

Derivatives Markets

THIRD EDITION



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ALWAYS LEARNING

Chapter 13 (Chapter 20 in the textbook)

Brownian Motion
and Itô's Lemma

PEARSON



Points to Note

1. Definition of the Standard Brownian motion. See P.3 – 4.
2. Stochastic Processes which are constructed from the standard Brownian motion. See P.5 – 11.
3. Modelling the correlated asset prices through correlated Brownian motions. See P.12 – 16.
4. Ito's lemma: univariate and multivariate versions. See P.17 – 27.
5. Sharpe ratios of two perfectly correlated assets. See P.28 – 30.



Brownian Motion

- A stochastic process is a random process that is a function of time.
- Brownian motion is a stochastic process that evolves in continuous time, with movements that are continuous.
 - A random walk can be generated by flipping a coin each period and moving one step, with the direction determined by whether the coin is heads or tails.
 - To generate Brownian motion, we would flip the coins infinitely fast and take infinitesimally small steps at each point.



Brownian Motion (cont'd)

$$Z(t) \sim N(0, t)$$

- Brownian motion is a continuous stochastic process, $Z(t)$, with the following characteristics:
 - $Z(0) = 0$.
 - $Z(t + s) - Z(t)$ is normally distributed with mean 0 and variance s .
 - $Z(t + s_1) - Z(t)$ is independent of $Z(t) - Z(t - s_2)$, where $s_1, s_2 > 0$. That is, nonoverlapping increments are independently distributed.
 - $Z(t)$ is continuous.
- These properties imply that $Z(t)$ is a martingale: a stochastic process for which

$$E[Z(t+s) | \{Z(u), 0 \leq u \leq t\}] = Z(t).$$



Arithmetic Brownian Motion

- With pure Brownian motion, the expected change in Z is 0, and the variance per unit time is 1. We can generalize this to allow an arbitrary variance and a nonzero mean SDE (stochastic differential equation)

$$dX(t) = \alpha dt + \sigma dZ(t) \quad (20.8)$$

- This process is called **arithmetic Brownian motion**

- α is the instantaneous mean per unit time.
- σ^2 is the instantaneous variance per unit time.
- The variable $X(t)$ is the sum of the individual changes dX . $X(t)$ is normally distributed, i.e., $X(T) - X(0) \sim N(\alpha T, \sigma^2 T)$.

$$dx(t) = \alpha dt + \sigma dz(t) \quad , \quad \alpha, \sigma \text{ are constants}$$

$$\int_0^t dx(s) = \int_0^t \alpha ds + \int_0^t \sigma dz(s)$$

$$x(t) - x(0) = \alpha t + \sigma [z(t) - z(0)]$$

$$x(t) = x(0) + \alpha t + \sigma z(t)$$



Arithmetic Brownian Motion (cont'd)

- An integral representation of equation (20.8) is

$$X(T) = X(0) + \int_0^T \alpha dt + \int_0^T \sigma dZ(t)$$

- Here are some properties of the process in equation (20.8)
 - $X(t)$ is normally distributed because it is a scaled Brownian process.
 - The random term is multiplied by a scale factor that enables us to specify the variance of the process.
 - The αdt term introduces a nonrandom *drift* into the process.



Arithmetic Brownian Motion (cont'd)

- Arithmetic Brownian motion has several drawbacks
 - There is nothing to prevent X from becoming negative, so it is a poor model for stock prices.
 - The mean and variance of changes in dollar terms are independent of the level of the stock price.
- Both of these criticisms will be eliminated with geometric Brownian motion.



The Ornstein-Uhlenbeck Process

- We can incorporate mean reversion by modifying the drift term

$$dX(t) = \lambda[a - X(t)] dt + \sigma dZ(t) \quad (20.9)$$

- This equation is called an **Ornstein-Uhlenbeck process**.

- The parameter λ measures the speed of the reversion:
If λ is large, reversion happens more quickly.
- In the long run, we expect X to revert toward a .
- As with arithmetic Brownian motion, X can still become negative.

$$dX(t) = \lambda[a - X(t)]dt + \sigma dZ(t) \quad (20.9)$$

$$\int_0^t dX(s) = \int_0^t \lambda [a - X(s)] ds + \int_0^t \sigma dZ(s)$$

$$X(t) - X(0) = \int_0^t \lambda a ds - \lambda \int_0^t X(s) ds + \sigma [Z(t) - Z(0)]$$

Interest rate

mean-reversion

$$\lambda [a - X(t)]$$

mean-level
?

$$\lambda > 0$$

If $a > X(t)$, $\lambda [a - X(t)] > 0 \Rightarrow$ drag the $X(t)$ up

If $a < X(t)$, $\lambda [a - X(t)] < 0 \Rightarrow$ drag $X(t)$ down

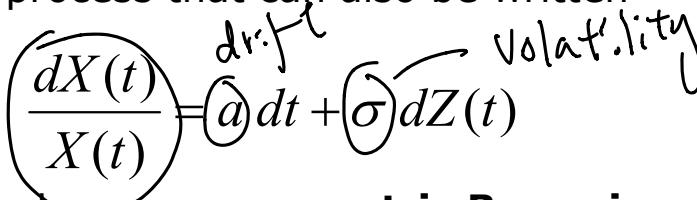


Geometric Brownian Motion

- An equation, in which the drift and volatility depend on the stock price, is called an **Itô process**.
 - Suppose we modify arithmetic Brownian motion to make the instantaneous mean and standard deviation proportional to $X(t)$

$$dX(t) = \alpha X(t) dt + \sigma X(t) dZ(t)$$

- This is an Itô process that can also be written

return 

$$\left(\frac{dX(t)}{X(t)} \right) = (\hat{\alpha}) dt + (\hat{\sigma}) dZ(t) \quad (20.11)$$

- This process is known as **geometric Brownian motion (GBM)**.



Geometric Brownian Motion (cont'd)

- The percentage change in the asset value is normally distributed with instantaneous mean α and instantaneous variance σ^2 .
- The integral representation for equation (20.11) is
NOT the solution

$$X(T) - X(0) = \int_0^T \alpha X(t) dt + \int_0^T \sigma X(t) dZ(t)$$

$$X(t) = X(0) \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)\right) \checkmark$$



Multiplication Rules

- We can simplify complex terms containing dt and dZ by using the following “multiplication rules”:

$$dt \times dZ = 0 \quad (20.15a)$$

$$(dt)^2 = 0 \quad (20.15b)$$

$$(dZ)^2 = dt \quad (20.15c)$$

$$d\bar{z} = \bar{z}(t+\Delta t) - \bar{z}(t) \sim N(0, \sigma^2)$$

$$\sigma_{\bar{z}} \approx \sqrt{\Delta t}$$

$$\begin{aligned}
 S_1(t) &: \frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 (dz_1(t)) = \frac{dW_1(t)}{\sqrt{2}} \\
 S_2(t) &: \frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2 (dz_2(t)) = \sum_{j=1}^2 \gamma_{2j} dW_j(t) \\
 &\vdots && \vdots \\
 S_n(t) &: \frac{dS_n(t)}{S_n(t)} = \alpha_n dt + \sigma_n (dz_n(t)) = \sum_{j=1}^n \gamma_{nj} dW_j(t)
 \end{aligned}$$

$z_1(t), z_2(t) \dots z_n(t)$ are **Correlated BMs**
 $w_1(t) \dots w_n(t)$ are **independent BMs**
 $\text{Corr}(z_i(t), z_j(t)) = c_{ij}(t)$

Modelling **correlated** assets by **independent** BMs

$$n=2$$

$$\begin{aligned}
 \frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 (dz_1(t)) \\
 \frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2 \rho (dz_1(t)) + \sigma_2 \sqrt{1-\rho^2} (dz_2(t))
 \end{aligned}$$

Additional stochastic

$z_1(t), z_2(t)$ are **independent BMs**

$n=3$

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \left(\sigma_{11} dW_1(t) \right)$$

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \left(\sigma_{21} dW_1(t) + \sigma_{22} dW_2(t) \right)$$

$$\frac{dS_3(t)}{S_3(t)} = \alpha_3 dt + \sigma_{31} dW_1(t) + \sigma_{32} dW_2(t) + \sigma_{33} dW_3(t)$$

$$\text{Cov}(\sigma_{11} W_1(t), \sigma_{21} W_1(t) + \sigma_{22} W_2(t))$$

$$= \text{Cov}(\sigma_{11} W_1(t), \sigma_{21} W_1(t)) + \text{Cov}(\sigma_{11} W_1(t), \sigma_{22} W_2(t))$$

$$\leq \sigma_{11} \sigma_{21} \text{Var}(W_1(t))$$

$$\leq \sigma_{11} \sigma_{21} t$$



Modeling Correlated Asset Prices

- Suppose that we have m asset processes

$$\frac{dX_i}{X_i} = (\alpha_i - \delta_i)dt + \sigma_i dZ_i \quad i = 1, \dots, m$$

The correlation between X_i and X_j will be generated by correlation between $Z_i(t)$ and $Z_j(t)$.

Next, we illustrate how we can create correlated diffusion processes by expressing dZ_i and dZ_j as sums of independent diffusions.



Modeling Correlated Asset Prices (cont'd)

- With $m = 2$ as an illustration:

Let $W_1(t)$ and $W_2(t)$ be independent Brownian motions and define

$$dZ_1(t) = dW_1(t)$$

$$dZ_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)$$

- This is the Cholesky decomposition

Consider

$$\begin{aligned} dZ_1(t)dZ_2(t) &= \rho dW_1(t)^2 + \sqrt{1 - \rho^2} dW_1(t)dW_2(t) \\ &= \rho dt + \sqrt{1 - \rho^2} dW_1(t)dW_2(t) \end{aligned}$$



Modeling Correlated Asset Prices (cont'd)

The independence of $W_1(t)$ and $W_2(t)$ implies that

$$E_t \{ [W_1(t+s) - W_1(t)][W_2(t+s) - W_2(t)] \} = 0$$

Using the differential notation, we can write

$$dW_1(t) \times dW_2(t) = 0$$

Therefore,

$$dZ_1(t)dZ_2(t) = \rho dt$$

← Correlation

$$\text{Given } dz_1(t) dz_2(t) = \rho dt$$

Want to find $w_1(t)$, $w_2(t)$, where they are independent BMS. to represent $z_1(t)$ and $z_2(t)$.

$$dz_1(t) = dw_1(t)$$

$$dz_2(t) = \rho dw_1(t) + \sqrt{1-\rho^2} dw_2(t)$$

Checking

$$\begin{aligned} & dz_1(t) \cdot dz_2(t) \\ &= [dw_1(t)] [\rho dw_1(t) + \sqrt{1-\rho^2} dw_2(t)] \\ &\subseteq \rho dw_1(t) dw_1(t) + \sqrt{1-\rho^2} dw_1(t) dw_2(t) \\ &= \rho dt + 0 = \rho dt \end{aligned}$$

~~$dw_1(t) dw_2(t)$~~



Modeling Correlated Asset Prices (cont'd)

In general, we can construct dZ_i , $i = 1, \dots, n$, as follows:

$$dZ_i(t) = \sum_{k=1}^n \lambda_{i,k} dW_k(t)$$

where we scale the coefficients so that

$$\sum_{k=1}^n \lambda_{i,k}^2 = 1$$



Modeling Correlated Asset Prices (cont'd)

Because the Brownian increments are jointly-normally distributed, their sum is normal. We also have

$$Var[dZ_i(t)] = Var\left(\sum_{k=1}^n \lambda_{i,k} dW_k(t)\right) = \sum_{k=1}^n \lambda_{i,k}^2 dW_k(t)^2 = dt$$

$$dZ_i(t)dZ_j(t) = \sum_{k=1}^n \lambda_{i,k} dW_k(t) \sum_{k=1}^n \lambda_{j,k} dW_k(t)$$

$$= \sum_{k=1}^n \lambda_{i,k} \lambda_{j,k} dt = \rho_{i,j} dt$$

Cholesky

decomposition

where

$$\rho_{i,j} = \sum_{k=1}^n \lambda_{i,k} \lambda_{j,k}$$



Itô's Lemma

- Suppose that the stock price, $S(t)$, follows the Itô process given by

$$dS(t) = [\hat{\alpha}[S(t), t] - \hat{\delta}[S(t), t]]dt + \hat{\sigma}[S(t), t]dZ(t)$$

- In this equation, the expected return, α , the dividend yield, δ , and the volatility, σ , can be functions of the stock price and time.
- If

$$\hat{\alpha}[S(t), t] = \alpha S(t), \quad \hat{\delta}[S(t), t] = \delta S(t), \quad \hat{\sigma}[S(t), t] = \sigma S(t),$$

then $S(t)$ follows geometric Brownian motion.

Given GBM

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$$

$Z(t)$ is a standard BM.

$$f(S(t), t)$$

e.g.;) $f(S(t), t) = \ln S(t)$

ii) $f(S(t), t) = S^2(t)$

iii) $f(S(t), t) = \sin(S(t))$

What is the SDE of $f(S(t), t)$?

By Itô's lemma -

$$\begin{aligned} df(S(t), t) &= \underbrace{\frac{\partial f}{\partial t} dt}_{S} + \underbrace{\frac{\partial f}{\partial S} ds}_{t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (ds)^2 \\ &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} (S(t) [\alpha dt + \sigma dZ(t)]) \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} [S(t) [\alpha dt + \sigma dZ(t)]]^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} S(t) [\alpha dt + \sigma dZ(t)] \\
 &\quad + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial S^2} \left[S^2(t) \left[\alpha^2 dt^2 + 2 \alpha dt \sigma dZ(t) \right. \right.} \\
 &\quad \quad \quad \left. \left. + \sigma^2 dZ(t)^2 \right] \right] \\
 &= \left[\frac{\partial f}{\partial t} + \alpha S(t) \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2(t) \frac{\partial^2 f}{\partial S^2} \right] dt \\
 &\quad + \sigma S(t) \frac{\partial f}{\partial S} dZ(t)
 \end{aligned}$$

Solution of GBM

$$\begin{aligned}
 \frac{dX(t)}{X(t)} &= \alpha dt + \sigma dZ(t) \\
 dX(t) &= \alpha X(t) dt + \sigma X(t) dZ(t)
 \end{aligned}$$

$$\text{Let } Y(t) = \ln X(t), t = \ln X(0).$$

By Itô's lemma,

$$dY(t) = ??$$

$$df \underset{\substack{u \\ s \\ g}}{(s,t)} = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial s} ds + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} (ds)^2$$

$$\frac{\partial x(t)}{\partial t} = 0, \quad \frac{\partial Y}{\partial X} = \frac{1}{X(t)}, \quad \frac{\partial^2 Y}{\partial X^2} = -\frac{1}{X^2(t)}$$

$$dx(t) \leq 0 \cdot dt + \frac{1}{X(t)} dx(t) + \frac{1}{2} \left(-\frac{1}{X^2(t)} \right) (dx(t))^2$$

$$= \frac{1}{X(t)} [x(t)] [\alpha dt + \sigma dz(t)]$$

$$- \frac{1}{2X^2(t)} [x^2(t)] [\alpha dt + \sigma dz(t)]^2$$

$$= \alpha dt - \frac{1}{2} \sigma^2 dt + \sigma dz(t)$$

$$= (\alpha - \frac{1}{2}\sigma^2)dt + \sigma dz(t)$$

$$X(t) = X(0) + (\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)$$

$$\ln X(t) = \ln X(0) + (\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)$$

$$X(t) = X(0) e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)}$$



Itô's Lemma (cont'd)

- $C[S(t), t]$ is the value of a derivative claim that is a function of the stock price.
- How can we describe the behavior of this claim in terms of the behavior of S ?



Itô's Lemma (cont'd)

- **Itô's Lemma (Proposition 20.1)**

- If $C[S(t), t]$ is a twice-differentiable function of $S(t)$, then the change in C is

$$\begin{aligned} dC(S, t) &= C_S dS + \frac{1}{2} C_{SS} (dS)^2 + C_t dt \\ &= \left[(\hat{\alpha}(S, t) - \hat{\delta}(S, t)) C_S + \frac{1}{2} \hat{\sigma}(S, t)^2 C_{SS} + C_t \right] dt + \sigma(S, t) C_S dZ \end{aligned}$$

- where $C_S = \partial C / \partial S$, $C_{SS} = \partial^2 C / \partial S^2$, and $C_t = \partial C / \partial t$
 - The terms in square brackets are the expected change in the option price.



Itô's Lemma (cont'd)

- In the case where $S(t)$ follows geometric Brownian motion, we have

$$dC(S, t) = \left[(\alpha - \delta)SC_S + \frac{1}{2}\sigma^2S^2C_{SS} + C_t \right]dt + \sigma SC_S dZ$$



Itô's Lemma (cont'd)

Example (The Black-Scholes Assumption of Stock Prices)

The expression for a lognormal stock price is

$$S(t) = S(0)e^{(\alpha - \delta - 0.5\sigma^2)t + \sigma Z(t)}$$

The stock price is a function of the Brownian process $Z(t)$. We can use Itô's Lemma to characterize the behavior of the stock as a function of $Z(t)$. We have

$$\frac{\partial S(t)}{\partial t} = \left(\alpha - \delta - \frac{1}{2}\sigma^2 \right) S(t); \quad \frac{\partial S(t)}{\partial Z(t)} = \sigma S(t); \quad \frac{\partial^2 S(t)}{\partial Z(t)^2} = \sigma^2 S(t)$$



Itô's Lemma (cont'd)

Itô's Lemma states that $dS(t)$ is given as

$$\begin{aligned} dS(t) &= \frac{\partial S(t)}{\partial t} dt + \frac{\partial S(t)}{\partial Z(t)} dZ(t) + \frac{1}{2} \frac{\partial^2 S(t)}{\partial Z(t)^2} [dZ(t)]^2 \\ &= (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t) \end{aligned}$$

This calculation demonstrates that a variable that follows geometric Brownian motion is lognormally distributed.



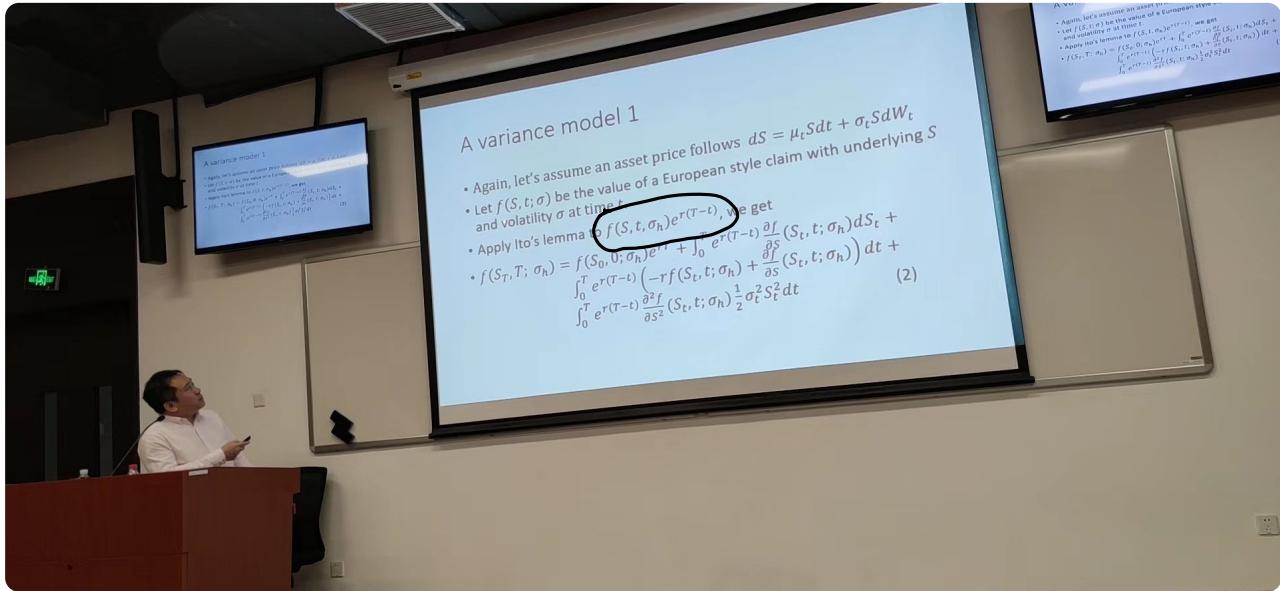
Itô's Lemma (cont'd)

Example

Let $Y(t) = \ln[S(t)]$. Then

$$d \ln[S(t)] = \frac{dS(t)}{S(t)} - \frac{1}{2} \frac{dS(t)^2}{S(t)^2} = \frac{dS(t)}{S(t)} - \frac{1}{2} \sigma^2 dt$$

This implies that continuously compounded returns – measured as $\ln[S(T)/S(0)]$ – are lower than the instantaneous return, $\alpha - \delta$, by the factor $0.5\sigma^2$.



$$g(S(t), t) = f(S(t), t) e^{r(T-t)}$$

$$\frac{\partial g}{\partial t}, \quad \frac{\partial g}{\partial S(t)}, \quad , \quad \frac{\partial^2 g}{\partial S^2(t)}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial t} e^{r(T-t)} - r e^{r(T-t)} f(S(t), t)$$

$$\frac{\partial g}{\partial S} = \frac{\partial f}{\partial S} e^{r(T-t)}, \quad \frac{\partial^2 g}{\partial S^2} = \frac{\partial^2 f}{\partial S^2} e^{r(T-t)}$$

$$dg = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial S} ds + \frac{1}{2} \frac{\partial^2 g}{\partial S^2} (ds)^2$$

Find $\int_0^t z(s) dz(s)$.

Let $Y(t) = \frac{1}{2} z^2(t)$

By Itô's lemma,

$$\begin{aligned} dY(t) &= \frac{\partial Y}{\partial t} dt + \frac{\partial Y}{\partial z} dz(t) + \frac{1}{2} \frac{\partial^2 Y}{\partial z^2} (dz(t))^2 \\ &= 2z(t) dz(t) + (dz(t))^2 \\ &= 2z(t) dz(t) + dt \\ &= 2z(t) dz(t) + ds \end{aligned}$$

$$\begin{aligned} \int_0^t dY(s) &= 2 \int_0^t z(s) dz(s) + \int_0^t ds \\ \int_0^t z(s) dz(s) &= \frac{1}{2} [Y(t) - Y(0)] - \frac{t}{2} \\ &= \frac{1}{2} z^2(t) - \frac{1}{2} t \end{aligned}$$



Multivariate Itô's Lemma

- A derivative may have a value depending on more than one price, in which case we can use a multivariate generalization of Itô's Lemma
- **Multivariate Itô's Lemma (Proposition 20.2)**
 - Suppose we have n correlated Itô processes

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i dt + \sigma_i dZ_i, \quad i = 1, \dots, n$$

- Denote the pairwise correlations as

$$dZ_i \times dZ_j = \rho_{i,j} dt$$



Multivariate Itô's Lemma (cont'd)

- If $C(S_1, \dots, S_n, t)$ is a twice-differentiable function of the S_i 's, we have

$$dC(S_1, \dots, S_n, t) = \sum_{i=1}^n C_{S_i} dS_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n dS_i dS_j C_{S_i S_j} + C_t dt$$

$$\begin{aligned} dC(S_1, S_2, t) &= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1} dS_1 + \\ &\quad \frac{\partial C}{\partial S_2} dS_2 + \frac{\partial^2 C}{\partial S_1 \partial S_2} dS_1 dS_2 + \\ &\quad \underbrace{\frac{1}{2} \frac{\partial^2 C}{\partial S_1^2} (dS_1)^2}_{\text{1}} + \underbrace{\frac{1}{2} \frac{\partial^2 C}{\partial S_2^2} (dS_2)^2}_{\text{2}} \end{aligned}$$



Multivariate Itô's Lemma (cont'd)

$$\frac{dX}{X} = \alpha dt + \sigma dZ$$

S_1, S_2 are GBMs

Example

Suppose $C(S_1, S_2) = S_1/S_2$. Then by Itô's Lemma we have

$$d\left(\frac{S_1}{S_2}\right) = dS_1\left(\frac{1}{S_2}\right) - dS_2\left(\frac{S_1}{S_2^2}\right) + 0.5 \left[2(dS_2)^2 \frac{S_1}{S_2^3} - 2dS_1 dS_2 \frac{1}{S_2^2} \right]$$

$$d\left(\frac{S_1}{S_2}\right) \frac{S_2}{S_1} = (\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dZ_1 - \sigma_2 dZ_2$$

GBM

$$\frac{d\left(\frac{S_1}{S_2}\right)}{\left(\frac{S_1}{S_2}\right)} = (\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \underline{\sigma_1 dZ_1} - \underline{\sigma_2 dZ_2}$$

ratio of two GBMs \Rightarrow GBM

product of two GBMs \Rightarrow GBM.



Multivariate Itô's Lemma (cont'd)

From the earlier discussion of correlated Itô's processes, we have

$$d\left(\frac{S_1}{S_2}\right) = \left(\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)dt + \hat{\sigma} dZ$$

where

$$\hat{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \quad \text{and} \quad dZ = (\sigma_1 dZ_1 - \sigma_2 dZ_2)/\sigma.$$

Remark:

Even if S_1 and S_2 have equal drifts (i.e., $\alpha_1 = \alpha_2$), the ratio of S_1 and S_2 will have generally have zero drift.



The Sharpe Ratio

- If asset i has (total) expected return α_i , the risk premium is defined as

$$\text{Risk premium}_i = \alpha_i - r$$

– where r is the risk-free rate.

- The **Sharpe ratio** for asset i is the risk premium, $\alpha_i - r$, per unit of volatility, σ_i

$$\text{Sharpe ratio}_i = \frac{\alpha_i - r}{\sigma_i} \quad (20.25)$$



The Sharpe Ratio (cont'd)

- We can use the Sharpe ratio to compare two perfectly correlated claims, such as a derivative and its underlying asset.
- Two assets that are perfectly correlated must have the same Sharpe ratio, or else there will be an arbitrage opportunity.
 - Consider the processes for two non-dividend paying stocks

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ \quad (20.26)$$

$$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ \quad (20.27)$$

- Because the two stock prices are driven by the same dZ , it must be the case that

$$(\alpha_1 - r) / \sigma_1 = (\alpha_2 - r) / \sigma_2$$

$$\text{Suppose } \frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}$$

Asset 1 Asset 2

Buy (1) Sell (2)

$$\begin{aligned} \text{Buy } \frac{r}{\sigma_1 S_1} \text{ units of (1)} \\ \text{Sell } \frac{1}{\sigma_2 S_2} \text{ units of (2)} \end{aligned} \quad \left. \begin{array}{l} \text{at time 0} \\ \text{Cost} = \frac{1}{\sigma_1 S_1} (S_1) - \frac{1}{\sigma_2 S_2} (S_2) \end{array} \right\}$$

$$\text{Profit of the portfolio at time } dt. = \frac{1}{\sigma_1} - \frac{1}{\sigma_2}$$

$$\begin{aligned} & \frac{1}{\sigma_1 S_1} dS_1 - \frac{1}{\sigma_2 S_2} dS_2 - \underbrace{r \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) dt}_{FV(\text{cost})} \\ &= \frac{1}{\sigma_1 S_1} S_1 [\alpha_1 dt + \sigma_1 dZ(t)] - \frac{1}{\sigma_2 S_2} [S_2] [\alpha_2 dt + \sigma_2 dZ(t)] \\ & \quad - r \left[\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right] dt \\ &= \left[\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right] dt > 0 \end{aligned}$$

arbitrage $\Rightarrow \frac{\alpha_1 - r}{\sigma_1} \leq \frac{\alpha_2 - r}{\sigma_2}$

$C(S(t), t)$: call option on non-dividend paying stock

$$\frac{dS(t)}{S(t)} = \alpha dt + (\sigma dZ(t))$$

σ stock volatility

$$\frac{dC(S(t), t)}{C(S(t), t)} = \left(\frac{\partial C}{\partial t} + \alpha S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt$$

$$+ \frac{\sigma S \frac{\partial C}{\partial S} dZ(t)}{C(S(t), t)}$$

σ_{call}

$$\sigma_{call} = \frac{\sigma S \left(\frac{\partial C}{\partial S} \right)}{C(S(t), t)} = \frac{\sigma S \Delta}{C} = \sigma \left(\frac{S \Delta}{C} \right) = \sigma \frac{\Delta}{C}$$

σ_{stock}

$$\frac{\tau_{call} - r}{\sigma_{call}} = \frac{\alpha - r}{\sigma_{stock}}$$

$$\tau_{call} - r = \frac{\sigma_{call}}{\sigma_{stock}} (\alpha - r) = \eta (\alpha - r)$$



The Sharpe Ratio (cont'd)

- The arbitrage is straightforward. Suppose that the Sharpe ratio of asset 1 is greater than that of asset 2. We then
 - Buy $1/(\sigma_1 S_1)$ shares of asset 1.
 - Short $1/(\sigma_2 S_2)$ shares of asset 2.
 - Invest (or borrow) $1/\sigma_2 - 1/\sigma_1$, by buying (or borrowing) the risk-free bond, which has the rate of return rdt .
- The return of the above portfolio is

$$\frac{1}{\sigma_1 S_1} dS_1 - \frac{1}{\sigma_2 S_2} dS_2 + \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1} \right) rdt = \left(\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right) dt > 0$$

So, the arbitrage profit is obtained.

Chapter 14

Martingale Pricing Theory



See Section 3.2 of “Mathematical Models of Financial Derivatives”, 2nd edition, by Yue Kuen KWOK, Springer Verlag, 2008.

Points to Note

1. What is the definition of the equivalent martingale measure? See P.3 – 6.
2. What is the relationship between the no-arbitrage price of a financial product and the risk-neutral probability? See P.8 – 13.
3. How do we change a measure in the expectation? Use the notation of the Randon-Nikodym derivative. See P.14 – 15.
4. Girsanov Theorem. See P.16 – 19.
5. Converting the dynamic of the asset price processes from the real probability to the risk-neutral probability. See P.20 – 25.
6. Derivation of the BS formula by using the change of numeraire. See P.26 – 38.
7. The Black-Scholes formula for the dividend-paying asset. See P.39 – 47.

Equivalent Martingale Measure and Risk Neutral Valuation

Under the continuous time framework, the investors are allowed to trade continuously in the financial market up to finite time T .

"Model Setup"

Consider the securities model, there are

$K + 1$ securities whose price processes are modeled by $M(t)$ and $S_m(t)$ (**non-dividend-paying assets**), where $m = 1, \dots, K$.

money market Account

The uncertainty of the market is described by the actual (real) probability measure (distribution) P .

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

We use $M(t)$ to denote the money market account process that starts at \$1 and grows at the deterministic risk-free interest rate $r(t)$, that is,

$$\boxed{dM(t) = r(t)M(t)dt \Leftrightarrow M(t) = e^{\int_0^t r(s)ds}}.$$

The discounted security price process $S_m^*(t)$ is defined by

$$S_m^*(t) = \frac{S_m(t)}{M(t)}, \quad m = 1, 2, \dots, K.$$

$$\simeq \text{PV}(S_m(t))$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

Suppose that Q_1 and Q_2 are probability measures (distributions) on the sample space Ω , Q_1 and Q_2 are equivalent if

$$Q_1(A) = 0 \Leftrightarrow$$

$$Q_2(A) = 0 \text{ for any } A \subset \Omega.$$

Imposs'ble
under Q_1

Imposs'ble
under Q_2

Eg

X is a random variable

$$Q_1(X < 20) = 0, \quad Q_1(20 \leq X \leq 80) = \frac{3}{4},$$

$\Pr(X < 20)$ under Q_1

$$Q_1(X > 80) = \frac{1}{4}.$$

$$Q_2(X < 20) = 0, \quad Q_2(20 \leq X \leq 80) = \frac{1}{3},$$

$$Q_2(X > 80) = \frac{2}{3}$$

$$Q_1 \neq Q_2$$

$$Q_1 \Leftrightarrow Q_2$$

$$Q_3(X < 10) = 0, \quad Q_3(10 \leq X < 18) = \frac{1}{8},$$

$$Q_3(X \geq 18) = \frac{7}{8}.$$

$$Q_3(X < 18) = \frac{1}{8}, \quad Q_1(X < 18) = 0$$

$$Q_1 \not\Leftarrow Q_3$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

A probability measure Q is said to be an equivalent martingale measure (or risk-neutral measure) to the real probability measure P if it satisfies

- Q is equivalent to P ; $Q \rightleftharpoons P$
- The discounted security price process $S_m^*(t)$, $m = 1, 2, \dots, K$, are martingales under Q , that is

$$E_t^Q [S_m^*(u)] = S_m^*(t) \quad \text{for all } 0 \leq t \leq u \leq T.$$

$$\mathbb{E}^Q [S_m^*(u) | \{S_m^*(s) : 0 \leq s \leq t\}] = S_m^*(t)$$

From Prof. Chen's Class

$$E[\underline{Z_n} | \underline{x_1, \dots, x_{n-1}}] = Z_{n-1}$$

information upto
time $n-1$

Binomial tree , $p^* = \frac{e^{rh} - d}{u - d}$

$$S_j^* = \frac{S_j}{e^{r(jh)}}$$

$$\text{Want to show: } E^*[S_{n+k}^* | \{S_0, S_1, \dots, S_n\}] = S_n^*$$

Proof

$$S_{n+k} = \begin{cases} u^k S_n \\ u^{k-1} d S_n \\ \vdots \\ d^k S_n \end{cases}$$

$$\begin{aligned} & E^*[S_{n+k}^* | \{S_0, \dots, S_n\}] \\ &= \frac{1}{e^{r(n+k)h}} \sum_{j=0}^k (p^*)^j ((1-p^*))^{k-j} u^j d^{k-j} S_n \\ &= \frac{S_n}{e^{r(n+k)h}} \sum_{j=0}^k (p^* u)^j [(1-p^*)d]^{k-j} \\ &= \frac{S_n}{e^{r(n+k)h}} [p^* u + (1-p^*)d]^k = \frac{S_n}{e^{rnh}} = S_n^* \end{aligned}$$

**

MFE5130 – Financial Derivatives

Class Activity (27-November-2019) (Solution)

Important Notes:

1. This class activity is counted toward your class participation score. **Fail** to hand in this class activity worksheet in the class will receive **0 score** for that class.
2. **0 mark** will be received if you leave the solution blank.

Name:	Student No.:
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Problem 1

You are given:

$$\frac{dS(t)}{S(t)} = 0.5dt + 0.4dZ(t) \quad G(t) = t^2 S(t) = f(t, S(t))$$

$Z(t)$ is a standard Brownian motion.

Find an expression for $dG(t)$.

$$\frac{\partial G}{\partial S} = t^2, \quad \frac{\partial^2 G}{\partial S^2} = 0$$

Solution

The expression for $G(t)$ is $t^2 S(t)$.

The partial derivatives are:

$$\frac{\partial G}{\partial t} = S(t) 2t$$

$$G_S = t^2, \quad G_{SS} = 0, \quad G_t = 2tS.$$

From Itô's lemma, we have:

$$\begin{aligned} dG(t) &= G_S dS(t) + \frac{1}{2} G_{SS} [dS(t)]^2 + G_t dt \\ &= t^2 dS(t) + 2tS(t) dt \\ &= t^2 [0.5S(t)dt + 0.4S(t)dZ(t)] + 2tS(t)dt \\ &= t^2 S(t) [0.5dt + 0.4dZ(t)] + t^2 S(t) \frac{2}{t} dt. \\ &= G(t) \left[\left(\frac{0.5t+2}{t} \right) dt + 0.4dZ(t) \right]. \end{aligned}$$

Problem 2

Let $X(t)$ be a stochastic process. The stochastic differential equation of $X(t)$ is given by

$$dX(t) = X(t) [\mu dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t)],$$

where $Z_1(t)$ and $Z_2(t)$ are two independent standard Brownian motions.

Solve the stochastic differential equation of $X(t)$.

Solution Hint : Let $\bar{X}(t) = (\ln X(t))$

$$f(t, s(t)) = [s^2(t) + 3s(t)] e^{rt}$$

$$\frac{\partial f}{\partial s(t)} = [2s(t) + 3] e^{rt}$$

$$\frac{\partial^2 f}{\partial (s(t))^2} = [2 + 0] e^{rt} = 2e^{rt}$$

$$\frac{\partial f}{\partial t} = [s^2(t) + 3s(t)] r e^{rt}$$

Let $Y(t) = \ln X(t)$.

From Itô's lemma, we have:

$$\begin{aligned}
 dY(t) &= \left(\frac{\partial Y}{\partial X} dX(t) + \frac{1}{2} \frac{\partial^2 Y}{\partial X^2} (dX(t))^2 + \frac{\partial Y}{\partial t} dt \right) \\
 &= \frac{1}{X(t)} dX(t) - \frac{1}{2} \frac{1}{X^2(t)} (dX(t))^2 \\
 &= \frac{X(t)}{X(t)} [\mu dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t)] - \frac{1}{2} \frac{X^2(t)}{X^2(t)} (\mu dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t))^2 \\
 &= \mu dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t) - \frac{1}{2} (\sigma_1^2 dt + \sigma_2^2 dt) \\
 &= \left(\mu - \frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 \right) dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_2(t).
 \end{aligned}$$

multipl. rule
 $(dt)^2 = 0$
 $dt dZ_i = 0$
 $(dZ_i)^2 = dt$
 $dZ_i dZ_j = 0$
 if $i \neq j$

So,

$$\begin{aligned}
 \int_0^t dY(s) &= \int_0^t \left(\mu - \frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 \right) ds + \int_0^t \sigma_1 dZ_1(s) + \int_0^t \sigma_2 dZ_2(s) \\
 Y(t) - Y(0) &= \left(\mu - \frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 \right) t + \sigma_1 Z_1(t) + \sigma_2 Z_2(t)
 \end{aligned}$$

$$\begin{aligned}
 \ln X(t) &= \ln X(0) + \left(\mu - \frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2 \right) t + \sigma_1 Z_1(t) + \sigma_2 Z_2(t) \\
 X(t) &= X(0) \exp \left(\left(\mu - \left(\frac{1}{2} \sigma_1^2 + \frac{1}{2} \sigma_2^2 \right) t + \sigma_1 Z_1(t) + \sigma_2 Z_2(t) \right) \right).
 \end{aligned}$$