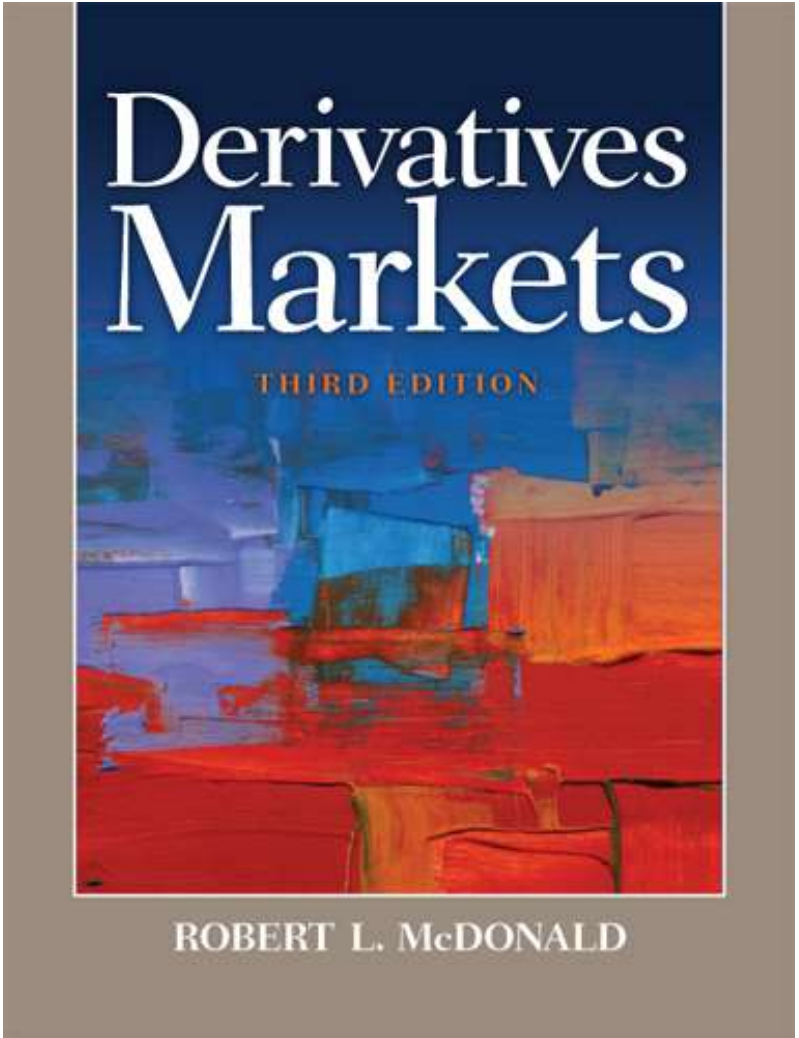


Chapter 9
(Chapter 10 in the textbook)

Binomial Option Pricing

Points to Note

1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13).
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 - 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)

Replicating portfolio Call option on S (Time to expiry: h)

$$\begin{tikzcd}[row sep=small] & uS \\ S \arrow[ur] \arrow[dr] & \\ & dS \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & C_u = \max(uS-K,0) \\ C \arrow[ur] \arrow[dr] & \\ & C_d = \max(dS-K,0) \end{tikzcd}$$

Replicating portfolio : $\Delta S + B$ ← lending Amt

Payoff $= \Delta e^{\delta h} S + e^{rh} B$

Find Δ and B to have

Payoff of replicating portfolio = payoff of call option

$$\Rightarrow \Delta = e^{-\delta h} \frac{C_u - C_d}{(u-d)S}, \quad B = e^{-rh} \frac{uC_d - dC_u}{u-d}$$

No arbitrage condition:

$$d < e^{rh} < u \qquad (1)$$

$$C = \Delta S + B = e^{-rh}\left[p^* C_u + (1-p^*) C_d \right]$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

Because of (1), $0 < p^* < 1$

probability

risk-free interest rate

$$C = e^{-rh} E[C_h]$$

risk-neutral expectation

$$\begin{tikzcd}
 & C_u \\
C \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\
 & C_d
\end{tikzcd}$$

Forward Tree

$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

John Hull's book

$$u = e^{\sigma\sqrt{h}}$$

$$d = e^{-\sigma\sqrt{h}}$$

σ: Volatility = standard deviation of the stock return.

Tutorial - Class Activity (Solution)

5 November, 2019

$$\begin{tikzcd}[row sep=small] & 120 \\ 100 \arrow[ur] \arrow[dr] & \\ & 105 \end{tikzcd}$$

$\Delta = ?\ ,\ B = ?$

Problem 1

10.11 Suppose $S_0 = \$100$, $K = \$50$, $r = 7.696\%$ (continuously compounded), $\delta = 0$, and $T = 1$.

a. Suppose that for $h = 1$, we have $u = 1.2$ and $d = 1.05$. What is the binomial option price for a call option that lives one period? Is there any problem with having $d > 1$?

b. Suppose now that $u = 1.4$ and $d = 0.6$. Before computing the option price, what is your guess about how it will change from your previous answer? Does it change? How do you account for the result? Interpret your answer using put-call parity.

c. Now let $u = 1.4$ and $d = 0.4$. How do you think the call option price will change from (a)? How does it change? How do you account for this? Use put-call parity to explain your answer.

Solution

a) We can calculate the option delta, B and the premium with our standard binomial pricing formulas:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

$$\Delta = 1$$

$$B = -46.296$$

$$price = 53.704$$

$$\begin{tikzcd}[row sep=small] & 120 \\ S \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\ & 105 \end{tikzcd}$$

It is no problem to have a d that is larger than one. The only restriction that we have imposed is that $d < e^{(r-\delta)h} = e^{(0.07696)1} = 1.08$, which is respected.

$d < e^{(r-\delta)h} < u$

b) We may expect the option premium to go down drastically because with a d equal to 0.6, the option is only slightly in the money in the down state at $t = 1$. However, the potential in the up state is even higher, and it is difficult to see what effect the change in u and d has on the risk-neutral probability. Let's have a look at put-call-parity. The key is the put option. A put option with a strike of 50 never pays off, neither in (a) nor in (b) because in (a) the lowest possible stock price is 105, and in (b) it is 60. Therefore, the put option has a value of zero. But then, the put-call-parity reduces to:

$$C = S - Ke^{-0.07696} = 100 - 50 \times 0.926 = 53.704.$$

Clearly, as long as the strike price is inferior to the lowest value the stock price can attain at expiration, the value of the call option is independent of u and d. Indeed, we can calculate:

$$\Delta = 1$$

$$B = -46.296$$

$$price = 53.704$$

$$C - P = S - Ke^{-rT} \Rightarrow C = P + S - Ke^{-rT}$$

$uS = 140$ $\quad K = 50$

$dS = 60$

$$S \begin{cases} 140 \Rightarrow 90 \\ 60 \Rightarrow 10 \end{cases}$$

$$P \begin{cases} 0 \\ 0 \end{cases} \Rightarrow P = 0$$

$$\boxed{\begin{array}{l} u = 1.9 \\ d = 0.51 \end{array}} \Rightarrow C = 53.704$$

c) Again, we are tempted to think in the wrong direction. You may think that, since the call option can now expire worthless in one state of the world, it must be worth less than in part (b). This is not correct. Let us use put-call-parity to see why.

Now, with $d = 0.4$, a stock price of 40 at $t = 1$ is admissible, and the corresponding put option has a positive value because it will pay off in one state of the world. We can use put-call-parity to see that:

$$C = S - Ke^{-0.07696} + P = 100 - 50 \times 0.926 + P = \boxed{53.704 + P} > \boxed{53.704.}$$

(a)

Indeed, we can calculate:

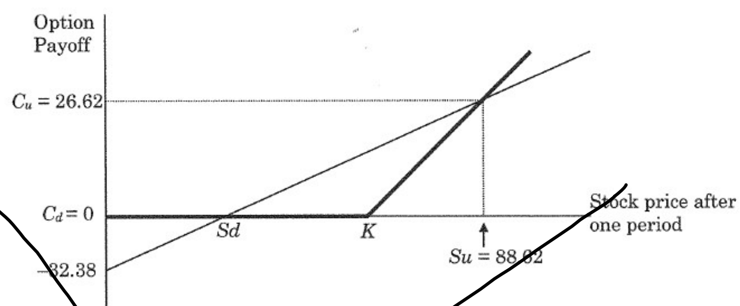
$$\Delta = 0.9$$

$$B = -33.333$$

$$price = 56.6666.$$

Problem 2

The graph below describes the payoffs of a European call option expiring in 1 year. The price evolution of the underlying stock follows a binomial tree with each period being 1 year in length. The underlying stock does not pay dividends. The current price of the underlying stock is \$60.



The continuously compounded risk-free rate of return is 9%. Calculate the current value of the option.

Solution

Since the dividend yield δ of the underlying stock is 0, Δ is given by

$$\Delta = \frac{C_u - C_d}{Su - Sd}.$$

Although we are not given the value of Sd, we can still find Δ by using the points (0, −32.38) and (88.62, 26.62) as follows:

$$\Delta = \frac{C_u - C_d}{Su - Sd} = \frac{26.62 - (-32.38)}{88.62 - 0} = \frac{59}{88.62} = 0.6658.$$

The intercept at the vertical axis is the value of the replicating portfolio if the stock price is 0 at the end of 1 year:

$$\Delta \times 0 + Be^{0.09} = -32.38$$
$$B = -29.5931$$

The option can be replicated by purchasing 0.6658 shares of stock and borrowing \$29.5931:

$$C = S_0\Delta + B = 60(0.6658) - 29.5931 = 10.3549.$$

Problem 3

The current price of Stock A is \$100, and the current price of Stock B is \$75. There are three scenarios for the prices of Stock A and Stock B at the end of one year, and each scenario is described in the table below.

	End of Year 1	
	Price of Stock A	Price of Stock B
Scenario 1	\$200	\$0
Scenario 2	\$50	\$0
Scenario 3	\$25	\$300

Stock A pays a discrete dividend of \$10 at the end of 6 months. Stock B pays dividends at a continuously compounded rate of 5%.

P_A is the price of a 1-year European put option on Stock A with a strike price of \$100.

C_B is the price of a 1-year European call option on Stock B with a strike price of \$200.

The continuously compounded risk-free rate of return is 8%. Calculate $P_A - C_B$.

Replicating portfolio of $P_A - C_B$ = ??

Solution

The end-of-year payoffs of the call and put options in each scenario are shown in the table below. The rightmost column is the payoff resulting from buying the put option and selling the call option.

$\max(100 - S_A(1), 0)$ $\max(S_B(1) - 200, 0)$

	End of Year 1				
	Price of Stock A	Price of Stock B	$P_A(100)$ Payoff	$C_B(200)$ Payoff	$P_A(100) - C_B(200)$ Payoff
Scenario 1	\$200	\$0	0	0	0
Scenario 2	\$50	\$0	50	0	50
Scenario 3	\$25	\$300	75	100	–25

K K

We need to determine the cost of replicating the payoffs in the rightmost column above. We can replicate those payoffs by determining the proper amount of Stock A, Stock B, and the risk-free asset to purchase.

A: Stock A B: Stock B , C: Amt of Lend:

Define

A: Number of shares of Stock A to purchase

B: Number of shares of Stock B to purchase

C: Amount to lend at the risk-free rate

Since Stock A pays a \$10 dividend at time 0.5, each share of Stock A that is purchased provides its holder with the final price of Stock A at time 1 plus the accumulated value of \$10. Since Stock B pays continuously compounded dividends of 5%, each share of Stock B purchased at time 0 grows to $e^{0.05}$ shares of Stock B at time 1.

We have 3 equations and 3 unknown variables:

Scenario 1: $\left(200+10e^{0.08(0.5)}\right)A+0B+Ce^{0.08}=0$

Scenario 2: $\left(50+10e^{0.08(0.5)}\right)A+0B+Ce^{0.08}=50$

Scenario 3: $\left(25+10e^{0.08(0.5)}\right)A+300e^{0.05}B+Ce^{0.08}=-25.$

Subtracting the first equation from the second equation allows us to solve for A:

$$\left(50+10e^{0.08(0.5)}\right)A-\left(200+10e^{0.08(0.5)}\right)A=50$$

$$A=\frac{50}{50+10e^{0.04}-200-10e^{0.04}}=-\frac{1}{3}.$$

The equation associated with Scenario 1 can now be used to find C:

by determining the proper amount of Stock A, Stock B, and the risk

A: Stock A   B: Stock B , C: Amt of Lending

Cost of replicating portfolio for $P_A - C_B$

$$= A\,S_A(0) + B\,S_B(0) + C$$

? A , ? B , ? C

Payoff of the replicating portfolio :

$$= A\left[S_A(1) + 10e^{(0.08)(0.5)}\right]$$

$$+ B e^{(0.05)(1)} S_B(1)$$

$$+ Ce^{0.08}$$

$$= \text{Payoff of } P_A - C_B$$

Scenario 1

$$A\left[200 + 10e^{(0.08)(0.5)}\right] + Be^{0.05}[0] + Ce^{0.08}$$

$$= 0$$

	End of Year 1				
	Price of Stock A	Price of Stock B	$P_A(100)$ Payoff	$C_B(200)$ Payoff	$P_A(100) - C_B(200)$ Payoff
Scenario 1	\$200	\$0	0	0	0
Scenario 2	\$50	\$0	50	0	50
Scenario 3	\$25	\$300	75	100	–25

need to determine the cost of replicating the payoffs in the rightmost column above. We can

$$\left(200+10e^{0.08(0.5)}\right)A+0B+Ce^{0.08}=0$$

$$C=-\left(200+10e^{0.08(0.5)}\right)Ae^{-0.08}=\left(200+10e^{0.08(0.5)}\right)\left(\frac{1}{3}\right)e^{-0.08}=64.7437.$$

We use the equation associated with Scenario 3 to find *B*:

$$\left(25+10e^{0.08(0.5)}\right)A+300e^{0.05}B+Ce^{0.08}=-25$$

$$-\left(25+10e^{0.08(0.5)}\right)\left(\frac{1}{3}\right)+300e^{0.05}B+64.7437e^{0.08}=-25$$

$$B=-0.2642.$$

The cost now of replicating the payoffs resulting from the put and selling the call is equal to the cost of establishing a position consisting of *A* shares of Stock A, *B* shares of Stock B, and *C* lent at the risk-free rate. Since the price of Stock A is \$100 and the price of Stock B is \$75, we have:

$$100A+75B+C=100\left(-\frac{1}{3}\right)+75(-0.2642)+64.7437=11.5954.$$

$$\boxed{P_A - C_B}$$