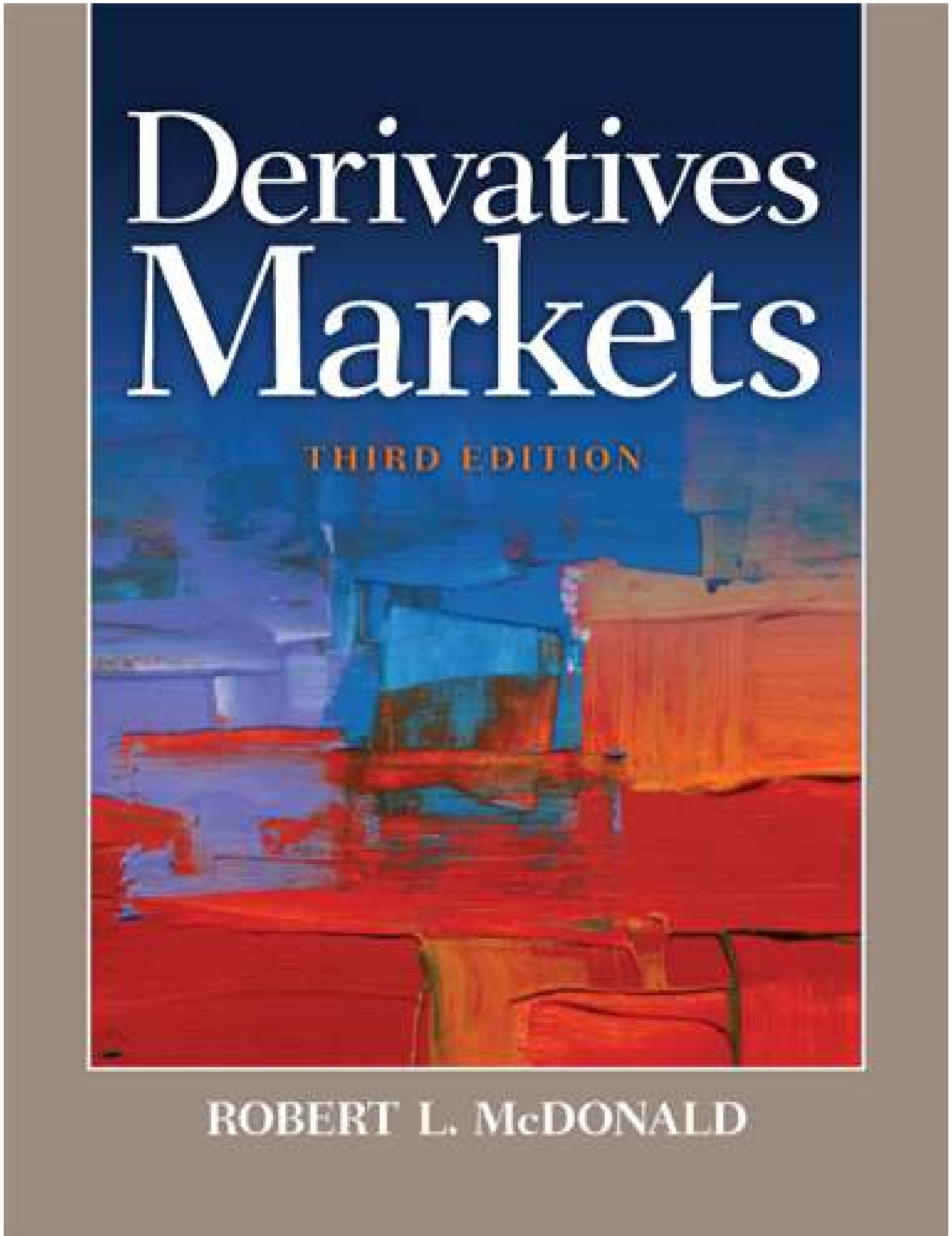
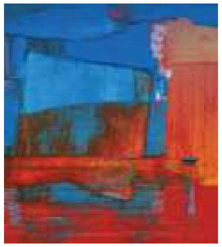


Chapter 8
(Chapter 9 in the textbook)

Parity and Other Option Relationships



Points to Note

1. Important relation: Put-call parity. See P.5
2. Generalized put-call parity on exchange options. See P.9
3. The relationship between call and put options on exchange rate. See P.15
4. Compare the prices of European and American options. See P.16
5. The upper and lower bounds of the option price. See P.17 – 18
6. Early exercises of American call and put options. See P.19 – 24
7. Relationship between time to expiration and option price. See P. 25
8. Relationship between strike prices and option prices. See P.26 - 30
9. Convexity property of option prices with respect to strike prices. See P.31

IBM Option Quotes

TABLE 9.1 IBM option prices, dollars per share, May 6, 2011. The closing price of IBM on that day was $168.89.

		Calls		Puts	
Strike	Expiration	Bid ($)	Ask ($)	Bid ($)	Ask ($)
160	June	10.05	10.15	1.16	1.20
165	June	6.15	6.25	2.26	2.31
170	June	3.20	3.30	4.25	4.35
175	June	1.38	1.43	7.40	7.55
160	October	14.10	14.20	5.70	5.80
165	October	10.85	11.00	7.45	7.60
170	October	8.10	8.20	9.70	9.85
175	October	5.80	5.90	12.40	12.55

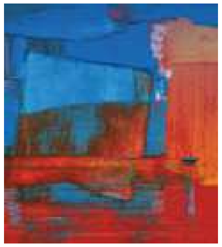
Source: Chicago Board Options Exchange.

Put-Call Parity

Important Note: Starting from here, the meaning of T in $F_{0,T}$ is changed to mean the maturity date of the forward contract.

Notations:

- $C(K, T)$ and $P(K, T)$ are the prices of a European call and put with the strike price K and the **time to expiration T** respectively;
- $F_{0,T}$ be the time 0 price of the forward contract with the **maturity date at time T**.



Put-Call Parity (cont'd)

For European options with the same strike price and time to expiration the parity relationship is

Call – put = *PV* (forward price – strike price)

Or

$$C(K,T) - P(K,T) = PV_{0,T}(F_{0,T} - K) = e^{-rT}(F_{0,T} - K)$$

- Intuition
  - Buying a call and selling a put with the strike equal to the forward price ($F_{0,T} = K$) creates a synthetic forward contract and hence must have a zero price.
- In general, put-call parity *fails* for American style options.

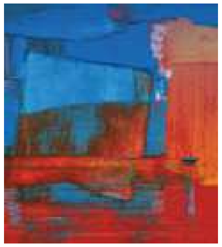
Parity for Options on Stocks

- If underlying asset is a stock and $PV_{0,T}(\text{Div})$ is the present value of the dividends payable over the life of the option, then $e^{-rT}F_{0,T} = S_0 - PV_{0,T}(\text{Div})$, therefore

$$C(K,T) = P(K,T) + [S_0 - \mathrm{PV}_{0,T}(\mathrm{Div})] - e^{-rT}(K)$$

- For index options, $S_0 - \mathrm{PV}_{0,T}(\mathrm{Div}) = S_0 e^{-\delta T}$, therefore

$$C(K,T) = P(K,T) + S_0 e^{-\delta T} - \mathrm{PV}_{0,T}(K)$$



Parity for Options on Stocks (cont'd)

- Examples 9.1 & 9.2
  - Price of a non-dividend-paying stock: \$40, r=8%, option strike price: \$40, time to expiration: 3 months, European call: \$2.78, European put: \$1.99. ➡ $\$2.78 = \$1.99 + \$40 - \$40e^{-0.08 \times 0.25}$.
  - Additionally, if the stock pays \$5 just before expiration, call: \$0.74, and put: \$4.85. ➡ $\$0.74 - \$4.85 = (\$40 - \$5e^{-0.08 \times 0.25}) - \$40e^{-0.08 \times 0.25}$.
- Synthetic security creation using parity
  - Synthetic stock: buy call, sell put, lend PV of strike and dividends.
  - Synthetic T-bill: buy stock, sell call, buy put.
  - Synthetic call: buy stock, buy put, borrow PV of strike and dividends.
  - Synthetic put: sell stock, buy call, lend PV of strike and dividends.

Generalized Parity and Exchange Options

- Suppose we have an option to exchange one asset for another.
- Let the underlying asset, asset A, have price S_t, and the strike asset, asset B, have the price Q_t.
- Let $F_{t,T}^{P}(S)$ denote the time *t* price of a prepaid forward on the underlying asset, paying S_T at time *T*.
- Let $F_{t,T}^{P}(Q)$ denote the time *t* price of a prepaid forward on the underlying asset, paying Q_T at time *T*.
- Let $C(S_t, Q_t, T - t)$ denote the time *t* price of an option with $T-t$ periods of expiration, which gives us the right to give up asset B in exchange for asset A.
- Let $P(S_t, Q_t, T - t)$ denote the time *t* price of an option with $T-t$ periods of expiration, which gives us the right to give up asset A in exchange for asset B.

Generalized Parity and Exchange Options (cont'd)

- At time T, we have

  $C(S_T, Q_T, 0) = \max(0, S_T - Q_T)$ and

  $P(S_T, Q_T, 0) = \max(0, Q_T - S_T)$

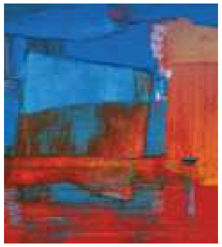
- Then for European options we have this form of the parity equation:

$$C(S_t, Q_t, T-t) - P(S_t, Q_t, T-t) = F_{t,T}^P(S) - F_{t,T}^P(Q)$$

Generalized Parity Relationship

TABLE 9.2 Payoff table demonstrating that there is an arbitrage opportunity unless $-C(S_t, Q_t, T-t) + P(S_t, Q_t, T-t) + F^P_{t,T}(S) - F^P_{t,T}(Q) = 0$.

Transaction	Time 0	Expiration: $S_T \le Q_T$	Expiration: $S_T > Q_T$
Buy call	$-C(S_t, Q_t, T-t)$	0	$S_T - Q_T$
Sell put	$P(S_t, Q_t, T-t)$	$S_T - Q_T$	0
Sell prepaid forward on A	$F^P_{t,T}(S)$	$-S_T$	$-S_T$
Buy prepaid forward on B	$-F^P_{t,T}(Q)$	Q_T	Q_T
Total	$-C(S_t, Q_t, T-t)$ $+P(S_t, Q_t, T-t)$ $+F^P_{t,T}(S) - F^P_{t,T}(Q)$	0	0



Currency Options

- A currency transaction involves the exchange of one kind of currency for another.
- The idea that calls can be relabeled as puts is commonplace in currency markets.
- A term sheet for a currency option might specify

  "EUR Call USD Put, AMT: EUR 100 million, USD 120 million"

  It says explicitly that the option can be viewed either as a call on the euro or a put on the dollar. Exercise of the option will entail an exchange of €100 million for $120 million.
- A call in one currency can be converted into a put in the other.

Currency Options (cont'd)

Example

Suppose the current exchange rate is x_0 = \$1.25/€. Consider the following two options:

1. A *1-year dollar-denominated call option* on euros with a strike price of \$1.20 and premium of \$0.06545. In 1 year, the owner of the option has the right to buy €1 for \$1.20. the payoff on this option, in dollars, is therefore

$$\max(0, x_1 - 1.20)$$

Currency Options (cont'd)

2. A *<u>1-year euro-denominated put option</u>* on dollars with a strike price of 1/1.20 = €0.833. The premium of this option is €0.04363. In 1 year the owner of this put has the right to give up \$1 and receive €0.833; the owner will exercise the put when \$1 is worth less than €0.833. The euro value of \$1 in 1 year will be $1/x_1$. Hence the payoff of this option is

$$\max\left(0, \frac{1}{1.2} - \frac{1}{x_1}\right)$$

BOTH the call and put options are exercised when $x_1 > 1.20$.

Currency Options (cont'd)

TABLE 9.3 The equivalence of buying a dollar-denominated euro call and a euro-denominated dollar put. In transaction I, we buy one dollar-denominated call option, permitting us to buy €1 for a strike price of $1.20. In transaction II, we buy 1.20 euro-denominated puts, each with a premium of €0.04363, and permitting us to sell $1 for a strike price of €0.833.

		Year 0		Year 1			
				$x_1 < 1.20$		$x_1 \geq 1.20$	
	Transaction	$	€	$	€	$	€
I:	Buy 1 euro call	−0.06545	—	0	0	−1.20	1
II:	Convert dollars to euros,	−0.06545	0.05236				
	buy 1.20 dollar puts		−0.05236	0	0	−1.20	1

Currency Options (cont'd)

In summary, we have

$$C_{\$}\left(x_0, K, T\right) = x_0 K P_f\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

where

$C_{\$}(x_0, K, T)$ is the price of a dollar-denominated foreign currency call with strike K, when the current exchange rate is x_0;

$P_f(1/x_0, 1/K, T)$ is the price of a foreign-currency-denominated dollar put with strike $1/K$, when the exchange rate is $1/x_0$.

Properties of Option Prices

- European versus American Options
  - Since an American option can be exercised at anytime, whereas a European option can only be exercised at expiration, an American option must always be at least as valuable as an otherwise identical European option

$$C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T)$$

$$P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T)$$

Properties of Option Prices (cont'd)

- Maximum and Minimum Option Prices
  - The price of a European call option:
    - Cannot be negative, because the call need not be exercised.
    - Cannot exceed the stock price, because the best that can happen with a call is that you end up owning the stock.
    - Must be at least as great as the price implied by put-call parity using a zero put value.

$$S > C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max[0, \mathrm{PV}_{0,T}(F_{0,T}) - \mathrm{PV}_{0,T}(K)]$$

Properties of Option Prices (cont'd)

- The price of a European put option:
  - Cannot be worth more than the undiscounted strike price, since that is the most it can ever be worth (if the stock price drops to 0, the put pays K at some point).
  - Must be at least as great as the price implied by put-call parity with a zero call value.

$$K > P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max[0, \mathrm{PV}_{0,T}(K) - \mathrm{PV}_{0,T}(F_{0,T})]$$

Properties of Option Prices (cont'd)

Early exercise for American options

<u>Calls on a non-dividend-paying stock</u>

Early exercise is not optimal if the price of an American call prior to expiration satisfies

$$C_{\text{Amer}}(S_t, K, T - t) > S_t - K$$

If this inequality holds, you would lose money by early-exercising (receiving $S_t - K$) as opposed to selling the option (receiving $C_{\text{Amer}}(S_t, K, T - t) > S_t - K$).

Properties of Option Prices (cont'd)

No early exercise for American call option on **non-dividend-paying** stock.

Proof

From the put-call parity, we have

$$C_{Eur}(S_t, K, T-t) = \underbrace{S_t - K}_{\text{Exercise value}} + \underbrace{P_{Eur}(S_t, K, T-t)}_{\text{Insurance against } S_T < K} + \underbrace{K\left(1 - e^{-r(T-t)}\right)}_{\text{Time value of money on } K}$$

$$> S_t - K$$

Since $C_{Amer} \geq C_{Eur}$, we have

$$C_{Amer} \geq C_{Eur} > S_t - K$$

♦

Properties of Option Prices (cont'd)

Early-exercising has the following effects:

1. Throw away the implicit put protection should the stock later move below the strike price.
2. Accelerate the payment of the strike price.
3. (**No early-exercise**) The possible loss from deferring receipt of the stock. However, when there is no dividends, we lose nothing by waiting to take physical possession of the stock.

Properties of Option Prices (cont'd)

Exercising calls just prior to a dividend

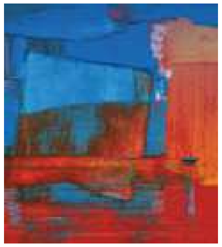
If the stock pays dividends, the parity relationship is

$$C(S_t, K, T-t) = P(S_t, K, T-t) + S_t - PV_{t,T}(Div) - PV_{t,T}(K)$$
$$= S_t - K + P(S_t, K, T-t) + K - PV_{t,T}(K) - PV_{t,T}(Div)$$

The early exercise is possible if

$$K - PV_{t,T}(K) - PV_{t,T}(Div) < 0$$
$$K - PV_{t,T}(K) < PV_{t,T}(Div)$$

If dividends do make early exercise rational, it will be optimal to exercise at the last moment before the ex-dividend date.



Properties of Option Prices (cont'd)

<u>Early exercise for puts (**non-dividend paying stock**)</u>

The put will never be exercised as long as *P* > *K* – *S*. Supposing that the stock pays no dividends, parity for the put is

$$P(S_t, K, T-t) = C(S_t, K, T-t) - S_t + PV_{t,T}(K)$$

The no-exercise condition, *P* > *K* – *S*, then implies

$$C(S_t, K, T-t) - S_t + PV_{t,T}(K) > K - S_t$$
$$C(S_t, K, T-t) > K - PV_{t,T}(K)$$

The early exercise is possible if the call is sufficiently valueless.

Properties of Option Prices (cont'd)

Early exercise for puts (**dividend paying stock**)

When the stock pays discrete dividend, the no-exercise condition, $P > K - S$, will be modified as

$$C(S_t, K, T-t) - S_t + PV_{t,T}(K) + PV_{t,T}(div) > K - S_t$$

$$C(S_t, K, T-t) > K - PV_{t,T}(K) - PV_{t,T}(div)$$

So, the stock dividends make American put harder to exercise earlier.

Properties of Option Prices (cont'd)

- Time to Expiration
  - An American option (both put and call) with more time to expiration is at least as valuable as an American option with less time to expiration. This is because the longer option can easily be converted into the shorter option by exercising it early.
  - A European call option on a non-dividend-paying stock will be at least as valuable as an otherwise identical option with a shorter time to expiration. This is because a European call on a non-dividend-paying stock has the same price as an otherwise identical American call.
  - European call and put options on dividend-paying stock **may be** less valuable or more valuable than an otherwise identical option with less time to expiration.

Properties of Option Prices (cont'd)

- Different strike prices ($K_1 < K_2 < K_3$), for both European and American options
  - A call with a low strike price is at least as valuable as an otherwise identical call with a higher strike price:

$$C(K_1) \geq C(K_2)$$

  - A put with a high strike price is at least as valuable as an otherwise identical call with a low strike price :

$$P(K_2) \geq P(K_1)$$

Properties of Option Prices (cont'd)

– The premium difference between otherwise identical calls with different strike prices cannot be greater than the difference in strike prices:

$$C(K_1) - C(K_2) \leq K_2 - K_1$$

If the calls are **European** calls, we can put a tighter restriction on the difference in call premiums, namely,

$$C(K_1) - C(K_2) \leq PV\left(K_2 - K_1\right)$$

Properties of Option Prices (cont'd)

– The premium difference for otherwise identical puts also cannot be greater than the difference in strike price:

$$P(K_2) - P(K_1) \leq K_2 - K_1$$

If the puts are **European** puts, we can put a tighter restriction on the difference in put premiums, namely,

$$P(K_2) - P(K_1) \leq PV(K_2 - K_1)$$

Properties of Option Prices (cont'd)

Example

Suppose we observe the call premium in Panel A of Table 9.6. These values violate the property that the premium difference cannot be greater than the difference of the strike prices. In this case, the arbitrage profit can be created.

Properties of Option Prices (cont'd)

TABLE 9.6 Panel A shows call option premiums for which the change in the option premium ($6) exceeds the change in the strike price ($5). Panel B shows how a bear spread can be used to arbitrage these prices. By lending the bear spread proceeds, we have a zero cash flow at time 0; the cash outflow at time T is always greater than $1.

Panel A

Strike	50	55
Premium	18	12

Panel B

		Expiration or Exercise		
Transaction	**Time 0**	$S_T < 50$	$50 \leq S_T \leq 55$	$S_T \geq 55$
Buy 55-strike call	−12	0	0	$S_T - 55$
Sell 50-strike call	18	0	$50 - S_T$	$50 - S_T$
Total	6	0	$50 - S_T$	−5

Properties of Option Prices (cont'd)

– Premiums decline at a decreasing rate for calls with progressively higher strike prices. The same is true for puts as strike prices decline (Convexity of option price with respect to strike price):

$$\frac{C(K_1)-C(K_2)}{K_2-K_1} \geq \frac{C(K_2)-C(K_3)}{K_3-K_2}$$

$$\frac{P(K_2)-P(K_1)}{K_2-K_1} \leq \frac{P(K_3)-P(K_2)}{K_3-K_2}$$

Properties of Option Prices (cont'd)

- Exercise and Moneyness
  - If it is optimal to exercise an option, it is also optimal to exercise an otherwise identical option that is more in-the-money.

Example

Suppose a call option on a dividend-paying stock has a strike price of \$50, and the stock price is \$70.

Also suppose that it is optimal to exercise the option. The option must sell for \$70 – \$50 = \$20.

What can we say about the premium of a 40-strike option?

Properties of Option Prices (cont'd)

Since

$$C(40) - C(50) \leq 50 - 40$$

$$C(40) \leq C(50) + 50 - 40 \leq 30$$

the 40-strike call is optimal to exercise.