Tutorial - Class Activity (Solution)

21 November, 2018

Problem 1

For a two-period binomial model for stock prices, you are given

(i) The length of each period is 1 year.
(ii) The current price of a non-dividend paying stock is $40.
(iii) $u = 1.05$ and $d = 0.9$.
(iv) The continuously compounded risk-free interest rate is 3%.

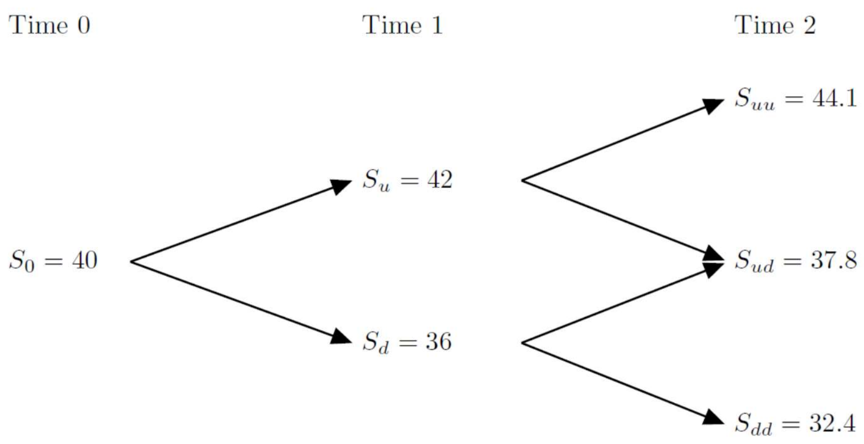
Consider Derivative X, which gives its holder the right, but not the obligation, to buy a $38-strike European put option at the end of the first year for $0.5. This put option is written on the stock and will mature at the end of the second year.

(a) Calculate the current price of Derivative X.
(b) Using the result of part (a), calculate the current price of Derivative Y, which is identical to Derivative X, except that it gives its holder the right to sell the same put option for $0.5 at the end of the first year.

Solution

(a)

The evolution of the stock price is given as follows:



The risk-neutral probability of an up move is

$$p^* = \frac{e^{0.03} - 0.9}{1.05 - 0.9} = 0.8697.$$

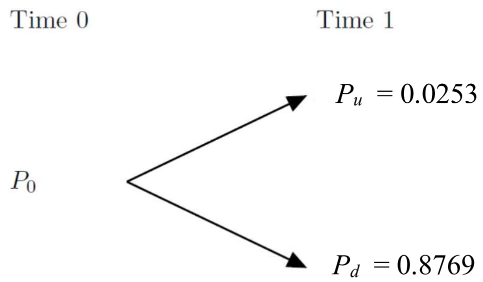
The possible time-2 payoffs of the 2-year put option are:

$$P_{uu} = \max(38 - 44.1, 0) = 0, \quad P_{ud} = \max(38 - 37.8, 0) = 0.2, \quad P_{dd} = \max(38 - 32.4, 0) = 5.6.$$

By risk-neutral pricing, the possible time-1 values of this put option are

$$P_u = e^{-0.03}(1 - p^*)(0.2) = 0.0253,$$
$$P_d = e^{-0.03}\left[p^*(0.2) + (1 - p^*)(5.6)\right] = 0.8769.$$

Here is the evolution of the put price:



As a call on the above put, Derivative X pays off only at the *d* note with a non-zero payoff of

$$V_d^X = \max(P_d - 0.5, 0) = 0.3769.$$

By risk-neutral pricing again, the time-0 price of Derivative X is

$$V_0^X = e^{-0.03}(1 - p^*)V_d^X = 0.0477.$$

(b)

Derivative Y is a 1-year European put option on the put option which matures 2 years from now. Derivative X and Y therefore form a call-put pair. To apply put-call parity, we need the time-0 price of the underlying asset, which is the 2-year put option:

$$P_0 = e^{-0.03}\left[0.0253p^* + 0.8769(1 - p^*)\right] = 0.1322.$$

By put-call parity for non-dividend paying underlying assets (the 2-year put in this case), we have

$$V_0^X - V_0^Y = P_0 - 0.5e^{-r}$$
$$0.0477 - V_0^Y = 0.1322 - 0.5e^{-0.03}$$
$$V_0^Y = 0.4007.$$

Problem 2

Assume the Black-Scholes framework. Consider a one-year at-the-money European call option on a stock.

You are given:

(i) The ratio of the call option price to the stock price is less than 10%.

(ii) The delta of the call option is 0.6.

(iii) The continuously compounded dividend yield of the stock is 2%.

(iv) The continuously compounded risk-free interest rate is 4.94%.

Determine the stock's volatility.

Solution

The value of delta can be used to determine d_1:

$$\begin{aligned}\Delta_{Call} &= e^{-\delta T}N(d_1)\\ 0.6 &= e^{-0.02\times 1}N(d_1)\\ 0.6121 &= N(d_1)\\ d_1 &= 0.28.\end{aligned}$$

The formula for d_1 is used to find a quadratic equation in terms of σ.

$$d_1 = \frac{\ln(S/K)+(r-\delta+0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$0.28 = \frac{\ln(S/S)+(0.0494-0.02+0.5\sigma^2)\times 1}{\sigma\sqrt{1}}$$

$$0.28 = \frac{(0.0294+0.5\sigma^2)}{\sigma}$$

$$0.28\sigma = 0.0294+0.5\sigma^2$$

$$0.5\sigma^2 - 0.28\sigma + 0.0294 = 0$$

$$\sigma^2 - 0.56\sigma + 0.0558 = 0.$$

The two solutions to the quadratic equation are found below:

$$\sigma = \frac{0.56 \pm \sqrt{(-0.56)^2 - 4(1)(0.0588)}}{2}$$

$$\sigma = 0.14 \quad \text{or} \quad 0.42.$$

The value of $N(d_2)$ depends on the value of σ:

$$\sigma = 0.14 \Rightarrow d_2 = d_1 - \sigma\sqrt{T} = 0.28 - 0.14 = 0.14 \Rightarrow N(d_2) = 0.5557.$$

$$\sigma = 0.42 \Rightarrow d_2 = d_1 - \sigma\sqrt{T} = 0.28 - 0.42 = -0.14 \Rightarrow N(d_2) = 1 - 0.5557 = 0.4443.$$

For an at-the-money option, $K = S$. We are given that the call price is less than 10% of the stock price.

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$\frac{C}{S} = e^{-0.02}N(d_1) - e^{-0.0494}N(d_2)$$

$$e^{-0.02}N(d_1) - e^{-0.0494}N(d_2) < 0.1$$

$$0.6 - e^{-0.0494}N(d_2) < 0.1$$

$$0.5 < e^{-0.0494}N(d_2)$$

$$0.52532 < N(d_2).$$

For the inequality above to be satisfied, it must be the case that:

$\sigma = 0.14$.

Problem 3

Assume the Black-Scholes framework. Consider a six-month financial derivative on a stock with the following payoff:

$$\text{Payoff} = \begin{cases} 85 & \text{if } 0 \le S(0.5) \le 85, \\ 170 - S(0.5) & \text{if } S(0.5) > 85. \end{cases}$$

where $S(0.5)$ is the stock price at $t = 0.5$.

You are given that

(i) The time-0 stock price is \$90.

(ii) The stock's volatility is 35%.

(iii) The stock's dividend yield is 5%.

(iv) The continuously compounded risk-free interest rate is 8%.

Calculate the time-0 contingent claim elasticity.

Solution

The financial derivative can be replicated with a portfolio consisting of a zero-coupon bond with the face value of \$85 and a short position in a European call option with a strike price of \$85.

To find the value of this financial derivative, we must find the value of the call option:

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + 0.5\sigma^2\right)(0.5)}{\sigma\sqrt{0.5}} = \frac{\ln\left(\frac{90}{85}\right) + \left(0.08 - 0.05 + 0.5 \times 0.35^2\right)(0.5)}{0.35\sqrt{0.5}}$$

$$= 0.42$$

$$d_2 = d_1 - \sigma\sqrt{0.5} = 0.42 - 0.35\sqrt{0.5} = 0.17.$$

We have:

$$N(d_1) = N(0.42) = 0.6628,$$

$$N(d_2) = N(0.17) = 0.5675.$$

The value of the European call option is:

$$C(90, 85, 0.5) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

$$= 90 \times e^{-0.05 \times 0.5} \times 0.6628 - 85 \times e^{-0.08 \times 0.5} \times 0.5675$$

$$= 11.8331.$$

The current value of the contingent claim is the price of the zero-coupon bond **minus** the value of the European call option:

$$V = 85e^{-0.08\times 0.5} - 11.8331 = 69.834.$$

The delta of the call option is:

$$\Delta_{Call} = e^{-\delta T} N\left(d_1\right) = e^{-0.05\times 0.5} \times 0.6628 = 0.6464.$$

The delta of the financial derivative is the delta of the zero-coupon bond **minus** the delta of the call option:

$$\Delta = 0 - 0.6464 = -0.6464.$$

The elasticity of the financial derivative is:

$$\Omega = \frac{S\Delta}{V} = \frac{90\left(-0.6464\right)}{69.834} = -0.8331.$$