MFE5130 – Financial Derivatives
First Term, 2019 – 20

Assignment 5 (Solution)

Additional Problem 1

The Sharpe ratio of Stock Q is greater than that of Stock X:

$$\frac{0.07-0.03}{0.12} > \frac{0.05-0.03}{0.11} \Rightarrow 0.3333 > 0.1818.$$

Since both assets follow GBM, a strategy that involves

i. purchasing $\frac{1}{\sigma_Q Q}$ shares of Stock Q and,

ii. selling $\frac{1}{\sigma_X X}$ shares of Stock X.

results an arbitrage profit of:

$$\left[\frac{\alpha_Q - r}{\sigma_Q} - \frac{\alpha_X - r}{\sigma_X}\right]dt = \left[\frac{0.07-0.03}{0.12} - \frac{0.05-0.03}{0.11}\right]dt = 0.1515dt.$$

Therefore a strategy that involves selling 1 share of Stock X results in an arbitrage profit of:

$$(\sigma_X X)(0.1515dt) = (0.11)(182)(0.1515)dt = 3.033dt.$$

Additional Problem 2

Let $\eta = \frac{0.05-0.09}{0.1} = -0.4$ and consider the Radon Nikodym derivative of Q with respect to P based on the information up to time t:

$$\frac{dQ}{dP} = \exp\left(0.4Z(t) - \frac{1}{2}(-0.4)^2 t\right) = \exp(0.4Z(t) - 0.08t).$$

Under the measure Q, the stochastic process

$$\tilde{Z}(t) = Z(t) - 0.4t$$

is a standard Brownian motion under Q by the Girsanov Theorem.

It is seen that when we set $\eta = -0.4$ then

$$0.05dt + 0.1dZ(t) = 0.05dt + 0.1\left(d\tilde{Z}(t) + 0.4dt\right) = 0.09dt + 0.1d\tilde{Z}(t).$$

Therefore, $S(t)$ is governed by

$$\frac{dS(t)}{S(t)} = 0.09dt + 0.1d\widetilde{Z}(t)$$

under measure Q.

<u>Additional Problem 3</u>

(a)

Let δ_X and δ_Y be the dividend yields of Stock X and Stock Y respectively.

Since $\delta_X = 0$, $r = 0.08$.

$\beta = r - \delta_Y = r - 0 = 0.08$.

For non-dividend paying stock, under the risk-neutral measure, β is the risk-free interest rate.

(b)

Using $Z_1(t)$ and $Z_2(t)$, $W_1(t)$ and $W_2(t)$ can be rewritten as

$$dW_1(t) = \sqrt{1-0.2^2}\,dZ_1(t) + 0.2dZ_2(t),$$
$$dW_2(t) = dZ_2(t).$$

The stochastic differential equations of $X(t)$ and $Y(t)$ can be rewritten as

$$\frac{dX(t)}{X(t)} = 0.08dt + 0.18\left(\sqrt{1-0.2^2}\,dZ_1(t) + 0.2dZ_2(t)\right) = 0.08dt + 0.1764dZ_1(t) + 0.036dZ_2(t),$$
$$\frac{dY(t)}{Y(t)} = 0.08dt + 0.42dZ_2(t).$$

Hence, $\alpha_1 = 0.1764$, $\alpha_2 = 0.036$ and $\alpha_3 = 0.42$.

(c)

Assume the money market account starts at \$1 at $t = 0$. Let $M(t)$ be the value of the money market account at time t.

$$\frac{dQ_X}{dQ} = \frac{X(t)}{X(0)} \Big/ \frac{M(t)}{M(0)}$$

$$= \frac{X(0)\exp\left[\left(0.08 - \frac{1}{2}(0.1764)^2 - \frac{1}{2}(0.036)^2\right)t + 0.1764Z_1(t) + 0.036Z_2(t)\right]}{X(0)\exp(0.08t)}$$

$$= \exp\left[-\frac{1}{2}(0.1764)^2 t - \frac{1}{2}(0.036)^2 t + 0.1764Z_1(t) + 0.036Z_2(t)\right].$$

(d)

From the Girsanov theorem, we have

$$\tilde{Z}_1(t) = Z_1(t) - 0.1764t,$$
$$\tilde{Z}_2(t) = Z_2(t) - 0.036t.$$

are independent standard Brownian motions under Q_X.

So, the dynamic of $Y(t)$ under Q_X is

$$\frac{dY(t)}{Y(t)} = 0.08dt + 0.42\left(d\tilde{Z}_2(t) + 0.036dt\right)$$
$$= 0.0951dt + 0.42d\tilde{Z}_2(t).$$

<u>Additional Problem 4</u>

From the given stochastic differential equation of $X(t)$, we have

$$X(t) = 100\exp\left(\left(0.05 - \frac{1}{2}(0.27)^2 - \frac{1}{2}(0.13)^2\right)t + 0.27Z_1(t) + 0.13Z_2(t)\right).$$

Let

$M(t) = e^{0.05t}$

Define a new measure Q_X by

$$
\begin{aligned}
\frac{dQ_X}{dP} &= \frac{\frac{X(t)}{X(0)}}{\frac{M(t)}{M(0)}} \\
&= \frac{\exp\left(\left(0.05-\frac{1}{2}(0.27)^2-\frac{1}{2}(0.13)^2\right)t+0.27Z_1(t)+0.13Z_2(t)\right)}{e^{0.05t}} \\
&= \exp\left(-\frac{1}{2}(0.27)^2 t-\frac{1}{2}(0.13)^2 t+0.27Z_1(t)+0.13Z_2(t)\right).
\end{aligned}
$$

From the Girsanov theorem, we have

$$
\begin{aligned}
\tilde{Z}_1(t) &= Z_1(t)-0.27t, \\
\tilde{Z}_2(t) &= Z_2(t)-0.13t,
\end{aligned}
$$

are independent standard Brownian motions under Q_X.

So, the dynamic of $Y(t)$ under Q_X is

$$
\begin{aligned}
\frac{dY(t)}{Y(t)} &= 0.08dt+0.32\left(d\tilde{Z}_2(t)+0.13dt\right) \\
&= 0.1216dt+0.32d\tilde{Z}_2(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Y(t) &= 100\exp\left(\left(0.1216-\frac{1}{2}(0.32)^2\right)t+0.32\tilde{Z}_2(t)\right) \\
&= 100\exp\left(0.0704t+0.32\tilde{Z}_2(t)\right).
\end{aligned}
$$

By using the theorem on change of numeraire, we have

$$
\begin{aligned}
E^P\left[X(2)\mathbf{I}_{\{Y(2)\le 100\}}\right] &= M(0)E^P\left[\frac{M(2)X(2)\mathbf{I}_{\{Y(2)\le 100\}}}{M(2)}\right] \\
&= X(0)E^{Q_X}\left[\frac{M(2)X(2)\mathbf{I}_{\{Y(2)\le 100\}}}{X(2)}\right] \\
&= X(0)E^{Q_X}\left[M(2)\mathbf{I}_{\{Y(2)\le 100\}}\right] \\
&= 100e^{0.05(2)}E^{Q_X}\left[\mathbf{I}_{\{Y(2)\le 100\}}\right] \\
&= 100e^{0.1}Q_X\left(Y(2)\le 100\right) \\
&= 100e^{0.1}Q_X\left(100\exp\left(0.0704\times 2+0.32\tilde{Z}_2(2)\right)\le 100\right) \\
&= 100e^{0.1}Q_X\left(Z\le -0.31\right) \\
&= 100e^{0.1}\left(1-N(0.31)\right) \\
&= 100e^{0.1}\left(1-0.6217\right) \\
&= 41.8086.
\end{aligned}
$$

<u>Additional Problem 5</u>

Let $Y(t) = S^2(t)$.

By Itô's lemma, we have

$$
\begin{aligned}
dY(t) &= \frac{\partial Y}{\partial S}dS(t)+\frac{1}{2}\frac{\partial^2 Y}{\partial S^2}\left(dS(t)\right)^2+\frac{\partial Y}{\partial t}dt \\
&= 2S(t)S(t)\left(0.08dt+0.3dZ(t)\right)+\frac{1}{2}(2)S^2(t)\left(0.08dt+0.3dZ(t)\right)^2+0\cdot dt \\
&= 2S^2(t)\left(0.08dt+0.3dZ(t)\right)+S^2(t)(0.3)^2\,dt \\
&= S^2(t)\left(0.25dt+0.6dZ(t)\right) \\
&= Y(t)\left(0.25dt+0.6dZ(t)\right).
\end{aligned}
$$

So, $Y(t)$ follows a GBM with the drift of 0.25 and volatility of 0.6. Also, $Y(0) = 64$.

Hence,

$$
\begin{aligned}
&E^P\left[\max\left(S^2(2)-60,0\right)\right] \\
&= E^P\left[\max\left(Y(2)-60,0\right)\right] \\
&= Y(0)e^{0.25(2)}N\left(\tilde{d}_1\right)-60N\left(\tilde{d}_2\right),
\end{aligned}
$$

where

$$\tilde{d}_1 = \frac{\ln\left(\frac{64}{60}\right) + \left(0.25 + \frac{1}{2}(0.6)^2\right)2}{0.6\sqrt{2}} = 1.09,$$

$$\tilde{d}_2 = \tilde{d}_1 - 0.6\sqrt{2} = 0.24.$$

So,

$$\begin{aligned}
&E^P\left[\max\left(S^2(2) - 60, 0\right)\right] \\
&= E^P\left[\max\left(Y(2) - 60, 0\right)\right] \\
&= 64e^{0.25(2)}N(1.09) - 60N(0.24) \\
&= 55.2792.
\end{aligned}$$

<u>Additional Problem 6</u>

Let $V(0)$ be the price of the financial derivative at time 0.

Since the dynamics of $X(t)$ and $Y(t)$ are under the risk-neutral measure and both stocks do not pay dividends, the risk-free interest rate $r = 4\%$.

Let $M(t)$ be the value of the money market account at time t.

$$M(t) = e^{rt} = e^{0.04t}.$$

By the risk-neutral valuation theorem,

$$\begin{aligned}
V(0) &= e^{-3(4\%)}E_0^Q\left[\max\left(X(3)Y^2(3) - 235X(3), 0\right)\right] \\
&= M(0)E_0^Q\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{M(3)}\right] \qquad (1)
\end{aligned}$$

We choose $X(t)$ as the numeraire, the corresponding Radon-Nikodym derivative of Q_X with respect to Q is

$$\begin{aligned}
\frac{dQ_X}{dQ} &= L(t) \\
&= \frac{X(t)}{X(0)} \bigg/ \frac{M(t)}{M(0)} \\
&= \frac{X(0)\exp\left[\left(0.04 - \frac{1}{2}(0.28)^2 - \frac{1}{2}(0.12)^2\right)t + 0.28Z_1(t) + 0.12Z_2(t)\right]}{X(0)\exp(0.04t)} \\
&= \exp\left[-\frac{1}{2}(0.28)^2 t - \frac{1}{2}(0.12)^2 t + 0.28Z_1(t) + 0.12Z_2(t)\right]
\end{aligned}$$

From the Girsanov theorem, we have

$$\tilde{Z}_1(t) = Z_1(t) - 0.28t,$$
$$\tilde{Z}_2(t) = Z_2(t) - 0.12t.$$

are independent standard Brownian motions under Q_X.

So, the dynamic of $Y(t)$ under Q_X is

$$\begin{aligned}\frac{dY(t)}{Y(t)} &= 0.04dt + 0.31\left(d\tilde{Z}_2(t) + 0.12dt\right) \\ &= 0.0772dt + 0.31d\tilde{Z}_2(t).\end{aligned}$$

Hence,

By Ito's lemma,

$$\begin{aligned}dY^2(t) &= 2Y(t)dY(t) + \left(dY(t)\right)^2 \\ &= 2Y(t)Y(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right) + Y^2(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right)^2 \\ &= 2Y^2(t)\left(0.0772dt + 0.31d\tilde{Z}_2(t)\right) + Y^2(t)(0.31)^2 dt \\ &= Y^2(t)\left(0.2505dt + 0.62d\tilde{Z}_2(t)\right).\end{aligned}$$

So,

$$\begin{aligned}Y^2(t) &= Y^2(0)\exp\left[\left(0.2505 - 0.5(0.62)^2\right)t + 0.62\tilde{Z}_2(t)\right] \\ &= 225\exp\left[0.0583t + 0.62\tilde{Z}_2(t)\right].\end{aligned}$$

By using the theorem on change of numeraire, we have

$$\begin{aligned}V(0) &= M(0)E_0^Q\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{M(3)}\right] \\ &= X(0)E_0^{Q_X}\left[\frac{X(3)\max\left(Y^2(3) - 235, 0\right)}{X(3)}\right] \\ &= 12E_0^{Q_X}\left[\max\left(Y^2(3) - 235, 0\right)\right] \\ &= 12\left[225e^{(0.2505)(3)}N\left(\tilde{d}_1\right) - 235N\left(\tilde{d}_2\right)\right].\end{aligned}$$

where

$$\tilde{d}_1 = \frac{\ln\left(\frac{225}{235}\right) + \left(0.2505 + 0.5\times 0.62^2\right)3}{0.62\sqrt{3}} = 1.2 \quad \text{and} \quad \tilde{d}_2 = \tilde{d}_1 - 0.62\sqrt{3} = 0.13.$$

So,

$$\begin{aligned} V(0) &= 12\left[225e^{0.2505(3)}N\left(\tilde{d}_1\right) - 235N\left(\tilde{d}_1\right)\right] \\ &= 12\left[225e^{0.7515}N(1.2) - 235N(0.13)\right] \\ &= 3509.7986. \end{aligned}$$

Additional Problem 7

From the given stochastic differential equations (SDEs), we have

$$S_1(t) = 3\exp\left(\left(0.08 - 0.5\times 0.28^2\right)t + 0.28Z_1(t)\right) = 3\exp\left(0.0408t + 0.28Z_1(t)\right) \text{ and}$$

$$S_2(t) = 5\exp\left(\left(0.05 - 0.5\times 0.35^2\right)t + 0.35Z_2(t)\right) = 5\exp\left(-0.0113t + 0.35Z_2(t)\right).$$

So,

$$\begin{aligned} S_1(2)S_2(2) &= 15\exp\left((0.0408 - 0.0113)(2) + 0.28Z_1(2) + 0.35Z_2(2)\right) \\ &= 15\exp\left(0.059 + 0.28Z_1(2) + 0.35Z_2(2)\right). \end{aligned}$$

We have

$$E\left[0.28Z_1(2) + 0.35Z_2(2)\right] = 0;$$

$$\begin{aligned} &Var\left[0.28Z_1(2) + 0.35Z_2(2)\right] \\ &= (0.28)^2 Var\left[Z_1(2)\right] + (0.35)^2 Var\left[Z_2(2)\right] + 2(0.28)(0.35)Cov\left(Z_1(2), Z_2(2)\right) \\ &= (0.28)^2(2) + (0.35)^2(2) + 2(0.28)(0.35)(-0.1)\sqrt{2}\sqrt{2} \\ &= 0.3626. \end{aligned}$$

So, $0.28Z_1(2) + 0.35Z_2(2) = \sqrt{0.3626}Z, \quad \text{where} \quad Z \sim N(0,1).$

$$\begin{aligned} \Pr\left(S_1(2)S_2(2) > 20\right) &= \Pr\left(15\exp\left(0.059 + \sqrt{0.3626}Z\right) > 20\right) \\ &= \Pr\left(Z > \frac{\ln\left(\frac{20}{15}\right) - 0.059}{\sqrt{0.3626}}\right) \\ &= \Pr(Z > 0.38) \\ &= 0.352. \end{aligned}$$

Question 14.20

a) Since all the options will be expiring at the same date t_1, we have the payoff of the chooser will be

$$\max[\max\left(S_{t_1} - K, 0\right), \max\left(K - S_{t_1}, 0\right)].$$

If $S_{t_1} < K$, then the payoff is $K - S_{t_1}$.

If $S_{t_1} > K$, then the payoff is $S_{t_1} - K$.

This is equivalent to a K-strike straddle.

b) Using put-call parity at t_1, the value of the as-you-like-it option at t_1 will be:

$$\max\left[C\left(S_{t_1}, K, T - t_1\right), C\left(S_{t_1}, K, T - t_1\right) + Ke^{-r(T-t_1)} - S_{t_1}e^{-\delta(T-t_1)}\right]$$

$$= C\left(S_{t_1}, K,\ T - t_1\right)\ +\ \max\ \left(0, Ke^{-r(T-t_1)} - S_{t_1}e^{-\delta(T-t_1)}\right)$$

$$= C\left(S_{t_1}, K, T - t_1\right) + e^{-\delta(T-t_1)}\max\left(0, Ke^{(\delta-r)(T-t_1)} - S_{t_1}\right).$$

The first term is the value of a call with strike K and maturity T; the second term is the payoff from holding $e^{-\delta(T-t_1)}$ put options that expire at t_1 with strike $Ke^{(\delta-r)(T-t_1)}$.

Question 23.1

The payoff of the COD can be expressed as

$$
\begin{aligned}
\text{Payoff} &= \begin{cases} 0 & \text{if } S(T) < K \\ S(T) - K - P & \text{if } S(T) \geq K \end{cases} \\
&= \left(S(T) - K - P\right)\mathbf{I}_{\{S(T)\geq K\}}.
\end{aligned}
$$

By the risk-neutral valuation theorem,

$$
\begin{aligned}
V(0) &= M(0)E^Q\left(\frac{\left(S(T)-K-P\right)\mathbf{I}_{\{S(T)\geq K\}}}{M(T)}\right) \\
&= e^{-rT}E^Q\left[\left(S(T)-K-P\right)\mathbf{I}_{\{S(T)\geq K\}}\right] \\
&= e^{-rT}E^Q\left[\left(S(T)-K\right)\mathbf{I}_{\{S(T)\geq K\}}\right] - e^{-rT}PE^Q\left[\mathbf{I}_{\{S(T)\geq K\}}\right].
\end{aligned}
$$

Since it is zero cost to buy COD, we have

$$0 = BSCall(S_0, K, \sigma, r, T, \delta) - P \times CashCall(S_0, K, \sigma, r, T, \delta).$$

P can then be solved from this equation.

a) Given the inputs and pricing the above options, P must satisfy

$$0 = 10.45 - P\,(0.5323),$$

which implies $P = 10.45/0.5323 = 19.632$.

b) At $t = 0$, the delta of the COD is

$$0.637 - 19.632 \times 0.01875 = 0.2689$$

and the gamma of the COD is

$$0.019 - 19.632\,(-0.00033) = 2.55\%.$$

c) As the option approaches maturity, the delta will explode when the option is at the money, making delta hedging difficult.