# Law of Large numbers:

Let, $X_1, X_2, \ldots, X_n$ be sequence of independent and identically destributed random variables with common mean $E[X] = \mu$ & varience $Var(X) = \sigma^2$.

Let, $S_n = X_1 + X_2 + X_3 + \cdots + X_n$. Then for all $\varepsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \to 0 \quad \& \quad n \to \infty$$

**Pf:** Proove using chebyshev's inequality.

$$E\left[\frac{S_n}{n}\right] = \frac{1}{n} E[X_1 + X_2 + \cdots + X_n] = \frac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n}\mu$$

$$= \mu.$$

$$\operatorname{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2} \operatorname{Var}(X_1 + X_2 + \cdots X_n)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \operatorname{Var}(X_i) = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n}$$

By chebychev's inequality:

$$P\left(\frac{S_n}{n} - \mu \,\middle|\, \geq \varepsilon\right) \leq \frac{\operatorname{Var}\left(\frac{S_n}{n}\right)}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n \cdot \varepsilon^2} \to 0 \text{ when } n \to \infty$$

# ☆ Chapter - 4

## Motivation problem:

Sampling proper q-colouring of a graph, Fix a set $\{1, 2, \ldots, q\}$. A proper q-coloring of graph $G = (V, E)$ is an assignment of colors to vertices $V_i$ subject to constant that neighbouring vertices do not have the same color.

## Reasons for sampling:

- Determine the size of a set $\quad P(S) = \dfrac{\# \text{ of outcomes}}{\text{total } \# \text{ of outcomes}}$
- Compute estimators (mean, variance) of the estimates.

# How do you sample a set?

For example: number of proper 3-colorings on peterson graph is 120.

- **Brute-force**: Identify all 120 proper 3-colorings, assign each a number & use a random number generator

- **Probabilistic approach**: Find random process $X_t$ that converges to $X$ that has distribution of proper $q$-coloring. (Markov's chain Monte carlo sampling)

# Markov Chains:

**Defn:** A sequence of random variables $X_t$ that satisfies the condition:

(memoryless property)

$$P(X_t = j \mid X_{t-1} = i_{t-1}, X_{t-2} = i_{t-2}, \dots, X_i = i, X_0 = i_0) = P(X_t = j \mid X_{t-1} = i_{t-1})$$

is called markov's chain.

**Defn:** The (one-stop) transition probabilities of a markov chain is defined by:

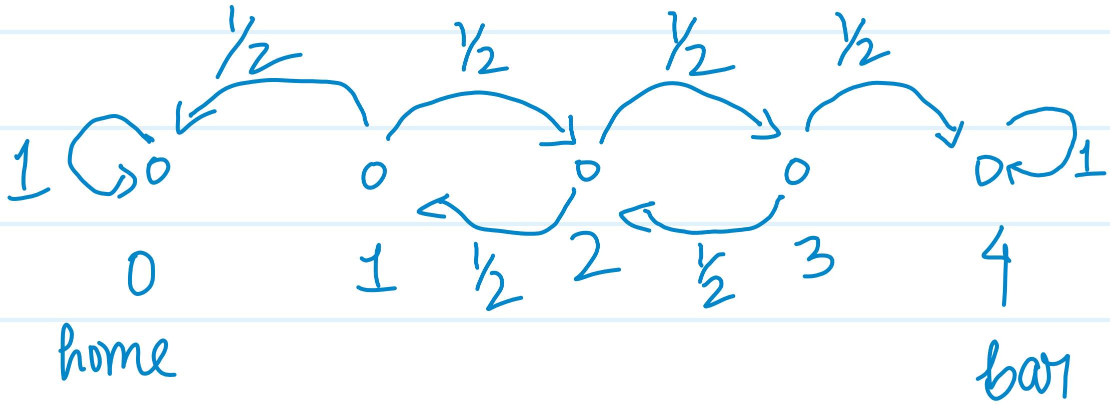
$$P_{ij} = P(X_t = j \mid X_{t-1} = i)$$ → clean up notation.

If $P_{ij}$ are entries of a matrix, i.e

$P = [P_{ij}]$, then $P$ is called transition matrix

**Example:** Drunkard's walk

Person walks along 4-block stretch. If she at corners 1, 2 or 3 she walks left or right w equal probability. She continues until she reaches corner 4 which is the bar, our corner 0, which is home




$$
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
2 & & & & & \\
3 & & & & & \\
4 & & & & &
\end{array}
$$

P