Chapter 14
Martingale Pricing Theory

See Section 3.2 of "Mathematical Models of Financial Derivatives", 2nd edition, by Yue Kuen KWOK, Springer Verlag, 2008.

Points to Note

1. What is the definition of the equivalent martingale measure? See P.3 – 6.
2. What is the relationship between the no-arbitrage price of a financial product and the risk-neutral probability? See P.8 – 13.
3. How do we change a measure in the expectation? Use the notation of the Randon-Nikodym derivative. See P.14 – 15.
4. Girsanov Theorem. See P.16 – 19.
5. Converting the dynamic of the asset price processes from the real probability to the risk-neutral probability. See P.20 – 25.
6. Derivation of the BS formula by using the change of numeraire. See P.26 – 38.
7. The Black-Scholes formula for the dividend-paying asset. See P.39 – 47.

Equivalent Martingale Measure and Risk Neutral Valuation

Under the continuous time framework, the investors are allowed to trade continuously in the financial market up to finite time T.

"Model setup"

Consider the securities model, there are

$K + 1$ securities whose price processes are modeled by $M(t)$ and $S_m(t)$ (**non-dividend-paying assets**), where $m = 1, \ldots, K$.

money market Account

The uncertainly of the market is described by the actual (real) probability measure (distribution) P.

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

We use $M(t)$ to denote the money market account process that starts at \$1 and grows at the *deterministic* risk-free interest rate $r(t)$, that is,

$$\boxed{dM(t) = r(t)M(t)\,dt \quad \Leftrightarrow \quad M(t) = e^{\int_0^t r(s)ds}.}$$

The discounted security price process $S_m^*(t)$ is defined by

$$S_m^*(t) = \frac{S_m(t)}{M(t)}, \qquad m = 1, 2, \cdots, K.$$

$\doteq PV(S_m(t))$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

Suppose that Q_1 and Q_2 are probability measures (distributions) on the sample space Ω, Q_1 and Q_2 are *equivalent* if

$$Q_1(A) = 0 \Leftrightarrow Q_2(A) = 0 \text{ for any } A \subset \Omega.$$

Impossible under Q_1

Impossible under Q_2

Eg X is a random variable

$Q_1(X<20)=0$, $Q_1(20\le X\le 80)=3/4$,

Pr$(X<20)$ under Q_1

$Q_1(X>80)=\frac{1}{4}$.

$Q_2(X<20)=0$, $Q_2(20\le X\le 80)=\frac{1}{3}$,

$Q_2(X>80)=2/3$ $\qquad Q_1\neq Q_2$

$Q_1 \Longleftrightarrow Q_2$

$Q_3(X<10)=0$, $Q_3(10\le X<19)=\frac{1}{8}$,

$Q_3(X\geq 19)=7/8$.

$Q_3(X<19)=\frac{1}{8}$, $Q_1(X<19)=0$

$Q_1 \not\Leftrightarrow Q_3$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

A probability measure Q is said to be an *equivalent martingale measure* (or *risk-neutral measure*) to the real probability measure P if it satisfies

i. Q is equivalent to P; $Q \rightleftarrows P$

ii. The discounted security price process $S_m^*(t)$, m = 1, 2, …, K, are martingales under Q, that is

$$E_t^Q[S_m^*(u)] = \boxed{S_m^*(t)} \quad \text{for all } 0 \le t \le u \le T.$$

$$E^Q\left[S_m^*(u) \,\middle|\, \{S_m^*(s) : 0 \le s \le t\}\right] = S_m^*(t)$$

From Prof. Chon's Class

$$E[Z_n \mid \boxed{X_1, \ldots X_{n-1}}] = Z_{n-1}$$

information up to time $n-1$

Binomial tree, $p^* = \dfrac{e^{rh} - d}{u - d}$

$$S_j^* = \frac{S_j}{e^{r(jh)}}$$

Want to show: $E^*[S_{n+k}^* \mid \{S_0, S_1, \ldots S_n\}] = S_n^*$

<u>Proof</u>

$$S_{n+k} = \begin{cases} u^k S_n \\ u^{k-1} d S_n \\ \vdots \\ d^k S_n \end{cases}$$

$$E^*[S_{n+k}^* \mid \{S_0, \ldots S_n\}]$$

$$= \frac{1}{e^{r(n+k)h}} \sum_{j=0}^{k} (p^*)^j (1-p^*)^{k-j} u^j d^{k-j} S_n$$

$$= \frac{S_n}{e^{r(n+k)h}} \sum_{j=0}^{k} (p^* u)^j [(1-p^*) d]^{k-j}$$

$$= \frac{S_n}{e^{r(n+k)h}} [p^* u + (1-p^*) d]^k = \frac{S_n}{e^{rnh}} = S_n^*$$

\#

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

In (ii), $E_t^Q(\cdot)$ is defined as the expectation with respect to the probability measure Q conditional on the information available up to time t.

For simplicity, we would write $E_0^Q(\cdot)$ as $E^Q(\cdot)$.

Remark:

a. Existence of Q $\Rightarrow$ Absence of arbitrage.

b. Absence of arbitrage + some technical conditions $\Rightarrow$ Existence of Q.

c. Complete market $\Leftrightarrow$ Q is unique.

Incomplete $\Rightarrow$ Q is not unique

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Theorem (Risk Neutral Valuation)

Assume that an equivalent martingale measure Q exists. Let $V(T)$ be the payoff of a contingent claim which can be generated by the $K + 1$ securities, the *no arbitrage price* of the contingent claim at time t, $V(t)$, is given by

discounted payoff

$$V(t) = M(t) E_t^Q \left[\frac{V(T)}{M(T)} \right].$$

$$M(t) = e^{\int_0^t r(s)ds}$$

$$\frac{V(t)}{M(t)} = E_t^Q \left[\frac{V(T)}{M(T)} \right]$$

$\Rightarrow \left\{ \frac{V(t)}{M(t)} : t \geq 0 \right\}$ is a martingale under Q.

Application of the Risk Neutral Valuation Theorem (Cont'd)

1. Zero-coupon bond (pays \$1 at the maturity T)

   $V(T) = 1$ $\quad V(T) = 1$

   By the risk neutral valuation theorem, we have

$$P(0,T) = M(0)\,E_0^Q\left[\frac{\overset{V(T)}{1}}{M(T)}\right] \qquad M(0) = 1$$

$$= E^Q\left[e^{-\int_0^T r(s)\,ds}\right]. \quad = \quad e^{-rT}$$

if $r(s) = r$ for all s

Application of the Risk Neutral Valuation Theorem (Cont'd)

2. Forward price

The payoff for the long position is

$V(T) = S(T) - K$ where K is the forward price.

By the risk neutral valuation theorem, we have

V(T)

$$V(0) = M(0)E^{Q}\left[\frac{S(T)-K}{M(T)}\right].$$

Since there is 0 cost to enter into a forward contract, $V(0) = 0$. This implies

$$\boxed{E^{Q}\left[\frac{S(T)-K}{M(T)}\right] = 0.}$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

So,

$$E^Q\left[\frac{K}{M(T)}\right] = E^Q\left[\frac{S(T)}{M(T)}\right]$$

Because $\frac{S(t)}{M(t)}$ is a martingale under Q

$$KE^Q\left[\frac{1}{M(T)}\right] = \frac{S(0)}{M(0)}$$

$$KP(0,T) = S(0)$$

$$K = \frac{S(0)}{P(0,T)}. = F_{0,T}$$

$$= Se^{rT}$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

3. Swap

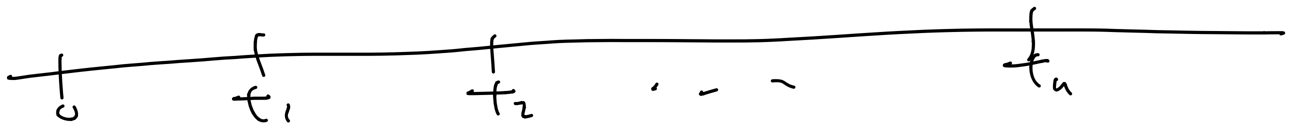
   Consider a swap with the fixed swap price payment dates on t_i, $i = 1, \ldots, n$.

   Let R be the fixed swap price.

   By the risk neutral valuation theorem, the value of the swap from the perspective of long party is given by

$$V(0) = M(0)E^Q\left[\sum_{i=1}^{n}\frac{\left(S(t_i) - R\right)}{M(t_i)}\right].$$

   Since the value of the swap is 0 at the inception date ($t = 0$), Payoff of swap (Long position, swap rate = R)



payoff $\quad (S(t_1)-R) \quad (S(t_2)-R) \quad \cdots \quad (S(t_n)-R)$

$$V(0) = M(0)\, E^Q\left[\sum_{i=1}^{n} \frac{(S(t_i)-R)}{M(t_i)}\right]$$

$$0 = E^Q\left[\sum_{i=1}^{n} \frac{(S(t_i)-R)}{M(t_i)}\right]$$

$\Rightarrow$ solve R

Application of the Risk Neutral Valuation Theorem (Cont'd)

$$R\sum_{i=1}^{n} E^{Q}\left[\frac{1}{M(t_i)}\right] = \sum_{i=1}^{n} E^{Q}\left[\frac{S(t_i)}{M(t_i)}\right]$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} S(0)$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} \frac{S(0)P(0,t_i)}{P(0,t_i)}$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} F_{0,t_i} P(0,t_i)$$

$$R = \frac{\sum_{i=1}^{n} F_{0,t_i} P(0,t_i)}{\sum_{i=1}^{n} P(0,t_i)}.$$

motivation for change of prob (measure)

Y : continuous r.v.

pdf of Y : $f(y)$

$$E[Y] = \int y\, f(y)\, dy$$

Let $g(y)$ be a pdf.

$$\{y : g(y) \neq 0\} = \{y : f(y) \neq 0\}$$

pdf

$$E[Y] = \int y\, f(y)\, dy = \int \boxed{y \frac{f(y)}{g(y)}}\, g(y)\, dy$$

$$= E\left[Y \frac{f(Y)}{g(Y)} \right]$$

r.v.

Change of prob

$$E^{f}[Y] = E^{g}\left[Y \frac{f(Y)}{g(Y)} \right]$$

$\dfrac{f(Y)}{g(Y)}$: ratio of two pdfs

① likelihood ratio

② Radon - Nikodym derivative

$$\frac{dQ_2}{dQ_1}$$

$f(Y)$

d

$g(Y)$

Change of Measure

Definition

The Randon-Nikodym derivative of Q_2 with respect to Q_1 based is denoted as

$$\frac{dQ_2}{dQ_1}$$

pdf

pdf

This notation can be understood as the ratio of the two probability density functions (pdfs).

Change of Measure (cont'd)

The expectations under measures Q_1 and Q_2 are related as follows:

$$E^{Q_2}\left[Y\right] = E^{Q_1}\left[\frac{dQ_2}{dQ_1}Y\right],$$

where Y is a random variable.

motivation for Girsanov thm

under a prob measure P, $Z(t)$ is a standard BM

$Z(t) \sim N(0, t)$

Consider $Y(t) = Z(t) + \mu t$ ← drift, $Y(t) \sim N(\mu t, t)$

$Y(t)$ is NOT a standard BM under P.

Question: Construct a prob measure Q to make $Y(t)$ to be a standard BM.

Procedure:

if under Q, $Z(t) \sim N(-\mu t, t)$, then $Y(t)$ is a standard BM

under P, pdf of $Z(t)$: $f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$

under Q, pdf of $Z(t)$: $g(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+\mu t)^2}{2t}}$

Radon - Nikodym derivative :

$$\frac{dQ}{dP} = \frac{g(x)}{f(x)}$$

$$= \exp\left(-\mu x - \frac{1}{2}\mu^2 t\right)$$

under P	under Q
$Z(t)$ is a standard BM	$Z(t)$ is NOT a standard BM BM with drift $-\mu t$

Change of Measure (Cont'd)

Girsanov Theorem (1-dimensional)

Let $Z(t)$ be *a* Brownian motion under the probability measure *P*. The Radon-Nikodym derivative of $\tilde{P}$ with respect to *P* is given by

$$\frac{d\tilde{P}}{dP} = \xi(t),$$

$$\frac{dQ}{dP} = \frac{g(x)}{f(x)}$$

$$= \exp\left(-\eta x - \frac{1}{2}\eta^2 t\right)$$

where

$$\xi(t) = \exp\left\{-\eta Z(t) - 0.5\eta^2 t\right\}.$$

Change of Measure (Cont'd)

Girsanov Theorem (1-dimensional) (Cont'd)

Under the new probability measure $\tilde{P}$, the process

$$\tilde{Z}(t) = Z(t) + \eta t,$$

$\Rightarrow$ $Z(t) = \tilde{Z}(t) - \eta t$

is a Brownian. = standard BM :

Change of Measure (Cont'd)

Girsanov Theorem (*K*-dimensional)

Let $Z_1(t)$, …, $Z_K(t)$ be K independent Brownian motions under the probability measure P. The Radon-Nikodym derivative of $\tilde{P}$ with respect to P is given by

$$\frac{d\tilde{P}}{dP} = \xi(t),$$

where

$$\xi(t) = \exp\left\{-\sum_{i=1}^{K} \eta_i Z_i(t) - 0.5t\sum_{i=1}^{K} \eta_i^2\right\}.$$

Change of Measure (Cont'd)

Girsanov Theorem (*K*-dimensional) (Cont'd)

Under the new probability measure $\widetilde{P}$, the process

$$\widetilde{Z}_i(t) = Z_i(t) + \eta_i t, \quad \text{for } i = 1, \ldots, K$$

is a Brownian motion and $\widetilde{Z}_1(t), \widetilde{Z}_1(t), \ldots, \widetilde{Z}_K(t)$ are independent.

P	$\tilde{P}$
$Z(t)$ is a standard BM	$Z(t)$ is a BM with drift $-\eta t$

$\tilde{Z}(t) = Z(t) + \eta t \longrightarrow$ $\tilde{Z}(t)$ is a standard BM

$\tilde{Z}(t)$ is a BM with drift ηt

By Girsanov thm, $\frac{d\tilde{P}}{dP} = \exp\left(-\eta Z(t) - \frac{1}{2}\eta^2 t\right)$

Black-Scholes Model Revisited

Assume the existence of a risk neutral measure Q under which all discounted price processes are martingales.

Also, assume that the securities model only has two tradable securities:

(i) Money market account: $M(t)$;

(ii) Non-dividend paying asset: $S(t)$.

Black-Scholes Model Revisited (Cont'd)

The price processes of the risky asset and the money market account under the real (physical) probability measure P are governed by

drift volatility

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t) \qquad S(t) = S(0)\exp((\mu - \tfrac{1}{2}\sigma^2)t + \sigma Z(t))$$

$$dM(t) = rM(t)dt \qquad E[S(t)] = S(0)e^{\mu t}$$

respectively, where μ is the expected return on the stock, r is the constant risk-free interest rate, $Z(t)$ is a Brownian motion under P.

Black-Scholes Model Revisited (Cont'd)

The price of the discounted risky asset is

$$S^*(t) = \frac{S(t)}{M(t)}.$$

$P \longrightarrow Q$

By Ito's lemma, the price process $S^*(t)$ becomes

drift

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\,dt + \sigma dZ(t)$$

$$= \sigma\left[dZ(t) + \frac{(\mu - r)}{\sigma}dt\right]$$

$$= \sigma d\left[Z(t) + \frac{(\mu - r)}{\sigma}t\right].$$

$S^*(t)$ is a martingale

SDE without drift $\Rightarrow$ $X(t)$ is a martingale

$$\frac{dX(t)}{X(t)} = \beta\, dZ(t) \Rightarrow X(t) = e^{-\frac{1}{2}\beta^2 t + \beta Z(t)}$$

$$\Downarrow$$

martingale

<u>under P</u>

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\, dt + \sigma\, dZ(t)$$

$$= \sigma\, d\left[Z(t) + \frac{\mu - r}{\sigma} t \right]$$

$$\Downarrow$$

Standard BM under a new prob.

By Girsanov thm,

$$\frac{dQ}{dP} = \exp\left(-\eta Z(t) - \frac{1}{2}\eta^2 t\right)$$

where $\eta = \frac{\mu - r}{\sigma}$

$\tilde{Z}(t) = Z(t) + \eta t$ is a standard BM under Q

under Q,

$$\frac{dS^*(t)}{S^*(t)} = \sigma\, d\tilde{Z}(t)$$ where $\tilde{Z}(t)$ is a Standard BM.

$S^*(t)$ is a martingale under Q.

Find SDE of $S(t)$ from the SDE of $S^*(t)$

$$S(t) = e^{rt} S^*(t)$$

$$dS(t) = \frac{\partial S(t)}{\partial S^*} dS^* + \frac{\partial S}{\partial t} dt + \cancel{\frac{1}{2}\frac{\partial^2 S}{\partial S^{*2}} (dS^*)^2}$$

$$= \underbrace{e^{rt} S^*}_{S(t)} dS^* + r \underbrace{e^{rt} S^*}_{S(t)} dt$$

$$= r S(t)\, dt + \sigma S(t)\, d\tilde{Z}(t)$$

(under Q)

Black-Scholes Model Revisited (Cont'd)

We would like to find the equivalent martingale measure Q under which $S^*(t)$ is a martingale. By the Girsanov Theorem, we choose

$$\xi(t) = \exp\left\{-\eta Z(t) - 0.5t\eta^2\right\} \quad \text{where } \eta = \frac{\mu - r}{\sigma},$$

Then

$$\tilde{Z}(t) = Z(t) + \frac{(\mu - r)}{\sigma} t$$

is a Brownian motion under the new measure constructed from the Girsanov Theorm. This measure is the required Q.

Black-Scholes Model Revisited (Cont'd)

Under the measure Q,

$$\frac{dS^*(t)}{S^*(t)} = \sigma d\widetilde{Z}(t).$$

Since $\widetilde{Z}(t)$ is a Brownian motion under Q, so $S^*(t)$ is a martingale under Q.

Under Q,

$$\frac{dS(t)}{S(t)} = r dt + \sigma d\widetilde{Z}(t)$$

where the drift rate equals the risk-free interest rate r.

Black-Scholes Model Revisited (Cont'd)

By the risk-neutral valuation theorem, the price of the call option $V(0)$ at time 0 is

$V(T)$: payoff of the call option

$$V(0) = e^{r\times 0} E_0^Q\left[\frac{\max(S(T)-K,0)}{e^{rT}}\right]$$

$$= e^{-rT} E_0^Q\left[\max(S(T)-K,0)\right]$$

So, the Black-Scholes formula can then be obtained. The details can refer to the hand-written notes "Derivation of BS formula".

Change of Numeraire

A *numeraire* is a positively priced **non-dividend-paying** asset which denominate other asset. A typical example of a numeraire is the currency of a country, because people usually measure other asset's price in terms of the unit of currency.

$$\frac{V(0)}{M(0)} = E^{Q}\left[\frac{V(T)}{M(T)}\right]$$

denominator

$\{X(t) : t \geq 0\}$ is a stochastic process

$$= E^{Q}\left[\frac{V(T)}{X(T)} X(T) \frac{1}{M(T)}\right]$$

$$= E^{Q}\left[\frac{V(T)}{X(T)} \frac{X(T)}{M(T)}\right]$$

$$V(0) = M(0) E^{Q}\left[\frac{V(T)}{X(T)} \frac{X(T)}{M(T)}\right]$$

$$= E^{Q}\left[\frac{V(T)}{X(T)} \frac{\frac{X(T)}{X(0)}}{\frac{M(T)}{M(0)}}\right] X(0)$$

Radon–Nikodym derivative

$$= E^{Q_X}\left[\frac{V(T)}{X(T)}\right] X(0)$$

$$\frac{V(0)}{X(0)} = E^{Q_X}\left[\frac{V(T)}{X(T)}\right] \Rightarrow \frac{V(t)}{X(t)} \text{ is a martingale under } Q_X$$

Tencent : RMB 300 $\Rightarrow$ Tencent = 10 shares of ABC

ABC : RMB 30

Change of Numeraire (Cont'd)

In the risk-neutral valuation, we take the money market account as the numeraire, when the security price is denominated with money market account it is a martingale under the risk-neutral measure Q.

Change of Numeraire (Cont'd)

Theorem (Change of Numeraire)

(*Geman, Karoui and Rochet (1995)*)

Let $X(t)$ be a positively priced non-dividend-paying asset such that $X(t)/M(t)$ is a martingale under Q.

Then there exists a probability measure Q_X defined by its Radon-Nikodym derivative with respect to Q

$$\frac{dQ_X}{dQ} = \left(\frac{X(t)}{X(0)}\right) \Big/ \left(\frac{M(t)}{M(0)}\right), \qquad 0 \le t \le T.$$

Change of Numeraire (Cont'd)

Theorem (Change of Numeraire) (Cont'd)

such that

$$V(0) = M(0)E^{Q}\left[\frac{V(T)}{M(T)}\right] = X(0)E^{Q_X}\left[\frac{V(T)}{X(T)}\right]$$

where $V(T)$ is defined in the payoff of a derivative security which satisfies the properties in Risk neutral Valuation Theorem.

Change of Numeraire (Cont'd)

Remark

The Change of Numeraire Theorem can be generalized for changing from the numeraire-measure pair $(N_1(t), Q_{N_1})$ to the other pair $(N_2(t), Q_{N_2})$:

$$N_1(0)E^{Q_{N_1}}\left[\frac{V(T)}{N_1(T)}\right] = N_2(0)E^{Q_{N_2}}\left[\frac{V(T)}{N_2(T)}\right]$$

and

$$\frac{dQ_{N_1}}{dQ_{N_2}} = \left(\frac{N_1(t)}{N_1(0)}\right) \Big/ \left(\frac{N_2(t)}{N_2(0)}\right), \qquad 0 \le t \le T.$$

Deriving the Black-Scholes Formula

Define the indicator function $\mathbf{I}_{\{S_T \geq K\}}$ as follows:

$$\mathbf{I}_{\{S_T \geq K\}} = \begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

We have the following result for the indicator function:

$$E_t^Q\left[\mathbf{I}_{\{S_T \geq K\}}\right] = Q\left(S_T \geq K \mid S_t = S\right)$$

$$\int_0^\infty 1_{\{x \geq K\}}\, g(x \mid S_t = S)\, dx$$

where $g(x \mid S_t = S)$ is the pdf of S_T conditional on $S_t = S$

$$= \int_K^\infty g(x \mid S_t = S)\, dx = \Pr(S_T \geq K \mid S_t = S)$$

Deriving the Black-Scholes Formula (Cont'd)

By the risk-neutral valuation theorem, the price of the call option at time 0 is

$V(\tau)$

$$
\begin{aligned}
V(0) &= e^{-rT} E_0^Q\left[\max\left(S_T - K, 0\right)\right] \\
&= e^{-rT} E_0^Q\left[\left(S_T - K\right)\mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} E_0^Q\left[K \mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} K E_0^Q\left[\mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} K Q\left(S_T \geq K \mid S_0 = S\right)
\end{aligned}
$$

$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$

$Z \sim N(0,1)$

$N(d_2)$

$$e^{-rT} E^{Q}[S_T \mathbb{1}_{\{S_T \geq K\}}]$$

Define a new measure Q_S,

$$\frac{dQ_S}{dQ} = \frac{\frac{S(t)}{S(0)}}{\frac{M(t)}{M(0)}}$$

By change of numeraire thm,

$$e^{-rT} E^{Q}[S_T \mathbb{1}_{\{S_T \geq K\}}] = E^{Q}\left[\frac{S_T \mathbb{1}_{\{S_T \geq K\}}}{M(T)}\right]$$

$$= S(0) E^{Q_S}\left[\frac{\cancel{S_T} \mathbb{1}_{\{S_T \geq K\}}}{\cancel{S_T}}\right]$$

$$= S(0) E^{Q_S}[\mathbb{1}_{\{S_T \geq K\}}]$$

$$= S(0) Q_S[S_T \geq K]$$

Δ

Find out the SDE of S_t under Q_S,

$$\frac{dQ_S}{dQ} = \frac{\frac{S(t)}{S(0)}}{\frac{M(t)}{M(0)}} = \frac{\frac{S(0) \exp((r - \frac{1}{2}\sigma^2)t + \sigma \tilde{Z}(t))}{S(0)}}{e^{rt}}$$

$$= \exp(-\frac{1}{2}\sigma^2 t + \sigma \tilde{Z}(t))$$

Under Q_S,

$W(t) = \tilde{Z}(t) - \sigma t$ is a standard BM under Q_S

Recall: under Q, $\frac{dS(t)}{S(t)} = r\,dt + \sigma\, d\tilde{Z}(t)$

Under Q_S,

$$\frac{dS(t)}{S(t)} = r\,dt + \sigma\left[dW(t) + \sigma\, dt\right]$$

$$= (r + \sigma^2)\,dt + \sigma\, dW(t)$$

$$\Rightarrow S(t) = S(0)\exp\left(\left(r + \sigma^2 - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$$

$$Q_S(S_T \geq K) = Q_S\left(S(0)\exp\left(\left(r + \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z\right) > K\right)$$

$$= N(d_1)$$

Deriving the Black-Scholes Formula (Cont'd)

For simplicity, we assume δ = 0, under Q,

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(r - 0.5\sigma^2\right)T, \sigma^2 T\right)$$

Therefore,

$$Q\left(S_T \geq K \mid S_0 = S\right) = N\left(d_2\right)$$

Deriving the Black-Scholes Formula (Cont'd)

To evaluate $E_0^Q[S_T \mathbf{I}_{\{S_T \geq K\}}]$, we use the change of numeraire technique by changing the numeraire-measure pair (M_t, Q) to the other pair (S_t, Q_S).

Deriving the Black-Scholes Formula (Cont'd)

Now

$$e^{-rT}E_0^Q[S_T\mathbf{I}_{\{S_T\geq K\}}] = E_0^Q\left[\frac{S_T\mathbf{I}_{\{S_T\geq K\}}}{e^{rT}}\right]$$

$$= SE_0^{Q_S}\left[\frac{S_T\mathbf{I}_{\{S_T\geq K\}}}{S_T}\right]$$

$$= SQ_s\left(S_T\geq K \mid S_0 = S\right)$$

Deriving the Black-Scholes Formula (Cont'd)

The Radon-Nikodym derivative of Q_S with respect to Q is

$$\frac{dQ_S}{dQ} = \left(\frac{S(t)}{S(0)}\right) \Big/ \left(\frac{M(t)}{M(0)}\right) = \left(\frac{S(t)}{S(0)}\right) \Big/ \left(\frac{e^{rt}}{e^{r \times 0}}\right)$$

$$= \exp\left(-0.5\sigma^2 t + \sigma Z_t\right)$$

Deriving the Black-Scholes Formula (Cont'd)

By the Girsanov Theorem, we have

$$\widetilde{Z}_t = Z_t - \sigma t$$

is a Brownian motion under Q_S.

Under Q_S, we have

$$\begin{aligned}\frac{dS_t}{S_t} &= r dt + \sigma dZ_t \\ &= r dt + \sigma\left(d\widetilde{Z}_t + \sigma dt\right) \\ &= \left(r + \sigma^2\right)dt + \sigma d\widetilde{Z}_t\end{aligned}$$

Deriving the Black-Scholes Formula (Cont'd)

By Itô's lemma, we have

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(r + 0.5\sigma^2\right)T, \sigma^2 T\right)$$

So,

$$Q_s\left(S_T \geq K \mid S_0 = S\right)$$
$$= N(d_1)$$

Combining all the terms, we have

$$V(0) = SN(d_1) - e^{-rT} KN(d_2).$$

Change of Numeraire (Summary)

$$V(0) = M(0)\, E^{Q}\left[\frac{V(T)}{M(T)}\right] = X(0)\, E^{Q_X}\left[\frac{V(T)}{X(T)}\right]$$

how to choose $X(T)$

$$\frac{dQ_X}{dQ} = \frac{\dfrac{X(t)}{X(0)}}{\dfrac{M(t)}{M(0)}}$$

Continuous Dividend Yield Models

- In the previous sections, we require discounted stock prices are martingale under the risk-neutral measure. *<u>This is the case provided the stock pays no dividend</u>*.
- The key feature of a risk-neutral measure is that it causes discounted portfolio values to be martingales, and that ensures the absence of arbitrage.

Continuous Dividend Yield Models (cont'd)

- In order for the discounted value of a portfolio that invests in a dividend-paying stock to be a martingale, the discounted value of the stock with the dividends reinvested must be a martingale, but the discounted stock price itself is **NOT** a martingale. That is, we require

$$V^*(t) = \frac{e^{qt} S(t)}{M(t)}, \Rightarrow \text{martingale under risk-neutral prob } Q$$

where q is the dividend yield, to be a martingale under Q.

Continuous Dividend Yield Models (cont'd)

The asset price dynamics is assumed to follow the GBM under real prob. P

$$\frac{dS(t)}{S(t)} = \rho dt + \sigma dZ(t).$$

ex-dividend

stock price

Continuous Dividend Yield Models (cont'd)

Suppose all the dividend yields received are used to purchase additional units of asset, then the value process $V(t)$ of holding 1 unit of asset initially is given by

$$V(t) = e^{qt} S(t),$$

where e^{qt} represents the growth factor in the number of units.

Continuous Dividend Yield Models (cont'd)

By Ito's lemma, we have

$$\frac{dV(t)}{V(t)} = \left(\rho + q\right)dt + \sigma dZ\left(t\right).$$

We would like to find the equivalent risk-neutral measure Q under which the

$$V^*(t) = \frac{V\left(t\right)}{M\left(t\right)}$$

is a Q-martingale.

G: martingale under Q.

under P,

$$dV^*(t) = V^*(t)\,[(\rho + q - r)\,dt + \sigma\, dZ(t)]$$

$$\frac{dV^*(t)}{V^*(t)} = \sigma\, d\left[\underbrace{Z(t) + \frac{\rho+q-r}{\sigma}t}_{\text{standard BM under } Q} \right]$$

By Girsanov thm,

$$\frac{dQ}{dP} = \exp\left(-\eta Z(t) - \frac{1}{2}\eta^2 t\right)$$

where $\eta = \frac{\rho + q - r}{\sigma}$

under Q, $\tilde{Z}(t) = Z(t) + \eta t$ is a standard BM.

$$\frac{dV^*(t)}{V^*(t)} = \sigma\, d\tilde{Z}(t)$$

So, $V^*(t)$ is a martingale under Q.

$$V^*(t) = \frac{e^{qt}S(t)}{e^{rt}} \Rightarrow S(t) = e^{(r-q)t}V^*(t)$$

By Itô's lemma,

$$\frac{dS(t)}{S(t)} = (r-q)\,dt + \sigma\, d\tilde{Z}(t).$$

Continuous Dividend Yield Models (cont'd)

We choose η in the Radon-Nikodym derivative to be

$$\eta = \frac{\rho + q - r}{\sigma}.$$

Now $\tilde{Z}(t)$ is a Brownian process under Q and

$$d\tilde{Z}(t) = dZ(t) + \frac{\rho + q - r}{\sigma} dt.$$

Also, $V(t)$ becomes Q-martingale since

$$\frac{dV^*(t)}{V^*(t)} = \sigma d\tilde{Z}(t).$$

Continuous Dividend Yield Models (cont'd)

The asset price $S(t)$ under Q becomes

$$\frac{dS(t)}{S(t)} = (r-q)\,dt + \sigma d\tilde{Z}(t).$$

Hence, the risk-neutral drift rate of $S(t)$ is $r - q$.

Equivalently, we have

$$\frac{d\left(e^{qt}S(t)\right)}{e^{qt}S(t)} = r\,dt + \sigma d\tilde{Z}(t).$$

$$\frac{dX(t)}{X(t)} = r\,dt + \sigma\, d\tilde{Z}(t)$$

where $X(t) = e^{qt}S(t)$

Continuous Dividend Yield Models (cont'd)

Call and put price formulas

The price of a European call option can be obtained as follows:

$$C = e^{-rT} E^Q\left[\max\left(S(T) - K, 0\right)\right]$$

$X(T)$

$$= e^{-qT} e^{-rT} E^Q\left[\max\left(e^{qT} S(T) - e^{qT} K, 0\right)\right] = e^{-(q+r)T} E^Q\left[\max\left(X(T) - e^{qT} K, 0\right)\right]$$

$$= e^{-qT}\left[S e^{q\times 0} N(d_1) - e^{-rT} e^{qT} K N(d_2)\right]$$

$$= S e^{-qT} N(d_1) - K e^{-rT} N(d_2),$$

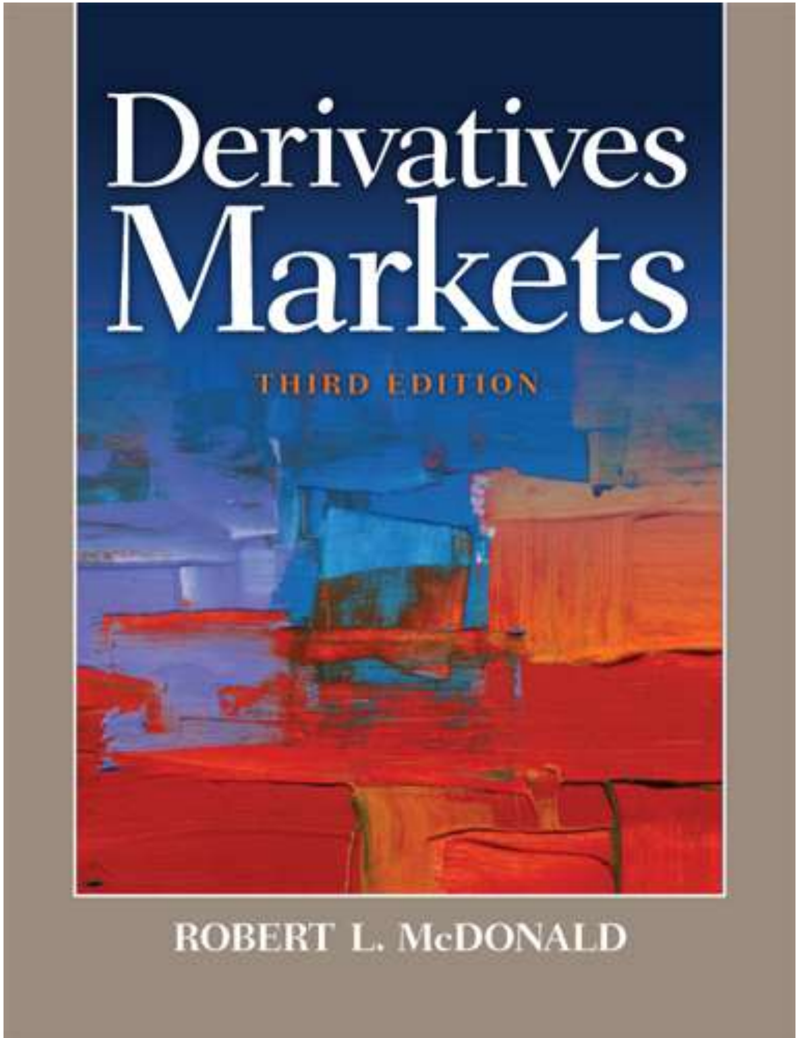
where

$$d_1 = \frac{\ln\left(\dfrac{S}{Ke^{qT}}\right) + \left(r + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\dfrac{S}{K}\right) + \left(r - q + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Continuous Dividend Yield Models (cont'd)

Similarly, the put price formula is given by

$$P = Ke^{-rT}N\left(-d_2\right) - Se^{-qT}N\left(-d_1\right).$$



Chapter 15
(Chapter 23 in the textbook)
Exotic Options

Points to Notes

1. What are the all-or-nothing options? See P. 3 – 4.
2. How are the all-or-nothing options related to the BS call and put options? See P. 5 – 9.
3. What are the Asian options? See P. 10 - 11
4. What are the differences between the arithmetic and geometric average? See P. 12 – 14.
5. What are the barrier options? See P. 15.
6. How are the barrier options related to the ordinary call and put options? See P. 16 – 18.

All-or-Nothing Options

Terminology

Notation	Meaning
Asset	Payment at expiration is one unit of the asset
Cash	Payment at expiration is $1
Call	Payment received if *S*(*T*) > *K*.
Put	Payment received if *S*(*T*) < *K*.

Definition

$$d_1 = \frac{\left[\ln\left(S(t)/K\right) + \left(r - \delta + 0.5\sigma^2\right)(T-t)\right]}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

All-or-Nothing Options

- Simple all-or-nothing options pay the holder a discrete amount of cash or a share if some particular event occurs.
- Cash-or-nothing
  - Call: pays \$1 if $S_T > K$ and zero otherwise

payoff $= \$1 \cdot 1_{\{S_T > K\}}$

$$\text{CashCall}(S, K, \sigma, r, T-t, \delta) = e^{-r(T-t)} N(d_2)$$

  - Put: pays \$1 if $S_T < K$ and zero otherwise

$$\text{CashPut}(S, K, \sigma, r, T-t, \delta) = e^{-r(T-t)} N(-d_2)$$

- Asset-or-nothing
  - Call: pays S_T (one unit share) if $S_T > K$ and zero otherwise

$$\text{AssetCall}(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)} N(d_1)$$

  - Put: pays S_T (one unit share) if $S_T < K$ and zero otherwise

$$\text{AssetPut}(S, K, \sigma, r, T-t, \delta) = Se^{-\delta(T-t)} N(-d_1)$$

① Payoff $= \$1 \cdot 1_{\{S_T > K\}}$

$$V(0) = E^Q\left[\frac{V(T)}{M(T)}\right] = e^{-rT} E^Q\left[1_{\{S_T > K\}}\right]$$

$$= e^{-rT} Q[S_T > K]$$

$$= e^{-rT} N(d_2)$$

② pay \$1 if $S_T < K$.

$$V(0) = e^{-rT} N(-d_2)$$

③ pay 1 share of stock if $S_T > K$.

$$\text{payoff} = S_T \, 1_{\{S_T > K\}}$$

$$V(0) = e^{-rT} E^Q\left[S_T \, 1_{\{S_T > K\}}\right]$$

$$= S_0 e^{-\delta T} N(d_1)$$

All-or-Nothing Options (cont'd)

- + 1 asset-or-nothing call option with strike price *K*
  – *K* cash-or-nothing call option with strike price *K*
  = 1 ordinary call option with strike price *K*

$$
\begin{aligned}
&\mathrm{BSCall}(S, K, \sigma, r, T-t, \delta) \\
&= \mathrm{AssetCall}(S, K, \sigma, r, T-t, \delta) - K \times \mathrm{CashCall}(S, K, \sigma, r, T-t, \delta) \\
&= Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)
\end{aligned}
$$

Asset Call = BS Call + K Cash Call

All-or-Nothing Options (cont'd)

- Similarly, a put option can be created by buying *K* cash-or-nothing puts, and selling 1 asset-or-nothing put

$$\text{BSPut}(S, K, \sigma, r, T-t, \delta)$$
$$= K \times \text{CashPut}(S, K, \sigma, r, T-t, \delta) - \text{AssetPut}(S, K, \sigma, r, T-t, \delta)$$

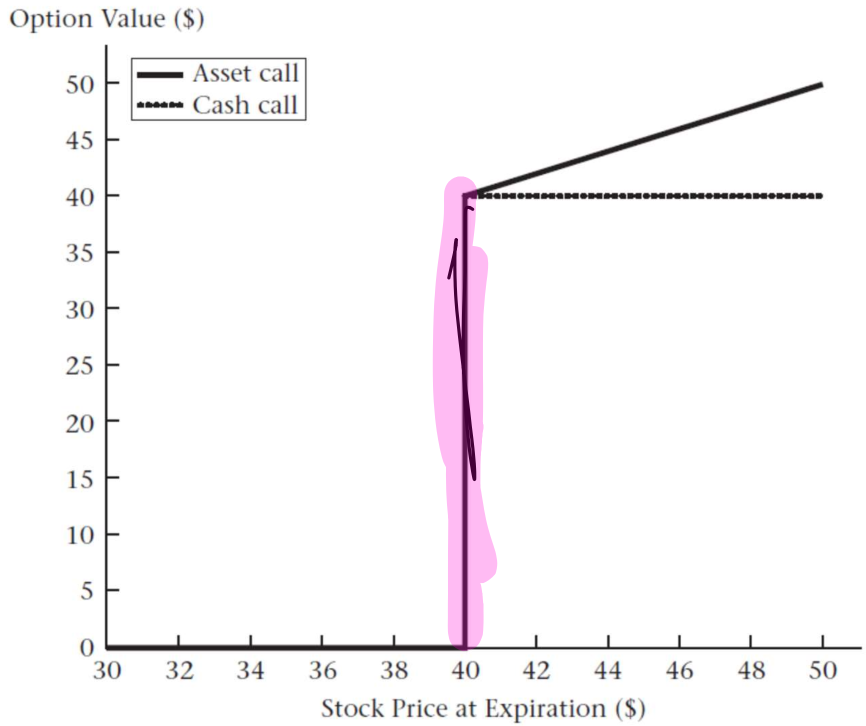
All-or-Nothing Options (cont'd)

- All-or-nothing options are easy to price but hard to hedge.
- Fig. 1 shows that a small swing in the stock price can determine whether the option is in- or out-of-the money, with the payoff changing discretely.
- Fig. 2 shows that hedging is straightforward and delta is well behaved when 3 months to expiration. However, with 2 minutes to expiration, the cash call delta at \$40 is 15. *For the at-the-money option, delta and gamma approach infinity at expiration because an arbitrarily small change in the price can result in a \$1 change in the option's value.*

All-or-Nothing Options (cont'd)

FIGURE 23.1

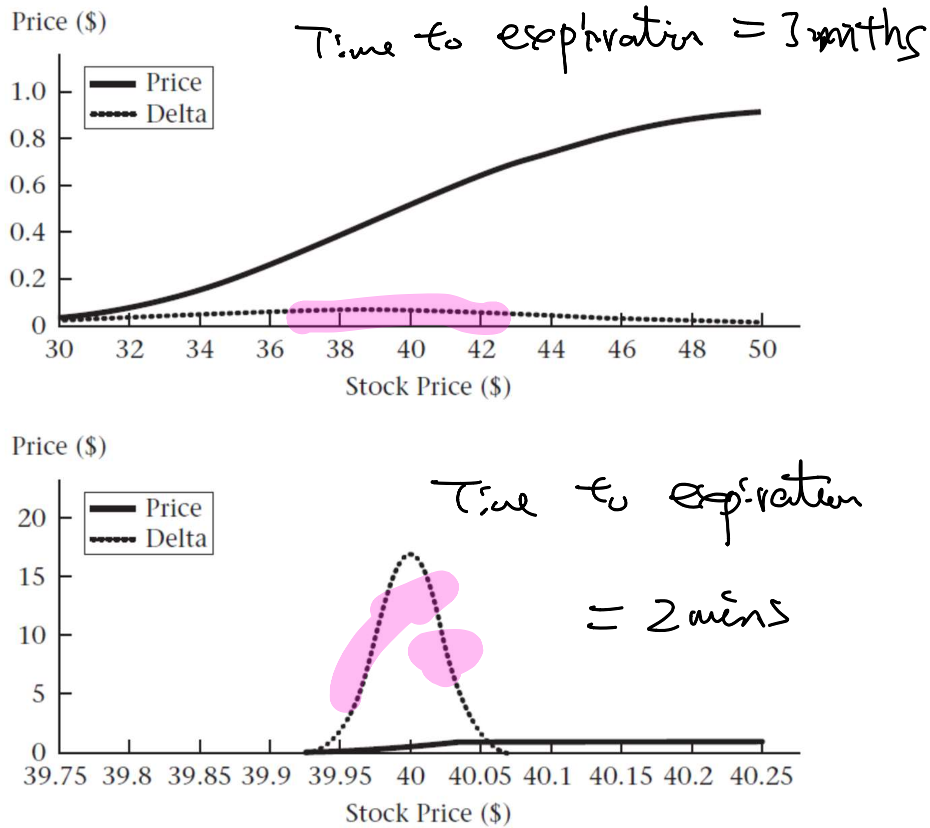
Payoff at maturity to one asset call and 40 cash calls. Assumes $K = \$40$, $\sigma = 0.30$, $r = 0.08$, and $\delta = 0$. The payoff to both is zero for $S < \$40$.



All-or-Nothing Options (cont'd)

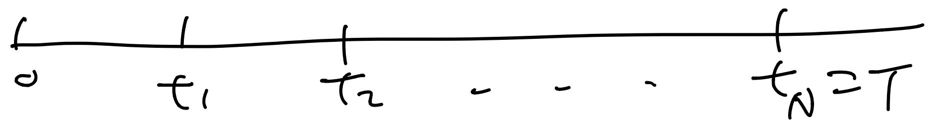
FIGURE 23.2

Price and delta of a cash call at two different times to expiration: 3 months (top panel) and 2 minutes (bottom panel). Assumes $K = \$40$, $\sigma = 0.30$, $r = 0.08$, and $\delta = 0$.



Average of stock price

Two types of average



① Arithmetic average

$$A(T) = \frac{\sum_{i=1}^{N} S(t_i)}{N}$$

② Geometric average

$$G(T) = \left[\prod_{i=1}^{N} S(t_i) \right]^{\frac{1}{N}}$$

payoff of Asian options :

① $\max(A(T) - K, 0)$

② $\max(G(T) - K, 0)$

③ $\max(K - A(T), 0)$

④ $\max(K - G(T), 0)$

By risk-neutral valuation

a) $V(0) = e^{-rT} E^{Q} [\max(A(T) - K, 0)]$

require pdf of $A(T)$

b) $V(0) = e^{-rT} E^{Q} [\max(G(T) - K, 0)]$

require the pdf of $G(T)$.

b) is easier to price than a)

$G(T)$ is GBM but $A(T)$ is not.

Asian Options

- The payoff of an Asian option is based on the average price over some period of time. An Asian options is an example of a path-dependent option.
- Situations when Asian options are useful:
  - When a business cares about the average exchange rate over time.
  - When a single price at a point in time might be subject to manipulation.
  - When price swings are frequent due to thin markets.

Asian Options (cont'd)

- Asian options are less valuable than otherwise equivalent ordinary options, since the averaged price of the underlying asset is less volatile than the asset price itself, and an option on a lower volatility asset is worth less.

Asian Options (cont'd)

- There are eight (2^3) basic kinds of Asian options:
  - Put or call.
  - Geometric or arithmetic average.
  - Average asset price is used in place of underlying. price or the strike price.
- Arithmetic versus geometric average:
  - Suppose we record the stock price every *h* periods from t = 0 to t = T.
  - Arithmetic average: Geometric average:

$$A(T) = \frac{1}{N}\sum_{i=1}^{N} S_{ih}$$

$$G(T) = (S_h \times S_{2h} \times \cdots \times S_{Nh})^{1/N}$$

Asian Options (cont'd)

- Average used as the asset price: Average price option
  - Geometric average price call = max [0, $G(T) - K$].
  - Geometric average price put = max [0, $K - G(T)$].
- Average used as the strike price: Average strike option
  - Geometric average strike call = max [0, $S_T - G(T)$].
  - Geometric average strike put = max [0, $G(T) - S_T$].

Asian Options (cont'd)

- All four options above could also be computed using arithmetic average instead of geometric average.
- Relatively simple pricing formulas exist for pricing European options on the geometric average but not for arithmetic average options.

MFE5130 – Financial Derivatives

Class Activity (4-December-2019) (Solution)

Important Notes:

1. This class activity is counted toward to your class participation score. **Fail** to hand in this class activity worksheet in the class will receive **0 score** for that class.
2. **0 mark** will be received if you leave the solution blank.

Name:	Student No.:

Problem 1

Given that $S(t)$ is a GBM (Geometric Brownian motion) which follows

$$\frac{dS(t)}{S(t)} = 0.06dt + 0.3dZ(t)$$

where $Z(t)$ is a standard Brownian motion under measure P.

Find another measure Q by specifying the Radon-Nikodym derivative of Q with respect to P, $\frac{dQ}{dP}$, such that $S(t)$ is governed by

??

$$\frac{dS(t)}{S(t)} = 0.02dt + 0.3d\tilde{Z}(t)$$

under the measure Q, where $\tilde{Z}(t)$ is a standard Brownian motion under Q.

Solution

Let $\eta = \frac{0.06-0.02}{0.3} = \frac{2}{15}$. and consider the Radon Nikodym derivative of Q with respect to P based on the information up to time t:

$$\frac{dQ}{dP} = \exp\left(-\frac{2}{15}Z(t) - \frac{1}{2}\left(\frac{2}{15}\right)^2 t\right) = \exp\left(-\frac{2}{15}Z(t) - \frac{2}{225}t\right).$$

$\eta = \frac{2}{15}$

Under the measure Q, the stochastic process

$$\boxed{\tilde{Z}(t) = Z(t) + \frac{2}{15}t}$$

is a standard Brownian motion under Q by the Girsanov Theorem.

It is seen that when we set $\eta = \frac{2}{15}$, then

$$Z(t) = \tilde{Z}(t) - \frac{2}{15}t$$

$$0.06dt + 0.3dZ(t) = 0.06dt + 0.3\left(d\tilde{Z}(t) - \frac{2}{15}dt\right) = 0.02dt + 0.3d\tilde{Z}(t).$$

Therefore, $S(t)$ is governed by

$$\frac{dS(t)}{S(t)} = 0.02dt + 0.3d\tilde{Z}(t)$$

under measure Q.

$$\frac{dS(t)}{S(t)} = 0.06\,dt + 0.3\,dZ(t)$$

$$\downarrow$$

$$\frac{dS(t)}{S(t)} = 0.02\,dt + 0.3\,d\tilde{Z}(t)$$

$$\frac{dS(t)}{S(t)} = 0.02\,dt - 0.02\,dt + 0.06\,dt + 0.3\,dZ(t)$$

$$= 0.02\,dt + 0.04\,dt + 0.3\,dZ(t)$$

$$= 0.02\,dt + 0.3\,d\Big[\underbrace{Z(t) + \underbrace{\frac{0.04}{0.3}}_{\text{drift}}t}_{\substack{\Downarrow \\ \tilde{Z}(t) \\ \text{Standard BM}}}\Big]$$

By Girsanov thm

$$\frac{dQ}{dP} = \exp\left(-\eta Z(t) - \frac{1}{2}\eta^2 t\right)$$

$$\tilde{Z}(t) = Z(t) + \underbrace{\eta}_{= \frac{0.04}{0.3}} t$$ is a standard BM under Q.