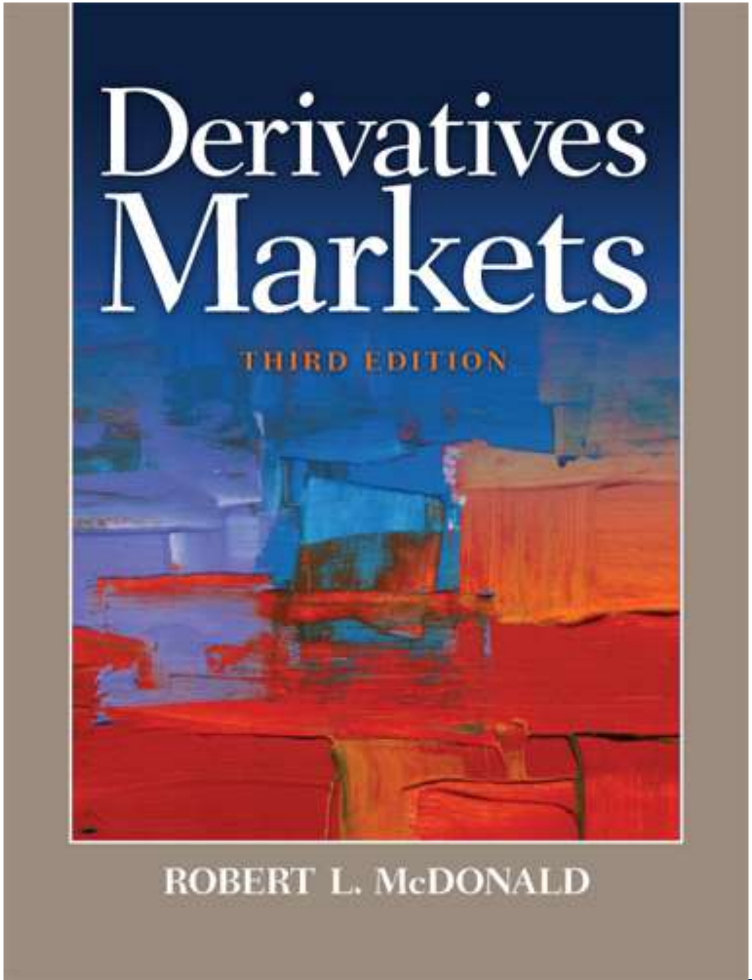


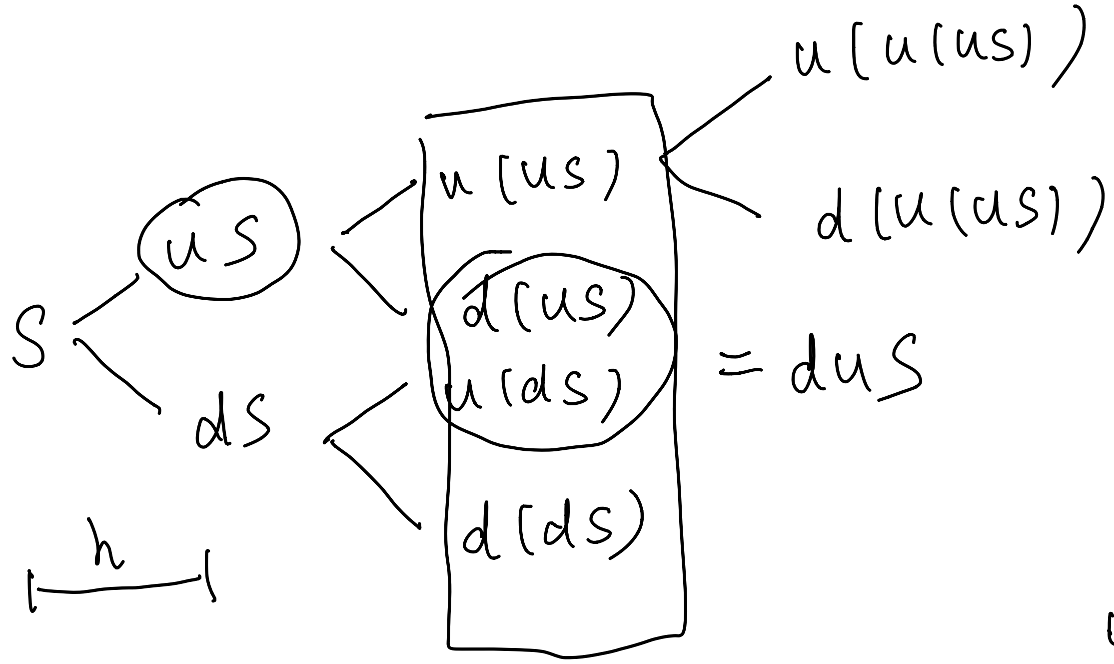
Chapter 9
(Chapter 10 in the textbook)

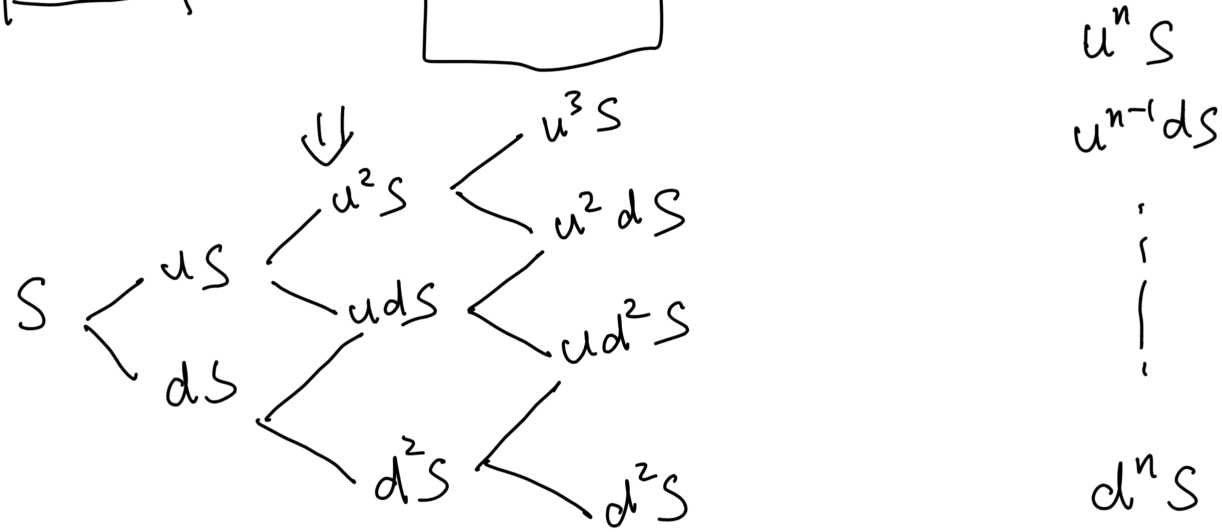
Binomial Option Pricing

Points to Note

1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13).
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 - 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)

possibilities 2 3 4

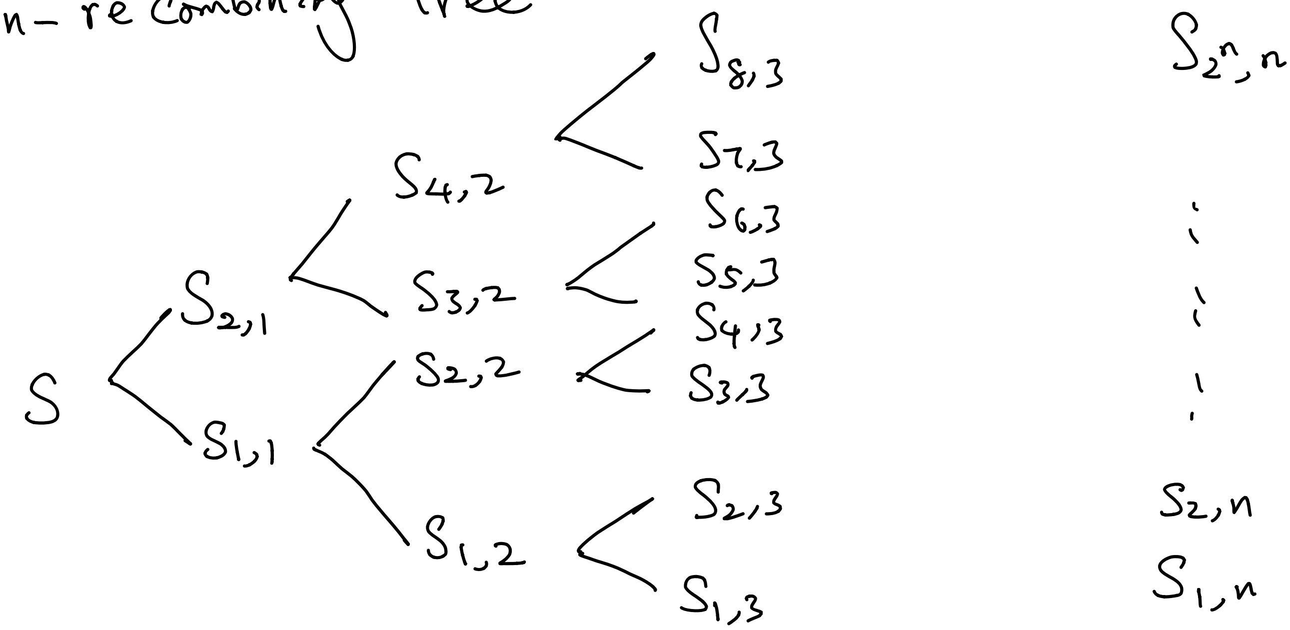




t 0 h 2h 3h ---- nh

No. of possibilities 2 3 4 --- (n+1)

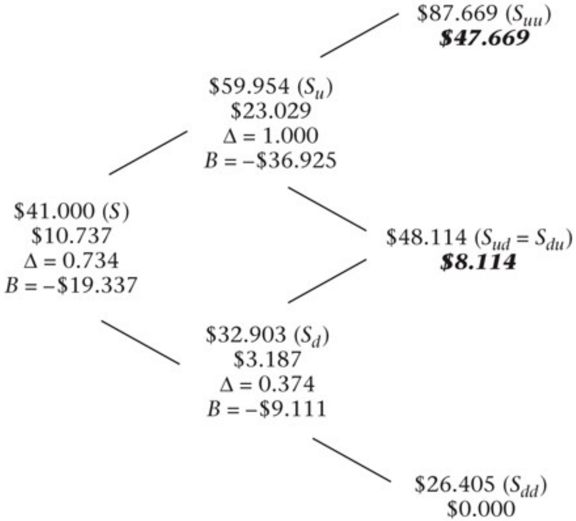
Non-recombining tree



t	0	h	2h	3h	---	nh
No. of possibilities		2	4	8	...	2^n

A Two-Period European Call

- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before



A Two-period European Call (cont'd)

- Note that an up move by the stock followed by a down move (S_{ud}) generates the same stock price as a down move followed by an up move (S_{du}). This is called a **recombining tree**. Otherwise, we would have a **nonrecombining tree**.

$$S_{ud} = S_{du} = u \times d \times \$41$$
$$= e^{(0.08+0.3)} \times e^{(0.08-0.3)} \$41 = \$48.114$$

Pricing the Call Option

- To price an option with two binomial periods, we work *backward* through the tree
  - *Year 2, Stock Price=$87.669*: since we are at expiration, the option value is *max* (0, *S* – *K*) = $47.669.
  - *Year 2, Stock Price=$48.114*: similarly, the option value is $8.114.
  - *Year 2, Stock Price=$26.405*: since the option is out of the money, the value is 0.

Pricing the Call Option (cont'd)

- *Year 1, Stock Price=\$59.954:*
  at this node, we compute the option value using equation (10.3), where *uS* is \$87.669 and *dS* is \$48.114

$$e^{-0.08}\left(\$47.669\times\frac{e^{0.08}-0.803}{1.462-0.803}+\$8.114\times\frac{1.462-e^{0.08}}{1.462-0.803}\right)=\$23.029$$

- *Year 1, Stock Price=\$32.903:*
  again using equation (10.3), the option value is \$3.187.
- *Year 0, Stock Price = \$41:*
  similarly, the option value is computed to be \$10.737.

Pricing the Call Option (cont'd)

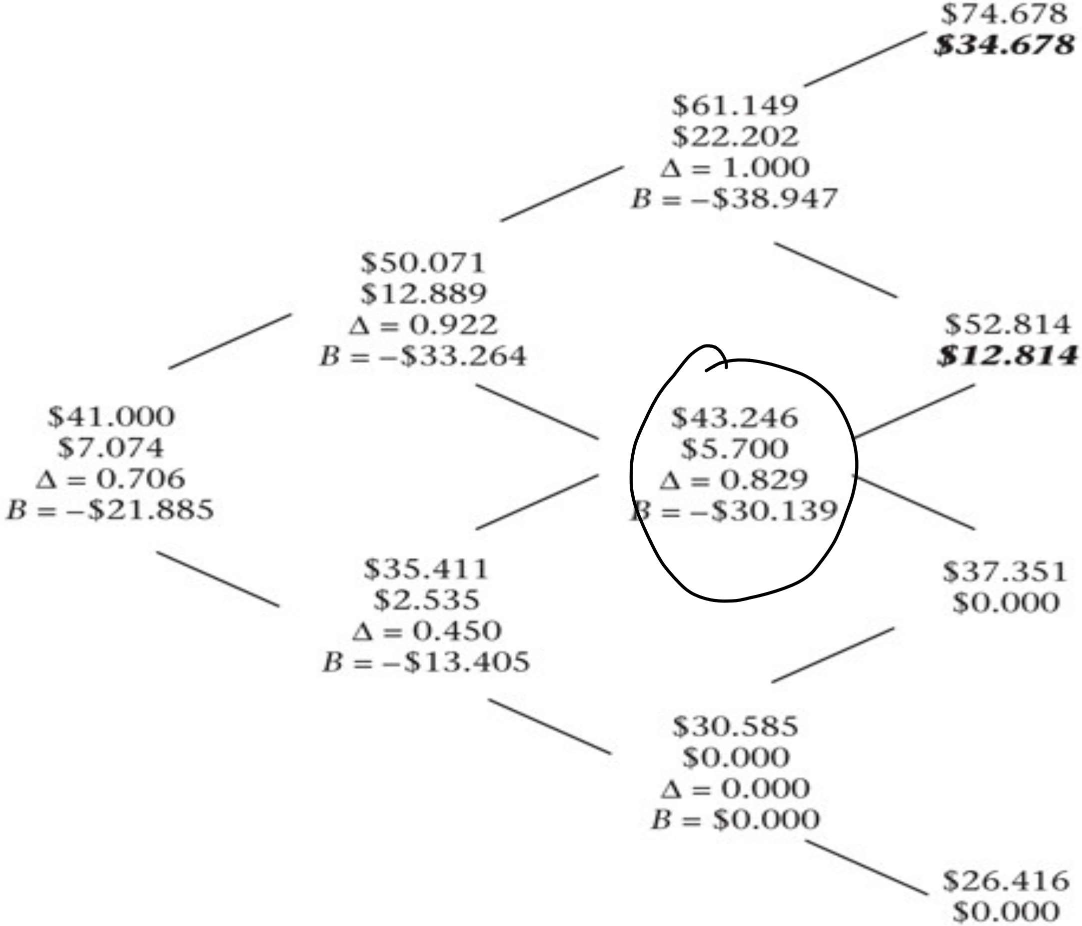
- Notice that
  - The option price is greater for the 2-year than for the 1-year option.
  - The option was priced by working backward through the binomial tree.
  - The option's Δ and B are different at different nodes. At a given point in time, Δ increases to 1 as we go further into the money.
  - Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than S – K; hence, we would not exercise even if the option had been American.

Many Binomial Periods

- Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.
  - Consider the previous example of the 1-year European call option.
  - Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$.
  - Assume that other inputs are the same as before (so, $r = 0.08$ and $\sigma = 0.3$).

Many Binomial Periods (cont'd)

- The stock price and option price tree for this option.


$74.678
$34.678
$61.149
$22.202
Δ = 1.000
B = −$38.947
$50.071
$12.889
Δ = 0.922
B = −$33.264
$52.814
$12.814
$41.000
$7.074
Δ = 0.706
B = −$21.885
$43.246
$5.700
Δ = 0.829
B = −$30.139
$35.411
$2.535
Δ = 0.450
B = −$13.405
$37.351
$0.000
$30.585
$0.000
Δ = 0.000
B = $0.000
$26.416
$0.000

Many Binomial Periods (cont'd)

- Note that since the length of the binomial period is shorter, *u* and *d* are closer to 1 before (u = 1.2212 and d = 0.8637 as opposed to 1.462 and 0.803 with h = 1).
  - The second-period nodes are computed as follows

$$S_u = \$41e^{0.08\times 1/3+0.3\sqrt{1/3}} = \$50.071$$

$$S_d = \$41e^{0.08\times 1/3-0.3\sqrt{1/3}} = \$35.411$$

  - The remaining nodes are computed similarly.

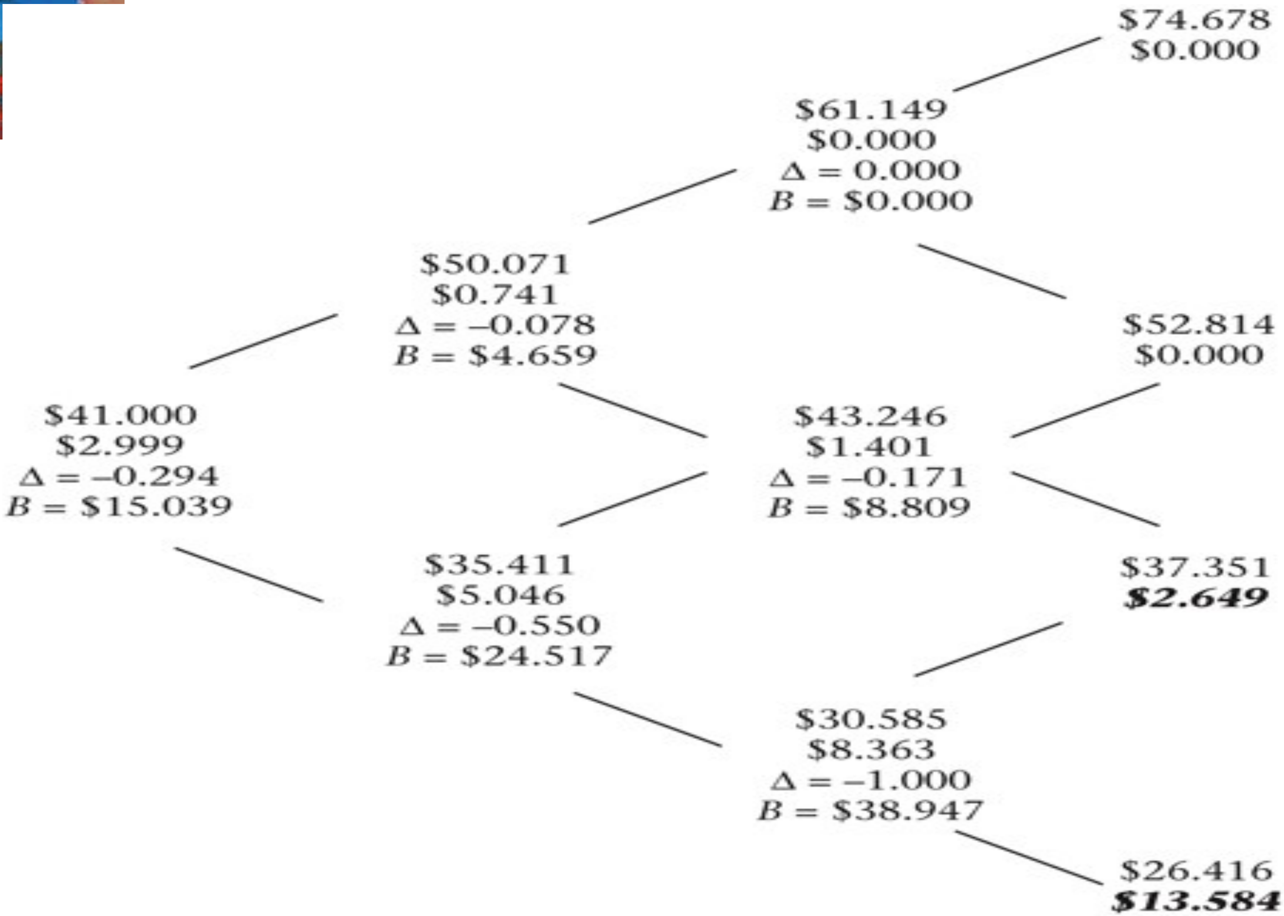
- Analogous to the procedure for pricing the 2-year option, the price of the three-period option is computed by working backward using equation (10.3). The option price is $7.074.

Put Options

- We compute put option prices using the same stock price tree and in almost the same way as call option prices.
- The only difference with a European put option occurs at expiration.
  - Instead of computing the price as $\max(0, S - K)$, we use $\max(0, K - S)$.

Put Options (cont'd)

- A binomial tree for a European put option with 1-year to expiration.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$2.999
Δ = –0.294
B = $15.039
$43.246
$1.401
Δ = –0.171
B = $8.809
$35.411
$5.046
Δ = –0.550
B = $24.517
$37.351
$2.649
$30.585
$8.363
Δ = –1.000
B = $38.947
$26.416
$13.584

General Formulation

- With loss of generality, consider an European call option on a non-dividend paying asset with the payoff of

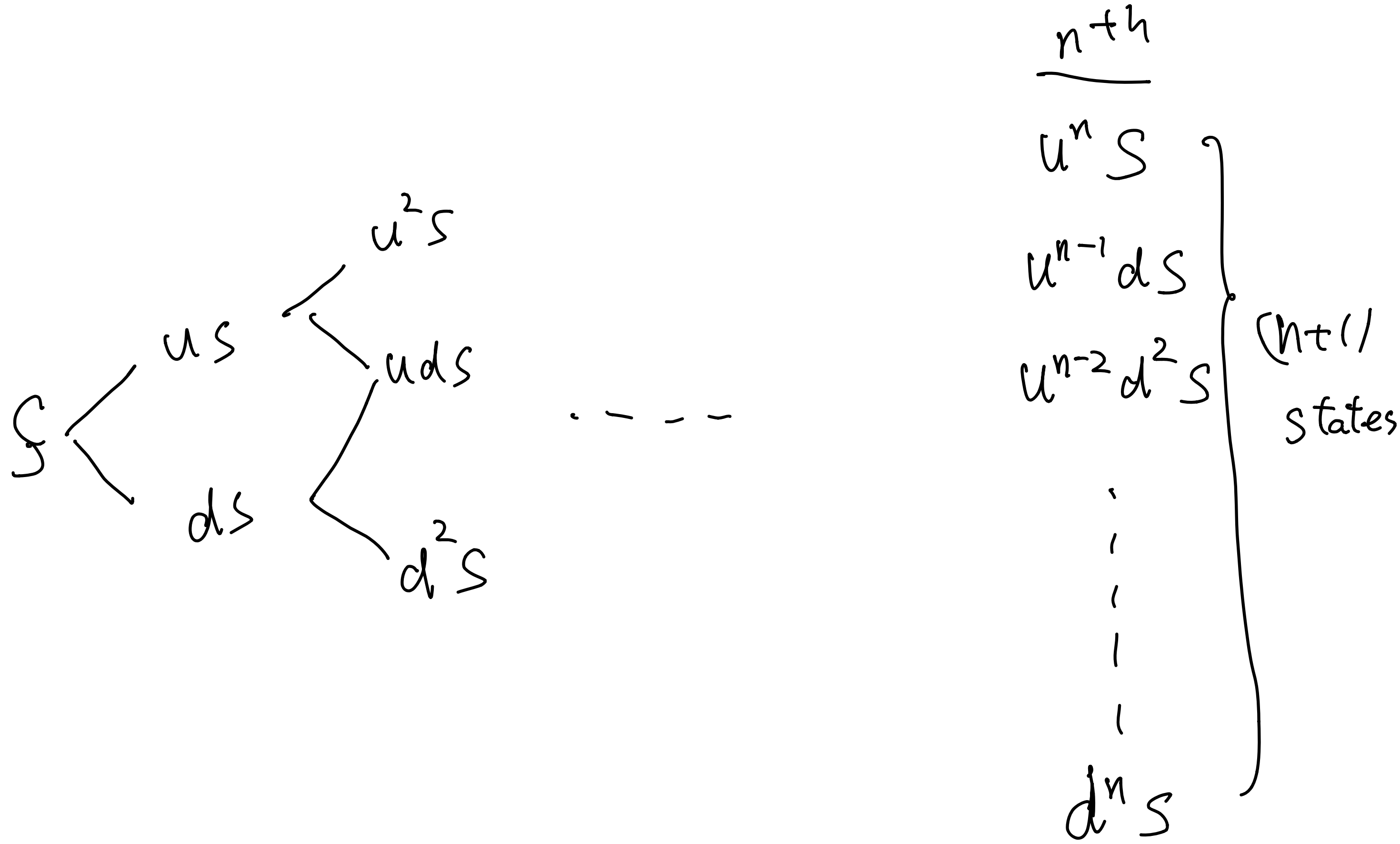
$$\max(S_T - K, 0)$$

- Let C be the call option price at time 0.
- In the n-period binomial tree, the risk-neutral probability of having j up-jumps and $(n - j)$ down-jumps is given by

$$C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j}$$

where

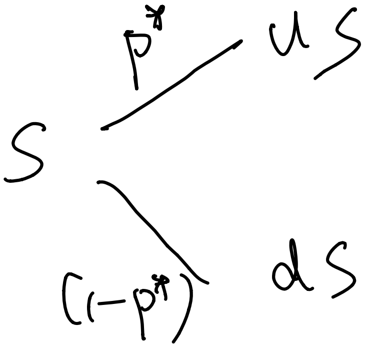
$$C_j^n = \frac{n!}{j!\left(n-j\right)!}.$$



$$C(0) = e^{-rT} E^*[\max(\underbrace{S_T}^{\text{r.v.}} - K, 0)]$$

$$= e^{-rT} E^*[\max(u^J d^{n-J} S - K, 0)]$$

J : r.v. no. of up-jump.



J : Binomial r.v.

$\Pr(J = 1) = p^*$

$\Pr(J = 0) = 1 - p^*$

Determine $\underset{\min}{\underset{\sim}{k}}$ for which $u^j d^{n-j} S > K$

$$u^k d^{n-k} S \overset{=}{\cancel{>}} K$$

$$k = \left\{ \text{smallest integer} \geq \left[\frac{\ln\left(\frac{K}{S d^n}\right)}{\ln\left(\frac{u}{d}\right)} \right] \right\}$$

$$C(0) = e^{-rT} \sum_{j=k}^{n} {}^{n}C_j (p^*)^j (1-p^*)^{n-j} (u^j d^{n-j} S - K)$$

General Formulation (cont'd)

- The corresponding payoff when *j* up-jumps and *n* – *j* down-jumps occur is seen to be

$$\max\left(u^j d^{n-j} S_0 - K, 0\right)$$

The call value obtained from the *n*-period binomial model is given by

$$C = e^{-rnh} \sum_{j=0}^{n} C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j} \max\left(u^j d^{n-j} S_0 - K, 0\right).$$

General Formulation (cont'd)

We define k to be the smallest nonnegative integer such that $u^k d^{n-k} S_0 \geq K$, that is

$$k \geq \frac{\ln \dfrac{K}{S_0 d^n}}{\ln \dfrac{u}{d}}.$$

Accordingly, we have

$$\max\left(u^j d^{n-j} S_0 - K, 0\right) = \begin{cases} 0 & \text{when } \; j < k \\ u^j d^{n-j} S_0 - K & \text{when } \; j \geq k. \end{cases}$$

General Formulation (cont'd)

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money. So,

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n,k,\tilde{p}\right) - Ke^{-rT} \Phi\left(n,k,p^*\right).$$

$Pr(J \geq k)$

where

$$\Phi\left(n,k,p\right) = \sum_{j=k}^{n} C_j^n \left(p\right)^j \left(1-p\right)^{n-j}, \quad \tilde{p} = \frac{up^*}{e^{rh}} \quad \text{and} \quad 1-\tilde{p} = \frac{d\left(1-p^*\right)}{e^{rh}}.$$

Note: When $n \to \infty$, the binomial tree model will converge to the Black-Scholes formula (see Binomial to BS.pdf for detail).

$$\Phi(n, k, p) = \Pr(J \geq k)$$

$$C = S_0 \underbrace{\tilde{\Pr}(J \geq k)}_{\text{w.r.t } \tilde{p}} - K e^{-rT} \underbrace{\Pr^*(J \geq k)}_{\text{w.r.t } p^*}$$

General Formulation (cont'd)

The *first term* gives the discounted expectation of the asset price at expiration given that the call expires in-the-money and the *second term* gives the present value of the expected cost incurred by exercising the call, where the expectation is taken under the risk-neutral measure.

The similar formulation can be obtained for the European put option.

American Options

- The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation (10.3).
- The value of the option if it is exercised is given by *max* (0, *S* – *K*) if it is a call and *max* (0, *K* – *S*) if it is a put.
- For an American put, the value of the option at a node is given by

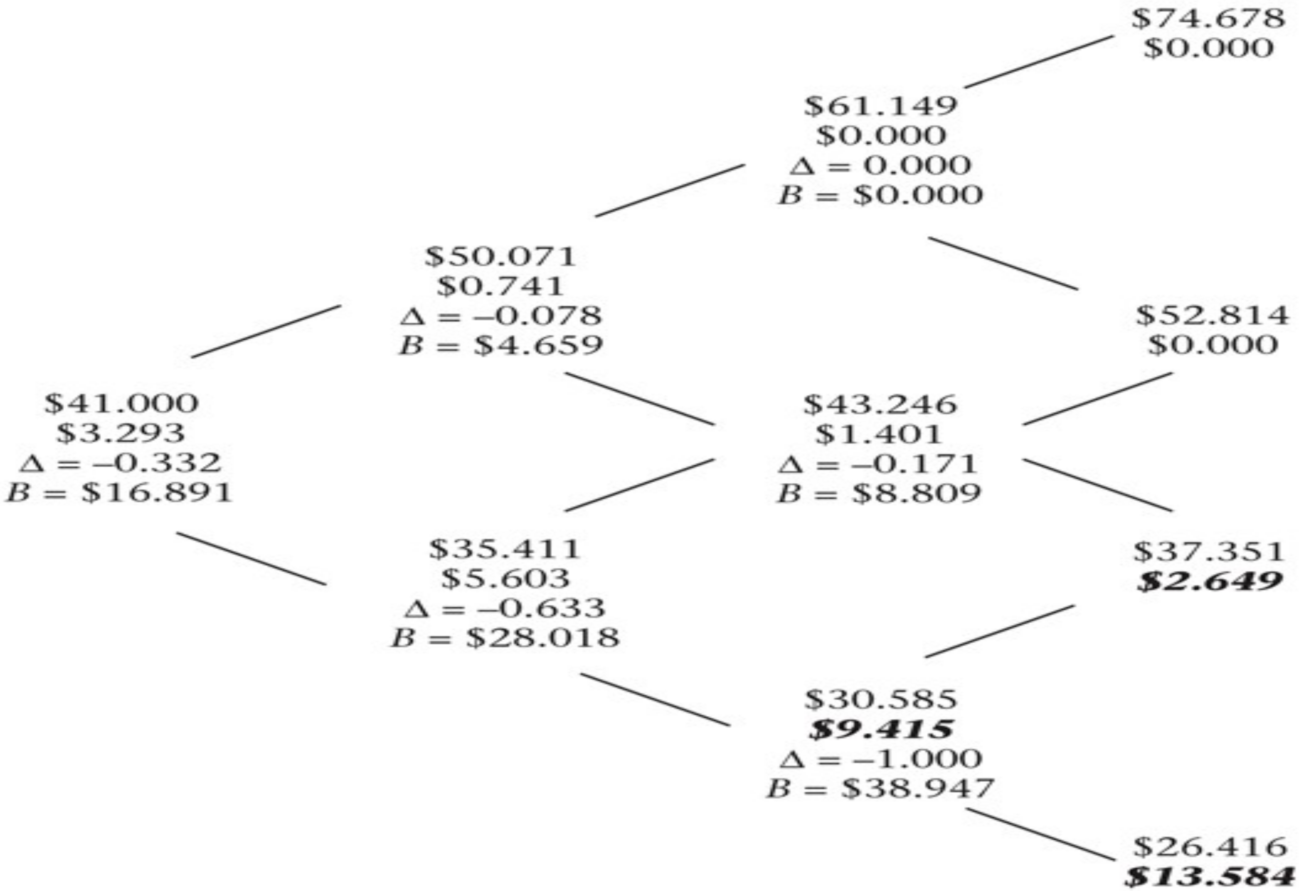
$$P(S,K,t) = \max\left(K - S, e^{-rh}\left[P(uS,K,t+h)p^* + P(dS,K,t+h)(1-p^*)\right]\right)$$

American Options (cont'd)

- The valuation of American options proceeds as follows:
  - At each node, we check for early exercise.
  - If the value of the option is greater when exercised, we assign that the **exercised** value to the node. Otherwise, we assign the value of the option **unexercised**.
  - We work backward through the tree as usual.

American Options (cont'd)

- Consider an American version of the put option valued in the previous example.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$3.293
Δ = –0.332
B = $16.891
$43.246
$1.401
Δ = –0.171
B = $8.809
$37.351
$2.649
$35.411
$5.603
Δ = –0.633
B = $28.018
$30.585
$9.415
Δ = –1.000
B = $38.947
$26.416
$13.584

American Options (cont'd)

- The only difference in the binomial tree occurs at the S_{dd} node, where the stock price is \$30.585. The American option at that point is worth \$40 – \$30.585 = \$9.415, its early-exercise value (as opposed to \$8.363 if unexercised). The greater value of the option at that node ripples back through the tree.

Options on Other Assets

- The binomial model developed thus far can be modified easily to price options on underlying assets other than non-dividend-paying stocks.
- The difference for different underlying assets is the construction of the binomial tree and the risk-neutral probability.
- We examine options on
  - Stock indexes
  - Currencies
  - Futures contracts
  - Commodities
  - Bonds

Options on a Stock Index

- Suppose a stock index pays continuous dividends at the rate δ.
- The procedure for pricing this option is equivalent to that of the first example, which was used for our derivation. Specifically
  - the up and down index moves are given

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

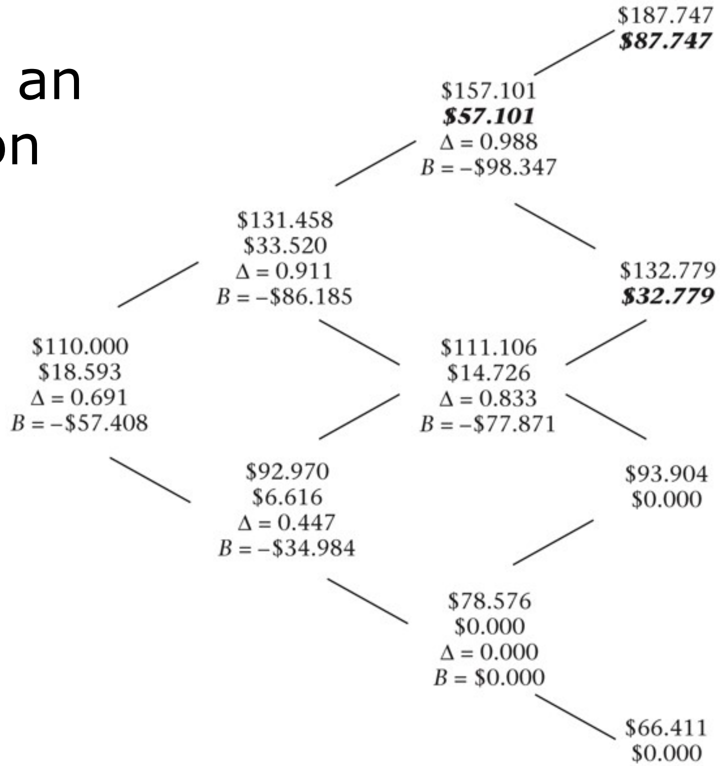
  - the replicating portfolio by equation (10.1) and (10.2).
  - the option price by equation (10.3).
  - the risk-neutral probability by equation (10.5).

Options on a Stock Index

- Given
  - S = \$110;
  - K = \$100;
  - σ = 0.30;
  - r = 0.05
  - T = 1 year
  - δ = 0.035
  - h = 0.333

Options on a Stock Index (cont'd)

- A binomial tree for an American call option on a stock index:



Options on Currencies

- With a currency with spot price x_0, the forward price is

$$F_{0,h} = x_0 e^{(r - r_f)h}$$

where r_f is the foreign interest rate.

- Thus, we construct the binomial tree using

$$ux = xe^{(r - r_f)h + \sigma\sqrt{h}}$$

$$dx = xe^{(r - r_f)h - \sigma\sqrt{h}}$$

$r_f \doteq \delta$

Underlying : Euro

Euro

use $\boxed{\text{USD}}$ to buy the underlying

denominated currency

r_{USD} (r)

risk-free rate

r_{Euro} — δ (r_f)

dividend yield

Options on Currencies (cont'd)

- Investing in a "currency" means investing in a money-market fund or fixed income obligation denominated in that currency.

- Taking into account interest on the foreign-currency denominated obligation, the two equations are

$$\Delta \times uxe^{r_f h} + e^{rh} \times B = C_u$$

$$\Delta \times dxe^{r_f h} + e^{rh} \times B = C_d$$

- The risk-neutral probability of an up move is

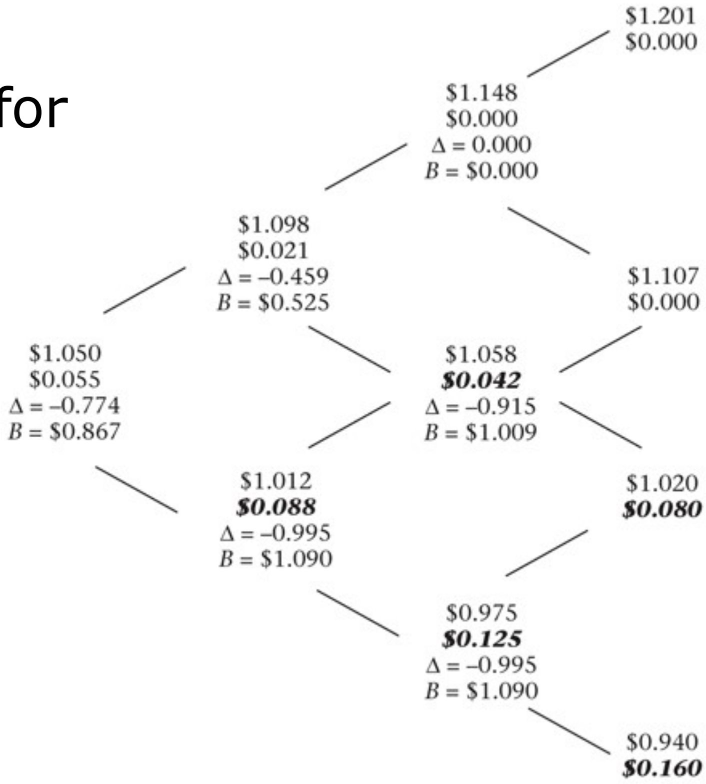
$$p^* = \frac{e^{(r-r_f)h} - d}{u - d} \qquad (10.20)$$

Options on Currencies (cont'd)

- Consider a dollar-denominated American put option on the euro, where
  - The current exchange rate is \$1.05/€ (*S*);
  - The strike is \$1.10/€ (*K*);
  - σ = 0.10;
  - The euro-denominated interest rate is 3.1% (δ);
  - The dollar-denominated rate is 5.5% (*r*).

Options on Currencies (cont'd)

- The binomial tree for the American put option on the euro



Options on Futures Contracts

- Assume the forward price is the same as the futures price.
- The nodes are constructed as

r

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

$\delta = r$

- We need to find the number of futures contracts, Δ, and the lending, *B*, that replicates the option.
  - A replicating portfolio must satisfy

$$\Delta \times (uF - F) + e^{rh} \times B = C_u$$
$$\Delta \times (dF - F) + e^{rh} \times B = C_d$$

$t=0$

Δ of Future Contract

$+$ lending $\$B$

$$\cancel{V(0) = \Delta F + B} \qquad V(0) = B$$

$$F \begin{cases} uF \\ dF \end{cases} \qquad u = e^{\sigma\sqrt{h}} \qquad d = e^{-\sigma\sqrt{h}}$$

mark-to-market.

$$V(h) = B\cdot e^{rh} + \boxed{(F_h - F)}\,\Delta \qquad F_h = \begin{cases} uF \\ dF \end{cases}$$

Options on Futures Contracts (cont'd)

- Solving for Δ and B gives

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d}\right)$$

Δ tells us how many futures contracts to hold to hedge the option, and B is simply the value of the option.

- We can again price the option using equation (10.3).

- The risk-neutral probability of an up move is given by

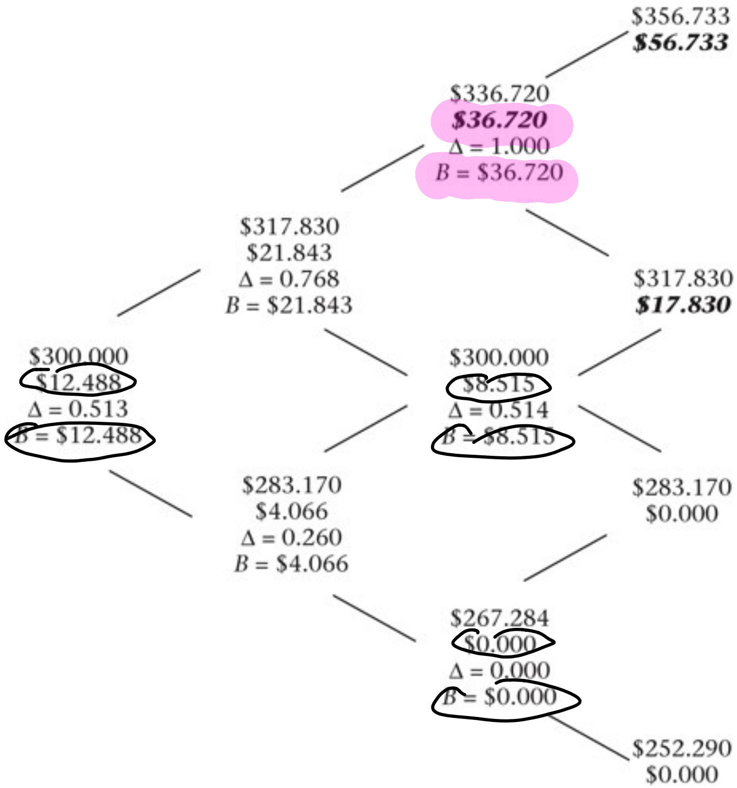
$$p^* = \frac{1-d}{u-d} \qquad (10.21)$$

Options on Futures Contracts (cont'd)

- The motive for early-exercise of an option on a futures contract is the ability to earn interest on the mark-to-market proceeds.
  - When an option is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price.

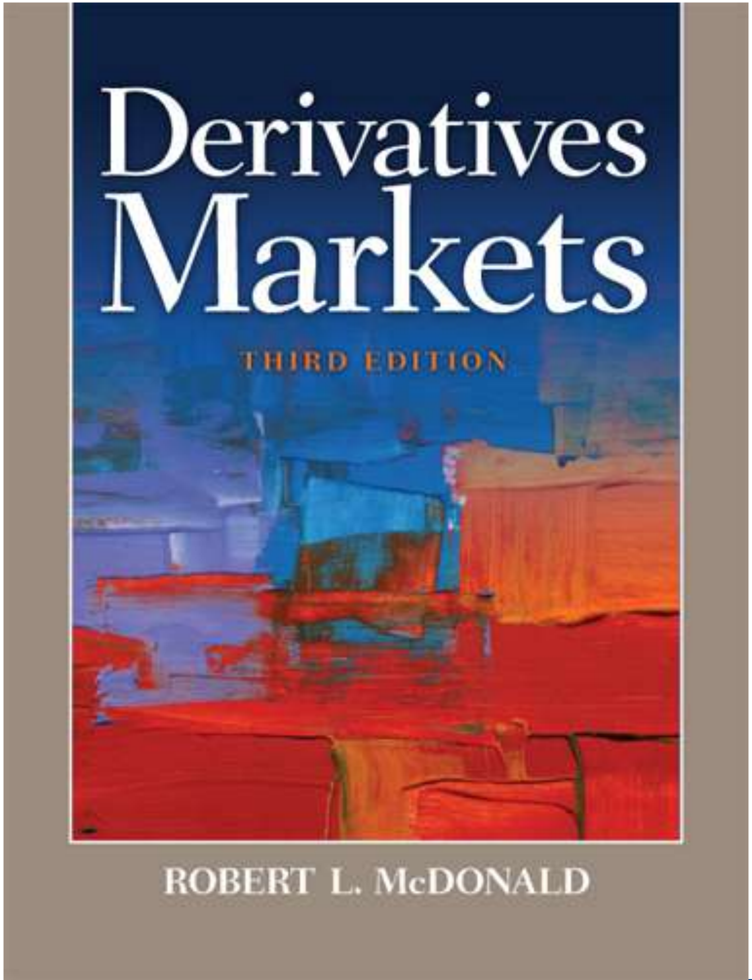
Options on Futures Contracts (cont'd)

- A tree for an American call option on a futures contract



Note

option price = B



Chapter 10
(Chapter 12 in the textbook)

The Black-Scholes Formula

Points to Note

1. What is the Black-Scholes formula for the European call and put options? (see P.3 – 5)
2. What are the assumptions of the Black-Scholes formula? (see P.6 – 7)
3. What is the relationship of the binomial model and the Black-Scholes formula? (see P.8 – 9)
4. The Black-Scholes formula for different underlying assets. (see P.10 – 15)
5. Option Greeks (see P.16 – 46)
6. Implied volatility (see P.47 – 52)

Black-Scholes Formula

- The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.
- Consider an European call (or put) option written on a stock.
- Assume that the stock pays dividend at the continuous rate δ.

Black-Scholes Formula (cont'd)

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Black-Scholes Formula (cont'd)

$C = \Delta S + B$

- Call Option price:

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T}N(d_1) \boxed{- Ke^{-rT}N(d_2)}$$

- Put Option price:

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where

$$d_1 = \frac{ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

N(*x*) is the cumulative distribution for standard normal random variable.

Black-Scholes Assumptions

- Assumptions about stock return distribution:
  - Continuously compounded returns on the stock are normally distributed and independent over time. (We assume there are no "jumps" in the stock price).
  - The volatility of continuously compounded returns is known and constant.
  - Future dividends are known, either as dollar amount or as a fixed dividend yield.

Black-Scholes Assumptions (cont'd)

- Assumptions about the economic environment:
  - The risk-free rate is known and constant.
  - There are no transaction costs or taxes.
  - It is possible to short-sell costlessly and to borrow at the risk-free rate.

Continuous Limits of the Binomial Model

By considering the general formulation of the call option price on a non-dividend stock under the binomial model,

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n, k, \tilde{p}\right) - Ke^{-rT} \Phi\left(n, k, p^*\right).$$

where $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$

$$\Phi\left(n,k,p\right) = \sum_{j=k}^{n} C_j^n \left(p\right)^j \left(1-p\right)^{n-j}, \quad \tilde{p} = \frac{up^*}{e^{rh}} \quad \text{and} \quad 1-\tilde{p} = \frac{d\left(1-p^*\right)}{e^{rh}}.$$

Continuous Limits of the Binomial Model (cont'd)

It can shown that

$$\lim_{n\to\infty}\left[S_0\Phi\left(n,k,\tilde{p}\right)-Ke^{-rT}\Phi\left(n,k,p^*\right)\right]=S_0N\left(d_1\right)-Ke^{-rT}N\left(d_2\right).$$

Recall that $\Phi\left(n,k,p^*\right)$ is the risk neutral probability that the number of upward moves in the asset price is greater than or equal to *k* in the *n*-period binomial model, where p^* is the risk neutral probability of an upward move.

(See Binomial tree to BS.pdf)