MFE5130 – Financial Derivatives
First Term, 2019 – 20

Assignment 4 (Solution)

Additional Problem 1

The gamma of the position to be hedged is:

$$-100(0.0521) = -5.21.$$

We can solve for the quantity, Q, of the \$55-strike call option that must be purchased to bring the hedged portfolio's gamma to zero:

$$-5.21 + 0.0441Q = 0$$

$$Q = 118.1406.$$

The delta of the position becomes:

$$-100(0.5824) + 118.1406(0.3769) = -13.7128.$$

The quantity of underlying stock that must be purchased, Q_S, is the opposite of the delta of the position being hedged:

$$Q_S = 13.7128.$$

The cost of purchasing the \$55-strike options and the stock is offset by the value of the \$50-strike options that are sold. The resulting cost of establishing the delta-hedged position is:

$$13.7128(50) + 118.1406(2.05) - 100(3.48) = 579.8288.$$

The cost of establishing the position earns the risk-free rate of interest, so we treat it as borrowing.

After 1 day, the value of the \$50-strike options has changed by:

$$-100(4.06 - 3.48) = -58.00.$$

After 1 day, the value of the \$55-strike options has changed by:

$$118.1406(2.43 - 2.05) = 44.8934.$$

After 1 day, the value of the shares of stock has changed by:

$$13.7128(51 - 50) = 13.7128.$$

After 1 day, the value of the funds that were borrowed at the risk-free rate has changed by:

$$-579.8288(e^{0.08/365} - 1) = -0.1271.$$

The sum of these changes is the overnight profit.

Component	Change
Gain on \$50-strike call	–58
Gain on \$55-strike call	44.8934
Gain on stock	13.7128
Interest	–0.1271
Overnight profit	0.4791

The overnight profit is \$0.4791.

Additional Problem 2

The delta-gamma approximation is

$$V\left(S\left(0\right)+\varepsilon,h\right)\approx V\left(S(0),0\right)+\varepsilon\Delta\left(S\left(0\right),0\right)+\frac{\varepsilon^2}{2}\Gamma\left(S\left(0\right),0\right).$$

The approximation above can be used to create a quadratic equation:

$$6.08=5.92-0.323\varepsilon+\frac{\varepsilon^2}{2}\left(0.015\right)$$

$$0=0.0075\varepsilon^2-0.323\varepsilon-0.16$$

$$\varepsilon=\frac{0.323\pm\sqrt{\left(-0.323\right)^2-4\left(0.0075\right)\left(-0.16\right)}}{2\left(0.0075\right)}$$

$$\varepsilon=-0.4898 \quad \text{or} \quad \varepsilon=43.5565.$$

The definition of ε can be rearranged to obtain an expression for $S(0)$:

$$\varepsilon=S\left(h\right)-S\left(0\right)\Rightarrow S\left(0\right)=S\left(h\right)+\varepsilon.$$

Since we have 2 possible values for ε, we have 2 possible values for $S(0)$:

$$\varepsilon=-0.4898\Rightarrow S\left(0\right)=86-\left(-0.4898\right)=86.4898,$$

$$\varepsilon=43.5565\Rightarrow S\left(0\right)=86-\left(43.5565\right)=42.4435.$$

Since (i) in the question tell us that the original stock price is greater than 75. Therefore, the initial stock price is 86.4898.

Additional Problem 3

The probability that Pete's investment outperforms Jim's investment is equal to the probability that a risk-free investment outperforms the stock:

$$\Pr\left[S_T<K\right]=N\left(-\hat{d}_2\right) \quad \text{where } K=100e^{rT}.$$

The value of $\hat{d}_2$ is:

$$\hat{d}_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(\alpha - \delta - 0.5\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{100e^{rT}}\right) + \left(\alpha - 0 - 0.5\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\left(\alpha - r - 0.5\sigma^2\right)\sqrt{T}}{\sigma}.$$

The probability that Pete's investment outperforms Jim's investment is:

$$\Pr\left[S_T < S_0 e^{rT}\right] = N\left(-\hat{d}_2\right) = N\left(\frac{0.5\sigma^2 - \alpha + r}{\sigma}\sqrt{T}\right).$$

The question tells us that this probability is equal to 50%:

$$N\left(\frac{0.5\sigma^2 - \alpha + r}{\sigma}\sqrt{T}\right) = 50\%.$$

This implies that the value in parentheses above is 0:

$$\frac{0.5\sigma^2 - \alpha + r}{\sigma}\sqrt{T} = 0$$

$$0.5\sigma^2 - \alpha + r = 0$$

$$0.5\sigma^2 - 0.1 + 0.08 = 0$$

$$\sigma^2 = 0.04$$

$$\sigma = 0.2.$$

Additional Problem 4

The partial expectation of the stock price in 3 months conditional on $S_T < K$ is given by:

$$\int_0^K S_{0.25} g\left(S_{0.25}; S_0\right) dS_{0.25} = E\left[S_{0.25} \middle| S_{0.25} < K\right] \times \Pr\left(S_{0.25} < K\right) = 67.69(1 - 0.6)$$

$$= 27.076$$

Since the conditions are mutually exclusive and exhaustive, we can add the two partial expectations to obtain the expected value of the stock price in 3 months:

$$E\left[S_T\right] = E\left[S_T \middle| S_T < K\right] \times P\left(S_T < K\right) + E\left[S_T \middle| S_T \ge K\right] \times P\left(S_T \ge K\right)$$

$$= 27.076 + 52.01$$

$$= 79.086.$$

The expected return on the stock is 17%, so,

$$e^{0.17\times 0.25} = \frac{e^{0.02\times 0.25}}{S_0} E[S_T]$$
$$S_0 = e^{-0.0375} E[S_T]$$
$$S_0 = e^{-0.0375} \times 79.086$$
$$S_0 = 76.18.$$

Question 20.1

If $y = \ln(S)$ then $S = e^y$ and $dy = \left(\frac{\alpha(S,t)}{S} - \frac{\sigma(S,t)^2}{2S^2} \right) dt + \frac{\sigma(S,t)}{S} dZ_t$.

a) $dy = \left(\frac{\alpha}{e^y} - \frac{\sigma^2}{2e^{2y}} \right) dt + \frac{\sigma}{e^y} dZ_t$.

b) $dy = \left(\frac{\lambda a}{e^y} - \lambda - \frac{\sigma^2}{2e^{2y}} \right) dt + \frac{\sigma}{e^y} dZ_t$.

c) $dy = \left(\alpha - \frac{\sigma^2}{2} \right) dt + \sigma dZ_t$

Question 20.12

Strategy: Buy $\frac{1}{\sigma_1 S_1}$ units of stock 1 and buy $\frac{1}{\sigma_2 S_2}$ units of stock 2. Finance the difference by borrowing at the risk-free interest rate. The return to the portfolio of the two assets and the borrowed amount I is:

$$\frac{1}{\sigma_1 S_1} dS_1 + \frac{1}{\sigma_2 S_2} dS_2 + Irdt$$

$$= \frac{1}{\sigma_1 S_1}\left(\alpha_1 S_1 dt + \sigma_1 S_1 dZ\right) + \frac{1}{\sigma_2 S_2}\left(\alpha_2 S_2 dt - \sigma_2 S_2 dZ\right) + \left(-\frac{1}{\sigma_1 S_1} S_1 - \frac{1}{\sigma_2 S_2} S_2 \right) rdt$$

$$= \frac{\alpha_1}{\sigma_1} dt + dZ + \frac{\alpha_2}{\sigma_2} dt - dZ - \frac{r}{\sigma_1} dt - \frac{r}{\sigma_2} dt$$

$$= \left(\frac{\alpha_1 - r}{\sigma_1} + \frac{\alpha_2 - r}{\sigma_2} \right) dt$$

Because the portfolio is self-financing and riskless, its return should be zero to avoid arbitrage. We have indeed demonstrated that the Sharpe ratios sum to zero.

Additional Problem 5

Since we are given the risk-free rate in both the U.S. and Great Britain, we have:

$$r - r^* = 0.06 - 0.09 = -0.03.$$

The forward price follows geometric Brownian motion:

$$\begin{aligned} G(t) &= S(t)e^{-0.03(T-t)} \\ &= S(0)e^{\left[0.05-0.5(0.3)^2\right]t+0.3Z(t)}e^{-0.03(T-t)} \\ &= S(0)e^{-0.03T}e^{\left[0.05-0.5(0.3)^2\right]t+0.3Z(t)}e^{0.03t} \\ &= G(0)e^{\left[0.05-0.5(0.3)^2\right]t+0.3Z(t)}. \end{aligned}$$

Since the forward price follows geometric Brownian motion, we can use the following equivalency:

$$dG(t) = \left(\hat{\alpha} - \hat{\delta}\right)G(t)\,dt + \hat{\sigma}G(t)\,dZ(t) \quad \Leftrightarrow \quad G(t) = G(0)e^{\left(\hat{\alpha}-\hat{\delta}-0.5\hat{\sigma}^2\right)t+\sigma Z(t)}$$

We use the ^ symbol above to avoid confusion with the parameters for $S(t)$.

Since we have the rightmost expression above, we can express the left side as:

$$dG(t) = 0.08G(t)\,dt + 0.3G(t)\,dZ(t) = G(t)\left[0.08dt + 0.3dZ(t)\right].$$

Alternative Solution (Using Itô's lemma)

The expression for $G(t)$ is therefore:

$$G(t) = S(t)e^{-0.03(T-t)}.$$

The partial derivatives are:

$$\begin{aligned} G_S &= e^{-0.03(T-t)}, \\ G_{SS} &= 0, \\ G_t &= S(t)(0.03)e^{-0.03(T-t)}. \end{aligned}$$

From Itô's lemma, we have

$$
\begin{aligned}
dG(t) &= G_S dS(t) + \frac{1}{2} G_{SS} \left[dS(t) \right]^2 + G_t dt \\
&= e^{-0.03(T-t)} dS(t) + \frac{1}{2}(0)\left[dS(t) \right]^2 + S(t)(0.03)e^{-0.03(T-t)} dt \\
&= e^{-0.03(T-t)} dS(t) + 0.03G(t)\,dt \\
&= e^{-0.03(T-t)} S(t)\left[0.05dt + 0.3dZ(t) \right] + 0.03G(t)\,dt \\
&= G(t)\left[0.05dt + 0.3dZ(t) \right] + 0.03G(t)\,dt \\
&= G(t)\left[0.08dt + 0.3dZ(t) \right].
\end{aligned}
$$

<u>Additional Problem 6</u>

A lognormal stock price implies that changes in the stock price follow geometric Brownian motion, and vice versa:

$$X(t) = X(0)e^{\left(\alpha_X - \delta_X - 0.5\sigma_X^2\right)t + \sigma_X Z(t)} \quad \Leftrightarrow \quad dX(t) = \left(\alpha_X - \delta_X\right) X(t)\,dt + \sigma_X X(t)\,dZ(t).$$

We can use the equation provided in the question to find values for α_X and σ_X in the price process for Stock X:

$$X(t) = 15e^{0.14+0.07t}e^{0.42Z(t)} \quad \Rightarrow \quad X(t) = 15e^{0.14}e^{0.07t+0.42Z(t)}.$$

$$
\begin{aligned}
X(0) &= 15e^{0.14} \\
\sigma_X &= 0.42 \\
0.07 &= \alpha_X - \delta_X - 0.5\sigma_X^2 \\
0.07 &= \alpha_X - 0 - 0.5(0.42)^2 \\
\alpha_X &= 0.1582.
\end{aligned}
$$

The price process for Stock X is therefore:

$$dX(t) = 0.1582X(t)\,dt + 0.42X(t)\,dZ(t).$$

From the price process for Stock Q, we observe that:

$\alpha_Q = 0.08$ and $\sigma_Q = 0.13$.

Stock X and Stock Q are perfectly correlated since they are driven by the same stochastic process, so they must have the same Sharpe ratio:

$$
\begin{aligned}
\frac{0.1582 - r}{0.42} &= \frac{0.08 - r}{0.13} \\
r &= 0.0449.
\end{aligned}
$$