t, which is a contradiction.

Second, I show Point 2. Suppose $v > v^*$. Then, the consumer chooses $a_0^* = 0$ in $t = 0$. Under a constant privacy policy, this implies that in $t = 1$, the consumer faces the identical problem of choosing a_1 as $t = 0$. Repeating this argument, we obtain $a_t^* = 0$ for all $t \in \mathbb{N}$. □

B Proof of Proposition 2

Proof. Suppose that the platform adopts the privacy policy $\boldsymbol{\gamma}^* = (\gamma_1^*, \gamma_2^*, \dots)$ defined in the proposition. I show that the consumer's equilibrium behavior induces $\sigma_t^2 = \frac{1}{\rho_t^*}$ for all $t \in \mathbb{N}$. The proof follows a simple induction. First, given $\rho_0^* = \frac{1}{\sigma_0^2}$ and γ_1^*, the myopic consumer chooses $a_1 = \bar{a}(\rho_0^*, \gamma_1^*)$ by construction. Given this activity level, the conditional variance of X at the end of $t = 1$ is given by $\frac{1}{\rho_1^*}$ by Bayes' rule. Second, suppose that at the beginning of t, $\sigma_{t-1}^2 = \frac{1}{\rho_{t-1}^*}$. Again, given γ_t^*, the consumer chooses $\bar{a}(\rho_{t-1}^*, \gamma_t^*)$, which leads to the conditional variance given by $\sigma_t^2 = \frac{1}{\rho_t^*}$ where $\rho_t^* = \rho_{t-1}^* + \frac{1}{\frac{1}{\bar{a}(\rho_{t-1}^*, \gamma_t^*)} + \gamma_t^*}$. This completes the proof of induction.

I show that $\boldsymbol{\gamma}^*$ is optimal for the platform. Take any privacy policy $\hat{\boldsymbol{\gamma}} := (\hat{\gamma}_1, \hat{\gamma}_2, \dots)$, and let $(\hat{\rho}_1, \hat{\rho}_2, \dots)$ denote the sequence of precisions induced by the consumer's equilibrium behavior. I show that $\rho_t^* \geq \hat{\rho}_t$ for all $t \in \mathbb{N}_+$ by induction. $\rho_0^* \geq \hat{\rho}_0$ holds with equality because both sides are equal to $\frac{1}{\sigma_0^2}$. Suppose that $\rho_{t-1}^* \geq \hat{\rho}_{t-1}$. Since $\bar{a}(\rho, \gamma)$ is increasing in ρ, $\bar{a}(\rho_{t-1}^*, \gamma) \geq \bar{a}(\hat{\rho}_{t-1}, \gamma)$. This, in turn, implies that

$$
\begin{aligned}
\rho_t^* &= \rho_{t-1}^* + \frac{1}{\arg\min_{\gamma \geq 0} \frac{1}{\bar{a}(\rho_{t-1}^*, \gamma)} + \gamma} \\
&\geq \hat{\rho}_{t-1} + \frac{1}{\frac{1}{\bar{a}(\hat{\rho}_{t-1}, \hat{\gamma}_t)} + \hat{\gamma}_t} \\
&= \hat{\rho}_t.
\end{aligned}
$$

Thus, we obtain $\rho_t^* \geq \hat{\rho}_t$ for all $t \in \mathbb{N}$, which implies that $\sigma_t^2(\boldsymbol{\gamma}^*) \leq \sigma_t^2(\hat{\boldsymbol{\gamma}})$ for all t, where $\sigma_t^2(\boldsymbol{\gamma})$ is the conditional variance of X at the end of period t given that the platform adopts a privacy policy $\boldsymbol{\gamma}$ and the consumer behaves optimally in each period. Therefore, for any discount factor δ_P, the platform's objective is maximized by $\boldsymbol{\gamma}^*$. $\square$

C Proof of Corollary 2

Proof. The first sentence follows from the proof of Proposition 2. The second part of the corollary follows from the fact that the optimal policy Proposition 2 does not depend on δ_P. Finally, I show that $\boldsymbol{\gamma}^*$ consists of an equilibrium when the platform chooses a privacy level in each period without commitment. Consider the following strategy profile: The consumer behaves according to $\bar{\alpha}(\rho, \gamma)$ on and off the paths of play. The platform chooses γ_t by solving (5) on and off the paths of play. Clearly, the resulting privacy levels are given by $\boldsymbol{\gamma}^*$ on path. To show that this is a subgame perfect equilibrium, suppose to the contrary that the platform can strictly increase its payoff by deviating to γ_t in some period t. Let $(\tilde{\sigma}_t^2, \tilde{\sigma}_{t+1}^2, \dots)$ and $(\hat{\sigma}_t^2, \hat{\sigma}_{t+1}^2, \dots)$ denote the sequences of conditional variances that would occur with and without the deviation, respectively. Then, the argument in the previous paragraph implies that $\hat{\sigma}_s^2 \le \tilde{\sigma}_s^2$ for all $s \ge t$. Thus, for any discount factor, the deviation cannot strictly increase the platform's payoff, which is a contradiction. $\square$

D Proof of Proposition 3

Proof. Suppose that the platform has chosen its optimal policy $\boldsymbol{\gamma}^*$. Let $(a_t^*)_t$ and $(\hat{\sigma}_t^2)_t$ denote the induced sequences of activity levels and precisions, respectively. First, I show $\lim_{t\to\infty} \hat{\sigma}_t^2 = 0$. Recall that the consumer's payoff in $t = 1$ is given by

$$U(v, \gamma_1, a) := u(a) - v\left(\sigma_0^2 - \frac{1}{\frac{1}{\sigma_0^2} + \frac{1}{\frac{1}{a} + \gamma_1}}\right).$$

$\lim_{\gamma_1\to\infty} U(v, \gamma_1, a^*) = u(a^*) > 0$. Thus, for a sufficiently large γ_1, the consumer chooses $a_1 > 0$. This implies that there is a γ^* such that if the platform chooses γ^*-constant privacy policy, then the induced sequence of conditional variances σ_t^2 satisfies $\lim_{t\to\infty} \sigma_t^2 = 0$. Then, $\lim_{t\to\infty} \hat{\sigma}_t^2 = 0$ holds because $\hat{\sigma}_t^2 \le \sigma_t^2$ for all t.

To show $\lim_{t\to\infty} a_t^* = a^*$, suppose to the contrary that there is a $\lambda > 0$ such that $a_t^* \le a^* - \lambda$ for infinitely many t's. Without loss of generality, suppose $a_t^* \le a^* - \lambda$ for all t. Than, we can take a convergent subsequence of (a_t^*) such that $\lim_{t\to\infty} a_t^* \le a^* - \lambda$. We can then apply the "proof-by-contradiction" part of $\lim_{t\to\infty} a_t^* = a^*$ in the proof of Proposition 1.

Finally, suppose to the contrary that there is a $\underline{\gamma} > 0$ such that $\gamma_t^* \geq \underline{\gamma}$ for infinitely many t's. To simplify exposition, suppose without loss of generality that $\gamma_t^* \geq \underline{\gamma}$ for all $t \geq 2$. Then, the minimized value in (5) converges to $\frac{1}{a^*} + \underline{\gamma}$. To show a contradiction, let T denote the first period such that $\bar{a}(\frac{1}{\sigma_{T-1}^2}, 0) > 0$. Then, $\bar{a}(\frac{1}{\sigma_{t-1}^2}, 0) > 0$ for any $t \geq T$. If the platform chooses $\gamma_t = 0$ instead of γ_t^* in period $t \geq T$, the minimized value is $\frac{1}{\bar{a}(\rho_{t-1}^*, 0)}$, which converges to $\frac{1}{a^*} < \frac{1}{a^*} + \underline{\gamma}$ for a sufficiently large a^*. This implies that for a sufficiently large t, the platform can strictly increase its payoff in period t by setting $\gamma_t = 0$, which is a contradiction. □

E Proof of Proposition 4

Proof. Consider competition with commitment. Below, I provide a proof that applies to any discount factors of the consumer. Suppose that both platforms commit to $\gamma_t^k = \bar{\gamma}$ for all t. Consider the following strategy of the consumer. On the equilibrium path, the consumer chooses platform A in all periods, i.e., $a_t^B = 0$ for all t. (Alternatively, the consumer can randomize between two platforms before $t = 1$, but cannot change which platform to use thereafter.) If platform k unilaterally deviates and chooses a different privacy policy, then the consumer chooses platform $-k$ in all $t \in \mathbb{N}$. In either case, the consumer chooses an activity level a_t^k optimally. This is an equilibrium. First, no platform can profitably deviate because the deviation leads to zero payoff. Second, consider the consumer's incentive after a unilateral deviation of (say) platform A. To show that it is optimal for the consumer to use platform B in any period $t \geq 1$, take any sequence of activity levels $(a_t^A, a_t^B)_{t\in\mathbb{N}}$. I show that the consumer is weakly better off under $(0, \max(a_t^A, a_t^B))_{t\in\mathbb{N}}$ than under $(a_t^A, a_t^B)_{t\in\mathbb{N}}$. First, the consumer obtains a weakly greater per-period payoff in $t = 1$ under $(0, \max(a_1^A, a_1^B))$, because platform B offers a weakly greater privacy level. Second, take $t = 2$. The change from (a_2^A, a_2^B) to $(0, \max(a_2^A, a_2^B))$ does not affect the consumer's gross utility $u(a)$ in period 2. Moreover, this change reduces the increment of the consumer's privacy cost in period 2, because the marginal cost of using platform k is increasing in $\sigma_{1,k}^2$, and $\sigma_{1,A}^2 \geq \sigma_{1,B}^2$ given that the consumer chose $(0, \max(a_1^A, a_1^B))$. Thus, the consumer is weakly better off under $(0, \max(a_2^A, a_2^B))$ in period 2. Thus, regardless of the consumer's discount factor, she never chooses platform A after its deviation.

Consider competition without commitment (here, I assume the myopic consumer). Suppose to the contrary that there is an equilibrium such that both platforms choose $\bar{\gamma}$ in all periods on the equilibrium path. By the same argument as competition *with* commitment, the myopic consumer chooses the same platform (say A) in all periods. Now, consider the following (t, λ)-*deviation* of platform A: In any period $s \neq t$, platform A chooses $\bar{\gamma}$, but in period t, it chooses $\bar{\gamma} - \lambda$. First, without deviation, the consumer's activity level on platform A converges to $a^* > 0$. Second, since the consumer's marginal privacy cost on platform k is decreasing in $\sigma^2_{t,k}$, the following holds: There exist $\bar{t} \in \mathbb{N}$ and $\bar{\lambda} > 0$ such that for any $t \geq \bar{t}$ and $\lambda \in (0, \bar{\lambda})$, if platform A adopts the (t, λ)-deviation, then the consumer still uses platform A in all periods. Third, there is a (t, λ) such that the (t, λ)-deviation strictly increases platform A's payoff for any period $s \geq t$. Indeed, without deviation, the variance of the joint noise term $\varepsilon^A_t + z^A_t$ converges to $\frac{1}{a^*} + \bar{\gamma}$. In contrast, if platform A adopts the (t, λ)-deviation, then the variance of the joint noise term in period t is equal to $\frac{1}{\bar{a}(\sigma^2_{t-1,A}, \bar{\gamma}-\lambda)} + \bar{\gamma} - \lambda$ for a large t. Note that conditional on that the consumer uses platform A in t, her activity level is given by $\bar{a}(\sigma^2_{t-1,A}, \bar{\gamma} - \lambda)$. Since $\bar{a}(\sigma^2_{t-1,A}, \bar{\gamma} - \lambda) \to a^*$, it holds that $\frac{1}{\bar{a}(\sigma^2_{t-1,A}, \bar{\gamma}-\lambda)} + \bar{\gamma} - \lambda \to \frac{1}{a^*} + \bar{\gamma} - \lambda$. Thus, we can conclude that there is a (t, λ) such that if platform A adopts the (t, λ)-deviation, then the variance of the joint noise term in period t becomes strictly smaller than without deviation. This benefits platform A in not only period t but all periods $s \geq t$, because the consumer's activity level in period $s \geq t$ is increasing in $\frac{1}{\sigma^2_{t,A}}$.

Finally, I prove the second part of Point 2. Consider the following equilibrium. In period t, the consumer chooses platform $k^* \in \arg\min_{k=A,B} \sigma^2_{t-1,k}$ given the conditional variance of each platform at the beginning of period t. If the consumer is indifferent between two platforms in $t = 1$, then she breaks a tie in favor of (say) platform A. If the consumer is indifferent in period $t \geq 2$, then she breaks a tie in favor of the platform that she chose most recently. Platform $-k^*$ chooses a privacy level $\bar{\gamma}$. Platform k^* chooses a privacy level γ^*, which solves

$$
\begin{aligned}
&\min_{\gamma} \frac{1}{\bar{a}(\frac{1}{\sigma^2_{t-1,k^*}}, \gamma)} + \gamma \\
\text{s.t.} \quad &\arg\max_{a \geq 0} u(a) - v[\sigma^2_{t-1,k^*} - \sigma^2_{t,k^*}(\gamma, a|\sigma^2_{t-1,k^*})] \\
&\geq \arg\max_{a \geq 0} u(a) - v[\sigma^2_{t-1,-k^*} - \sigma^2_{t,-k^*}(\bar{\gamma}, a|\sigma^2_{t-1,-k^*})],
\end{aligned} \tag{17}
$$

where $\sigma^2_{t,k}(\gamma, a|\sigma^2_{t-1,k})$ represents the platform k's information given its privacy level, the consumer's activity level, and the conditional variance in the previous period. The set of γ's that satisfy the constraint (17) is non-empty, because $\gamma = \bar{\gamma}$ satisfies it. I show that this strategy profile is an equilibrium. First, it is optimal for the consumer to choose platform $k^* \in \arg\min_{k=A,B} \sigma^2_{t-1,k}$. Indeed, if platform k^* solves the above minimization problem, then its constraint implies that

$$\begin{aligned}&\arg\max_a u(a) - v[\sigma_0^2 - \sigma^2_{t,k^*}(\gamma^*, a|\sigma^2_{t-1,k^*})] - v[\sigma_0^2 - \sigma^2_{t-1,-k^*}]\\ \geq &\arg\max_a u(a) - v[\sigma_0^2 - \sigma^2_{t,-k^*}(\bar{\gamma}, a|\sigma^2_{t-1,-k^*})] - v[\sigma_0^2 - \sigma^2_{t-1,k^*}],\end{aligned}$$

This implies that it is optimal for the consumer to choose $a_t^{k^*} = \bar{a}(\frac{1}{\sigma^2_{t-1,k^*}}, \gamma^*)$ and $a_t^{-k^*} = 0$. Second, it is optimal for platform $-k^*$ to choose $\bar{\gamma}$. Indeed, the consumer never uses platform $-k^*$ after period t as long as platform k^* follows the above strategy, and thus any privacy level is optimal.

Next, I show that the strategy of platform k^* is optimal. I consider two cases. First, suppose that platform k^* deviates and chooses γ', and as a result, (17) is violated. Then, in that period, the consumer chooses platform $-k^*$. If $\sigma^2_{t,k^*} \geq \sigma^2_{t,-k^*}$, then the consumer never chooses platform k^*, and thus the deviation cannot be strictly profitable. If $\sigma^2_{t,k^*} < \sigma^2_{t,-k^*}$, then the consumer chooses platform k^* from period $t+1$ on. Let $\gamma'_{t+1}, \gamma'_{t+2}, \dots$ denote the sequence of privacy levels of platform k^* after the deviation. Then, platform k^* could choose $\gamma_t = \gamma'_{t+1}$, $\gamma_{t+1} = \gamma'_{t+2}, \dots$ to increase its payoff. This is a contradiction. Second, suppose that platform k^* deviates and chooses γ', but (17) holds. This simply reduces the accuracy of the signal in period t, i.e., $\frac{1}{\bar{a}\left(\frac{1}{\sigma^2_{t-1,k^*}}, \gamma\right)} + \gamma$, and reduces the consumer's activity levels in all future periods.

Finally, if all players follow the above strategies, then the consumer uses platform A in all periods, and platform A offers privacy levels γ_t^A converging to 0. We can prove this by contradiction, following the same argument as the case of a monopoly platform. □

References

Acemoglu, Daron, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar (2019), "Too much data: Prices and inefficiencies in data markets." Working Paper 26296, National Bureau of Economic Research.

Becker, Gary S and Kevin M Murphy (1988), "A theory of rational addiction." *Journal of political Economy*, 96, 675–700.

Bergemann, Dirk, Alessandro Bonatti, and Tan Gan (2019), "The economics of social data." Cowles foundation discussion paper 2203, Yale University.

Bonatti, Alessandro and Gonzalo Cisternas (Forthcoming), "Consumer scores and price discrimination." *Review of Economic Studies*.

Chiou, Lesley and Catherine Tucker (2017), "Search engines and data retention: Implications for privacy and antitrust." Technical report, National Bureau of Economic Research.

Fainmesser, Itay P, Andrea Galeotti, and Ruslan Momot (2019), "Digital privacy." *Available at SSRN*.

Prufer, Jens and Christoph Schottmüller (2017), "Competing with big data." *Tilburg Law School Research Paper*.

Srinivasan, Dina (2019), "The antitrust case against facebook: A monopolist's journey towards pervasive surveillance in spite of consumers' preference for privacy." *Berkeley Business Law Journal*, 16, 39.

Tucker, Catherine (2018), "Privacy, algorithms, and artificial intelligence." *The Economics of Artificial Intelligence: An Agenda*.