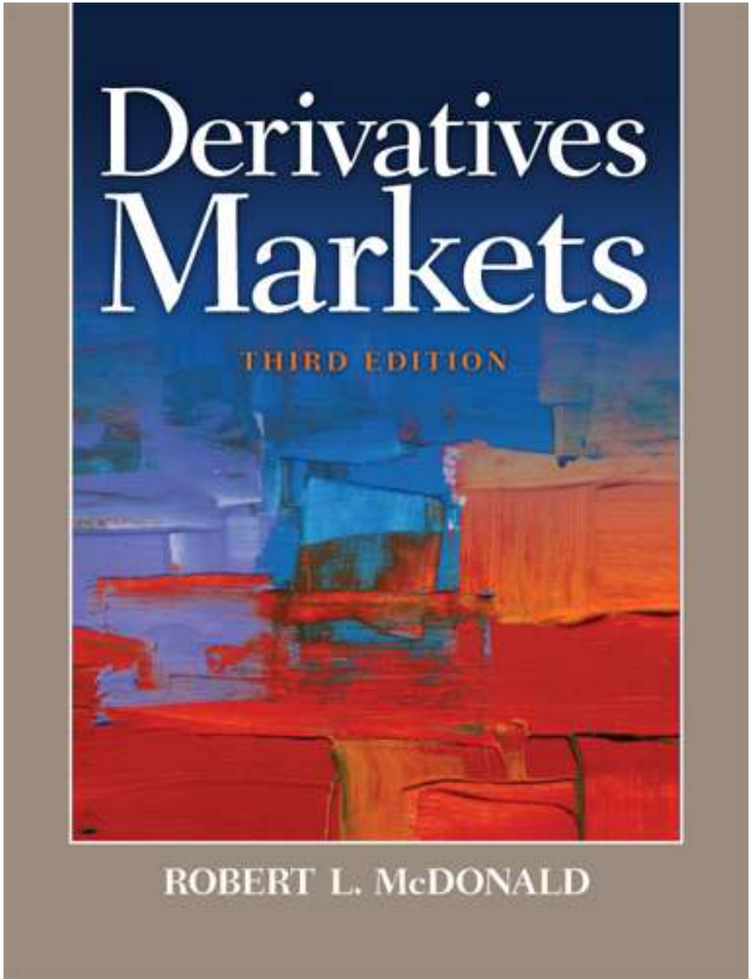

Chapter 13
(Chapter 20 in the textbook)

Brownian Motion and Itô’s Lemma

Points to Note

1. Definition of the Standard Brownian motion. See P.3 – 4.
2. Stochastic Processes which are constructed from the standard Brownian motion. See P.5 – 11.
3. Modelling the correlated asset prices through correlated Brownian motions. See P.12 – 16.
4. Ito’s lemma: univariate and multivariate versions. See P.17 – 27.
5. Sharpe ratios of two perfectly correlated assets. See P.28 – 30.

BM $\longrightarrow$ Stochastic processes

SDE

Geometric Brownian motion

$\{X(t) : t \geq 0\}$

$$\frac{dX(t)}{X(t)} = \alpha \, dt + \sigma \, dZ(t) \quad \text{— } (1)$$

where $Z(t)$ is a standard BM $\quad Z(t) \sim N(0, t)$

- assumption in B-S
- $X(t)$ follows lognormal.
- By Ito's lemma, solution of (1)

$$X(t) = X(0) \exp\left((\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t) \right)$$

$$\ln\left(\frac{X(t)}{X(0)}\right) = (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma Z(t), \quad Z(t) \sim N(0, t)$$

$$= (\alpha - \tfrac{1}{2}\sigma^2)t + \sigma W \sqrt{t} \quad \text{where } W \sim N(0,1)$$

$$\frac{X(t)}{X(0)} \sim LN\left((\alpha - \tfrac{1}{2}\sigma^2)t, \; \sigma^2 t \right)$$

or $\ln\left(\frac{X(t)}{X(0)}\right) \sim N\left((\alpha - \tfrac{1}{2}\sigma^2)t, \; \sigma^2 t \right)$

GBM $\Leftrightarrow$ LN

Itô's lemma

* 1-D
* n-D

$C(t, S(t))$

Eg. $F(t, S(t)) = e^{rt}\sqrt{S(t)} = e^{rt}\sqrt{S}$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2} (dS)^2$$

$\because$ by multiplication rules = $\begin{cases} (dt)^2 = 0 \\ dt\, dz = 0 \\ (dz)^2 = dt \end{cases}$

$$= (\quad) dt + (\quad) dz(t)$$

$F(t, S_1(t), S_2(t))$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S_1} dS_1 + \frac{\partial F}{\partial S_2} dS_2$$
$$\frac{1}{2}\frac{\partial^2 F}{\partial S_1^2} (dS_1)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial S_2^2} (dS_2)^2 + \frac{\partial^2 F}{\partial S_1 \partial S_2} dS_1 dS_2$$

$\because$ multiplication rules = $\begin{cases} (dt)^2 = 0 \\ dt\, dz_i(t) = 0 \\ dz_i dz_j(t) = \begin{cases} dt & i = j \\ \rho_{ij}\, dt & i \neq j \end{cases} \end{cases}$

$$= (\quad) dt + (\quad) dz_1(t) + (\quad) dz_2(t)$$

Chapter 14
Martingale Pricing Theory

See Section 3.2 of "Mathematical Models of Financial Derivatives", 2nd edition, by Yue Kuen KWOK, Springer Verlag, 2008.

Points to Note

1. What is the definition of the equivalent martingale measure? See P.3 – 6.
2. What is the relationship between the no-arbitrage price of a financial product and the risk-neutral probability? See P.8 – 13.
3. How do we change a measure in the expectation? Use the notation of the Randon-Nikodym derivative. See P.14 – 15.
4. Girsanov Theorem. See P.16 – 19.
5. Converting the dynamic of the asset price processes from the real probability to the risk-neutral probability. See P.20 – 25.
6. Derivation of the BS formula by using the change of numeraire. See P.26 – 38.
7. The Black-Scholes formula for the dividend-paying asset. See P.39 – 47.

$Q_1 \overset{?}{\Leftrightarrow} Q_2$, Impossible in Q_1 $\Leftrightarrow$ Impossible in Q_2

$Pr(\text{Win}) = \frac{1}{2}$ $\quad\Leftrightarrow\quad$ $Pr(\text{Win}) = \frac{2}{3}$

$Pr(\text{loss}) = \frac{1}{2}$ $\qquad\qquad$ $Pr(\text{loss}) = \frac{1}{3}$

$\not\Updownarrow$

$Pr(\text{Win}) = 1$

$Pr(\text{loss}) = 0$

Binomial Tree

Real prob (p) $\longrightarrow$ risk-neutral prob (p^*)

$$\begin{tikzcd}
 & uS \\
S \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & dS
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & uS \\
S \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\
 & dS
\end{tikzcd}$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

A probability measure Q is said to be an *equivalent martingale measure* (or *risk-neutral measure*) to the real probability measure P if it satisfies

i. Q is equivalent to P; $Q \Leftrightarrow P$

ii. The discounted security price process $S_m^*(t)$, m = 1, 2, …, K, are martingales under Q, that is

$$E_t^Q\left[S_m^*(u)\right] = \boxed{S_m^*(t)} \quad \text{for all } 0 \le t \le u \le T.$$

$$E^Q\left[S_m^*(u) \,\middle|\, \{S_m^*(s) : 0 \le s \le t\}\right] = S_m^*(t)$$

Tutorial - Class Activity

3 Dec, 2019 (Solution)

Problem 1

You are given:

$$\frac{dS_1(t)}{S_1(t)} = \mu_1 dt + \sigma_1 dZ_1(t),$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2 dt + \sigma_2 dZ_2(t),$$

$G(t) = (e^{-rt})(S_1)(S_2)$

"t" S_1 S_2

where μ_1, μ_2, σ_1 and σ_2 are constants, $Z_1(t)$ and $Z_2(t)$ are correlated Brownian motions with $dZ_1(t)dZ_2(t) = \rho dt$.

Let $G(t) = e^{-rt}S_1(t)S_2(t)$.

Solve the SDE of $G(t)$.

Find $G(t) = ??$

Find the stochastic differential equation (SDE) of $G(t)$.

Solution

The expression for $G(t)$ is $e^{-rt}S_1(t)S_2(t)$.

The partial derivatives are:

$$\boxed{\begin{aligned} &G_{S_1} = S_2(t)e^{-rt}, \quad G_{S_2} = S_1(t)e^{-rt}, \quad G_t = -rS_1(t)S_2(t)e^{-rt}, \\ &G_{S_1S_2} = e^{-rt}, G_{S_1S_1} = G_{S_2S_2} = 0. \end{aligned}}$$

From Itô's lemma, we have:

$$\begin{aligned} dG(t) &= G_{S_1}dS_1 + G_{S_2}dS_2 + \frac{1}{2}\left(G_{S_1S_1}(dS_1)^2 + 2G_{S_1S_2}(dS_1)(dS_2) + G_{S_2S_2}(dS_2)^2\right) + G_t dt \\ &= S_2(t)e^{-rt}S_1(t)\left(\mu_1 dt + \sigma_1 dZ_1(t)\right) + S_1(t)e^{-rt}S_2(t)\left(\mu_2 dt + \sigma_2 dZ_1(t)\right) + \\ &\quad e^{-rt}S_1(t)\left(\mu_1 dt + \sigma_1 dZ_1(t)\right)S_2(t)\left(\mu_2 dt + \sigma_2 dZ_1(t)\right) - rS_1(t)S_2(t)e^{-rt}dt \\ &= G(t)\left[\left(\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2\right)dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_1(t)\right]. \Rightarrow \underline{\text{GBM}} \end{aligned}$$

$=0$

$=0$

Let $H(t) = \ln G(t)$

$$dH(t) = \frac{\partial H}{\partial G} dG(t) + \frac{1}{2}\frac{\partial^2 H}{\partial G^2}(dG(t))^2$$

$$= [\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2]dt + \sigma_1 dz_1(t) + \sigma_2 dz_2(t)$$

$$- \frac{1}{2}[(\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2)dt + \sigma_1 dz_1(t) + \sigma_2 dz_2(t)]^2$$

$$= (\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2)dt + \sigma_1 dz_1(t) + \sigma_2 dz_2(t)$$

$$- \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2^2) dt$$

$$= (\mu_1 + \mu_2 - r - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2) dt + \sigma_1 dz_1(t)$$

$$+ \sigma_2 dz_2(t)$$

$$G(t) = G(0) \exp((\mu_1 + \mu_2 - r - \frac{1}{2}\sigma_1^2 - \frac{1}{2}\sigma_2^2)t + \sigma_1 z_1(t) + \sigma_2 z_2(t))$$

Problem 2

Consider two non-dividend-paying assets X and Y, whose prices are driven by the same standard Brownian motion $Z(t)$. You are given that the assets X and Y satisfy the stochastic differential equations:

$$\frac{dX(t)}{X(t)} = 0.09dt + 0.16dZ(t),$$

$$\frac{dY(t)}{Y(t)} = Gdt + HdZ(t),$$

where G and H are constants.

You are also given:

(i) $d\ln\left[Y(t)\right] = 0.07dt + \sigma dZ(t)$.

$d\ln Y(t) = \overset{0.07}{\overset{\parallel}{\boxed{\left(G - \tfrac{1}{2}H^2\right)}}}dt + \overset{\sigma}{\overset{\uparrow}{H}}dZ(t)$

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) $\sigma < 0.3$.

Determine the values of G and H.

Solution

By Itô's Lemma, we have

$$d\ln\left[Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ(t).$$

The arithmetic Brownian motion provided in (i) for $d\ln[Y(t)]$ allows us to find an expression for H and G:

$$d\ln\left[Y(t)\right] = 0.07dt + \sigma dZ(t) \quad \text{and} \quad d\ln\left[Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ(t)$$

$$\Rightarrow \quad G - 0.5H^2 = 0.07 \quad \text{and} \quad H = \sigma.$$

Since X and Y have the same source of randomness, $dZ(t)$, they must have the same Sharpe ratio:

Sharpe X $\quad \dfrac{0.09 - 0.05}{0.16} = \dfrac{G - 0.05}{H} \quad$ Sharpe Y

$$0.25 = \frac{G - 0.05}{H}$$

$$0.25 = \frac{0.07 + 0.5H^2 - 0.05}{H}$$

$$0.25H = 0.02 + 0.5H^2$$

$$0.5H^2 - 0.25H + 0.02 = 0$$

We use the quadratic formula to solve for H:

$$H = \frac{0.25 \pm \sqrt{(-0.25)^2 - 4(0.5)(0.02)}}{2(0.5)} = 0.1 \quad \text{or} \quad 0.4.$$

Since we are given that $\sigma < 0.3$ and we know that $H = \sigma$, it must be the case that

$$H = 0.1.$$

We can now find the value of G:

$$0.25 = \frac{G - 0.05}{H}$$

$$0.25 = \frac{G - 0.05}{0.1}$$

$$G = 0.075.$$

$$S^*(1) = \frac{S(1)}{M(1)}$$

$$E^Q[S^*(1) \mid S(0)] = S^*(0) = \frac{S(0)}{M(0)}$$

Q is the risk-neutral prob.

Problem 3

Consider a securities model with the money market account $M(t)$ and a risky asset $S(t)$. Suppose that $M(0) = 1$ and $S(0) = 3$. At $t = 1$, $M(1) = 1.5$ and under the real probability $S(1)$ has three possible values which are are given by the following vector

$$S(1) = \begin{pmatrix} 6 \\ 4.5 \\ 3 \end{pmatrix}.$$

Trinomial tree

$S^*(1)$ to be a martingale

Determine the risk-neutral probability of this security model. Is this risk-neutral probability unique?

Solution

Because the risk-neutral probability is equivalent to the real probability. So, there are three possible states of $S(1)$ under the risk-neutral probability. Let q_1, q_2 and q_3 be the risk-neutral probabilities for $S(1) = 6$, 4.5 and 3 respectively.

By the definition of the risk-neutral probability, we have

$$E^Q\left[\frac{S(1)}{M(1)}\right] = \frac{S(0)}{M(0)} = S(0).$$

So,

$$q_1 \frac{6}{1.5} + q_2 \frac{4.5}{1.5} + q_3 \frac{3}{1.5} = 3,$$

$$q_1 + q_2 + q_3 = 1.$$

Let Q be the risk-neutral prob.

① $Q \Leftrightarrow P$ (real prob)

$q_1, q_2, q_3 \Rightarrow q_1 + q_2 + q_3 = 1$

② $E^Q[S^*(1) \mid S^*(0)] = S^*(0)$ martingale property of Q

$$\Rightarrow q_1 \frac{6}{M(1)} + q_2 \frac{4.5}{M(1)} + q_3 \frac{3}{M(1)}$$

$$= \frac{S(0)}{M(0)} = \frac{3}{1} = 3$$

Let $q_1 = \lambda$, $\Rightarrow q_2 = 1 - 2\lambda$, $q_3 = \lambda$

$$\boxed{0 < \lambda < \frac{1}{2}}$$

Binomial tree $\quad p^* = \dfrac{e^{rh} - d}{u - d}$

Derive p^* from the definition of risk-neutral prob.

$$\begin{tikzcd} & uS \\ S \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\ & dS \end{tikzcd}$$

$$E^Q[S^*(1)] = S^*(0)$$

$$\boxed{p^* \frac{uS}{e^{rh}} + (1-p^*) \frac{dS}{e^{rh}} = S}$$

or

$$\begin{aligned} 4q_1 + 3q_2 + 2q_3 &= 3, \\ q_1 + q_2 + q_3 &= 1. \end{aligned}$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $q_1 = \lambda$, $q_2 = 1 - 2\lambda$, $q_3 = \lambda$, where λ is a free parameter. In order that all q_i, $i = 1, 2, 3$, are all strictly positive. We must have $0 < \lambda < 1/2$.