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Assignment - 3

Calculate MAP estimation of Gaussian (μ, σ) where prior distribution of μ, σ is two diff. gaussian

Let the data $X = \{x_1, x_2, \dots, x_n\}$

likelihood $p(X|\mu,\sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)$

let our independent Gaussian prior for μ and σ be

$$p(\mu) = \frac{1}{\sqrt{2\pi}\sigma_\mu} \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_\mu^2}\right)$$

$$p(\sigma) = \frac{1}{\sqrt{2\pi}\sigma_\sigma} \exp\left(-\frac{(\sigma-\mu_\sigma)^2}{2\sigma_\sigma^2}\right)$$

The log-posterior is given by

$$p(\mu,\sigma|X) \propto p(X|\mu,\sigma)\, p(\mu)\, p(\sigma)$$

Taking logarithmic & dropping const. terms, the log-posterior becomes

$$L(\mu,\sigma) = \log p(X|\mu,\sigma) + \log p(\mu) + \log p(\sigma)$$

$$= -n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{1}{2\sigma_\mu^2}(\mu-\mu_0)^2 - \frac{1}{2\sigma_\sigma^2}(\sigma-\cancel{\mu_\sigma})^2$$

For μ

Differentiating L with r.t μ:

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) - \frac{(\mu-\mu_0)}{\sigma_\mu^2}$$

setting this derivative to zero:

$$\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right) - \frac{(\mu-\mu_0)}{\sigma_\mu^2} = 0$$

$$\frac{n\mu}{\sigma^2} + \frac{\mu}{\sigma_\mu^2} = \frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu_0}{\sigma_\mu^2}$$

For σ

Differentiate L with respect to σ:

$$\frac{\partial L}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{(\sigma-\mu_\sigma)}{\sigma_\sigma^2}$$

Setting the derivative to zero,

$$-\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3} - \frac{(\sigma-\mu_\sigma)}{\sigma_\sigma^2} = 0$$

$$-n\sigma^2 + \sum_{i=1}^{n}(x_i-\mu)^2 - \frac{\sigma^3(\sigma-\mu_\sigma)}{\sigma_\sigma^2} = 0$$

We obtained the implicit equation eq^n for σ:

$$\frac{1}{\sigma_\sigma^2}\sigma^4 - \frac{\mu_\sigma\sigma^3}{\sigma_\sigma^2} + n\sigma^2 - \sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$\therefore$ MAP estimates

For μ : $$\mu_{MAP} = \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{1}{\sigma_\mu^2}\mu_0}{\frac{n}{\sigma^2} + \frac{1}{\sigma_\mu^2}}$$

where $\sigma = \sigma_{MAP}$

For σ :

$$\frac{1}{\sigma_\sigma^2}\sigma^4 - \frac{\mu_\sigma}{\sigma_\sigma^2}\sigma^3 + n\sigma^2 - \sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

for $\sigma = \sigma_{MAP}$