Tutorial - Class Activity (Solution)

19 November, 2019

Problem 1

You use the following information to construct a binomial forward tree for modeling the price movements of a stock:

(i) The length of each period is 1 year.

(ii) The current stock price is 190.

(iii) The stock's volatility is 30%.

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(v) The continuously compounded risk-free interest rate is 3%.

Calculate the price of a 3-year 150-250 European strangle consisting of purchasing of one 150-strike put and one 250-strike call.

Solution

In a forward tree,

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.03-0.05)1+0.3\sqrt{1}} = 1.3231,$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.03-0.05)1-0.3\sqrt{1}} = 0.7261.$$

The risk-neutral probability of an up move is

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{(0.03-0.05)1}-0.7261}{1.3231-0.7261} = 0.4256.$$

Then the four possible 1-year stock prices are

$$S_{uuu} = 440.08, \quad S_{uud} = 241.5101, \quad S_{udd} = 132.5376, \quad S_{ddd} = 72.7349$$

and the corresponding payoffs of the 150-250 strangle (being max(S(3) – 250, 0) + max(150 – S(3), 0)) are

$$V_{uuu} = 190.08, \quad V_{uud} = 0, \quad V_{udd} = 17.4624, \quad V_{ddd} = 77.2651.$$

By the risk-neutral pricing, the current price of the strangle is

$$V_0 = e^{-rT}\left[\left(p^*\right)^3 V_{uuu} + 3\left(p^*\right)^2\left(1-p^*\right)V_{uud} + 3p^*\left(1-p^*\right)^2 V_{udd} + \left(1-p^*\right)^3 V_{ddd}\right]$$
$$= e^{-(0.03)(3)}[(0.4256)^3(190.08) + 3(0.4256)(1-0.4256)^2(17.4624)$$
$$+(1-0.4256)^3(77.2651)]$$
$$= 33.498.$$

Problem 2

For a two-period binomial model for stock prices, you are given

(i) The length of each period is 1 year.
(ii) The current price of a non-dividend paying stock is \$40.
(iii) $u = 1.05$ and $d = 0.9$.
(iv) The continuously compounded risk-free interest rate is 3%.

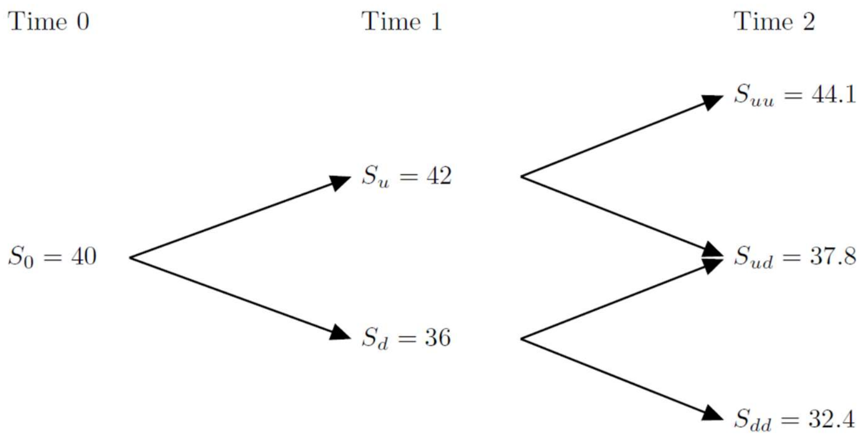
Consider Derivative X, which gives its holder the right, but not the obligation, to buy a \$38-strike European put option at the end of the first year for \$0.5. This put option is written on the stock and will mature at the end of the second year.

(a) Calculate the current price of Derivative X.
(b) Using the result of part (a), calculate the current price of Derivative Y, which is identical to Derivative X, except that it gives its holder the right to sell the same put option for \$0.5 at the end of the first year.

Solution

(a)

The evolution of the stock price is given as follows:



The risk-neutral probability of an up move is

$$p^* = \frac{e^{0.03} - 0.9}{1.05 - 0.9} = 0.8697.$$

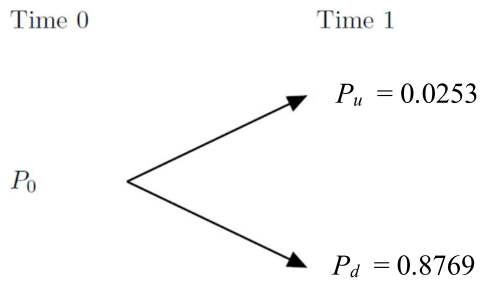
The possible time-2 payoffs of the 2-year put option are:

$$P_{uu} = \max(38 - 44.1, 0) = 0, \quad P_{ud} = \max(38 - 37.8, 0) = 0.2, \quad P_{dd} = \max(38 - 32.4, 0) = 5.6.$$

By risk-neutral pricing, the possible time-1 values of this put option are

$$P_u = e^{-0.03}\left(1-p^*\right)\left(0.2\right) = 0.0253,$$
$$P_d = e^{-0.03}\left[p^*\left(0.2\right)+\left(1-p^*\right)\left(5.6\right)\right] = 0.8769.$$

Here is the evolution of the put price:



As a call on the above put, Derivative X pays off only at the *d* note with a non-zero payoff of

$$V_d^X = \max\left(P_d - 0.5, 0\right) = 0.3769.$$

By risk-neutral pricing again, the time-0 price of Derivative X is

$$V_0^X = e^{-0.03}\left(1-p^*\right)V_d^X = 0.0477.$$

(b)

Derivative Y is a 1-year European put option on the put option which matures 2 years from now. Derivative X and Y therefore form a call-put pair. To apply put-call parity, we need the time-0 price of the underlying asset, which is the 2-year put option:

$$P_0 = e^{-0.03}\left[0.0253p^* + 0.8769\left(1-p^*\right)\right] = 0.1322.$$

By put-call parity for non-dividend paying underlying assets (the 2-year put in this case), we have

$$\begin{aligned} V_0^X - V_0^Y &= P_0 - 0.5e^{-r} \\ 0.0477 - V_0^Y &= 0.1322 - 0.5e^{-0.03} \\ V_0^Y &= 0.4007. \end{aligned}$$

Problem 3

The price of a stock is \$72. The dividend yield of the stock is 3%. The price evolution of the stock follows the forward tree with each period being 1 year in length. The volatility of the stock is 23%. The continuously compounded risk-free interest rate is 8%.

Consider a 2-year American option with the exercised payoff

$$\max\left(9.1 - \sqrt{S(t)}, 0\right),$$

where $S(t)$ is the price of the stock at time t for $0 \le t \le 2$.

Calculate the current price of the American option.

Solution

Under the binomial (forward) tree, the values of u and d are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.03)1+0.23\sqrt{1}} = 1.3231.$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0.03)1-0.23\sqrt{1}} = 0.8353.$$

The risk-neutral probability of an upward movement is:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.08-0.03)(1)} - 0.8353}{1.3231 - 0.8353} = 0.4427.$$

	node uu	node $ud = du$	node dd
S	126.0427	79.5734	50.2363
Option value = $\max\left(9.1-\sqrt{S},0\right)$	0	0.1796	2.0122

	$t = 0$	node u	node d
S	72	95.2632	60.1416
Continuation value (CV)	0.7296	0.0924	1.1086
Value of early exercise = $\max\left(9.1-\sqrt{S},0\right)$ (EE)	0.6147	0	1.3449
Option value = max(CV, EE)	0.7296	0.0924	**1.3449**

If the stock price initially moves down, then the resulting option price is \$1.3449. This price is in bold type above to indicate that it is optimal to exercise early at this node:

$$\text{Early exercise value} = 9.1 - \sqrt{60.1416} = 1.3449.$$

The exercise value of 1.3449 is greater than the value of hold the option, which is:

$$\text{Continuation value} = e^{-0.08(1)}\left[(0.4427)(0.1796) + (1-0.4427)(2.0122)\right] = 1.1086.$$

The current continuation value of the American option is:

$$e^{-0.08(1)}\left[(0.4427)(0.0924) + (1-0.4427)(1.3449)\right] = 0.7296.$$

The current early exercise value of the American option is:

$$9.1 - \sqrt{72} = 0.6147.$$

Therefore, the current value of the American option is 0.7296.