Tutorial - Class Activity

28 November, 2018 (Solution)

Problem 1

The price of a stock is \$60. The stock does not pay dividends. The continuously compounded risk-free rate of return is 9%. A \$55-strike European put option has a price of \$1.60. The delta of the put option is –0.2377. A \$60-strike European put option has a price of \$3.50. The delta of the put option is –0.4146.

A market-maker enters into a put ratio spread where the market-maker buys 100 of the \$55-strike puts and sells 200 of the \$60-strike puts. The market-maker delta-hedges the position.

One day later, the price of the stock is \$62, the price of the \$55-strike European put option is \$1.17, and the price of the \$60-strike European put option is \$2.73. Calculate the overnight profit for the market-maker.

Solution

From the market-maker's perspective, the initial value of the position to be hedged is:

$$100(1.60) - 200(3.50) = -540.00.$$

The delta of the position to be hedged, from the perspective of the market-maker, is:

$$100(-0.2377) - 200(-0.4146) = 59.15.$$

To delta hedge the position, the market-maker shorts 59.15 shares of stock. The value of this position is:

$$-59.15(60) = -\$3{,}549.00.$$

The market-maker receives \$540.00 for entering into the put ratio spread. The market-maker also receives \$3,549 for the stock that it shorts. The sum of the proceeds is lent at the risk-free rate of return:

$$540 + 3{,}549 = \$4{,}089$$

The initial position, from the perspective of the market-maker, is:

Component	Value
Options	–540
Shares	–3,549
Risk-free asset	4,089
Net	0

After 1 day, the value of the option has changed by:

$$100(1.17 - 1.6) - 200(2.73 - 3.5) = 111.$$

After 1 day, the value of the shares of stock has changed by:

$$-59.15(62 - 60) = -118.3.$$

After 1 day, the value of the funds that were lent at the risk-free rate has changed by:

$$4{,}089(e^{0.09/365} - 1) = 1.0084.$$

The sum of these changes is the overnight profit:

Component	Change
Gain on options	111
Gain on stock	–118.3
Interest	1.0084
Overnight profit	–6.2916

The position experienced an overnight loss of $6.2916.

Problem 2

You are given the following information for a stock:

(i) The current time is 0, and the current price of the stock is $S(0) = 75$.

(ii) The stock price at time t is denoted by $S(t)$.

(iii) $\ln\left(\frac{S(t)}{S(0)}\right) \sim N(0.035t, 0.09t)$.

(iv) The stock pays continuously compounded dividends at an annual rate of 3%.

An investor purchases a share of stock today. The investor will reinvest the dividends paid by the stock. Determine the median value of the investor's position at the end of 4 years.

Solution

Let m be the median of the stock price at the end of 4 years.

$$\Pr\left(S(4) < m\right) = 0.5$$

$$\Pr\left(\ln\left(\frac{S(4)}{S(0)}\right) < \ln\left(\frac{m}{S(0)}\right)\right) = 0.5$$

$$\Pr\left(Z < \frac{\ln\left(\frac{m}{75}\right) - 0.035(4)}{\sqrt{(0.09)(4)}}\right) = 0.5, \quad \text{where } Z \sim N(0,1)$$

$$\frac{\ln\left(\frac{m}{S(0)}\right) - 0.035(4)}{\sqrt{(0.09)(4)}} = 0$$

$$\begin{aligned} m &= 75e^{(0.035)(4)} \\ &= 86.2705. \end{aligned}$$

Since the investor reinvests the dividends, the investor has the following quantity of shares at the end of 4 years:

$$e^{\delta t} = e^{0.03\times 4} = 1.1275.$$

The median value of the investor's position at the end of 4 years is the quantity of shares owned times the median stock price:

$$1.1275 \times 86.2705 = 97.27.$$

Problem 3

Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is 80.
(ii) The stock's volatility is 25%.
(iii) The stock has nonzero dividend yield.
(iv) The continuously compounded risk-free interest rate is 6%.
(v) $E[S(1)] = 84.2069$.

Calculate $E[S(1)|S(1) < 80]$.

Solution

We are given from (v) that

$$E\left[S(1)\right] = S(0)e^{(\alpha-\delta)T} = 80e^{(\alpha-\delta)} = 84.2069, \quad \text{or} \quad \alpha - \delta = 0.0513.$$

Then

$$\hat{d}_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta + 0.5\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln(1) + \left(0.0513 + 0.5(0.25)^2\right)1}{0.25\sqrt{1}} = 0.33,$$

$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T} = 0.33 - 0.25\sqrt{1} = 0.08,$$

$N\left(-\hat{d}_1\right) = 0.3707,$

$N\left(-\hat{d}_2\right) = 0.4681,$

And

$$\begin{aligned} E\left[S(1) \middle| S(1) < 80\right] &= S(0)e^{(\alpha-\delta)T} \frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)} \\ &= 84.2069\left(\frac{0.3707}{0.4681}\right) \\ &= 66.6855. \end{aligned}$$