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Numerical Solution for a System of Fractional Differential Equations with Applications in Fluid Dynamics and Chemical Engineering

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Abstract:
In this paper, a Haar wavelets based numerical method to solve a system of linear or nonlinear fractional differential equations has been proposed. Numerous nontrivial test examples along with practical problems from fluid dynamics and chemical engineering have been considered to illustrate applicability of the proposed method. We have derived a theoretical error bound which plays a crucial role whenever the exact solution of the system is not known and also it guarantees the convergence of approximate solution to exact solution.

Keywords: Haar wavelets, Caputo fractional differential operator, system of fractional differential equations, higher order fractional differential equation

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1 Introduction

Researchers in almost all branches of science and engineering Podlubny (1999) have been modeling the real life problems with the help of fractional calculus. Like problems in viscoelastic Atanackovic and Stankovic (2002) and Khan (2009), falling body problem through air Fa (2005), fluid dynamics problem Khan and Wang (2009), Magneto-thermoelasticity problem Ezzat (2011) etc. have been successfully modeled in the form of a fractional differential equation. Also, there exists real life models involving system of fractional differential equations. Some of these models like fractional SIRC model and influenza A Khader and Babatin (2014), fractional Dengue model Diethelm (2013), fractional HIV model Ding and Ye (2009) etc. have been recently proposed models in the field of biology. We know the fact that there does not exist a general method to solve a system of integer order differential equations, similar is the case with system of fractional differential equations. Further, if the system of fractional differential equations is nonlinear, then it gets more difficult or many times not practically possible to find an analytical solution. This limitation has captured attention of researchers and a number of methods have been proposed which include fractional difference method, differential transform method, Adomian decomposition method and homotopy analysis method, predictor-corrector method Diethelm, Ford, and Freed (2002) and Diethelm (2010), fractional Adams method Diethelm, Ford, and Freed (2004) etc. While in this paper, we have developed the Haar wavelets based method to solve it. So first we will give a brief literature review related to Haar wavelets. Even though the Haar wavelets were proposed in the year 1909 by Haar, but significant development took place after the year 1997 in which Haar wavelets operational matrices were proposed by Chen and Hsiao Chen and Hsiao (1997). Development of operational matrices for Haar wavelets was not all of a sudden. The operational matrices based on Walsh functions were initially derived in 1977 by Cheng and Tsay Cheng and Tsay (1977). One of its unique feature was that it performs like an integrator in the time domain. Subsequently various operational matrices were derived based on different orthogonal functions like Laguerre Hwang and Shih (1981), Chebyshev Paraskevopoulos (1983), Legendre Chang and Wang (1984) and Fourier Paraskevopoulos, Sparcis, and Monroursos (1985) etc. These operational matrices were supported on the whole interval. So these matrices were having global support which was a drawback for analyzing the systems having sudden variation in time. One can overcome this drawback with the help of wavelets having local compact support. The simplest wavelet among these is Haar wavelet as it is made up of piecewise constants and one can integrate these functions easily over an arbitrary interval. Due to its simple structure over the other

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wavelets, the researchers interest have been continuously increasing in recent years and a lot of numerical methods based on Haar wavelets have been developed like time-varying singular nonlinear systems Hsiao and Wang (1999), nonlinear Fredholm integral equations of the second kind Babolian and Shahsavaran (2009), evaluation of Hankel transforms Pandey, Singh, and Singh (2010), second-order boundary-value problems Aziz Siraj-ul-Islam and Šarler (2010), singularly perturbed boundary-value problems Pandit and Kumar (2014), Burger's equation Jiwari (2012, 2015) , Mittal and Pandit (2017), Poisson equations and biharmonic equations Zhi and Yong-yan (2011), non linear coupled differential equation Mittal and Pandit (2017), fractional differential equation Chen, Yi, and Yu (2012), Volterra integro-differential equations Siraj-ul-Islam and Fayyaz (2013), fractional nonlinear differential equations Saeed and Rehman (2013), nonlinear Fredholm and Volterra integral equations Siraj-ul-Islam (2013) and integro-differential equations of first and higher orders Siraj-ul-Islam and Al-Fhaid (2014), Riccati differential equation Li et al. (2014), fractional nonlinear partial differential equations Saeed and Rehman (2015) etc.

In recent years, researchers have proposed various other wavelets like Chebyshev wavelets, Legendre wavelets Razzaghi and Yousefi (2001), CAS wavelets, Daubechies wavelets and Gegenbauer wavelets etc. Chebyshev wavelets have been used to solve systems of nonlinear singular fractional Volterra integro-differential equations Heydari et al. (2014), fractional differential equations Wang and Fan (2012), fractional nonlinear Fredholm integro-differential equations Zhu and Fan (2012), nonlinear systems of Volterra integral equations Biazar and Ebrahimi (2012) and nonlinear fractional-order Volterra integro-differential equations Zhu and Fan (2013). While Legendre wavelets, CAS wavelets and Gegenbauer wavelets are used to solve fractional differential equations Rehman and Khan (2011), nonlinear Fredholm integro-differential equations of fractional order Saeedi et al. (2011) and fractional differential equations Rehman and Saeed (2015) respectively.

In this paper, we solve the following system of fractional differential equations

$$\begin{array}{rcl} \mathscr{D}_*^{\alpha_1} u_1(t) & = & \mathscr{F}_1(t, u_1(t), u_2(t), \cdots, u_d(t)), \\ \mathscr{D}_*^{\alpha_2} u_2(t) & = & \mathscr{F}_2(t, u_1(t), u_2(t), \cdots, u_d(t)), \\ \vdots & & \vdots \\ \mathscr{D}_*^{\alpha_d} u_d(t) & = & \mathscr{F}_d(t, u_1(t), u_2(t), \cdots, u_d(t)), \end{array} \tag{1}$$

with initial conditions

$$u_l^{(j)}(0) = u_{l_i}, \tag{2}$$

where u_{l_i} are given real constants, $\mathscr{F}_l(\cdot)$ are known linear or nonlinear functions, $u_l(t)$ are the unknown functions and $\mathscr{D}_*^{\alpha_l}$ are Caputo fractional differential operators such that $l = 1, 2.., d;\ \ i = 0, 1, 2, ..., (m_l - 1);\ \ m_l - 1 < \alpha_l \le m_l;\ m_l, d \in \mathbb{N}$. The symbol $\mathbb{N}$ is used to denote the set of positive integers throughout this paper.

2 Preliminaries

In the preliminaries, the important definitions and results are stated which are used in this paper.

2.1 Fractional differential and integral operator

Among various definitions of a fractional derivative of order $\alpha > 0$, the two widely used definitions are the Riemann-Liouville and Caputo Miller and Ross (1993), Oldham and Spanier (1974), and Samko, Kilbas, and Marichev (1993). The Caputo fractional derivatives of order αGejji and Jafari (2007) is defined as

$$\mathscr{D}_*^{\alpha} u(x) = \mathscr{J}^{m-\alpha} \mathscr{D}^m u(x), \quad 0 \le m - 1 < \alpha \le m,\ m \in \mathbb{N}, \tag{3}$$

where $u(x) \in C^m[a, b];\ a, b \in \mathbb{R};\ \mathscr{D}^m$ is the standard differential operator of integer order m and $\mathscr{J}^{m-\alpha}$ is the Riemann-Liouville integral operator of order $(m - \alpha)$ such that the Riemann-Liouville fractional integral operator of order αMiller and Ross (1993) is defined as

$$\mathscr{J}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} u(t) dt, x > 0, \mathscr{J}^0 u(x) = u(x), \tag{4}$$

where $u(x)$ is piecewise continous on $(0, \infty)$ and integrable on any subinterval of $[0, \infty)$.

The relation between the Riemann-Liouville integral operator and Caputo fractional differential operator is given by the following lemma Chen, Yi, and Yu (2012)

Lemma

If $m-1 < \alpha \leq m$; $m \in \mathbb{N}$, then $\mathscr{D}_*^\alpha \mathscr{J}^\alpha u(x) = u(x)$, and

$$\mathscr{J}^\alpha \mathscr{D}_*^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0. \tag{5}$$

2.2 Haar wavelets

The Haar wavelet function Chen, Yi, and Yu (2012) and Li et al. (2014) can be defined on the real line as

$$\mathscr{H}(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right), \\ -1, & t \in \left[\frac{1}{2}, 1\right), \\ 0, & t \in (-\infty, 0] \cup [1, \infty). \end{cases} \tag{6}$$

For all t on $[0, 1)$, define

$$h_i(t) = \begin{cases} 1, & i = 0, \\ 2^{\frac{j}{2}} \mathscr{H}(2^j t - k), & i = 1, 2, \cdots; (\text{i.e. } i = 2^j + k : k = 0, 1, \cdots, 2^j - 1; j = 0, 1, \cdots). \end{cases} \tag{7}$$

The sequence $\{h_i(t)\}_{i=0}^{\infty}$ forms a complete orthonormal system Chen, Yi, and Yu (2012) and Li et al. (2014) in $\mathscr{L}^2[0, 1)$. Therefore, any function $u(t) \in \mathscr{L}^2[0, 1)$ can be expanded in a Haar wavelet as

$$u(t) = \sum_{i=0}^{\infty} c_i h_i(t). \tag{8}$$

And, on truncating this series for $u(t)$, we get the corresponding approximation of $u(t)$ as

$$u(t) \approx \sum_{i=0}^{k-1} c_i h_i(t) = C_{k\times1}^T \mathscr{H}_{k\times1}(t), \tag{9}$$

where the coefficients c_i are given by inner product $\langle u(t), h_i(t)\rangle = \int_0^1 u(t)h_i(t)dt, \;\; i = 1, 2, \cdots, (k-1)$, $C_{k\times1} = [c_0, c_1, \cdots, c_{k-1}]^T$ and $\mathscr{H}_{k\times1} = [h_0, h_1, \cdots, h_{k-1}]^T$ and superscript T denotes the transpose of the matrix.

3 Methodology for the solution

The proposed method of solution is based on the assumption that unique solution for system eq. (1) exists. First let us consider the approximation of fractional derivative with the help of Haar wavelets as

$$\begin{array}{lll} \mathscr{D}_*^{\alpha_1} u_1(t) & \approx \mathscr{D}_*^{\alpha_1} u_{1k}(t) & =^1 C_{1\times k}^T \mathscr{H}_{k\times1}(t), \quad m_1 - 1 < \alpha_1 \leq m_1, \\ \mathscr{D}_*^{\alpha_2} u_2(t) & \approx \mathscr{D}_*^{\alpha_2} u_{2k}(t) & =^2 C_{1\times k}^T \mathscr{H}_{k\times1}(t), \quad m_2 - 1 < \alpha_2 \leq m_2, \\ \vdots & \vdots & \\ \mathscr{D}_*^{\alpha_d} u_d(t) & \approx \mathscr{D}_*^{\alpha_d} u_{dk}(t) & =^d C_{1\times k}^T \mathscr{H}_{k\times1}(t), \quad m_d - 1 < \alpha_d \leq m_d. \end{array} \tag{10}$$

On operating fractional integral operator $\mathscr{J}^\alpha(\cdot)$ in the above system eq. (10), we get

$$\begin{array}{ll} u_1(t) & \approx^1 C_{1\times k}^T \mathscr{P}_{k\times k}^\alpha \mathscr{H}_{k\times1}(t) + \sum_{i=0}^{m_1-1} u_1^{(i)}(0^+)\frac{t^i}{i!}, \\ u_2(t) & \approx^2 C_{1\times k}^T \mathscr{P}_{k\times k}^\alpha \mathscr{H}_{k\times1}(t) + \sum_{i=0}^{m_2-1} u_2^{(i)}(0^+)\frac{t^i}{i!}, \\ \vdots & \qquad \vdots \\ u_d(t) & \approx^d C_{1\times k}^T \mathscr{P}_{k\times k}^\alpha \mathscr{H}_{k\times1}(t) + \sum_{i=0}^{m_d-1} u_d^{(i)}(0^+)\frac{t^i}{i!}, \end{array} \tag{11}$$

where $\mathscr{P}^{\alpha}_{k\times k}$ is fractional operational matrix Chen, Yi, and Yu (2012) of Haar wavelets. Now we will implement residual based Galerkin's method to solve the system eq. (1). For this, the residual error can be defined as

$$\epsilon_i\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right);\quad i=1,2,\cdots,d, \tag{12}$$

which occurs when we substitute the approximate solution eq. (11) defined with the help of Haar wavelets in each equation of system eq. (1). More precisely, the residual error can be defined through the following equations:

$$\begin{array}{rcl}
{}^{1}C^{T}_{1\times k}\mathscr{H}_{k\times 1}(t) & = & \mathscr{F}_1\left(t,{}^{1}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_1-1}u_1^{(i)}(0^+)\frac{t^i}{i!},\cdots,{}^{d}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_d-1}u_d^{(i)}(0^+)\frac{t^i}{i!}\right)\\
& + & \epsilon_1\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right),\\
{}^{2}C^{T}_{1\times k}\mathscr{H}_{k\times 1}(t) & = & \mathscr{F}_2\left(t,{}^{1}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_1-1}u_1^{(i)}(0^+)\frac{t^i}{i!},\cdots,{}^{d}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_d-1}u_d^{(i)}(0^+)\frac{t^i}{i!}\right)\\
& + & \epsilon_2\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right),\\
\vdots & & \qquad\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
{}^{d}C^{T}_{1\times k}\mathscr{H}_{k\times 1}(t) & = & \mathscr{F}_d\left(t,{}^{1}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_1-1}u_1^{(i)}(0^+)\frac{t^i}{i!},\cdots,{}^{d}C^{T}_{1\times k}\mathscr{P}^{\alpha}_{k\times k}\mathscr{H}_{k\times 1}(t)+\sum\limits_{i=0}^{m_d-1}u_d^{(i)}(0^+)\frac{t^i}{i!}\right)\\
& + & \epsilon_d\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right).
\end{array} \tag{13}$$

Now incorporating the initial conditions eq. (2) and then insisting that the residual error

$$\epsilon_i\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right);\quad i=1,2,\cdots,d,$$

in above system eq. (13) vanishes at k number of collocation points $t_i=\frac{2i-1}{2k}; i=1,2,\cdots,k$. Hence we obtain $d\times k$ number of linear or nonlinear algebraic equations

$$\epsilon_i\left(t,\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k}\right)=0;\quad i=1,2,\cdots,d, \tag{14}$$

in $d\times k$ number of unknowns

$$\mathscr{C}_{1,0},\mathscr{C}_{1,1}\cdots,\mathscr{C}_{1,k};\quad \mathscr{C}_{2,0},\mathscr{C}_{2,1}\cdots,\mathscr{C}_{2,k};\quad \cdots;\quad \mathscr{C}_{d,0},\mathscr{C}_{d,1}\cdots,\mathscr{C}_{d,k},$$

depending on whether the given system eq. (1) is linear or nonlinear.

4 Error analysis

We embark on error analysis by first defining a function space $\mathscr{L}^2_d$ as

$$\mathscr{L}^2_d=\left\{u(x)=(u_1(x),u_2(x),...,u_d(x))^T : u_j(x)\in\mathscr{L}^2[0,1); j=1,2,\cdots,d\right\}, \tag{15}$$

which is a Hilbert space equipped with following norm $\|\,.\,\|_{\mathscr{L}^2_d}$ and inner product $\langle .,.\rangle_{\mathscr{L}^2_d}$

$$\|\,u\,\|_{\mathscr{L}^2_d}\ =\left(\frac{1}{\mathscr{N}}\sum_{j=1}^{d}\|\,u_j\,\|^2_{\mathscr{L}^2}\right)^{\frac{1}{2}}\quad \text{for } u(x)\in\mathscr{L}^2_d, \tag{16}$$

$$\langle u,v\rangle_{\mathscr{L}^2_d}\ =\frac{1}{\mathscr{N}}\sum_{j=1}^{d}\langle u_j,v_j\rangle_{\mathscr{L}^2}\qquad \text{for } u(x),v(x)\in\mathscr{L}^2_d, \tag{17}$$

where $\mathscr{N}$ is a positive constant and $\mathscr{L}^2[0,1)=\{u_j:[0,1)\to\mathbb{R}:\int_0^1[u_j(x)]^2dx<\infty, j=1,2,\cdots,d\}$ is a Hilbert Space of all real valued functions which are square integrable over the interval $[0,1)$. This Hilbert space is equipped with the norm $\|\cdot\|^2_{\mathscr{L}^2}$ and inner product $\langle .,.\rangle_{\mathscr{L}^2}$ which are defined as:

$$\|u_j\|_{\mathscr{L}^2} = \Big(\int_0^1 [u_j(x)]^2 dx\Big)^{1/2} : u_j(x) \in \mathscr{L}^2; j = 1, 2, \cdots, d,$$

$$\langle u_j, v_j\rangle_{\mathscr{L}^2} = \int_0^1 [u_j(x)v_j(x)]dx : u_j(x), v_j(x) \in \mathscr{L}^2, j = 1, 2, \cdots, d.$$

We assume that $u_l^{(m_l)}(t)$ is continuous and bounded on $(0, 1)$ where the superscript m_l in braces denotes derivative of order m_l for function $u_l(t)$. This derivative notation is used throughout the paper. This implies that there exists $\mathscr{M}_l > 0$ for all $t \in (0, 1)$ such that $|u_l^{(m_l)}(t)| \le \mathscr{M}_l$, $l = 1, 2, ..., d$. Defining $\mathscr{M} = \max_{1\le l\le d} \mathscr{M}_l$, we get

$$|u_l^{(m_l)}(t)| \le \mathscr{M}, \forall \;\; l = 1, 2, ..., d. \tag{18}$$

For convenience entitling some more expressions as

$$\begin{aligned}
\mathscr{M}_{min} &= \min_{1\le l\le d}\{m_l - \alpha_l\},\\
\alpha_{min} &= \min_{1\le l\le d}\{\alpha_l\},\\
\Gamma_{(m-\alpha)_{min}} &= \min_{1\le l\le d}\{\Gamma(m_l - \alpha_l)\},\\
\Gamma_{\alpha_{min}} &= \min_{1\le l\le d}\{\Gamma(\alpha_l)\},
\end{aligned}$$

which will be used later.

The function $\mathscr{D}_*^\alpha u(t)$ can be expressed in terms of scaling Haar function $h_0(t)$ and Haar wavelets $h_i(t); i = 1, 2, \cdots, \infty$ as

$$\mathscr{D}_*^\alpha u(t) = \sum_{i=0}^{\infty} \mathscr{C}_i h_i(t), \tag{19}$$

where $\mathscr{C}_i$ are defined with the help of inner porduct as $\mathscr{C}_i = \langle \mathscr{D}_*^\alpha u(t), h_i(t)\rangle_{\mathscr{L}_d^2}$.

Now the function $\mathscr{D}_*^\alpha u(t)$ can be approximated with the help of Haar wavelets as $\mathscr{D}_*^\alpha u(t) \approx \mathscr{D}_*^\alpha u_k(t) = \left[\mathscr{D}_*^{\alpha_1} u_{1k}(t), \mathscr{D}_*^{\alpha_2} u_{2k}(t), ..., \mathscr{D}_*^{\alpha_d} u_{dk}(t)\right]^T = \left[\sum_{i=0}^{k-1} \mathscr{C}_{1,i}h_i(t), \sum_{i=0}^{k-1} \mathscr{C}_{2,i}h_i(t), ..., \sum_{i=0}^{k-1} \mathscr{C}_{d,i}h_i(t)\right]^T$, which implies

$$\mathscr{D}_*^\alpha u(t) - \mathscr{D}_*^\alpha u_k(t) = \left[\sum_{i=k}^{\infty} \mathscr{C}_{1,i}h_i(t), \sum_{i=k}^{\infty} \mathscr{C}_{2,i}h_i(t), ..., \sum_{i=k}^{\infty} \mathscr{C}_{d,i}h_i(t)\right]^T. \tag{20}$$

The following theorem gives an error bound for fractional derivative by using Haar wavelets.

Theorem 1

Suppose that the functions $\mathscr{D}_*^\alpha u_k(t)$ obtained by using Haar wavelet are the approximation of $\mathscr{D}_*^\alpha u(t)$, then an upper bound of the error with respect to $\mathscr{L}_d^2$-norm is given by

Case I: $m_l - 1 < \alpha_l < m_l$, $l = 1, 2 \cdots, d$

$$\|\mathscr{D}_*^\alpha u - \mathscr{D}_*^\alpha u_k\|_{\mathscr{L}_d^2} \le \frac{\mathscr{M}\sqrt{d}}{\sqrt{\mathscr{N}\left(1 - 2^{-2\mathscr{M}_{min}}\right)}\mathscr{M}_{min}\Gamma_{(m-\alpha)_{min}} k^{\mathscr{M}_{min}}}.$$

Case II: *$\alpha_l = m_l$, $l = 1, 2 \cdots, d$, We assume that $u_l^{m_l+1}(t)$ is continuous and bounded on $[0, 1)$*

$$\|\mathscr{D}_*^\alpha u - \mathscr{D}_*^\alpha u_k\|_{\mathscr{L}_d^2} \le \frac{\mu\sqrt{d}}{k\sqrt{3\mathscr{N}}},$$

where $\mu > 0$ for all $t \in [0, 1)$ such that

$$|u_l^{m_l+1}(t)| \le \mu, \; l = 1, 2, \cdots, d.$$

Proof

The error in approximation of function $\mathscr{D}_*^{\alpha} u(t)$ by Haar wavelets with respect to $\mathscr{L}_d^2$-norm is given by

$$\|\mathscr{D}_*^{\alpha} u - \mathscr{D}_*^{\alpha} u_k\|_{\mathscr{L}_d^2} = \left[\frac{1}{\mathcal{N}} \sum_{l=1}^{d} \int_0^1 (\mathscr{D}_*^{\alpha_l} u_l(t) - \mathscr{D}_*^{\alpha_l} u_{lk}(t))^2 dt\right]^{\frac{1}{2}}. \tag{21}$$

Using eq. (20), we get

$$\|\mathscr{D}_*^{\alpha} u - \mathscr{D}_*^{\alpha} u_k\|_{\mathscr{L}_d^2}^2 = \frac{1}{\mathcal{N}} \sum_{l=1}^{d} \sum_{i=2^{\beta+1}}^{\infty} \sum_{j=2^{\beta+1}}^{\infty} \mathscr{C}_{l,i}\mathscr{C}_{l,j} \int_0^1 h_i(t)h_j(t)dt.$$

The orthonormality of the sequence $\{h_i(t)\}$ on $[0,1)$ implies that $\int_0^1 \mathscr{H}_k(t)\mathscr{H}_k^T(t)dt = I_k$, where I_k is an identity matrix of order k, hence one can obtain

$$\|\mathscr{D}_*^{\alpha} u - \mathscr{D}_*^{\alpha} u_k\|_{\mathscr{L}_d^2}^2 = \frac{1}{\mathcal{N}} \sum_{l=1}^{d} \sum_{j=\beta+1}^{\infty} \sum_{i=2^j}^{2^{j+1}-1} \mathscr{C}_{l,i}^2, \tag{22}$$

where $\mathscr{C}_{l,i} = \langle \mathscr{D}_*^{\alpha_l} u_l(t), h_i(t)\rangle = \int_0^1 \mathscr{D}_*^{\alpha_l} u_l(t) 2^{\frac{j}{2}} \mathscr{H}(2^j t - k)dt$, $k = 0, 1, .., 2^j - 1; j = 0, 1, 2....$, which gives

$$\mathscr{C}_{l,i} = 2^{\frac{j}{2}} \{ \int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} \mathscr{D}_*^{\alpha_l} u_l(t)dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} \mathscr{D}_*^{\alpha_l} u_l(t)dt\}. \tag{23}$$

Using mean value theorem of integrals, there exists t_{l_1}, t_{l_2} such that

$$k2^{-j} \le t_{l_1} \le \left(k + \frac{1}{2}\right) 2^{-j}, \quad \left(k + \frac{1}{2}\right) 2^{-j} \le t_{l_2} \le (k+1)\, 2^{-j},$$

which gives

$$\mathscr{C}_{l,i} = 2^{\frac{j}{2}} \left(\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} \mathscr{D}_*^{\alpha_l} u_l(t_{l_1})dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} \mathscr{D}_*^{\alpha_l} u_l(t_{l_2})dt \right),$$

hence we obtain

$$\mathscr{C}_{l,i}^2 = 2^{-(j+2)} \left(\mathscr{D}_*^{\alpha_l} u_l(t_{l_1}) - \mathscr{D}_*^{\alpha_l} u_l(t_{l_2}) \right)^2. \tag{24}$$

From the definition of Caputo fractional differential, we have

$$|\mathscr{D}_*^{\alpha_l} u_l(t_{l_1}) - \mathscr{D}_*^{\alpha_l} u_l(t_{l_2})| = \frac{1}{\Gamma(m_l - \alpha_l)} \left| \int_0^{t_{l_1}} \frac{u_l^{(m_l)} T}{(t_{l_1} - T)^{\alpha_l - m_l + 1}} dT - \int_0^{t_{l_2}} \frac{u_l^{(m_l)} T}{(t_{l_2} - T)^{\alpha_l - m_l + 1}} dT \right|.$$

On simplifying it further and using eq. (18), one can obtain

$$|\mathscr{D}_*^{\alpha_l} u_l(t_{l_1}) - \mathscr{D}_*^{\alpha_l} u_l(t_{l_2})| \le \frac{\mathscr{M}}{(m_l - \alpha_l)\Gamma(m_l - \alpha_l)} \left[\left(t_{l_1}^{m_l - \alpha_l} + (t_{l_2} - t_{l_1})^{m_l - \alpha_l} - t_{l_2}^{m_l - \alpha_l} \right) + (t_{l_2} - t_{l_1})^{m_l - \alpha_l} \right].$$

Since $t_{l_1} < t_{l_2}$, we get

$$|\mathscr{D}_*^{\alpha_l} u_l(t_{l_1}) - \mathscr{D}_*^{\alpha_l} u_l(t_{l_2})| \le \frac{2\mathscr{M}}{(m_l - \alpha_l)\Gamma(m_l - \alpha_l)} (t_{l_2} - t_{l_1})^{m_l - \alpha_l},$$

which finally leads to

$$\left(\mathscr{D}_*^{\alpha_l} u_l(t_{l_1}) - \mathscr{D}_*^{\alpha_l} u_l(t_{l_2}) \right)^2 \le \frac{4\mathscr{M}^2}{\mathscr{M}_{min}^2 \Gamma_{(m-\alpha)_{min}}^2} \frac{1}{2^{2j(\mathscr{M}_{min})}}. \tag{25}$$

Substituting eq. (25) in eq. (24), we get

$$\mathscr{C}_{l,i}^2 \le \frac{2^{-j}\mathscr{M}^2}{\mathscr{M}_{min}^2 \Gamma_{(m-\alpha)_{min}}^2} \frac{1}{2^{2j\mathscr{M}_{min}}}. \tag{26}$$

Substituting eq. (26) in eq. (22), we get

$$\|\mathscr{D}_*^\alpha u - \mathscr{D}_*^\alpha u_k\|^2_{\mathscr{L}^2_d} \leq \frac{1}{\mathcal{N}} \sum_{l=1}^{d} \sum_{j=\beta+1}^{\infty} \frac{\mathscr{M}^2}{\mathscr{M}^2_{min}\Gamma^2_{(m-\alpha)_{min}}} \frac{1}{2^{2j(\mathscr{M}_{min})}}. \tag{27}$$

Let $k = 2^{\beta+1}$ and simplifying the eq. (27), we finally obtain the error bound for fractional derivative as

$$\|\mathscr{D}_*^\alpha u - \mathscr{D}_*^\alpha u_k\|_{\mathscr{L}^2_d} \leq \frac{\mathscr{M}\sqrt{d}}{\sqrt{\mathcal{N}\left(1-2^{-2\mathscr{M}_{min}}\right)}\mathscr{M}_{min}\Gamma_{(m-\alpha)_{min}} k^{\mathscr{M}_{min}}}. \tag{28}$$

Case II: One can proceed in a similar way as in ***Case I*** to obtain the given result.□

Theorem 2

Suppose that the functions $u_k(t)$ obtained by using haar wavelet are the approximation of $u(t)$, then an upper bound of the error with respect to $\mathscr{L}^2_d$-norm is

Case I: $m_l - 1 < \alpha_l < m_l$, $l = 1, 2 \cdots, d$

$$\|u - u_k\|_{\mathscr{L}^2_d} \leq \frac{\mathscr{M}\sqrt{d}}{\sqrt{\mathcal{N}\left(1-2^{-2\mathscr{M}_{min}}\right)}\,\alpha_{min}\,\Gamma_{\alpha_{min}}\,\mathscr{M}_{min}\,\Gamma_{(m-\alpha)_{min}} k^{\mathscr{M}_{min}}}.$$

Case II: $\alpha_l = m_l$, $l = 1, 2 \cdots, d$,

$$\|\mathscr{D}_*^\alpha u - \mathscr{D}_*^\alpha u_k\|_{\mathscr{L}^2_d} \leq \frac{\mu\sqrt{d}}{\alpha_{min}\,\Gamma_{\alpha_{min}}\,\sqrt{3\mathcal{N}}\,k}.$$

Proof

The error in approximation of function $u(t)$ by Haar wavelets with respect to $\mathscr{L}^2_d$-norm is given by

$$\|u - u_k\|_{\mathscr{L}^2_d} = \|\mathscr{J}^\alpha \mathscr{D}_*^\alpha (u-u_k)\|_{\mathscr{L}^2_d} \leq \|\mathscr{J}^\alpha\|_{op}\ \|\mathscr{D}_*^\alpha (u-u_k)\|_{\mathscr{L}^2_d}, \tag{29}$$

where the operator $\mathscr{J}^\alpha = diag[\mathscr{J}^{\alpha_1}, \mathscr{J}^{\alpha_2}, ..., \mathscr{J}^{\alpha_d}]$ and its operator norm $\|\cdot\|_{op}$ is defined as

$$\|\mathscr{J}^\alpha\|_{op} = \sup_{1\leq l\leq d} \|\mathscr{J}^{\alpha_l}\|_{op}, \tag{30}$$

such that

$$\|\mathscr{J}^{\alpha_l}\|_{op} = \sup\left\{\|\mathscr{J}^{\alpha_l} f\|_{\mathscr{L}^2}\ :\ f \in \mathscr{L}^2[0,1),\ \|f\|_{\mathscr{L}^2} = 1\right\}. \tag{31}$$

To find the operator norm, we consider

$$\|\mathscr{J}^{\alpha_l} f\|^2_{\mathscr{L}^2} = \int_0^1 (\mathscr{J}^{\alpha_l} f(t))^2 dt.$$

Using boundedness of fractional integral operator $\mathscr{J}^{\alpha_l}$, we get

$$\|\mathscr{J}^{\alpha_l}\|_{op} \leq \frac{1}{\alpha_l \Gamma(\alpha_l)},$$

which implies

$$\|\mathscr{J}^\alpha\|_{op} \leq \frac{1}{\alpha_{min}\Gamma_{\alpha_{min}}}. \tag{32}$$

Substituting eqs. (28) and (32) in eq. (29), we finally obtain an error bound of Haar wavelet approximation to the solution of the system of fractional differential equation

$$\|u(t) - u_k(t)\|_{\mathscr{L}^2_d} \leq \frac{\mathscr{M}\sqrt{d}}{\sqrt{\mathcal{N}\left(1-2^{-2\mathscr{M}_{min}}\right)}\,\alpha_{min}\,\Gamma_{\alpha_{min}}\,\mathscr{M}_{min}\,\Gamma_{(m-\alpha)_{min}} k^{\mathscr{M}_{min}}}. \tag{33}$$

Case II: One can proceed in a similar way as in ***Case I*** to obtain the given result.□

5 Illustrative examples

Now we implement the proposed methodology described in **section 3**. through nontrivial test examples.

Example 1

Let the system of two linear fractional differential equations Erturk and Momani (2008) be given by

$$\mathscr{D}_*^{\alpha_1} u_1(t) \quad = u_1(t) + u_2(t), \tag{34}$$

$$\mathscr{D}_*^{\alpha_2} u_2(t) \quad = -u_1(t) + u_2(t), \tag{35}$$

with the initial conditions

$$u_1(0) = 0, u_2(0) = 1. \tag{36}$$

Further, for the special case $\alpha_1 = \alpha_2 = 1$, the exact solution is $u_1(t) = e^t \sin(t), u_2(t) = e^t \cos(t)$.

Figure 1 shows approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ with $\alpha_1 = \alpha_2 = 1$ and $\alpha_1 = 0.7, \alpha_2 = 0.9$ for $k = 128$ in **Example 1** together with the actual solution $u_1(t)$ and $u_2(t)$ for $\alpha_1 = \alpha_2 = 1$. Since in the case $\alpha_1 = \alpha_2 = 1$, the exact solution is known, we could compare the approximate and actual solution which justifies that our proposed method is giving tangible results. Figures 2 and 3 represent the approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ respectively for different values of k in the special case when $\alpha_1 = 0.7, \alpha_2 = 0.9$ in **Example 1**.

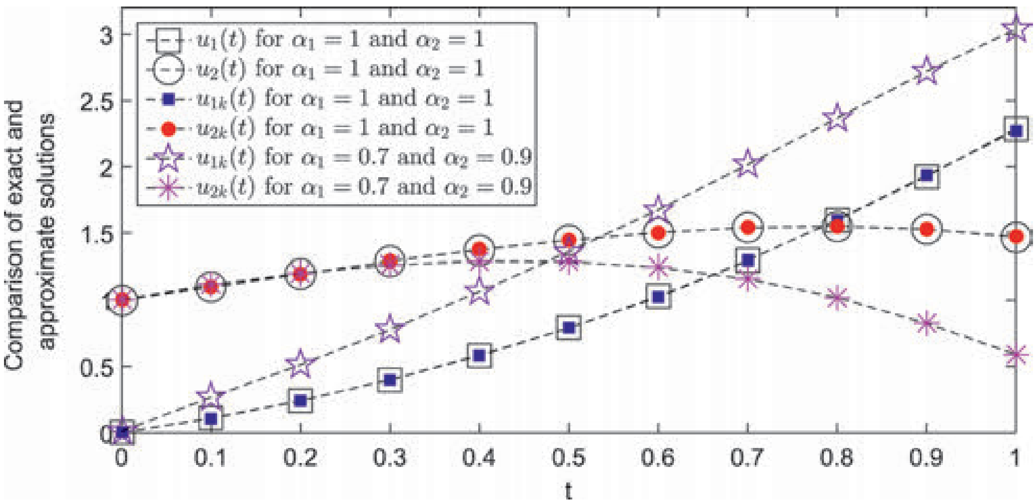

Figure 1 It shows approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ with $\alpha_1 = \alpha_2 = 1$ and $\alpha_1 = 0.7, \alpha_2 = 0.9$ for $k = 128$ in **Example 1**.

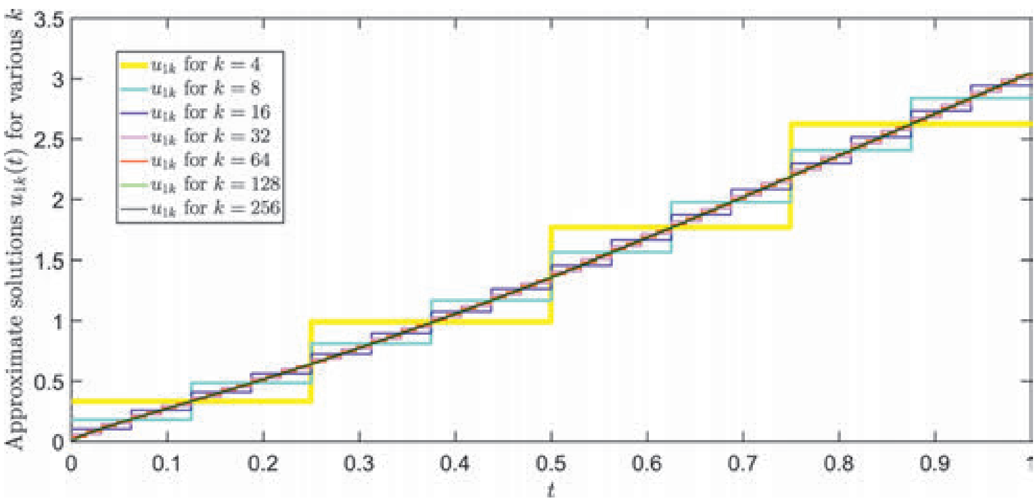

Figure 2 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.7, \alpha_2 = 0.9$ in **Example 1**.

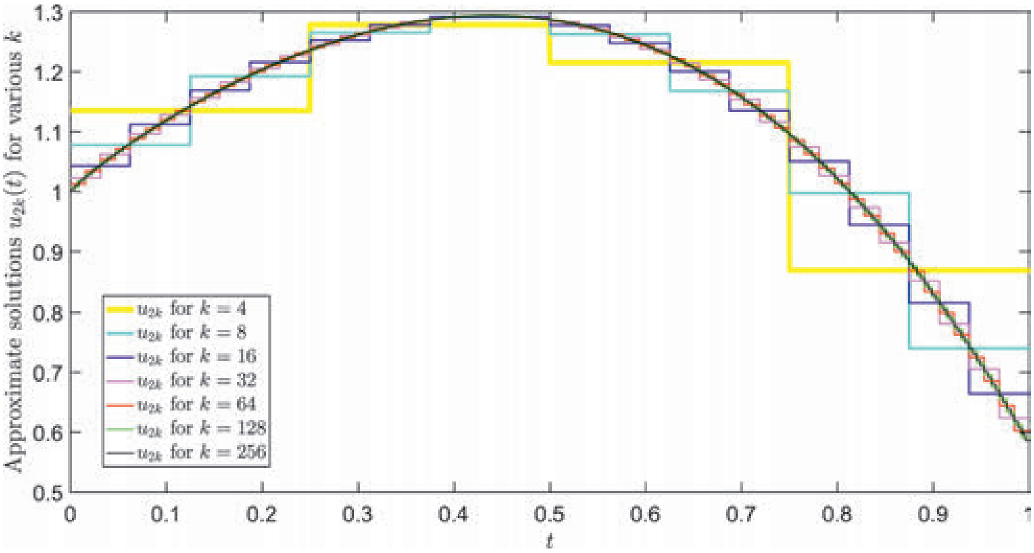

Figure 3 It represents the approximate solutions $u_{2k}(t)$ for different values of k in the special case when $\alpha_1 = 0.7, \alpha_2 = 0.9$ in **Example 1**.

Figures 4 and 5 represent the approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ respectively for different values of k and $\alpha_1 = \alpha_2 = \alpha$ ranging from 0.7 to 0.9 in **Example 1**.

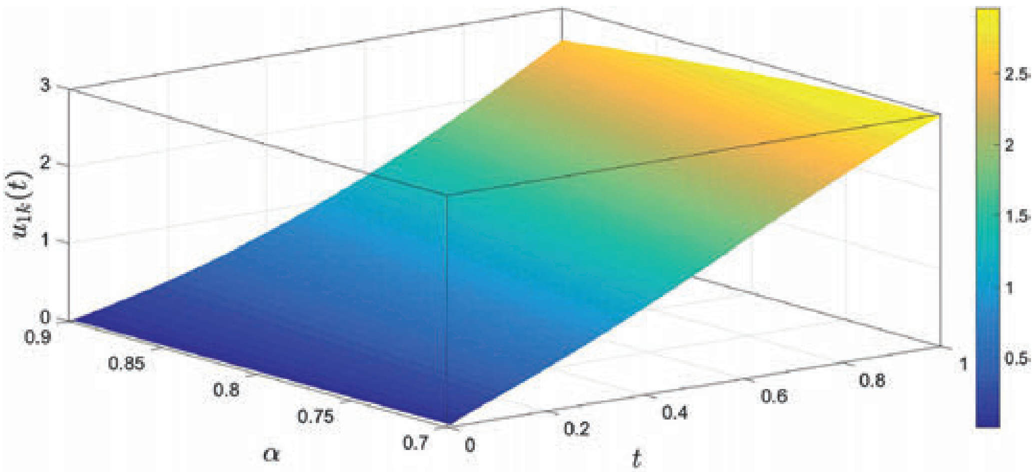

Figure 4 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.7 to 0.9 in **Example 1**.

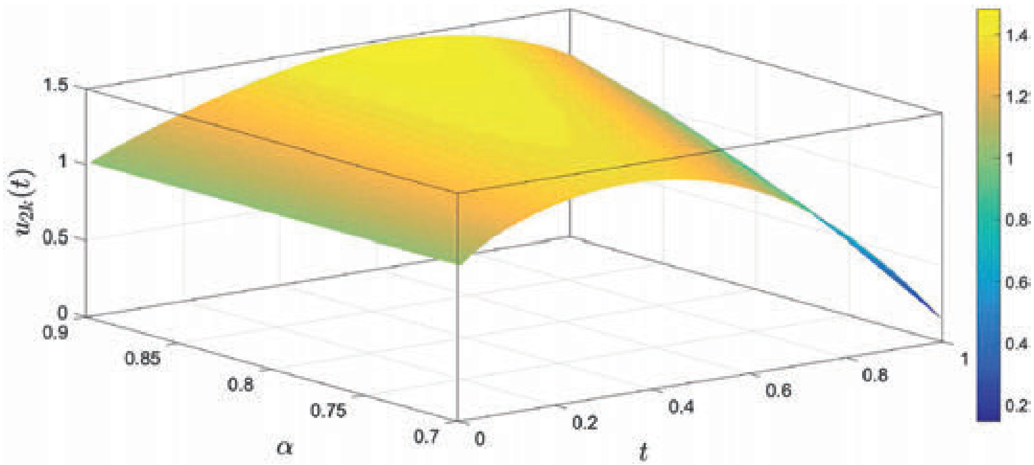

Figure 5 It represents the approximate solutions $u_{2k}(t)$ for different values of k and α from 0.7 to 0.9 in **Example 1**.

[1]Table 1 and Table 2 shows the comparison of the theoretical error bounds with actual errors for **Example 1**. having $\alpha_1 = \alpha_2 = 1$ and $\alpha_1 = 0.7, \alpha_2 = 0.9$ respectively.

Table 1 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}_d^2$-norm for the special case $\alpha_1 = 1, \alpha_2 = 1$ of **Example 1**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	Error bound of $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	Error bound of $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}_d^2}$	Error bound of $\|u - u_k\|_{\mathscr{L}_d^2}$
2	3.46E-04	1.55E-03	9.49E-05	2.28E-03	5.08E-06	4.55E-05
4	1.75E-04	7.75E-04	4.99E-05	1.14E-03	2.57E-06	2.28E-05
8	8.75E-05	3.88E-04	2.53E-05	5.69E-04	1.29E-06	1.14E-05
16	4.38E-05	1.94E-04	1.27E-05	2.85E-04	6.44E-07	5.69E-06
32	2.19E-05	9.69E-05	6.34E-06	1.42E-04	3.22E-07	2.85E-06
64	1.09E-05	4.85E-05	3.15E-06	7.11E-05	1.60E-07	1.42E-06
128	5.30E-06	2.42E-05	1.54E-06	3.56E-05	7.81E-08	7.11E-07
256	2.37E-06	1.21E-05	6.88E-07	1.78E-05	3.49E-08	3.56E-07

Table 2 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}_d^2$-norm for the special case $\alpha_1 = 0.7, \alpha_2 = 0.9$ of **Example 1**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	Error bound of $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	Error bound of $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}_d^2}$	Error bound of $\|u - u_k\|_{\mathscr{L}_d^2}$
2	4.40E-04	1.03E-02	1.58E-04	6.97E-03	6.61E-06	2.06E-04
4	2.20E-04	9.60E-03	8.68E-05	6.50E-03	3.35E-06	1.92E-04
8	1.10E-04	8.96E-03	4.44E-05	6.06E-03	1.68E-06	1.79E-04
16	5.52E-05	8.36E-03	2.23E-05	5.66E-03	8.42E-07	1.67E-04
32	2.76E-05	7.80E-03	1.12E-05	5.28E-03	4.21E-07	1.56E-04
64	1.37E-05	7.27E-03	5.56E-06	4.93E-03	2.10E-07	1.45E-04
128	6.71E-06	6.79E-03	2.71E-06	4.60E-03	1.02E-07	1.36E-04
256	3.00E-06	6.33E-03	1.21E-06	4.29E-03	4.58E-08	1.27E-04

Example 2

Consider the system of two nonlinear fractional differential equations Erturk and Momani (2008) be given by

$$\mathscr{D}_*^{\alpha_1} u_1(t) \quad = u_1(t) + u_2^2(t), \tag{37}$$

$$\mathscr{D}_*^{\alpha_2} u_2(t) \quad = u_1(t) + 5u_2(t), \tag{38}$$

with the initial conditions

$$u_1(0) = 0, u_1'(0) = 1, u_2(0) = 0, u_2'(0) = 1, u_2''(0) = 1. \tag{39}$$

For $\alpha_1 = 1.3, \alpha_2 = 2.4$, the approximate solutions have been calculated. Figures 6 and 7 show the approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ respectively for different values of k in the special case when $\alpha_1 = 1.3, \alpha_2 = 2.4$ in **Example 2**.

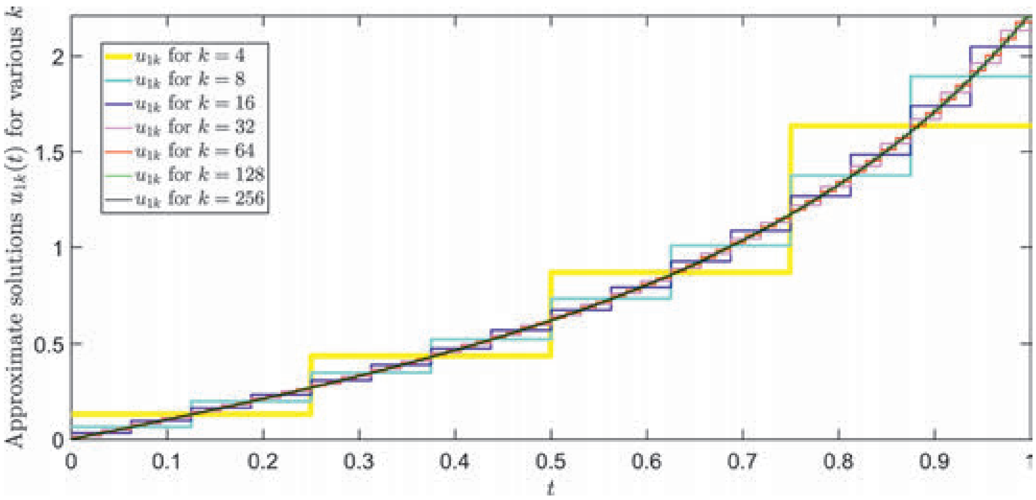

Figure 6 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 1.3, \alpha_2 = 2.4$ in **Example 2**.

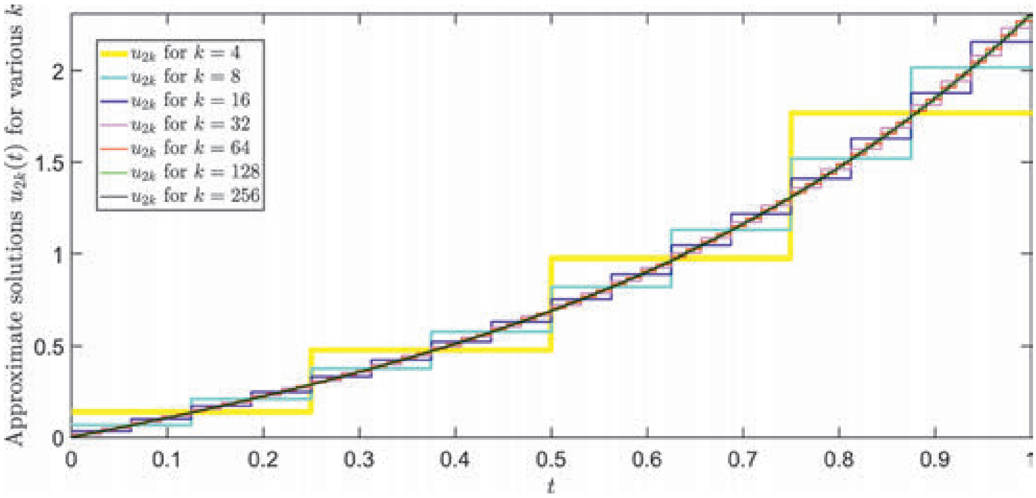

Figure 7 It represents the approximate solutions $u_{2k}(t)$ for different values of k in the special case when $\alpha_1 = 1.3, \alpha_2 = 2.4$ in **Example 2**.

Figures 8 and 9 represent the approximate solutions $u_{1k}(t)$ and $u_{2k}(t)$ respectively for different values of k and $\alpha_1 = \alpha_2 = \alpha$ ranging from 1 to 1.5 in **Example 2**.

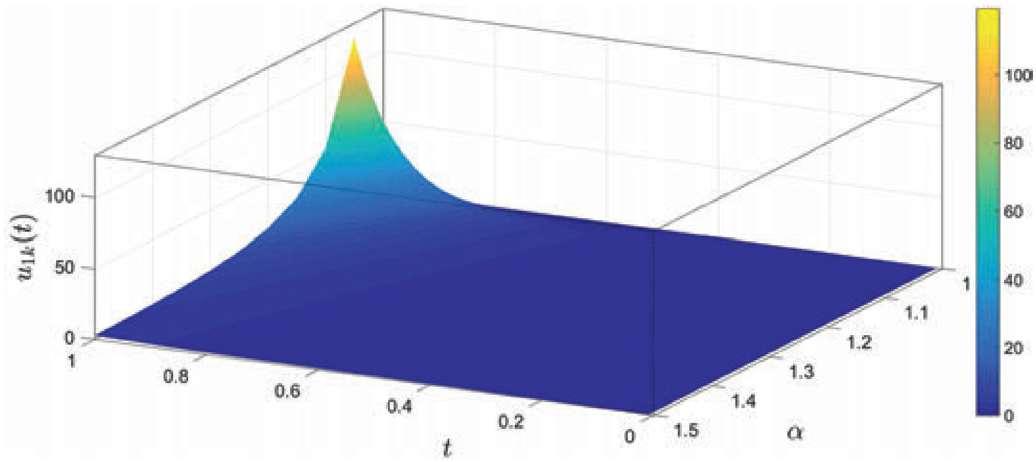

Figure 8 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 1 to 1.5 in **Example 2**.

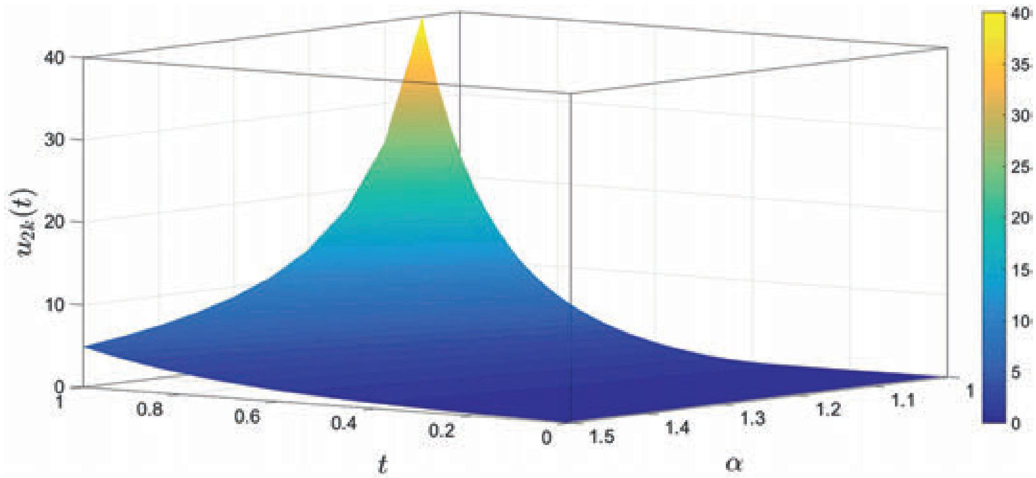

Figure 9 It represents the approximate solutions $u_{2k}(t)$ for different values of k and α from 1 to 1.5 in **Example 2**.

Table 3 shows the comparison of the theoretical error bounds with actual errors for **Example 2**. having $\alpha_1 = 1.3, \alpha_2 = 2.4$.

Table 3 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}^2_d$-norm for the special case $\alpha_1 = 1.3, \alpha_2 = 2.4$ of **Example 2**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}^2_d}$	**Error bound of** $\|u - u_k\|_{\mathscr{L}^2_d}$
2	3.42E-04	2.46E-03	3.55E-04	4.50E-03	6.97E-06	9.00E-05
4	1.81E-04	1.62E-03	1.83E-04	2.97E-03	3.65E-06	5.93E-05
8	9.23E-05	1.07E-03	9.25E-05	1.96E-03	1.85E-06	3.92E-05
16	4.63E-05	7.06E-04	4.63E-05	1.29E-03	9.27E-07	2.58E-05
32	2.32E-05	4.66E-04	2.31E-05	8.52E-04	4.63E-07	1.70E-05
64	1.15E-05	3.07E-04	1.15E-05	5.62E-04	2.30E-07	1.12E-05
128	5.62E-06	2.03E-04	5.61E-05	3.71E-04	1.12E-07	7.42E-06
256	2.51E-06	1.34E-04	2.51E-05	2.45E-04	5.02E-08	4.89E-06

Example 3

Let us consider the system of three nonlinear fractional differential equations Erturk and Momani (2008) be given by

$$\mathscr{D}_*^{\alpha_1} u_1(t) \quad = 2u_2^2(t), \quad 0 < \alpha_1 \le 1, \tag{40}$$

$$\mathscr{D}_*^{\alpha_2} u_2(t) \quad = t u_1(t), \quad 0 < \alpha_2 \le 1, \tag{41}$$

$$\mathscr{D}_*^{\alpha_3} u_3(t) \quad = u_2(t) u_3(t), \quad 0 < \alpha_3 \le 1, \tag{42}$$

with the initial conditions

$$u_1(0) = 0, u_2(0) = 1, u_3(0) = 1. \tag{43}$$

For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the approximate solutions have been calculated. Figures 10, 11 and 12 show the approximate solutions $u_{1k}(t), u_{2k}(t)$ and $u_{3k}(t)$ respectively for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in **Example 3**.

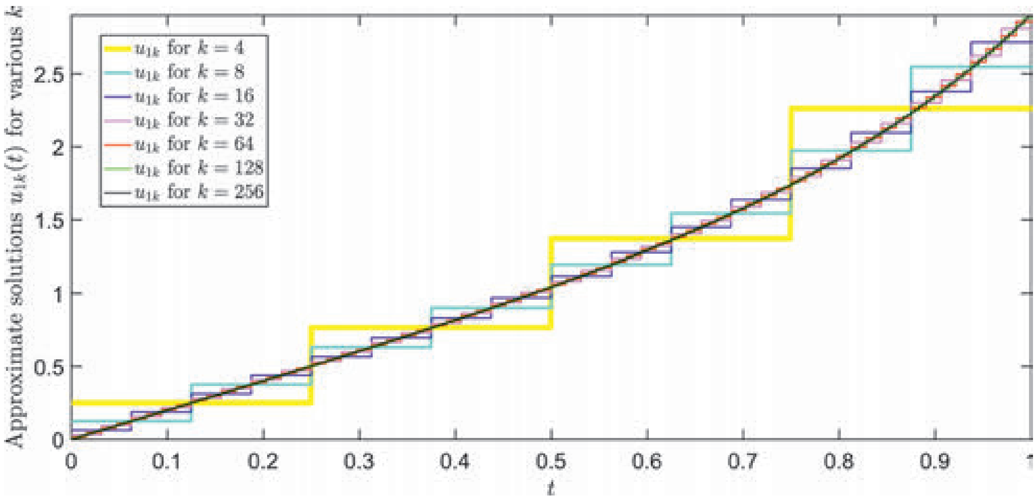

Figure 10 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in **Example 3**.

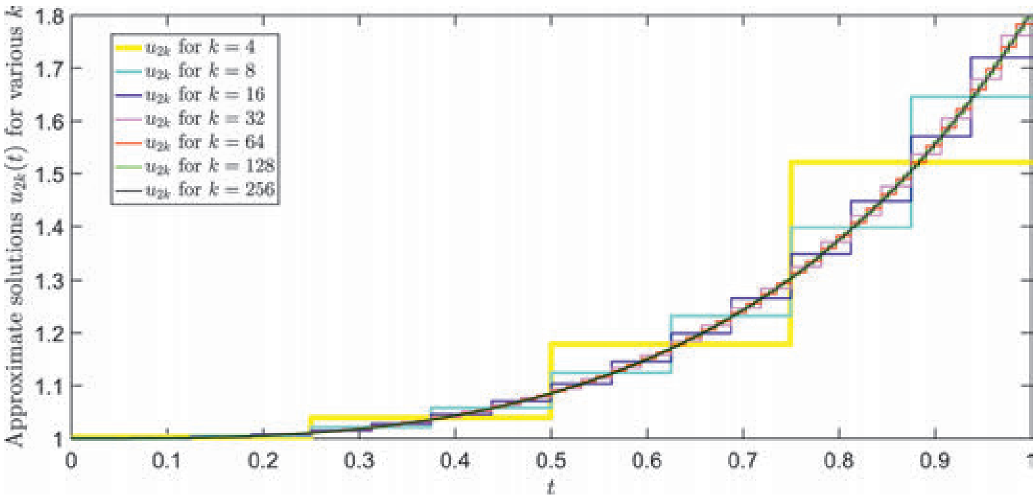

Figure 11 It represents the approximate solutions $u_{2k}(t)$ for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in **Example 3**.

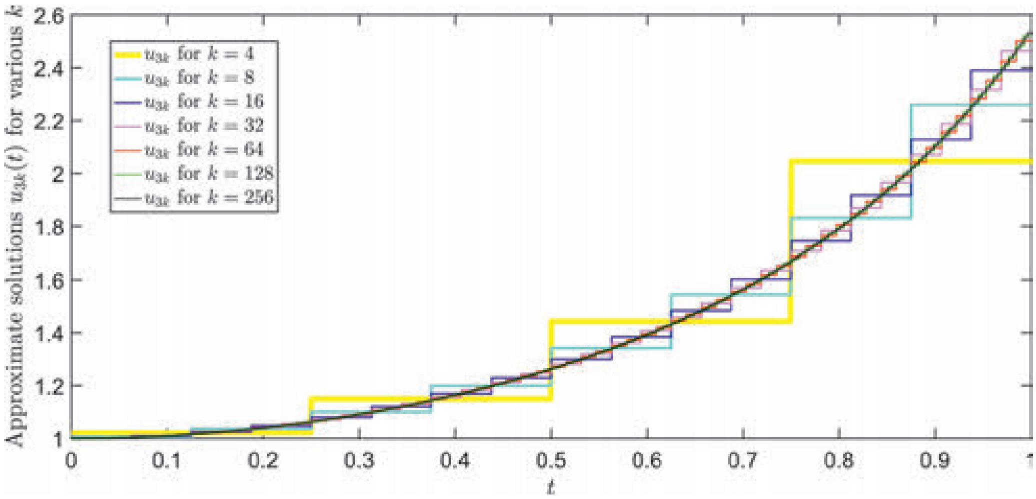

Figure 12 It represents the approximate solutions $u_{3k}(t)$ for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in **Example 3**.

Figures 13, 14 and 15 represent the approximate solutions $u_{1k}(t)$, $u_{2k}(t)$ and $u_{3k}(t)$ respectively for different values of k and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ ranging from 0.7 to 1 in **Example 3**.

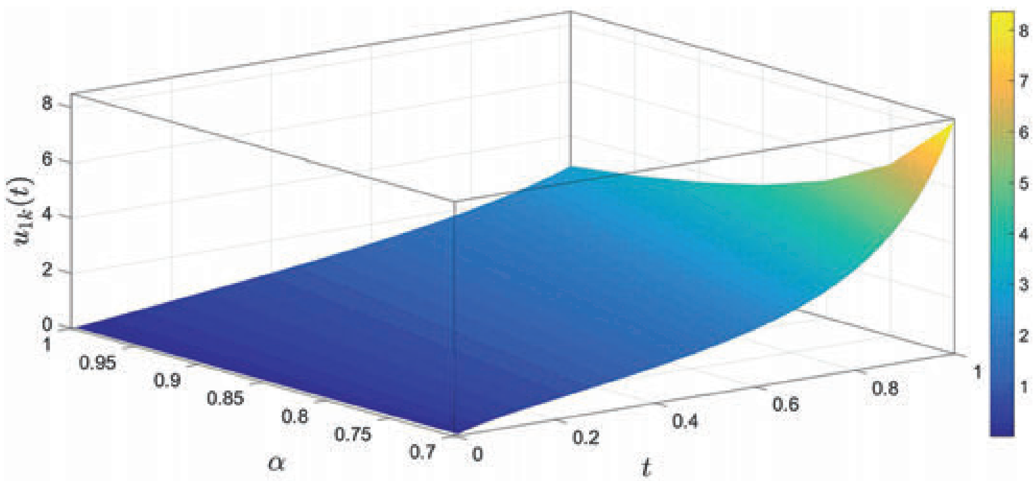

Figure 13 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.7 to 1 in **Example 3**.

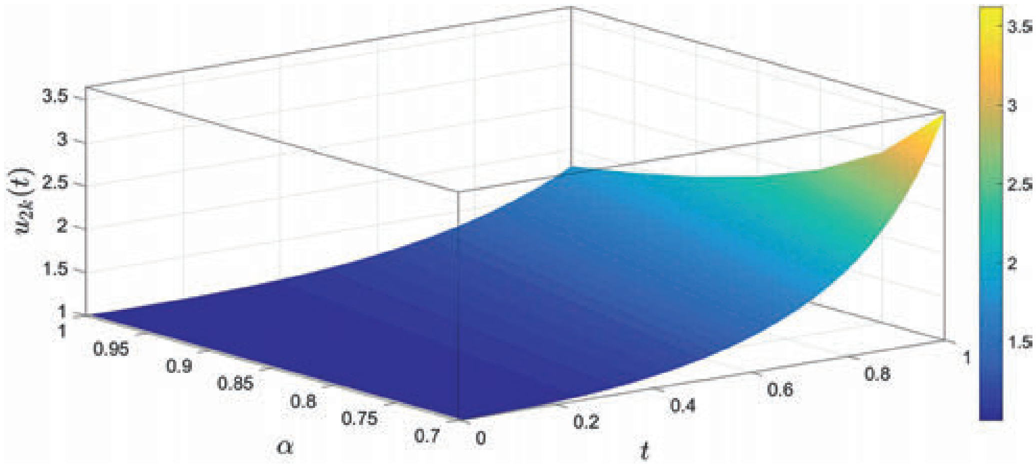

Figure 14 It represents the approximate solutions $u_{2k}(t)$ for different values of k and α from 0.7 to 1 in **Example 3**.

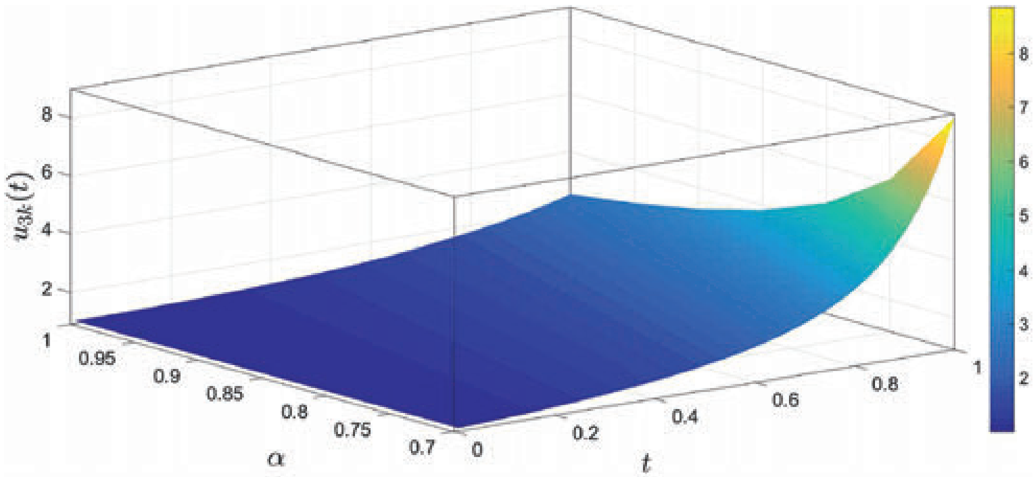

Figure 15 It represents the approximate solutions $u_{3k}(t)$ for different values of k and α from 0.7 to 1 in **Example 3**.

Table 4 shows the comparison of the theoretical error bounds with actual errors for **Example 3**. having $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Table 4 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}_d^2$-norm for the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ of **Example 3**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	Error bound of $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	Error bound of $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	Error bound of $\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}_d^2}$	Error bound of $\|u - u_k\|_{\mathscr{L}_d^2}$
2	4.26E-04	1.03E-02	1.45E-04	4.66E-03	3.46E-04	9.94E-03	8.03E-06	2.54E-04
4	2.22E-04	5.17E-03	7.92E-05	2.33E-03	1.83E-04	4.97E-03	4.21E-06	1.27E-04
8	1.12E-04	2.59E-03	4.06E-05	1.17E-03	9.29E-05	2.48E-03	2.14E-06	6.34E-05
16	5.63E-05	1.29E-03	2.04E-05	5.83E-04	4.66E-05	1.24E-03	1.07E-06	3.17E-05
32	2.82E-05	6.47E-04	1.02E-05	2.91E-04	2.33E-05	6.21E-04	5.37E-07	1.58E-05
64	1.40E-05	3.23E-04	5.07E-06	1.46E-04	1.16E-05	3.11E-04	2.67E-07	7.92E-06
128	6.83E-06	1.62E-04	2.48E-06	7.28E-05	5.65E-06	1.55E-04	1.30E-07	3.96E-06
256	3.05E-06	8.09E-05	1.11E-06	3.64E-05	2.53E-06	7.76E-05	5.82E-08	1.98E-06

Example 4

Let us consider the homogeneous differential equation Momani and Al-Khaled (2005) of order 4α is given by

$$\mathscr{D}_*^{4\alpha} u(t) - 2\mathscr{D}_*^{2\alpha} u(t) + u(t) = 0, \quad 0 < \alpha \leq 2, \tag{44}$$

with the initial conditions

$$u(0) = 0, u'(0) = 1. \tag{45}$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4\alpha$ and

$$u(t) = u_1, \mathscr{D}_*^{\alpha_1} u_1(t) = u_2, \mathscr{D}_*^{\alpha_2} u_2(t) = u_3, \mathscr{D}_*^{\alpha_3} u_3(t) = u_4, \tag{46}$$

then the eq. (44) is reduced to following system of fractional differential equations:

$$\mathscr{D}_*^{\alpha_1} u_1(t) \quad = u_2(t), \;\; 0 < \alpha_1 \leq 1, \tag{47}$$

$$\mathscr{D}_*^{\alpha_2} u_2(t) \quad = u_3(t), \;\; 0 < \alpha_2 \leq 1, \tag{48}$$

$$\mathscr{D}_*^{\alpha_3} u_3(t) \quad = u_4(t), \;\; 0 < \alpha_3 \leq 1, \tag{49}$$

$$\mathscr{D}_*^{\alpha_4} u_4(t) \quad = 2u_3(t) - u_1(t), \;\; 0 < \alpha_4 \leq 1, \tag{50}$$

with the initial conditions

$$u_1(0) = 0, u_2(0) = 0, u_3(0) = 1, u_4(0) = 0. \tag{51}$$

If we choose $\alpha_1 = 0.5, \alpha_2 = 0.5, \alpha_3 = 0.5, \alpha_4 = 0.5$, then the exact solution of equation eq. (44) is $u(t) = te^t$. Figure 16 shows the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.5$ in **Example 4**.

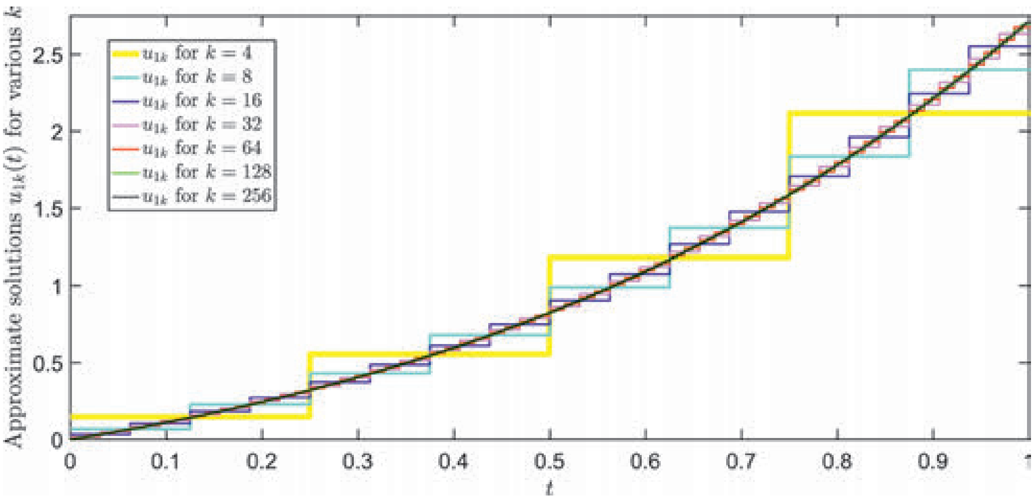

Figure 16 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.5, \alpha_2 = 0.5, \alpha_3 = 0.5, \alpha_4 = 0.5$ in **Example 4**.

Figure 17 represent the approximate solutions $u_{1k}(t)$ for different values of k and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ ranging from 0.5 to 0.8 in **Example 4**. Table 5 shows the comparison of the theoretical error bounds with actual errors for **Example 4**. having $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.5$.

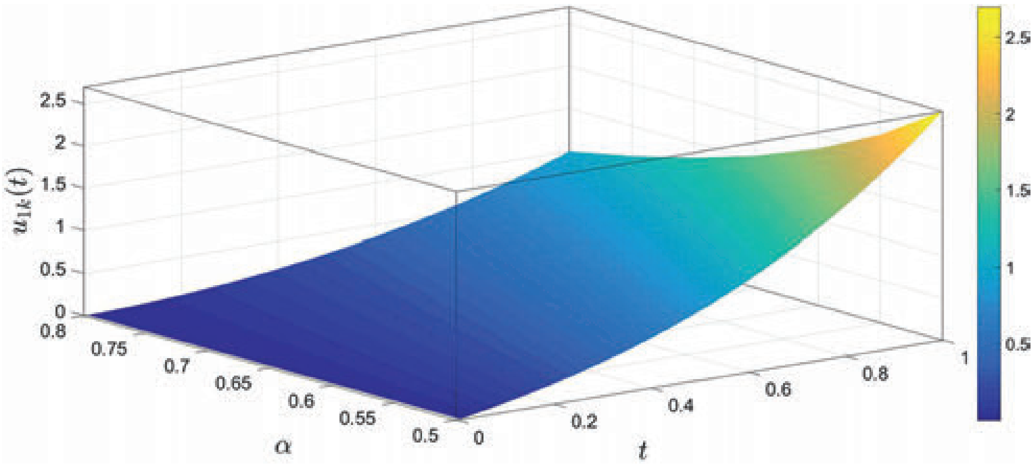

Figure 17 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 4**.

Table 5 Error in solution $u_1(t)$ and corresponding error bound measured with respect to $\mathscr{L}^2$-norm for the special case $\alpha = 0.5$ of **Example 4**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$
2	4.20E-04	5.08E-03
4	2.15E-04	3.59E-03
8	1.08E-04	2.54E-03
16	5.40E-05	1.79E-03
32	2.70E-05	1.27E-03
64	1.34E-05	8.97E-04
128	6.55E-06	6.35E-04
256	2.93E-06	4.49E-04

Example 5

(**The problem on motion of a large plate in a Newtonian fluid**)

Let us consider the nonhomogeneous differential equation Momani and Al-Khaled (2005) of order 4α is given by

$$\mathbb{A}\mathscr{D}_*^{4\alpha}u(t) + \mathbb{B}\mathscr{D}_*^{3\alpha}u(t) + \mathbb{C}u(t) = f(t), \;\; 0 < \alpha \le 1, \;\; 0 \le t \le 1, \tag{52}$$

(52)

with the initial conditions

$$u(0) = 0, u'(0) = 1. \tag{53}$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4\alpha$ and

$$u(t) = u_1, \mathscr{D}_*^{\alpha_1}u_1(t) = u_2, \mathscr{D}_*^{\alpha_2}u_2(t) = u_3, \mathscr{D}_*^{\alpha_3}u_3(t) = u_4, \tag{54}$$

then the eq. (52) is reduced to following system of fractional differential equations:

$$\mathscr{D}_*^{\alpha_1}u_1(t) \;\; = u_2(t), \;\; 0 < \alpha_1 \le 1, \tag{55}$$

$$\mathscr{D}_*^{\alpha_2}u_2(t) \;\; = u_3(t), \;\; 0 < \alpha_2 \le 1, \tag{56}$$

$$\mathscr{D}_*^{\alpha_3}u_3(t) \;\; = u_4(t), \;\; 0 < \alpha_3 \le 1, \tag{57}$$

$$\mathscr{D}_*^{\alpha_4}u_4(t) \;\; = -\frac{\mathbb{C}}{\mathbb{A}}u_1(t) - \frac{\mathbb{B}}{\mathbb{A}}u_4(t) + \frac{1}{\mathbb{A}}f(t), \;\; 0 < \alpha_4 \le 1, \tag{58}$$

with the initial conditions

$$u_1(0) = 1, u_2(0) = 0, u_3(0) = 1, u_4(0) = 0. \tag{59}$$

If we choose $\alpha = 0.5$, then eq. (52) reduces to inhomogeneous BagleyTorvik equation:

$$\mathbb{A}\mathscr{D}_*^2 u(t) + \mathbb{B}\mathscr{D}_*^{\frac{3}{2}} u(t) + \mathbb{C}u(t) = f(t), \quad 0 \le t \le 1, \tag{60}$$

with the same initial conditions eq. (59), where $\mathbb{A} = \mathbb{M}, \mathbb{B} = 2S\sqrt{\nu\rho}, \mathbb{C} = K$.

This problem describes the motion of a large plate of the surface S and mass $\mathbb{M}$ in a Newtonian fluid of viscosity ν and density ρ. The plate is hanged with a massless string of stiffness K. The loading force is represented by function $f(t)$. For comparison purpose with the results already present in literature, the values chosen are: $\mathbb{A} = \mathbb{B} = \mathbb{C} = 1, f(t) = \mathbb{C}(t+1)$. The exact solution of equation eq. (44) is $u(t) = t + 1$.

We have calculated the approximate solution of eq. (60) by first converting it in the system eqs. (55, 56, 57,58) on choosing $\alpha_1 = 0.5, \alpha_2 = 0.5, \alpha_3 = 0.5, \alpha_4 = 0.5$. Figure 18 represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.5$ in **Example 5**.

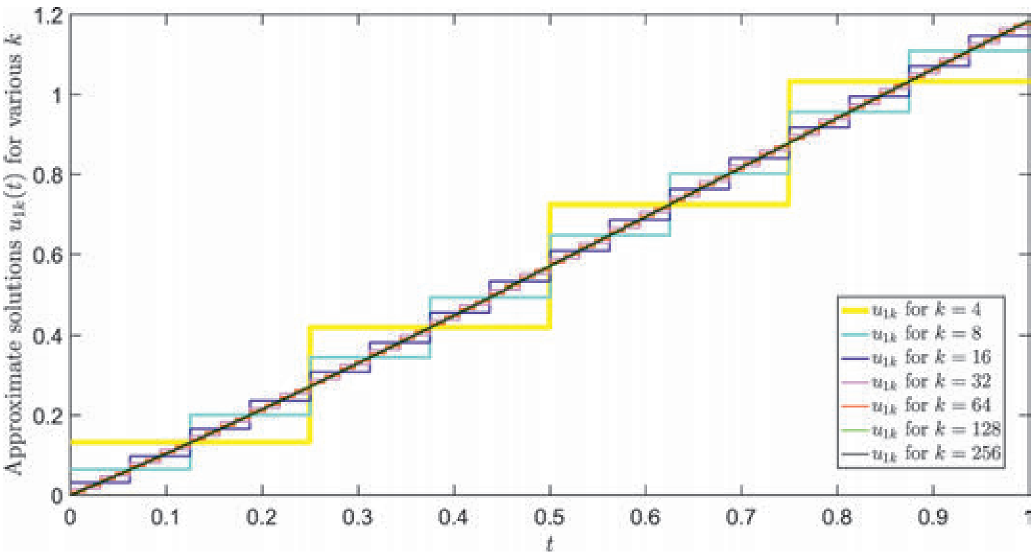

Figure 18 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.5, \alpha_2 = 0.5, \alpha_3 = 0.5, \alpha_4 = 0.5$ in **Example 5**.

Figure 19 represent the approximate solutions $u_{1k}(t)$ for different values of k and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ ranging from 0.5 to 0.8 in **Example 5**.

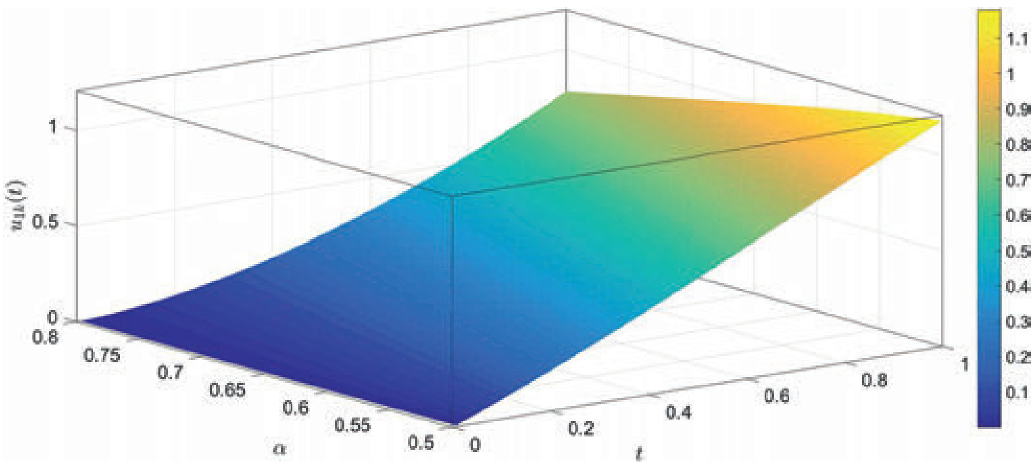

Figure 19 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 5**.

Table 6 shows the comparison of the theoretical error bounds with actual errors for **Example 5**. having $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.5$.

Table 6 Error in solution $u_1(t)$ w.r.t. $\mathscr{L}^2$-norm and the upper bound of solution for **Example 5**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$
2	1.72E-04	1.72E-03
4	8.57E-05	1.22E-03
8	4.28E-05	8.60E-04
16	2.14E-05	6.08E-04
32	1.07E-05	4.30E-04
64	5.31E-06	3.04E-04
128	2.59E-06	2.15E-04
256	1.16E-06	1.52E-04

Example 6

(The problem on unsteady motion of a particle in a viscous fluid)

Consider the following differential equation Momani and Al-Khaled (2005) :

$$\mathscr{D}_*^1 u(t) + a_1 \mathscr{D}_*^\alpha u(t) + a_2 u(t) = f(t), \quad 0 < \alpha \leq 1, \quad 0 \leq t, \tag{61}$$

with initial condition $y(0) = c$, where a_1 and a_2 are positive constants and c is an arbitrary constant.

This fractional differential equation arises in fluid dynamics when the unsteady motion of a particle accelerates in a viscous fluid under the force of gravity. We have considered f(t)=1 and a vanishing initial velocity $c = 0$. Further, consider the ordinary Basset problem for $\alpha = 0.5$. Then the eq. (61) can be reduced in the following system of fractional differential equation:

$$\mathscr{D}_*^{0.5} u_1(t) = u_2(t), \tag{62}$$

$$\mathscr{D}_*^{0.5} u_2(t) = -a_2 u_1(t) - a_1 u_2(t) + 1, \tag{63}$$

with the initial conditions

$$u_1(0) = 0, u_2(0) = J_{1/2}(1), \tag{64}$$

where $u_1(t) = u(t)$.

Using the formulation of the problem given by Mainardi (1997), we let

$$a_1 = \left(\frac{9}{1+2\chi}\right)^{\frac{1}{2}}, \tag{65}$$

$$a_2 = 1, \tag{66}$$

where $\chi = 0.5, 10, 100$.

Figures 20, 21 and 22 show the approximate solutions $u_{1k}(t)$ with $\chi = 0.5, 10, 100$ respectively for different values of k in the special case when $\alpha_1 = \alpha_2 = 0.5$ in **Example 6**.

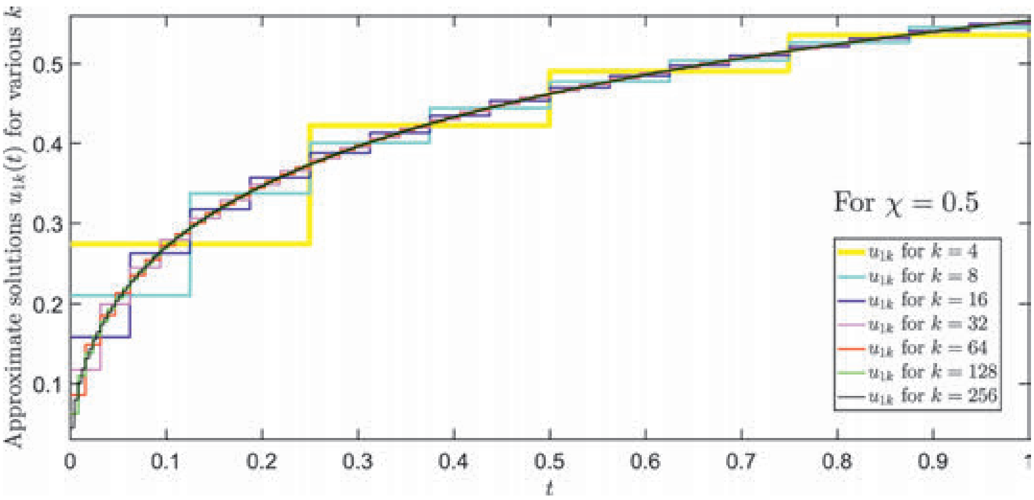

Figure 20 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.5, \alpha_2 = 0.5$ and $\chi = 0.5$ in **Example 6**.

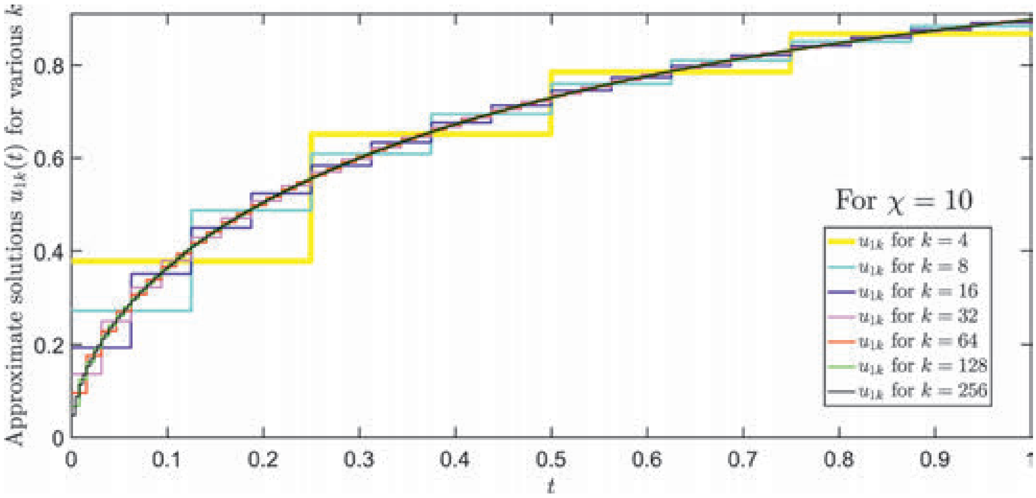

Figure 21 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.5, \alpha_2 = 0.5$ and $\chi = 10$ in **Example 6**.

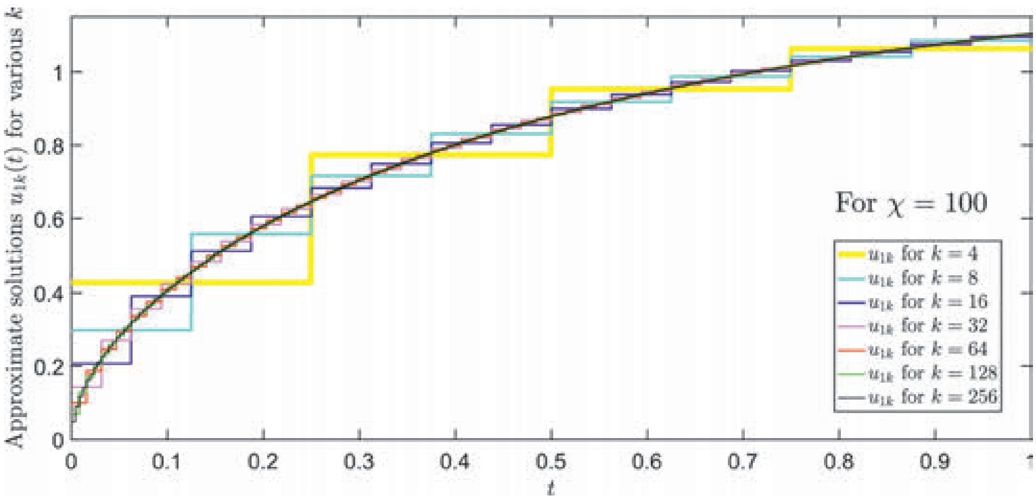

Figure 22 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 0.5, \alpha_2 = 0.5$ and $\chi = 100$ in **Example 6**.

Figures 23, 24 and 25 represent the approximate solutions $u_{1k}(t)$ for $\chi = 0.5, 10, 100$ respectively for different values of k and $\alpha_1 = \alpha_2 = \alpha$ ranging from 0.5 to 0.8 in **Example 6**.

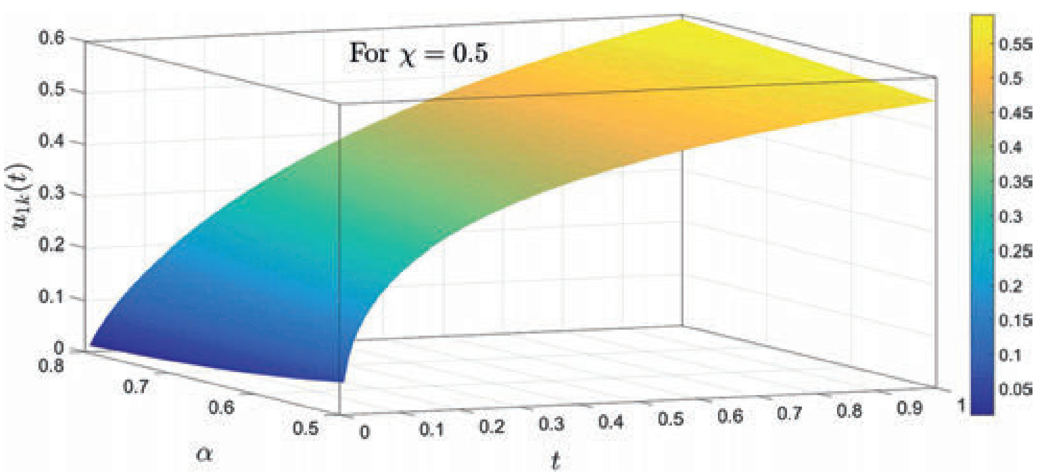

Figure 23 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 6**.

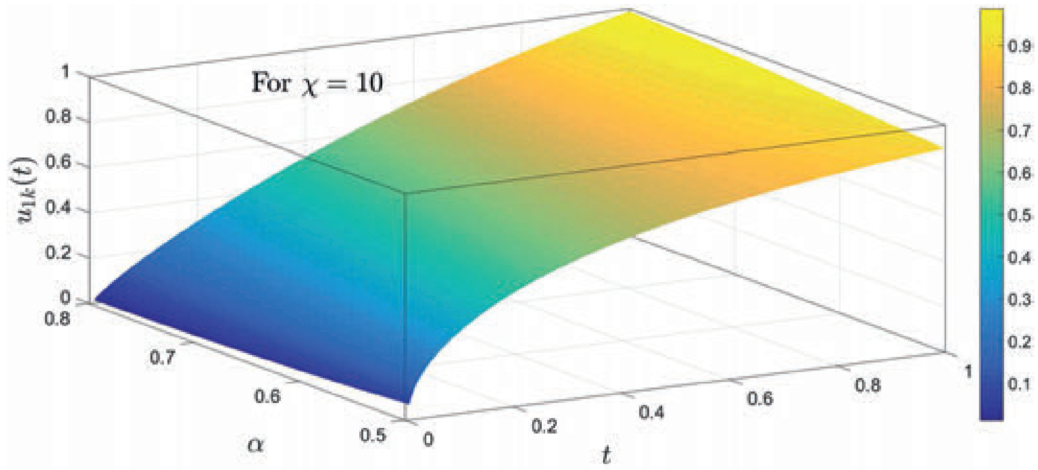

Figure 24 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 6**.

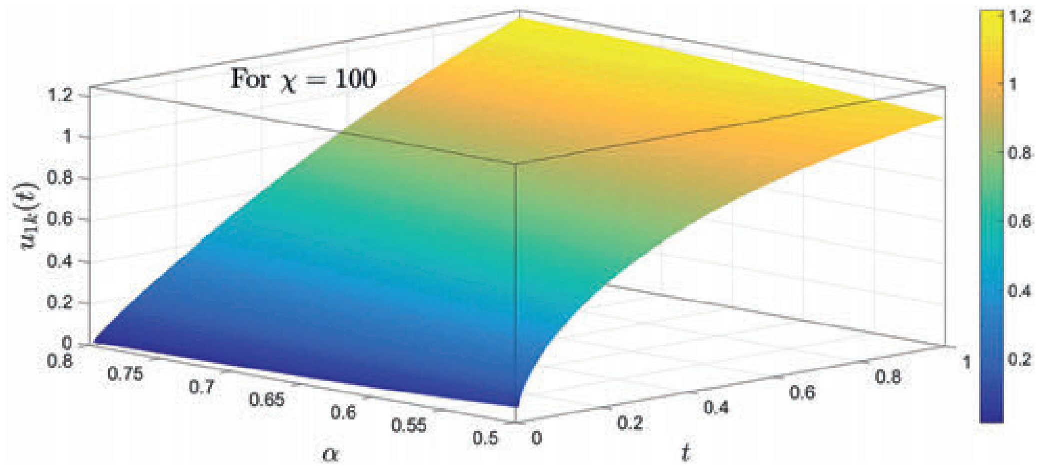

Figure 25 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 6**.

Table 7 shows the comparison of the theoretical error bounds with actual errors for **Example 6**. having $\alpha_1 = \alpha_2 = 0.5$ for each value of $\chi = 0.5, 10, 100$.

Table 7 Error in solution $u_1(t)$ w.r.t. $\mathscr{L}^2$-norm and the upper bound of solution for **Example 6**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ for $\chi = 0.5$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ **for** $\chi = 0.5$	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ for $\chi = 10$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ **for** $\chi = 10$	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ for $\chi = 100$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$ **for** $\chi = 100$
2	6.97E-05	1.18E-03	1.23E-04	1.32E-03	1.55E-04	1.48E-03
4	4.29E-05	8.34E-04	7.19E-05	9.30E-04	8.81E-05	1.05E-03
8	2.55E-05	5.90E-04	4.02E-05	6.58E-04	4.82E-05	7.39E-04
16	1.47E-05	4.17E-04	2.20E-05	4.65E-04	2.58E-05	5.23E-04
32	8.28E-06	2.95E-04	1.18E-05	3.29E-04	1.36E-05	3.70E-04
64	4.55E-06	2.09E-04	6.24E-06	2.32E-04	7.12E-06	2.61E-04
128	2.41E-06	1.48E-04	3.20E-06	1.64E-04	3.62E-06	1.85E-04
256	1.15E-06	1.04E-04	1.49E-06	1.16E-04	1.67E-06	1.31E-04

Example 7

(**Chemical Reactor Problem**)

There are two reactors in series where a reaction $A \rightarrow B$ occurs. The reactors are not in steady state but properly mixed. Now if we consider $CA_0, CA_1 = u_1, CA_2 = u_3$ are the concentration of A at the inlet of I reactor, at the outlet of the I reactor (and inlet of the II) and outlet of the second reactor respectively. Similarly, $CB_1 = u_2, CB_2 = u_4$ is are the concentration of B at the outlet of the I reactor and outlet of the II reactor respectively. The following system of fractional differential equations have been resulted from The unsteady-state mass balance for each stirred tank reactor Khan, Aray, and Mahmood (2010) :

$$\mathscr{D}_*^{\alpha} u_1(t) \quad = \frac{CA_0 - u_1(t)}{\tau} - \beta u_1(t), \tag{67}$$

$$\mathscr{D}_*^{\alpha} u_2(t) \quad = \frac{-u_2(t)}{\tau} + \beta u_1(t), \tag{68}$$

$$\mathscr{D}_*^{\alpha} u_3(t) \quad = \frac{u_1(t) - u_3(t)}{\tau} - \beta u_3(t), \tag{69}$$

$$\mathscr{D}_*^{\alpha} u_4(t) \quad = \frac{u_2(t) - u_4(t)}{\tau} - \beta u_4(t), \tag{70}$$

where the residence time for each reactor is τ, the rate constant for reaction of A to produce B is β.

If $CA_0 = 10$, $\beta = 0.1/\text{min}$ and $\tau = 5$ min and the considered initial conditions are given by $u_1(0) = 0, u_2(0) = 0, u_3(0) = 0, u_4(0) = 0$. Figures. 26, 27, 28 and 29 represents the approximate solutions $u_{1k}(t), u_{2k}(t), u_{3k}(t)$ and $u_{4k}(t)$ respectively for different values of k and $\alpha = 1$ for **Example 7**.

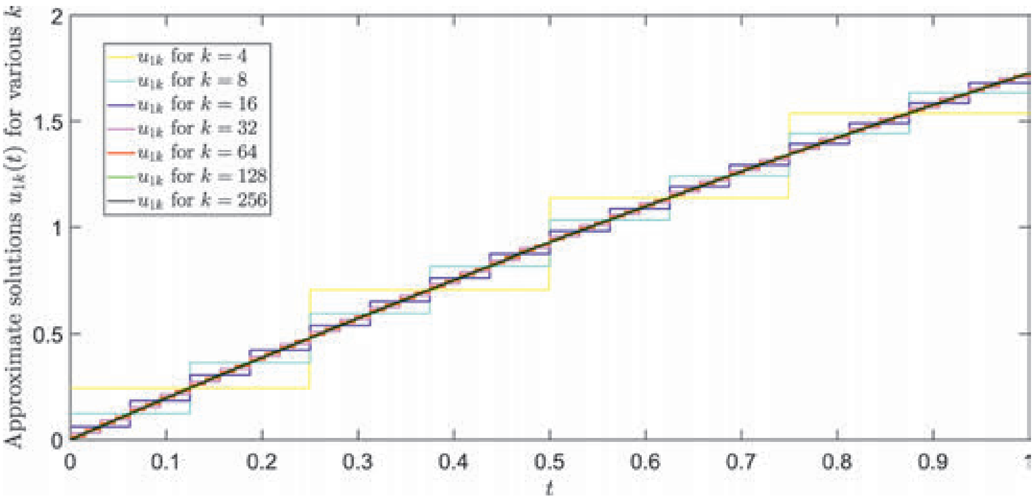

Figure 26 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha = 1$ in **Example 7**.

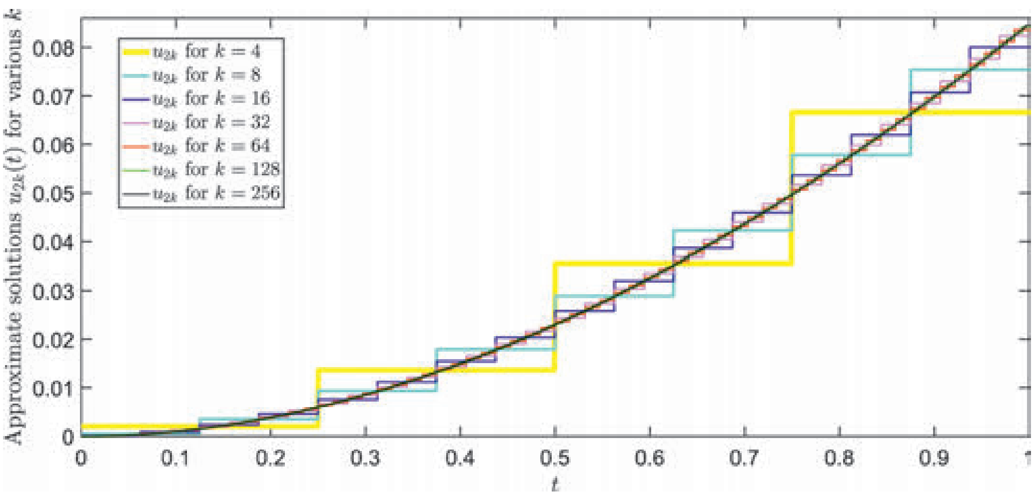

Figure 27 It represents the approximate solutions $u_{2k}(t)$ for different values of k in the special case when $\alpha = 1$ in **Example 7**.

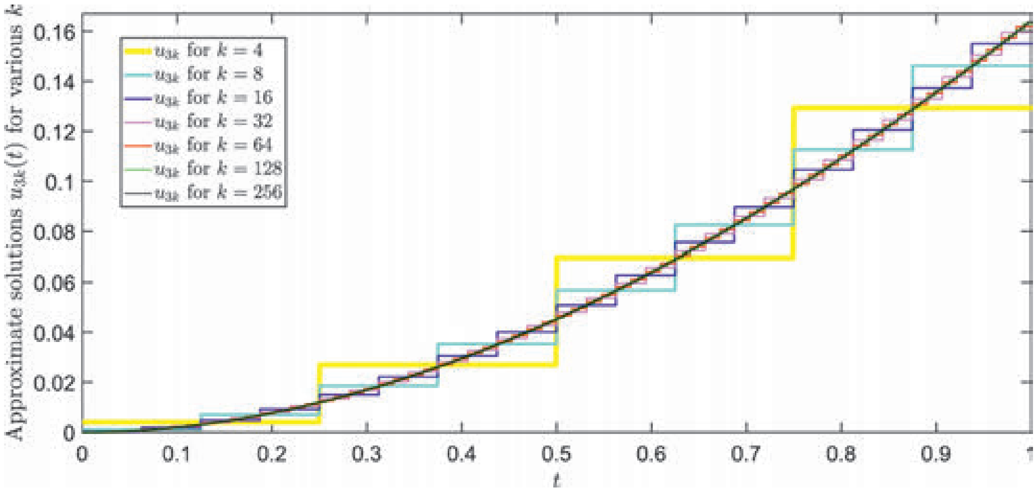

Figure 28 It represents the approximate solutions $u_{3k}(t)$ for different values of k in the special case when $\alpha = 1$ in **Example 7**.

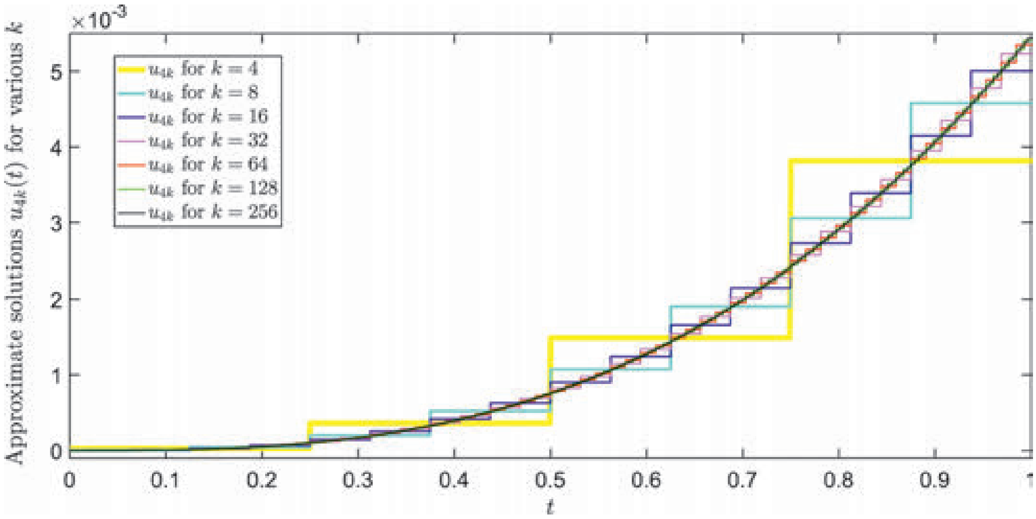

Figure 29 It represents the approximate solutions $u_{4k}(t)$ for different values of k in the special case when $\alpha = 1$ in **Example 7**.

Table 8 shows the comparison of the theoretical error bounds with actual errors for **Example 7**. having $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

Table 8 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}^2_d$-norm for the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ of **Example 7**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	$\|u_4 - u_{4k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_4 - u_{4k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}^2_d}$	**Error bound of** $\|u - u_k\|_{\mathscr{L}^2_d}$
2	2.50E-04	3.00E-04	1.36E-05	9.98E-05	2.62E-05	1.99E-04	9.76E-07	1.33E-05	3.56E-06	8.47E-06
4	1.25E-04	1.50E-04	6.89E-06	4.99E-05	1.33E-05	9.97E-05	5.10E-07	6.63E-06	1.78E-06	4.24E-06
8	6.26E-05	7.49E-05	3.46E-06	2.49E-05	6.67E-06	4.99E-05	2.58E-07	3.32E-06	8.91E-07	2.12E-06
16	3.13E-05	3.74E-05	1.73E-06	1.25E-05	3.34E-06	2.49E-05	1.29E-07	1.66E-06	4.45E-07	1.06E-06
32	1.56E-05	1.87E-05	8.64E-07	6.23E-06	1.67E-06	1.25E-05	6.49E-08	8.29E-07	2.22E-07	5.30E-07
64	7.76E-06	9.36E-06	4.30E-07	3.12E-06	8.28E-07	6.23E-06	3.20E-08	4.14E-07	1.11E-07	2.65E-07
128	3.79E-06	4.68E-06	2.10E-07	1.56E-06	4.04E-07	3.12E-06	1.64E-08	2.07E-07	5.39E-08	1.32E-07
256	1.69E-06	2.34E-06	9.37E-08	7.79E-07	1.81E-07	1.56E-06	7.01E-09	1.04E-07	2.41E-08	6.62E-08

Figures 30, 31, 32 and 33 represents the approximate solutions $u_{1k}(t)$, $u_{2k}(t)$, $u_{3k}(t)$ and $u_{4k}(t)$ respectively for different values of k and α ranging from 0.5 to 0.8 in **Example 7**.

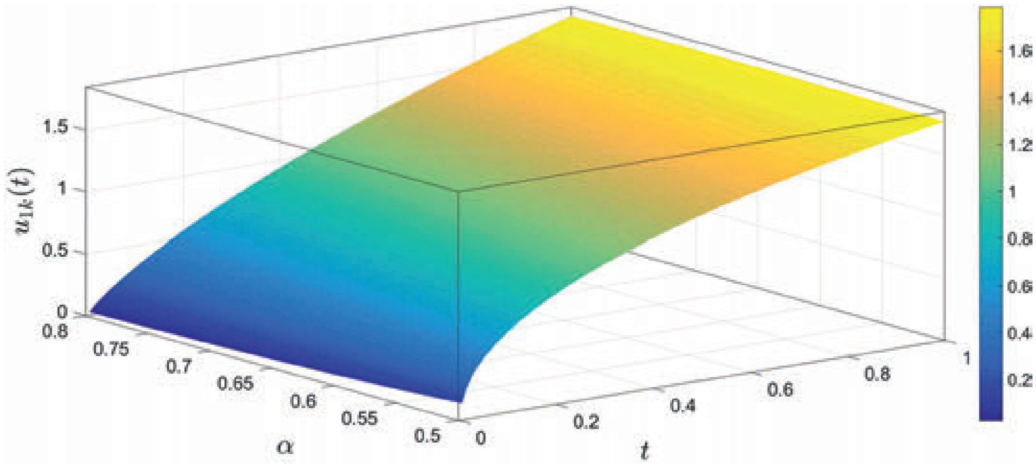

Figure 30 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 7**.

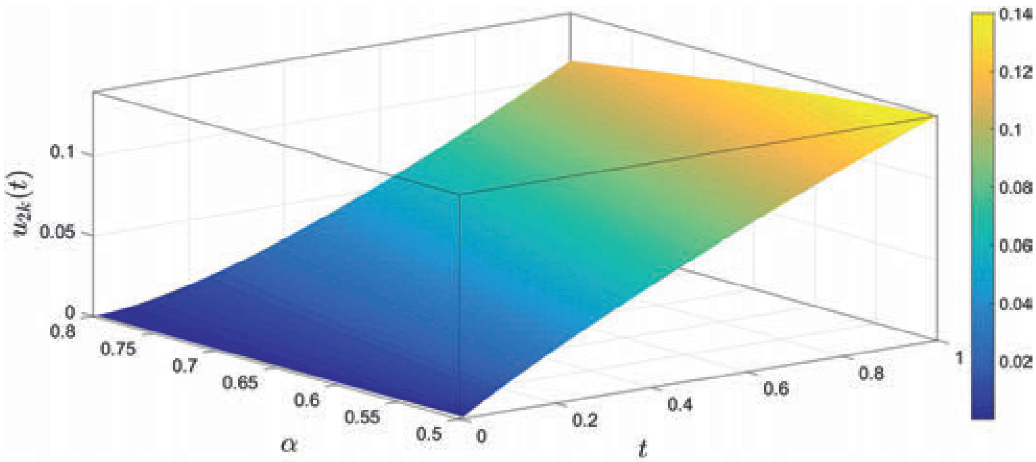

Figure 31 It represents the approximate solutions $u_{2k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 7**.

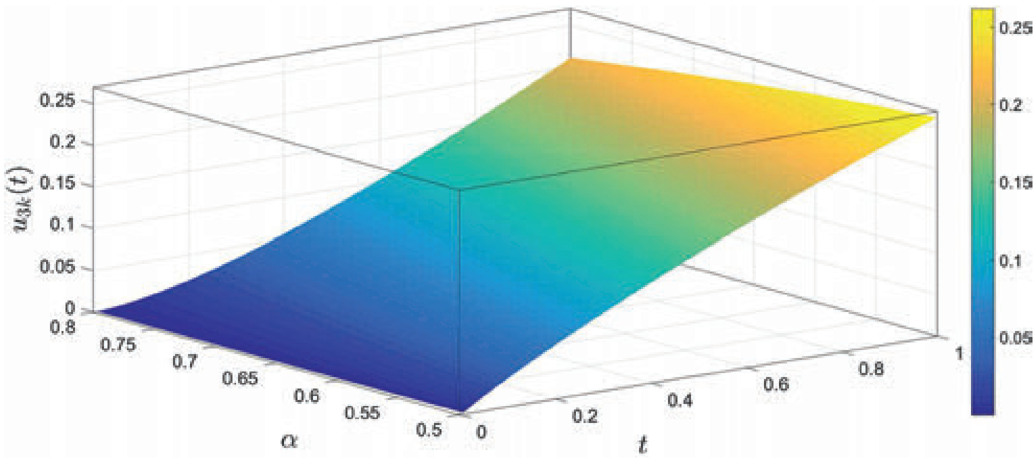

Figure 32 It represents the approximate solutions $u_{3k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 7**.

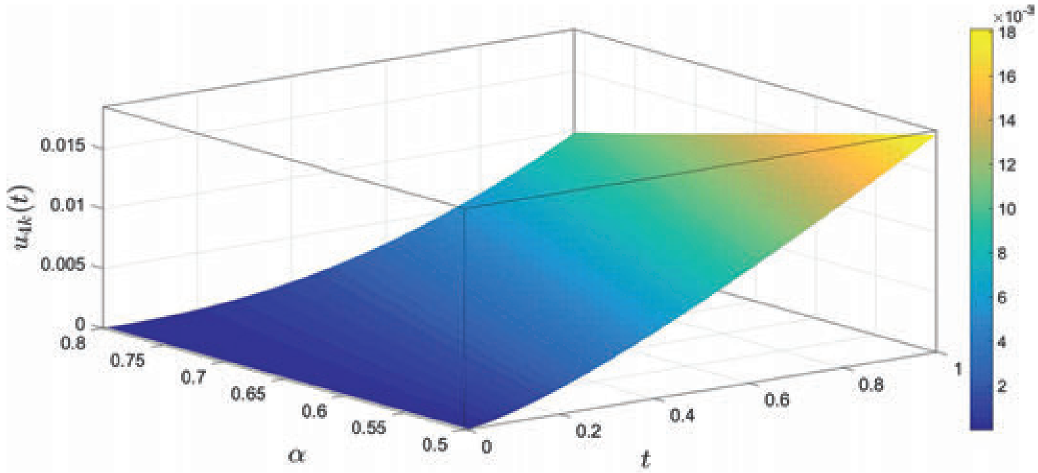

Figure 33 It represents the approximate solutions $u_{4k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 7**.

Example 8

(Chemical reaction problem)

Consider the following differential equation Khan, Aray, and Mahmood (2010) :

$$\mathscr{D}_*^{\alpha} u_1(t) \quad = -u_1, \tag{71}$$

$$\mathscr{D}_*^{\alpha} u_2(t) \quad = u_1 - u_2^2, \tag{72}$$

$$\mathscr{D}_*^{\alpha} u_3(t) \quad = u_2^2, \tag{73}$$

Table 9 shows the comparison of the theoretical error bounds with actual errors for **Example 8**. having $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Figures 34, 35 and 36 represents the approximate solutions $u1_k(t), u_{2k}(t)$ and $u_{3k}(t)$ respectively for different values of k and $\alpha = 1$ for **Example 8**.

Table 9 Error in solution $u(t)$ and corresponding error bound measured with respect to $\mathscr{L}_d^2$-norm for the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ of **Example 8**.

k	$\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_1 - u_{1k}\|_{\mathscr{L}^2}$	$\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_2 - u_{2k}\|_{\mathscr{L}^2}$	$\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	**Error bound of** $\|u_3 - u_{3k}\|_{\mathscr{L}^2}$	$\|u - u_k\|_{\mathscr{L}_d^2}$	**Error bound of** $\|u - u_k\|_{\mathscr{L}_d^2}$
2	9.37E-05	4.98E-04	8.01E-05	5.50E-04	2.18E-05	1.80E-04	1.77E-06	1.35E-05
4	4.73E-05	2.49E-04	4.08E-05	2.75E-04	1.11E-05	8.99E-05	8.97E-07	6.74E-06
8	2.37E-05	1.24E-04	2.05E-05	1.38E-04	5.55E-06	4.49E-05	4.50E-07	3.37E-06
16	1.19E-05	6.22E-05	1.03E-05	6.88E-05	2.78E-06	2.25E-05	2.25E-07	1.68E-06
32	5.92E-06	3.11E-05	5.12E-06	3.44E-05	1.39E-06	1.12E-05	1.12E-07	8.42E-07
64	2.94E-06	1.55E-05	2.55E-06	1.72E-05	6.89E-07	5.62E-06	5.59E-08	4.21E-07
128	1.44E-06	7.77E-06	1.24E-06	8.60E-06	3.36E-07	2.81E-06	2.73E-08	2.11E-07
256	6.42E-07	3.89E-06	5.56E-07	4.30E-06	1.50E-07	1.40E-06	1.22E-08	1.05E-07

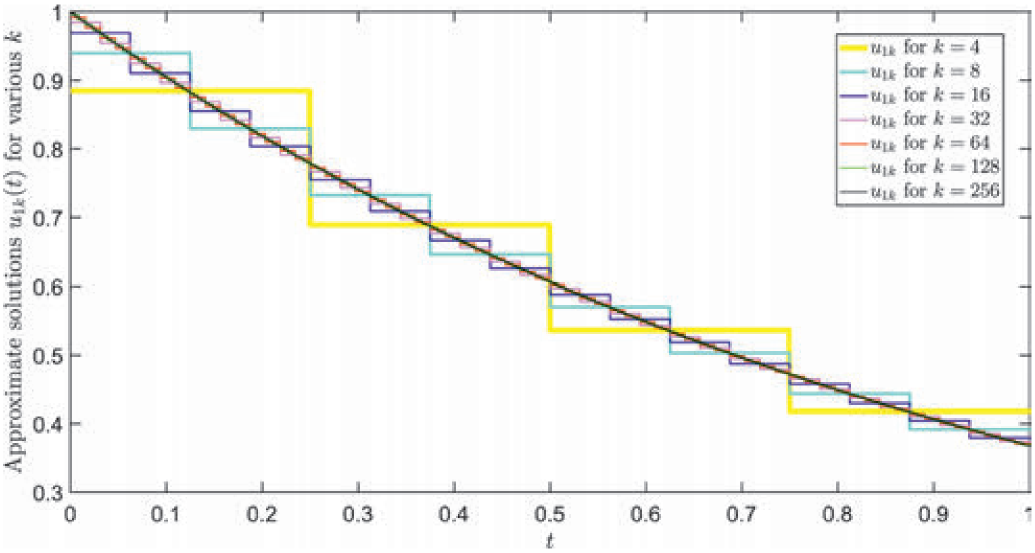

Figure 34 It represents the approximate solutions $u_{1k}(t)$ for different values of k in the special case when $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = 1$ in **Example 8**.

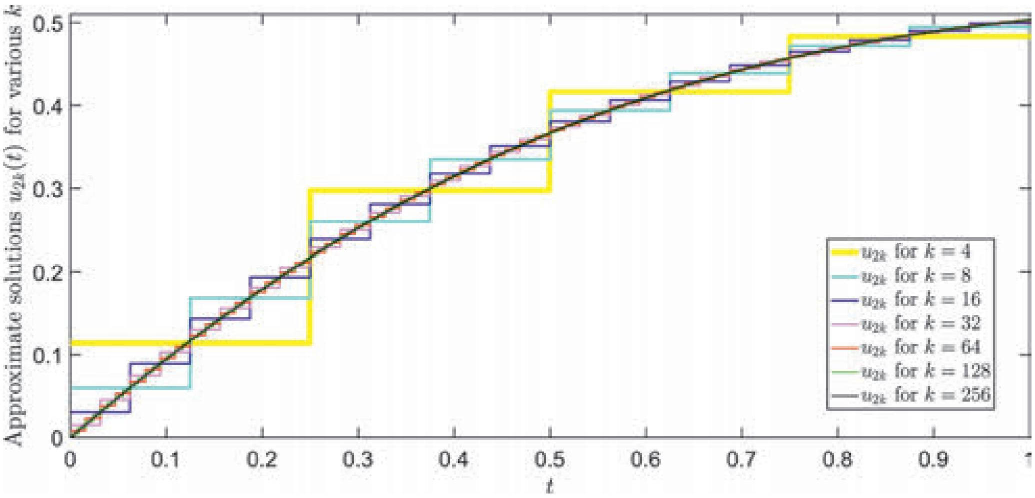

Figure 35 It represents the approximate solutions $u_{2k}(t)$ for different values of k in the special case when $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = 1$ in **Example 8**.

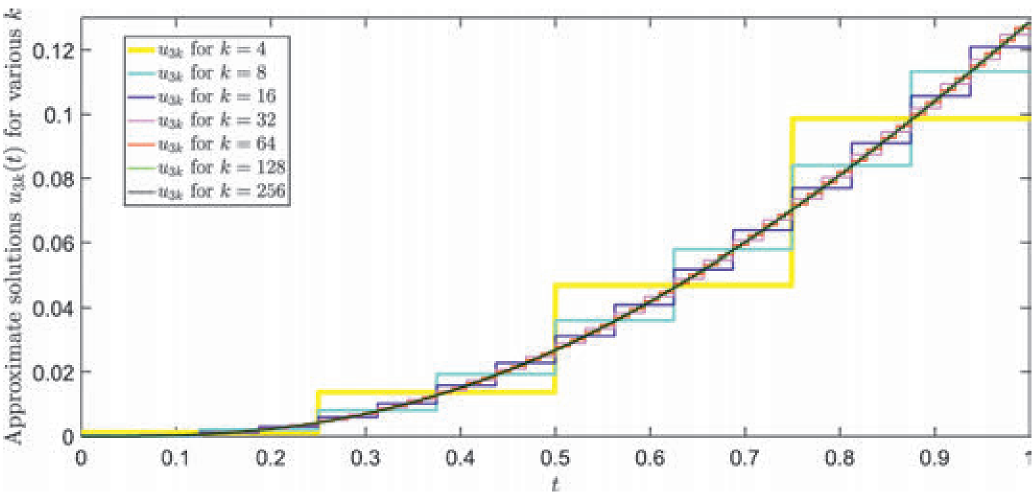

Figure 36 It represents the approximate solutions $u_{3k}(t)$ for different values of k in the special case when $\alpha_1 = 1, \alpha_2 = 1$ and $\alpha_3 = 1$ in **Example 8**.

Figures 37, 38 and 39 represents the approximate solutions $u_{1k}(t)$, $u_{2k}(t)$ and $u_{3k}(t)$ respectively for different values of k and α ranging from 0.5 to 0.8 in **Example 8**.

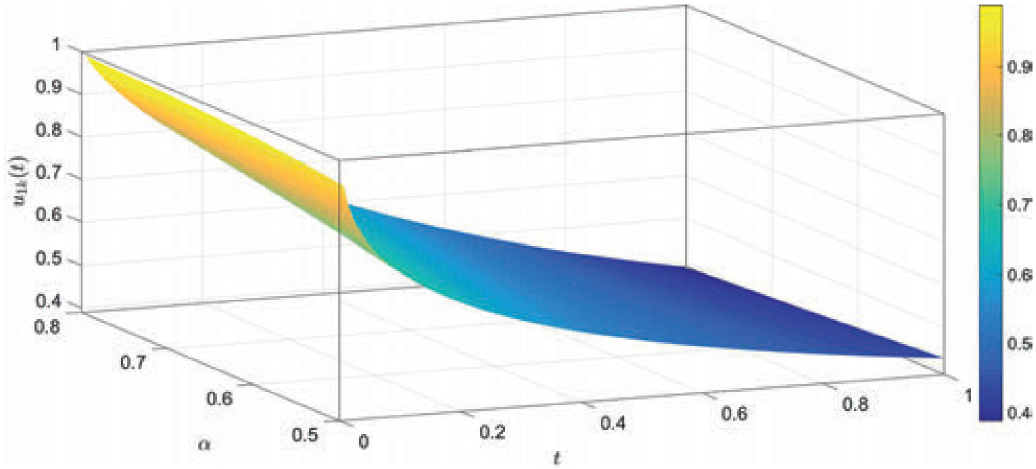

Figure 37 It represents the approximate solutions $u_{1k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 8**.

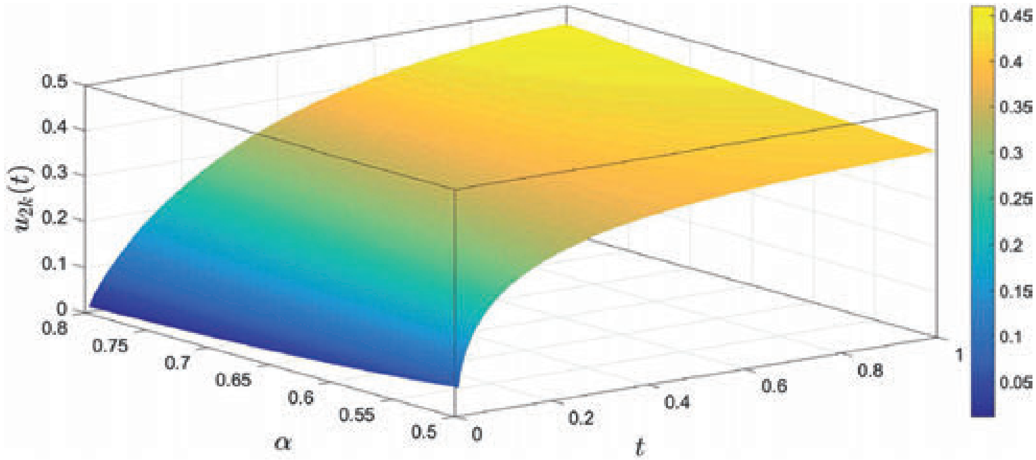

Figure 38 It represents the approximate solutions $u_{2k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 8**.

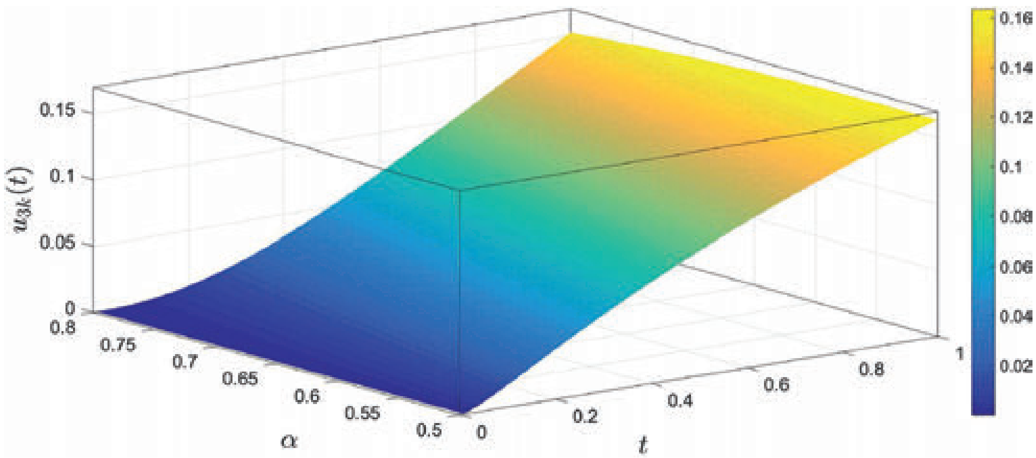

Figure 39 It represents the approximate solutions $u_{3k}(t)$ for different values of k and α from 0.5 to 0.8 in **Example 8**.

Conclusion

A numerical method is proposed to solve the system of fractional differential equations by using Haar wavelets. The existing methods to solve the system of fractional differential equations include differential transform method [14], Adomain decomposition method (ADM) [32], variational iteration method and homotopy perturbation method [26] etc. The results of proposed method are in good agreement with those obtained by the above mentioned methods. The advantage of Haar wavelets based method over the above methods is that they are piecewise constants and orthonormal functions with compact support. It makes the method highly computer oriented. Nontrivial test examples are considered to illustrate the applicability of method. Both linear (**Example 1, 7**) and nonlinear (**Example 2,3,8**) system of fractional differential equations are solved by the method. Among these, **Example 7** and **Example 8** are the problems arising in the field of Chemical Engineering. Also, first and higher order fractional differential equations (**Example 4,5,6**) have also been solved by converting it into a system of fractional differential equations. These include fluid dynamics fractional models (**Example 5,6**), where the solution is not known. The most significant result is the derived theoretical error bound which have been verified through all the examples even in the case where the exact solution is unknown. The theoretical error bound can be used for finding how many number of Haar basis functions will be sufficient to give the desired accuracy in the approximate solution.

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Notes

[1]**Remark**: In all the tables, E represents 10 e.g. 4.10E-04 = 4.10×10^{-4}

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