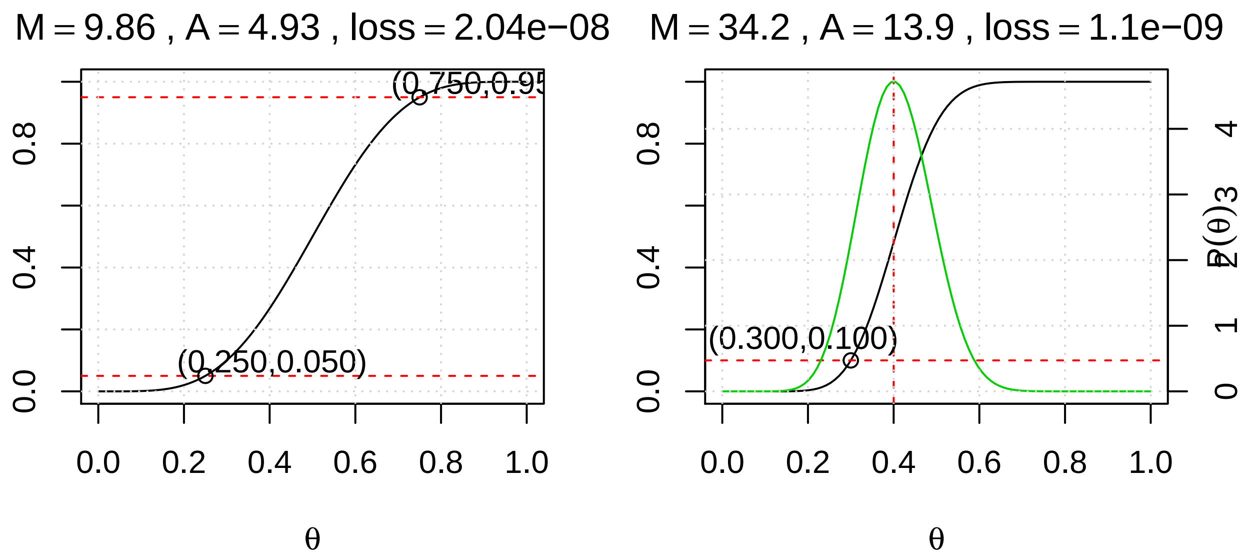

Figure 1: The shape of the beta distribution constrianed under the respective conditions. Left: Constraint 1. Right: Constraint 2. (green: PDF, black: CDF)

Contents

0.1 Finding priors for the beta distribution

Beta distribution is the natural prior for a binomial/bernoulli distribution. Considering distribution of a binomial variable $X \mid \theta \sim Binom(\theta)$, in order to make its marginalisd distribution $P(X) = \int P(X \mid \theta)P(\theta).d\theta$ analytically tractable, one of the choice is to assume $\theta \sim Beta(M, \alpha)$, so that:

$$
\begin{aligned}
P(\theta) &= \frac{x^{\alpha-1}(1-x)^{M-\alpha-1}}{B(\alpha, M-\alpha)} \\
E(\theta) &= \frac{\alpha}{M}
\end{aligned}
$$

0.1.1 Constraint 1:

$$
\begin{aligned}
P(\theta \leq 0.25) = P(\theta \geq 0.75) &= 0.05 \\
P(\theta \leq 0.75) &= 0.95
\end{aligned}
$$

Fitted result: $\theta \sim$ Beta(4.933, 4.932)

0.1.2 Constraint 2

$$
\begin{aligned}
\text{argmax}_{\theta}(P(\theta)) &= 0.4 \\
P(\theta \leq 0.3) &= 0.1
\end{aligned}
$$

Fitted result: $\theta \sim$ Beta(13.863, 20.295)

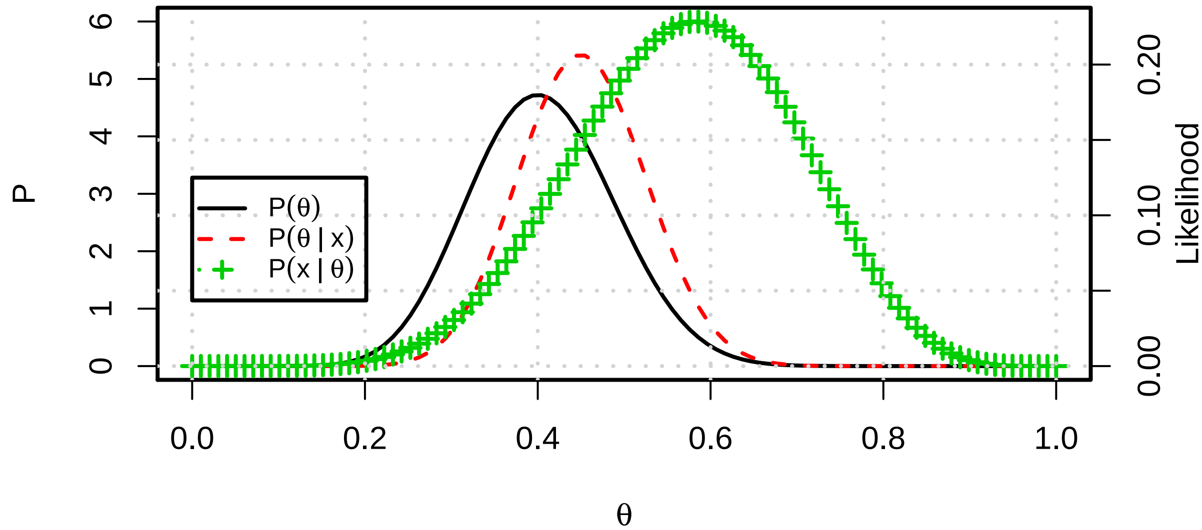

Figure 2: Inference of posterior distribution on parameter θ given mutatble sequence: 011100101101

0.1.3 Basics for Bayesian inference

$$\text{likelihood} : f(\theta) = P(x \mid \theta)$$
$$\text{prior} : f(\theta) = P(\theta)$$
$$\text{posterior} : f(\theta) = P(\theta \mid x) = \frac{P(x \mid \theta)P(\theta)}{P(x)}$$
$$\text{marginal likelihood} : P(x) = \int P(x \mid \theta)P(\theta).d\theta$$

The observed toss sequence is: 011100101101

Assume each coin toss is independent from each other, the likelihood of an observed sequence is only dependent on the total number of heads and not the sequence it occurred in. Denoting the coin toss as a sequence $\{x_i\}$ where $x_i \in \{0, 1\}$, we have

$$\#\text{head} = \mathbb{1}\{x_i = 1\}$$
$$\#\text{head} \sim Binom(|\{x_i\}|, \theta)$$

Combining with the prior $\theta \sim \text{Beta}(13.863, 20.295\ )$, I calculated the marginal likelihood numerically to be $P(x) = 0.11$ and derived the posterior distribution accordingly (see figure 2). MLE is obtained at $\hat{\theta} = 0.45$