## Tutorial - Class Activity

## 8 October, 2019 (Solution)

Problem 1

**5.10** The S&R index spot price is 1100 and the continuously compounded risk-free rate is 5%. You observe a 9-month forward price of 1129.257.

**a.** What dividend yield is implied by this forward price?

**b.** Suppose you believe the dividend yield over the next 9 months will be only 0.5%. What arbitrage would you undertake?

**c.** Suppose you believe the dividend yield will be 3% over the next 9 months. What arbitrage would you undertake?

## Solution

a) We plug the continuously compounded interest rate, the forward price, the initial index level and the time to expiration in years into the valuation formula and solve for the dividend yield:

$$F_{0,T} = S_0 \times e^{(r-\delta)\times T}$$

$$\Leftrightarrow \quad \frac{F_{0,T}}{S_0} = e^{(r-\delta)\times T}$$

$$\Leftrightarrow \quad \ln\left(\frac{F_{0,T}}{S_0}\right) = (r-\delta)\times T$$

$$\Leftrightarrow \quad \delta = r - \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right)$$

$$\Rightarrow \quad \delta = 0.05 - \frac{1}{0.75}\ln\left(\frac{1129.257}{1100}\right) = 0.05 - 0.035 = 0.015$$

b) With a dividend yield of only 0.005, you expect that the forward price would be:

$$F_{0,T} = S_0 \times e^{(r-\delta)\times T} = 1{,}100 \times e^{(0.05-0.005)\times 0.75} = 1{,}100 \times 1.0343 = 1{,}137.759$$

Therefore, if we think the dividend yield is 0.005, we consider the observed forward price of \$1,129.257 to be too cheap. We will therefore buy the forward and create a synthetic short forward, capturing a certain amount of \$8.502. We engage in a **reverse cash and carry arbitrage**:

| Description | Today | In nine months |
|---|---|---|
| Long forward | 0 | $S_T - \$1{,}129.257$ |
| Sell short tailed position in index | $1,100 × 0.99626 = $1,095.88 | $-S_T$ |
| Lend $1,095.88 | −$1,095.88 | $1,137.759 |
| TOTAL | 0 | $8.502 |

c) With a dividend yield of 0.03, you expect that the forward price would be:

$$F_{0,T} = S_0 \times e^{(r-\delta)\times T} = 1{,}100 \times e^{(0.05-0.03)\times 0.75} = 1{,}100 \times 1.01511 = 1{,}116.62$$

Therefore, if we think the dividend yield is 0.03, we consider the observed forward price of $1,129.257 to be too expensive. We will therefore sell the forward and create a synthetic long forward, capturing a certain amount of $12.637. We engage in a **cash and carry arbitrage**:

| Description | Today | In nine months |
|---|---|---|
| Short forward | 0 | $\$1{,}129.257 - S_T$ |
| Buy tailed position in index | −$1,100 × .97775 = −$1,075.526 | $S_T$ |
| Borrow $1,075.526 | $1,075.526 | −$1,116.62 |
| TOTAL | 0 | $12.637 |

Problem 2

**5.13** Verify that going long a forward contract and lending the present value of the forward price creates a payoff of one share of stock when

**a.** The stock pays no dividends.

**b.** The stock pays discrete dividends.

**c.** The stock pays continuous dividends.

## Solution

a)

| Description | Today | At expiration of the contract |
|---|---|---|
| Long forward | 0 | $S_T - F_{0,T} = S_T - S_0 e^{rT}$ |
| Lend $S_0$ | $-S_0$ | $S_0 e^{rT}$ |
| Total | $-S_0$ | $S_T$ |

In the first row, we made use of the forward price equation if the stock does not pay dividends. We see that the total aggregate position is equivalent to the payoff of one stock.

b) In the case of discrete dividends, we have:

| Description | Today | At expiration of the contract |
|---|---|---|
| Long forward | 0 | $S_T - F_{0,T} = S_T - S_0 e^{rT} + \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$ |
| Lend $S_0 - \sum_{i=1}^{n} e^{-rt_i} D_{t_i}$ | $-S_0 + \sum_{i=1}^{n} e^{-rt_i} D_{t_i}$ | $+S_0 e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$ |
| Total | $-S_0 + \sum_{i=1}^{n} e^{-rt_i} D_{t_i}$ | $S_T$ |

In the first row, we made use of the forward price equation if the stock pays discrete dividends. We see that the total aggregate position is equivalent to the payoff of one stock at time *T*.

c) In the case of a continuous dividend, we have to tail the position initially. We, therefore, create a synthetic share at the time of expiration T of the forward contract.

| Description | Today | At expiration of the contract |
|---|---|---|
| Long forward | 0 | $S_T - F_{0,T} = S_T - S_0 e^{(r-\delta)T}$ |
| Lend $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $S_0 e^{(r-\delta)T}$ |
| Total | $-S_0 e^{-\delta T}$ | $S_T$ |

In the first row, we made use of the forward price equation if the stock pays a continuous dividend. We see that the total aggregate position is equivalent to the payoff of one stock at time *T*.

Problem 3

Suppose the current stock price is \$30.58 and the continuously compounded risk-free interest rate is 6%. The stock pays dividend of \$1.8 and \$2.5 at the end of 3 months and at the end of 6 months respectively. You observe an 8-month forward contract with forward price \$29.15. Is there an arbitrage opportunity on the forward contract? If so, describe the strategy to realize profit and find the accumulated arbitrage profits at the end of 8 months.

**Solution**

The theoretical forward price = $(30.58 - 1.8e^{-6\%\times 0.25} - 2.5e^{-6\%\times 0.5})\, e^{6\% \times (8/12)} = 27.4573$.

Now, we have the observed market forward price, \$29.15, is higher than the theoretical forward price. So, the strategy to realize the arbitrage profit is to short the forward contract and long the synthetic forward.

| Transactions | Cash Flows | | | |
|---|---|---|---|---|
| | t = 0 | t = 0.25 | t = 0.5 | t = 8/12 |
| Short one forward | 0 | 0 | 0 | 29.15 $-S_{8/12}$ |
| Buy one share of the stock | −30.58 | 0 | 0 | $S_{8/12}$ |
| Borrow \$30.58 at t = 0 | 30.58 | 0 | 0 | $-30.58e^{(0.06)(8/12)}$ $= -31.828$ |
| Receive the dividend (\$1.8) at t = 0.25 | 0 | 1.8 | 0 | 0 |
| Lend \$1.8 at t =0.25 | 0 | −1.8 | 0 | $1.8e^{(0.06)(5/12)} = 1.8456$ |
| Receive the dividend (\$2.5) at t = 0.5 | 0 | 0 | 2.5 | 0 |
| Lend \$2.5 at t =0.5 | 0 | 0 | −2.5 | $2.5e^{(0.06)(2/12)} = 2.5251$ |
| Total | 0 | 0 | 0 | 1.6927 |

This position requires no initial investment, has no stock price risk, and has a strictly positive payoff. We have exploited the mispricing with a pure arbitrage strategy. The accumulated arbitrage profits at the end of 8 months is \$1.6927.