MFE5130 – Financial Derivatives

First Term, 2017-18

Final Examination (Solution)

Question 1

The financial derivative can be replicated with a portfolio consisting of a long position in a zero-coupon bond with the face value of $80 and a short position in a European put option with a strike price of $80.

To find the value of this financial derivative, we must find the value of the put option:

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + 0.5\sigma^2\right)(0.5)}{\sigma\sqrt{0.5}} = \frac{\ln\left(\frac{90}{80}\right) + \left(0.06 - 0.03 + 0.5\times 0.3^2\right)(0.5)}{0.3\sqrt{0.5}}$$

$$= 0.73$$

$$d_2 = d_1 - \sigma\sqrt{0.5} = 0.73 - 0.3\sqrt{0.5} = 0.52.$$

We have:

$$N(-d_1) = N(-0.73) = 0.2327$$

$$N(-d_2) = N(-0.52) = 0.3015.$$

The value of the European put option is:

$$P(90, 80, 0.5) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

$$= 80\times e^{-0.06\times 0.5}\times 0.3015 - 90\times e^{-0.03\times 0.5}\times 0.2327$$

$$= 2.7759.$$

The current value of the financial derivative is the price of the zero-coupon bond **minus** the value of the European put option:

$$V = 80e^{-0.06\times 0.5} - 2.7759 = 74.8597.$$

The delta of the put option is:

$$\Delta_{Put} = -e^{-\delta T}N(-d_1) = -e^{-0.03\times 0.5}\times 0.2327 = -0.2292.$$

The delta of the financial derivative is the delta of the zero-coupon bond **minus** the delta of the put option:

$$\Delta = 0 - (-0.2292) = 0.2292.$$

The elasticity of the financial derivative is:

$$\Omega = \frac{S\Delta}{V} = \frac{90(0.2292)}{74.8597} = 0.2756.$$

Question 2

(a)

Under the binomial (forward) tree, the values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.04-0.03)(1/3)+0.3\sqrt{1/3}} = 1.1931.$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.04-0.03)(1/3)-0.3\sqrt{1/3}} = 0.8438.$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.04-0..03)(1/3)} - 0.8438}{1.1931 - 0.8438} = 0.4567.$$

	node u^3	node u^2d	node ud^2	node d^3
S	152.8527	108.1025	76.4537	54.0706
Option value	0	0	9.5463	31.9294

	node uu	node $ud = du$	node dd
S	128.1139	90.6064	64.0799
Continuation value	0	5.1178	21.4195
Value of early exercise	0	0	21.9201
Option value	0	5.1178	**21.9201**

	t = 0	node u	node d
S	90	107.3790	75.9420
Continuation value	8.7729	2.7437	14.0578
Value of early exercise	0	0	10.0580
Option value	8.7729	2.7437	14.0578

So, the price of the American option at time 0 is \$8.7729.

(b)

The value of the delta of the American option at time 0 is

$$\Delta = e^{-\delta h}\frac{V_u - V_d}{S(u-d)} = e^{-0.03(1/3)}\frac{2.7437 - 14.0578}{90(1.1931 - 0.8438)} = -0.3563.$$

Question 3

The market-maker buys 200 of $90-strike 91-day call, sells 150 of $95-strike 91-day call and sells 90 of $100-strike 91-day call.

The gamma of the position to be hedged is:

$$200 \times 0.035 - 150 \times 0.0381 - 90 \times 0.0297 = -1.388.$$

We can now solve the quantity, Y, of the $95-strike 180-day call option that must be purchased to bring the hedged portfolio's gamma to zero:

$$\begin{aligned} -1.388 + 0.0208Y &= 0 \\ Y &= 66.7308. \end{aligned}$$

The delta of the position becomes:

$$200 \times 0.7015 - 150 \times 0.5646 - 90 \times 0.4388 + 66.7308 \times 0.5958 = 55.8762.$$

The quantity of the underlying stock that must be purchased, X, is the opposite of the delta of the position being hedged:

$$X = -55.8762.$$

Question 4

Let $V(0)$ be the price of the financial derivative at time 0.

Since the dynamics of $X(t)$ and $Y(t)$ are under the risk-neutral measure and both stocks do not pay dividends, the risk-free interest rate $r = 4\%$.

Let $M(t)$ be the value of the money market account at time t.

$$M(t) = e^{rt} = e^{0.04t}.$$

By the risk-neutral valuation theorem,

$$\begin{aligned} V(0) &= e^{-3(4\%)} E_0^Q\left[\max\left(X(3)Y^2(3) - 235X(3), 0\right)\right] \\ &= M(0) E_0^Q\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{M(3)}\right] \qquad (1) \end{aligned}.$$

We choose $X(t)$ as the numeraire, the corresponding Radon-Nikodym derivative of Q_X with respect to Q is

$$\begin{aligned} \frac{dQ_X}{dQ} &= L(t) \\ &= \frac{X(t)}{X(0)} \Big/ \frac{M(t)}{M(0)} \\ &= \frac{X(0)\exp\left[\left(0.04 - \frac{1}{2}(0.28)^2 - \frac{1}{2}(0.12)^2\right)t + 0.28Z_1(t) + 0.12Z_2(t)\right]}{X(0)\exp(0.04t)} \\ &= \exp\left[-\frac{1}{2}(0.28)^2 t - \frac{1}{2}(0.12)^2 t + 0.28Z_1(t) + 0.12Z_2(t)\right] \end{aligned}$$

From the Girsanov theorem, we have

$$\tilde{Z}_1(t) = Z_1(t) - 0.28t,$$
$$\tilde{Z}_2(t) = Z_2(t) - 0.12t.$$

are independent standard Brownian motions under Q_X.

So, the dynamic of $Y(t)$ under Q_X is

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= 0.04dt + 0.31\left(d\tilde{Z}_2(t) + 0.12dt\right) \\ &= 0.0772dt + 0.31d\tilde{Z}_2(t). \end{aligned}$$

Hence,

By Ito's lemma,

$$dY^2(t) = 2Y(t)\,dY(t) + \left(dY(t)\right)^2$$
$$= 2Y(t)Y(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right) + Y^2(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right)^2$$
$$= 2Y^2(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right) + Y^2(t)(0.31)^2\,dt$$
$$= Y^2(t)\left(0.2505dt + 0.62d\tilde{Z}_2(t)\right).$$

So,

$$Y^2(t) = Y^2(0)\exp\left[\left(0.2505 - 0.5(0.62)^2\right)t + 0.62\tilde{Z}_2(t)\right]$$
$$= 225\exp\left[0.0583t + 0.62\tilde{Z}_2(t)\right].$$

By using the theorem on change of numeraire, we have

$$V(0) = M(0)E_0^Q\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{M(3)}\right]$$
$$= X(0)E_0^{Q_X}\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{X(3)}\right]$$
$$= 12E_0^{Q_X}\left[\max\left(Y^2(3) - 235, 0\right)\right]$$
$$= 12\left[225e^{(0.2505)(3)}N\left(\tilde{d}_1\right) - 235N\left(\tilde{d}_2\right)\right].$$

where

$$\tilde{d}_1 = \frac{\ln\left(\frac{225}{235}\right) + \left(0.2505 + 0.5\times 0.62^2\right)3}{0.62\sqrt{3}} = 1.2 \quad \text{and} \quad \tilde{d}_2 = \tilde{d}_1 - 0.62\sqrt{3} = 0.13.$$

So,

$$V(0) = 12\left[225e^{0.2505(3)}N\left(\tilde{d}_1\right) - 235N\left(\tilde{d}_1\right)\right]$$
$$= 12\left[225e^{0.7515}N(1.2) - 235N(0.13)\right]$$
$$= 3509.7986.$$

Question 5

From the given stochastic differential equations (SDEs), we have

$$S_1(t) = 4\exp\left(\left(0.06 - 0.5\times 0.42^2\right)t + 0.42Z_1(t)\right) = 4\exp\left(-0.0282t + 0.42Z_1(t)\right) \quad \text{and}$$
$$S_2(t) = 2\exp\left(\left(0.03 - 0.5\times 0.19^2\right)t + 0.19Z_2(t)\right) = 2\exp\left(0.012t + 0.19Z_2(t)\right).$$

So,

$$\begin{aligned}\frac{S_1(3)}{S_2(3)} &= 2\exp\left(\left(-0.0282 - 0.012\right)(3) + 0.42Z_1(3) - 0.19Z_2(3)\right)\\ &= 2\exp\left(-0.1206 + 0.42Z_1(3) - 0.19Z_2(3)\right).\end{aligned}$$

We have

$$E\left[0.42Z_1(3) - 0.19Z_2(3)\right] = 0;$$

$$\begin{aligned}&Var\left[0.42Z_1(3) - 0.19Z_2(3)\right]\\ &= (0.42)^2 Var\left[Z_1(3)\right] + (-0.19)^2 Var\left[Z_2(3)\right] + 2(0.42)(-0.19)Cov\left(Z_1(3), Z_2(3)\right)\\ &= (0.42)^2(3) + (-0.19)^2(3) + 2(0.42)(-0.19)(0.2)\sqrt{3}\sqrt{3}\\ &= 0.5417\end{aligned}$$

So, $0.42Z_1(3) - 0.19Z_2(3) = \sqrt{0.5417}Z$, where $Z \sim N(0,1)$.

$$\begin{aligned}\Pr\left(S_1(3) > 3S_2(3)\right) &= \Pr\left(\frac{S_1(3)}{S_2(3)} > 3\right)\\ &= \Pr\left(2\exp\left(-0.1206 + 0.42Z_1(3) - 0.19Z_2(3)\right) > 3\right)\\ &= \Pr\left(2\exp\left(-0.1206 + \sqrt{0.5417}Z\right) > 3\right)\\ &= \Pr\left(Z > \frac{\ln\left(\frac{3}{2}\right) + 0.1206}{\sqrt{0.5417}}\right)\\ &= \Pr\left(Z > 0.71\right)\\ &= 0.2389.\end{aligned}$$