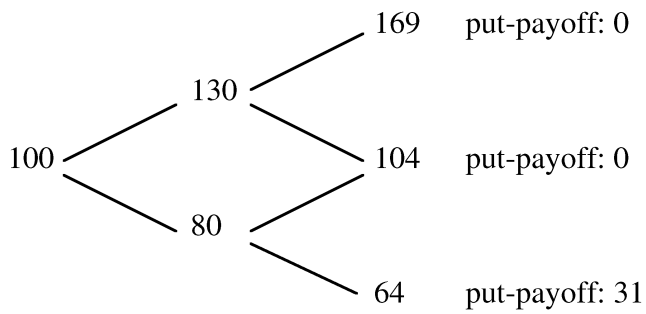
MFE5130 – Financial Derivatives
First Term, 2019 – 20

Assignment 3 (Solution)

Question 10.8

The stock prices evolve according to the following picture:



Since we have two binomial steps, and a time to expiration of one year, h is equal to 0.5. Therefore, we can calculate with the usual formulas for the respective nodes:

$t = 0, S = 100$	$t = 0.5, S = 80$	$t = 0.5, S = 130$
$\Delta = -0.3088$	$\Delta = -0.775$	$\Delta = 0$
$B = 38.569$	$B = 77.4396$	$B = 0$
$price = 7.6897$	$price = 15.4396$	$price = 0$

Question 10.18

$$u = e^{(1\%-5\%)(0.3333)+10\%\times\sqrt{0.3333}} = 1.0454, \quad d = e^{(1\%-5\%)(0.3333)-10\%\times\sqrt{0.3333}} = 0.9314.$$

a) We now have to inverse the interest rates: We have a yen-denominated option, therefore, the dollar interest rate becomes the foreign interest rate. With these changes, and equipped with an exchange rate of Y120/$ and a strike of Y120, we can proceed with our standard binomial procedure.

	node u^3	node u^2d	node ud^2	node d^3
x	137.0981	122.1472	108.8267	96.9589
Call premium	17.0981	2.1472	0	0

	node uu	node $ud = du$	node dd
x	131.1493	116.8423	104.1003
delta	0.9835	0.1585	0

B	−119.6007	−17.4839	0
call premium	9.3756	1.0391	0

Using these call premia at all previous nodes yields:

	$t = 0$	node d	node u
x	120	111.7678	125.4483
delta	0.3283	0.0802	0.5733
B	−36.6885	−8.4614	−66.8456
call premium.	2.7116	0.5029	5.0702

The price of the European yen-denominated call option is $2.7116.

For the American call option, the binomial approach yields:

	node uu	node $ud = du$	node dd
x	131.1493	116.8423	104.1003
delta	0.9835	0.1585	0
B	−119.6007	−17.4839	0
call premium	9.3756	1.0391	0
value of early exercise	11.1439	0	0

Using the maximum of the call premium and the value of early exercise at the previous nodes and at the initial node yields:

	$t = 0$	node d	node u
x	120	111.7678	125.4483
delta	0.3899	0.0802	0.6949
B	−43.6568	−8.4614	−81.2441
call premium	3.1257	0.5029	5.9259
value of early exercise	0	0	5.4483

Question 10.20

a) We have to use the formulas of the textbook to calculate the tree for the futures price and the prices of the options. Remember that while it is possible to calculate a delta, the option price is just the value of B because it does not cost anything to enter into a futures contract.

$$u = e^{30\% \times \sqrt{0.3333}} = 1.1891, \quad d = e^{-30\% \times \sqrt{0.3333}} = 0.841, \quad p^* = \frac{1-d}{u-d} = 0.4568.$$

Consider a European put option with strike price of 1,000.

	node u^3	node u^2d	node ud^2	node d^3
Futures price (F)	1681.338	1189.139	841.0278	594.8233
Put premium = max(1,000 – F, 0)	0	0	158.9722	405.1767

	node uu	node $ud = du$	node dd
Futures price (F)	1413.959	1000.033	707.281
Put premium	0	84.9264	287.8724

	$t = 0$	node u	node d
Futures price	1000	1189.1	841
Put premium	122.9212	45.3695	191.9408

The current price of the European put option is $122.9212.

Because of the put-call parity, we have that the prices of the European call and put must be equal.

b) Consider an American put option with strike price of 1,000.

	node u^3	node u^2d	node ud^2	node d^3
Futures price (F)	1681.338	1189.139	841.0278	594.8233
Put premium = max(1,000 – F, 0)	0	0	158.9722	405.1767

	node uu	node $ud = du$	node dd
Futures price (F)	1413.959	1000.033	707.281
Continuation value (CV)	0	84.9264	287.8724
Value of Early Exercise (EE)	0	0	292.719
Put premium = max(CV, EE)	0	84.9264	**292.719**

	$t = 0$	node u	node d
Futures price	1000	1189.1	841

Continuation value (CV)	124.3044	45.3695	194.53
Value of Early Exercise (EE)	0	0	159
Put premium = max (CV, EE)	124.3044	45.3695	194.53

Consider an American call option with strike price of 1,000.

	node u^3	node u^2d	node ud^2	node d^3
Futures price (*F*)	1681.338	1189.139	841.0278	594.8233
Call premium = max(*F* – 1,000, 0)	681.3384	189.1394	0	0

	node *uu*	node *ud* = *du*	node *dd*
Futures price (*F*)	1413.959	1000.033	707.281
Continuation value (CV)	407.1335	84.9708	0
Value of Early Exercise (EE)	413.9588	0.0331	0
Call premium = max(CV, EE)	**413.9588**	84.9708	0

	$t = 0$	node *u*	node *d*
Futures price	1000	1189.1	841
Continuation value (CV)	124.3332	231.3641	38.1731
Value of Early Exercise (EE)	0	189.1	0
Call premium = max (CV, EE)	124.3332	231.3641	38.1731

c) We have the following time 0 replicating portfolios:

$$\Delta = \frac{V_u - V_d}{F(u-d)}, \quad B = e^{-rh}\left(V_u \frac{1-d}{u-d} + V_d \frac{u-1}{u-d}\right)$$

For the European put option:

Sell 0.4211 futures contracts.

Lend 122.9212

For the European call option:

Buy 0.5462 futures contracts.

Lend 122.9212

Question 12.7

a) $C(100, 95, 0.3, 0.08, 0.75, 0.03) = \$14.3863$

b) $S(\text{new}) = 100 \times \exp(-0.03 \times 0.75) = \$97.7751$

$K(\text{new}) = 95 \times \exp(-0.08 \times 0.75) = \$89.4676$

$C(97.7751, 89.4676, 0.3, 0, 0.75, 0) = \$14.3863$

This is a direct application of equation (12.5) of the main text. As the dividend yield enters the formula only to discount the stock price, we can take care of it by adapting the stock price before we plug it into the Black-Scholes formula. Similarly, the interest rate is only used to discount the strike price, which we did when we calculated K(new). Therefore, we can calculate the Black- Scholes call price by using S(new) and K(new) and by setting the interest rate and the dividend yield to zero.

Additional Problem 1

The value of h is 1/12 since the intervals are monthly periods. The values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)+0.2\sqrt{1/12}} = 1.0639,$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)-0.2\sqrt{1/12}} = 0.9478.$$

The risk-neutral probability of an upward movement is :

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.02)(1/12)} - 0.9478}{1.0639 - 0.9478} = 0.4856.$$

Let k be the largest integer such that $K \geq u^k d^{n-k} S_0$, that is

$$k \leq \frac{\ln\left(\frac{K}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)} = \frac{\ln\left(\frac{102}{130(0.9478)^{10}}\right)}{\ln\left(\frac{1.0639}{0.9478}\right)} = 2.5404.$$

So, $k = 2$.

The put option value is then given by

$$P=102e^{-(0.07)\left(\frac{10}{12}\right)}\sum_{j=0}^{2}C_j^{10}\left(p^*\right)^j\left(1-p^*\right)^{10-j}-130\sum_{j=0}^{2}C_j^{10}\left(p^*\right)^j\left(1-p^*\right)^{10-j}\frac{u^j d^{10-j}}{e^{(0.07)\left(\frac{10}{12}\right)}}$$

$$=102e^{-(0.07)\left(\frac{10}{12}\right)}\left(C_0^{10}(0.4856)^0(0.5144)^{10}+C_1^{10}(0.4856)^1(0.5144)^9+C_2^{10}(0.4856)^2(0.5144)^8\right)-$$

$$\frac{130}{e^{(0.07)\left(\frac{10}{12}\right)}}[C_0^{10}(0.4856)^0(0.5144)^{10}(1.0639)^0(0.9478)^{10}+C_1^{10}(0.4856)^1(0.5144)^9(1.0639)^1(0.9478)^9$$

$$+C_2^{10}(0.4856)^2(0.5144)^8(1.0639)^2(0.9478)^8]$$

$$=102e^{-(0.07)\left(\frac{10}{12}\right)}(0.0656)-\frac{130}{e^{(0.07)\left(\frac{10}{12}\right)}}(0.0471)$$

$$=0.536.$$

Additional Problem 2

The prepaid forward price of the stock is:

$$F_{0,T}^P(S)=S_0-PV_{0,T}(Div)=77-7e^{-0.1(0.25)}=70.1728.$$

The volatility of the stock is:

$$\sigma=0.2743.$$

The values of d_1 and d_2 are

$$d_1=\frac{\ln\left(\frac{F_{0,T}^P(S)}{F_{0,T}^P(K)}\right)+0.5\sigma^2T}{\sigma\sqrt{T}}=\frac{\ln\left(\frac{70.1728}{73e^{-0.1(0.5)}}\right)+0.5(0.2743)^2(0.5)}{0.2743\sqrt{0.5}}=0.15,$$

$$d_2=d_1-\sigma\sqrt{T}=0.15-0.2743\sqrt{0.5}=-0.04.$$

We have

$$N(-d_1)=N(-0.15)=0.4404,$$

$$N(-d_2)=N(0.04)=0.5160.$$

The value of the put option is

$$P=Ke^{-rT}N(-d_2)-\left[S_0-PV_{0,T}(Div)\right]N(-d_1)$$

$$=73e^{-0.1(0.5)}(0.5160)-(70.1728)(0.4404)$$

$$=4.9268.$$

Additional Problem 3

At time 0, the values of d_1 and d_2 are:

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{30}{30}\right) + 0.5\sigma^2 T}{\sigma\sqrt{T}} = 0.5\sigma\sqrt{T},$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.5\sigma\sqrt{T} - \sigma\sqrt{T} = -0.5\sigma\sqrt{T}.$$

We can find $N(-d_1)$ in terms of $N(-d_2)$:

$$N(-d_1) = N\left(-0.5\sigma\sqrt{T}\right),$$

$$N(-d_2) = N\left(0.5\sigma\sqrt{T}\right),$$

$$N(-d_1) = 1 - N(-d_2).$$

We can substitute this value of $N(-d_1)$ into the time 0 formula for the price in order to find d_2:

$$P = Ke^{-rT}N(-d_2) - Fe^{-rT}N(-d_1)$$

$$3.4 = 30e^{-(0.08)(0.75)}N(-d_2) - 30e^{-(0.08)(0.75)}N(-d_1)$$

$$3.4 = 30e^{-(0.08)(0.75)}\left[N(-d_2) - \left(1 - N(-d_2)\right)\right]$$

$$\frac{3.4}{30}e^{0.06} = 2N(-d_2) - 1$$

$$N(-d_2) = 0.5602$$

$$d_2 = -0.15.$$

We can now find σ:

$$d_2 = -0.5\sigma\sqrt{T}$$

$$-0.15 = -0.5\sigma\sqrt{0.75}$$

$$\sigma = 0.3464.$$

After 3 months, the new values of d_1 and d_2 are:

$$d_1 = \frac{\ln\left(\frac{27}{30}\right) + 0.5(0.3464)^2(0.5)}{0.3464\sqrt{0.5}} = -0.31,$$

$$d_2 = -0.31 - 0.3464\sqrt{0.5} = -0.55.$$

We can look up the values of $N(-d_1)$ and $N(-d_2)$ from the normal distribution table:

$$N(-d_1) = N(0.31) = 0.6217,$$

$$N(-d_2) = N(0.55) = 0.7088.$$

After 3 months, the value of the put option is:

$$P = 30e^{-(0.08)(0.5)} \times 0.7088 - 27e^{-(0.08)(0.5)} \times 0.6217$$

$$= 4.3025.$$

Additional Problem 4

Since the Sharpe ratio of the call option is equal to the Sharpe ratio of the stock, we can find σ_{Stock}:

$$\frac{\alpha - r}{\sigma_{Stock}} = 0.24$$

$$\frac{0.15 - 0.09}{\sigma_{Stock}} = 0.24$$

$$\sigma_{Stock} = 0.25$$

The values of d_1 and d_2 are:

$$d_1 = \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(105/100) + \left(0.09 - 0.07 + 0.5(0.25)^2\right)1}{0.25\sqrt{1}}$$

$$= 0.40$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.40 - 0.25\sqrt{1} = 0.15.$$

The value of the call option is

$$C(S = 105, K = 100, T = 1) = 105e^{-0.07}N(0.40) - 100e^{-0.09}N(0.15) = 13.021.$$

The delta of the call option is:

$$\Delta_{Call} = e^{-0.07}N(0.40) = 0.6111.$$

So,

$$\sigma_{Call} = \sigma_{Stock} \times |\Omega_{Call}| = 0.25 \times \frac{105 \times 0.6111}{13.021} = 1.232.$$

Additional Problem 5

We can use put-call parity to determine the value of the put option:

$$C + Ke^{-rT} = S_0e^{-\delta T} + P$$

$$5.1 + 60e^{-0.06(0.5)} = 60e^{-0.06(0.5)} + P$$

$$P = 5.1.$$

The value of the portfolio is

$$V_{Port} = 2C + P = 2(5.1) + 5.1 = 15.3.$$

The delta of the put option is:

$$\Delta_{Put} = \Delta_{Call} - e^{-\delta T} = 0.5277 - e^{-0.06(0.5)} = -0.4427.$$

The elasticity of call and put options are

$$\Omega_{call} = \frac{S\Delta_{call}}{C} = \frac{60\left(0.5277\right)}{5.1} = 6.2082.$$

$$\Omega_{put} = \frac{S\Delta_{put}}{P} = \frac{60\left(-4427\right)}{5.1} = -5.2082.$$

The elasticity of the portfolio is:

$$\begin{aligned}\Omega_{Port} &= \frac{2C}{V_{Port}}\Omega_{call} + \frac{P}{V_{Port}}\Omega_{put} \\ &= \frac{2\left(5.1\right)}{15.3}6.2082 + \frac{5.1}{15.3}\left(-5.2082\right) \\ &= 2.4027.\end{aligned}$$

The expected return of the portfolio is:

$$\gamma_{Port} = \left(\alpha - r\right)\Omega_{Port} + r = \left(0.22 - 0.06\right)\left(2.4027\right) + 0.06 = 0.444.$$