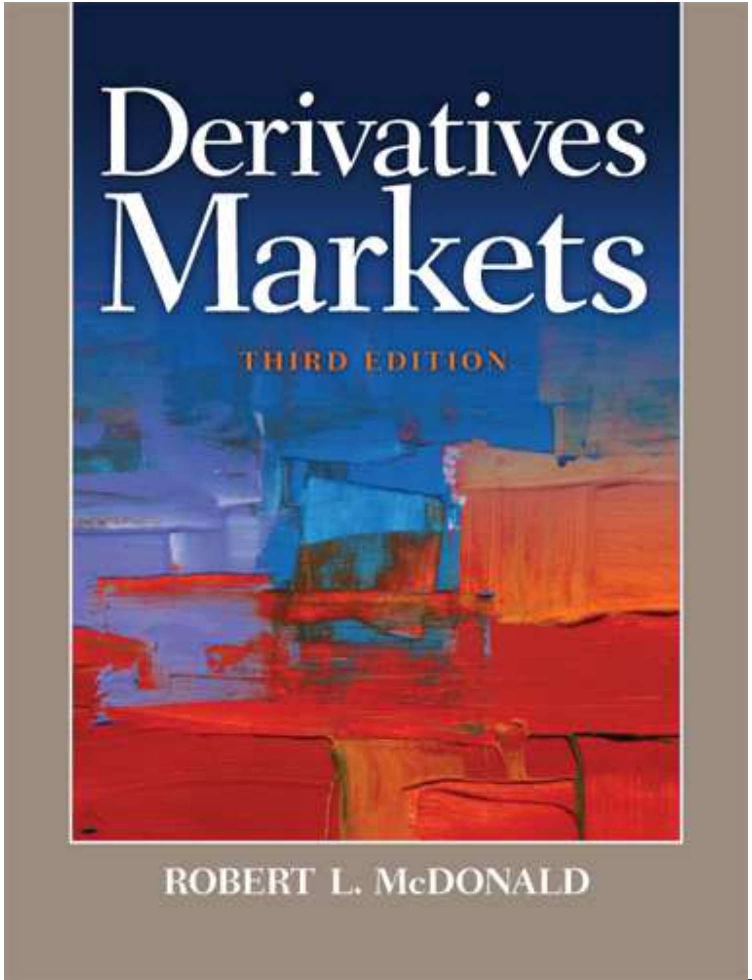


Chapter 11
(Chapter 13 in the textbook)

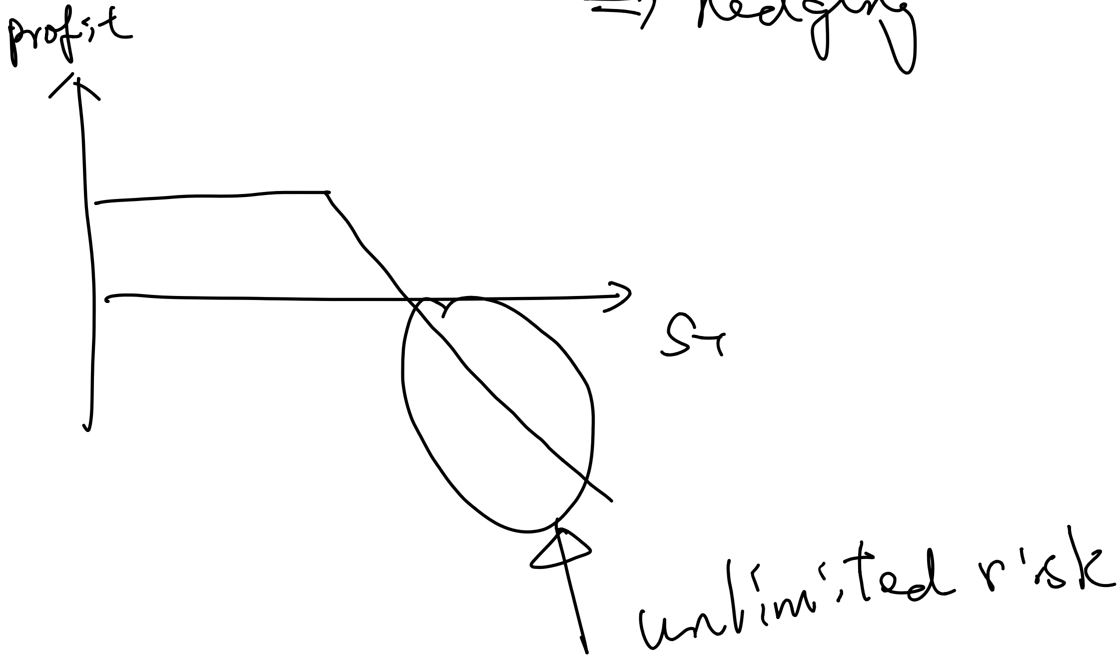
Market-Making and Delta-Hedging

Call option

<u>Short</u>

market - marker

$\Rightarrow$ hedging



Points to Note

1. The delta-gamma approximation of the option price. See P.10 – 11.
2. How does the delta-hedging work? See P.12 – 19.
3. The relationship between the delta hedging and the Greek letters. See P.18 – 24.
4. The definition of the "Greek" neutral portfolio. See P.26.
5. Construction of the "Greek" neutral portfolio. See P.26 – 31.
6. Determine Greek for the binomial tree. See P.32 – 34.

Delta - gamma approx.

(Taylor Expansion)

$V(S, T)$

$S \longrightarrow S + \varepsilon$

$$V(S+\varepsilon, T) \approx V(S, \varepsilon) + \underbrace{\underbrace{\frac{\partial V}{\partial S}}_{\Delta}\varepsilon + \frac{1}{2}\underbrace{\boxed{\frac{\partial^2 V}{\partial S^2}}}_{\Gamma}\varepsilon^2}_{\text{“}\Delta-\Gamma\text{ approximation”}}$$

Accuracy of $\Delta - \Gamma$ approximation.

What Do Market Makers Do?

- Provide immediacy by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price).
- Generate inventory as needed by short-selling.
- Profit by charging the bid-ask spread.

What Do Market Makers Do? (cont'd)

- The position of a market-maker is the result of whatever order flow arrives from customers.
- Proprietary trading, which is conceptually distinct from market-making, is trading to express an investment strategy. Proprietary traders typically expect their positions to be profitable depending upon whether the market goes up or down.

Market-Maker Risk

- Market makers attempt to hedge in order to avoid the risk from their arbitrary positions due to customer orders.
- Market-makers can control risk by *delta-hedging*. The market-maker computes the option delta and takes an offsetting position in shares. We say that such a position is *delta-hedged*.
- In general a delta-hedged position is not a zero-value position: The cost of the shares required to hedge is not the same as the cost of the options. Because of the cost difference, the market-maker must invest capital to maintain a delta-hedged position.

Market-Maker Risk (cont'd)

- Delta-hedged positions should expect to earn risk-free return.
- If a customer wishes to buy a 91-day call option, the market-maker fills this order by selling a call option. To be specific, see Table 13.1.
  - Because delta is negative, the risk of the market-maker who has written a call is that the stock price will **rise**.
  - The figure (just after Table 13.1) graphs the overnight profit of the *unhedged* written call option as a function of the stock price, against the profit of the option at expiration.

Market-Maker Risk (cont'd)

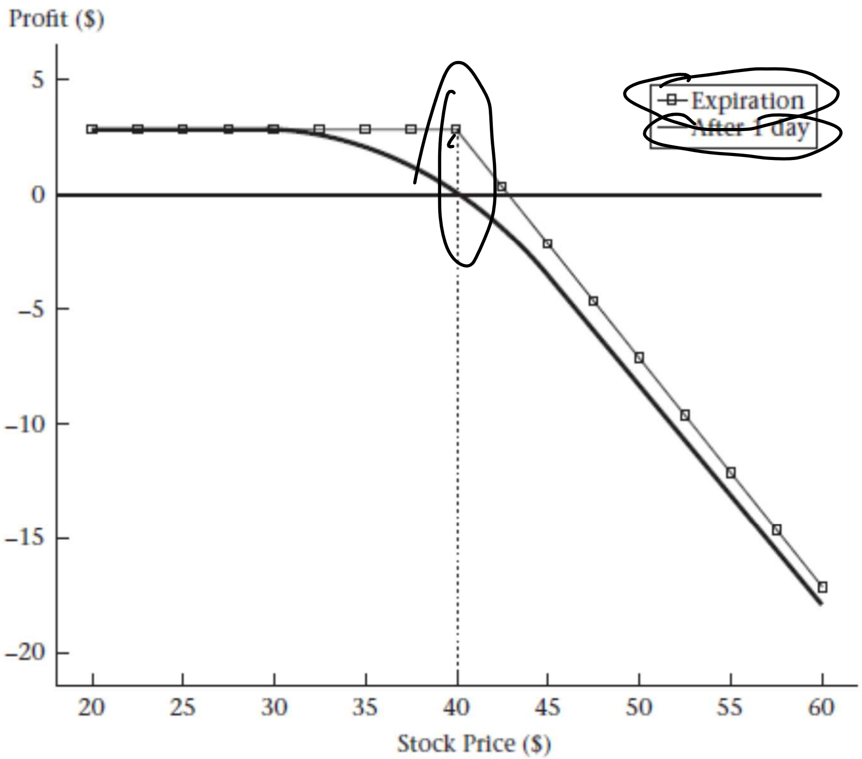
TABLE 13.1 Price and Greek information for a call option with $S = \$40$, $K = \$40$, $\sigma = 0.30$, $r = 0.08$ (continuously compounded), $T - t = 91/365$, and $\delta = 0$.

	Purchased	Written
Call price	2.7804	−2.7804
Delta	0.5824	−0.5824
Gamma	0.0652	−0.0652
Theta	−0.0173	0.0173

Market-Maker Risk (cont'd)

FIGURE 13.1

Depiction of overnight and expiration profit from writing a call option on one share of stock, if the market-maker is unhedged.



Market-Maker Risk (cont'd)

- Delta (Δ) and gamma (Γ) as measures of exposure
  - Suppose Δ is 0.5824, when S = \$40 (Table 13.1 and Figure 13.1).
  - A \$0.75 increase in stock price would be expected to increase option price by \$0.4368 (= \$0.75 x 0.5824).
  - The actual increase in the option's value is higher: \$0.4548.
  - This discrepancy occurs because Δ increases as stock price increases. Using the smaller Δ at the lower stock price ***understates*** the actual change.
  - Similarly, using the original Δ ***overstates*** the change in the option value as a response to a stock price decline.
  - Using Γ in addition to Δ improves the approximation of the option value change.

Market-Maker Risk (cont'd)

- Δ–Γ approximations
  - Using the Δ–Γ approximation the accuracy can be improved a lot

$$C(S_{t+h}) = C(S_t) + \varepsilon\Delta(S_t) + \frac{1}{2}\varepsilon^2\Gamma(S_t)$$

  - Example 13.1: *S*: \$40 ➡ \$40.75, *C*: \$2.7804 ➡ \$3.2352, Γ: 0.0652
    - Using Δ approximation
      C(\$40.75) = *C*(\$40) + 0.75 x 0.5824 = \$3.2172
    - Using Δ–Γ approximation
      C(\$40.75) = *C*(\$40) + 0.75 x 0.5824 + 0.5 x 0.75^2 x 0.0652 = \$3.2355
    - Similarly, for a stock price decline to \$39.25, the true option price is \$2.3622. The Δ approximation gives \$2.3436, and the Δ–Γ approximation gives \$2.3619.

Market-Maker Risk (cont'd)

- Δ–Γ approximation (cont'd)

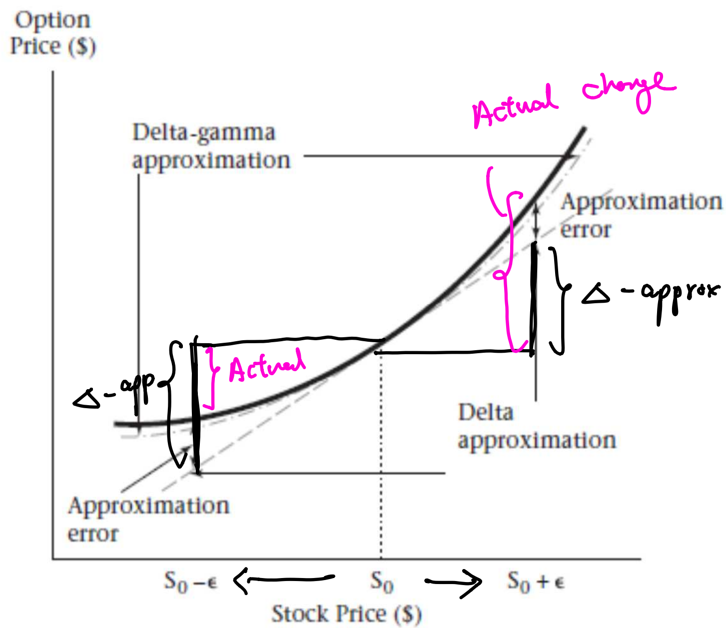


FIGURE 13.3

Delta- and delta-gamma approximations of option price. The true option price is represented by the bold line, and approximations by dashed lines.

Delta-Hedging

- Delta hedging for 2 days: (daily rebalancing and mark-to-market):

100 units

Consider the 40-strike call option described in Table 13.1, written on 100 shares of stocks.

- Day 0: Share price = \$40, call price is \$2.7804, and $\Delta = 0.5824$
  - Sell call written on 100 shares for \$278.04, and buy 58.24 shares (=100×0.5824).
  - Net investment: (58.24x\$40) – \$278.04 = \$2051.56.
  - At 8%, overnight financing charge is \$0.45 [= \$2051.56x($e^{0.08/365}$-1)].

Δ sell call = -58.24

Buy stock

option selling

Delta-Hedging (cont'd)

– Day 1: If share price = \$40.5, call price is \$3.0621, and Δ = 0.6142

- Overnight profit/loss:

Gain on 58.24 shares	= 58.24 × (\$40.50 – \$40) = \$29.12	Profit
Gain on written call option	= \$278.04 – \$306.21 = –\$28.17	Loss
Interest	= $-(e^{0.08/365} - 1) \times \$2051.56 = -\$0.45$	
Overnight profit	**= 29.12 – 28.17 – 0.45 = \$0.50.**	

- Since delta has increased, we must buy 61.42 – 58.24 = 3.18 additional shares. This transaction requires an investment of \$40.5 × 3.18 = \$128.79.

– Day 2: If share price = \$39.25, call price is \$2.3282.

- Overnight profit/loss: – \$76.78 + $\boxed{\$73.39}$ – \$0.48 = – \$3.87.

Stock

option = −100 (2.3282−3.0621)

= (58.24 + 3.18)(39.25 − 40.5)

Interest

$$= -(\underbrace{2051.56}_{\text{Day 0}} + \underbrace{128.79}_{\text{Day 1}})(e^{\frac{8\%}{365}} - 1)$$

$$= -0.48$$

Delta-Hedging (cont'd)

- Delta hedging for several days

TABLE 13.2 Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares.

	Day 0	1	2	3	4	5
Stock ($)	40.00	40.50	39.25	38.75	40.00	40.00
Call ($)	278.04	306.21	232.82	205.46	271.04	269.27
100 × delta	58.24	61.42	53.11	49.56	58.06	58.01
Investment ($)	2051.58	2181.30	1851.65	1715.12	2051.35	2051.29
Interest ($)		−0.45	−0.48	−0.41	−0.38	−0.45
Capital gain ($)		0.95	−3.39	0.81	−3.62	1.77
Daily profit ($)		0.50	−3.87	0.40	−4.00	1.32

Delta-Hedging (cont'd)

- Let Δ_i denote the option delta on day i, S_i the stock price, C_i the option price, and MV_i the market value of the portfolio.
- Borrowing capacity on day i is $MV_i = \Delta_i S_i - C_i$.
- The result of the previous example can be *generalized* to

$$
\begin{aligned}
&\text{Net cash flow of from day } i-1 \text{ to day } i \\
&= \Delta_{i-1}\left(S_i - S_{i-1}\right) + \left(C_{i-1} - C_i\right) - \left(e^{rh} - 1\right)\mathrm{MV}_{i-1} \\
&= \Delta_i S_i - C_i - \left(\Delta_{i-1} S_{i-1} - C_{i-1}\right) - S_i\left(\Delta_i - \Delta_{i-1}\right) - \left(e^{rh} - 1\right)\mathrm{MV}_{i-1} \\
&= \boxed{\mathrm{MV}_i - \mathrm{MV}_{i-1}} - S_i\left(\Delta_i - \Delta_{i-1}\right) - \left(e^{rh} - 1\right)\mathrm{MV}_{i-1}.
\end{aligned}
$$

interest

Change of market value of hedged portfolio

Change of stock position

Delta-Hedging (cont'd)

- Hence, as time passes, there are three sources of cash flow into and out of the portfolio:
  - **Borrowing**: Our borrowing capacity equals the market value of securities in the portfolio; hence, borrowing capacity changes as the net value of the position changes.
  - **Purchase or sale of shares**: We buy or sell shares as necessary to maintain delta-neutrality.
  - **Interest**: We pay interest on the borrowed amount.

Delta-Hedging (cont'd)

– In our last scenario, we have

$$\mathrm{MV}_1 - \mathrm{MV}_0 - S_1\left(\Delta_1 - \Delta_0\right) - rh\mathrm{MV}_0$$
$$= \$2181.3 - \$2051.56 - \$128.79 - \$0.45 = \$0.50$$

This value is equal to the overnight profit we calculated between day 0 and day 1.

Delta-Hedging (cont'd)

- Delta hedging for several days (cont.)

  Γ: For the largest moves in the stock price, the market-maker loses money. For small moves in the stock price, the market-maker makes money. The loss for large moves results from Γ:

  - As the stock prices rises, the delta of the call increases and the (shorting) call loses money *faster than* the stock makes money.
  - As the stock price falls, the delta of the call decreases and the (shorting) call makes money more *slowly than* the fixed stock position loses money.

  In effect, the market-maker becomes unhedged net long as the stock price falls and unhedged net short as the stock prices rises. The losses on days 2 and 4 are attributable to Γ.

Delta-Hedging (cont'd)

- Delta hedging for several days (cont.)

  θ: If a day passes with no change in the stock price, the option becomes cheaper. This time decay works to the benefit of the market-maker who could unwind the position more cheaply. Time decay is especially evident in the profit on day 5, but is also responsible for the profit on days 1 and 3.

  - Interest cost: In order to hedge, the market-maker must purchase stock. The net carrying cost is a component of the overall cost.

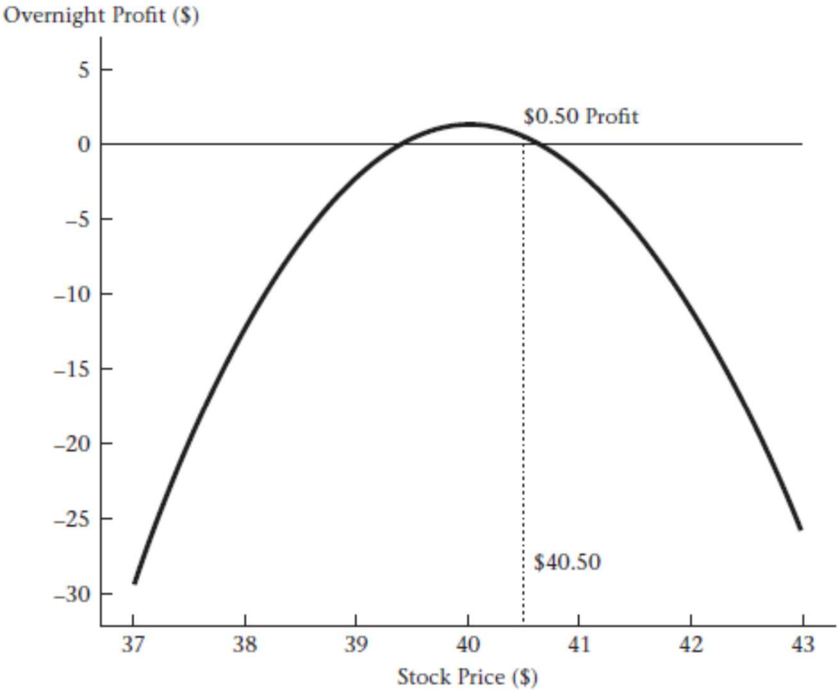
Delta-Hedging (cont'd)

- ✓ The figure on the next page shows that overnight market-maker profit on day 1 as a function of the stock price on day 1.
- ✓ The graph verifies that the delta-hedging market-maker who has written a call wants small stock price moves and can suffer substantial loss with a big move.

Delta-Hedging (cont'd)

FIGURE 13.2

Overnight profit as a function of the stock price for a delta-hedged market-maker who has written a call.



Mathematics of Δ-Hedging

$f(x+h, y+k) = f(x,y) + \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} h^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} k^2 + \frac{\partial^2 f}{\partial x \partial y} hk$

- θ: Accounting for time

$C(S_{t+h}, T-t-h)$ Taylor series at $(S_t, T-t)$

$$= C(S_t, T-t) + \varepsilon\Delta(S_t, T-t) + \frac{1}{2}\varepsilon^2\Gamma(S_t, T-t) + h\theta(S_t, T-t)$$

where $\varepsilon = S_{t+h} - S_t$.

TABLE 13.4 Predicted option price over a period of 1 day, assuming stock price move of \$0.75, using equation (13.6). Assumes that $\sigma = 0.3$, $r = 0.08$, $T - t = 91$ days, and $\delta = 0$, and the initial stock price is \$40.

	Starting Price	$\epsilon\Delta$	$\frac{1}{2}\epsilon^2\Gamma$	θh	Option Price 1 Day Later ($h = 1$ day) Predicted	Actual
$S_{t+h} = \$40.75$	\$2.7804	0.4368	0.0183	−0.0173	\$3.2182	\$3.2176
$S_{t+h} = \$39.25$	\$2.7804	−0.4368	0.0183	−0.0173	\$2.3446	\$2.3452

Mathematics of Δ-Hedging (cont'd)

- Market-maker's profit when the stock price changes by ε over an interval *h*:

$$\overbrace{\Delta(S_{t+h} - S_t)}^{\text{Change in value of stock}} - \overbrace{\left[\Delta(S_{t+h} - S_t) + \frac{1}{2}(S_{t+h} - S_t)^2\Gamma + \theta h\right]}^{\text{Change in value of option}} - \overbrace{rh[\Delta S_t - C(S_t)]}^{\text{Interest expense}}$$

$$= -\left(\underbrace{\frac{1}{2}\varepsilon^2\Gamma}_{\text{The effect of }\Gamma} + \underbrace{\theta h}_{\text{The effect of }\theta} + \underbrace{rh[\Delta S_t - C(S_t)]}_{\text{Interest cost}}\right)$$

Mathematics of Δ-Hedging (cont'd)

- Note that Δ, Γ and θ are computed at t.
- For simplicity, the subscript "t" is omitted in the above equation.
- Since θ is negative, time decay benefits the market-maker, whereas interest and gamma work against the market-maker.

Construction of the Delta and Gamma Neutral Portfolio

TABLE 13.6 Prices and Greeks for 40-strike call, 45-strike call, and the (gamma-neutral) portfolio resulting from selling the 40-strike call for which $T - t = 0.25$ and buying 1.2408 45-strike calls for which $T - t = 0.33$. By buying 17.49 shares, the market-maker can be both delta- and gamma-neutral. Assumes $S = \$40$, $\sigma = 0.3$, $r = 0.08$, and $\delta = 0$.

	40-Strike Call	45-Strike Call	Sell 40-Strike Call, Buy 1.2408 45-Strike Calls
Price ($)	2.7847	1.3584	−1.0993
Delta	0.5825	0.3285	−0.1749
Gamma	0.0651	0.0524	0.0000
Vega	0.0781	0.0831	0.0250
Theta	−0.0173	−0.0129	0.0013
Rho	0.0513	0.0389	−0.0031

Construction of the Delta and Gamma Neutral Portfolio (cont'd)

A portfolio is said to be **delta neutral** if the delta of the portfolio is 0.

The same definition is applied to other Greeks such as gamma neutral, vega neutral, etc.

Consider the market-maker in our previous example, he would like to delta-gamma hedge his position (selling 100 40-strike call). That is, he needs to construct a **delta and gamma neutral** portfolio.

The gamma of his delta-hedged portfolio

$$\boxed{-100\left(0.0651\right)=-6.51.}$$

Construction of the Delta and Gamma Neutral Portfolio (cont'd)

We need to find the quantity, Q, of the 45-strike call option that the market-maker must be purchased to make the portfolio to be gamma neutral:

$$-6.51 + Q \times \Gamma_{C(45)} = 0$$

0 gamma

$$Q = \frac{6.51}{0.0524}$$

$$= 124.24.$$

Construction of the Delta and Gamma Neutral Portfolio (cont'd)

After we have made the portfolio to be gamma neutral, the delta of the portfolio will be changed. The delta of the gamma neutral portfolio becomes:

$$-100\Delta_{C(40)} + 124.24\Delta_{C(45)}$$

$$= -100(0.5825) + 124.24(0.3285)$$

$$\boxed{= -17.44.}$$

Δ port

Construction of the Delta and Gamma Neutral Portfolio (cont'd)

The quantity of the underlying stock that must be purchased, Q_S, in order to make the gamma neutral portfolio to be delta neutral again is equal to the opposite of the delta of the gamma neutral portfolio:

$$Q_S = 17.44.$$

In summary, for both delta and gamma hedging 100 units of 40-strike call we have sold, we need to

i) buy 124.24 of the 45-strike call option **and**

ii) buy 17.44 shares of stock.

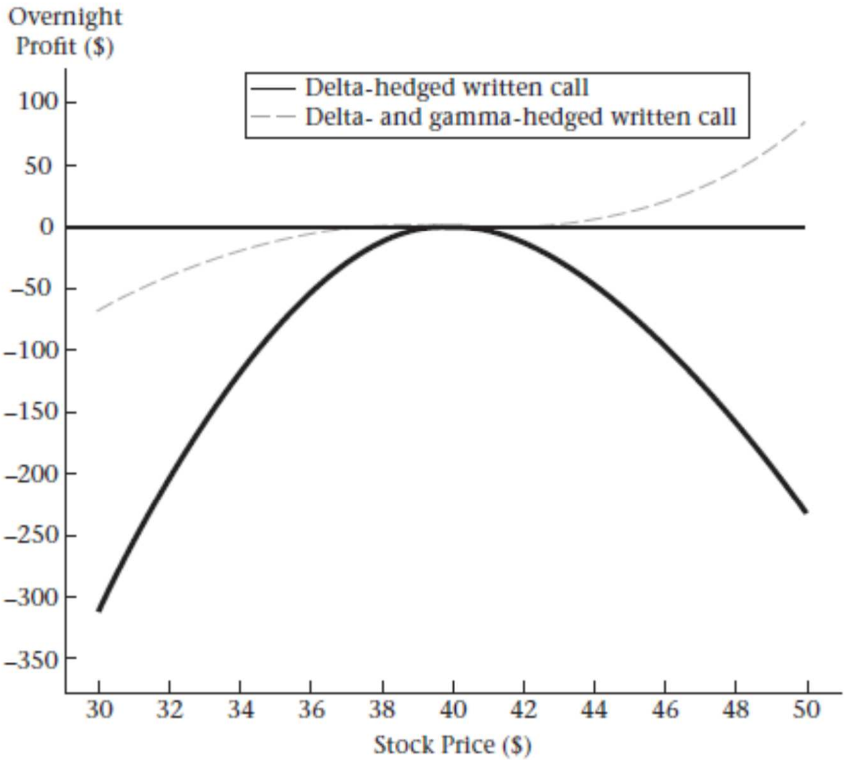
Construction of the Delta and Gamma Neutral Portfolio (cont'd)

- The following figure shows that the delta-hedged position has the problem that large moves of the stock price always cause losses. The delta-gamma-hedged position loses less if there is a large move down, and can make money if the stock price increases.

Construction of the Delta and Gamma Neutral Portfolio (cont'd)

FIGURE 13.4

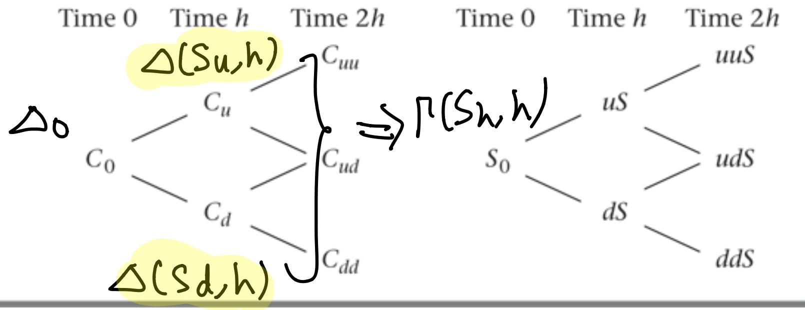
Comparison of 1-day holding period profit for delta-hedged position described in Table 13.2 and delta- and gamma-hedged position described in Table 13.6.



Greeks In The Binomial Model

FIGURE 13.5

Option price and stock price trees, assuming that the stock can move up or down u or d each period.



1) $\Delta(S,0)$ ✓

2) $\Gamma(S,0) = \dfrac{\partial \Delta(S,0)}{\partial S} \approx \dfrac{\Delta(S+\varepsilon,0) - \Delta(S,0)}{\varepsilon}$

$\Gamma(S,0) \approx \Gamma(S_h,h) \approx \dfrac{\Delta(S_u+\varepsilon, h) - \Delta(S_h,h)}{\varepsilon}$

Greeks In The Binomial Model (cont'd)

- *Delta* at the initial node is computed as

$$\Delta(S,0) = e^{-\delta h}\frac{C_u - C_d}{uS - dS}$$

- *Gamma* at time *h* is computed as

$$\Gamma(S_h, h) = \frac{\Delta(uS,h) - \Delta(dS,h)}{uS - dS} \approx \frac{\partial \Delta(S,h)}{\partial S}$$

It is a reasonably well approximation of $\Gamma(S_0, 0)$.

Greeks In The Binomial Model (cont'd)

- Define

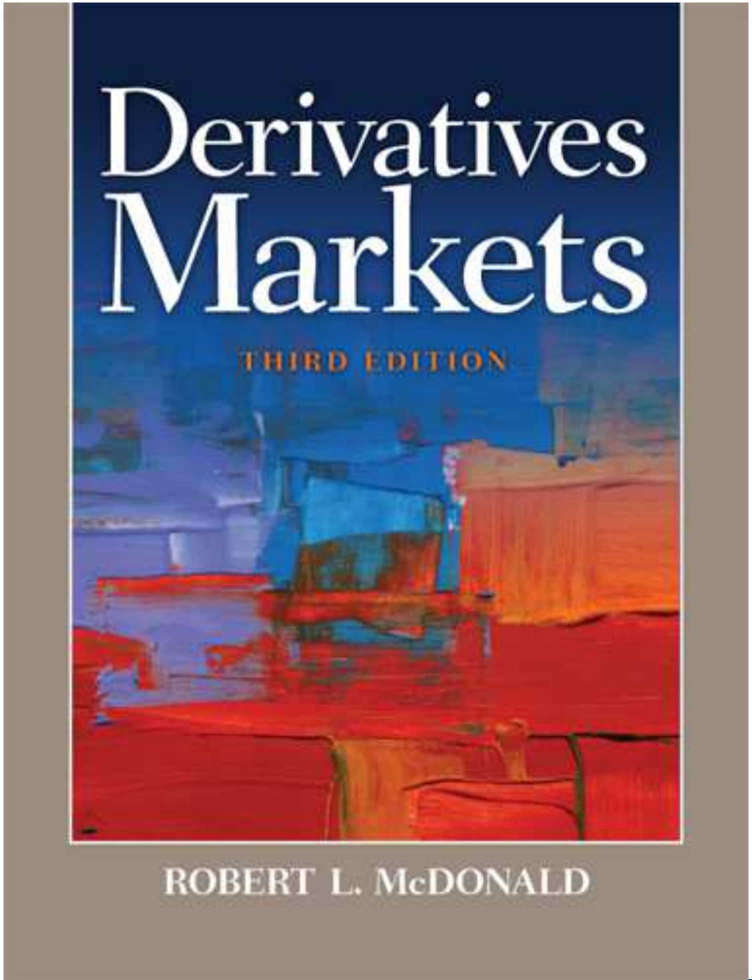
$$\varepsilon = udS - S$$

Taylor series at $(S, 0)$

$\theta(S, 0)$ can be obtained as follows:

$$C(udS, 2h) = \boxed{C(S,0) + \varepsilon\Delta(S,0) + \frac{1}{2}\varepsilon^2\Gamma(S,0) + 2h\theta(S,0)}$$

$$\theta(S,0) = \frac{C(udS,2h) - \varepsilon\Delta(S,0) - \frac{1}{2}\varepsilon^2\Gamma(S,0) - C(S,0)}{2h}$$



Chapter 12
(Chapter 18 in the textbook)

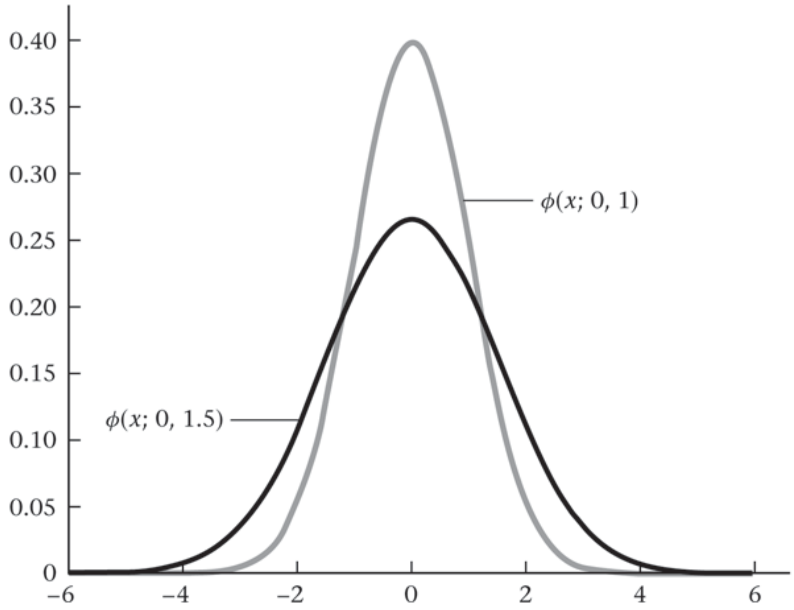
The Lognormal Distribution

Points to Note

1. Definition of the lognormal distribution. See P.10.
2. Properties of lognormal random variables. See P.10.
3. The expectation and variance of the lognormal random variable. See P.11.
4. The lognormal model of stock prices. See P.14 – 15.
5. Some results of the lognormal distribution. See P.17 – 20.
6. Estimating the parameters of a lognormal distribution. See P.21 – 23.

The Normal Distribution

- Normal distribution (or density) $\Phi(x;\mu,\sigma) \equiv \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



The Normal Distribution (cont'd)

- Normal density is symmetric about the mean μ :

$$\Phi(\mu + a; \mu, \sigma) = \Phi(\mu - a; \mu, \sigma)$$

- If a random variable *x* is normally distributed with mean μ and standard deviation, σ then $x \sim N(\mu, \sigma^2)$
- Use *z* to represent a random variable that has a standard normal distribution: $z \sim N(0,1)$

The Normal Distribution (cont'd)

- The value of the cumulative normal distribution function $N(a)$ equals to the probability $P(z<a)$ of a number z drawn from the normal distribution to be less than a.

$$N(a) \equiv \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

The Normal Distribution (cont'd)

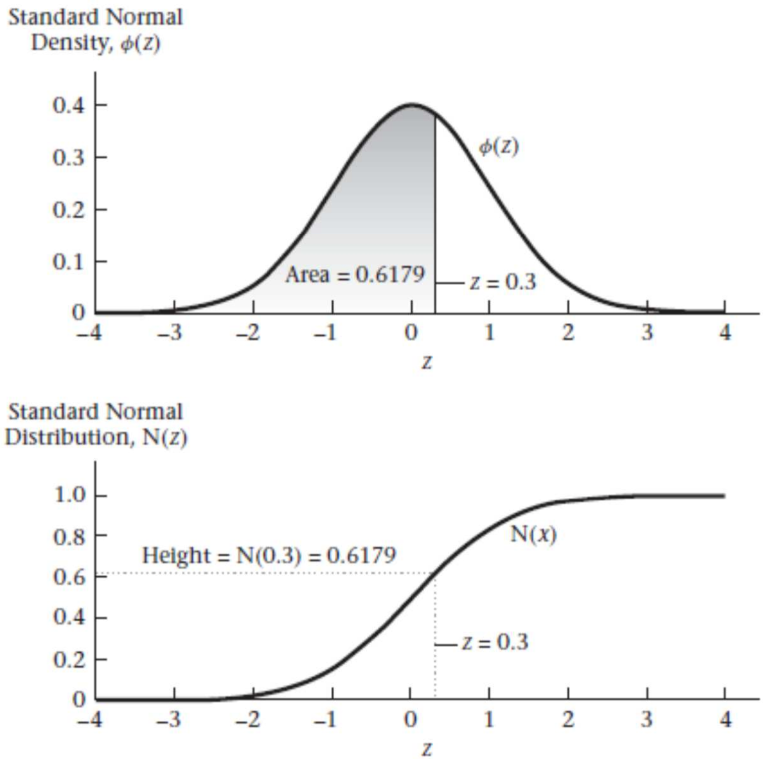


FIGURE 18.2

Top panel: Area under the normal curve to the left of 0.3. *Bottom panel:* Cumulative normal distribution. The height at $x = 0.3$, given by $N(0.3)$, is 0.6179.

The Normal Distribution (cont'd)

- The probability that a number drawn from the standard normal distribution will be between a and $-a$:

$$Prob\ (z < -a) = N(-a)$$

$$Prob\ (z < a) = N(a)$$

therefore

$$Prob\ (-a < z < a) = N(a) - N(-a) =$$

$$N(a) - [1 - N(a)] = 2 \cdot N(a) - 1$$

- Example: $Prob\ (-0.3 < z < 0.3) = 2 \cdot 0.6179 - 1 = 0.2358$.

The Normal Distribution (cont'd)

- Converting a normal random variable to standard normal
  - If $x \sim N(\mu, \sigma^2)$, then $z \sim N(0,1)$ if $z = \dfrac{x-\mu}{\sigma}$
- Converting a standard normal variable to a normal variable
  - If $z \sim N(0,1)$, then $x \sim N(\mu, \sigma^2)$ if $x = \mu + \sigma z$

- Example 18.2: Suppose $x \sim N(3,25)$ and $z \sim N(0,1)$ then

$$\frac{x-3}{5} \sim N(0,1) \quad \text{and} \quad 3 + 5 \times z \sim N(3,25)$$

The Normal Distribution (cont'd)

- The sum of normal random variables is also

$$\sum_{i=1}^{n} \omega_i x_i \sim N\left(\sum_{i=1}^{n} \omega_i \mu_i , \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij} \right)$$

where x_i, $i = 1,\ldots,n$, are n random variables, with mean $E(x_i) = \mu_i$, variance $Var(x_i) = \sigma_i^2$, covariance $Cov(x_i,x_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

$X \sim LN(\mu, \sigma^2) \Rightarrow \ln(X) \sim N(\mu, \sigma^2)$

The Lognormal Distribution

- A random variable x is **lognormally distributed** if $\ln(x)$ is normally distributed
  - If x is normal, and $\ln(y) = x$ (or $y = e^x$), then y is lognormal.
  - If continuously compounded stock *returns* are *normal* then the stock *price* is *lognormally* distributed.
- Product of lognormal variables is lognormal
  - If x_1 and x_2 are normal, then $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$ are lognormal.
  - The product of y_1 and y_2: $y_1 \times y_2 = e^{x_1} \times e^{x_2} = e^{x_1 + x_2}$.
  - Since $x_1 + x_2$ is normal, $e^{x_1 + x_2}$ is lognormal.
  - **Note**: the sum of lognormal variables is NOT lognormal.

$X_1 \sim LN(\mu_1, \sigma_2^2)$, $X_2 \sim LN(\mu_2, \sigma_2^2)$

X_1, X_2 are independent.

$Y = X_1 X_2$

$\ln(Y) = \ln X_1 + \ln X_2$

$\ln(X_1) \sim N(\mu_1, \sigma_1^2)$, $\ln(X_2) \sim N(\mu_2, \sigma_2^2)$

$\Rightarrow \ln(Y) \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$\Rightarrow Y \sim LN(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$V = X_1 + X_2$

V is NOT Lognormal

$X \sim LN(\mu, \sigma^2)$

$E[X] = ?$ $\quad$ $Var(X) = ?$

$E[X] = E[e^Y]$ where $Y \sim N(\mu, \sigma^2)$

$= e^{\mu + \frac{1}{2}\sigma^2}$ (moment generating fct of normal dist)

$Var(X) = E[X^2] - (E[X])^2$

The Lognormal Distribution (cont'd)

- If *ln y* ~ $N(m,v^2)$, the lognormal density function of *y* is

$$g(y;m,v) \equiv \frac{1}{yv\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-m]}{v}\right)^2}$$

- If *x* ~ $N(m,v^2)$, then

$$E(e^x) = e^{m+\frac{1}{2}v^2}$$

$$\mathrm{Var}(e^x) = e^{2m+v^2}\left(e^{v^2}-1\right)$$

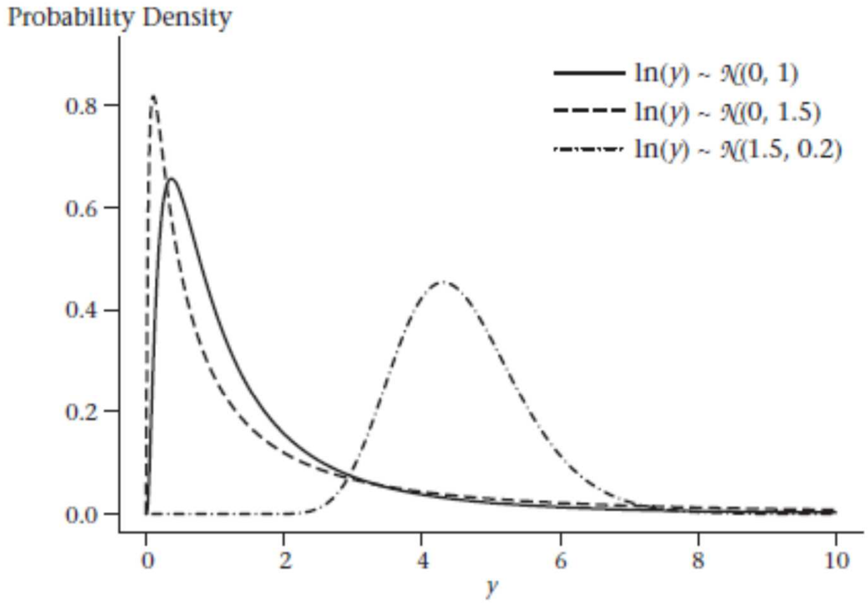
$Y \sim LN(\mu, \sigma^2)$

$X = \ln Y$

The Lognormal Distribution (cont'd)

FIGURE 18.3

Graph of the lognormal density for y, where $\ln(y) \sim \mathcal{N}(0, 1)$, $\ln(y) \sim \mathcal{N}(0, 1.5)$, and $\ln(y) \sim \mathcal{N}(1.5, 0.2)$.



A Lognormal Model of Stock Prices

- If the stock price S_t is lognormal, then $S_t / S_0 = e^x$, where x, the continuously compounded return from 0 to t, is normally distributed.

- If $R(t, s)$ is the continuously compounded return from t to s, and, $t_0 < t_1 < t_2$, then $R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$.

- From 0 to T, $E[R(0,T)] = n\alpha_h$, and $Var[R(0,T)] = n\sigma_h^2$, where $\alpha_h = E[R((i - 1)h, ih)]$ and $\sigma_h^2 = Var[R((i - 1)h, ih)]$. Here, $R((i - 1)h, ih)$ are uncorrelated.

- If returns are *i.i.d.*, the mean and variance of the continuously compounded returns are proportional to time.

$X \sim LN(\mu, \sigma^2)$ $\ln X \sim N(\mu, \sigma^2)$

A Lognormal Model of Stock Prices (cont'd)

$\Rightarrow S_t \sim LN(\beta, \gamma)$

- If we assume that

log return

$$\ln(S_t / S_0) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t]$$

$\beta = (\alpha - \delta - \frac{1}{2}\sigma^2)t + \ln S_0$

then $\ln(S_t / S_0) = (\alpha - \delta - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z$

and therefore $S_t = S_0 e^{(\alpha - \delta - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}$

$\gamma = \sigma^2 t$

$$
\begin{aligned}
E(S_t) &= E\left(S_0 e^{(\alpha-\delta-0.5\sigma^2)t+\sigma\sqrt{t}Z}\right) \\
&= S_0 e^{(\alpha-\delta-0.5\sigma^2)t} E\left(e^{\sigma\sqrt{t}Z}\right) \\
&= S_0 e^{(\alpha-\delta-0.5\sigma^2)t} e^{0.5\sigma^2 t} \\
&= S_0 e^{(\alpha-\delta)t}
\end{aligned}
$$

$E[S_t] = S_0 e^{(\alpha - \delta)t}$

Expected return

$$\ln E\left(\frac{S_t}{S_0}\right) = (\alpha - \delta)t$$

A Lognormal Model of Stock Prices (cont'd)

- The expression $\alpha - \delta$ is called the continuously compounded expected rate of stock-price appreciation on the stock.
- The median stock price – the value such that 50% of the time prices will be above or below that value – is obtained by setting $Z = 0$ in S_t. The median is thus

$$E(S_t)e^{-0.5\sigma^2 t}$$

A Lognormal Model of Stock Prices (cont'd)

Example

Suppose that the stock price today is \$100, the expected rate of return on the stock is α = 10%/year, and σ = 30%/year. If the stock is lognormally distributed, then we have

$$S_2 = \$100e^{\left(0.1-0.5\times0.3^2\right)2+\sigma\sqrt{2}Z}$$

Thus,

$$E\left(S_2\right) = \$100e^{(0.1)(2)} = \$122.14.$$

and the median is

$$\$100e^{\left(0.1-0.5\times0.3^2\right)\times2} = \$111.63.$$

Lognormal Probability Calculations

- Probabilities

  If

$$\ln(S_t / S_0) \sim N\left[\left(\alpha - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right], \quad \text{or,}$$

$$\ln(S_t) \sim N\left[\ln(S_0) + \left(\alpha - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right]$$

  then

$$\Pr(S_t < K) = N\left(-\hat{d}_2\right)$$

  where

$$\hat{d}_2 = \frac{\left(\alpha - \delta - 0.5\sigma^2\right)t + \ln(S_0 / K)}{\sigma\sqrt{t}}$$

$S_t \sim \ln(\beta, \gamma)$ $\quad \ln S_t \sim N(\beta, \gamma)$

$\boxed{Pr(S_t < K)} \Rightarrow Pr(S_T < K)$

$= Pr(\ln S_t < \ln K)$

$= Pr\left(Z < \dfrac{\ln K - \beta}{\sqrt{\gamma}}\right)$ $\quad Z \sim N(0,1)$

$$\boxed{\begin{aligned} \beta &= \ln S_0 + (r - \delta - \tfrac{1}{2}\sigma^2)t \\ \gamma &= \sigma^2 t \end{aligned}}$$

$= N(-\hat{d}_2)$

$$\hat{d}_2 = \frac{(\alpha - \delta - \frac{1}{2}\sigma^2)t + \ln(S_0/K)}{\sigma\sqrt{t}}$$

Lognormal Probability Calculations (cont'd)

- Given a call option expires in the money, what is the expected stock price?

The **partial** expectation of S_t, conditional on $S_t < K$, is defined as

$$\int_0^K S_t g(S_t; S_0) dS_t = S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln(K) - \left[\ln(S_0) + \left(\alpha - \delta + 0.5\sigma^2\right)t\right]}{\sigma\sqrt{t}}\right)$$

$$= S_0 e^{(\alpha-\delta)t} N\left(-\hat{d}_1\right)$$

where $g(S_t; S_0)$ is the probability density of S_t conditional on S_0, and $\hat{d}_1$ is the Black-Scholes d_1 with α replacing r.

put $= K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$

Lognormal Probability Calculations (cont'd)

The probability that $S_t < K$ is $N(-\hat{d}_2)$. Thus, the expectation of S_t conditional on $S_t < K$ is

$$E(S_t \mid S_t < K) = S_0 e^{(\alpha-\delta)t} \frac{N(-\hat{d}_1)}{N(-\hat{d}_2)}$$

Similarly, we obtain

$$E(S_t \mid S_t > K) = S_0 e^{(\alpha-\delta)t} \frac{N(\hat{d}_1)}{N(\hat{d}_2)}$$

$E[S_t \mid S_t < K]$

$$g_{S_t|S_t<K}(x) = \frac{g(x)}{Pr(S_t<K)} \qquad 0 < x < K$$

$E[S_t \mid S_t < K]$

$$= \int_0^K x \, g_{S_t|S_t<K}(x) \, dx$$

$$= \int_0^K x \frac{g(x)}{Pr(S_t<K)} \, dx$$

$$= \frac{1}{Pr(S_t<K)} \boxed{\int_0^K x \, g(x) \, dx} \quad ?$$

Partial Expectation

Lognormal Probability Calculations (cont'd)

- The Black-Scholes formula—the price of a call option is

$$C(S,K,\sigma,r,t,\delta) = e^{-rt}\int_K^{\infty}(S_t - K)\,g^*(S_t;S_0)\,dS_t$$
$$= e^{-rt}E^*(S_t - K \mid S > K)\times \Pr{}^*(S > K)$$
$$= e^{-\delta t}SN(d_1) - Ke^{-rt}N(d_2)$$

See "Derivation of BS formula.pdf".

where g^* denote the risk-neutral lognormal probability density, E^* denote the expectation taken with respect to risk-neutral probabilities, and $\Pr^*$ denote those probabilities. Under g^*,

$$\ln(S_t/S_0) \sim N\left[\left(r - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right].$$

Estimating the Parameters of a Lognormal Distribution

- When stocks are lognormally distributed, the price S_t evolves from the previous price observed at time $t - h$, according to

$$S_t = S_{t-h} e^{\left(\alpha-\delta-\sigma^2/2\right)h+\sigma\sqrt{h}z}$$

Thus

$$E\left(\ln\left(\frac{S_t}{S_{t-h}}\right)\right) = \left(\alpha - \delta - \sigma^2/2\right)h$$

$$Var\left(\ln\left(\frac{S_t}{S_{t-h}}\right)\right) = \sigma^2 h$$

Estimating the Parameters of a Lognormal Distribution (cont'd)

TABLE 18.2 Hypothetical weekly stock price observations and corresponding weekly continuously compounded returns, $\ln(S_t/S_{t-1})$

Week	Price ($)	$\ln(S_t/S_{t-1})$
1	100	—
2	105.04	0.0492
3	105.76	0.0068
4	108.93	0.0295
5	102.50	–0.0608
6	104.80	0.0222
7	104.13	–0.0064

Estimating the Parameters of a Lognormal Distribution (cont’d)

- Example 18.8:
  - The mean of the second column is 0.006745 and the standard deviation is 0.038208.
  - $h = 1/52$.
  - Annualized standard deviation

$$= 0.038208 \times \sqrt{52} = 0.2755$$

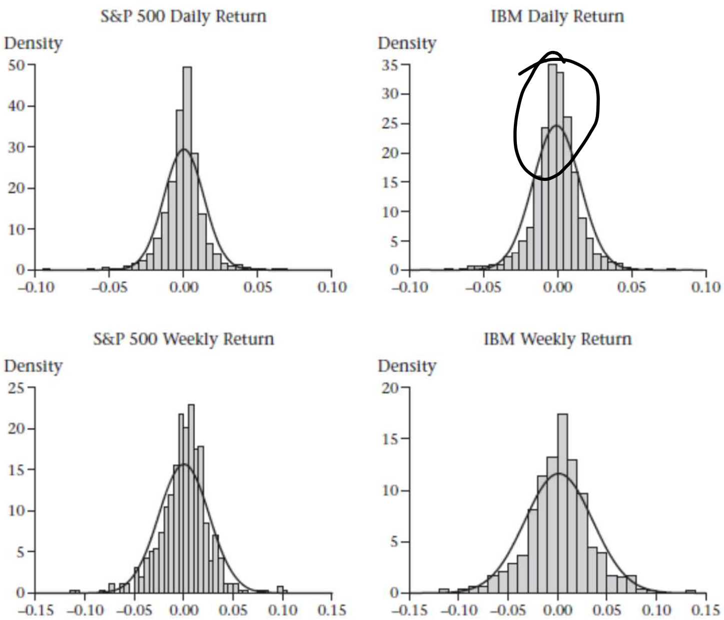
  - Annualized expected return

$$=0.006745 \times 52 + 0.5 \times 0.2755^2 = 0.3887$$

How Are Asset Prices Distributed?

FIGURE 18.4

Histograms for daily and weekly returns on the S&P 500 index and IBM, from July 1, 2001 to July 1, 2011.



How Are Asset Prices Distributed? (cont'd)

- None of the histograms appears exactly normal. All of the histograms exhibit a peak around 0; the presence or absence of this peakedness is referred to as *kurtosis* (a measure of how sharp the peak of the distribution is).
- The graph displays *leptokurtosis* (small, thin and delicate).
- Kurtosis for the S&P and IBM are 8.03 and 9.54 for daily returns, and 4.68 and 5.21 for weekly returns.

How Are Asset Prices Distributed? (cont'd)

- Accompanying the peaks are fat tails, large returns that occur more often would be predicted by the lognormal model.
- The normal probability plot can be used for assessing normality. If the data points lie along the straight line in the graph, the data are consistent with a normal distribution. If the data plot is curved, the data are less likely to have come from a normal distribution.

FIGURE 18.5

Normal probability plots for daily and weekly returns on the S&P 500 index and IBM from July 1, 2001 to July 1, 2011.