Partial Expectation

Show

$$\int_0^K S_t\, g(S_t ; S_0)\, dS_t = S_0\, e^{(\alpha-\delta)t}\, N(-\hat{d}_1)$$

where $$\hat{d}_1 = \frac{\ln(S_t/K) + (\alpha - \delta + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

Proof :

Since $\ln S_t \sim N(\ln S_0 + (\alpha - \delta - \frac{1}{2}\sigma^2)t,\ \sigma^2 t)$,

$$g(S_t ; S_0) = \frac{1}{S_t\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{1}{2}\left(\frac{\ln S_t - m}{\sigma\sqrt{t}}\right)^2}$$

where $m = \ln S_0 + (\alpha - \delta - \frac{1}{2}\sigma^2)t$

$$\int_0^K S_t\, g(S_t : S_0)\, dS_t$$

$$= \int_0^K \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}(\ln S_t - m)^2\right) dS_t$$

Let $y = \ln S_t$.

$$\int_0^K S_t\, g(S_t ; S_0)\, dS_t$$

$$= \int_{-\infty}^{\ln K} \frac{e^y}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}(y-m)^2\right) dy \quad \left(\text{because } \frac{dy}{dS_t} = \frac{1}{S_t}\right)$$

$$= \int_{-\infty}^{\ln K} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}\left((y-m)^2 - 2y\sigma^2 t\right)\right) dy$$

Consider

$$\begin{aligned}(y-m)^2 - 2y\sigma^2 t &= y^2 - 2y(m+\sigma^2 t) + m^2 \\ &= \left(y - (m+\sigma^2 t)\right)^2 - 2m\sigma^2 t - \sigma^4 t^2 \\ &= \left(y - (m+\sigma^2 t)\right)^2 - 2\sigma^2 t\left(m + \frac{\sigma^2 t}{2}\right) \\ &= \left(y - (m+\sigma^2 t)\right)^2 - 2\sigma^2 t\left(\ln S_0 + (\alpha-\delta)t\right)\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\ln K} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}\left((y-m)^2 - 2y\sigma^2 t\right)\right) dy$$

$$= \int_{-\infty}^{\ln K} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}\left(\left(y-(m+\sigma^2 t)\right)^2 - 2\sigma^2 t\left(\ln S_0 + (\alpha-\delta)t\right)\right)\right) dy$$

$$= S_0 e^{(\alpha-\delta)t} \int_{-\infty}^{\ln K} \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}\left(y-(m+\sigma^2 t)\right)^2\right) dy$$

$$= S_0 e^{(\alpha-\delta)t} \Pr\left(Z < \frac{\ln K - (m+\sigma^2 t)}{\sigma\sqrt{t}}\right) \quad \text{where } Z \sim N(0,1)$$

$$= S_0 e^{(\alpha-\delta)t} \Pr\left(Z < \frac{\ln K - \ln S_0 - (\alpha - \delta + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)$$

$$= S_0 e^{(\alpha-\delta)t} \Pr\left(Z < -\frac{\ln(S_0/K) + (\alpha-\delta+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)$$

$$= S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1).$$