

Derivatives Markets

THIRD EDITION



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Chapter 9 (Chapter 10 in the textbook)

Binomial Option Pricing



Points to Note

1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13).
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 – 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)



Introduction to Binomial Option Pricing

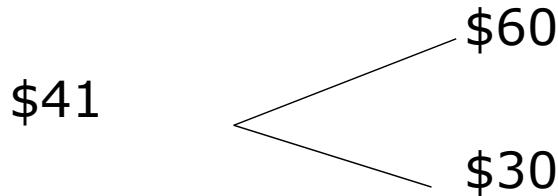
- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the stock or other underlying asset.
- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution—that is, the asset price in each period can move only up or down by a specified amount.
- The binomial model is often referred to as the “Cox-Ross-Rubinstein pricing model”.



A One-Period Binomial Tree

- Example

- Consider a European call option on the stock of XYZ, with a \$40 strike and 1 year to expiration. $K = 40$, $T = 1$
- XYZ does not pay dividends, and its current price is \$41. $S = 41$
- The continuously compounded risk-free interest rate is 8%. $r = 8\%$
- The following figure depicts possible stock prices over 1 year, i.e., a binomial tree





Computing the Option Price

- Next, consider two portfolios:
 - *Portfolio A*: buy one call option.
 - *Portfolio B*: buy 2/3 shares of XYZ and borrow \$18.462 at the risk-free rate.
- Costs ($t=0$) ??
 - *Portfolio A*: the call premium, which is unknown.
 - *Portfolio B*: $2/3 \times \$41 - \$18.462 = \$8.871$.



Computing the Option Price (cont'd)

- Payoffs:

- Portfolio A: *Call option*



Payoff

- Portfolio B:

2/3 purchased shares

Repay loan of \$18.462

Total payoff

Stock Price in 1 Year

	\$30	\$60
	\$20	\$40
-	\$20	-\$20
	0	\$20

$$\text{Cost}_A = \text{Cost}_B$$

$$\text{Call option price} = \frac{2}{3}(41) - 18.462 = 8.871 \quad \#$$

outstanding questions:

1) How do we construct B??

Why $\frac{2}{3}$? "Δ"

Why 18.462? "B"

2) 60 why??

41 30 why??

Call \Leftrightarrow $\left[\frac{2}{3} \text{ share of XYZ} + \text{leaving of } 18.462 \right]$

Synthet. call



option delta



Computing the Option Price (cont'd)

- Portfolios A and B have the same payoff. Therefore
 - Portfolios A and B should have the same cost. Since Portfolio B costs \$8.871, the price of one option must be \$8.871.

The idea that positions that have the same payoff should have the same cost is called the law of one price.



Computing the Option Price (cont'd)

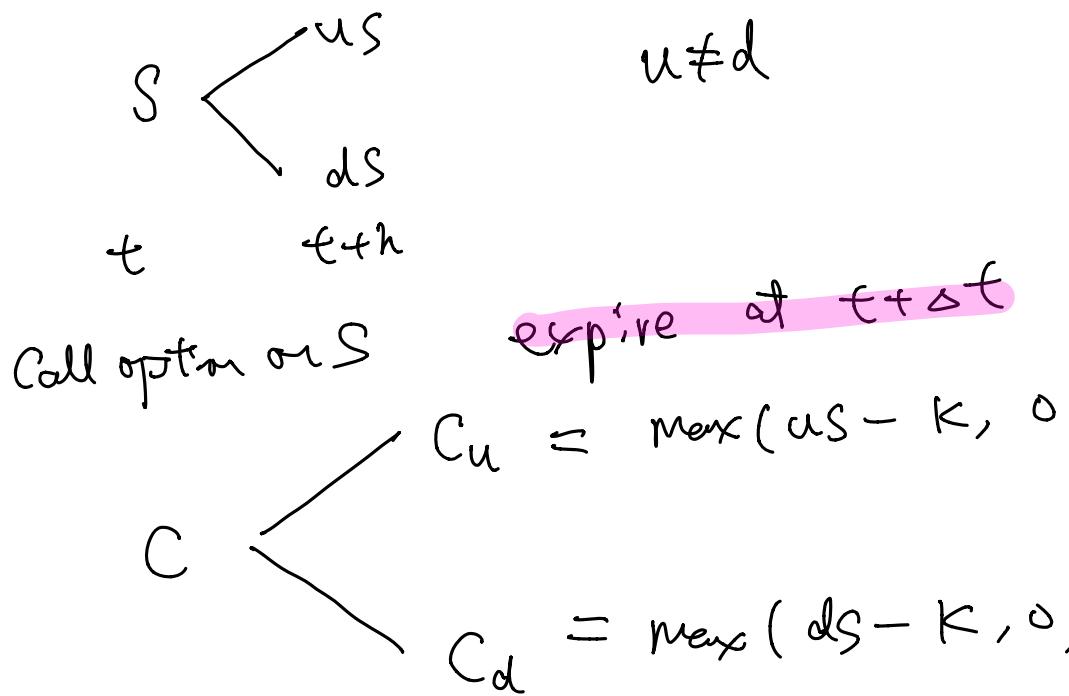
- There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a ***synthetic call*** or **replicating portfolio of the call option**.
- One option has the risk of $2/3$ shares. The value $2/3$ is the *delta* (Δ) of the option: the number of shares that replicates the option payoff.



The Binomial Solution

(How do we get “ $\frac{2}{3}$ ” and “18.462”?)

- How do we find a replicating portfolio consisting of **buying Δ shares of stock** and a **dollar amount B in lending**, such that the portfolio imitates the option whether the stock rises or falls?
 - Suppose that the **stock has a continuous dividend yield of δ** , which is reinvested in the stock. Thus, if you buy one share at time t , at time $t+h$ you will have $e^{\delta h}$ shares.
 - If the length of a period is h , the interest factor per period is e^{rh} .
 - **uS** denotes the stock price when the price **goes up**, and **dS** denotes the stock price when the price **goes down**.
 - The up and down movements of the stock price reflect the **ex-dividend** price.



Portfolio B: Δ share of stock + level \$B.

At $t+h$,
 Payoff of portfolio B = $\Delta e^{S_{t+h}} + e^{r_h} B$

We want to have portfolio B to be a replicating portfolio of call option.

At $t+h$

$$\Delta e^{g_h u s} + e^{r_h B} = C_u$$

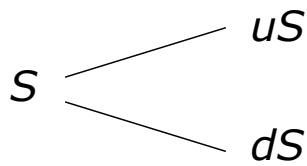
$$\Delta e^{g_h} ds + e^{r_h} B = cd$$

\Rightarrow solve Δ and B in terms of Cu, Cd .

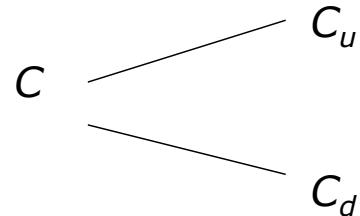


The Binomial Solution (cont'd)

- Stock price tree:



- Corresponding tree for the value of the option:



- Note that u (d) in the stock price tree is interpreted as one plus the rate of capital gain (loss) on the stock if it goes up (down).
- The value of the **replicating portfolio** at time h , with stock price S_h , is

$$\Delta S_h e^{\delta h} + e^{rh} B$$



The Binomial Solution (cont'd)

- At the prices $S_h = uS$ and $S_h = dS$, a successful replicating portfolio will satisfy

$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$

$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$

- Solving for Δ and B gives

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u - d)} \quad (10.1)$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u - d} \quad (10.2)$$



The Binomial Solution (cont'd)

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S + B$.

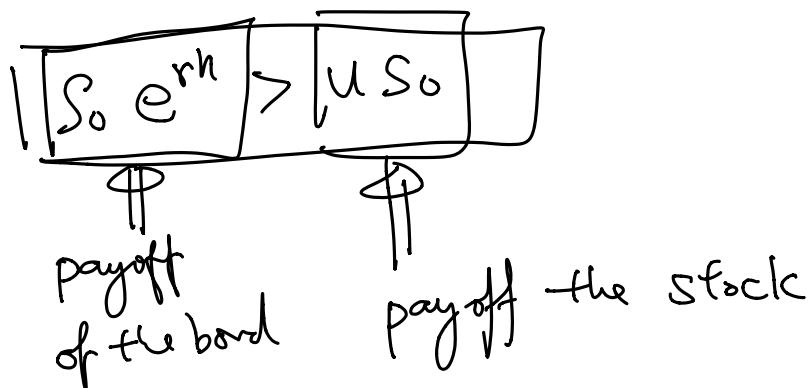
$$\Delta S + B = e^{-rh} \left(C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d} \right) \quad (10.3)$$

- The no-arbitrage condition is

$$u > e^{(r-\delta)h} > d \quad (10.4)$$

If $e^{rh} > u > d$, what is the arbitrage portfolio??

Bond + Stock



\Rightarrow buy the bond + sell the stock
w/ fw face value
 $= S_0 e^{rh}$

	$t=0$	$t=h$
buy the bond	$-S_0$	$S_0 e^{rh}$
Sell the stock	$+S_0$	$-u S_0$ or $-d S_0$
total	0	$S_0 e^{rh} - u S_0 > 0$ or $S_0 e^{rh} - d S_0 > 0$

If $u > d > e^{rh}$, your own Ex.

Violation : ① $e^{rh} > u > d$ or ② $u > d > e^{rh}$



The Binomial Solution (cont'd)

- Suppose that $\delta = 0$. If the condition were violated, we would
 - Short the stock to hold bonds (if $e^{rh} \geq u$), or
 - we borrow to buy the stock (if $d \geq e^{rh}$).

Either way, we would earn an arbitrage profit.

Home exercise: The case of $\delta \neq 0$ is your exercise.



The Binomial Solution (cont'd)

Example

Use the information in P.3, consider a call option had a strike price of \$40 and 1 year to expiration. Calculate the price of the call option.

Here, we have $h = 1$.

$$C_u = \$60 - \$40 = \$20; \quad C_d = \$0$$

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \quad (10.1)$$

$$\Delta = \frac{\$20 - 0}{\$41 \times (1.4634 - 0.7317)} = \frac{2}{3}$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u - d} \quad (10.2)$$

$$B = e^{-0.08} \frac{1.4634 \times \$0 - 0.7317 \times \$20}{1.4634 - 0.7317} = \textcolor{pink}{-\$18.462}$$



The Binomial Solution (cont'd)

Hence the option price is given by

$$\Delta S + B = \frac{2}{3} \times \$41 - \$18.462 = \$8.871$$

Alternatively, the option price can also obtained by

$$\begin{aligned}\Delta S + B &= e^{-0.08} \left(\$20 \times \frac{e^{0.08} - 0.7317}{1.4634 - 0.7317} + \$0 \times \frac{1.4634 - e^{0.08}}{1.4634 - 0.7317} \right) \\ &= \$8.871\end{aligned}$$



Arbitraging a Mispriced Option

- If the observed option price differs from its theoretical price, arbitrage is possible.

Example

- If a call option is **overpriced**, we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option.
- If a call option is **underpriced**, we buy the option. To hedge the risk associated with the possibility of the stock price falling at expiration, we sell a synthetic option at the same time.



Risk-Neutral Pricing

- We can interpret the terms $(e^{(r-\delta)h} - d)/(u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ as probabilities.
- Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \quad (10.5)$$

- Then equation (10.3) can then be written as

$$C = e^{-rh} [p^* C_u + (1 - p^*) C_d] \quad (10.6)$$

We call p^* is the **risk-neutral probability** of an increase in the stock price.

- The pricing procedure illustrated in Eq. (10.6) is called risk-neutral valuation.

$$C = \Delta S + B = e^{-rh} \left(C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d} \right) \quad (10.3)$$

$u > e^{(r-\delta)h} > d$

$$0 < p^* = \frac{e^{(r-\delta)h} - d}{u - d} < 1$$

↑
"probability"

$$C = e^{-rh} [p^* C_u + (1 - p^*) C_d]$$

p^* : prob for $S \rightarrow uS$

$1 - p^*$: prob for $S \rightarrow dS$

p^* : risk-neutral prob.
??

$$C = e^{-rh} E^* [C_{t+h}]$$



Continuously Compounded Returns

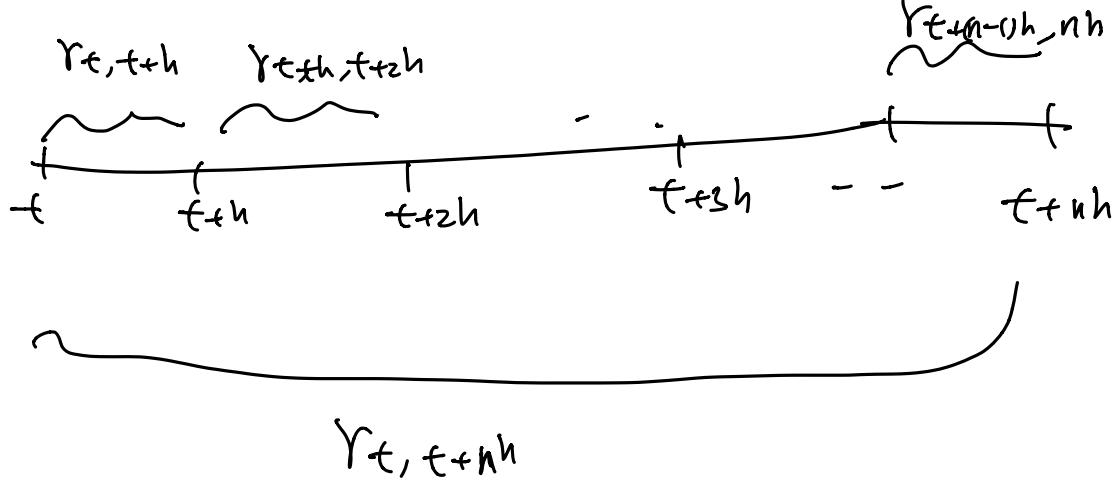
- Some important properties of continuously compounded returns:
 - ◆ The logarithmic function computes returns from prices. The continuously compounded return between t and $t + h$, $r_{t,t+h}$ is then
 - ◆ The exponential function computes prices from returns

$$r_{t,t+h} = \ln(S_{t+h}/S_t) \quad \text{or} \quad S_{t+h} = S_t e^{r_{t,t+h}}$$

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

- ◆ Continuously compounded returns are additive

$$r_{t,t+nh} = \sum_{i=1}^n r_{t+(i-1)h, t+ih}$$



$$r_{t,t+nh} = \sum_{i=1}^n r_{t+(i-1)h,t+ih}$$

$$S_t e^{r_{t,t+nh}} = S_t e^{\sum_{i=1}^n Y_{t+(i-1)h,t+ih}}$$

$r_{t+(i-1)h,h}$ "random"



Volatility

- The volatility of an asset, defined as the standard deviation of continuously compounded returns.
- Suppose the continuously compounded return over month i is $r_{\text{monthly},i}$. Since returns are additive, the annual return is

$$r_{\text{annual}} = \sum_{i=1}^{12} r_{\text{monthly},i}$$

- The variance of the annual return is

.. Volatility of stock

$$\text{Var}(r_{\text{annual}}) = \text{Var}\left(\sum_{i=1}^{12} r_{\text{monthly},i}\right) \quad (10.13)$$

A hand-drawn diagram illustrating a sequence of stock prices over time. The vertical axis on the left lists time points $t=0$, 1, 2, 3, \vdots , 7. To the right of each time point is a price value: 100, 110, 121, 133.1, 7, and 1. A large bracket is drawn vertically along the right side of the list, spanning from the top value down to the bottom value. Inside this bracket, the text "Volatility of stock = 10%" is written twice, once near the top and once near the bottom, indicating that each individual price movement has a 10% standard deviation.



The Standard Deviation of Continuously Compounded Returns (cont'd)

- Suppose that returns are uncorrelated over time and that each month has the same variance of returns. Then from equation (10.13) we have

$$\sigma^2 = 12 \times \sigma_{\text{monthly}}^2 \quad \text{Var}(X_1 + X_2)$$

where σ^2 denotes the annual variance.

- The **volatility** of the asset is then given by[†] $\sqrt{\text{Var}(X_1) + \text{Var}(X_2)}$

annual $\sigma = \sigma_{\text{monthly}} \sqrt{12}$ monthly

- To generalize this formula, if we split the year into n periods of length h (so that $h = 1/n$), the standard deviation over the period of length h , σ_h , is

$$\sigma_h = \sigma \sqrt{h} \quad (10.15)$$



Constructing u and d

- *In the absence of uncertainty*, the stock price next period must equal the forward price. The formula for the forward price is

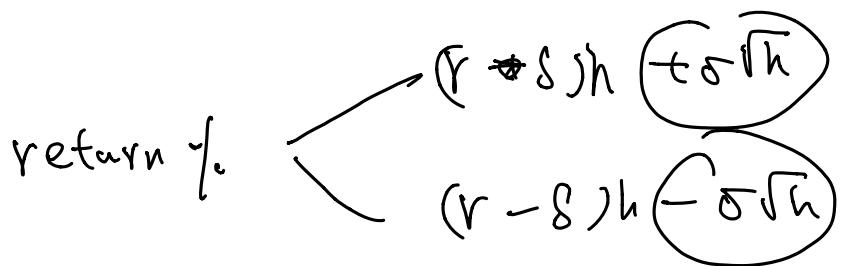
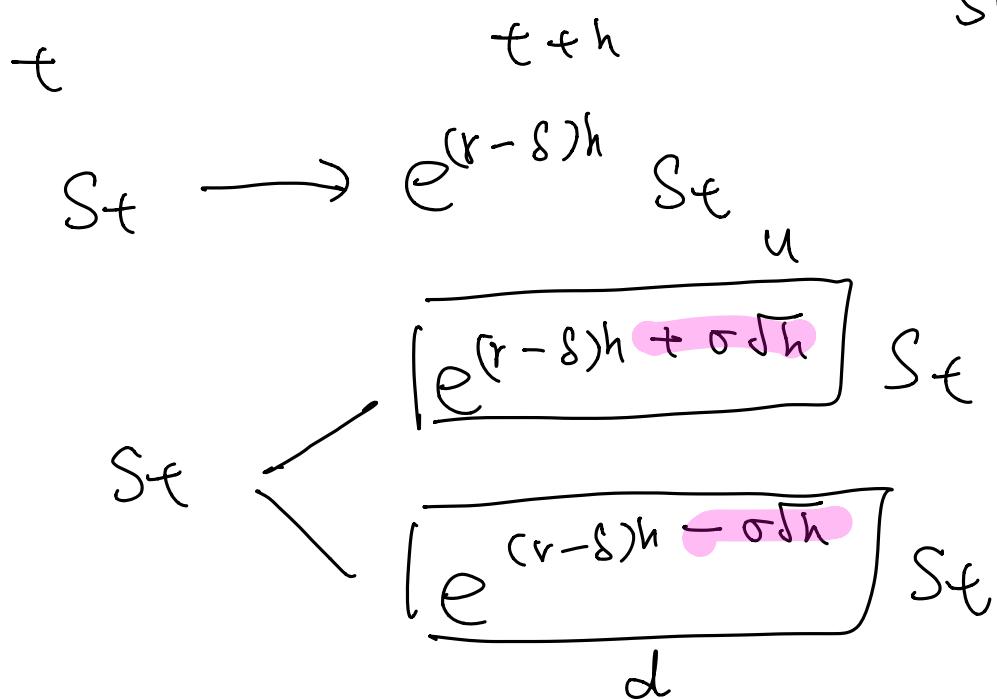
$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

Thus, without uncertainty we must have

$$S_{t+h} = F_{t,t+h}$$

The rate of return on the stock must be the risk-free rate.

Assume No uncertainty σ : volatility of stock



$h \rightarrow 0$ Binomial tree \rightarrow BS formula.

risk-free return $\pm \sigma\sqrt{h}$



Constructing u and d (cont'd)

- With uncertainty, the stock price evolution is

$$\begin{aligned} uS_t &= F_{t,t+h} e^{+\sigma\sqrt{h}} \\ dS_t &= F_{t,t+h} e^{-\sigma\sqrt{h}} \end{aligned} \tag{10.17}$$

where σ is the annualized standard deviation of the continuously compounded return, and $\sigma\sqrt{h}$ is standard deviation over a period of length h .

- We can also rewrite (10.17) as

$$\begin{aligned} u &= e^{(r-\delta)h+\sigma\sqrt{h}} \\ d &= e^{(r-\delta)h-\sigma\sqrt{h}} \end{aligned} \tag{10.9}$$

- We refer to a tree constructed using equation (10.9) as a "forward tree."



Estimating Historical Volatility

- We need to decide what value to assign to σ , which we cannot observe directly.
- One possibility is to measure σ by computing the standard deviation of continuously compounded historical returns.
 - Volatility computed from historical stock returns is **historical volatility**.
 - For example, calculate the standard deviation with weekly data, then annualize the result by using equation (10.15).
 - For option pricing, it is generally the volatility of the price excluding dividends that matters.



Estimating Historical Volatility (cont'd)

- Example

TABLE 10.1

Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

Date	S&P 500 Price	$\ln(S_t/S_{t-1})$	IBM Price	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	-0.02363	125.27	-0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	-0.03408	129.83	-0.01103
8/18/2010	1094.16	0.00430	129.39	-0.00338
8/25/2010	1055.33	-0.03613	125.27	-0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
(weekly) (annualized)	Standard deviation	0.02800	0.02486	0.02486
	Standard deviation $\times \sqrt{52}$	0.20194	0.17926	0.17926



One-Period Example with a Forward Tree

- Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%.
- The continuously compounded risk-free interest rate is 8%.
- $S = 41$, $r = 0.08$, $\delta = 0$, $\sigma = 0.30$, and $h = 1$.
- Use these inputs to
 - calculate the final stock prices.
 - calculate the final option values.
 - calculate Δ and B .
 - calculate the option price.



One-Period Example with a Forward Tree (cont'd)

- Calculate the final stock prices

$$\begin{aligned} uS &= \$41e^{(0.08-0)\times 1 + 0.3 \times \sqrt{1}} = \$59.954 & u &= \frac{\$59.954}{S} = \frac{59.954}{41} = 1.4623 \\ dS &= \$41e^{(0.08-0)\times 1 - 0.3 \times \sqrt{1}} = \$32.903 & d &= \frac{\$32.903}{S} = \frac{32.903}{41} = 0.8025 \end{aligned}$$

- Calculate the final option values

$$C_u = \max(uS - K, 0) = \max(59.954 - 40, 0) = 19.954$$

$$C_d = \max(dS - K, 0) = \max(32.903 - 40, 0) = 0$$

- Calculate Δ and B

$$\Delta = \frac{19.954 - 0}{\$41 \times (1.4623 - 0.8025)} = 0.7376$$

$$B = e^{-0.08} \frac{1.4623 \times \$0 - 0.8025 \times \$19.954}{1.4623 - 0.8025} = -\$22.405$$

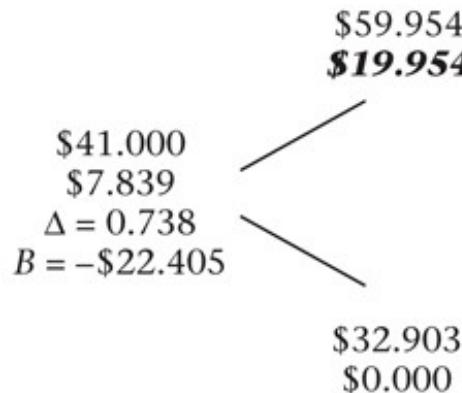
- Calculate the option price

$$\Delta S + B = 0.7376 \times 41 - \$22.405 = \$7.839$$



One-Period Example with a Forward Tree (cont'd)

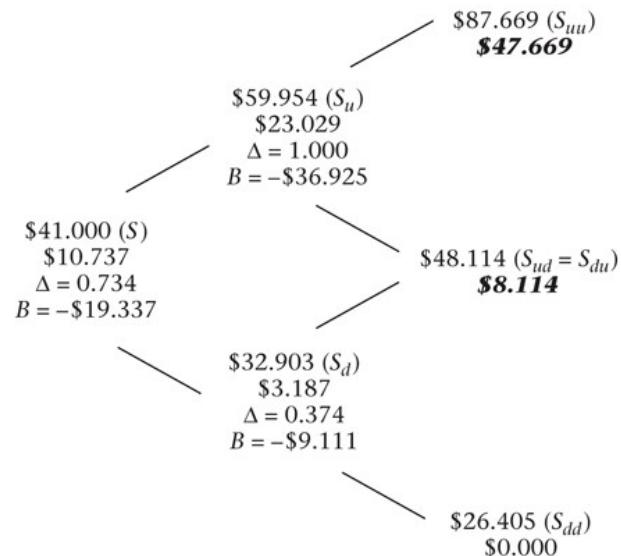
- The following figure depicts the possible stock prices and option prices over 1 year, i.e., a **binomial tree**

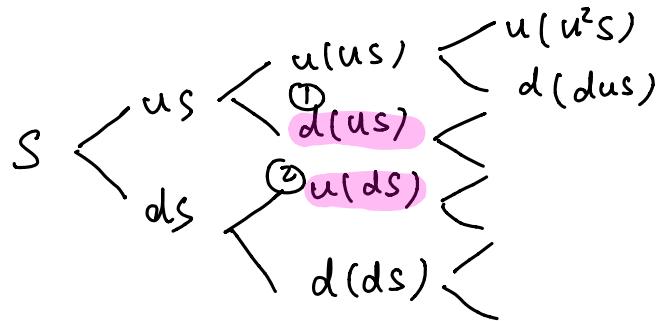




A Two-Period European Call

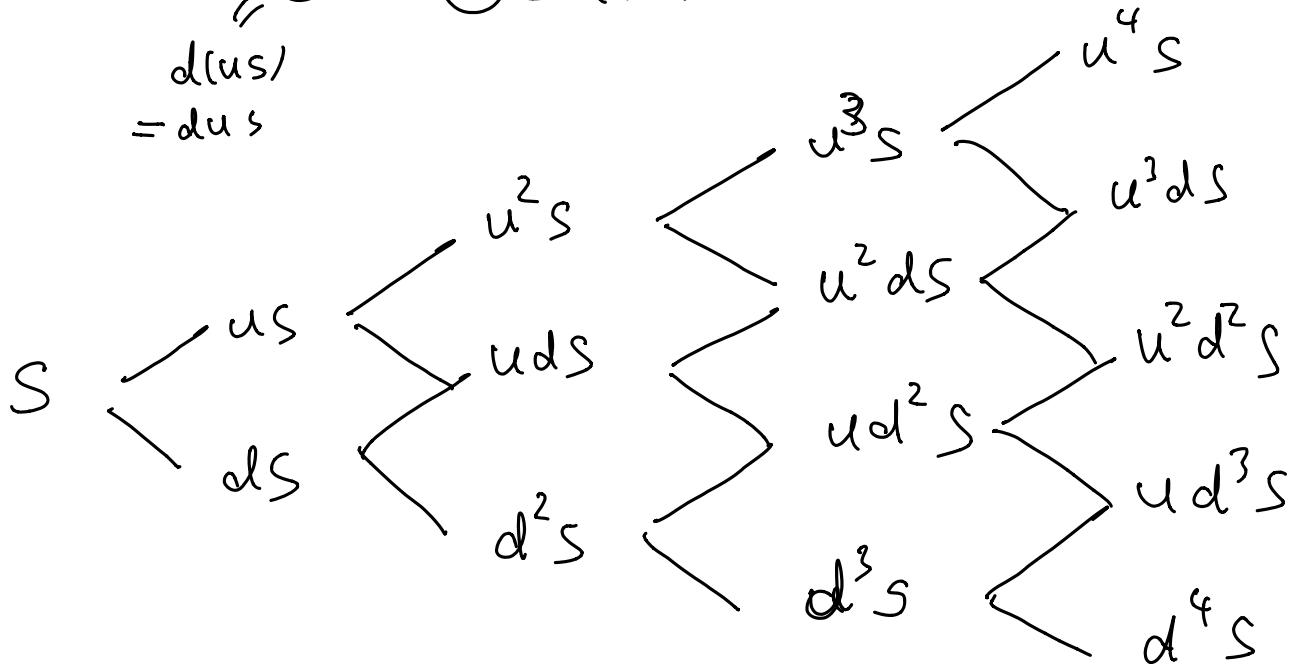
- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before





$$\textcircled{1} = \textcircled{2} = u(ds) = uds$$

$$d(us) \\ = dus$$



"4-step tree"

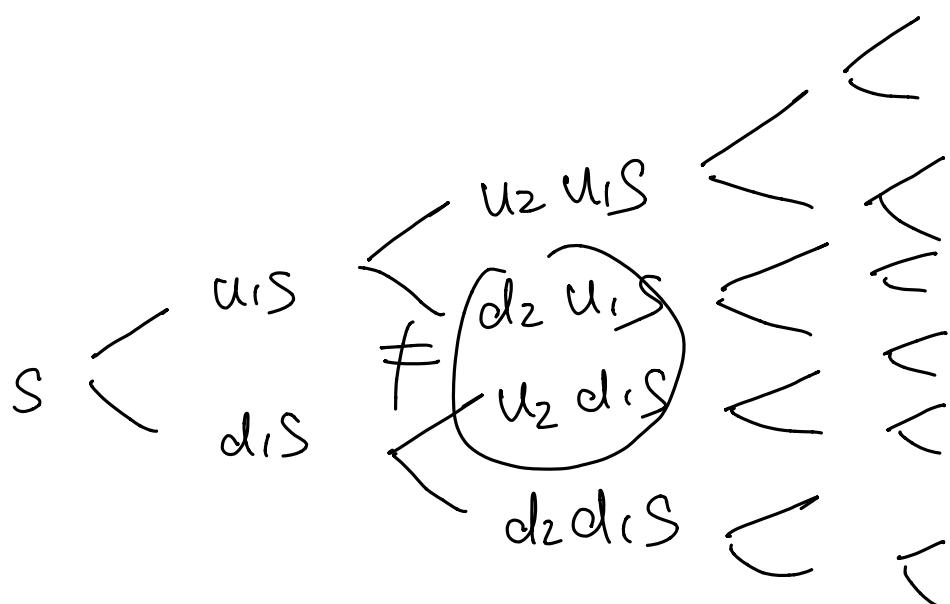
no of
node

1 2

3

4

5



1

2_{2^1}

4_2

8

16

32

After 100 time steps

tree(1) : 101 nodes

tree(2) : 2^{100} nodes

∅∅ =
recombing tree

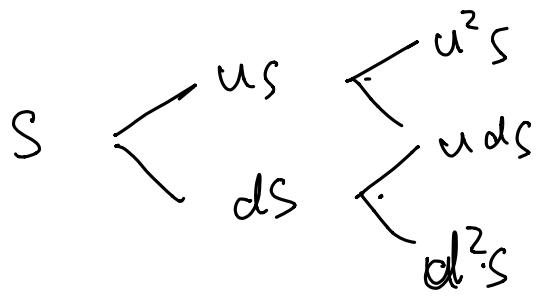
: non-recombing
tree



A Two-period European Call (cont'd)

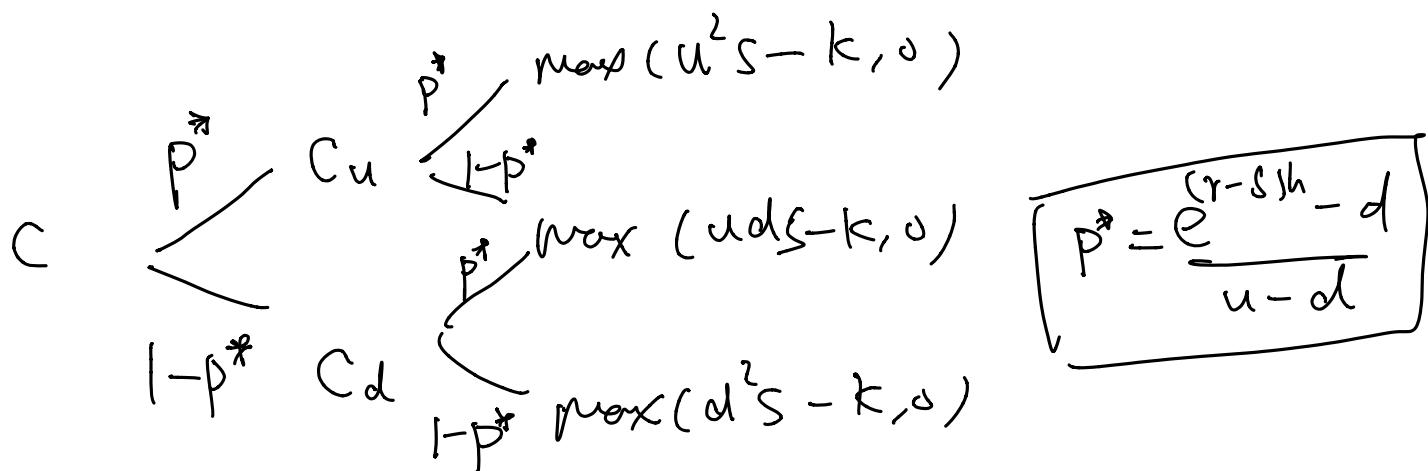
- Note that an up move by the stock followed by a down move (S_{ud}) generates the same stock price as a down move followed by an up move (S_{du}). This is called a **recombining tree**. Otherwise, we would have a **nonrecombining tree**.

$$\begin{aligned}S_{ud} &= S_{du} = u \times d \times \$41 \\&= e^{(0.08+0.3)} \times e^{(0.08-0.3)} \$41 = \$48.114\end{aligned}$$



Call option: K , time to expiration $2\Delta t$

option tree $2\Delta t$



$$C_h = e^{-r\Delta t} E [C_{2h}]$$

$$C_u = e^{-r\Delta t} \left[p^* \max(u^2 S - K, 0) + (1-p^*) \max(u d S - K, 0) \right]$$

$$C_d = e^{-r\Delta t} \left[p^* \max(u d S - K, 0) + (1-p^*) \max(d^2 S - K, 0) \right]$$

option price at $t=0$

$$C = e^{-r\Delta t} \left[p^* C_u + (1-p^*) C_d \right]$$



Pricing the Call Option

- To price an option with two binomial periods, we work *backward* through the tree
 - Year 2, Stock Price=\$87.669:
since we are at expiration, the option value is $\max(0, S - K) = \$47.669$.
 - Year 2, Stock Price=\$48.114:
similarly, the option value is \$8.114.
 - Year 2, Stock Price=\$26.405:
since the option is out of the money, the value is 0.



Pricing the Call Option (cont'd)

- *Year 1, Stock Price=\$59.954:*
at this node, we compute the option value
using equation (10.3), where uS is \$87.669
and dS is \$48.114

$$e^{-0.08} \left(\$47.669 \times \frac{e^{0.08} - 0.803}{1.462 - 0.803} + \$8.114 \times \frac{1.462 - e^{0.08}}{1.462 - 0.803} \right) = \$23.029$$

- *Year 1, Stock Price=\$32.903:*
again using equation (10.3), the option value is \$3.187.
- *Year 0, Stock Price = \$41:*
similarly, the option value is computed to be \$10.737.



Pricing the Call Option (cont'd)

- Notice that
 - The option price is greater for the 2-year than for the 1-year option.
 - The option was priced by working backward through the binomial tree.
 - The option's Δ and B are different at different nodes. At a given point in time, Δ increases to 1 as we go further into the money.
 - Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than $S - K$; hence, we would not exercise even if the option had been American.

MFE5130 – Financial Derivatives
Class Activity (30-October-2019) (Solution)

Important Notes:

1. This class activity is counted toward your class participation score. **Fail** to hand in this class activity worksheet in the class will receive **0 score** for that class.
2. **0 mark** will be received if you leave the solution blank.

Name:	Student No.:
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Problem 1

$$S_0 \quad \sigma = 27\%$$

The price of a share of stock is \$65. The stock pays dividends at a continuously compounded rate of 5%. The stock's volatility is 27%. The price evolution of the stock follows the forward tree with each period being 1 year in length.

The continuously compounded risk-free interest rate is 8%. A 1-year European put option on the stock has a strike price of \$63. The market price of the put option is \$6.00. $k = 63$

An arbitrageur constructs a strategy involving the purchase or sale of exactly one of the European put options.

Determine the accumulated arbitrage profits at the end of one year.

market
price

Cost of Replicating portfolio.

Solution

$$\Delta S_u + \beta$$

The values of u and d are

"forward tree"

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.08-0.05)(1)+0.27\sqrt{1}} = 1.34986,$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.08-0.05)(1)-0.27\sqrt{1}} = 0.78663.$$

$$\Delta = e^{-rh} \frac{C_u - C_d}{S(u-d)}$$

The possible stock prices are therefore:

$$65u = 65(1.34986) = 87.7409,$$

$$65d = 65(0.78663) = 51.1310.$$

$$\beta = e^{-rh} \frac{u C_d - d C_u}{u - d}$$

If the stock price moves up, then the option pays \$0. If the stock price moves down, then the option pays \$11.8692.

$$\begin{array}{ccc}
 87.7409 & = uS & 0 \\
 S & 65 & V \\
 51.1310 & = dS & 11.8690
 \end{array}
 \quad \max(k - uS, 0) \quad \max(k - dS, 0)$$

To replicate the option, the investor must purchase delta shares of the underlying stock:

$$\Delta = e^{-\delta h} \frac{V_u - V_d}{S(u-d)} = e^{-0.05(1)} \frac{0 - 11.8690}{87.7409 - 51.131} = -0.3084.$$

To replicate the option, the investor must lend B :

$$B = e^{-rh} \frac{uV_d - dV_u}{u - d} = e^{-0.08(1)} \frac{1.34986(0) - 0.78663(11.869)}{1.34986 - 0.78663} = 26.2587.$$

The purchase of the option can therefore be replicated by selling 0.3084 shares of stock and lending \$26.2587.

Therefore, the price of the put option should be:

$$V = S_0 \Delta + B = 65(-0.3084) + 26.2587 = 6.2127.$$

*theoretical price
of put option*

But the **market price of the option is \$6.00**. Therefore an arbitrageur can earn an arbitrage profit by **purchasing the option for \$6.00 and synthetically selling it for \$6.2127**.

Buy market put + sell synthetic put

The synthetic sale of the option requires that the arbitrageur buy 0.3084 shares of stock and borrow \$26.2587.

		$t = 1$	
Transaction	$t = 0$	$S_1 = 87.7409$	$S_1 = 51.1310$
Buy the put option	-6.000	0	11.8690
Sell the synthetic put option	6.2127	0	-11.8690
Total	0.2127	0	0

The accumulated arbitrage profits at the end of one year = $0.2127e^{0.08} = 0.2304$.

Sell a unit of stock
 + Borrowing \$13