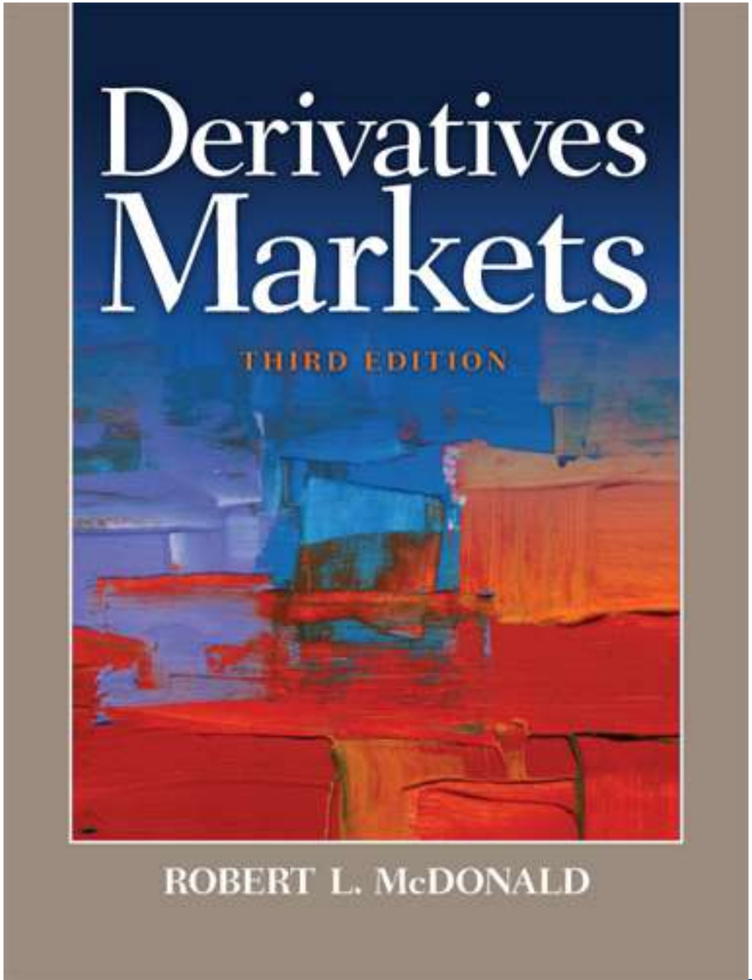


25-Oct-2018

Chapter 9
(Chapter 10 in the textbook)

Binomial Option Pricing

Forward contract

$$F_{t,T} = S_t e^{(r-\delta)(T-t)}$$

Pricing option ; specify the prob. dist. of S_t

Define $\{S_t : t \geq 0\}$

<u>Method 1</u>

Binomial tree : Discrete version of B-S formula

Assumption on $\{S_n : n = 0, 1, 2, \cdots\}$

Points to Note

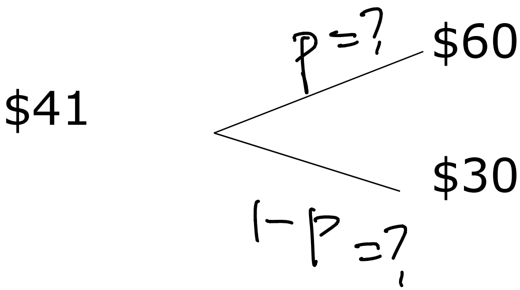
1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13).
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 - 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)

Introduction to Binomial Option Pricing

- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the stock or other underlying asset.
- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution—that is, the asset price in each period can move only up or down by a specified amount.
- The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model".

A One-Period Binomial Tree

- Example K T $S_0 = 41$
  - Consider a European call option on the stock of XYZ, with a $40 strike and 1 year to expiration.
  - XYZ does not pay dividends, and its current price is $41. $\delta = 0$
  - The continuously compounded risk-free interest rate is 8%. r
  - The following figure depicts possible stock prices over 1 year, i.e., a binomial tree



Computing the Option Price

- Next, consider two portfolios:
  - *Portfolio A*: buy one call option.
  - *Portfolio B*: buy 2/3 shares of XYZ and borrow $18.462 at the risk-free rate.
- Costs
  - *Portfolio A*: the call premium, which is unknown.
  - *Portfolio B*: $\underbrace{2/3 \times \$41}_{\text{XYZ stock}} - \underbrace{\$18.462}_{\text{borrowing}} = \$8.871$.

Computing the Option Price (cont'd)

- Payoffs:
  - *Portfolio A*:

	Stock Price in 1 Year	
	$30	**$60**
Payoff	0	$20

  - *Portfolio B*:

	Stock Price in 1 Year	
	$30	**$60**
2/3 purchased shares	$20	$40
Repay loan of $18.462	– $20	–$20
Total payoff	0	$20

Computing the Option Price (cont'd)

- Portfolios A and B have the same payoff. Therefore
  - Portfolios A and B should have the same cost. Since Portfolio B costs $8.871, the price of one option must be $8.871.

  The idea that positions that have the same payoff should have the same cost is called the *law of one price*.

Computing the Option Price (cont'd)

- There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a ***synthetic* call** or **replicating portfolio of the call option**.
- One option has the risk of 2/3 shares. The value 2/3 is the *delta* (Δ) of the option: the number of shares that replicates the option payoff.

Replicating portfolio

1) 2/3 shares of stock $\Longleftrightarrow$ Long a call option ($K=45$)

2) Borrowing \$18.462

The Binomial Solution

Cost $= \Delta S_0 + B$

- How do we find a replicating portfolio consisting of **buying Δ shares of stock** and a **dollar amount *B* in lending**, such that the portfolio imitates the option whether the stock rises or falls?
  - Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time t, at time $t+h$ you will have $e^{\delta h}$ shares.
  - If the length of a period is h, the interest factor per period is e^{rh}.
  - uS denotes the stock price when the price goes up, and dS denotes the stock price when the price goes down.
  - The up and down movements of the stock price reflect the *ex-dividend* price.

$$
\begin{array}{ccc}
 & & uS \\
S & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\
 & & dS \\
 & & \\
t & & t+h
\end{array}
\qquad u > d
$$

Replicating portfolio

1) Δ units of stock

2) Lending \$B

$$\text{Cost} = \Delta S + B$$

Payoff of the replicating portfolio at $t+h$

$$= \Delta S_h e^{\delta h} + B e^{rh}$$

$$= \begin{cases} \Delta u S e^{\delta h} + B e^{rh} \\ \Delta d S e^{\delta h} + B e^{rh} \end{cases}$$

The Binomial Solution (cont'd)

- Stock price tree:

$$\begin{tikzcd}[row sep=small] & uS \\ S \arrow[ur, dash] \arrow[dr, dash] & \\ & dS \end{tikzcd}$$

- Corresponding tree for the value of the option:

$$\begin{tikzcd}[row sep=small] & C_u \\ C \arrow[ur, dash] \arrow[dr, dash] & \\ & C_d \end{tikzcd}$$

  – Note that *u* (*d*) in the stock price tree is interpreted as one plus the rate of capital gain (loss) on the stock if it goes up (down).

- The value of the **replicating portfolio** at time *h*, with stock price S_h, is

$$\Delta S_h e^{\delta h} + e^{rh} B$$

The Binomial Solution (cont'd)

- At the prices $S_h = uS$ and $S_h = dS$, a successful replicating portfolio will satisfy

$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$

$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$

- Solving for Δ and B gives

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \qquad (10.1)$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u-d} \qquad (10.2)$$

The Binomial Solution (cont'd)

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S+B$.

$$C = \Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d}\right) \quad (10.3)$$

- The no-arbitrage condition is

$$u > e^{(r-\delta)h} > d \quad (10.4)$$

Proof of (10.4)

$\delta = 0 \Rightarrow (10.4) \quad (u > e^{rh} > d)$

Suppose $u \leq e^{rh}$, arbitrage??

$\Rightarrow \quad uS \leq Se^{rh}$

Value of stock $\leq$ Value of bond at t+h

$\Rightarrow$ buy the bond + short the stock.

	t=0	t=h
short stock	S	$-S_h$
Buy the bond face Value $= Se^{rh}$	$-Se^{rh}\cdot e^{-rh} = -S$	Se^{rh}
total	0	$Se^{rh} - S_h$

Since $uS \leq Se^{rh}$, $dS \leq uS$ $\Rightarrow$ $Se^{rh} - S_h \geq 0$

$uS \quad dS$

The Binomial Solution (cont'd)

- Suppose that $\delta = 0$. If the condition were violated, we would
  - Short the stock to hold bonds (if $e^{rh} \geq u$), <u>or</u>
  - we borrow to buy the stock (if $d \geq e^{rh}$).

  Either way, we would earn an *<u>arbitrage profit</u>*.

<u>Home exercise:</u> The case of $\delta \neq 0$ is your exercise.

The Binomial Solution (cont'd)

Example

Use the information in P.3, consider a call option had a strike price of \$40 and 1 year to expiration. Calculate the price of the call option.

Here, we have h = 1.

C_u = \$60 - \$40 = \$20; C_d = \$0

$$\Delta = \frac{\$20-0}{\$41\times\left(1.4634-0.7317\right)} = \frac{2}{3}$$

$$B = e^{-0.08}\,\frac{1.4634\times\$0-0.7317\times\$20}{1.4634-0.7317} = -\$18.462$$

The Binomial Solution (cont'd)

Hence the option price is given by

$$\Delta S + B = \frac{2}{3} \times \$41 - \$18.462 = \$8.871$$

Alternatively, the option price can also obtained by

$$\Delta S + B = e^{-0.08}\left(\$20 \times \frac{e^{0.08} - 0.7317}{1.4634 - 0.7317} + \$0 \times \frac{1.4634 - e^{0.08}}{1.4634 - 0.7317}\right)$$

$$= \$8.871$$

Arbitraging a Mispriced Option

- If the observed option price differs from its theoretical price, arbitrage is possible.

  Example

  – If a call option is **overpriced**, we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option.

  – If a call option is **underpriced**, we buy the option. To hedge the risk associated with the possibility of the stock price falling at expiration, we sell a synthetic option at the same time.

Risk-Neutral Pricing

- We can interpret the terms $(e^{(r-\delta)h} - d)/(u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ as probabilities.

$$d \leq e^{(r-\delta)h} \leq u$$

- Let

$$0 \leq p^* = \frac{e^{(r-\delta)h} - d}{u-d} \leq 1 \qquad (10.5)$$

- Then equation (10.3) can then be written as

$$C = e^{-rh}[p^* C_u + (1 - p^*)C_d] \qquad (10.6)$$

We call p^* is the **risk-neutral probability** of an increase in the stock price.

- The pricing procedure illustrated in Eq. (10.6) is called *risk-neutral valuation*.

$$C = e^{-rh} E^*[C_u] \; : \; \text{risk neutral Valuation}$$

E^* : expectation with respect to P^*.

E^* : risk-neutral expectation

Continuously Compounded Returns

- Some important properties of continuously compounded returns:
  - ◆ The logarithmic function computes returns from prices. The continuously compounded return between t and $t + h$, $r_{t,t+h}$ is then

$$r_{t,t+h} = \ln\left(S_{t+h} / S_t\right)$$

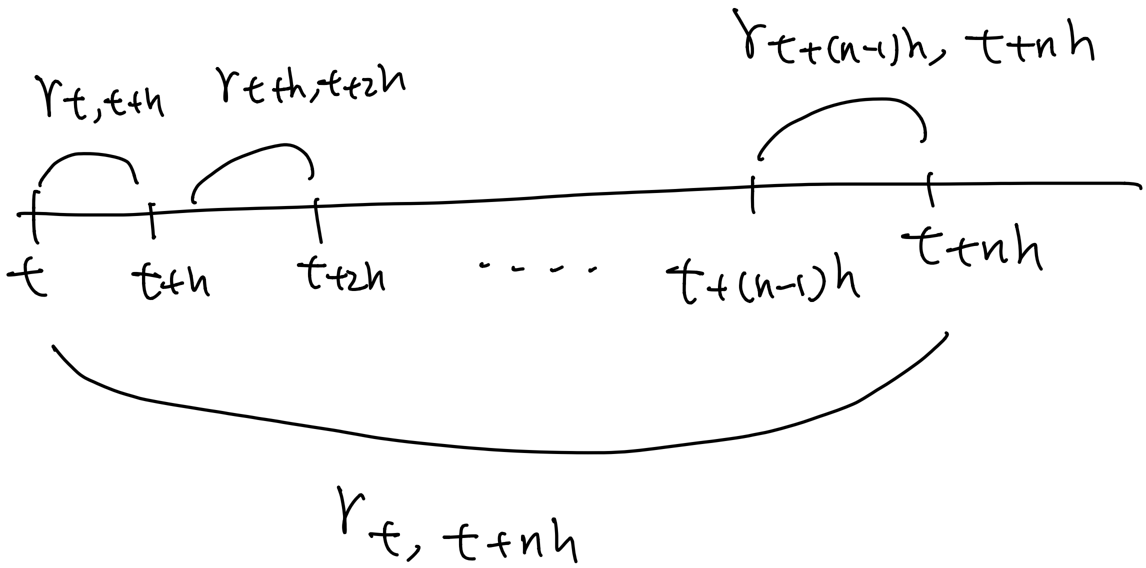
  - ◆ The exponential function computes prices from returns

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

$r_{t,t+h}$

  - ◆ Continuously compounded returns are additive

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$



Additive $\Rightarrow \displaystyle\sum_{i=1}^{(n-1)} r_{t+(i-1)h,\, t+ih} = r_{t,\, t+nh}$

Volatility

- The volatility of an asset, defined as the standard deviation of continuously compounded returns.
- Suppose the continuously compounded return over month *i* is $r_{\text{monthly},i}$. Since returns are additive, the annual return is

$$r_{\text{annual}} = \sum_{i=1}^{12} r_{\text{monthly},i}$$

$$\text{Var}(r_{\text{annual}}) = \sum_{a=1}^{12} \text{Var}(r_{\text{monthly},i})$$

- The variance of the annual return is

$$\text{Var}(r_{\text{annual}}) = \text{Var}\left(\sum_{i=1}^{12} \text{r}_{\text{monthly},i}\right) \qquad (10.13)$$

The Standard Deviation of Continuously Compounded Returns (cont'd)

- Suppose that returns are uncorrelated over time and that each month has the same variance of returns. Then from equation (10.13) we have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

where σ^2 denotes the annual variance.

- The **volatility** of the asset is then given by

$$\sigma = \sigma_{\text{monthly}}\sqrt{12}$$

- To generalize this formula, if we split the year into n periods of length h (so that $h = 1/n$), the standard deviation over the period of length h, σ_h, is

$$\sigma_h = \sigma\sqrt{h} \qquad (10.15)$$

Constructing *u* and *d* (Forward Tree)

- *In the absence of uncertainty*, the stock price next period must equal the forward price. The formula for the forward price is

$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

Thus, without uncertainly we must have

$$S_{t+h} = F_{t,t+h}$$

The rate of return on the stock must be the risk-free rate.

Constructing *u* and *d* (cont'd)

- *With uncertainty*, the stock price evolution is

$$uS_t = F_{t,t+h}\, e^{+\sigma\sqrt{h}}$$
$$dS_t = F_{t,t+h}\, e^{-\sigma\sqrt{h}} \qquad (10.17)$$

where σ is the annualized standard deviation of the continuously compounded return, and $\sigma\sqrt{h}$ is standard deviation over a period of length h.

- We can also rewrite (10.17) as

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} \qquad (10.9)$$

  – We refer to a tree constructed using equation (10.9) as a "forward tree."

Estimating Historical Volatility

- We need to decide what value to assign to σ, which we cannot observe directly.
- One possibility is to measure σ by computing the standard deviation of continuously compounded historical returns.
  - Volatility computed from historical stock returns is **historical volatility**.
  - For example, calculate the standard deviation with weekly data, then annualize the result by using equation (10.15).
  - For option pricing, it is generally the volatility of the price excluding dividends that matters.

Estimating Historical Volatility (cont'd)

- Example

TABLE 10.1 Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

	S&P 500		IBM	
Date	**Price**	$\ln(S_t/S_{t-1})$	**Price**	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	−0.02363	125.27	−0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	−0.03408	129.83	−0.01103
8/18/2010	1094.16	0.00430	129.39	−0.00338
8/25/2010	1055.33	−0.03613	125.27	−0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

One-Period Example with a Forward Tree

- Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%.
- The continuously compounded risk-free interest rate is 8%.
- $S = 41$, $r = 0.08$, $\delta = 0$, $\sigma = 0.30$, and $h = 1$.
- Use these inputs to
  - calculate the final stock prices.
  - calculate the final option values.
  - calculate Δ and B.
  - calculate the option price.

One-Period Example with a Forward Tree (cont'd)

- Calculate the final stock prices

$$uS = \$41e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = \$59.954$$

$$dS = \$41e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = \$32.903$$

$$\Rightarrow$$

$$u = \frac{\$59.954}{S} = \frac{59.954}{41} = 1.4623$$

$$d = \frac{\$32.903}{S} = \frac{32.903}{41} = 0.8025$$

- Calculate the final option values

$$C_u = \max(uS - K, 0) = \max(59.954 - 40, 0) = 19.954$$

$$C_d = \max(dS - K, 0) = \max(32.903 - 40, 0) = 0$$

- Calculate Δ and B

$$\Delta = \frac{19.954 - 0}{\$41\times(1.4632 - 0.8025)} = 0.7376$$

$$B = e^{-0.08}\frac{1.4632\times\$0 - 0.8025\times\$19.954}{1.4632 - 0.8025} = -\$22.405$$

- Calculate the option price

$$\Delta S + B = 0.7376\times 41 - \$22.405 = \$7.839$$

One-Period Example with a Forward Tree (cont'd)

- The following figure depicts the possible stock prices and option prices over 1 year, i.e., a **binomial tree**