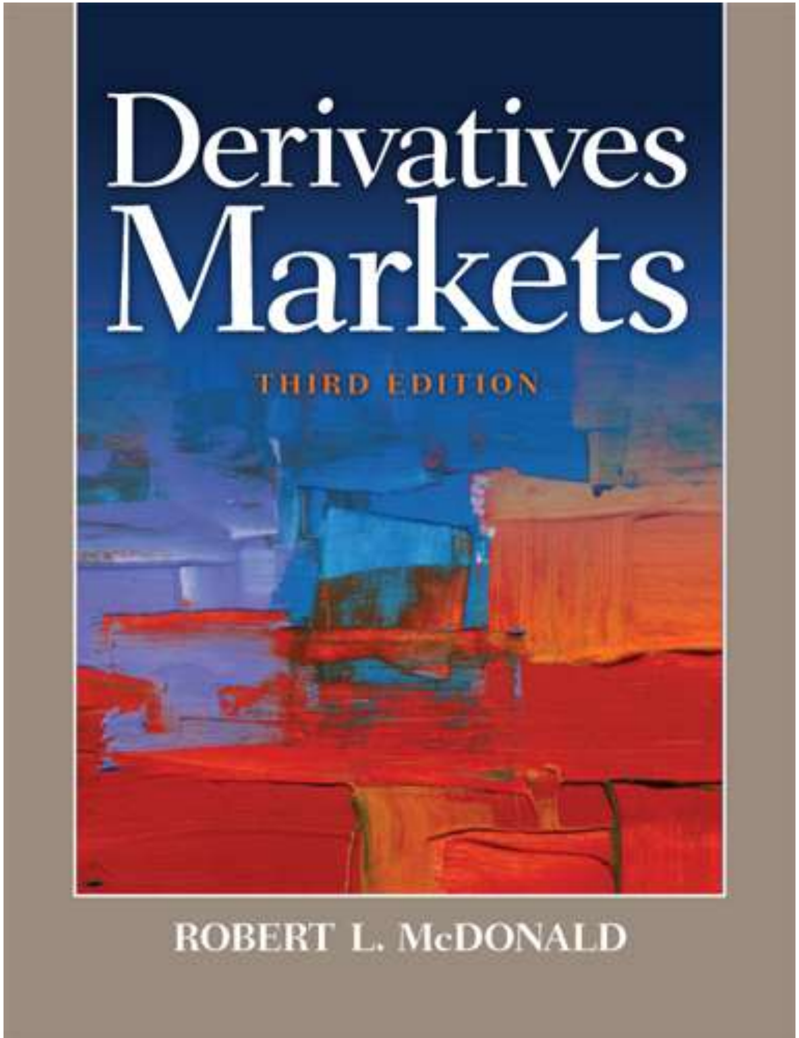


Chapter 9
(Chapter 10 in the textbook)

Binomial Option Pricing

Points to Note

1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13). $d < e^{(r-\delta)h} < u$
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 - 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)

Call option on S and time to expiry of h

$$
\begin{tikzcd}[row sep=small]
 & uS \\
S \arrow[ur] \arrow[dr] & \\
 & dS
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
 & C_u = \max(uS-K, 0) \\
C \arrow[ur] \arrow[dr] & \\
 & C_d = \max(dS-K, 0)
\end{tikzcd}
$$

$t=0$ $t=h$ $t=0$ $t=h$

Replicating portfolio: at $t=0$

$$C = \Delta S + B \leftarrow \text{lending Amt}$$

Δ ↑ no. of unit of underlying

<u>Risk-neutral valuation:</u>

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right]$$

where $$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$ artificial

$0 < p^* < 1$ because of $d < e^{(r-\delta)h} < u$

$$\begin{tikzcd}[row sep=small] & uS \\ S \arrow[ur, "p^*"] \arrow[dr, "(1-p^*)"'] & \\ & dS \end{tikzcd}$$

$t=0$ $\qquad\qquad$ $t=h$

p^* derived from replication.

p^* : risk-neutral prob. (measure)

$$E^*[S_h] = p^* Su + (1-p^*) Sd$$

$$= S e^{(r-\delta)h}$$

Expected return $= r - \delta$ $\qquad\qquad r$

risk-free

$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

$$\mathrm{Var}\left(\underbrace{\frac{S_h - S}{S}}_{\text{return}}\right) = \sigma^2 \Delta t \qquad (\text{up to order of } \Delta t)$$

$$Var\left(\frac{S_h - S}{S}\right) = Var\left(\frac{S_h}{S}\right)$$

$$= E\left[\left(\frac{S_h}{S}\right)^2\right] - \left(E\left[\frac{S_h}{S}\right]\right)^2$$

$$= p^* u^2 + (1-p^*) d^2$$

$$- [p u^* + (1-p^*) d]^2$$

$$u = e^{(r-\delta)h + \sigma\sqrt{h}}$$

$$d = e^{(r-\delta)h - \sigma\sqrt{h}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

neglect

$$= \sigma^2 \Delta t + O(\Delta t^2)$$

$$\approx \sigma^2 \Delta t$$

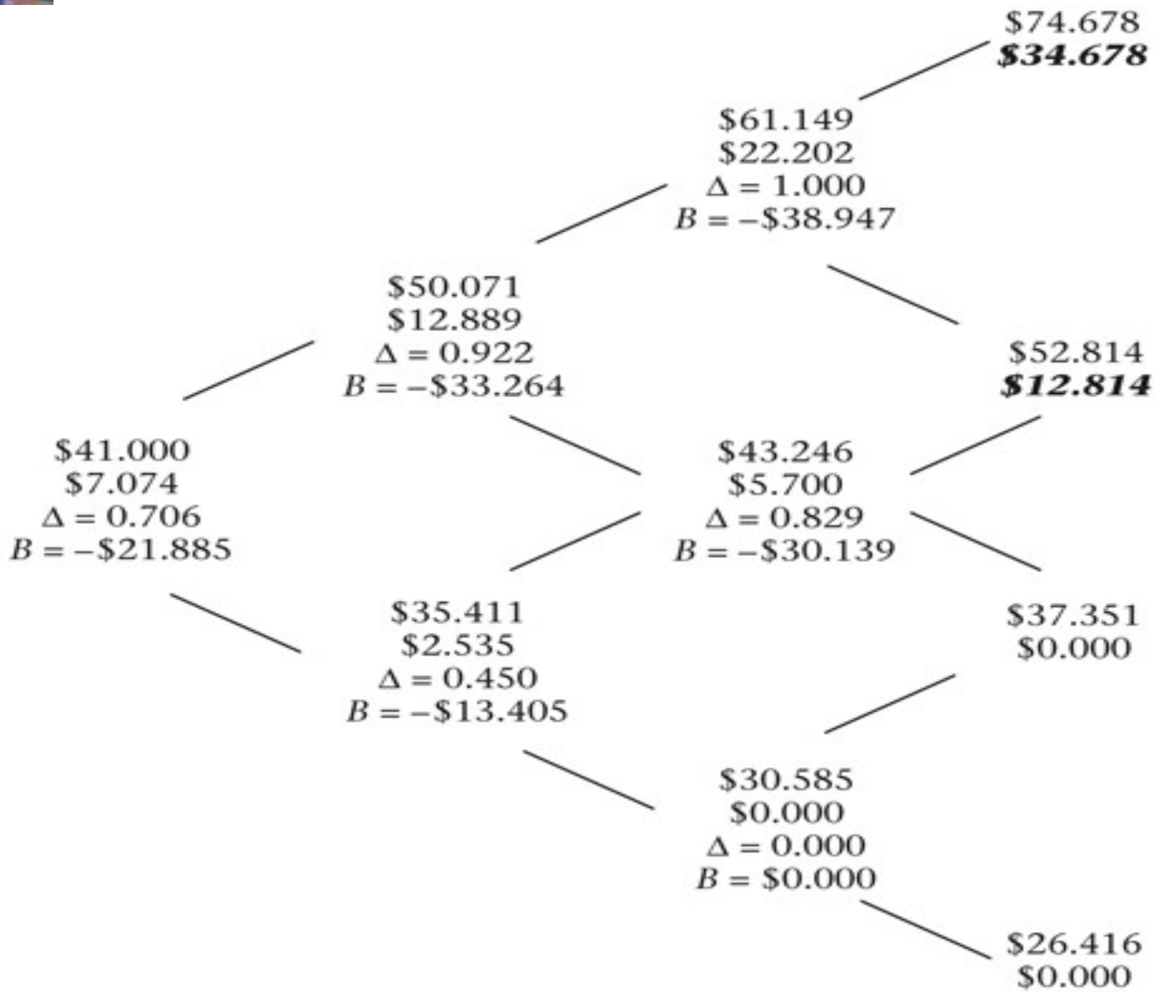
$$\begin{tikzcd}
 & & & u^3S \\
 & & u^2S \arrow[ur] \arrow[dr] & \\
 & uS \arrow[ur] \arrow[dr] & & u^2dS \\
S \arrow[ur] \arrow[dr] & & udS \arrow[ur] \arrow[dr] & \\
 & dS \arrow[ur] \arrow[dr] & & ud^2S \\
 & & d^2S \arrow[ur] \arrow[dr] & \\
 & & & d^3S
\end{tikzcd}$$

Many Binomial Periods

- Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.
  - Consider the previous example of the 1-year European call option.
  - Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$.
  - Assume that other inputs are the same as before (so, $r = 0.08$ and $\sigma = 0.3$).

Many Binomial Periods (cont'd)

- The stock price and option price tree for this option.


$74.678
$34.678
$61.149
$22.202
Δ = 1.000
B = –$38.947
$50.071
$12.889
Δ = 0.922
B = –$33.264
$52.814
$12.814
$41.000
$7.074
Δ = 0.706
B = –$21.885
$43.246
$5.700
Δ = 0.829
B = –$30.139
$35.411
$2.535
Δ = 0.450
B = –$13.405
$37.351
$0.000
$30.585
$0.000
Δ = 0.000
B = $0.000
$26.416
$0.000

Many Binomial Periods (cont’d)

- Note that since the length of the binomial period is shorter, *u* and *d* are closer to 1 before (u = 1.2212 and d = 0.8637 as opposed to 1.462 and 0.803 with h = 1).
  - The second-period nodes are computed as follows

$$S_u = \$41e^{0.08\times 1/3+0.3\sqrt{1/3}} = \$50.071$$

$$S_d = \$41e^{0.08\times 1/3-0.3\sqrt{1/3}} = \$35.411$$

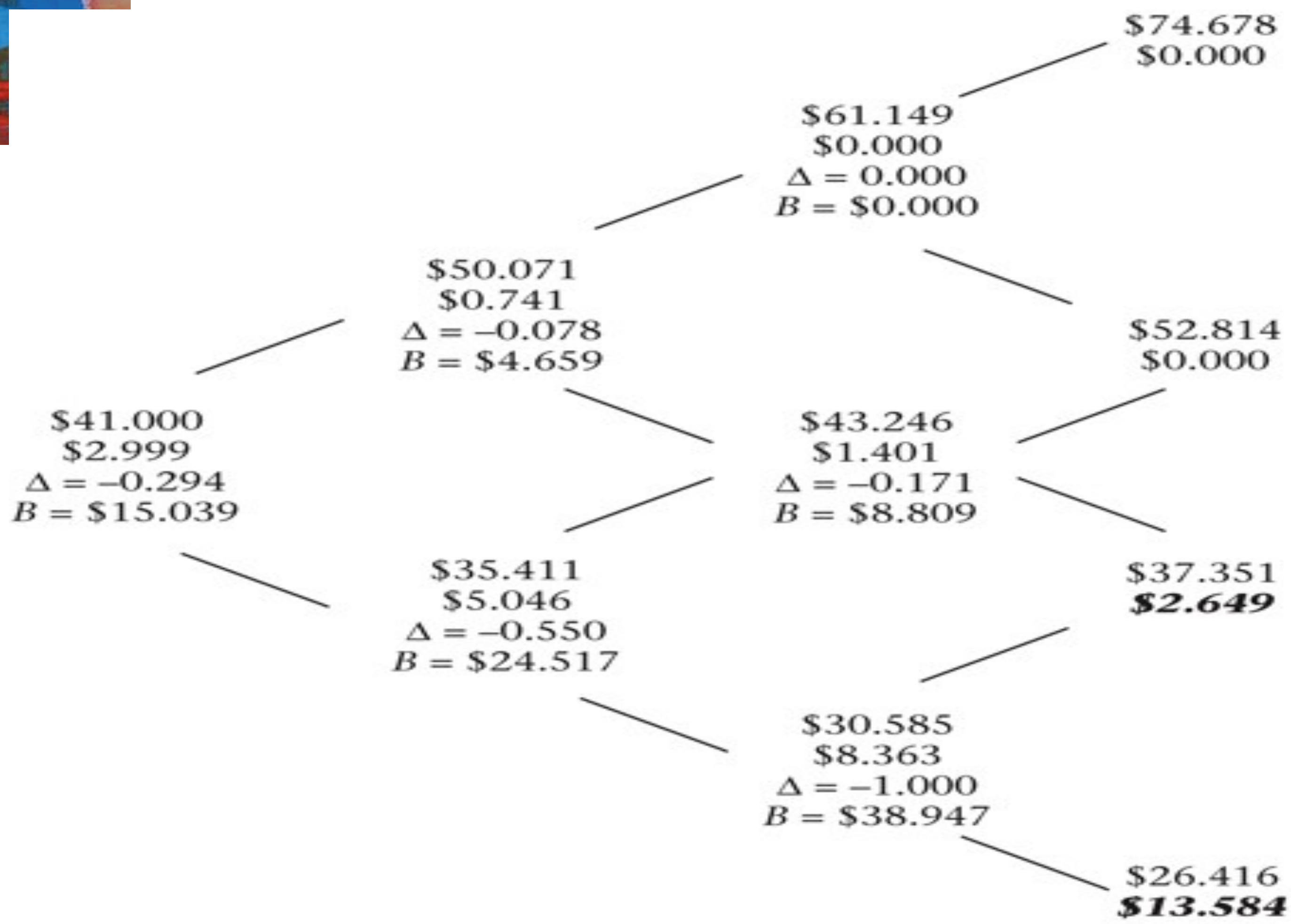
  - The remaining nodes are computed similarly.
- Analogous to the procedure for pricing the 2-year option, the price of the three-period option is computed by working backward using equation (10.3). The option price is $7.074.

Put Options

- We compute put option prices using the same stock price tree and in almost the same way as call option prices.
- The only difference with a European put option occurs at expiration.
  - Instead of computing the price as $\max(0, S - K)$, we use $\max(0, K - S)$.

Put Options (cont’d)

- A binomial tree for a European put option with 1-year to expiration.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$2.999
Δ = –0.294
B = $15.039
$43.246
$1.401
Δ = –0.171
B = $8.809
$35.411
$5.046
Δ = –0.550
B = $24.517
$37.351
$2.649
$30.585
$8.363
Δ = –1.000
B = $38.947
$26.416
$13.584

General Formulation

- With loss of generality, consider an European call option on a non-dividend paying asset with the payoff of

$$\max(S_T - K, 0)$$

- Let C be the call option price at time 0.

- In the n-period binomial tree, the risk-neutral probability of having j up-jumps and $(n - j)$ down-jumps is given by

$$C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j}$$

where

$$C_j^n = \frac{n!}{j!\left(n-j\right)!}.$$

$$S \begin{cases} uS \\ dS \end{cases} \cdots$$

$$S_n: \quad \begin{array}{c} u^n S \\ u^{n-1} d S \\ u^{n-2} d^2 S \\ \vdots \\ d^n S \end{array}$$

$$C = e^{-rT} E^*[\max(S_n - K, 0)]$$

$$= e^{-rT} \sum_{j=0}^{n} \boxed{{}^nC_j \, (p^*)^j (1-p^*)^{n-j}} \max(u^j d^{n-j} S_0 - K, 0)$$

$${}^nC_j = \frac{n!}{(n-j)!\,j!}$$

n steps :

j steps : up

$n-j$ steps : down

General Formulation (cont'd)

- The corresponding payoff when *j* up-jumps and *n* – *j* down-jumps occur is seen to be

$$\max\left(u^j d^{n-j} S_0 - K, 0\right)$$

The call value obtained from the *n*-period binomial model is given by

$$C = e^{-rnh} \sum_{j=0}^{n} C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j} \max\left(u^j d^{n-j} S_0 - K, 0\right).$$

$= 0$

when $u^j d^{n-j} S_0 < K$

General Formulation (cont'd)

We define k to be the smallest nonnegative integer such that $u^k d^{n-k} S_0 \geq K$, that is

$$k \geq \frac{\ln \frac{K}{S_0 d^n}}{\ln \frac{u}{d}}.$$

solve j from

$u^j d^{n-j} S_0 > K$

Accordingly, we have

$$\max\left(u^j d^{n-j} S_0 - K, 0\right) = \begin{cases} 0 & \text{when } \; j < k \\ u^j d^{n-j} S_0 - K & \text{when } \; j \geq k. \end{cases}$$

General Formulation (cont'd)

The integer *k* gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money. So,

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n,k,\tilde{p}\right) - Ke^{-rT} \Phi\left(n,k,p^*\right).$$

where

$$\Phi\left(n,k,p\right) = \sum_{j=k}^{n} C_j^n \left(p\right)^j \left(1-p\right)^{n-j}, \quad \tilde{p} = \frac{up^*}{e^{rh}} \quad \text{and} \quad 1-\tilde{p} = \frac{d\left(1-p^*\right)}{e^{rh}}.$$

Note: When $n \to \infty$, the binomial tree model will converge to the Black-Scholes formula (see Binomial to BS.pdf for detail).

General Formulation (cont'd)

The *first term* gives the discounted expectation of the asset price at expiration given that the call expires in-the-money and the *second term* gives the present value of the expected cost incurred by exercising the call, where the expectation is taken under the risk-neutral measure.

The similar formulation can be obtained for the European put option.

American Options

- The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation (10.3).
- The value of the option if it is exercised is given by *max* (0, *S* – *K*) if it is a call and *max* (0, *K* – *S*) if it is a put.
- For an American put, the value of the option at a node is given by

$$P(S,K,t) = \max\left(\underbrace{\boxed{K - S}}_{\text{Early Ex}},\ \underbrace{e^{-rh}\left[P(uS,K,t+h)\,p^* + P(dS,K,t+h)(1-p^*)\right]}_{\text{Continuation}}\right)$$

American Options (cont'd)

- The valuation of American options proceeds as follows:
  - At each node, we check for early exercise.
  - If the value of the option is greater when exercised, we assign that the **exercised** value to the node. Otherwise, we assign the value of the option **unexercised**.
  - We work backward through the tree as usual.

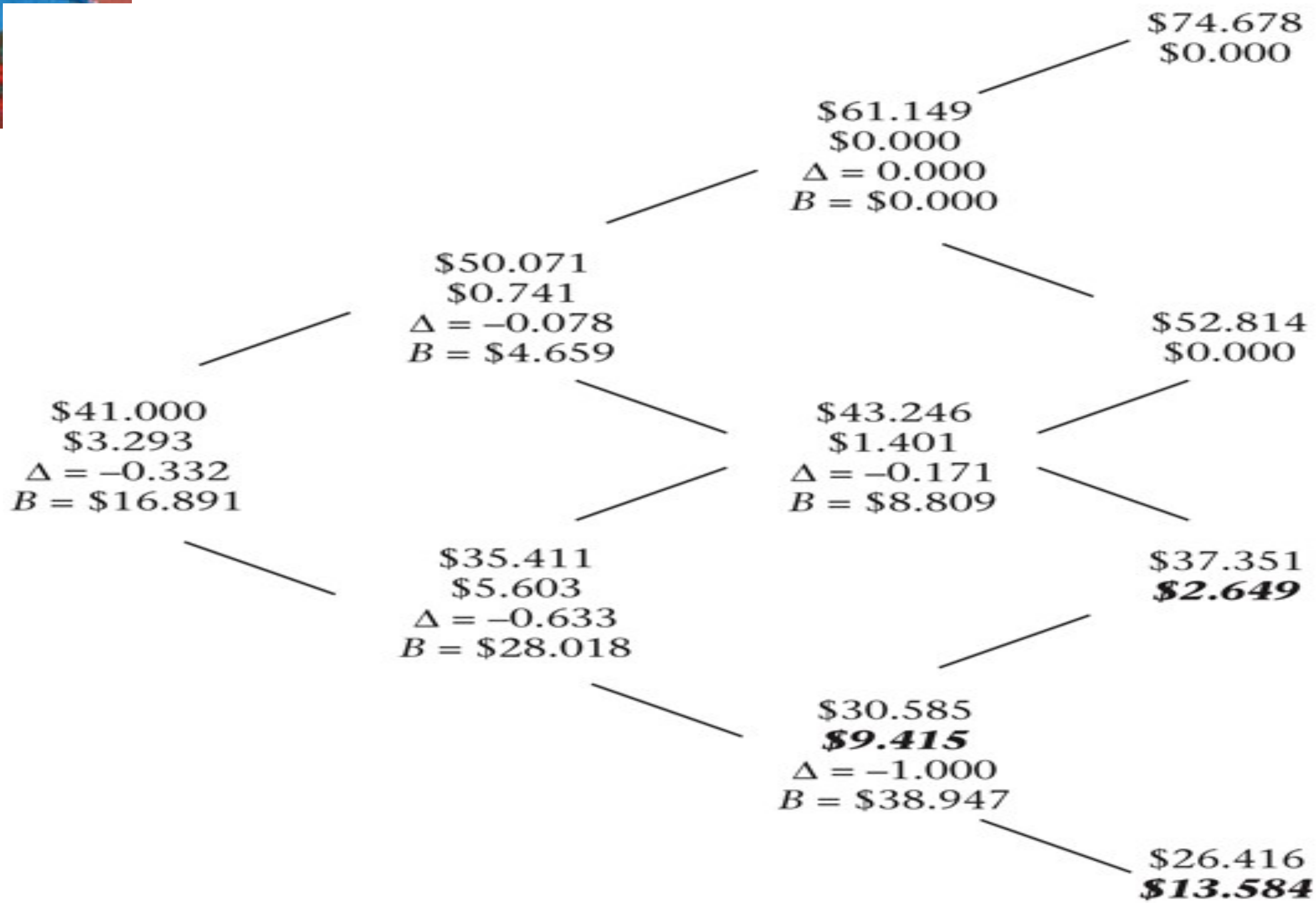
Optimally:

$$C_{Amer} \geq \max(S_{Ex} - K, 0)$$

$$P_{Amer} \geq \max(K - S_{Ex}, 0)$$

American Options (cont'd)

- Consider an American version of the put option valued in the previous example.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$3.293
Δ = –0.332
B = $16.891
$43.246
$1.401
Δ = –0.171
B = $8.809
$35.411
$5.603
Δ = –0.633
B = $28.018
$37.351
$2.649
$30.585
$9.415
Δ = –1.000
B = $38.947
$26.416
$13.584

American Options (cont'd)

- The only difference in the binomial tree occurs at the S_{dd} node, where the stock price is \$30.585. The American option at that point is worth \$40 – \$30.585 = \$9.415, its early-exercise value (as opposed to \$8.363 if unexercised). The greater value of the option at that node ripples back through the tree.

Options on Other Assets

- The binomial model developed thus far can be modified easily to price options on underlying assets other than non-dividend-paying stocks.
- The difference for different underlying assets is the construction of the binomial tree and the risk-neutral probability.
- We examine options on
  - Stock indexes = stock with dividend yield
  - Currencies = $\delta = r_f$
  - Futures contracts
  - Commodities
  - Bonds

Options on a Stock Index

- Suppose a stock index pays continuous dividends at the rate δ.
- The procedure for pricing this option is equivalent to that of the first example, which was used for our derivation. Specifically
  - the up and down index moves are given

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

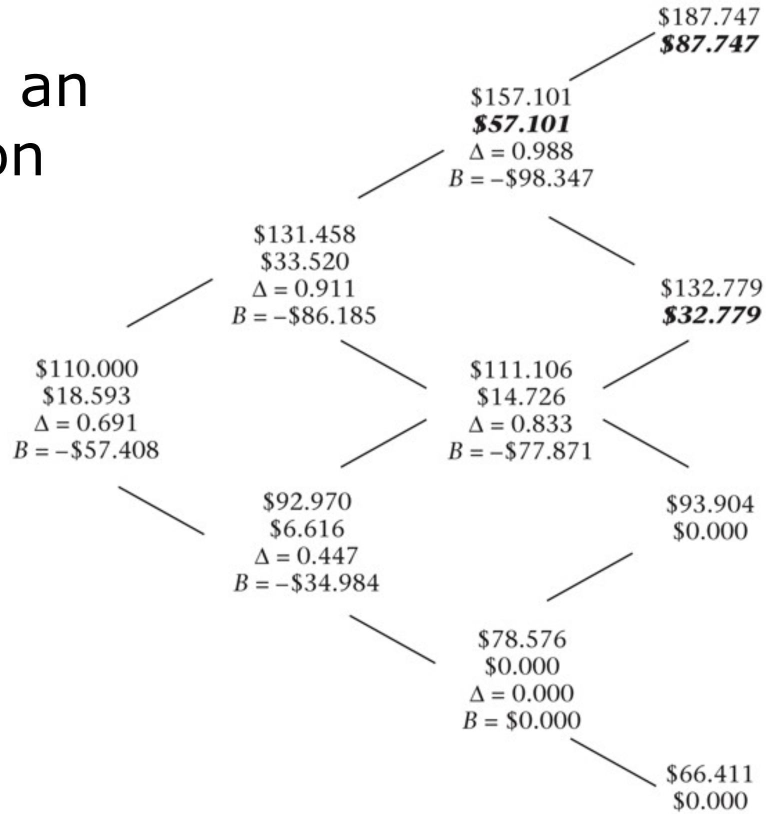
  - the replicating portfolio by equation (10.1) and (10.2).
  - the option price by equation (10.3).
  - the risk-neutral probability by equation (10.5).

Options on a Stock Index

- Given
  - S = \$110;
  - K = \$100;
  - σ = 0.30;
  - r = 0.05
  - T = 1 year
  - δ = 0.035
  - h = 0.333

Options on a Stock Index (cont'd)

- A binomial tree for an American call option on a stock index:



Options on Currencies

- With a currency with spot price x_0, the forward price is

$$F_{0,h} = x_0 e^{(r - r_f)h}$$

where r_f is the foreign interest rate.

- Thus, we construct the binomial tree using

$$ux = xe^{(r - r_f)h + \sigma\sqrt{h}}$$

$$dx = xe^{(r - r_f)h - \sigma\sqrt{h}}$$

$\delta = r_f$

Options on Currencies (cont'd)

- Investing in a "currency" means investing in a money-market fund or fixed income obligation denominated in that currency.
- Taking into account interest on the foreign-currency denominated obligation, the two equations are

$$\Delta \times uxe^{r_f h} + e^{rh} \times B = C_u$$

$$\Delta \times dxe^{r_f h} + e^{rh} \times B = C_d$$

- The risk-neutral probability of an up move is

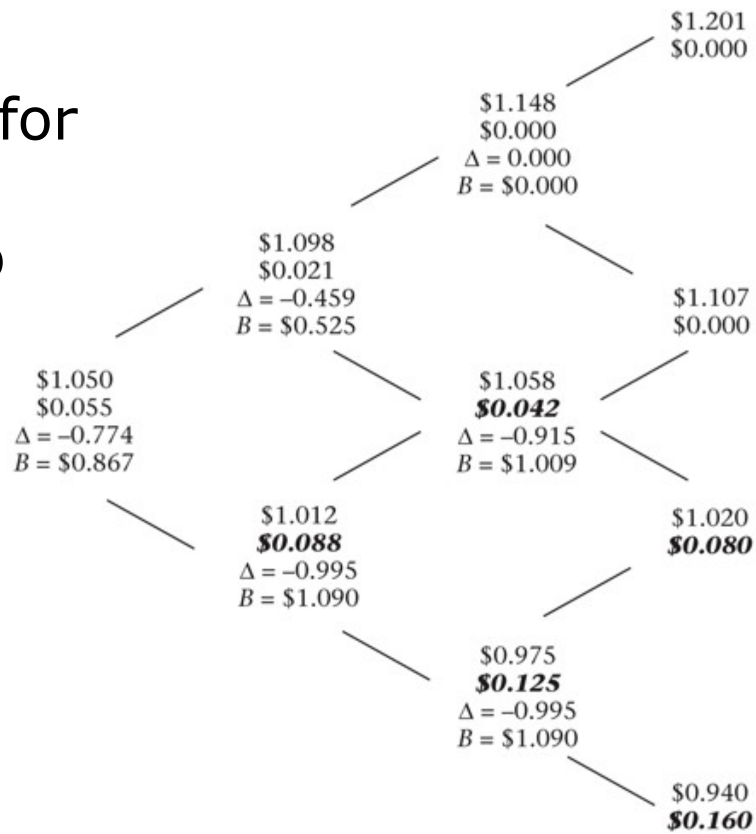
$$p^* = \frac{e^{(r-r_f)h} - d}{u - d} \qquad (10.20)$$

Options on Currencies (cont'd)

- Consider a dollar-denominated American put option on the euro, where
  - The current exchange rate is \$1.05/€ (S);
  - The strike is \$1.10/€ (K);
  - $\sigma = 0.10$;
  - The euro-denominated interest rate is 3.1% (δ);
  - The dollar-denominated rate is 5.5% (r).

Options on Currencies (cont'd)

- The binomial tree for the American put option on the euro



Options on Futures Contracts

- Assume the forward price is the same as the futures price.
- The nodes are constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

- We need to find the number of futures contracts, Δ, and the lending, *B*, that replicates the option.
  - A replicating portfolio must satisfy

$$\Delta \times (uF - F) + e^{rh} \times B = C_u$$
$$\Delta \times (dF - F) + e^{rh} \times B = C_d$$

Derive u and d

Let $F_{t,T}$ be the price of T- year forward at time t.

$$F_{t,T} = S_t e^{(r-\delta)(T-t)}$$

$$F_{t,T} \longrightarrow F_{t+h,T} \text{ ??}$$

$$F_{t+h,T} = S_{t+h} e^{(r-\delta)(T-(t+h))}$$

$$S_t \begin{cases} S_t e^{(r-\delta)h+\sigma\sqrt{h}} \\ S_t e^{(r-\delta)h-\sigma\sqrt{h}} \end{cases}$$ (forward tree)

$$F_{t+h,T} = \begin{cases} S_t e^{(r-\delta)h+\sigma\sqrt{h}} \cdot e^{(r-\delta)(T-(t+h))} \\ S_t e^{(r-\delta)h-\sigma\sqrt{h}} \, e^{(r-\delta)(T-(t+h))} \end{cases}$$

$$= \begin{cases} e^{\sigma\sqrt{h}} S_t e^{(r-\delta)(T-t)} \\ e^{-\sigma\sqrt{h}} S_t e^{(r-\delta)(T-t)} \end{cases}$$

$=$

$$= \begin{cases} e^{\sigma\sqrt{h}} F_{t,T} & \Rightarrow u = e^{\sigma\sqrt{h}} \\ e^{-\sigma\sqrt{h}} F_{t,T} & \Rightarrow d = e^{-\sigma\sqrt{h}} \end{cases}$$

<u>Replicating portfolio</u>

Long Δ unit of Forward (futures)

Lend \$B.

At $t = t+h$, $F = F_{t,T}$

$$\Delta \times \underbrace{(uF - F)}_{\text{Because of mark to market}} + Be^{rh} = C_u \quad (1)$$

$$\Delta \times (dF - F) + Be^{rh} = C_d \quad (2)$$

Solve Δ and B from above two Eqs. (1) & (2)

Options on Futures Contracts (cont'd)

- Solving for Δ and B gives

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d}\right)$$

Δ tells us how many futures contracts to hold to hedge the option, and B is simply the value of the option.

- We can again price the option using equation (10.3).

- The risk-neutral probability of an up move is given by

$$p^* = \frac{1-d}{u-d} \qquad (10.21)$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d}\right)$$

$$= e^{-rh}\left[p^* C_u + (1-p^*) C_d \right]$$

$$= e^{-rh} E^*[C_h]$$

In the stock case

$$C = \Delta S + B$$

In the futures case

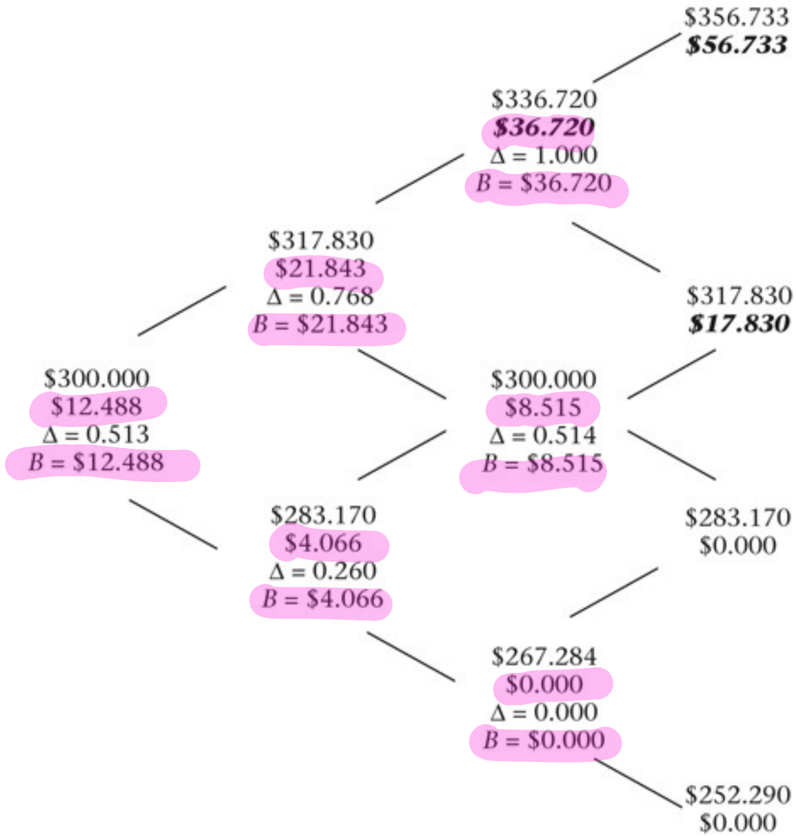
$$C = B$$

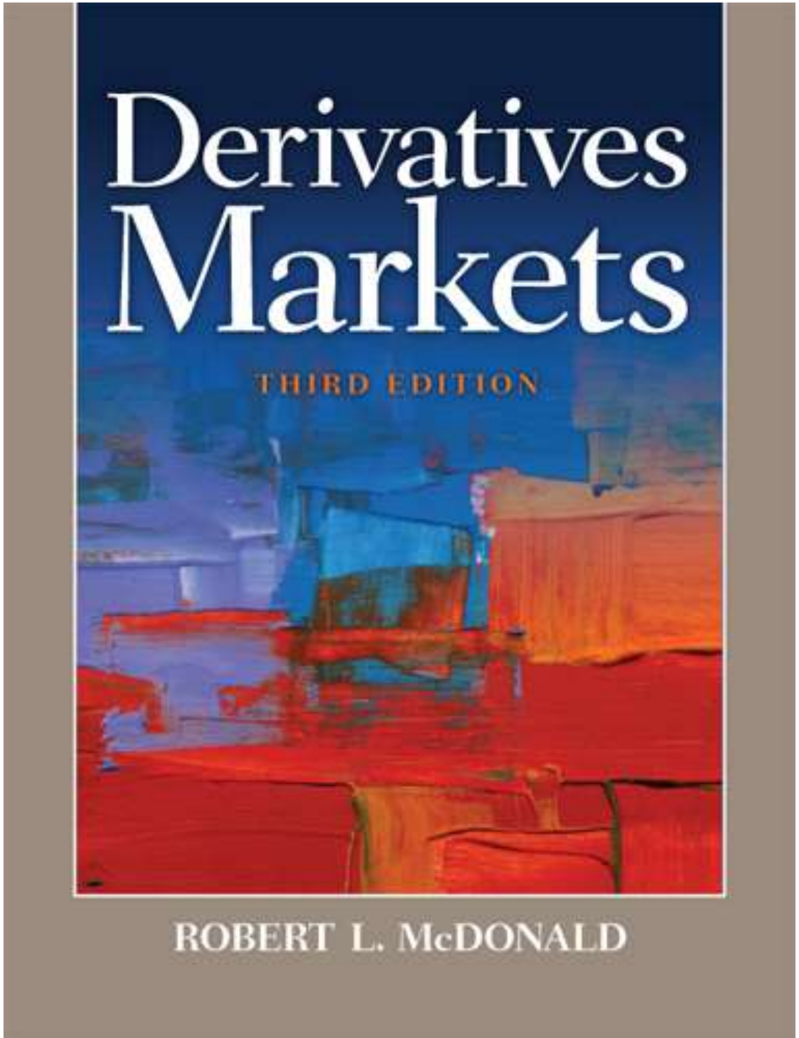
Options on Futures Contracts (cont'd)

- The motive for early-exercise of an option on a futures contract is the ability to earn interest on the mark-to-market proceeds.
  - When an option is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price.

Options on Futures Contracts (cont'd)

- A tree for an American call option on a futures contract





Chapter 10
(Chapter 12 in the textbook)

The Black-Scholes Formula

Points to Note

1. General understanding of the option price (see P.3 – 4)
2. What is the Black-Scholes formula for the European call and put options? (see P.5 – 7)
3. What are the assumptions of the Black-Scholes formula? (see P.8 – 9)
4. What is the relationship of the binomial model and the Black-Scholes formula? (see P.10 – 11)
5. The Black-Scholes formula for different underlying assets. (see P.12 – 17)
6. Option Greeks (see P.18 – 48)
7. Implied volatility (see P.49 – 54)

General Understanding of Option Price

- Option price = Intrinsic value + Time value

  where

  Intrinsic value:

  the value of an option would have if it were exercised today.

  Time value:

  the amount by which the price of an option exceeds the intrinsic value.

General Understanding of Option Price (cont'd)

$$C(S,K,T) = \underbrace{\boxed{\max(S-K,0)}}_{\text{Intrinsic Value}} + \text{Time value} \ ??$$

$$P(S,K,T) = \overbrace{\max(K-S,0)} + \text{Time value} \ ??$$

option price > Intrinsic Value

Black-Scholes Formula

- The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.
- Consider an European call (or put) option written on a stock.
- Assume that the stock pays dividend at the continuous rate δ.

Black-Scholes Formula (cont'd)

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Black-Scholes Formula (cont'd)

- Call Option price: Hedging

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

- Put Option price:

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

where

$$d_1 = \frac{ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

$N(x)$ is the cumulative distribution for standard normal random variable.

$$C(S,K,\sigma,r,T,\delta) = S\underbrace{e^{-\delta T} N(d_1)}_{n_1} - Ke^{-rT}\underbrace{N(d_2)}_{n_2}$$

$$= n_1 S - n_2 \, PV(K)$$

$\Rightarrow$ replicating portfolio of call option

$\Rightarrow$ Long n_1 ? units of stock

+ short n_2 ? unit of T-year zero-coupon bond with face value = K.

Black-Scholes Assumptions

- Assumptions about stock return distribution:
  - Continuously compounded returns on the stock are normally distributed and independent over time. (We assume there are no "jumps" in the stock price).
  - The volatility of continuously compounded returns is known and constant.
  - Future dividends are known, either as dollar amount or as a fixed dividend yield.

Black-Scholes Assumptions (cont'd)

- Assumptions about the economic environment:
  - The risk-free rate is known and constant.
  - There are no transaction costs or taxes.
  - It is possible to short-sell costlessly and to borrow at the risk-free rate.

Continuous Limits of the Binomial Model (cont'd)

It can shown that

$$\lim_{n\to\infty}\left[S_0\Phi\left(n,k,\tilde{p}\right)-Ke^{-rT}\Phi\left(n,k,p^*\right)\right]=S_0N\left(d_1\right)-Ke^{-rT}N\left(d_2\right).$$

Recall that $\Phi\left(n,k,p^*\right)$ is the risk neutral probability that the number of upward moves in the asset price is greater than or equal to *k* in the *n*-period binomial model, where p^* is the risk neutral probability of an upward move.

(See Binomial tree to BS.pdf)

MFE5130 – Financial Derivatives

Class Activity (12-November-2019) (Solution)

Important Notes:

1. This class activity is counted toward to your class participation score. **Fail** to hand in this class activity worksheet in the class will receive **0 score** for that class.
2. **0 mark** will be received if you leave the solution blank.

Name:	Student No.:

Problem 1 $S_0 = 130$ $\sigma = 35\%$

The current price of a stock is \$130. The volatility of the stock is 35%. The dividend yield of the stock is 2%. $\delta = 2\%$

The continuously compounded risk-free interest rate is 7%. $r = 7\%$

An 8-month European call option on the stock has a strike price of \$247. $K = 247$

The option is priced using the forward tree with 8 periods.

Calculate the value of the European call option.

Solution

The value of *h* is 1/12 since the intervals are monthly periods. The values of *u* and *d* are:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)+0.35\sqrt{1/12}} = 1.1109,$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.07-0.02)(1/12)-0.35\sqrt{1/12}} = 0.9077.$$

The risk-neutral probability of an upward movement is :

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.07-0.02)(1/12)} - 0.9077}{1.1109 - 0.9077} = 0.4748.$$

Let k be the smallest integer such that $u^k d^{n-k} S_0 \geq K$, that is

$$\boxed{k \geq \frac{\ln\left(\frac{K}{S_0 d^n}\right)}{\ln\left(\frac{u}{d}\right)}} = \frac{\ln\left(\frac{247}{130(0.9077)^8}\right)}{\ln\left(\frac{1.1109}{0.9077}\right)} = 7.0124.$$

So, $k = 8$.

The option value is then given by

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n,k,\tilde{p}\right) - Ke^{-rT} \Phi\left(n,k,p^*\right).$$

$n = 8$

$$
\begin{aligned}
C &= S_0 C_8^8 \left(p^*\right)^8 \left(1-p^*\right)^0 \frac{u^8 d^0}{e^{rT}} - K e^{-rT} C_8^8 \left(p^*\right)^8 \left(1-p^*\right)^0 \\
&= 130 e^{-0.07(8/12)} \left(0.4748\right)^8 \left(1.1109\right)^8 - 247 e^{-0.07(8/12)} \left(0.4748\right)^8 \\
&= 0.1344.
\end{aligned}
$$