2009基礎

大問１

(1)$0 < x < 1$の時

$log(1-x) < 0 \therefore |log(1-x)| = -log(1-x)$

$f(x) = \frac{x}{1-x} + log(1-x)$とおく

$f^{'}(x) = \frac{1}{(x-1)^2} + \frac{1}{x-1} = \frac{1}{(x-1)^2}(1-(1-x)) = \frac{x}{(1-x)^2}$

$\therefore 0 = f(0) \leq f(x) \forall x \in (0,1)$

$\Rightarrow -log(1-x) \leq \frac{x}{1-x}$

(2)$logb_n = \Sigma_{i=1}^{n} log(1-a_n)$

- $log(1-x) \leq -x$
  $\because f(x) = -x - log(1-x)$について$f^{'}(x) = -1 + \frac{1}{1-x} > 0 \Rightarrow 0 = f(0) \leq f(x) (\forall x \in (0,1))$

$\therefore logb_n \leq \Sigma_{i=1}^{n} - a_i \leq -\Sigma_{i=1}^{n} \frac{1}{2i+1} \leq -\Sigma_{i=1}^{n} \frac{1}{2i+2} \rightarrow -\infty (as \quad n \rightarrow \infty)$

よって、$logb_n \rightarrow -\infty \quad as \quad n \rightarrow \infty \Leftrightarrow b_n \rightarrow 0 (as \quad n \rightarrow \infty)$

(3)

$\Sigma_{n=1}^{\infty} a_n$が収束$\Leftrightarrow \lim_{n \rightarrow \infty} a_n = 0$

$\therefore \exists N \in \mathbb{N} \quad s.t. \quad \forall n \geq N \quad |a_n| < \frac{1}{2}$

$$\begin{aligned}
|logb_n| &= \Sigma_{i=1}^{n} |log(1-a_n)| \\
&\leq \Sigma_{i=1}^{n} \frac{a_i}{1-a_i} \\
&\leq \Sigma_{i=1}^{N} \frac{a_i}{1-a_i} + \Sigma_{i=N+1}^{n} \frac{a_i}{1-a_i} \\
&\leq \Sigma_{i=1}^{N} \frac{a_i}{1-a_i} + \frac{1}{2} \Sigma_{i=N+1}^{n} a_i
\end{aligned}$$

大問２

(1)

$x = \frac{u+v}{2}, y = \frac{u-v}{2}$と変数変換すると、$u = x+y, v = x-y$より、

$$\begin{aligned}
|J| &= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} \\
&= |-\frac{1}{4} - \frac{1}{4}| \\
&= \frac{1}{2}
\end{aligned}$$

$|x-y| = |v|, x^2 + y^2 = \frac{1}{2}(u^2 + v^2)$

$$\begin{aligned}
\therefore (与式) &= \iint_{\mathbb{R}^2} |v|^{2n-1} e^{-\frac{1}{2}(u^2+v^2)} \frac{1}{2} dudv \\
&= (\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du)(\int_{0}^{\infty} v^{2n-1} e^{\frac{v^2}{2}} dv) \\
&= \sqrt{2\pi} 2^{n-1} (n-1)!
\end{aligned}$$

大問３

(1)

$$
\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 2-\lambda & 2 & -3 \\ 2 & 5-\lambda & -6 \\ 2 & 4 & -5-\lambda \end{vmatrix} \\
&= \begin{vmatrix} 1-\lambda & 2 & -3 \\ 1-\lambda & 5-\lambda & -6 \\ 1-\lambda & 4 & -5-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -\lambda & -3 \\ -\lambda & -2-\lambda \end{vmatrix} \\
&= -\lambda(1-\lambda)^2
\end{aligned}
$$

$$\therefore \text{固有値} 0, 1$$

$$W_{A,0} = KerA = Ker\begin{pmatrix} 2 & 2 & -3 \\ 0 & 3 & -3 \\ 0 & 2 & -2 \end{pmatrix} = \langle\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\rangle$$

$$W_{A,1} = Ker(A-I) = Ker\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 2 & 4 & -6 \end{pmatrix} = \langle\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\rangle$$

(2)

$P = \begin{pmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$とすると$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$となる

(3)

- 対角行列同士の行列の積は可換なので、
  $(Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ)$
  $\Leftrightarrow Q^{-1}ABQ = Q^{-1}BAQ$
  $\Rightarrow AB = BA$
- AB, BAの$(1,3)$成分はそれぞれ、$2-3b, -7 \quad \therefore b = 3$
- (2)のPについて

$$
\begin{aligned}
BP &= \left[B\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, B\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, B\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right] \\
&= \left[\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}\right] \\
&= \left[-\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, 2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right] \\
&= P\begin{pmatrix} 0 & 2 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
\therefore P^{-1}BP &= \begin{pmatrix} 0 & 2 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\end{aligned}
$$

$C = \begin{pmatrix} 0 & 2 \\ -1 & 5 \end{pmatrix}$について$R^1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$とすると$(R^1)^{-1}CR^1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$R = \begin{pmatrix} R^1 & 0 \\ 0 & 1 \end{pmatrix}$として$Q = PR$とすると

$Q^{-1}BQ = R^{-1}P^{-1}BPR = R^{-1}\begin{pmatrix} 0 & 2 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$Q^{1}AQ = R^{-1}P^{-1}APR = R^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}R = \begin{pmatrix}R^{1}&0\\0&1\end{pmatrix}^{-1}\begin{pmatrix}I&0\\0&1\end{pmatrix}\begin{pmatrix}R^{1}&0\\0&1\end{pmatrix} = \begin{pmatrix}R^{1}&0\\0&1\end{pmatrix}\begin{pmatrix}R^{1}&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$B = \begin{pmatrix}2&1&-1\\0&0&2\\0&-1&2\end{pmatrix}$$

$$\begin{aligned}|B-\lambda I| &= \begin{vmatrix}2-\lambda&1&-1\\0&-\lambda&2\\0&-1&3-\lambda\end{vmatrix}\\ &= (2-\lambda)(\lambda^2-3\lambda+2)\\ &= (2-\lambda)^2(\lambda-1)\end{aligned}$$

$$\begin{aligned}W_{B,1} &= Ker\begin{pmatrix}1&0&-1\\0&-1&2\\0&-1&2\end{pmatrix}\\ &= \langle\begin{pmatrix}-1\\2\\1\end{pmatrix}\rangle\\ W_{B,2} &= Ker\begin{pmatrix}0&1&-1\\0&-2&2\\0&-1&1\end{pmatrix}\\ &= \langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\rangle\end{aligned}$$

大問４

(1)

$V = \langle\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix}\rangle$dimV=3, 基底は左の三つ

また、

$A\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}1&0\\0&0\end{pmatrix}A = \begin{pmatrix}0&0\\-1&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$

$A\begin{pmatrix}0&0\\0&1\end{pmatrix}-\begin{pmatrix}0&0\\0&1\end{pmatrix}A = \begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\-1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

$A\begin{pmatrix}0&1\\1&0\end{pmatrix}-\begin{pmatrix}0&1\\1&0\end{pmatrix}A = \begin{pmatrix}1&0\\0&-1\end{pmatrix}-\begin{pmatrix}-1&0\\0&1\end{pmatrix} = \begin{pmatrix}2&0\\0&-2\end{pmatrix}$

よって$\forall B = \lambda_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+\lambda_2\begin{pmatrix}0&0\\0&1\end{pmatrix}+\lambda_3\begin{pmatrix}0&1\\1&0\end{pmatrix}\in V$について、

$F(B) = \begin{pmatrix}2\lambda_3&-\lambda_1+\lambda_2\\-\lambda_1+\lambda_2&-2\lambda_3\end{pmatrix}$

(2)

$ImF = \langle\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\rangle$

$kerF = \langle\begin{pmatrix}1&0\\0&1\end{pmatrix}\rangle$

(3)

$V = \begin{pmatrix}a&b\\b&c\end{pmatrix}\simeq\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{C}^3$

$F(V) \simeq C(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = \begin{pmatrix} 2b \\ -a+c \\ -2b \end{pmatrix}$

Cを$\mathbb{R}$の標準基底で行列化すると$C = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}$

$det(C - \lambda I) = -\lambda^3 - 2\lambda - 2\lambda = -\lambda(\lambda^2 + 4)$

固有値$0, 2i, -2i$

$W_{C,0} = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle \to W_{F,0} = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$

$W_{C,2i} = ker(C - 2iI) = ker \begin{pmatrix} -2i & 2 & 0 \\ -1 & -2i & 1 \\ 0 & -2 & -2i \end{pmatrix} = \langle \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \rangle \to W_{F,2i} = \langle \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \rangle$

$W_{C,-2i} = ker(C + 2iI) = ker \begin{pmatrix} 2i & 2 & 0 \\ -1 & 2i & 1 \\ 0 & -2 & 2i \end{pmatrix} = \langle \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix} \rangle \to W_{F,2i} = \langle \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \rangle$

大問５

$z = f(x, y)$より

$x^2 + y^2 + f(x,y)^2 - af(x,y) + 2af(x,y) = 4a + 2$

両辺xで偏微分して

$2x + 3f(x,y)^2 f_x - ayf(x,y) - ayf_x + 2a = 0$

$(x, y) = (1, 1)$を代入すると$a = -1$

両辺yで変微分し$(x, y) = (1, 1)$を代入して$a = -1$が得られる

$f_{xx}(1,1) = -2, f_{yy}(1,1) = -2, f_{yx}(1,1) = f_{xy}(1,1) = 0$

よりヘッセ行列$H_f(1,1) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$detH_f(1,1) = 4 > 0, trH_f(1,1) = -4 < 0$より$H_f$は負定値

よって$(1, 1)$は極大値になる