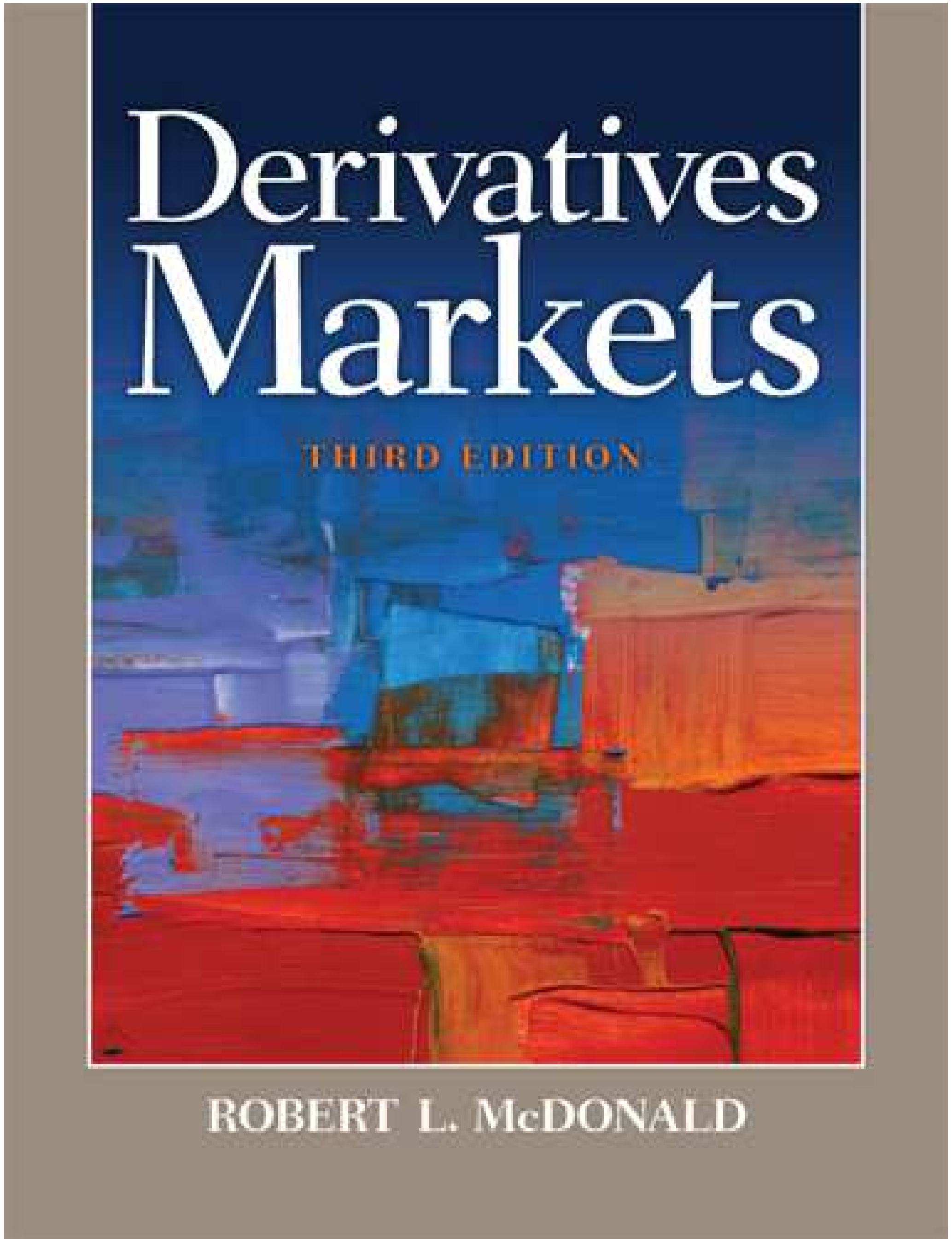

Chapter 13
(Chapter 20 in the textbook)

Brownian Motion and Itô’s Lemma

Points to Note

1. Definition of the Standard Brownian motion. See P.3 – 4.
2. Stochastic Processes which are constructed from the standard Brownian motion. See P.5 – 11.
3. Modelling the correlated asset prices through correlated Brownian motions. See P.12 – 16.
4. Ito’s lemma: univariate and multivariate versions. See P.17 – 28.
5. Sharpe ratios of two perfectly correlated assets. See P.29 – 31.

Brownian Motion

- A *stochastic process* is a random process that is a function of time.
- Brownian motion is a stochastic process that evolves in continuous time, with movements that are continuous.
  - A random walk can be generated by flipping a coin each period and moving one step, with the direction determined by whether the coin is heads or tails.
  - To generate Brownian motion, we would flip the coins infinitely fast and take infinitesimally small steps at each point.

Brownian Motion (cont'd)

- Brownian motion is a continuous stochastic process, $Z(t)$, with the following characteristics:
  - $Z(0) = 0$.
  - $Z(t + s) - Z(t)$ is normally distributed with mean 0 and variance s.
  - $Z(t + s_1) - Z(t)$ is independent of $Z(t) - Z(t - s_2)$, where $s_1, s_2 > 0$. That is, nonoverlapping increments are independently distributed.
  - $Z(t)$ is continuous.
- These properties imply that $Z(t)$ is a martingale: a stochastic process for which

$$E[Z(t+s) \mid \{Z(u), 0 \leq u \leq t\}] = Z(t).$$

Arithmetic Brownian Motion

- With pure Brownian motion, the expected change in Z is 0, and the variance per unit time is 1. We can generalize this to allow an arbitrary variance and a nonzero mean

$$dX(t) = \alpha\, dt + \sigma\, dZ(t) \tag{20.8}$$

- This process is called **arithmetic Brownian motion**
  - α is the instantaneous mean per unit time.
  - σ^2 is the instantaneous variance per unit time.
  - The variable $X(t)$ is the sum of the individual changes dX. $X(t)$ is normally distributed, i.e., $X(T) - X(0) \sim N(\alpha T, \sigma^2 T)$.

Arithmetic Brownian Motion (cont'd)

- An integral representation of equation (20.8) is

$$X(T) = X(0) + \int_0^T \alpha \, dt + \int_0^T \sigma \, dZ(t)$$

- Here are some properties of the process in equation (20.8)
  - $X(t)$ is normally distributed because it is a scaled Brownian process.
  - The random term is multiplied by a scale factor that enables us to specify the variance of the process.
  - The αdt term introduces a nonrandom *drift* into the process.

Arithmetic Brownian Motion (cont’d)

- Arithmetic Brownian motion has several drawbacks
  - There is nothing to prevent X from becoming negative, so it is a poor model for stock prices.
  - The mean and variance of changes in dollar terms are independent of the level of the stock price.
- Both of these criticisms will be eliminated with geometric Brownian motion.

The Ornstein-Uhlenbeck Process

- We can incorporate mean reversion by modifying the drift term

$$dX(t) = \lambda[a - X(t)]\,dt + \sigma\,dZ(t) \qquad (20.9)$$

- This equation is called an **Ornstein-Uhlenbeck process.**
  - The parameter λ measures the speed of the reversion: If λ is large, reversion happens more quickly.
  - In the long run, we expect *X* to revert toward α.
  - As with arithmetic Brownian motion, *X* can still become negative.

Geometric Brownian Motion

- An equation, in which the drift and volatility depend on the stock price, is called an **Itô process**.
  - Suppose we modify arithmetic Brownian motion to make the instantaneous mean and standard deviation proportional to $X(t)$

$$dX(t) = \alpha\, X(t)\, dt + \sigma\, X(t) dZ(t)$$

  - This is an Itô process that can also be written

$$\frac{dX(t)}{X(t)} = a\, dt + \sigma\, dZ(t) \qquad (20.11)$$

  - This process is known as **geometric Brownian motion (GBM)**.

Geometric Brownian Motion (cont'd)

- The percentage change in the asset value is normally distributed with instantaneous mean α and instantaneous variance σ^2.

- The integral representation for equation (20.11) is

$$X(T) - X(0) = \int_0^T \alpha\, X(t)dt + \int_0^T \sigma\, X(t)dZ(t)$$

Multiplication Rules

- We can simplify complex terms containing *dt* and *dZ* by using the following “multiplication rules”:

$$dt \times dZ = 0 \tag{20.15a}$$

$$(dt)^2 = 0 \tag{20.15b}$$

$$(dZ)^2 = dt \tag{20.15c}$$

Modeling Correlated Asset Prices

- Suppose that we have *m* asset processes

$$\frac{dX_i}{X_i} = (\alpha_i - \delta_i)dt + \sigma_i dZ_i \qquad i = 1, \ldots, m$$

The correlation between X_i and X_j will be generated by correlation between $Z_i(t)$ and $Z_j(t)$.

Next, we illustrate how we can create *correlated* diffusion processes by expressing dZ_i and dZ_j as *sums* of *independent* diffusions.

Modeling Correlated Asset Prices (cont'd)

- With m = 2 as an illustration:

Let $W_1(t)$ and $W_2(t)$ be *independent* Brownian motions and define

$$dZ_1(t) = dW_1(t)$$

$$dZ_2(t) = \rho dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)$$

  – This is the Cholesky decomposition

Consider

$$\begin{aligned} dZ_1(t)dZ_2(t) &= \rho dW_1(t)^2 + \sqrt{1-\rho^2}\, dW_1(t)dW_2(t) \\ &= \rho dt + \sqrt{1-\rho^2}\, dW_1(t)dW_2(t) \end{aligned}$$

Modeling Correlated Asset Prices (cont'd)

The independence of $W_1(t)$ and $W_2(t)$ implies that

$$E_t\left\{\left[W_1(t+s)-W_1(t)\right]\left[W_2(t+s)-W_2(t)\right]\right\}=0$$

Using the differential notation, we can write

$$dW_1(t) \times dW_2(t) = 0$$

Therefore,

$$dZ_1(t)dZ_2(t)=\rho dt$$

Modeling Correlated Asset Prices (cont'd)

In general, we can construct dZ_i, i = 1, …, n, as follows:

$$dZ_i(t) = \sum_{k=1}^{n} \lambda_{i,k} dW_k(t)$$

where we scale the coefficients so that

$$\sum_{k=1}^{n} \lambda_{i,k}^2 = 1$$

Modeling Correlated Asset Prices (cont'd)

Because the Brownian increments are jointly-normally distributed, their sum is normal. We also have

$$Var[dZ_i(t)] = Var\left(\sum_{k=1}^{n} \lambda_{i,k} dW_k(t)\right) = \sum_{k=1}^{n} \lambda_{i,k}^2 dW_k(t)^2 = dt$$

$$dZ_i(t) dZ_j(t) = \sum_{k=1}^{n} \lambda_{i,k} dW_k(t) \sum_{k=1}^{n} \lambda_{j,k} dW_k(t)$$

$$= \sum_{k=1}^{n} \lambda_{i,k} \lambda_{j,k} dt = \rho_{i,j} dt$$

where

$$\rho_{i,j} = \sum_{k=1}^{n} \lambda_{i,k} \lambda_{j,k}$$

Itô's Lemma

- Suppose that the stock price, $S(t)$, follows the Itô process given by

$$dS(t) = \left[\hat{\alpha}[S(t),t] - \hat{\delta}[S(t),t] \right] dt + \hat{\sigma}[S(t),t]\, dZ(t)$$

- In this equation, the expected return, α, the dividend yield, δ, and the volatility, σ, can be functions of the stock price and time.

- If

$$\hat{\alpha}[S(t),t] = \alpha S(t), \quad \hat{\delta}[S(t),t] = \delta S(t), \quad \hat{\sigma}[S(t),t] = \sigma S(t),$$

then $S(t)$ follows geometric Brownian motion.

Itô's Lemma (cont'd)

- $C[S(t), t]$ is the value of a derivative claim that is a function of the stock price.
- How can we describe the behavior of this claim in terms of the behavior of S?

Itô's Lemma (cont'd)

- **Itô's Lemma (Proposition 20.1)**
  - If $C[S(t), t]$ is a twice-differentiable function of $S(t)$, then the change in C is

$$dC(S,\ t) = C_S dS + \frac{1}{2} C_{SS} (dS)^2 + C_t dt$$

$$= \left[(\hat{\alpha}(S,t) - \hat{\delta}(S,t))C_S + \frac{1}{2} \hat{\sigma}(S,t)^2 C_{SS} + C_t \right] dt + \sigma(S,t) C_S dZ$$

    - where $C_S = \partial C/\partial S$, $C_{SS} = \partial^2 C/\partial S^2$, and $C_t = \partial C/\partial t$
  - The terms in square brackets are the expected change in the option price.

Itô’s Lemma (cont’d)

- **Proof (Proposition 20.1)**

  Proposition 20.1 can be proved by applying Itô’s lemma and the multiplication rule successively.

Itô's Lemma (cont'd)

- In the case where $S(t)$ follows geometric Brownian motion, we have

$$dC(S,t) = \left[(\alpha - \delta) S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + C_t \right] dt + \sigma S C_S dZ$$

Itô's Lemma (cont'd)

Example (The Black-Scholes Assumption of Stock Prices)

The expression for a lognormal stock price is

$$S(t) = S(0)e^{\left(\alpha - \delta - 0.5\sigma^2\right)t + \sigma Z(t)}$$

The stock price is a function of the Brownian process *Z*(*t*). We can use Itô's Lemma to characterize the behavior of the stock as a function of *Z*(*t*). We have

$$\frac{\partial S(t)}{\partial t} = \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)S(t); \quad \frac{\partial S(t)}{\partial Z(t)} = \sigma S(t); \quad \frac{\partial^2 S(t)}{\partial Z(t)^2} = \sigma^2 S(t)$$

Itô’s Lemma (cont’d)

Itô’s Lemma states that $dS(t)$ is given as

$$dS(t) = \frac{\partial S(t)}{\partial t}dt + \frac{\partial S(t)}{\partial Z(t)}dZ(t) + \frac{1}{2}\frac{\partial^2 S(t)}{\partial Z(t)^2}[dZ(t)]^2$$

$$= (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)$$

This calculation demonstrates that a variable that follows geometric Brownian motion is lognormally distributed.

Itô’s Lemma (cont’d)

Example

Let $Y(t) = \ln[S(t)]$. Then

$$d\ln[S(t)] = \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{dS(t)^2}{S(t)^2} = \frac{dS(t)}{S(t)} - \frac{1}{2}\sigma^2 dt$$

This implies that continuously compounded returns – measured as $\ln[S(T)/S(0)]$ – are lower than the instantaneous return, $\alpha - \delta$, by the factor $0.5\sigma^2$.

Multivariate Itô's Lemma

- A derivative may have a value depending on more than one price, in which case we can use a multivariate generalization of Itô's Lemma
- **Multivariate Itô's Lemma (Proposition 20.2)**
  - Suppose we have *n* correlated Itô processes

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i dt + \sigma_i dZ_i, \quad i = 1, \ldots, n$$

    - Denote the pairwise correlations as

$$dZ_i \times dZ_j = \rho_{i,j} dt$$

Multivariate Itô’s Lemma (cont’d)

- If $C(S_1, \ldots, S_n, t)$ is a twice-differentiable function of the S_i’s, we have

$$dC(S_1, \ldots, S_n, t) = \sum_{i=1}^{n} C_{S_i} dS_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} dS_i dS_j C_{S_i S_j} + C_t dt$$

Multivariate Itô's Lemma (cont'd)

Example

Suppose $C(S_1, S_2) = S_1/S_2$. Then by Itô's Lemma we have

$$d\left(\frac{S_1}{S_2}\right) = dS_1\left(\frac{1}{S_2}\right) - dS_2\left(\frac{S_1}{S_2^2}\right) + 0.5\left[2(dS_2)^2\frac{S_1}{S_2^3} - 2dS_1dS_2\frac{1}{S_2^2}\right]$$

$$d\left(\frac{S_1}{S_2}\right)\frac{S_2}{S_1} = (\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dZ_1 - \sigma_2 dZ_2$$

Multivariate Itô's Lemma (cont'd)

From the earlier discussion of correlated Itô's processes, we have

$$d\left(\frac{S_1}{S_2}\right)\frac{S_2}{S_1} = \left(\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)dt + \hat{\sigma}\,dZ$$

where

$$\hat{\sigma} = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \quad \text{and} \quad dZ = \left(\sigma_1 dZ_1 - \sigma_2 dZ_2\right)/\,\hat{\sigma}.$$

Remark:

Even if S_1 and S_2 have equal drifts (i.e., $\alpha_1 = \alpha_2$), the ratio of S_1 and S_2 will have generally have zero drift.

The Sharpe Ratio

- If asset *i* has (total) expected return α_i, the risk premium is defined as

$$\text{Risk premium}_i = \alpha_i - r$$

  - where *r* is the risk-free rate.
- The **Sharpe ratio** for asset *i* is the risk premium, $\alpha_i - r$, per unit of volatility, σ_i

$$\text{Sharpe ratio}_i = \frac{\alpha_i - r}{\sigma_i} \qquad (20.25)$$

The Sharpe Ratio (cont'd)

- We can use the Sharpe ratio to compare two perfectly correlated claims, such as a derivative and its underlying asset.
- Two assets that are perfectly correlated must have the same Sharpe ratio, or else there will be an arbitrage opportunity.
  - Consider the processes for two non-dividend paying stocks

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ \qquad (20.26)$$

$$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ \qquad (20.27)$$

  - Because the two stock prices are driven by the same *dZ*, it must be the case that

$$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$$

The Sharpe Ratio (cont'd)

- The arbitrage is straightforward. Suppose that the Sharpe ratio of asset 1 is greater than that of asset 2. We then
  - Buy $1/(\sigma_1 S_1)$ shares of asset 1.
  - Short $1/(\sigma_2 S_2)$ shares of asset 2.
  - Invest (or borrow) $1/\sigma_2 - 1/\sigma_1$, by buying (or borrowing) the risk-free bond, which has the rate of return *rdt*.
- The return of the above portfolio is

$$\frac{1}{\sigma_1 S_1} dS_1 - \frac{1}{\sigma_2 S_2} dS_2 + \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right) rdt = \left(\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right) dt > 0$$

So, the arbitrage profit is obtained.