BS formula

Assumption of stock price dynamic.

- Lognormal $\Leftrightarrow$ GBM

Under <u>risk-neutral prob.</u>

$$\frac{dS(t)}{S(t)} = (r-q)\,dt + \sigma\, dZ(t)$$

$$C = e^{-rT} E^{Q}\left[(S(T) - K)^{+} \right]$$

$$= S(0)\, e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

<u>Steps</u>

1) Change from $P \longrightarrow Q$
   (by Girsanov thm)

2) Use Risk-neutral valuation formula

$$V(0) = e^{-rT} E^{Q}\left[V(T) \right]$$

3) Make use of Change of Numeraire to simplify the derivation of BS

Solution of GBM

a) $$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma\, dZ(t)$$

$$X(t) = X(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z(t) \right)$$

b) $$\frac{dX(t)}{X(t)} = \mu\, dt + \sigma_1\, dZ_1(t) + \sigma_2\, dZ_2(t)$$

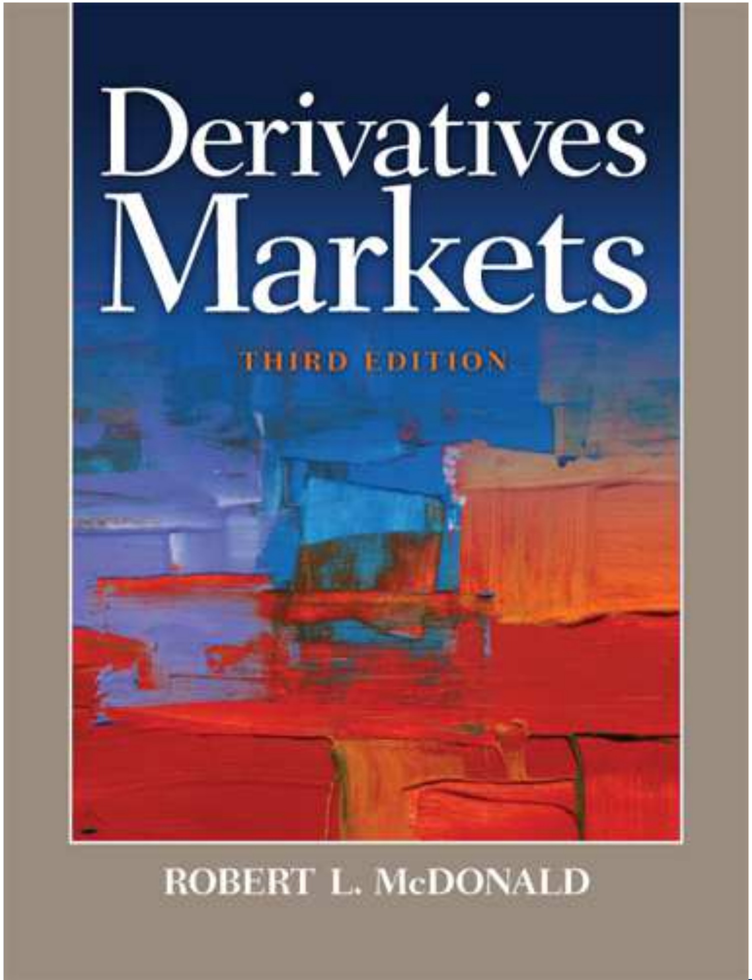
where $Z_1(t)$ and $Z_2(t)$ are independent.

Let $Y(t) = \ln X(t)$

$$dY(t) = \frac{\partial Y}{\partial t} dt + \frac{\partial Y}{\partial X} dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX)^2$$

$$= \left(\mu - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right) dt + \sigma_1\, dZ_1(t) + \sigma_2\, dZ_2(t)$$

$$X(t) = X(0) \exp\left(\left(\mu - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2\right) t + \sigma_1 Z_1(t) + \sigma_2 Z_2(t) \right)$$



Chapter 15

(Chapter 23 in the textbook)

Exotic Options

Points to Notes

1. What are the all-or-nothing options? See P. 3 – 4.
2. How are the all-or-nothing options related to the BS call and put options? See P. 5 – 9.
3. What are the Asian options? See P. 10 - 11
4. What are the differences between the arithmetic and geometric average? See P. 12 – 14.
5. What are the barrier options? See P. 15.
6. How are the barrier options related to the ordinary call and put options? See P. 16 – 18.

All-or-Nothing Options

Terminology

Notation	**Meaning**
Asset	Payment at expiration is one unit of the asset
Cash	Payment at expiration is $1
Call	Payment received if $S(T) > K$.
Put	Payment received if $S(T) < K$.

Definition

$$d_1 = \frac{\left[\ln\left(S(t)/K\right) + \left(r - \delta + 0.5\sigma^2\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

All-or-Nothing Options

- Simple all-or-nothing options pay the holder a discrete amount of cash or a share if some particular event occurs.
- Cash-or-nothing
  - Call: pays \$1 if $S_T > K$ and zero otherwise

$$\text{CashCall}(S, K, \sigma, r, T-t, \delta) = e^{-r(T-t)}N(d_2)$$

  - Put: pays \$1 if $S_T < K$ and zero otherwise

$$\text{CashPut}(S, K, \sigma, r, T-t, \delta) = e^{-r(T-t)}N(-d_2)$$

- Asset-or-nothing
  - Call: pays S_T (one unit share) if $S_T > K$ and zero otherwise

dividend yield.

$$\text{AssetCall}(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)}N(d_1)$$

  - Put: pays S_T (one unit share) if $S_T < K$ and zero otherwise

$$\text{AssetPut}(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)}N(-d_1)$$

Cash - Call

$$\text{Payoff} = \begin{cases} \$1 & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$$

By risk - neutral valuation,

$$V(0) = e^{-rT} E^Q[V(T)]$$
$$= e^{-rT} E^Q[I\{S_T > K\}]$$
$$= e^{-rT} \Pr(S_T > K)$$
$$= e^{-rT} N(d_2)$$

Cash - put

$$\text{Payoff} = I\{S_T < K\}$$

Asset call

$$\text{Payoff} = S_T 1_{\{S_T > K\}}$$

$$V(0) = e^{-rT} E^Q[S_T 1_{\{S_T > K\}}]$$
$$= e^{-qT} S(0) N(d_1)$$

Payoff of Asset call $-$ K Payoff of cash call

$$= S_T 1_{\{S_T > K\}} - K 1_{\{S_T > K\}}$$

$$= \max(S_T - K, 0)$$

All-or-Nothing Options (cont'd)

- + 1 asset-or-nothing call option with strike price *K*
  – *K* cash-or-nothing call option with strike price *K*
  = 1 ordinary call option with strike price *K*

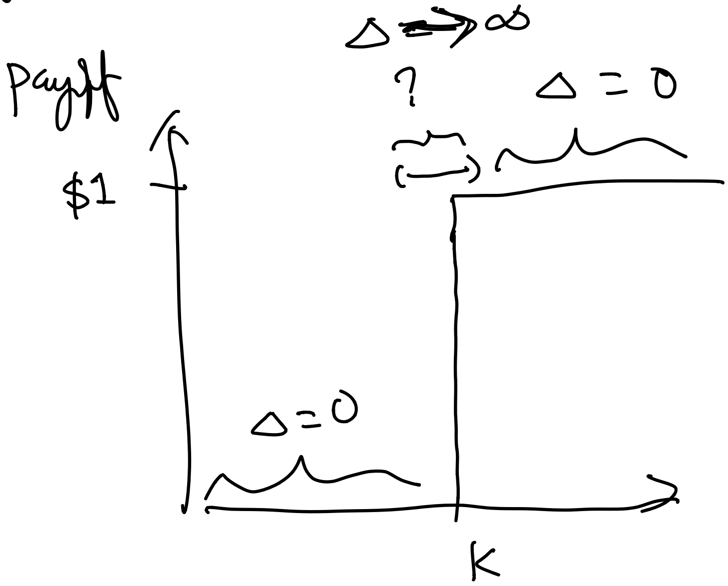
$$
\begin{aligned}
&\mathrm{BSCall}(S, K, \sigma, r, T-t, \delta) \\
&= \mathrm{AssetCall}(S, K, \sigma, r, T-t, \delta) - K \times \mathrm{CashCall}(S, K, \sigma, r, T-t, \delta) \\
&= Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}
$$

All-or-Nothing Options (cont'd)

- Similarly, a put option can be created by buying *K* cash-or-nothing puts, and selling 1 asset-or-nothing put

$$\begin{aligned}&\text{BSPut}(S, K, \sigma, r, T-t, \delta)\\&= K \times \text{CashPut}(S, K, \sigma, r, T-t, \delta) - \text{AssetPut}(S, K, \sigma, r, T-t, \delta)\end{aligned}$$

Payoff of Cash-Call



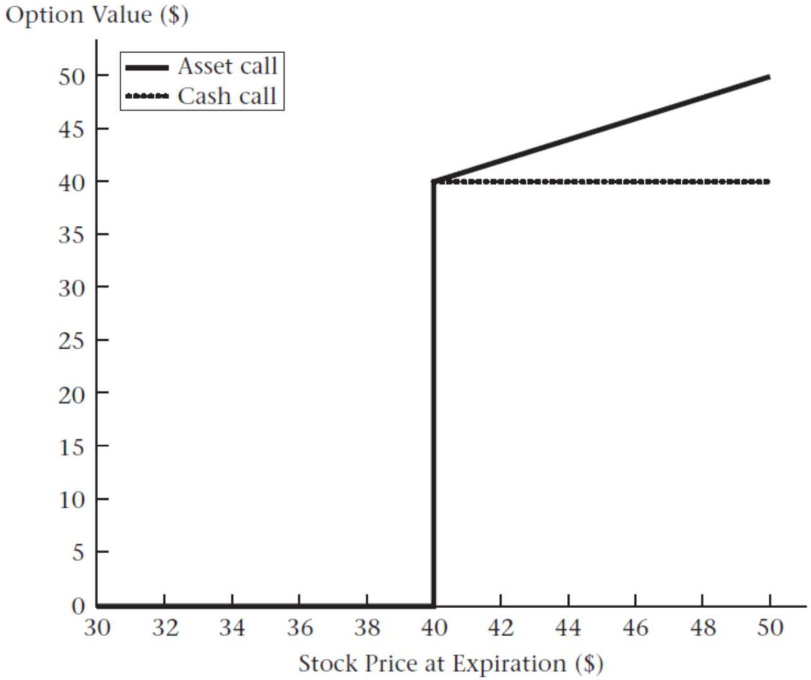
All-or-Nothing Options (cont'd)

- All-or-nothing options are easy to price but hard to hedge.
- Fig. 1 shows that a small swing in the stock price can determine whether the option is in- or out-of-the money, with the payoff changing discretely.
- Fig. 2 shows that hedging is straightforward and delta is well behaved when 3 months to expiration. However, with 2 minutes to expiration, the cash call delta at $40 is 15. *For the at-the-money option, delta and gamma approach infinity at expiration because an arbitrarily small change in the price can result in a $1 change in the option's value.*

All-or-Nothing Options (cont'd)

FIGURE 23.1

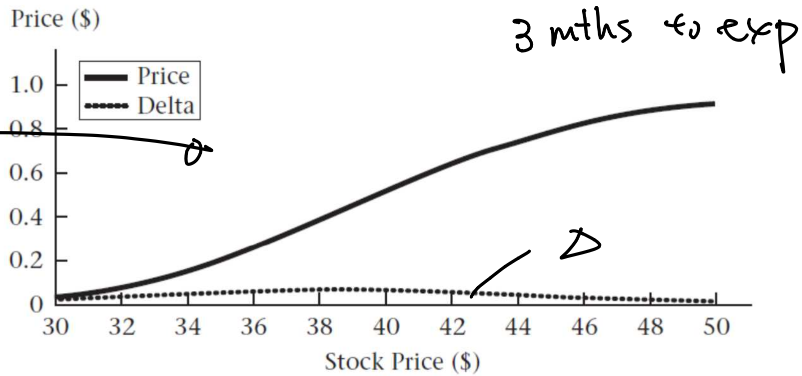
Payoff at maturity to one asset call and 40 cash calls. Assumes $K = \$40$, $\sigma = 0.30$, $r = 0.08$, and $\delta = 0$. The payoff to both is zero for $S < \$40$.

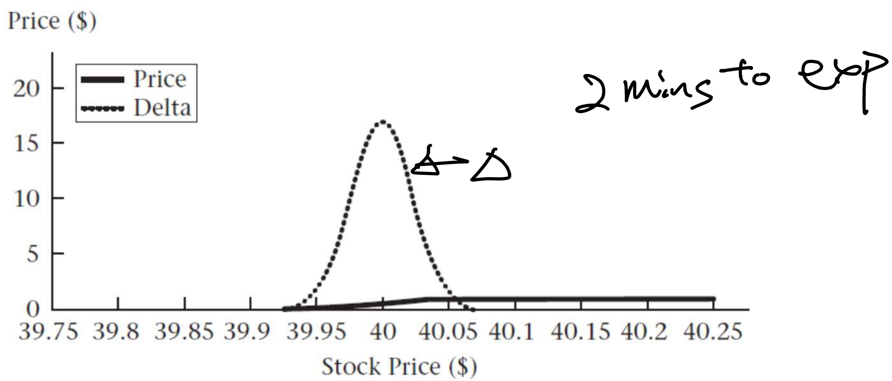


All-or-Nothing Options (cont'd)

FIGURE 23.2

Price and delta of a cash call at two different times to expiration: 3 months (top panel) and 2 minutes (bottom panel). Assumes $K = \$40$, $\sigma = 0.30$, $r = 0.08$, and $\delta = 0$.





Asian Options

Call : $\max(S_T - K, 0)$

- The payoff of an Asian option is based on the average price over some period of time. An Asian options is an example of a path-dependent option.
- Situations when Asian options are useful:
  - When a business cares about the average exchange rate over time.
  - When a single price at a point in time might be subject to manipulation.
  - When price swings are frequent due to thin markets.

Asian Options (cont'd)

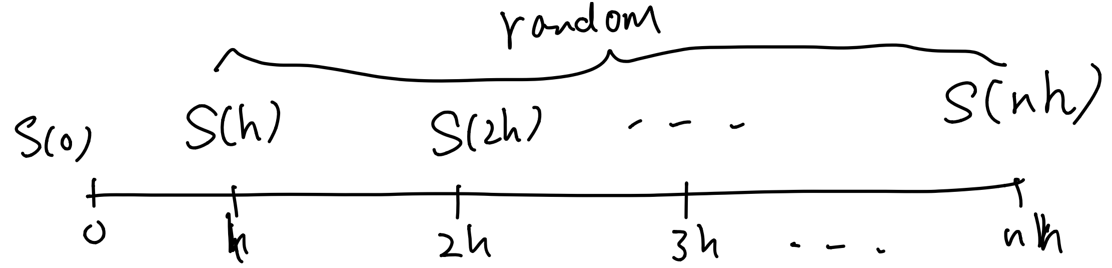
- Asian options are less valuable than otherwise equivalent ordinary options, since the averaged price of the underlying asset is less volatile than the asset price itself, and an option on a lower volatility asset is worth less.

Asian Options (cont'd)

- There are eight (2^3) basic kinds of Asian options:
  - Put or call.
  - Geometric or arithmetic average.
  - Average asset price is used in place of underlying. price or the strike price.
- Arithmetic versus geometric average:
  - Suppose we record the stock price every h periods from $t = 0$ to $t = T$.
  - Arithmetic average: Geometric average:

$$A(T) = \frac{1}{N}\sum_{i=1}^{N} S_{ih}$$

$$G(T) = (S_h \times S_{2h} \times \cdots \times S_{Nh})^{1/N}$$



$$A(T) = \frac{\sum_{i=1}^{n} S(ih)}{n}$$

$A(T)$ is stochastic

$$G(T) = \left(S(h) \times S(2h) \cdots S(nh) \right)^{\frac{1}{n}}$$

$G(T)$ is stochastic

$$\left.\begin{array}{l} \text{payoff} = \max(A(T) - K, 0) \\ \text{payoff} = \max(G(T) - K, 0) \end{array}\right\} \text{path dependent.}$$

To price : joint dist of $S(h)$, $S(2h)$, ... $S(nh)$

①

$$\underbrace{\max(A(T) - K, 0)}_{\text{cheaper}}$$

②

$$\max(S(T) - K, 0)$$

Asian Options (cont'd)

- Average used as the asset price: Average price option
  - Geometric average price call = max $[0, G(T) - K]$.
  - Geometric average price put = max $[0, K - G(T)]$.
- Average used as the strike price: Average strike option
  - Geometric average strike call = max $[0, S_T - G(T)]$.
  - Geometric average strike put = max $[0, G(T) - S_T]$.

Asian Options (cont'd)

- All four options above could also be computed using arithmetic average instead of geometric average.
- Relatively simple pricing formulas exist for pricing European options on the geometric average but not for arithmetic average options.

Barrier Options

- The payoff depends on whether over the option life the underlying price reaches a specified level, called the *barrier*.
  - Path-dependent.
  - Since barrier puts and calls never pay more than standard puts and calls, they are no more expensive than standard puts and calls.
  - Widely used in practice.

Barrier Options (cont'd)

- Barrier puts and calls

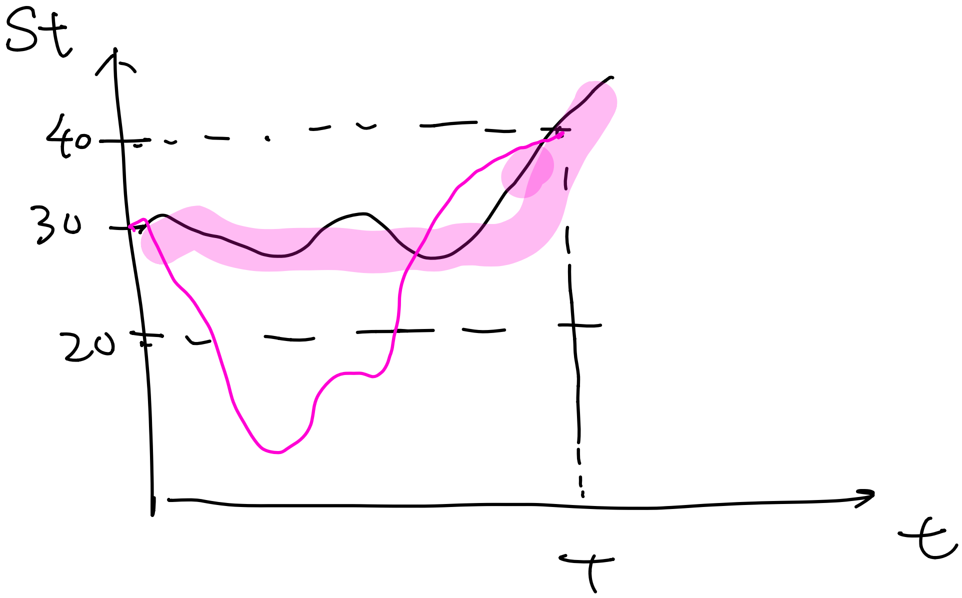
  die

  - Knock-out options: *go out of* existence (are "knocked-out")
    - down-and-out: if the asset price *falls* to reach the barrier.
    - up-and-out: if the asset price *rises* to reach the barrier.
  - Knock-in options: *come into* existence (are "knocked-in")
    - down-and-in: if the asset price *falls* to reach the barrier.
    - up-and-in: if the asset price *rises* to reach the barrier.
  - The important parity relation for barrier options is

$$\text{"Knock} - \text{in" option} + \text{"Knock} - \text{out" option} = \text{Ordinary option}$$

  - Rebate options: make a fixed payment if the asset price reaches the barrier
    - down rebates: if the asset price *falls* to reach the barrier.
    - up rebates: if the asset price *rises* to reach the barrier.

down-and-out



$\max(S_T - 38, 0) = 2$

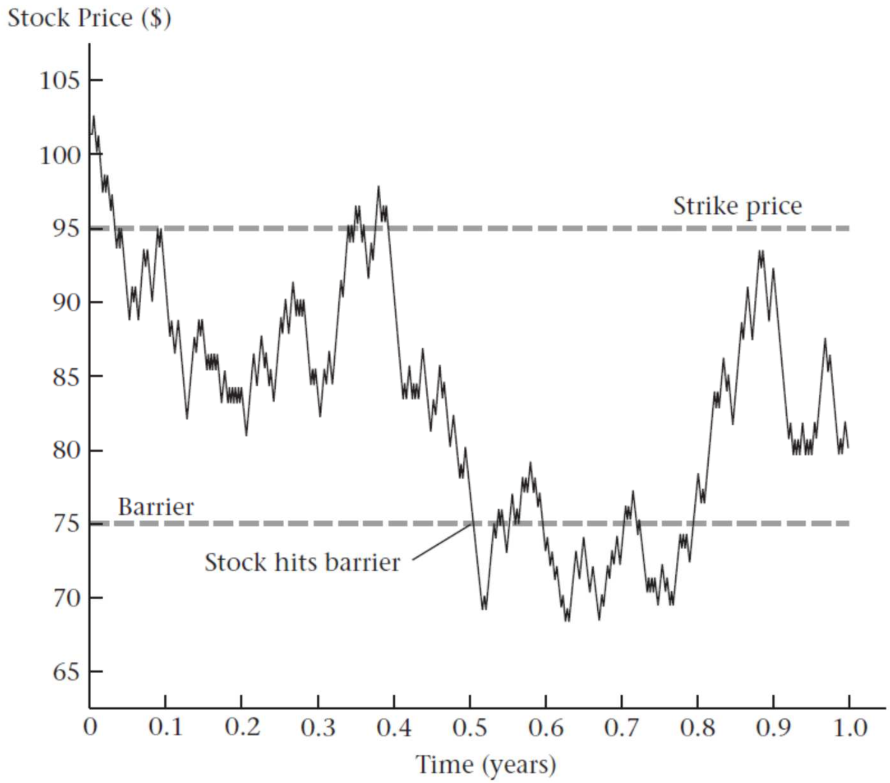
knock-out (barrier = \$20) payoff = 2
payoff = 0

Knock-in (barrier = \$20) payoff = 0
payoff = 2

Barrier Options (cont'd)

FIGURE 14.1

Illustration of a price path where the initial stock price is \$100 and the barrier is \$75. At $t = 0.5$, the stock hits the barrier.



Barrier Options (cont'd)

TABLE 14.3 Premiums of standard, down-and-in, and up-and-out currency put options with strikes K. The column headed "standard" contains prices of ordinary put options. Assumes $x_0 = 0.9$, $\sigma = 0.1$, $r_\$ = 0.06$, $r_€ = 0.03$, and $t = 0.5$.

	Standard	Down-and-In Barrier ($)		Up-and-Out Barrier ($)		
Strike ($)	**($)**	**0.8000**	**0.8500**	**0.9500**	**1.0000**	**1.0500**
$K = 0.8$	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
$K = 0.9$	0.0188	0.0066	0.0167	0.0174	0.0188	0.0188
$K = 1.0$	0.0870	0.0134	0.0501	0.0633	0.0847	0.0869