Chapter 14
Martingale Pricing Theory

See Section 3.2 of “Mathematical Models of Financial Derivatives”, 2nd edition, by Yue Kuen KWOK, Springer Verlag, 2008.

Points to Note

1. What is the definition of the equivalent martingale measure? See P.3 – 6.
2. What is the relationship between the no-arbitrage price of a financial product and the risk-neutral probability? See P.8 – 13.
3. How do we change a measure in the expectation? Use the notation of the Randon-Nikodym derivative. See P.14 – 15.
4. Girsanov Theorem. See P.16 – 19.
5. Converting the dynamic of the asset price processes from the real probability to the risk-neutral probability. See P.20 – 25.
6. Derivation of the BS formula by using the change of numeraire. See P.26 – 38.
7. The Black-Scholes formula for the dividend-paying asset. See P.39 – 47.

Equivalent Martingale Measure and Risk Neutral Valuation

Under the continuous time framework, the investors are allowed to trade continuously in the financial market up to finite time T.

Consider the securities model, there are

$K + 1$ securities whose price processes are modeled by $M(t)$ and $S_m(t)$ (**non-dividend-paying assets**), where $m = 1, \ldots, K$.

The uncertainly of the market is described by the actual (real) probability measure (distribution) P.

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

We use $M(t)$ to denote the money market account process that starts at \$1 and grows at the <u>*deterministic*</u> risk-free interest rate $r(t)$, that is,

$$dM(t) = r(t)M(t)\,dt \quad \Leftrightarrow \quad M(t) = e^{\int_0^t r(s)ds}.$$

The discounted security price process $S_m^*(t)$ is defined by

$$S_m^*(t) = \frac{S_m(t)}{M(t)}, \qquad m = 1, 2, \cdots, K.$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

Suppose that Q_1 and Q_2 are probability measures (distributions) on the sample space Ω, Q_1 and Q_2 are *equivalent* if

$$Q_1(A) = 0 \Leftrightarrow Q_2(A) = 0 \text{ for any } A \subset \Omega.$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

A probability measure *Q* is said to be an <u>*equivalent martingale measure*</u> (or <u>*risk-neutral measure*</u>) to the real probability measure *P* if it satisfies

i. *Q* is equivalent to *P*;

ii. The discounted security price process $S_m^*(t)$, $m = 1, 2, \ldots, K$, are martingales under *Q*, that is

$$E_t^Q[S_m^*(u)] = S_m^*(t) \qquad \text{for all } 0 \le t \le u \le T.$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

In (ii), $E_t^Q(\cdot)$ is defined as the expectation with respect to the probability measure *Q* conditional on the information available up to time *t*.

For simplicity, we would write $E_0^Q(\cdot)$ as $E^Q(\cdot)$.

Remark:

a. Existence of *Q* $\Rightarrow$ Absence of arbitrage.
b. Absence of arbitrage + some technical conditions $\Rightarrow$ Existence of *Q*.
c. Complete market $\Leftrightarrow$ *Q* is unique.

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Theorem (Risk Neutral Valuation)

Assume that an equivalent martingale measure Q exists. Let $V(T)$ be the payoff of a contingent claim which can be generated by the $K + 1$ securities, the *no arbitrage price* of the contingent claim at time t, $V(t)$, is given by

$$V(t) = M(t) E_t^Q \left[\frac{V(T)}{M(T)} \right].$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

1. Zero-coupon bond (pays \$1 at the maturity *T*)

   $V(T) = 1$

   By the risk neutral valuation theorem, we have

$$P(0,T) = M(0)E^Q\left[\frac{1}{M(T)}\right]$$

$$= E^Q\left[e^{-\int_0^T r(s)ds}\right].$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

2. Forward price

The payoff for the long position is

$V(T) = S(T) - K$ where K is the forward price.

By the risk neutral valuation theorem, we have

$$V(0) = M(0)E^Q\left[\frac{S(T)-K}{M(T)}\right].$$

Since there is 0 cost to enter into a forward contract, $V(0) = 0$. This implies

$$E^Q\left[\frac{S(T)-K}{M(T)}\right] = 0.$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

So,

$$E^Q\left[\frac{K}{M(T)}\right] = E^Q\left[\frac{S(T)}{M(T)}\right]$$

$$KE^Q\left[\frac{1}{M(T)}\right] = \frac{S(0)}{M(0)}$$

$$KP(0,T) = S(0)$$

$$K = \frac{S(0)}{P(0,T)}.$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

3. Swap

   Consider a swap with the fixed swap price payment dates on t_i, $i = 1, \ldots, n$.

   Let R be the fixed swap price.

   By the risk neutral valuation theorem, the value of the swap from the perspective of long party is given by

$$V(0) = M(0) E^Q \left[\sum_{i=1}^{n} \frac{\left(S(t_i) - R \right)}{M(t_i)} \right].$$

   Since the value of the swap is 0 at the inception date ($t = 0$),

Application of the Risk Neutral Valuation Theorem (Cont'd)

$$R\sum_{i=1}^{n} E^Q\left[\frac{1}{M(t_i)}\right] = \sum_{i=1}^{n} E^Q\left[\frac{S(t_i)}{M(t_i)}\right]$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} S(0)$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} \frac{S(0)P(0,t_i)}{P(0,t_i)}$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} F_{0,t_i} P(0,t_i)$$

$$R = \frac{\sum_{i=1}^{n} F_{0,t_i} P(0,t_i)}{\sum_{i=1}^{n} P(0,t_i)}.$$

Change of Measure

Definition

The Randon-Nikodym derivative of Q_2 with respect to Q_1 based is denoted as

$$\frac{dQ_2}{dQ_1}$$

This notation can be understood as the ratio of the two probability density functions (pdfs).

Change of Measure (cont'd)

The expectations under measures Q_1 and Q_2 are related as follows:

$$E^{Q_2}\left[Y\right] = E^{Q_1}\left[\frac{dQ_2}{dQ_1}Y\right],$$

where Y is a random variable.

Change of Measure (Cont'd)

Girsanov Theorem (1-dimensional)

Let $Z(t)$ be *a* Brownian motion under the probability measure P. The Radon-Nikodym derivative of $\tilde{P}$ with respect to P is given by

$$\frac{d\tilde{P}}{dP} = \xi(t),$$

where

$$\xi(t) = \exp\left\{-\eta Z(t) - 0.5\eta^2 t\right\}.$$

Change of Measure (Cont'd)

Girsanov Theorem (1-dimensional) (Cont'd)

Under the new probability measure $\widetilde{P}$, the process

$$\tilde{Z}(t) = Z(t) + \eta t,$$

is a Brownian.

Change of Measure (Cont'd)

Girsanov Theorem (*K*-dimensional)

Let $Z_1(t)$, …, $Z_K(t)$ be K independent Brownian motions under the probability measure P. The Radon-Nikodym derivative of $\tilde{P}$ with respect to P is given by

$$\frac{d\tilde{P}}{dP} = \xi(t),$$

where

$$\xi(t) = \exp\left\{-\sum_{i=1}^{K} \eta_i Z_i(t) - 0.5t \sum_{i=1}^{K} \eta_i^2\right\}.$$

Change of Measure (Cont'd)

Girsanov Theorem (*K*-dimensional) (Cont'd)

Under the new probability measure $\widetilde{P}$, the process

$$\widetilde{Z}_i(t) = Z_i(t) + \eta_i t, \quad \text{for } i = 1, \ldots, K$$

is a Brownian motion and $\widetilde{Z}_1(t), \widetilde{Z}_1(t), \ldots, \widetilde{Z}_K(t)$ are independent.

Black-Scholes Model Revisited

Assume the existence of a risk neutral measure Q under which all discounted price processes are martingales.

Also, assume that the securities model only has two tradable securities:

(i) Money market account: $M(t)$;

(ii) Non-dividend paying asset: $S(t)$.

Black-Scholes Model Revisited (Cont'd)

The price processes of the risky asset and the money market account under the real (physical) probability measure *P* are governed by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dZ(t)$$

$$dM(t) = rM(t)dt$$

respectively, where μ is the expected return on the stock, *r* is the constant risk-free interest rate, *Z*(*t*) is a Brownian motion under *P*.

Black-Scholes Model Revisited (Cont'd)

The price of the discounted risky asset is

$$S^*(t) = \frac{S(t)}{M(t)}.$$

By Ito's lemma, the price process $S^*(t)$ becomes

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r)\,dt + \sigma\,dZ(t)$$

$$= \sigma\left[dZ(t) + \frac{(\mu - r)}{\sigma}dt\right]$$

$$= \sigma d\left[Z(t) + \frac{(\mu - r)}{\sigma}t\right].$$

Black-Scholes Model Revisited (Cont'd)

We would like to find the equivalent martingale measure *Q* under which $S^*(t)$ is a martingale. By the Girsanov Theorem, we choose

$$\xi(t) = \exp\left\{-\eta Z(t) - 0.5 t \eta^2\right\} \quad \text{where } \eta = \frac{\mu - r}{\sigma},$$

Then

$$\widetilde{Z}(t) = Z(t) + \frac{(\mu - r)}{\sigma} t$$

is a Brownian motion under the new measure constructed from the Girsanov Theorm. This measure is the required *Q*.

Black-Scholes Model Revisited (Cont'd)

Under the measure *Q*,

$$\frac{dS^*(t)}{S^*(t)} = \sigma d\widetilde{Z}(t).$$

Since $\widetilde{Z}(t)$ is a Brownian motion under *Q*, so $S^*(t)$ is a martingale under *Q*.

Under *Q*,

$$\frac{dS(t)}{S(t)} = r dt + \sigma d\widetilde{Z}(t)$$

where the drift rate equals the risk-free interest rate *r*.

Black-Scholes Model Revisited (Cont'd)

By the risk-neutral valuation theorem, the price of the call option *V*(0) at time 0 is

$$V(0) = e^{r\times 0} E_0^Q\left[\frac{\max\left(S(T) - K, 0\right)}{e^{rT}}\right]$$
$$= e^{-rT} E_0^Q\left[\max\left(S(T) - K, 0\right)\right]$$

So, the Black-Scholes formula can then be obtained. The details can refer to the hand-written notes "Derivation of BS formula".

Change of Numeraire

A *numeraire* is a positively priced **non-dividend-paying** asset which denominate other asset. A typical example of a numeraire is the currency of a country, because people usually measure other asset's price in terms of the unit of currency.

Change of Numeraire (Cont'd)

In the risk-neutral valuation, we take the money market account as the numeraire, when the security price is denominated with money market account it is a martingale under the risk-neutral measure Q.

Change of Numeraire (Cont'd)

Theorem (Change of Numeraire)

(*Geman, Karoui and Rochet (1995)*)

Let $X(t)$ be a positively priced non-dividend-paying asset such that $X(t)/M(t)$ is a martingale under Q.

Then there exists a probability measure Q_X defined by its Radon-Nikodym derivative with respect to Q

$$\frac{dQ_X}{dQ} = \left(\frac{X(t)}{X(0)}\right) \bigg/ \left(\frac{M(t)}{M(0)}\right), \qquad 0 \le t \le T.$$

Change of Numeraire (Cont'd)

Theorem (Change of Numeraire) (Cont'd)

such that

$$V(0) = M(0) E^{Q}\left[\frac{V(T)}{M(T)}\right] = X(0) E^{Q_X}\left[\frac{V(T)}{X(T)}\right]$$

where $V(T)$ is defined in the payoff of a derivative security which satisfies the properties in Risk neutral Valuation Theorem.

Change of Numeraire (Cont'd)

Remark

The Change of Numeraire Theorem can be generalized for changing from the numeraire-measure pair $(N_1(t), Q_{N_1})$ to the other pair $(N_2(t), Q_{N_2})$:

$$N_1(0)E^{Q_{N_1}}\left[\frac{V(T)}{N_1(T)}\right]=N_2(0)E^{Q_{N_2}}\left[\frac{V(T)}{N_2(T)}\right]$$

and

$$\frac{dQ_{N_1}}{dQ_{N_2}}=\left(\frac{N_1(t)}{N_1(0)}\right)\Bigg/\left(\frac{N_2(t)}{N_2(0)}\right),\qquad 0\le t\le T.$$

Deriving the Black-Scholes Formula

Define the indicator function $\mathbf{I}_{\{S_T \geq K\}}$ as follows:

$$\mathbf{I}_{\{S_T \geq K\}} = \begin{cases} 1 & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

We have the following result for the indicator function:

$$E_t^Q\left[\mathbf{I}_{\{S_T \geq K\}}\right] = Q\left(S_T \geq K \mid S_t = S\right)$$

Deriving the Black-Scholes Formula (Cont'd)

By the risk-neutral valuation theorem, the price of the call option at time 0 is

$$
\begin{aligned}
V(0) &= e^{-rT} E_0^Q\left[\max\left(S_T - K, 0\right)\right] \\
&= e^{-rT} E_0^Q\left[\left(S_T - K\right)\mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} E_0^Q\left[K \mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} K E_0^Q\left[\mathbf{I}_{\{S_T \geq K\}}\right] \\
&= e^{-rT} E_0^Q\left[S_T \mathbf{I}_{\{S_T \geq K\}}\right] - e^{-rT} K Q\left(S_T \geq K \mid S_0 = S\right)
\end{aligned}
$$

Deriving the Black-Scholes Formula (Cont'd)

For simplicity, we assume $\delta = 0$, under Q,

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(r - 0.5\sigma^2\right)T, \sigma^2 T\right)$$

Therefore,

$$Q(S_T \geq K \mid S_0 = S) = N(d_2)$$

Deriving the Black-Scholes Formula (Cont'd)

To evaluate $E_0^Q[S_T \mathbf{I}_{\{S_T \geq K\}}]$, we use the change of numeraire technique by changing the numeraire-measure pair (M_t, Q) to the other pair (S_t, Q_S).

Deriving the Black-Scholes Formula (Cont'd)

Now

$$
\begin{aligned}
e^{-rT} E_0^Q[S_T \mathbf{I}_{\{S_T \geq K\}}] &= E_0^Q\left[\frac{S_T \mathbf{I}_{\{S_T \geq K\}}}{e^{rT}}\right] \\
&= S E_0^{Q_S}\left[\frac{S_T \mathbf{I}_{\{S_T \geq K\}}}{S_T}\right] \\
&= S Q_s\left(S_T \geq K \mid S_0 = S\right)
\end{aligned}
$$

Deriving the Black-Scholes Formula (Cont'd)

The Radon-Nikodym derivative of Q_S with respect to Q is

$$\frac{dQ_S}{dQ} = \left(\frac{S(t)}{S(0)}\right) \Big/ \left(\frac{M(t)}{M(0)}\right) = \left(\frac{S(t)}{S(0)}\right) \Big/ \left(\frac{e^{rt}}{e^{r\times 0}}\right)$$

$$= \exp\left(-0.5\sigma^2 t + \sigma Z_t\right)$$

Deriving the Black-Scholes Formula (Cont'd)

By the Girsanov Theorem, we have

$$\tilde{Z}_t = Z_t - \sigma t$$

is a Brownian motion under Q_S.

Under Q_S, we have

$$\begin{aligned}\frac{dS_t}{S_t} &= r\,dt + \sigma dZ_t \\ &= r\,dt + \sigma\left(d\tilde{Z}_t + \sigma dt\right) \\ &= \left(r + \sigma^2\right)dt + \sigma d\tilde{Z}_t\end{aligned}$$

Deriving the Black-Scholes Formula (Cont'd)

By Itô's lemma, we have

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(r + 0.5\sigma^2\right)T, \sigma^2 T\right)$$

So,

$$Q_s\left(S_T \geq K \mid S_0 = S\right)$$
$$= N(d_1)$$

Combining all the terms, we have

$$V(0) = SN(d_1) - e^{-rT} KN(d_2).$$

Continuous Dividend Yield Models

- In the previous sections, we require discounted stock prices are martingale under the risk-neutral measure. *This is the case provided the stock pays no dividend*.
- The key feature of a risk-neutral measure is that it causes discounted portfolio values to be martingales, and that ensures the absence of arbitrage.

Continuous Dividend Yield Models (cont'd)

- In order for the discounted value of a portfolio that invests in a dividend-paying stock to be a martingale, the discounted value of the stock with the dividends reinvested must be a martingale, but the discounted stock price itself is **NOT** a martingale. That is, we require

$$V^*(t) = \frac{e^{qt} S(t)}{M(t)},$$

where *q* is the dividend yield, to be a martingale under *Q*.

Continuous Dividend Yield Models (cont'd)

The asset price dynamics is assumed to follow the GBM

$$\frac{dS(t)}{S(t)} = \rho dt + \sigma dZ(t).$$

Continuous Dividend Yield Models (cont'd)

Suppose all the dividend yields received are used to purchase additional units of asset, then the value process $V(t)$ of holding 1 unit of asset initially is given by

$$V(t) = e^{qt} S\left(t\right),$$

where e^{qt} represents the growth factor in the number of units.

Continuous Dividend Yield Models (cont'd)

By Ito's lemma, we have

$$\frac{dV(t)}{V(t)} = (\rho + q)\,dt + \sigma\,dZ(t).$$

We would like to find the equivalent risk-neutral measure *Q* under which the

$$V^*(t) = \frac{V(t)}{M(t)}$$

is a *Q*-martingale.

Continuous Dividend Yield Models (cont'd)

We choose η in the Radon-Nikodym derivative to be

$$\eta = \frac{\rho + q - r}{\sigma}.$$

Now $\tilde{Z}(t)$ is a Brownian process under *Q* and

$$d\tilde{Z}(t) = dZ(t) + \frac{\rho + q - r}{\sigma} dt.$$

Also, *V*(*t*) becomes *Q*-martingale since

$$\frac{dV^*(t)}{V^*(t)} = \sigma d\tilde{Z}(t).$$

Continuous Dividend Yield Models (cont'd)

The asset price $S(t)$ under Q becomes

$$\frac{dS(t)}{S(t)} = (r - q)\,dt + \sigma d\tilde{Z}(t).$$

Hence, the risk-neutral drift rate of $S(t)$ is $r - q$.

Equivalently, we have

$$\frac{d\left(e^{qT} S(t)\right)}{e^{qT} S(t)} = r\,dt + \sigma d\tilde{Z}(t).$$

Continuous Dividend Yield Models (cont'd)

Call and put price formulas

The price of a European call option can be obtained as follows:

$$
\begin{aligned}
C &= e^{-rT} E^{Q}\left[\max\left(S(T)-K,0\right)\right] \\
&= e^{-qT} e^{-rT} E^{Q}\left[\max\left(e^{qT} S(T)-e^{qT} K,0\right)\right] \\
&= e^{-qT}\left[Se^{q\times 0} N(d_1)-e^{-rT} e^{qT} KN(d_2)\right] \\
&= Se^{-qT} N(d_1)-Ke^{-rT} N(d_2),
\end{aligned}
$$

where

$$
d_1 = \frac{\ln\left(\frac{S}{Ke^{qT}}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S}{K}\right)+\left(r-q+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1-\sigma\sqrt{T}.
$$

Continuous Dividend Yield Models (cont'd)

Similarly, the put price formula is given by

$$P = Ke^{-rT}N\left(-d_2\right) - Se^{-qT}N\left(-d_1\right).$$