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Event-triggered second-moment stabilisation under action-dependent Markov packet drops

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Abstract
This paper considers the problem of the second-moment stabilisation of a scalar linear plant with process noise. It is assumed that the sensor communicates with the controller over an unreliable channel, whose state evolves according to a Markov chain, with the transition matrix on a timestep depending on whether there is a transmission on that timestep. Under such a setting, an event-triggered transmission policy is proposed which meets the objective of exponential convergence of the second moment of the plant state to an ultimate bound. Further, upper bounds on the transmission fraction of the proposed policy are provided. The results are illustrated through an example scenario of control in the presence of a battery-equipped energy-harvesting sensor. The proposed control design as well as the analytical guarantees are verified through simulations for the example scenario.

1 | INTRODUCTION

In the literature, the problem of control over time-varying action-dependent channels has been understudied. This paper addresses this gap using the approach of event triggering for controlling a scalar linear system over an unreliable action-dependent Markov channel.

1.1 | Literature review

Last two decades have seen extensive research on various issues and design methods in networked control systems (NCS) [1–4]. One such area is event-triggered control [5–9], which has been applied in numerous contexts for various control goals. However, the volume of work on event-triggered control in a stochastic setting is still not as considerable as in the deterministic setting. Some early works in the stochastic setting include [10–13]. Several papers that consider event-triggered transmissions under stochastic packet drops exist in the context of estimation [14], linear quadratic Gaussian (LQG) control [15–17], non-linear systems [18], multi-loop control of linear systems [19, 20] and stabilisation [21–23]. However, these works consider only independent and identically distributed (i.i.d.) packet drops. An exception in the works on event-triggered control is our previous paper [24], which considers Markov packet drops.

Even in the literature on NCS, a very common assumption is that the packet drops are i.i.d. across time. However, in order to better capture time-correlation effects in networks, there has been recent consideration of packet-drop probabilities evolving according to a Markov chain. Some recent works considering Markov packet drops include stability of Kalman filtering over networks [25, 26], channel selection for control of multi-loop non-linear systems [27], and mean-square stabilisation with quantised feedback [28, 29]. Beyond packet drops, some other works on NCS with Markovian channels include [30] for Kalman filtering with Markov inter-reception times, control under Markov missing data [31], mean-square stabilisation with the channel data rate evolving as a Markov chain [32] and over a noisy fading channel where the evolution of fading gain is Markovian [33, 34] as well as in the context of control over vehicular ad hoc networks [35, 36] (see also references therein).

In the literature on communication systems, Markov models for channels have a long history, starting with the works of Gilbert [37] and Elliott [38]. Reference [39] is a relatively recent survey on Markov modelling of fading channels. Channels whose properties depend on past actions also serve as useful models for communication systems as well as for other applications. Some examples in the communication literature include [40], which considers streaming in buffer-enabled wireless networks, and [41], which is on communication in underwater acoustic channels. Action-dependent Markov processes also model systems of other communication channels. Reference [42] is a recent survey on models and research works on systems whose operation depends on a 'utilisation-dependent component' such as queueing in action-dependent servers [43], iterative learning algorithms and systems with energy-harvesting (EH) components, among other applications. Reference [44] considers a communication system powered by an EH battery, modelled as an action-dependent Markov channel. This model shares significant conceptual commonality with the model we use for simulations in Section 6.

1.2 | Contributions

The major contributions of this paper are as follows.

- We consider the problem of second-moment stabilisation over a channel with action-dependent Markov packet drops. To the best of our knowledge, such channels have not been considered before in the context of NCS. For example, the works [28, 29] consider Markov packet drops without dependence on past transmission actions. We provide a necessary condition on the plant dynamics and the channel parameters for our transmission policy to achieve the control objective. This necessary condition is similar to the conditions often found in the data rate limited control [45] and NCS in general.
- The proposed event-triggered transmission policy is similar in spirit to our earlier work [21, 24]. However, [21] consider only i.i.d. Bernoulli packet drops and [24] consider Markov packet drops. In contrast, here we consider action-dependent Markov packet drops, which results in a coupling of the evolution of the plant and channel states. This aspect makes the analysis necessary for providing theoretical guarantees on performance significantly more challenging. In particular, the two main analytical contributions in this part are theoretical guarantee on the second-moment stability and an upper bound on the fraction of timesteps, over a time horizon, on which a transmission occurs under the event-triggered policy.
- We model the problem of control with a battery-equipped EH sensor using the proposed action-based Markov channel framework and illustrate our proposed event-triggered policy and results through simulations. This example also demonstrates the wider applicability of our model, beyond the problem of control over wireless communication channels.

1.3 | Notation

We let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of real numbers, integers, natural numbers and non-negative integers, respectively. We use the standard font for scalar quantities while boldface for vectors and matrices. The notations $\mathbf{1}$, $\boldsymbol{\delta}_i$ and $\mathbf{I}$ denote the vector with all 1s, the vector whose ith entry takes the values 1 and 0 everywhere else, and the identity matrix, respectively, of appropriate dimensions. We use $\rho(\mathbf{A})$ to denote the spectral radius of a real square matrix $\mathbf{A}$. We denote the space of probability vectors (i.e. vectors with non-negative entries that sum to 1) of n dimensions as $\mathbb{P}^n$. The notation $\Pr[\cdot]$ denotes the probability of an event. We denote a generic transmission policy using $\mathcal{T}$, and $\mathbb{E}_{\mathcal{T}}[\,]$ represents expectation of a random variable under a given transmission policy $\mathcal{T}$. We denote the cardinality of a finite set $\mathcal{S}$ as $|\mathcal{S}|$. For integers a and b, we let that $[a,b]_{\mathbb{Z}}$, $(a,b)_{\mathbb{Z}}$ and $(a,b]_{\mathbb{Z}}$ represent the finite sets $[a,b] \cap \mathbb{Z}$, $(a,b) \cap \mathbb{Z}$ and $(a,b] \cap \mathbb{Z}$, respectively. For random variables X, Y and Z, the *tower property of conditional expectation* is

$$\mathbb{E}[\,\mathbb{E}[X \mid Y, Z] \mid Y\,] = \mathbb{E}[X \mid Y].$$

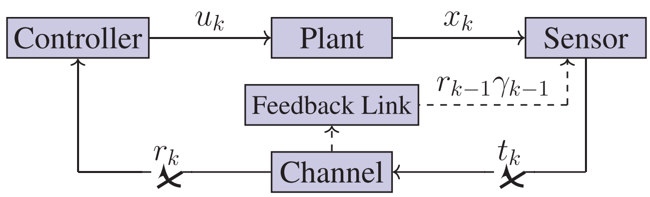

FIGURE 1 Schematic of the system under consideration

2 | SYSTEM DESCRIPTION

In this section, we describe the plant, channel, controller and the control objective. A schematic of the system is provided in Figure 1.

2.1 | Plant and controller model

Consider a scalar linear plant with process noise

$$x_{k+1} = a x_k + u_k + v_k, \quad x_k, u_k, v_k \in \mathbb{R}, \ \forall k \in \mathbb{N}_0. \tag{1}$$

The parameter a is the inherent gain of the plant, which we assume is unstable, that is, $|a| > 1$. The variables x_k, u_k and v_k are the plant state, the control input and the process noise, respectively, at timestep $k \in \mathbb{N}_0$. We assume that v_k is i.i.d. across timesteps k and independent of all the other system variables. Its distribution has zero mean and finite variance, that is, $\mathbb{E}[v_k] = 0$, $\mathbb{E}[v_k^2] =: M < \infty$.

At each timestep, a sensor perfectly measures the plant state and decides on whether to transmit a packet to the controller. The sensor's transmission decision on timestep k is t_k, where

$$t_k := \begin{cases} 1, & \text{if sensor transmits at } k \\ 0, & \text{if sensor does not transmit at } k. \end{cases}$$

The sensor determines t_k at each timestep k according to an event-triggered *transmission policy* on the basis of plant state and all the information available on timestep k. Even if the sensor transmits a packet at timestep k ($t_k = 1$), the packet may be dropped by the communication channel according to a packet-drop model which we describe in Section 2.2. We let r_k be the reception indicator:

$$r_k := \begin{cases} 1, & \text{if } t_k = 1 \text{ and packet received} \\ 0, & \text{if } t_k = 1 \text{ and packet dropped} \\ 0, & \text{if } t_k = 0. \end{cases}$$

The controller uses the *controller state*, $\hat{x}_k^+$, to generate the input $u_k := L\hat{x}_k^+$, where L is such that $\bar{a} := (a + L) \in (-1, 1)$. The controller state $\hat{x}_k^+$ itself evolves as

$$\hat{x}_k^+ = \begin{cases} x_k, & \text{if } r_k = 1 \\ \hat{x}_k, & \text{if } r_k = 0, \end{cases} \tag{2}$$

where $\hat{x}_k := \bar{a}\hat{x}_{k-1}^+$ is the *estimate* of the plant state given past data. Corresponding to the controller state and plant state estimates, we define the *estimation error* z_k and *controller state error* z_k^+ as

$$z_k := x_k - \hat{x}_k, \quad z_k^+ := x_k - \hat{x}_k^+. \tag{3}$$

The two quantities differ only on successful reception times. It is possible to write the plant state evolution as

$$x_{k+1} = ax_k + L\hat{x}_k^+ + v_k = \bar{a}x_k - Lz_k^+ + v_k, \tag{4a}$$

$$\hat{x}_{k+1} = \bar{a}\hat{x}_k^+. \tag{4b}$$

Equations (2)–(4) compositely describe the evolution of the plant state, controller state and the estimate of plant state.

2.2 | Channel model

We model the communication channel as an action-dependent *finite state–space Markov channel*. The channel can be in one among a finite number of states on each timestep. The state of the channel on a given timestep describes the quality of service it provides. Here, the channel state on a timestep determines the packet-drop probability on that timestep. We denote the *channel state* at timestep k by $\gamma_k \in \{1, \dots, n\}$, with n a finite positive integer. We assume that the probability distribution of γ_{k+1} depends on γ_k and t_k, the transmission decision on timestep k. Thus, the evolution of the channel is an action-dependent Markov process. We let $p_{ij}^{(0)}$ and $p_{ij}^{(1)}$ denote the probabilities of the channel state transitioning from j to i given t_k is equal to 0 and 1, respectively. Thus,

$$p_{ij}^{(0)} := \Pr\left[\gamma_{k+1} = i \mid \gamma_k = j, t_k = 0\right],$$

$$p_{ij}^{(1)} := \Pr\left[\gamma_{k+1} = i \mid \gamma_k = j, t_k = 1\right].$$

We let $\mathbf{P}_0$ and $\mathbf{P}_1$ be column-stochastic matrices, whose (i, j)th elements are $p_{ij}^{(0)}$ and $p_{ij}^{(1)}$, respectively. We model the unreliability of the channel through a packet-drop probability e_i for each element i of the channel state–space. Thus, if on timestep k the channel state $\gamma_k = i$ and if the sensor transmits a packet then the channel drops it with probability $e_i \in [0, 1]$ and it communicates the packet successfully to the controller with probability $(1 - e_i)$, that is,

$$r_k := \begin{cases} 1, & \text{w.p. } (1 - e_{\gamma_k}) \text{ if } t_k = 1 \\ 0, & \text{w.p. } e_{\gamma_k} \text{ if } t_k = 1 \\ 0, & \text{if } t_k = 0, \end{cases}$$

where 'w.p.' stands for 'with probability'. Thus, the packet drops on each timestep is Bernoulli, though *not* i.i.d. We collect the probabilities of packet drops across all possible channel states in the vector $\mathbf{e} := [e_1, \ e_2, \ \dots, \ e_n]^T \in [0, 1]^n$. Correspondingly, we define the transmission success probability vector $\mathbf{d}$ as $\mathbf{d} := \mathbf{1} - \mathbf{e}$.

2.3 | Sensor's information pattern

Next, we describe the information available to the sensor to make the transmission decisions t_k. Apart from the plant state x_k that the sensor can measure perfectly on each timestep k, we assume that if a successful reception occurs on timestep k, then the controller acknowledges it by relaying the reception indicator variable r_k and the channel state γ_k over an error-free feedback channel. However, the sensor may use this channel feedback information only on subsequent timesteps.

To describe all the information available to the sensor on timestep k more formally, we first introduce the variables R_k and R_k^+ to track the *latest reception time before* and *latest reception time until* timestep k, respectively. Thus,

$$R_k := \max_i \{i < k : r_i = 1\}, \ R_k^+ := \max_i \{i \le k : r_i = 1\}.$$

The variable R_k is useful for the sensor's decision making while R_k^+ is helpful in the analysis. Further, we let S_j for $j \in \mathbb{N}_0$ be the jth successful random reception time, that is,

$$S_0 = 0, \ S_{j+1} := \min\left\{k > S_j : r_k = 1\right\}, \ \forall j \in \mathbb{N},$$

where without loss of generality, we have assumed that the zeroth successful reception occurs on timestep 0.

From the controller feedback, the sensor knows R_k and γ_{R_k} before deciding t_k, from which the sensor can utilise the channel evolution model to obtain the probability distribution of the channel state $\mathbf{p}_k \in \mathbb{P}^n$ given R_k, γ_{R_k} and all the transmission decisions from R_k to $k-1$, that is,

$$\mathbf{p}_k(i) := \Pr\left[\gamma_k = i \mid R_k,\ \gamma_{R_k},\ \{t_w\}_{R_k}^{k-1}\right],$$

where $\mathbf{p}_k(i)$ is the ith element of the vector $\mathbf{p}_k$. Letting

$$\mathbf{p}_k^+ := \begin{cases} \mathbf{p}_k, & \text{if } r_k = 0 \\ \boldsymbol{\delta}_{\gamma_k}, & \text{if } r_k = 1, \end{cases}$$

we can obtain $\mathbf{p}_k$ recursively as

$$\mathbf{p}_{k+1} = \begin{cases} \mathbf{P}_0\mathbf{p}_k^+, & \text{if } t_k = 0 \\ \mathbf{P}_1\mathbf{p}_k^+, & \text{if } t_k = 1. \end{cases} \tag{5}$$

In the following remark, we discuss about the case when channel state feedback may not be error-free.

*Remark 2.1*Value of $\mathbf{p}_k^+$ under erroneous channel state feedback. The probability distribution $\mathbf{p}_k$ represents the belief of the sensor about the true value of channel state γ_k, which evolves based on the action-dependent Markov transition matrix and the intermittently available feedback through $\mathbf{p}_k^+$. Under perfect channel state feedback, on a reception timestep ($r_k = 1$), the sensor knows the value of γ_k and therefore updates the intermediate belief $\mathbf{p}_k^+$ to $\boldsymbol{\delta}_{\gamma_k}$, else ($r_k = 0$) it uses the current belief $\mathbf{p}_k$ for the same. In case of imperfect channel feedback, the channel state information acquired from the controller can be represented via a probability distribution $\hat{\mathbf{p}}_k$, and the value of $\mathbf{p}_k^+$ can be set to $\hat{\mathbf{p}}_k$ when $r_k = 1$. The analysis can then be suitably modified.

We denote by I_k the information available to the sensor about the controller's knowledge of plant state before transmission while we use I_k^+ to denote the information available to the sensor after channel state feedback (if any). Thus, $I_k^+ = I_k$ when $r_k = 0$, and I_k^+ contains r_k and γ_k over I_k when $r_k = 1$. In other words,

$$I_k := \{k, x_k, z_k, R_k, x_{R_k}, \mathbf{p}_k, t_{k-1}, r_{k-1}\gamma_{k-1}\}, \tag{6a}$$

$$I_k^+ := \{k, x_k, z_k^+, R_k^+, x_{R_k^+}, \mathbf{p}_k^+, t_k, r_k\gamma_k\}. \tag{6b}$$

Note that the channel state feedback by the controller is represented as $r_{k-1}\gamma_{k-1}$ and $r_k\gamma_k$ in I_k and I_k^+, respectively. If $r_k = 1$ then $r_k\gamma_k = \gamma_k$, and if $r_k = 0$ then $r_k\gamma_k = 0$ and thus no channel state feedback is available. Note that $\{I_k\}_{k\in\mathbb{N}_0}$ and $\{I_k^+\}_{k\in\mathbb{N}_0}$ are action-dependent Markov processes. In particular, the probability distribution of I_k conditioned on $\{I_s, t_s\}_{s=0}^{k-1}$ can be shown to be the same as the one conditioned on $\{I_{k-1}, t_{k-1}\}$. Similarly, $\{I_k^+\}$ is 'sufficient information' to determine the distribution of I_{k+1}^+ given all the past information

2.4 | Control objective

Given the plant and the controller models in Section 2.1, the only decision making left to be designed is the sensor's transmission policy $\mathcal{T}$, which determines t_k for each timestep k. In particular, we seek to design a feedback transmission policy using the available information I_k on timestep k. The *offline control objective* that we seek to guarantee is *the second-moment stabilisation of the plant state to an ultimate bound exponentially*. Formally, we want to ensure

$$\mathbb{E}_{\mathcal{T}}\left[x_k^2 \mid I_0^+\right] \leq \max\{c^{2k}x_0^2, B\},\ \forall k \in \mathbb{N}_0, \tag{7}$$

which is to have the second moment of the plant state decay exponentially at least at a rate of c^2 until it settles to the ultimate bound B. We assume that the convergence rate parameter $c^2 \in (\bar{a}^2, 1)$. Note that (7) prescribes the restriction on the plant state evolution in an offline fashion, in terms of only the initial information. However, a recursive formulation of the control objective is more conducive to designing a feedback transmission policy.

To design a feedback transmission policy, we need to define an online version of the control objectives which is conditioned upon the information sets $I_{R_k}^+$ that become available to the sensor through feedback received from the channel. First, we define the *performance function* h_k for every timestep k as follows:

$$h_k := x_k^2 - \max\{c^{2(k-R_k)}x_{R_k}^2, B\}.$$

Then, the *online objective* is to ensure

$$\mathbb{E}_{\mathcal{T}}\left[h_k \mid I_{R_k}^+\right] \leq 0, \quad \forall k \in \mathbb{N}_0. \tag{8}$$

We borrow Lemma III.1 from [21], which demonstrates that any transmission policy that satisfies the online objective also satisfies the offline objective.

Lemma 2.1(Sufficiency of the online objective [21]).. *If a transmission policy $\mathcal{T}$ satisfies the online objective (8), then it also satisfies the offline objective (7).* □

Note that in the control objective (7), the sources of randomness that determine the expectation are the transmission policy $\mathcal{T}$, the random channel behaviour and the process noise. The transmission policy and the random channel behaviour determine the successful reception times while the process noise affects the evolution of the performance function during the inter-reception times. As the online objective (8) is essentially a condition on the evolution of the performance function during the inter-reception times, Lemma 2.1 continues to hold in the setting of this paper.

3 | TWO-STEP DESIGN OF TRANSMISSION POLICY

Designing a transmission policy so that the described system meets the control objective (7) or even the stricter online objective (8) poses many challenges. The main challenge stems from the random packet drops, which makes the necessity of a transmission on timestep k dependent on future transmission decisions. Further, the future evolution of the channel state depends on all the past and current transmission decisions. Thus, the transmission decisions t_k cannot be made in a myopic manner and instead must be made by evaluating their impact on the channel and the control objective over a sufficiently long time frame. To tackle this problem, we adopt a two-step design procedure. This general design principle is similar to that in [21], wherein the reader can find a more detailed discussion about this procedure as well as its merits. We now describe the two steps of the design procedure.

In the first step, for each timestep k, we consider a family of *nominal policies* with *look-ahead parameter* $D \in \mathbb{N}$. A nominal policy with parameter D involves a 'hold-off' period of D timesteps from k to $k+D-1$ during which $t_k = 0$, and then there is perpetual transmission, that is $t_k = 1$ for all timesteps after $k+D-1$. Thus, letting $\mathcal{T}_k^D$ be the nominal policy with parameter D, we have

$$\mathcal{T}_k^D : t_i = \begin{cases} 0, & \text{if } i \in \{k, k+1, \dots, k+D-1\} \\ 1, & \text{for } i \geq k+D. \end{cases} \tag{9}$$

In the second step of the design procedure, we construct the event-triggered policy, $\mathcal{T}_{et}^D$, using the nominal policies as building blocks. Given (9), one can reason that if the nominal policy with parameter $D \in \mathbb{N}$ satisfies the online objective from the current timestep k, then a transmission on the current timestep is not necessary to meet the online objective. Further, if the online objective cannot be met from timestep k using the nominal policy $\mathcal{T}_k^D$ then it may be necessary to transmit on timestep k. This forms the basis for the construction of the event-triggered policy, which we detail next.

First, we need a method to check if the nominal policy $\mathcal{T}_k^D$ satisfies the online objective from timestep k. For this, we define the *look-ahead function*, $\mathcal{G}_k^D$, as the expected value of the performance function h_k at the next successful reception timestep $k = S_{j+1}$ under the nominal policy, that is,

$$\mathcal{G}_k^D := \mathbb{E}_{\mathcal{T}_k^D}\left[h_{S_{j+1}} \mid I_k, S_j = R_k\right]. \tag{10}$$

We can evaluate $\mathcal{G}_k^D$ as a total expectation, over all possible values of S_{j+1}, as

$$\mathcal{G}_k^D = \textstyle\sum_{w=D}^{\infty} \mathbb{E}_{\mathcal{T}_k^D}\Big[h_{S_{j+1}} \mid I_k, \dots$$
$$\cdots S_j = R_k, S_{j+1} = k+w\Big]\,\Omega_D(w, \mathbf{p}_k), \tag{11}$$

where $\Omega_D(w, \mathbf{p})$ is the probability of the event that the first successful reception after timestep k is at timestep $k+w$ under the nominal policy $\mathcal{T}_k^D$ and given $\mathbf{p}_k$, the probability distribution of the channel state at time k, conditioned on the information at time R_k. Formally,

$$\Omega_D(w, \mathbf{p}) := \Pr[S_{j+1} = k+w \mid \mathcal{T} = \mathcal{T}_k^D, \dots$$
$$\cdots \mathbf{p}_k = \mathbf{p}, S_j = R_k]. \tag{12}$$

The closed form of $\Omega_D(w, \mathbf{p})$ is given as follows:

$$\Omega_D(w, \mathbf{p}) = \mathbf{d}^T (\mathbf{P}_1 \mathbf{E})^{(w-D)} \mathbf{P}_0^{(D)} \mathbf{p}, \tag{13}$$

where $\mathbf{E}$ is the diagonal matrix with elements of $\mathbf{e}$ on its main diagonal. The explanation of (13) is as follows – the probability vector $\mathbf{p}$, when left-multiplied by $\mathbf{P}_0^{(D)}$ provides the probability vector of the channel state immediately after the hold-off period, which is of D timesteps. The said vector when left-multiplied by $(\mathbf{P}_1\mathbf{E})^{(w-D)}$ provides the probabilities of, subsequent to the hold-off period, making a transmission attempt $(w-D)$ times successively but failing to achieve reception on every attempt. Finally, left-multiplication by $\mathbf{d}^T$ gives the probability of finally having a successful reception on the $(k+w)^{\text{th}}$ timestep. Thus, (13) is the closed form of $\Omega_D(w, \mathbf{p})$ defined in (12).

3.1 | The event-triggered policy

The main idea behind the proposed event-triggered policy is the following. A negative sign of the look-ahead function $\mathcal{G}_k^D$ indicates that it is not 'necessary' to transmit on timestep k as there exists a transmission sequence (given by the nominal policy) that meets the objective at least on the next random reception timestep. However, if the sign of $\mathcal{G}_k^D$ is non-negative, it means that the sensor cannot afford to hold off transmission for D timesteps from the current timestep k, and still ensure that the online objective is not violated on some future timestep. In the proposed event-triggered transmission policy, the sensor evaluates $\mathcal{G}_k^D$ at every timestep k, and when it turns non-negative the sensor keeps transmitting on every timestep until a successful reception occurs, and then the sensor again waits for $\mathcal{G}_k^D$ to turn non-negative. The event-triggered transmission policy may be described formally as follows:

$$\mathcal{T}_{et}^D : t_k = \begin{cases} 0, & \text{if } k \in \{R_k+1, \dots, \tau_k - 1\} \\ 1, & \text{if } k \in \{\tau_k, \dots, Z_k\}, \end{cases} \tag{14}$$

where τ_k is the first timestep after R_k when $\mathcal{G}_k^D \geq 0$ and Z_k is the first timestep, after R_k, on which there is a successful reception, that is,

$$\tau_k := \min\{m > R_k \,:\, \mathcal{G}_m^D \geq 0\},$$

$$Z_k := \min\{m > R_k \,:\, R_m^+ = m\}.$$

Note that the event-triggered policy is described recursively in terms of R_k, the latest reception time before k, and the look-ahead function $\mathcal{G}_k^D$. As a result, the policy in (14) is valid for all time $k \geq 0$. In the analysis of the policy (14) in the sequel, it is useful to refer to the jth reception time, denoted by S_j. Similarly, we let

$$T_j := \min\{m > S_j \, : \, \mathcal{G}_m^D \geq 0\}.$$

So, if $S_j = R_k$ then $T_j = \tau_k$ and $S_{j+1} = Z_k$.

One can think of the policy (14) as operating in one of the two modes: 'do not transmit' or 'transmit'. The policy switches from the first mode to the second at a time k exactly when $\mathcal{G}_k^D \geq 0$ for the first time after the last successful reception. After a successful reception, the policy shifts back to the 'do not transmit' mode. Thus, from this perspective, $\mathcal{G}_k^D \geq 0$ can be thought of as the event-triggering rule.

4 | IMPLEMENTATION AND PERFORMANCE GUARANTEES

In this section, we describe the implementation details of the proposed event-triggered policy, and analyse the system under this policy through several intermediate results. At the end of the section, we provide sufficient conditions on the ultimate bound B and the look-ahead parameter D such that the system meets the online objective (and the offline objective) under the event-triggered policy.

4.1 | Closed-form expression of the look-ahead criterion

For implementation of the event-triggered policy (14), we need an easy method to compute the look-ahead function $\mathcal{G}_k^D$. In particular, we provide here a closed-form expression of the look-ahead function. We begin by expanding the expectation term in (11) as follows [46]:

$$\begin{aligned}\mathbb{E}\left[h_{S_{j+1}} \mid I_k, S_j = R_k, S_{j+1} = k + w\right] &= \bar{a}^{2w} x_k^2 \\ &+ 2\bar{a}^w (a^w - \bar{a}^w) x_k z_k + (a^{2w} - 2a^w \bar{a}^w + \bar{a}^{2w}) z_k^2 \\ &+ \bar{M}(a^{2w} - 1) - \max\{c^{2w} c^{2(k-R_k)} x_{R_k}^2, B\}.\end{aligned} \tag{15}$$

From (11) and (15), it is evident that convergence of $\mathcal{G}_k^D$ requires the convergence of infinite series of the form

$$\begin{aligned} g_D(b, \mathbf{p}) &:= \textstyle\sum_{w=D}^{\infty} b^w \Omega_D(w, \mathbf{p}) \\ &= b^D \textstyle\sum_{w=D}^{\infty} b^{(w-D)} \mathbf{d}^T (\mathbf{P}_1 \mathbf{E})^{(w-D)} \mathbf{P}_0^{(D)} \mathbf{p}, \end{aligned} \tag{16}$$

with $\mathbf{p} \in \mathbb{P}^n$, and $D \in \mathbb{N}$ and for values of b equal to $\bar{a}^2$, c^2, a^2, $\bar{a}a$ and 1, which satisfy

$$0 < \bar{a}^2 < c^2 < 1 < a^2, \quad |\bar{a}a| < a^2. \tag{17}$$

Each of the terms $g_D(b, \mathbf{p})$ involves an infinite matrix geometric series. The criteria for convergence and the closed form of $g_D(b, \mathbf{p})$ for these values of b would allow us to determine the same for $\mathcal{G}_k^D$. For the same, we use the well-known result that for a non-negative matrix $\mathbf{K}$, the infinite matrix geometric series $\sum_{i=0}^{\infty} \mathbf{K}^i$ converges to $(\mathbf{I} - \mathbf{K})^{-1}$ if and only if $\rho(\mathbf{K}) < 1$.

To obtain a closed-form expression of $g_D(b, \mathbf{p})$, first note that

$$g_D(b, \mathbf{p}) = b^D \mathbf{d}^T \left[\textstyle\sum_{w=0}^{\infty} (b\mathbf{P}_1 \mathbf{E})^w\right] \mathbf{P}_0^{(D)} \mathbf{p}.$$

If $\rho(b\mathbf{P}_1\mathbf{E}) < 1$ then we obtain

$$g_D(b, \mathbf{p}) = b^D \mathbf{d}^T (\mathbf{I} - b\mathbf{P}_1 \mathbf{E})^{-1} \mathbf{P}_0^{(D)} \mathbf{p}. \tag{18}$$

In the following result, we apply the convergence criterion of a matrix geometric series to provide a necessary and sufficient condition for $\mathcal{G}_k^D$ to be well defined.

Lemma 4.1. *$\mathcal{G}_k^D$ converges for all probability vectors $\mathbf{p}_k$* if and only if $a^2\rho(\mathbf{P}_1\mathbf{E}) < 1$.

Proof. From (11)–(12) and (15)–(16), we see that an expansion of $\mathcal{G}_k^D$ involves terms such as $g_D(b, \mathbf{p})$ with b equal to $\bar{a}^2$, $a\bar{a}$, a^2 and c^2. Using (17) and noting that $\rho(b_1\mathbf{P}_1\mathbf{E}) > \rho(b_2\mathbf{P}_1\mathbf{E})$ when $|b_1| > |b_2|$, we can state that $\rho(a^2\mathbf{P}_1\mathbf{E}) > \rho(\tilde{b}\mathbf{P}_1\mathbf{E})$ for $\tilde{b}$ equal to $\bar{a}^2$, $a\bar{a}$ and c^2. Thus, $\rho(a^2\mathbf{P}_1\mathbf{E}) = a^2\rho(\mathbf{P}_1\mathbf{E}) < 1$ is a necessary and sufficient condition for convergence of $\mathcal{G}_k^D$. □

We now proceed to give a closed-form expression of the look-ahead function $\mathcal{G}_k^D$ in the following lemma.

Lemma 4.2 (Closed form of the look-ahead function)**.** *Suppose that $a^2\rho(\mathbf{P}_1\mathbf{E}) < 1$. The following is a closed-form expression of the look-ahead function $\mathcal{G}_k^D$:*

$$\begin{aligned}\mathcal{G}_k^D &= g_D(\bar{a}^2, \mathbf{p}_k) x_k^2 + 2\big(g_D(a\bar{a}, \mathbf{p}_k) - g_D(\bar{a}^2, \mathbf{p}_k)\big) x_k z_k \\ &+ \big(g_D(a^2, \mathbf{p}_k) + g_D(\bar{a}^2, \mathbf{p}_k) - 2g_D(a\bar{a}, \mathbf{p}_k)\big) z_k^2 \\ &+ \bar{M}\big(g_D(a^2, \mathbf{p}_k) - g_D(1, \mathbf{p}_k)\big) - \big(B f_D(1, \mathbf{p}_k) + \\ &+ N_k\big[g_D(c^2, \mathbf{p}_k) - f_D(c^2, \mathbf{p}_k)\big]\big),\end{aligned}$$

where $\bar{M} := M(a^2 - 1)^{-1}$, $N_k := c^{2(k-R_k)} x_{R_k}^2$, the closed form of the function $g_D(b, \mathbf{p})$ is given in (18), while $f_D(b, \mathbf{p})$ is given by

$$f_D(b, \mathbf{p}) := b^\mu \mathbf{d}^T (\mathbf{P}_1 \mathbf{E})^{(\mu-D)} (\mathbf{I} - b\mathbf{P}_1 \mathbf{E})^{-1} \mathbf{P}_0^{(D)} \mathbf{p}.$$

Finally, μ is defined as follows:

$$\mu := \max\left\{D, \left\lceil \frac{\log(x_{R_k}^2/B)}{\log(1/c^2)} \right\rceil - (k - R_k)\right\}. \tag{19}$$

Proof. Most terms in the closed form of $\mathcal{G}_k^D$ follow directly from (11), the series expansion of $\mathcal{G}_k^D$, the closed form of $\Omega_D(w, \mathbf{p})$ in (13), the expansion of the expectation term (15), the definition (16) and the closed-form (18) of $g_D(b, \mathbf{p})$. We only need to simplify

$$\textstyle\sum_{w=D}^{\infty} \max\{c^{2w} c^{2(k-R_k)} x_{R_k}^2, B\}\, \Omega_D(b, \mathbf{p}_k).$$

We split this summation into two parts based on if $c^{2w} N_k$ is larger or smaller than B. Observe that μ, defined in (19), is the smallest integer $w \geq D$ such that $B \geq c^{2w} N_k$. Then,

$$\begin{aligned}
&\sum_{w=D}^{\infty} \max\{c^{2w} c^{2(k-R_k)} x_{R_k}^2, B\}\, \Omega_D(w, \mathbf{p}_k) \\
&= g_D(c^2, \mathbf{p}_k)\, N_k + \textstyle\sum_{w=\mu}^{\infty} (B - c^{2w} N_k) \Omega_D(w, \mathbf{p}_k) \\
&\overset{[r1]}{=} B f_D(1, \mathbf{p}_k) + N_k \big[g_D(c^2, \mathbf{p}_k) - f_D(c^2, \mathbf{p}_k)\big],
\end{aligned}$$

where we obtain [r1] by observing that

$$\begin{aligned}
\textstyle\sum_{w=\mu}^{\infty} b^w \Omega_D(w, \mathbf{p}) &= \textstyle\sum_{w=\mu}^{\infty} b^w \mathbf{d}^T (\mathbf{P}_1 \mathbf{E})^{(w-D)} \mathbf{P}_0^{(D)} \mathbf{p} \\
&= b^{\mu} \mathbf{d}^T (\mathbf{P}_1 \mathbf{E})^{(\mu - D)} \textstyle\sum_{w=0}^{\infty} (b \mathbf{P}_1 \mathbf{E})^w \mathbf{P}_0^{(D)} \mathbf{p} = f_D(b, \mathbf{p}),
\end{aligned}$$

assuming $\rho(b\mathbf{P}_1\mathbf{E}) < 1$. With this we obtain the complete closed-form expression of the look-ahead function $\mathcal{G}_k^D$. □

Note that the closed form of $\mathcal{G}_k^D$ is a third-degree polynomial of the plant state x_k, error z_k, and individual elements of $\mathbf{p}_k$, and is amenable for online computation. Furthermore, note that the look-ahead function $\mathcal{G}_k^D$ possesses a mathematical structure consisting of a linear operator with unit-dimensional rowspace acting on the stochastic vector $\mathbf{p}_k$.

4.2 | Necessary condition on the ultimate bound B

We now seek a necessary condition on the ultimate bound B for there to exist a transmission policy that satisfies the online objective. To this end, we introduce the *open-loop performance function*, $H(w, y)$, which we define as the expectation of the performance function $h_{S_{j+1}}$ conditioned upon $I_{S_j}^+$ and the event that $S_{j+1} = S_j + w$ and $x_{S_j}^2 = y$, that is,

$$H(w, y) := \mathbb{E}\Big[h_{S_{j+1}} \,|\, I_{S_j}^+, x_{S_j}^2 = y, S_{j+1} = S_j + w\Big]. \tag{20}$$

Note that $H(w, x_{S_j}^2)$ is very similar to (15) except that H is conditioned upon $I_{S_j}^+$ and defined for the special case of $k = S_j$. Thus, the closed form of $H(w, x_{S_j}^2)$ may be obtained from (15) by replacing k with S_j, x_k with x_{S_j} and z_k with $z_{S_j}^+ = 0$ and R_k with $R_{S_j}^+ = S_j$. Hence we have

$$H(w, x_{S_j}^2) = \bar{a}^{2w} x_{S_j}^2 + \bar{M}(a^{2w} - 1) - \max\{c^{2w} x_{S_j}^2, B\}. \tag{21}$$

Note that $H(w, x_{S_j}^2) < 0$ indicates that given the information $I_{S_j}^+$, the online objective is satisfied on timestep $S_j + w$. Conversely, a positive sign implies that the online objective is expected to be violated on timestep $S_j + w$. Using this observation, we demonstrate in the following proposition that for B less than a critical B_0, there exists *no* transmission policy that can satisfy the online objective.

Proposition 4.1 Necessary condition on the ultimate bound for meeting the online objective. *If $B < B_0 := \frac{\bar{M}\log(a^2)}{\log(c^2/\bar{a}^2)}$ then no transmission policy satisfies the online objective.*

Proof. The proof relies on demonstrating that $H(w, y) > 0$ for all $w \in \mathbb{N}$ and for all $y \in (B, B_0)$. This implies that if $x_{S_j}^2 \in (B, B_0)$, then the system would violate the online objective on the *very* next timestep after Sj. From (21), note that for a fixed y, the function $H(w, y)$ can be written as

$$H(w, y) = \begin{cases} l_1(w, y), & \text{if } w \leq w_{**}(y) \\ l_2(w, y), & \text{if } w > w_{**}(y), \end{cases}$$

with $l_1(w, y) := \bar{a}^{2w} y + \bar{M}(a^{2w} - 1) - c^{2w} y$ and $l_2(w, y) := \bar{a}^{2w} y + \bar{M}(a^{2w} - 1) - B$, where $w_{**}(y) := \frac{\log(y/B)}{\log(1/c^2)}$ is such that $l_1(w_{**}(y), y) = l_2(w_{**}(y), y)$. Now, it suffices to prove the following two claims.

Claim (a): $l_1(w, y) > 0$ for all $w \in \mathbb{N}$ for $y \in (B, B_0)$.

Claim (b): $l_2(w, y) > 0$ for all $w \in \mathbb{N}$ for $y \in (B, B_0)$.

First, note that $l_1(0, y) = 0$ for all values of y. Next, evaluating the partial of $l_1(w, y)$ with respect to w at $w = 0$ and for $y \in (B, B_0)$, we obtain

$$\begin{aligned}
\frac{\partial l_1(0, y)}{\partial w} &= \log(\bar{a}^2/c^2) y + \bar{M}\log(a^2) \\
&\overset{[r1]}{>} \log(\bar{a}^2/c^2) B_0 + \bar{M}\log(a^2) \overset{[r2]}{=} 0.
\end{aligned}$$

Note that we have used the fact that $\bar{a}^2 < c^2$ to obtain [r1], and used the definition of B_0 in [r2]. Since $l_1(w, y)$ is a quasi-convex function of w [21, Lemma IV.8], it is increasing for all $w > 0$, which proves claim (a).

Now, we prove claim (b). We first derive a function $g(w)$ that is a lower bound on $l_2(w, y)$ for $w \geq 0$ and $y \in (B, B_0)$.

$$l_2(w, y) = \bar{a}^{2w} y - \bar{M}(a^{2w} - 1) - B$$

$$>\bar{a}^{2w}y - y + \bar{M}(a^{2w} - 1)$$

$$\overset{[r3]}{>} \frac{B_0}{c^{2w}}(\bar{a}^{2w} - 1) + \bar{M}(a^{2w} - 1) =: g(w),$$

where in [r3], we have used the fact that $\bar{a}^2 < 1$, $c^2 < 1$ and $w \geq 0$. Note that $g(w)$ is strictly convex in w because

$$\frac{\partial^2 g(w)}{\partial w^2} = B_0 \frac{\bar{a}^{2w}}{c^{2w}} \log^2(\bar{a}^2/c^2) + \bar{M}a^{2w}\log^2(a^2) > 0.$$

The partial derivative of $g(w)$ evaluated at $w = 0$ is

$$\frac{\partial g(0)}{\partial w} = B_0 \log(\bar{a}^2/c^2) + \bar{M}\log(a^2) \overset{[r4]}{=} 0,$$

where in [r4] we have used the definition of B_0. Since $g(0) = 0$, $g(w)$ has slope 0 at $w = 0$ and g is strictly convex in w, we conclude that $h_2(w, y) > g(w) > 0$ for all $w \in \mathbb{N}$, which proves claim (b) and thus concluding the proof. □

Proposition 4.1 demonstrates that $B > B_0$ is a *necessary* condition on B for a transmission policy to satisfy the online objective. Note that this is a necessary condition on B even under the setting of [21, 24], where no such condition is provided. In the following subsection, we further analyse the open-loop performance function $H(w, y)$ to find a *sufficient* criterion on B and D that guarantees that the online objective is met under the event-triggered policy.

4.3 | The performance-evaluation function, $\mathcal{J}^D_{S_j}$

For the purpose of analysing system performance between any two successive reception times S_j and S_{j+1}, we define the *performance-evaluation function*, $\mathcal{J}^D_{S_j}$. Its definition is similar to that of $\mathcal{G}^D_k$ in (10), though we define $\mathcal{J}^D_{S_j}$ only for $k = S_j$ (successful reception times) and condition upon the information set $I^+_{S_j}$ instead of I_{S_j}. In particular, we let

$$\mathcal{J}^D_{S_j} := \mathbb{E}_{\mathcal{T}^{D-1}_{S_j+1}}\left[h_{S_{j+1}} \mid I^+_{S_j}\right] = \textstyle\sum_{w=D}^{\infty} H(w, x^2_{S_j})\tilde{\Omega}_D(w, \gamma_{S_j}). \tag{22}$$

Here, $\tilde{\Omega}_D(w, \gamma)$ is the probability of getting a successful reception w timesteps after S_j starting with channel state γ on S_j under the nominal policy $\mathcal{T}^{D-1}_{S_j+1}$. The purpose of the function $\tilde{\Omega}_D(w, \gamma)$ is analogous to that of $\Omega_D(w, \mathbf{p})$ in $\mathcal{G}^D_k$, and is formally defined as

$$\tilde{\Omega}_D(w, \gamma) := \Pr[S_{j+1} = S_j + w \mid \mathcal{T} = \mathcal{T}^{D-1}_{S_j+1}, \gamma_{S_j} = \gamma]. \tag{23}$$

The closed form of $\tilde{\Omega}_D(w, \gamma)$ can be obtained in a manner similar to the closed form of $\Omega_D(w, \mathbf{p})$, and is given as

$$\tilde{\Omega}_D(w, \gamma) = \mathbf{d}^T(\mathbf{P}_1\mathbf{E})^{(w-D)}\mathbf{P}_0^{(D-1)}\mathbf{P}_1\boldsymbol{\delta}_\gamma. \tag{24}$$

Note that in (24), the probability function $\tilde{\Omega}_D(w, \gamma)$ takes the channel state γ as an argument instead of a probability distribution $\mathbf{p}$, since our assumed channel state feedback mechanism stipulates perfect feedback, that is, $\mathbf{p}_{S_j} = \boldsymbol{\delta}_{\gamma_{S_j}}$, and thus $\mathbf{p}_{S_j}$ is a deterministic function of γ_{S_j}. Before proceeding, we discuss conceptual and structural differences between $\mathcal{G}^D_k$ and $\mathcal{J}^D_k$ in the following remark.

Remark 4.1 (Differences between $\mathcal{G}^D_k$ and $\mathcal{J}^D_{S_j}$). The core difference between the look-ahead criterion $\mathcal{G}^D_k$ and the performance-evaluation function $\mathcal{J}^D_{S_j}$ is that while $\mathcal{G}^D_k$ is computed onboard the sensor on every timestep k for the purpose of determining t_k according to the event-triggered policy, $\mathcal{J}^D_{S_j}$ is used as an analytical tool for evaluation of inter-reception performance between timesteps S_j and S_{j+1}. Note that the expectation in $\mathcal{G}^D_k$ is conditioned upon the nominal policy $\mathcal{T}^D_k$, while the expectation in $\mathcal{J}^D_{S_j}$ is conditioned upon the nominal policy $\mathcal{T}^{D-1}_{S_j+1}$ (as opposed to $\mathcal{T}^D_{S_j}$ in the i.i.d. case [21] and in the Markov channel case in [24]). The reason for doing this is that in case of non-action-dependent channels ($\mathbf{P}_0 = \mathbf{P}_1$), once γ_{S_j} is known, the resulting closed form of the probability function $\tilde{\Omega}_D(w, \gamma)$ is the same irrespective of whether we condition the probability in (23) upon nominal policy $\mathcal{T}^{D-1}_{S_j+1}$ or $\mathcal{T}^D_{S_j}$. However, this is not true for the action-dependent Markov channels, since the stipulation that $t_{S_j} = 1$ leads to calculation of belief on timestep $S_j + 1$ as $\mathbf{p}_{S_j+1} = \mathbf{P}_1\boldsymbol{\delta}_{\gamma_{S_j}}$ instead of $\mathbf{p}_{S_j+1} = \mathbf{P}_0\boldsymbol{\delta}_{\gamma_{S_j}}$. This is visible in the closed form of $\tilde{\Omega}_D(w, \gamma)$ in (24), and obviously this would not be an issue if $\mathbf{P}_0 = \mathbf{P}_1$, as aforementioned.

For a well-chosen value of B, it can be shown that the open-loop performance function possesses the property of *sign monotonicity*. This property is an important characteristic of $H(w, y)$ and will prove useful in later results.

Proposition 4.2 (Sign behaviour of the open-loop performance function [21, Proposition IV.6].). *There exists a $B^* \geq B_0$ with B_0 defined in Proposition 4.1 such that if $B > B^*$, then $H(w, y) > 0$ implies $H(s, y) > 0$ for all $s \geq w$.* □

The value of B^* defined in Proposition 4.2 can be numerically computed using the procedure in the Appendix, which is based on the proof of Lemma IV.13 in [21]. We now provide a closed-form expression of the performance-evaluation function $\mathcal{J}^D_{S_j}$, similar to the closed form of $\mathcal{G}^D_k$ in Lemma 4.2.

Lemma 4.3 (Closed form of performance-evaluation function). *Suppose that $a^2\rho(\mathbf{P}_1\mathbf{E}) < 1$. A closed form of the performance-evaluation* *function* $\mathcal{J}^D_{S_j}$ *is given as*

$$\mathcal{J}^D_{S_j} := \tilde{g}_D(\bar{a}^2,\gamma_{S_j})x^2_{S_j} + \bar{M}\Big[\tilde{g}_D(a^2,\gamma_{S_j}) - \tilde{g}_D(1,\gamma_{S_j})\Big] - \Big[B\tilde{f}_D(1,\gamma_{S_j}) + x^2_{S_j}\Big(\tilde{g}_D(c^2,\gamma_{S_j}) - \tilde{f}_D(c^2,\gamma_{S_j})\Big)\Big],$$

where

$$\tilde{f}_D(b,\gamma) := b^{\nu}\mathbf{d}^T(\mathbf{P}_1\mathbf{E})^{(\nu-D)}(\mathbf{I}-b\mathbf{P}_1\mathbf{E})^{-1}\mathbf{P}_0^{(D-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma},$$

$$\tilde{g}_D(b,\gamma) := b^{D}\mathbf{d}^T(\mathbf{I}-b\mathbf{P}_1\mathbf{E})^{-1}\mathbf{P}_0^{(D-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma},$$

and finally, ν *is defined as*

$$\nu := \max\left\{D,\left\lceil\frac{\log(x^2_{S_j}/B)}{\log(1/c^2)}\right\rceil\right\}.$$

Proof. Recall the infinite series expansion of $\mathcal{J}^D_{S_j}$ in (22). To evaluate it, we substitute $H(w,x^2_{S_j})$ with its closed form from (21) and that of $\tilde{\Omega}_D(w,\gamma_{S_j})$ from (24). Correspondingly, we get an expression that is the sum of multiple infinite series, as in the derivation of $\mathcal{G}^D_k$ in Lemma 4.2. To evaluate said terms, we define the summation functions $\tilde{f}_\theta(b,\gamma)$ and $\tilde{g}_\theta(b,\gamma)$ given in the statement of the lemma and which are analogous to $f_\theta(b,\mathbf{p})$ and $g_\theta(b,\mathbf{p})$, respectively, and used for obtaining the expression for $\mathcal{G}^D_k$. Proceeding exactly like in Lemma 4.2, we obtain the expression for $\mathcal{J}^D_{S_j}$. □

The next result is concerned with the expected value of $\mathcal{G}^D_{k+1}$ after no transmission or after successful reception and the channel state feedback on timestep k. Note that this result is valid for *any* transmission policy $\mathcal{T}$.

Proposition 4.3 (Expected value of look-ahead function on next timestep)**.** *Let* $\mathcal{T}$ *be any transmission policy. Then,*

1. $\mathbb{E}_{\mathcal{T}}[\mathcal{G}^D_{k+1} \mid I_k, t_k = 0] = \mathcal{G}^{D+1}_k$;
2. $\mathbb{E}_{\mathcal{T}}[\mathcal{G}^D_{k+1} \mid I_k, r_k = 1, \gamma_k] = \mathcal{J}^{D+1}_{S_j}$*, where* $S_j = k$.

Proof. **1**: Note that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{T}}\big[\mathcal{G}^D_{k+1} \mid I_k, t_k = 0\big]\\
&\overset{[r1]}{=} \mathbb{E}_{\mathcal{T}}\Big[\mathbb{E}_{\mathcal{T}^D_{k+1}}\Big[h_{S_{j+1}} \mid I_{k+1}, S_j = R_{k+1}\Big] \mid I_k, t_k = 0\Big],\\
&\overset{[r2]}{=} \mathbb{E}_{\mathcal{T}^{D+1}_k}\Big[\mathbb{E}_{\mathcal{T}^D_{k+1}}\Big[h_{S_{j+1}} \mid I_{k+1}, S_j = R_k\Big] \mid I_k, t_k = 0\Big],\\
&\overset{[r3]}{=} \mathbb{E}_{\mathcal{T}^{D+1}_k}\Big[h_{S_{j+1}} \mid I_k, t_k = 0, S_j = R_k\Big] = \mathcal{G}^{D+1}_k,
\end{aligned}$$

where [r1] follows from (10), while in [r2] we can replace the policy $\mathcal{T}$ with $\mathcal{T}^{D+1}_k$ because the event $t_k = 0$ is consistent with the policy $\mathcal{T}^{D+1}_k$ on time step k and once $t_k = 0$ is fixed the expected value of $\mathcal{G}^D_{k+1}$ is independent of the transmission policy used on subsequent timesteps. In [r2], we also use the fact that if $t_k = 0$ then $R_{k+1} = R_k$. Finally, [r3] uses the fact that $\{I_k, t_k\}$ is *sufficient information*and then the tower property.

2: For proving this part, we observe that I_k and the additional information that $r_k = 1$ and γ_k implies the knowledge of I^+_k. Considering this fact and proceeding with a similar methodology as the proof of claim 1, we observe that

$$\begin{aligned}
&\mathbb{E}_{\mathcal{T}}\big[\mathcal{G}^D_{k+1} \mid I_k, r_k = 1, \gamma_k\big]\\
&= \mathbb{E}_{\mathcal{T}}\Big[\mathbb{E}_{\mathcal{T}^D_{k+1}}\Big[h_{S_{j+1}} \mid I_{k+1}, S_j = R_{k+1}\Big] \mid I^+_k, r_k = 1\Big],\\
&= \mathbb{E}_{\mathcal{T}^D_{k+1}}\Big[\mathbb{E}_{\mathcal{T}^D_{k+1}}\Big[h_{S_{j+1}} \mid I_{k+1}, S_j = R_{k+1}\Big] \mid I^+_k, S_j = k\Big],\\
&= \mathbb{E}_{\mathcal{T}^{(D+1)-1}_{k+1}}\Big[h_{S_{j+1}} \mid I^+_k, S_j = k\Big] = \mathcal{J}^{D+1}_{S_j}.
\end{aligned}$$

□

Remark 4.2 ((Comparison with [21]).). Note that the statement of part 1 of Proposition 4.3 differs from Proposition IV.4(a) (first part) of [21] which considers the expected value of $\mathcal{G}^D_{k+1}$ in the setting of a channel with i.i.d. Bernoulli packet drops, in that we condition $\mathcal{G}^D_{k+1}$ upon the stricter condition that $t_k = 0$ as opposed to $r_k = 0$ in [21]. This is because if the probabilities of channel state transition are action dependent, then on a timestep with a transmission but no reception (i.e. $t_k = 1, r_k = 0$) the expected value of the look-ahead criterion on the next timestep cannot be written in terms of *either* $\mathcal{G}^{D+1}_k$ or $\mathcal{J}^{D+1}_k$, as opposed to i.i.d. Bernoulli packet-drop channel where $\mathbb{E}_{\mathcal{T}}[\mathcal{G}^D_{k+1}|I_k, r_k = 0] = \mathcal{G}^{D+1}_k$ holds. However, due to the robustness of the event-triggered policy design, this does not preclude utilisation of the event-triggered policy in the current case, as will be demonstrated in the proof of Theorem 4.1.

We use Proposition 4.2, Lemma 4.3 and Proposition 4.3 to give a *sufficient* condition on ultimate bound B, and the look-ahead parameter D under which the online objective is met. First, we give a sufficient condition to ensure $\mathcal{J}^\theta_{S_j}$ is negative.

Proposition 4.4 (Sufficient condition for performance-evaluation function to be negative)**.** *Suppose* $B \geq B_0 = \frac{\bar{M}\log(a^2)}{\log(c^2/\bar{a}^2)}$. *Consider the vector valued function* $\mathbf{Q}(\theta) : \mathbb{N}_0 \to \mathbb{R}^n$ *given by*

$$\mathbf{Q}(\theta) := \big[\mathcal{Z}_\theta(\bar{a}^2) - \mathcal{Z}_\theta(c^2)\big]\frac{B}{c^{2\theta}} + \bar{M}\big[\mathcal{Z}_\theta(a^2) - \mathcal{Z}_\theta(1)\big],$$

wherein $\mathcal{Z}_\theta(b) := b^\theta\mathbf{d}^T(\mathbf{I}-b\mathbf{P}_1\mathbf{E})^{-1}$. *If* $\mathbf{Q}(D) < \mathbf{0}$ *(elementwise), for some* $D \in \mathbb{N}$*, then* $\mathcal{J}^\theta_{S_j} < 0, \forall x_{S_j} \in \mathbb{R}$ *and* $\forall\theta \in [1, D]_{\mathbb{Z}}$.

We provide the proof of Proposition 4.4 in the Appendix. Next, we consolidate the results so far to provide a theoretical guarantee that the event-triggered policy satisfies the online objective (8).

Theorem 4.1 (Performance guarantee of the event-triggered policy)**.** *If $B > B^*$ (see Appendix) and the look-ahead parameter D satisfies the condition $\mathbf{Q}(D) < \mathbf{0}$, then the event-triggered policy (14) guarantees that the online objective (8), and therefore the original offline objective (7), are met.*

Proof. Given Lemma 2.1, it suffices to show that the online objective (8) is met by the event-triggered policy. We centre the proof around the following two claims.

Claim (a): For any $j \in \mathbb{N}_0$, $\mathbb{E}_{\mathcal{T}_{et}^D}[h_{S_{j+1}} \mid I_{S_j}^+] \le 0$ implies $\mathbb{E}_{\mathcal{T}_{et}^D}[h_k \mid I_{S_j}^+] \le 0$ for all $k \in [S_j, S_{j+1}]_{\mathbb{Z}}$.

Claim (b): For any $j \in \mathbb{N}_0$, $\mathbb{E}_{\mathcal{T}_{et}^D}[h_{S_{j+1}} \mid I_{S_j}^+] < 0$.

These two claims guarantee that the online objective is met, as

$$\begin{aligned} &\mathbb{E}_{\mathcal{T}_{et}^D}\left[h_k \mid I_0^+\right] \\ &= \mathbb{E}_{\mathcal{T}_{et}^D}\left[\dots \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathbb{E}_{\mathcal{T}_{et}^D}\left[h_k \mid I_{S_j}^+\right] \mid I_{S_{j-1}}^+\right] \dots \mid I_0^+\right], \end{aligned}$$

where $\{S_j\}$ are the random reception times and $S_j = R_k^+$.

To prove Claim (a), we note that by the definition of open-loop performance function $H(w, y)$ in (20), we have

$$\mathbb{E}_{\mathcal{T}_{et}^D}\left[h_k \mid I_{S_j}^+\right] = H(k - S_j, x_{S_j}^2), \ \forall k \in [S_j, S_{j+1}]_{\mathbb{Z}}.$$

If $\mathbb{E}_{\mathcal{T}_{et}^D}[h_{S_{j+1}} \mid I_{S_j}^+] = H(S_{j+1} - S_j, x_{S_j}^2) < 0$, then the sign monotonicity property of the open-loop performance function (Proposition 4.2) implies $H(k - S_j, x_{S_j}^2) \le 0$ for all $k \in [S_j, S_{j+1}]_{\mathbb{Z}}$, which proves Claim (a).

We now prove Claim (b). It can be seen from Proposition 4.3 that for all $k \in (S_j, T_j)_{\mathbb{Z}}$,

$$\begin{aligned} &\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{k+1}^D \mid k \in (S_j, T_j)_{\mathbb{Z}},\ I_{S_j}^+\right] \\ &\overset{[r1]}{=} \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{k+1}^D \mid I_k, t_k = 0\right] \mid I_{S_j}^+\right] \\ &\overset{[r2]}{=} \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_k^{D+1} \mid I_{S_j}^+\right], \end{aligned} \tag{25}$$

where [r1] is obtained using the tower property and the fact that $t_k = 0$ for $k \in (S_j, T_j)_{\mathbb{Z}}$, while [r2] is obtained from Proposition 4.3. Furthermore, Proposition 4.3 (b) implies that

$$\begin{aligned} \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{S_j+1}^D \mid I_{S_j}^+\right] &= \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{S_j+1}^D \mid I_{S_j}, r_{S_j} = 1, \gamma_{S_j}\right] \\ &= \mathcal{J}_{S_j}^{D+1}. \end{aligned} \tag{26}$$

Next, we condition the expected value of $h_{S_{j+1}}$ over information from timestep T_j as well as timestep S_j and using the tower property of conditional expectations, we obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{T}_{et}^D}\left[h_{S_{j+1}} \mid I_{S_j}^+\right] &\overset{[r3]}{=} \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathbb{E}_{\mathcal{T}_{T_j}^0}\left[h_{S_{j+1}} \mid I_{T_j}, S_j = R_{T_j}\right] \mid I_{S_j}^+\right] \\ &= \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{T_j}^0 \mid I_{S_j}^+\right], \end{aligned} \tag{27}$$

where the inner expectation in [r3] is conditioned under the nominal policy $\mathcal{T}_{T_j}^0$ since for all timesteps $k \in [T_j, S_{j+1}]_{\mathbb{Z}}$, we have transmissions ($t_k = 1$). We consider two cases: $T_j \le S_j + D$ and $T_j > S_j + D$. In the first case, since $t_k = 0$ for $k \in (S_j, T_j)_{\mathbb{Z}}$, we use (25) and (26) to write (27) as

$$\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{T_j}^0 \mid I_{S_j}^+\right] = \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{S_j+1}^{T_j - S_j - 1} \mid I_{S_j}^+\right] = \mathcal{J}_{S_j}^{T_j - S_j},$$

where Proposition 4.4 ensures that if $T_j - S_j \le D$ then $\mathcal{J}_{S_j}^{T_j - S_j} < 0$. We now consider the second case in which $T_j > S_j + D$. Since we have $t_k = 0$ for $k \in (S_j, T_j)_{\mathbb{Z}}$, we use (25) to write (27) as

$$\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{T_j}^0 \mid I_{S_j}^+\right] = \mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{T_j - D}^D \mid I_{S_j}^+\right] < 0,$$

since $\mathcal{G}_k^D$ is negative, by definition, for $k \in (S_j, T_j)_{\mathbb{Z}}$. This proves Claim (b), and hence also the result. □

We conclude this section by commenting on the extension of the event-triggered policy to vector systems.

Remark 4.3 (Extension to vector systems). The event-triggered policy for control objective (7) can easily be extended to a general vector system of the form $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{v}_k$, with $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbb{E}[\mathbf{v}_k] = \mathbf{0}$, and $\mathbb{E}[\mathbf{v}_k \mathbf{v}_k^T] = \mathbf{M} = \mathbf{M}^T > 0$, with the control objective being to find a policy $\mathcal{T}$ such that $\mathbb{E}_{\mathcal{T}}[\mathbf{x}_k^T \mathbf{x}_k \mid I_0^+] \le \max\{c^{2k} \mathbf{x}_0^T \mathbf{x}_0, B\}$. The control scheme could be $\mathbf{u}_k = \mathbf{L}\hat{\mathbf{x}}_k$ (similar to $u_k = L\hat{x}_k$ in the scalar case), with $(\mathbf{A} + \mathbf{BL})$ being Schur stable. There are two primary approaches towards the vector case extension. The first approach is applicable when it is possible to decompose the vector system into n scalar subsystems, and correspondingly obtain n look-ahead criteria $(\mathcal{G}_k^{(D,1)}, \dots, \mathcal{G}_k^{(D,n)})$ on every timestep. We can then use the largest value of the n look-ahead criteria so obtained in the triggering condition (14), thereby creating an event-triggered policy that can stabilise the worst-case mode of the system, and can thus stabilise the entire system. The second approach involves a scalarisation of the vector system using any appropriate l_p norm of the state variables and matrices involved in various calculations. This approach has been considered for vector systems in the Bernoulli packet-drop channel system in [21], and can easily be extended for the present case.

5 | TRANSMISSION FRACTION

This section analyses the efficiency of the proposed event-triggered transmission policy in terms of the fraction of times the sensor transmits ($t_k = 1$) over a given time horizon. First, we introduce the *transmission fraction* up to timestep K as

$$\mathcal{F}^K := \frac{\mathbb{E}_{\mathcal{T}_{et}^D}\left[\sum_{i=1}^{K} t_i \,\middle|\, I_0^+\right]}{\mathbb{E}_{\mathcal{T}_{et}^D}\left[K \,\middle|\, I_0^+\right]},$$

wherein the stopping timestep K could itself be a random variable. We call the limit of $\mathcal{F}^K$ when $K \to \infty$ as the *asymptotic transmission fraction*, denoted by $\mathcal{F}^\infty$.

We also consider another type of transmission fraction which we call the *transmission fraction up to state* $\mathcal{X}$, and denote it with $\mathcal{F}_\mathcal{X}$. It is defined as the transmission fraction up to the first reception timestep such that the squared plant state is lesser than $\mathcal{X}$. That is,

$$\mathcal{F}_\mathcal{X} := \frac{\mathbb{E}_{\mathcal{T}_{et}^D}\left[\sum_{i=1}^{S_j} t_i \,\middle|\, I_0^+,\ \{x_{S_l}^2\}_{l=0}^{j-1} \geq \mathcal{X},\ x_{S_j}^2 < \mathcal{X}\right]}{\mathbb{E}_{\mathcal{T}_{et}^D}\left[S_j \,\middle|\, I_0^+,\ \{x_{S_l}^2\}_{l=0}^{j-1} \geq \mathcal{X},\ x_{S_j}^2 < \mathcal{X}\right]}.$$

In the following remark, we discuss the conceptual difference between $\mathcal{F}^\infty$ and $\mathcal{F}_\mathcal{X}$, and the advantages of having a closed-form upper bound for both.

Remark 5.1 (Comparison between $\mathcal{F}^\infty$ and $\mathcal{F}_\mathcal{X}$). The asymptotic transmission fraction $\mathcal{F}^\infty$ denotes the fraction of timesteps the sensor transmits under the event-triggered policy over an infinite horizon. An upper bound on $\mathcal{F}^\infty$ is therefore useful in determining the worst-case channel utilisation over a long period of time. Note that the system behaviour captured by $\mathcal{F}^\infty$ is dominated by the timesteps when the second-moment plant state x_k^2 is under the ultimate bound B since $\mathcal{F}^\infty$ is defined over the infinite horizon $k \in [1, \infty)_\mathbb{Z}$. However, prior to the timestep $k = \log(Bx_0^{-2})\log(c^2)^{-1}$ the control envelope $\max\{c^{2k}x_0^2, B\}$ decays exponentially, and the transmission fraction to state $\mathcal{X}$, $\mathcal{F}_\mathcal{X}$, is useful in capturing the transmission fraction during this transient period. □

In Theorem 5.1, we give an upper bound on $\mathcal{F}_\mathcal{X}$ that only involves plant and channel parameters, and $\mathcal{X}$. Then, we derive an upper bound on the asymptotic transmission fraction $\mathcal{F}^\infty$ as a corollary.

Theorem 5.1 (Upper bound on $\mathcal{F}_\mathcal{X}$). *Suppose $\mathbf{Q}(D) < \mathbf{0}$ for a given value of D. The transmission fraction up to state $\mathcal{X}$ is upper bounded by*

$$\mathcal{F}_\mathcal{X} \leq \frac{C^{(1)}}{C^{(0)} + C^{(1)}},$$

where

$$C^{(0)} := \max_{\mathcal{B}\in\mathbb{N}_0}\{\mathcal{B} \mid \mathbf{Q}_\mathcal{X}(D + \mathcal{B}) < \mathbf{0}\}$$

$$\mathbf{Q}_\mathcal{X}(\theta) := \left[\mathcal{Z}_\theta(\bar{a}^2) - \mathcal{Z}_\theta(c^2)\right]\max\{\mathcal{X}, Bc^{-2\theta}\}$$

$$+ \bar{M}\left[\mathcal{Z}_\theta(a^2) - \mathcal{Z}_\theta(1)\right],$$

with $\mathcal{Z}_\theta(b)$ as defined in Proposition 4.4, while $C^{(1)}$ is given by

$$C^{(1)} = \max_{i\in[1,n]_\mathbb{Z}}\{\mathbf{d}^T(\mathbf{P}_1\mathbf{E})(\mathbf{I} - \mathbf{P}_1\mathbf{E})^{-2}\boldsymbol{\delta}_i\}.$$

Proof. We find an upper bound on $\mathcal{F}_\mathcal{X}$ by first considering the time horizon between two successive reception times, and then extending the analysis to an arbitrary number of inter-reception cycles. For $j \in \mathbb{N}_0$, we let Δ_j be the time horizon $(S_j, S_{j+1}]_\mathbb{Z}$. Further, throughout this proof, we use the shorthand $\boldsymbol{\Pi}_\theta(\gamma_{S_j}) := \mathbf{P}_0^{(\theta-1)}\mathbf{P}_1\delta_{\gamma_{S_j}}$ for notational convenience.

Using the structure of the event-triggered policy, we split Δ_j into two parts as $\Delta_j^{(0)} := (S_j, T_j)_\mathbb{Z}$ and $\Delta_j^{(1)} := [T_j, S_{j+1}]_\mathbb{Z}$. Hence, for $k \in \Delta_j^{(0)}$, no transmission occurs ($t_k = 0$) while for each $k \in \Delta_j^{(1)}$, a transmission occurs ($t_k = 1$). Now, consider the following two claims.

Claim (a): $\mathbb{E}_{\mathcal{T}_{et}^D}[|\Delta_j^{(0)}| \mid I_{S_j}^+,\ x_{S_j}^2 > \mathcal{X}] \geq C^{(0)}$.

Claim (b): $\mathbb{E}_{\mathcal{T}_{et}^D}[|\Delta_j^{(1)}| \mid I_{S_j}^+] \leq C^{(1)}$, for all $x_{S_j} \in \mathbb{R}$.

Supposing the two claims are true, consider the transmission fraction during the jth horizon, Δ_j, conditioned on $I_{S_j}^+$. We note that it satisfies the inequality in (28) since the transmission fraction is increasing in the term $\mathbb{E}_{\mathcal{T}_{et}^D}[|\Delta_j^{(1)}| \mid I_{S_j}^+]$, and decreasing in the term $\mathbb{E}_{\mathcal{T}_{et}^D}[|\Delta_j^{(0)}| \mid I_{S_j}^+,\ x_{S_j}^2 > \mathcal{X}]$. Now, as this upper bound is independent of the state of the system as long as $x_{S_j}^2 > \mathcal{X}$, we obtain the upper bound on $\mathcal{F}_\mathcal{X}$, stated in the result. Thus, all that remains now is to prove Claims (a) and (b).

To prove Claim (a), we start by demonstrating that, for a given value of $\theta \in \mathbb{N}$ and under the assumption that $x_{S_j}^2 \geq \mathcal{X}$, $\mathcal{J}_{S_j}^\theta \leq \mathbf{Q}_\mathcal{X}(\theta)\Pi_\theta(\gamma_{S_j})$. To this end, we consider two cases, $\mathcal{X} \in \Lambda_1 = [0, Bc^{-2\theta})$ and $\mathcal{X} \in \Lambda_2 = [Bc^{-2\theta}, \infty)$, respectively. If $\mathcal{X} \in \Lambda_1$, then we have

$$\mathcal{J}_{S_j}^\theta \leq \mathcal{R}_j(\theta) = \mathbf{Q}(\theta)\Pi_\theta(\gamma_{S_j}) = \mathbf{Q}_\mathcal{X}(\theta)\Pi_\theta(\gamma_{S_j}),$$

where the inequality is from Claim (a) of Proposition 4.4, the first equality from (A1) and the second equality from the fact that $\mathcal{X} \in \Lambda_1$. Now, consider the case of $x_{S_j}^2 \geq \mathcal{X} \in \Lambda_2$. Recall from the proof of Claim (a) of Proposition 4.4 that $\mathcal{J}_{S_j}^\theta$ can be upper bounded as given in (29):

$$\frac{\mathbb{E}_{\mathcal{T}_{et}^D}\left[|\Delta_j^{(1)}|\,\Big|\,I_{S_j}^+\right]}{\mathbb{E}_{\mathcal{T}_{et}^D}\left[|\Delta_j^{(0)}|\,\Big|\,I_{S_j}^+,\,x_{S_j}^2>\mathcal{X}\right]+\mathbb{E}_{\mathcal{T}_{et}^D}\left[|\Delta_j^{(1)}|\,\Big|\,I_{S_j}^+\right]}\le\frac{C^{(1)}}{C^{(0)}+C^{(1)}} \tag{28}$$

$$\mathcal{J}_{S_j}^{\theta}\le\left[\tilde{g}_{\theta}(\bar{a}^2,\gamma_{S_j})-\tilde{g}_{\theta}(c^2,\gamma_{S_j})\right]x_{S_j}^2+\bar{M}[\tilde{g}_{\theta}(a^2,\gamma_{S_j})-\tilde{g}_{\theta}(1,\gamma_{S_j})]$$

$$\overset{[r1]}{=}\left[\left(\mathcal{Z}_{\theta}(\bar{a}^2)-\mathcal{Z}_{\theta}(c^2)\right)\max\{\mathcal{X},Bc^{-2\theta}\}+\bar{M}\left(\mathcal{Z}_{\theta}(a^2)-\mathcal{Z}_{\theta}(1)\right)\right]$$

$$\Pi_{\theta}(\gamma_{S_j})\overset{[r2]}{=}\mathbf{Q}_{\mathcal{X}}(\theta)\Pi_{\theta}(\gamma_{S_j}), \tag{29}$$

$$C_i^{(1)}:=\mathbb{E}_{\mathcal{T}_{et}^D}[w\,|\,S_{j+1}=T_j+w,\gamma_{T_j}=i]=\mathbf{d}^T\left[\textstyle\sum_{s=0}^{\infty}s(\mathbf{P}_1\mathbf{E})^s\right]\boldsymbol{\delta}_i$$
$$=\mathbf{d}^T(\mathbf{P}_1\mathbf{E})(\mathbf{I}-\mathbf{P}_1\mathbf{E})^{-2}\boldsymbol{\delta}_i, \tag{30}$$

where [r1] is a result of (A1) and the facts that $\mathcal{Z}_{\theta}(\bar{a}^2)-\mathcal{Z}_{\theta}(c^2)<\mathbf{0}$ and $x_{S_j}^2\ge\mathcal{X}\ge Bc^{-2\theta}$, and [r2] uses the definition of $\mathbf{Q}_{\mathcal{X}}(\theta)$. Thus, we have demonstrated that for any given $\mathcal{X}\ge 0$, if $x_{S_j}^2\ge\mathcal{X}$ then $\mathcal{J}_{S_j}^{\theta}\le\mathbf{Q}_{\mathcal{X}}(\theta)\Pi_{\theta}(\gamma_{S_j})$.

Now, suppose $x_{S_j}^2\ge\mathcal{X}$ and $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})<\mathbf{0}$ for some $\mathcal{B}\in\mathbb{N}_0$, where D is the operational value of the look-ahead parameter. Then, through a recursive application of Proposition 4.3 $\mathcal{B}$ times, we get

$$\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{G}_{S_j+\mathcal{B}}^D|I_{S_j}^+\right]=\mathbb{E}_{\mathcal{T}_{et}^D}\left[\mathcal{J}_{S_j}^{D+\mathcal{B}}|I_{S_j}^+\right]\le\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})<\mathbf{0}. \tag{31}$$

Hence, from the design of the event-triggered policy (14), it follows that $T_j>S_j+\mathcal{B}$, or in other words, no transmission takes place at least $\mathcal{B}$ timesteps from S_j, *in expectation*. Thus,

$$\mathbb{E}_{\mathcal{T}_{et}^D}\left[|\Delta_j^{(0)}|\Big|I_0^+,x_{S_j}^2\ge\mathcal{X}\right]\ge C^{(0)}.$$

Now, consider Claim (b). Note that $t_k=1$ for all $k\in\Delta_j^{(1)}$, and from the event-triggered policy, $\mathbb{E}_{\mathcal{T}_{et}^D}[|\Delta_j^{(1)}|]$ is simply the expected number of timesteps for reception under a string of continuous transmission attempts, starting from timestep T_j and channel state γ_{T_j}. To capture the same, we define the constant $C_i^{(1)}$ for $i\in[1,n]_{\mathbb{Z}}$ in (30). We bound $|\Delta_j^{(1)}|$ by simply choosing the highest value of $C_i^{(1)}$ among $i\in[1,n]_{\mathbb{Z}}$, thereby showing that $C^{(1)}$ is indeed an upper bound on $|\Delta_j^{(1)}|$. This proves Claim (b) and the result. □

Remark 5.2 (Trade-off between control performance and transmission fraction). Suppose for a given value of $\mathcal{X}$ and some $\psi\in\mathbb{N}$, we have $\mathbf{Q}_{\mathcal{X}}(\psi)<\mathbf{0}$ but $\mathbf{Q}_{\mathcal{X}}(\psi+1)\boldsymbol{\delta}_i\ge 0$ for at least one $i\in[1,n]_{\mathbb{Z}}$. Then if the operational value of the look-ahead parameter is D, we note that $D+\mathcal{B}=\psi$. The system designer can either choose a high value of D (conservative control) but this results in a lower value of $\mathcal{B}$, and thus a larger upper bound on $\mathcal{F}_{\mathcal{X}}$. Conversely, a lower value of D (aggressive control) leads to a higher value of $\mathcal{B}$, and thus a smaller upper bound on $\mathcal{F}_{\mathcal{X}}$.

We show in the following result that an upper bound on the asymptotic transmission fraction, $\mathcal{F}^{\infty}$ can be obtained by setting $\mathcal{X}=Bc^{-2D}$ in the upper bound of $\mathcal{F}_{\mathcal{X}}$ provided in Theorem 5.1.

Corollary 5.1 (Upper bound on asymptotic transmission fraction)**.** *The asymptotic transmission fraction $\mathcal{F}^{\infty}$ is upper bounded by*

$$\mathcal{F}^{\infty}\le\frac{C^{(1)}}{C_{\infty}^{(0)}+C^{(1)}},$$

where $C_{\infty}^{(0)}:=\max_{\mathcal{B}\in\mathbb{N}_0}\{\mathcal{B}\,|\,\mathbf{Q}(D+\mathcal{B})<\mathbf{0}\}$ and $C^{(1)}$ is as defined in Theorem 5.1.

Proof. The proof is similar to that of Theorem 5.1 except for one key difference. We note that in Theorem 5.1, $C^{(0)}$ was obtained as the $\mathcal{B}$-maximiser of $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})$ under the constraint that $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})<\mathbf{0}$. This ensured that the transmission fraction over the horizon $(S_j,S_{j+1}]_{\mathbb{Z}}$ is upper bounded by $C^{(1)}(C^{(0)}+C^{(1)})^{-1}$, under the assumption that $x_{S_j}^2\ge\mathcal{X}$. In case of asymptotic transmission fraction, we know that said upper bound on transmission fraction over the horizon $(S_j,S_{j+1}]_{\mathbb{Z}}$ has to hold for all $j\in\mathbb{N}_0$, and equivalently for all $x_{S_j}^2>0$. Thus we derive the term $C_{\infty}^{(0)}$ by first maximising $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})$ over all possible values of $\mathcal{X}$ and then choosing the largest value of $\mathcal{B}$ such that $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})<\mathbf{0}$ and setting $C_{\infty}^{(0)}$ equal to said value.

The former maximisation is carried out because $\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})\boldsymbol{\Pi}_{D+\mathcal{B}}(\gamma_{S_j})$ acts as an upper bound on $\mathcal{J}_{S_j}^{D+\mathcal{B}}$, which we want to be negative so that (31) is valid. Thus, we let

$$C_{\infty}^{(0)}:=\max_{\mathcal{B}\in\mathbb{N}_0}\{\mathcal{B}\,|\max_{\mathcal{X}\in\mathbb{R},\,\mathcal{X}\ge 0}\{\mathbf{Q}_{\mathcal{X}}(D+\mathcal{B})\}<\mathbf{0}\}$$
$$=\max_{\mathcal{B}\in\mathbb{N}_0}\{\mathcal{B}\,|\,\mathbf{Q}(D+\mathcal{B})<\mathbf{0}\},$$

which follows from the fact that $c^2>\bar{a}^2$ and the definitions of $\mathbf{Q}_{\mathcal{X}}(\theta)$ and $\mathbf{Q}(\theta)$. The rest of the proof follows along similar lines as that of Theorem 5.1. □

6 | ILLUSTRATIVE EXAMPLE

In this section, we validate our transmission policy design through simulations. In this section, we illustrate the wider applicability of our channel model and our proposed design method with a model-based example. We consider control with a battery-powered EH sensor, and the state of charge (SoC) of the said battery constitutes the 'channel' state. The channel state evolves according to a *linear saturated system* with noise, which fits in the action-dependent Markov channel framework.

6.1 | Energy-harvesting sensor

In this subsection, we model an EH sensor with a battery. The amount of energy harvested by the sensor is assumed to be stochastic, and a lack of enough energy collected by the sensor could lead to failure of transmissions. We model the SoC of the battery as a discrete valued quantity in the set $[0, \bar{s}]_{\mathbb{Z}}$, where $\bar{s} > 0$ represents the maximum SoC. We let $S_k \in [0, \bar{s}]_{\mathbb{Z}}$ denote the battery SoC on timestep k, which also is the 'channel' state in our framework. On every timestep, the battery first provides energy for transmission if required ($t_k = 1$), and then harvests energy according to an arrival process $\{Z_k\}_{k=1}^{\infty}$, which we assume to be i.i.d. We let $\eta \in \mathbb{N}$ be the energy cost of making a successful transmission, and if there is less than η units of energy in the battery, the transmission fails and no energy is extracted from the battery. The above dynamics can be represented with a linear saturated system as

$$S_k^+ = \left\{ \begin{array}{ll} S_k, & \text{if } S_k < t_k\eta \\ S_k - t_k\eta, & \text{if } S_k \geq t_k\eta, \end{array} \right) \tag{32a}$$

$$S_{k+1} = \min\{S_k^+ + Z_k, \bar{s}\}, \ \forall k \in \mathbb{N}_0, \tag{32b}$$

where S_k^+ is the intermediate state after possibly a transmission, which utilises energy from the battery. We now derive the Markov transition matrices $\mathbf{P}_0$ and $\mathbf{P}_1$. From (32), we can obtain the (i, j)th element of $\mathbf{P}_0$ and $\mathbf{P}_1$, with $t_k = 0$ and $t_k = 1$, respectively, as

$$\Pr\left[S_{k+1} = s^{(i)} \mid S_k = s^{(j)}, t_k\right]$$

$$= \begin{cases} \Pr\left[Z_k = s^{(i)} - s^{(j)}\right], & \text{if } s^{(j)} < t_k\eta \text{ and } s^{(i)} < \bar{s} \\ \Pr\left[Z_k = s^{(i)} - (s^{(j)} - t_k\eta)\right], & \text{if } s^{(j)} \geq t_k\eta \text{ and } s^{(i)} < \bar{s} \\ \Pr\left[Z_k \geq s^{(i)} - s^{(j)}\right], & \text{if } s^{(j)} < t_k\eta \text{ and } s^{(i)} = \bar{s} \\ \Pr\left[Z_k \geq s^{(i)} - (s^{(j)} - t_k\eta)\right], & \text{if } s^{(j)} \geq t_k\eta \text{ and } s^{(i)} = \bar{s}, \end{cases}$$

where $s^{(i)} \in [0, \bar{s}]_{\mathbb{Z}}$ is the ith discrete level that the battery SoC could be in. For the purpose of simulations, let Z_k belong to a Poisson distribution with arrival rate $\lambda > 0$. Thus, $\Pr[Z_k = q] = \exp(-\lambda)\lambda^q(q!)^{-1}$ for $q \geq 0$, and $\Pr[Z_k = q] = 0$ for any $q < 0$. In order to determine the packet-drop probabilities, that is, the vector $\mathbf{e}$, we note that for any state s, if $s < \eta$ then the probability of packet drop is 1, otherwise it is 0. We write this formally as

$$\mathbf{e}(j) = \begin{cases} 1, & \text{if } s^{(j)} < \eta \\ 0, & \text{if } s^{(j)} \geq \eta, \end{cases}$$

where $\mathbf{e}(j)$ represents the jth element of the vector $\mathbf{e}$.

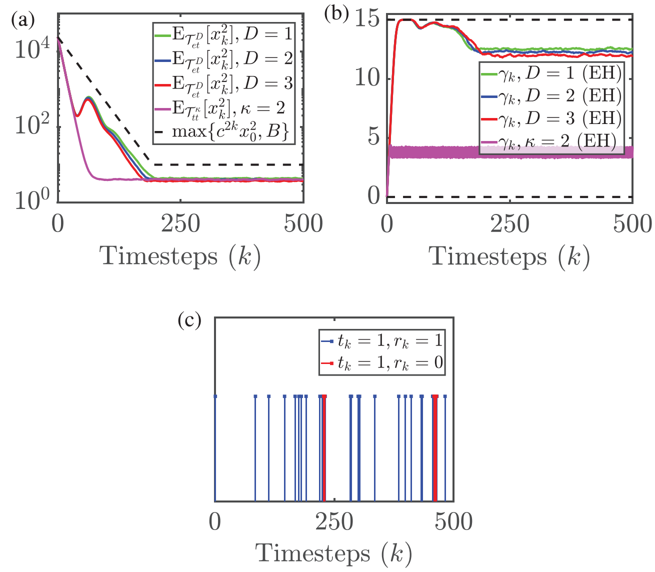

FIGURE 2 Evolution of the empirical mean of the plant and 'channel' states *Note*: (a) The empirical evolution of the second-moment plant state for the energy-harvesting (EH) sensor channel model. (b) The evolution of empirical mean of state of charge (SoC) of the battery attached under the EH sensor. In both figures, trajectories are provided for various values of look-ahead parameter D, and for the times-triggered policy with $\kappa = 2$. (c) The stem plot of transmissions and receptions under event-triggered transmissions for one realisation.

6.2 | Simulation results

For the EH sensor model, we choose the parameters $\bar{s} = 15$, $\eta = 8$ and $\lambda = 0.85$, while for the plant parameters, we choose the values $a = 1.05$, $c = 0.98$, $\bar{a} = 0.95c$, $M = 0.25$, $B = 10$ and $x_0 = 15.5B$. From the calculations presented in the Appendix, we find that $B^* = 2.32$, and therefore the condition $B > B^*$ is satisfied. We carried out simulations using MATLAB. In order to generate empirical results, we simulate the system evolution 5000 times, followed by taking an average of these results. For the channel, we set the initial state $\gamma_0 = 1$ for all simulated trajectories, that is, the battery starts off completely discharged.

The simulation results are presented in Figures 2 and 3. In particular, Figure 2(a) shows the evolution of the empirical mean of the plant state for different values of the look-ahead parameter D. We note that a higher value of D leads to more 'aggressive' control as described in Remark 5.2. Figure 2(b) shows the evolution of the empirical mean of the battery SoC. In order to compare performance of the event-triggered policy with a periodic time-triggered policy, we also include in Figure 2(a) and (b) the evolution of plant and channel state under policy $\mathcal{T}_{tt}^{\kappa}$, which sets $t_k = 1$ for every k which is an integer multiple of $\kappa \in \mathbb{N}_0$, and $t_k = 0$ otherwise. It is interesting to note in Figure 2(b) that the battery SoC (channel state) settles to a constant value after initial transient behaviour, and this constant value is smaller for larger values of D, that is, a higher value of D expends more energy from the battery. The benefit in terms of energy savings in the EH battery under the proposed policy over periodic time-triggered policies is evident from Figure 2(b). In

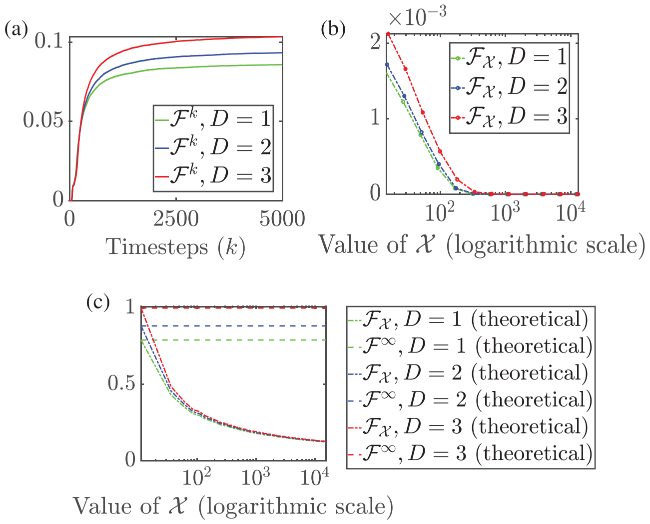

FIGURE 3 Simulation results and theoretical upper bounds on the transmission fractions $\mathcal{F}^k$ and $\mathcal{F}_{\mathcal{X}}$ for various values of look-ahead parameter *DNote*: Subparts (a) and (b) show the empirical mean values of $\mathcal{F}^k$ and $\mathcal{F}_{\mathcal{X}}$ respectively, while subpart (c) shows the theoretical upper bounds on $\mathcal{F}^k$ and $\mathcal{F}_{\mathcal{X}}$.

order to demonstrate the pattern of transmission times under the event-triggered policy, we display a stem plot of transmission and reception for one realisation of system evolution under event-triggered transmissions in Figure 2(c).

Figure 3(a) shows the empirical value of the transmission fraction $\mathcal{F}^k$ for both models for 5000 timesteps, and it can be seen that $\mathcal{F}^k$ reaches a steady-state value for large k, with greater values of D leading to higher asymptotic values of $\mathcal{F}^k$. Figure 3(b) shows the empirical value of $\mathcal{F}_{\mathcal{X}}$ generated during the simulation, while Figure 3(c) shows the theoretical upper bounds on both $\mathcal{F}_{\mathcal{X}}$ (given in Theorem 5.1) and $\mathcal{F}^\infty$ (given in Corollary 5.1). From Figure 3(c), it can be seen that the theoretical upper bound on $\mathcal{F}^\infty$ is similar to the theoretical upper bound on $\mathcal{F}_{\mathcal{X}}$ for $\mathcal{X} = Bc^{-2D}$, as noted in the proof of Theorem 5.1. As expected, both empirical values of $\mathcal{F}^k$ and $\mathcal{F}_{\mathcal{X}}$, and their respective upper bounds are greater for larger values of D, which demonstrates the trade-off between performance and transmission fraction, as discussed in Remark 5.2.

7 | CONCLUSION

This paper considers an NCS consisting of a scalar linear plant with process noise and non-collocated sensor and controller. Further, the sensor communicates over a time-varying channel whose state evolves according to an action-dependent Markov process. The state of the channel determines the probability with which a packet transmitted by the sensor is dropped. In this setting, we have designed an event-triggered transmission policy that guarantees the second-moment stabilisation of the plant state at a desired rate of convergence to an ultimate bound. We also derived upper bounds on the transient and the asymptotic transmission fraction, the fraction of timesteps on which the sensor transmits. We have verified and illustrated our analysis and theoretical guarantees through simulations in an example scenario, in which we considered the problem of control with an EH and battery-equipped sensor. Future work includes incorporation of imperfect measurement of plant and channel state, application of the proposed action-dependent Markov channel framework to control over a shared channel and over channels that are queuing processes.

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REFERENCES

1. Hespanha, J.P., Naghshtabrizi, P., Xu, Y.: A survey of recent results in networked control systems. Proc. IEEE 95(1), 138–162 (2007)
2. Park, P., et al.: Wireless network design for control systems: a survey. IEEE Commun. Surv. Tutorials 20(2), 978–1013 (2017)
3. Matveev, A.S., Savkin, A.V.: Estimation and Control over Communication Networks. Birkhäuser, Boston, MA (2009)
4. Yüksel, S., Başar, T.: Stochastic Networked Control Systems: Stabilization and Optimization Under Information Constraints. Systems & Control: Foundations and Applications Series, pp. 1–482.Springer, New York (2013)
5. Lemmon, M.: Event-triggered feedback in control, estimation, and optimization. In: Bemporad, A., Heemels, M., Johansson, M. (eds.) Networked Control Systems, pp. 293–358.Springer, London (2010)
6. Heemels, W.P.M.H., Johansson, K.H., Tabuada, P.: An introduction to event-triggered and self-triggered control. IEEE Conference on Decision and Control, Maui, HI, pp. 3270–3285 (2012)
7. Tolić, D., Hirche, S.: Networked Control Systems with Intermittent Feedback. CRC Press, Boca Raton, FL (2017)
8. Zou, L., Wang, Z.D., Zhou, D.H.: Event-based control and filtering of networked systems: a survey. Int. J. Autom. Comput. (2017) 14(3), 239–253
9. Peng, C., Li, F.: A survey on recent advances in event-triggered communication and control. Inf. Sci. 457, 113–125 (2018)
10. Astrom, K.J., Bernhardsson, B.M.: Comparison of Riemann and Lebesgue sampling for first order stochastic systems. Proceedings of the 41st IEEE Conference on Decision and Control, vol. 2, pp. 2011–2016, Las Vegas, NV (2002)
11. Henningsson, T., Johannesson, E., Cervin, A.: Sporadic event-based control of first-order linear stochastic systems. Automatica 44(11), 2890–2895 (2008)
12. Rabi, M., Johansson, K.H.: Scheduling packets for event-triggered control. European Control Conference, pp. 3779–3784.IEEE, (2009)
13. Molin, A., Hirche, S.: On the optimality of certainty equivalence for event-triggered control systems. IEEE Trans. Autom. Control 58(2), 470–474 (2013)
14. Leong, A.S., Dey, S., Quevedo, D.E.: Sensor scheduling in variance based event triggered estimation with packet drops. IEEE Trans. Autom. Control 62(4), 1880–1895 (2017)
15. Blind, R., Allgöwer, F.: The performance of event-based control for scalar systems with packet losses. IEEE Conference on Decision and Control, Maui, HI, pp. 6572–6576 (2012)
16. Leong, A.S., et al.: Event-based transmission scheduling and LQG control over a packet dropping link. IFAC PapersOnLine 50(1), 8945–8950 (2017)
17. Demirel, B., et al.: On the trade-off between communication and control cost in event-triggered dead-beat control. IEEE Trans. Autom. Control 62(6), 2973–2980 (2017)

18. Dolk, V., Heemels, M.: Event-triggered control systems under packet losses. Automatica 80, 143–155 (2017)
19. Molin, A., Hirche, S.: Price-based adaptive scheduling in multi-loop control systems with resource constraints. IEEE Trans. Autom. Control 59(12), 3282–3295 (2014)
20. Mamduhi, M.H., et al.: Error-dependent data scheduling in resource-aware multi-loop networked control systems. Automatica 81, 209–216 (2017)
21. Tallapragada, P., Franceschetti, M., Cortés, J.: Event-triggered second-moment stabilization of linear systems under packet drops. IEEE Trans. Autom. Control 63(8), 2374–2388 (2018)
22. Quevedo, D.E., et al.: Stochastic stability of event-triggered anytime control. IEEE Trans. Autom. Control 59(12), 3373–3379 (2014)
23. Anderson, R.P., Milutinović, D., Dimarogonas, D.V.: Self-triggered sampling for second-moment stability of state-feedback controlled SDE systems. Automatica 54, 8–15 (2015)
24. Bose, S., Tallapragada, P.: Event-triggered second moment stabilization under Markov packet drops. Indian Control Conference, New Delhi, India, pp. 113–118 (2019)
25. You, K., Fu, M., Xie, L.: Mean square stability for Kalman filtering with Markovian packet losses. Automatica 47(12), 2647–2657 (2011)
26. Wu, J., et al.: Kalman filtering over Gilbert-Elliott channels: stability conditions and critical curve. IEEE Trans. Autom. Control 63(4), 1003–1017 (2018)
27. Long, Y., et al.: Stochastic channel allocation for nonlinear systems with Markovian packet dropout. J. Syst. Sci. Complex. 31(1), 22–37 (2018)
28. You, K., Xie, L.: Minimum data rate for mean square stabilizability of linear systems with Markovian packet losses. IEEE Trans. Autom. Control 56(4), 772–785 (2011)
29. Okano, K., Ishii, H.: Stabilization of uncertain systems with finite data rates and Markovian packet losses. IEEE Trans. Control Network Syst. 1(4), 298–307 (2014)
30. Wu, J., et al.: An improved stability condition for Kalman filtering with bounded Markovian packet losses. Automatica 62, 32–38 (2015)
31. Mu, X., Zheng, B.: Containment control of second-order discrete-time multi-agent systems with Markovian missing data. IET Control Theory Appl. 9(8), 1229–1237 (2015)
32. Minero, P., Coviello, L., Franceschetti, M.: Stabilization over Markov feedback channels: the general case. IEEE Trans. Autom. Control 58(2), 349–362 (2013)
33. Xu, L., Xie, L., Xiao, N.: Mean-square stabilization over Gaussian finite-state Markov channels. IEEE Trans. Control Network Syst. 5(4), 1830–1840 (2018)
34. Su, L., Gupta, V., Chesi, G.: Stabilization of linear systems across a time-varying AWGN fading channel. IEEE Trans. Autom. Control 65, 4902-4907 (2020)
35. Guo, G., Wang, L.: Control over medium-constrained vehicular networks with fading channels and random access protocol: a networked systems approach. IEEE Trans. Veh. Technol. 64(8), 3347–3358 (2015)
36. Wen, S., Guo, G.: Sampled-data control for connected vehicles with markovian switching topologies and communication delay. IEEE Trans. Intell. Transp. Syst. 21(7), 2930–2942 (2020)
37. Gilbert, E.N.: Capacity of a burst-noise channel. Bell Syst. Tech. J. 39(5), 1253–1265 (1960)
38. Elliott, E.O.: Estimates of error rates for codes on burst-noise channels. Bell Syst. Tech. J. 42(5), 1977–1997 (1963)
39. Sadeghi, P., et al.: Finite-state Markov modeling of fading channels – a survey of principles and applications. IEEE Signal Process Mag. 25(5), 57–80 (2008)
40. Guo, Y., et al.: Buffer-aware streaming in small-scale wireless networks: a deep reinforcement learning approach. IEEE Trans. Veh. Technol. 68(7), 6891–6902 (2019)
41. Wang, C., et al.: Reinforcement learning-based adaptive transmission in time-varying underwater acoustic channels. IEEE Access 6, 2541–2558 (2018)
42. Jog, V., La, R.J., Martins, N.C.: Channels, learning, queueing and remote estimation systems with a utilization-dependent component. arXiv:190504362 (2019)
43. Lin, M., Martins, N.C., La, R.J.: Queueing subject to action-dependent server performance: utilization rate reduction. arXiv:200208514 (2020)
44. Tutuncuoglu, K., et al.: The binary energy harvesting channel with a unit-sized battery. IEEE Trans. Inf. Theory 63(7), 4240–4256 (2017)
45. Franceschetti, M., Minero, P.: Elements of information theory for networked control systems. In: Como, G., Bernhardsson, B., Rantzer, A. (eds.) Information and Control in Networks, vol. 450, pp. 3–37.Springer, New York (2014)
46. Tallapragada, P., Franceschetti, M., Cortés, J.: Event-triggered stabilization of scalar linear systems under packet drops. Allerton Conference on Communications, Control and Computing, Monticello, IL, pp. 1173–1180 (2016)

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APPENDIX

Proof of Proposition 4.4

Proof. We structure the proof in the form of two claims.

Claim (a): For the look-ahead parameter $\theta \in \mathbb{N}$, the performance-evaluation function $\mathcal{J}_{S_j}^{\theta}$ is uniformly (in x_{S_j}) upper bounded as $\mathcal{J}_{S_j}^{\theta} \leq \mathcal{R}_j(\theta)$, where

$$\mathcal{R}_j(\theta) := \Big[\tilde{g}_{\theta}(\bar{a}^2, \gamma_{S_j}) - \tilde{g}_{\theta}(c^2, \gamma_{S_j})\Big] \frac{B}{c^{2\theta}} + \bar{M}\Big[\tilde{g}_{\theta}(a^2, \gamma_{S_j}) - \tilde{g}_{\theta}(1, \gamma_{S_j})\Big].$$

Claim (b): For $\theta \in \mathbb{N}$, $\mathbf{Q}(\theta) < \mathbf{0}$ implies $\mathcal{R}_j(\theta) < 0, \forall j \in \mathbb{N}_0$. Further, $\mathbf{Q}(D) < \mathbf{0}$ for $D \in \mathbb{N}$ implies $\mathbf{Q}(\theta) < \mathbf{0}, \forall \theta \in [1, D]_{\mathbb{Z}}$.

Note that if Claims (a) and (b) are valid, then the result of Proposition 4.4 follows. For proving Claim (a), we partition the possible values of $x_{S_j}^2$ into two sets:

$$\Lambda_1 := [0, Bc^{-2\theta}), \quad \Lambda_2 := [Bc^{-2\theta}, \infty),$$

and demonstrate that $\mathcal{J}_{S_j}^{\theta} < \mathcal{R}_j(\theta)$ in each case. The proof is centred around the following two sub-claims, which establish bounds on some important terms of the closed form of $\mathcal{J}_{S_j}^{\theta}$ from Lemma 4.2.

Claim (a1): If $x_{S_j}^2 \in \Lambda_1$, then $B\tilde{f}_{\theta}(1, \gamma_{S_j}) \geq \frac{B}{c^{2\theta}}\tilde{g}_{\theta}(c^2, \gamma_{S_j})$.

Claim (a2): If $x_{S_j}^2 \in \Lambda_2$, then $B\tilde{f}_{\theta}(1, \gamma_{S_j}) \geq x_{S_j}^2 \tilde{f}_{\theta}(c^2, \gamma_{S_j})$.

To prove Claim (a1), we recall the term ν in the closed form of $\tilde{f}_{\theta}(b, \gamma)$ in Lemma 4.2 and note that $\nu = \theta$ when $x_{S_j}^2 < Bc^{-2\theta}$. Thus, $\tilde{g}_{\theta}(b, \gamma_{S_j}) = \tilde{f}_{\theta}(b, \gamma_{S_j})$ when $x_{S_j}^2 \in \Lambda_1$. Now observe that

$$Bc^{-2\theta}\tilde{g}_{\theta}(c^2, \gamma_{S_j}) \overset{[r1]}{=} \frac{B}{c^{2\theta}} c^{2\theta} \mathbf{d}^T (\mathbf{I} - c^2 \mathbf{P}_1 \mathbf{E})^{-1} \mathbf{P}_0^{(\theta-1)} \mathbf{P}_1 \boldsymbol{\delta}_{\gamma_{S_j}}$$

$$\overset{[r2]}{\leq} B\mathbf{d}^T (\mathbf{I} - \mathbf{P}_1 \mathbf{E})^{-1} \mathbf{P}_0^{(\theta-1)} \mathbf{P}_1 \boldsymbol{\delta}_{\gamma_{S_j}} = B\tilde{f}_{\theta}(1, \gamma_{S_j}),$$

where [r1] uses the definition of $\tilde{g}_\theta(b, \gamma_{S_j})$, and [r2] follows from the facts that $c^2 < 1$ and $(\mathbf{I} - c^2\mathbf{P}_1\mathbf{E})^{-1} = \sum_{w=0}^{\infty}(c^2\mathbf{P}_1\mathbf{E})^w < \sum_{w=0}^{\infty}(\mathbf{P}_1\mathbf{E})^w = (\mathbf{I} - \mathbf{P}_1\mathbf{E})^{-1}$, where the inequality is element-wise. This completes the proof of Claim (a1).

To prove Claim (a2), we establish an upper bound on $c^{2\nu}$ under the assumption that $x_{S_j}^2 \in \Lambda_2$. Note that

$$c^{2\nu} = c^{2\max\left\{\theta, \left\lceil \frac{\log(x_{S_j}^2/B)}{\log(1/c^2)} \right\rceil\right\}} \leq c^{2\left\lceil \frac{\log(B/x_{S_j}^2)}{\log(c^2)} \right\rceil} \leq \frac{B}{x_{S_j}^2},$$

where we have again used the fact that $c^2 < 1$. From this bound, one can upper bound $x_{S_j}^2 \tilde{f}_\theta(c^2, \gamma_{S_j})$ as

$$\begin{aligned} x_{S_j}^2 \tilde{f}_\theta(c^2, \gamma_{S_j}) &\leq B\mathbf{d}^T(\mathbf{P}_1\mathbf{E})^{(\nu-\theta)}(\mathbf{I} - c^2\mathbf{P}_1\mathbf{E})^{-1}\mathbf{P}_0^{(\theta-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma_{S_j}} \\ &\leq B\mathbf{d}^T(\mathbf{P}_1\mathbf{E})^{(\nu-\theta)}(\mathbf{I} - \mathbf{P}_1\mathbf{E})^{-1}\mathbf{P}_0^{(\theta-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma_{S_j}} \\ &= B\tilde{f}_\theta(1, \gamma_{S_j}). \end{aligned}$$

This concludes the proof of Claim (a2).

Now, we recall the closed form of $\mathcal{J}_{S_j}^\theta$. If $x_{S_j}^2 \in \Lambda_1$, we have $\tilde{f}_\theta(c^2, \gamma_{S_j}) - \tilde{g}_\theta(c^2, \gamma_{S_j}) = 0$ and $x_{S_j}^2 < Bc^{-2\theta}$, while $\tilde{g}_\theta(\bar{a}^2, \gamma_{S_j}) \geq 0$. These facts along with Claim (a) imply that $\mathcal{J}_{S_j}^\theta \leq \mathcal{R}_j(\theta)$ when $x_{S_j}^2 \in \Lambda_1$. In the case that $x_{S_j}^2 \in \Lambda_2$, we rearrange the closed form of $\mathcal{J}_{S_j}^\theta$ as

$$\begin{aligned} \mathcal{J}_{S_j}^\theta =& [\tilde{g}_\theta(\bar{a}^2, \gamma_{S_j}) - \tilde{g}(c^2, \gamma_{S_j})]x_{S_j}^2 + \bar{M}[\tilde{g}_\theta(a^2, \gamma_{S_j}) \\ & - \tilde{g}_\theta(1, \gamma_{S_j})] - [B\tilde{f}_\theta(1, \gamma_{S_j}) - x_{S_j}^2\tilde{f}_\theta(c^2, \gamma_{S_j})]. \end{aligned}$$

Then using Claim (b), the fact that $\tilde{g}_\theta(\bar{a}^2, \gamma_{S_j}) < \tilde{g}_\theta(c^2, \gamma_{S_j})$ (since $\bar{a}^2 < c^2$), and lastly the fact that $x_{S_j}^2 \geq Bc^{-2\theta}$, we conclude that $\mathcal{J}_{S_j}^\theta \leq \mathcal{R}_j(\theta)$ when $x_{S_j}^2 \in \Lambda_2$. Thus, $\mathcal{R}_j(\theta)$ uniformly upper bounds $\mathcal{J}_{S_j}^\theta$ for all $x_{S_j}^2 \in [0, \infty)$.

We start the proof of Claim (b) by noting that $\mathcal{R}_j(\theta)$ can be written as

$$\mathcal{R}_j(\theta) = \mathbf{Q}(\theta)\mathbf{P}_0^{(\theta-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma_{S_j}}. \tag{A1}$$

From the element-wise non-negativity of $\mathbf{P}_0^{(\theta-1)}\mathbf{P}_1\boldsymbol{\delta}_{\gamma_{S_j}}$ for all $\theta \in \mathbb{N}$ and $\gamma_{S_j} \in [1, n]_{\mathbb{Z}}$, we conclude that a *sufficient* condition to ensure $\mathcal{R}_j(D) < 0$ for a given D and all $j \in \mathbb{N}_0$ is to ensure that $\mathbf{Q}(D) < \mathbf{0}$. We now show that every element of $\mathbf{Q}(\theta)$ is monotonically increasing in θ, and thus, $\mathbf{Q}(D) < \mathbf{0}$ ensures $\mathbf{Q}(\theta) < \mathbf{0}$ for $\theta \in [1, D]_{\mathbb{Z}}$. The first and the second derivatives of $\mathbf{Q}(\theta)$ with respect to θ are

$$\frac{\mathrm{d}\mathbf{Q}(\theta)}{\mathrm{d}\theta} = \frac{B}{c^{2\theta}}\log\left(\frac{\bar{a}^2}{c^2}\right)\mathcal{Z}_\theta(\bar{a}^2) + \bar{M}\log(a^2)\mathcal{Z}_\theta(a^2)$$

$$\frac{\mathrm{d}^2\mathbf{Q}(\theta)}{\mathrm{d}\theta^2} = \frac{B}{c^{2\theta}}\log^2\left(\frac{\bar{a}^2}{c^2}\right)\mathcal{Z}_\theta(\bar{a}^2) + \bar{M}\log^2(a^2)\mathcal{Z}_\theta(a^2).$$

Note that each element of the second derivative is strictly positive. Thus, each element of $\mathbf{Q}(\theta)$ is strictly convex in θ. Also, note that the first derivative of $\mathbf{Q}(\theta)$ at $\theta = 0$ is

$$\frac{\mathrm{d}\mathbf{Q}(\theta)}{\mathrm{d}\theta} \overset{[r1]}{>} B\log\left(\frac{c^2}{\bar{a}^2}\right)\left[\mathcal{Z}_0(a^2) - \mathcal{Z}_0(\bar{a}^2)\right] > 0,$$

where [r1] follows from the fact that $B \geq B_0$. Since each element of $\mathbf{Q}(\theta)$ is strictly convex for $\theta \in \mathbb{R}$ and increasing at $\theta = 0$, it follows that each element of $\mathbf{Q}(\theta)$ is monotonically increasing for $\theta \geq 0$. Thus, $\mathbf{Q}(D) < 0$ implies $\mathbf{Q}(\theta) < 0$, and thereby $\mathcal{J}_{S_j}^\theta < 0$ for all $\theta \in [1, D]_{\mathbb{Z}}$. $\square$

Procedure to compute a sufficient lower bound B^* on the ultimate bound B

Here, we provide a procedure to compute the lower bound B^* on B, referred to in Proposition 4.2. This procedure is based on the proof of Lemma IV.13 in [21] and we present it here for completeness. First, we define the following constants:

$$P_1 := \log(a^2/\bar{a}^2),\ P_2 := \log(a^2c^2/\bar{a}^2),\ P_3 := \log(1/c^2),$$

$$P_4 := \log\left(\frac{\log(1/\bar{a}^2)}{\bar{M}\log(a^2)}\right).$$

Then, consider the following functions of B:

$$\begin{aligned} &U(B) := e^{\frac{P_3P_4}{P_2}}B^{\frac{P_1}{P_2}}, \quad w_{**}(B) := \frac{\log(B)}{P_2} + \frac{P_4}{P_2}, \\ &Y(B) := \bar{a}^{2w_{**}(B)}U(B) + \bar{M}a^{2w_{**}(B)}, \\ &F_{**}(B) := Y(B) - \bar{M} - B. \end{aligned}$$

The function $F_{**}(B)$ is strictly concave in B [21, Lemma IV.13]. Thus, it has at most two zeroes, one of which is $B_0 = \frac{\bar{M}\log(a^2)}{\log(c^2/\bar{a}^2)}$. There is another zero $B_z > B_0$ of $F_{**}(B)$ only if $F_{**}(B)$ is increasing at $B = B_0$. Such a B_z can be found numerically. We let

$$B^* := \begin{cases} B_0, & \text{if } F_{**}(B) \text{ is non-increasing at } B = B_0 \\ B_z, & \text{otherwise.} \end{cases}$$

We can also generalise our results to the case when there is no process noise ($\bar{M} = 0$). For this scenario, note that a more basic definition of the function $F_{**}(.)$ is given in Equation (24 b) in our previous work [21]. From this definition, it is easy to see that when $\bar{M} = 0$, $F_{**}(y) := \bar{a}^{2\sigma}y - B$, where σ is such that $c^{2\sigma}y = B$. Thus, in particular, we see that $\sigma = 0$ when $y = B$ and hence $F_{**}(B) = 0$ for all $B \geq 0$. Further, note that $B_0 = 0$ if $\bar{M} = 0$. Hence, if $\bar{M} = 0$, we can choose $B = B^* = B_0 = 0$, which then guarantees asymptotic stability for the plant state to zero.