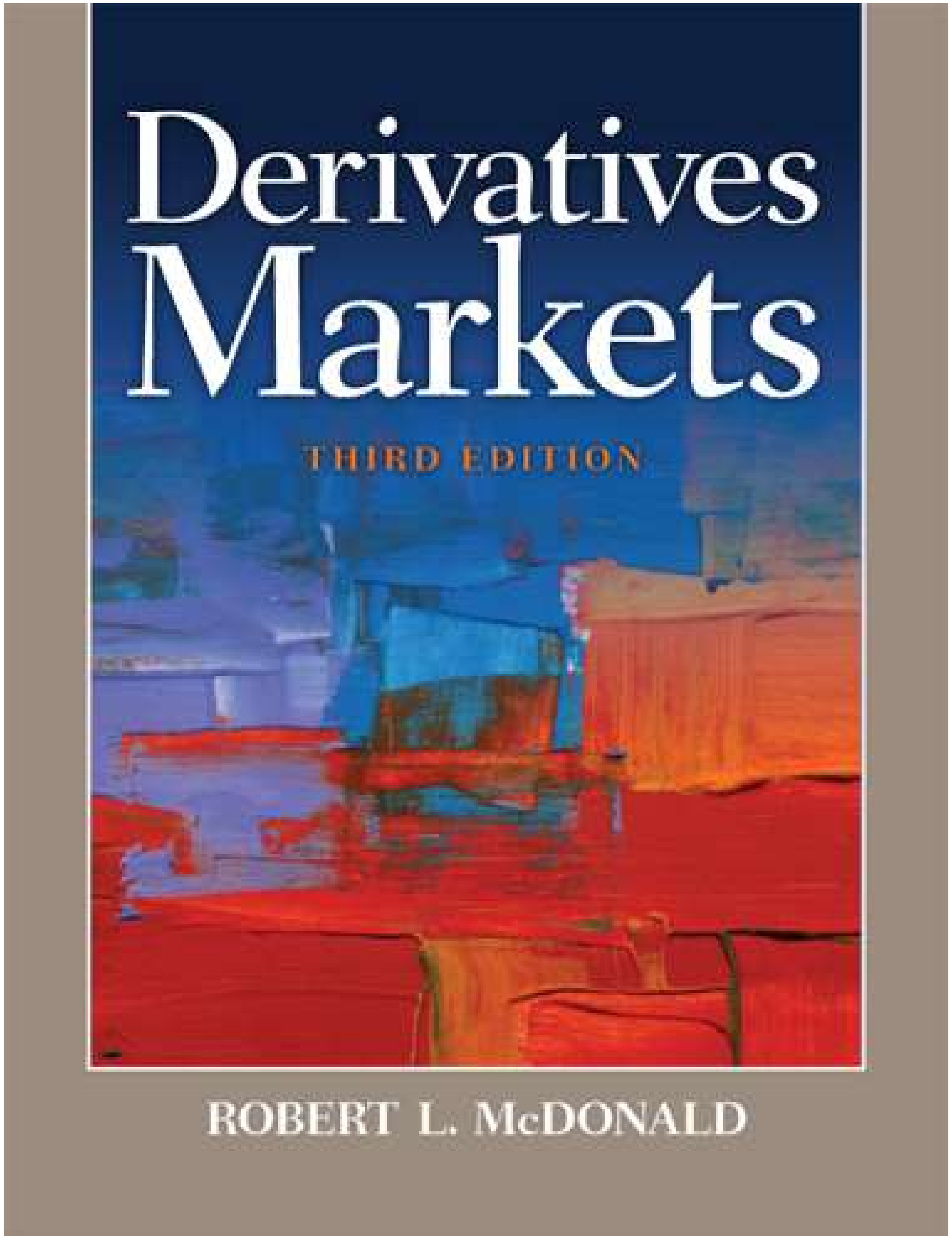
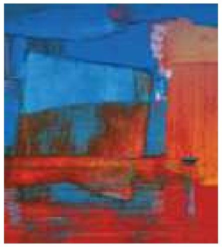


Chapter 9
(Chapter 10 in the textbook)

Binomial Option Pricing



Points to Note

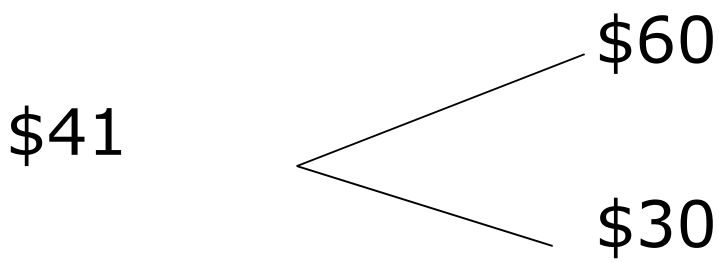
1. Under the one-period binomial model, determine the replicating portfolio of the call option. (see P.9 - 11)
2. What is the no-arbitrage condition for the one-period binomial tree? (see P.12 – 13).
3. Risk-neutral pricing (or valuation). (see P.17)
4. Definition of the volatility. (see P.18 – 20)
5. Construction of the one-period binomial (forward) tree. (see P.21 – 22)
6. Pricing the European call under the two-period forward tree. (see P.28 - 32)
7. Many binomial-period model. (see P. 33 – 44)
8. Pricing of American options. (see P. 45 – 49)
9. Options on other assets. (see P. 50 – 61)

Introduction to Binomial Option Pricing

- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the stock or other underlying asset.
- The binomial option pricing model assumes that the price of the underlying asset follows a binomial distribution—that is, the asset price in each period can move only up or down by a specified amount.
- The binomial model is often referred to as the "Cox-Ross-Rubinstein pricing model".

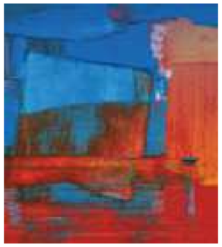
A One-Period Binomial Tree

- Example
  - Consider a European call option on the stock of XYZ, with a $40 strike and 1 year to expiration.
  - XYZ does not pay dividends, and its current price is $41.
  - The continuously compounded risk-free interest rate is 8%.
  - The following figure depicts possible stock prices over 1 year, i.e., a binomial tree



Computing the Option Price

- Next, consider two portfolios:
  - *Portfolio A*: buy one call option.
  - *Portfolio B*: buy 2/3 shares of XYZ and borrow \$18.462 at the risk-free rate.
- Costs
  - *Portfolio A*: the call premium, which is unknown.
  - *Portfolio B*: $2/3 \times \$41 - \$18.462 = \$8.871$.



Computing the Option Price (cont'd)

- Payoffs:
  - *Portfolio A*:

	Stock Price in 1 Year	
	$30	**$60**
Payoff	0	$20

  - *Portfolio B*:

	Stock Price in 1 Year	
	$30	**$60**
2/3 purchased shares	$20	$40
Repay loan of $18.462	– $20	–$20
Total payoff	0	$20

Computing the Option Price (cont'd)

- Portfolios A and B have the same payoff. Therefore
  - Portfolios A and B should have the same cost. Since Portfolio B costs \$8.871, the price of one option must be \$8.871.

    The idea that positions that have the same payoff should have the same cost is called the *law of one price*.

Computing the Option Price (cont'd)

– There is a way to create the payoff to a call by buying shares and borrowing. Portfolio B is a ***synthetic*** **call** or **replicating portfolio of the call option**.

– One option has the risk of 2/3 shares. The value 2/3 is the *delta* (Δ) of the option: the number of shares that replicates the option payoff.

The Binomial Solution

- How do we find a replicating portfolio consisting of **buying Δ shares of stock** and a **dollar amount *B* in lending**, such that the portfolio imitates the option whether the stock rises or falls?
  - Suppose that the stock has a continuous dividend yield of δ, which is reinvested in the stock. Thus, if you buy one share at time t, at time $t+h$ you will have $e^{\delta h}$ shares.
  - If the length of a period is h, the interest factor per period is e^{rh}.
  - uS denotes the stock price when the price goes up, and dS denotes the stock price when the price goes down.
  - The up and down movements of the stock price reflect the <u>*ex-dividend*</u> price.

The Binomial Solution (cont'd)

- Stock price tree:

$$
\begin{array}{ccc}
 & & uS \\
S & & \\
 & & dS
\end{array}
$$

- Corresponding tree for the value of the option:

$$
\begin{array}{ccc}
 & & C_u \\
C & & \\
 & & C_d
\end{array}
$$

  – Note that u (d) in the stock price tree is interpreted as one plus the rate of capital gain (loss) on the stock if it goes up (down).

- The value of the **replicating portfolio** at time h, with stock price S_h, is

$$\Delta S_h e^{\delta h} + e^{rh} B$$

The Binomial Solution (cont'd)

- At the prices $S_h = uS$ and $S_h = dS$, a successful replicating portfolio will satisfy

$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$

$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$

- Solving for Δ and B gives

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u-d)} \qquad (10.1)$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u-d} \qquad (10.2)$$

The Binomial Solution (cont'd)

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S+B$.

$$\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d} \right) \quad (10.3)$$

- The no-arbitrage condition is

$$u > e^{(r-\delta)h} > d \quad (10.4)$$

The Binomial Solution (cont'd)

- Suppose that $\delta = 0$. If the condition were violated, we would
  - Short the stock to hold bonds (if $e^{rh} \geq u$), <u>or</u>
  - we borrow to buy the stock (if $d \geq e^{rh}$).

  Either way, we would earn an *<u>arbitrage profit</u>*.

<u>Home exercise:</u> The case of $\delta \neq 0$ is your exercise.

The Binomial Solution (cont'd)

Example

Use the information in P.3, consider a call option had a strike price of \$40 and 1 year to expiration. Calculate the price of the call option.

Here, we have $h = 1$.

C_u = \$60 - \$40 = \$20; C_d = \$0

$$\Delta = \frac{\$20 - 0}{\$41 \times (1.4634 - 0.7317)} = \frac{2}{3}$$

$$B = e^{-0.08} \frac{1.4634 \times \$0 - 0.7317 \times \$20}{1.4634 - 0.7317} = -\$18.462$$

The Binomial Solution (cont'd)

Hence the option price is given by

$$\Delta S + B = \frac{2}{3} \times \$41 - \$18.462 = \$8.871$$

Alternatively, the option price can also obtained by

$$\Delta S + B = e^{-0.08}\left(\$20 \times \frac{e^{0.08} - 0.7317}{1.4634 - 0.7317} + \$0 \times \frac{1.4634 - e^{0.08}}{1.4634 - 0.7317} \right)$$

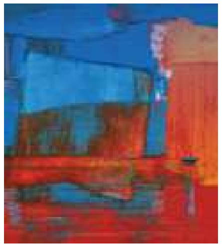
$$= \$8.871$$

Arbitraging a Mispriced Option

- If the observed option price differs from its theoretical price, arbitrage is possible.

  Example

  - If a call option is **overpriced**, we can sell the option. However, the risk is that the option will be in the money at expiration, and we will be required to deliver the stock. To hedge this risk, we can buy a synthetic option at the same time we sell the actual option.
  - If a call option is **underpriced**, we buy the option. To hedge the risk associated with the possibility of the stock price falling at expiration, we sell a synthetic option at the same time.



Risk-Neutral Pricing

- We can interpret the terms $(e^{(r-\delta)h} - d)/(u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ as probabilities.
- Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} \qquad (10.5)$$

- Then equation (10.3) can then be written as

$$C = e^{-rh}[p^* C_u + (1 - p^*) C_d] \qquad (10.6)$$

We call p^* is the **risk-neutral probability** of an increase in the stock price.

- The pricing procedure illustrated in Eq. (10.6) is called *risk-neutral valuation*.

Continuously Compounded Returns

- Some important properties of continuously compounded returns:
  - The logarithmic function computes returns from prices. The continuously compounded return between t and $t + h$, $r_{t,t+h}$ is then

$$r_{t,t+h} = \ln\left(S_{t+h} / S_t\right)$$

  - The exponential function computes prices from returns

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

  - Continuously compounded returns are additive

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

Volatility

- The volatility of an asset, defined as the standard deviation of continuously compounded returns.
- Suppose the continuously compounded return over month i is $r_{monthly,i}$. Since returns are additive, the annual return is

$$r_{annual} = \sum_{i=1}^{12} r_{monthly,i}$$

- The variance of the annual return is

$$\mathrm{Var}(r_{annual}) = \mathrm{Var}\left(\sum_{i=1}^{12} \mathrm{r}_{monthly,i}\right) \qquad (10.13)$$

The Standard Deviation of Continuously Compounded Returns (cont'd)

- Suppose that returns are uncorrelated over time and that each month has the same variance of returns. Then from equation (10.13) we have

$$\sigma^2 = 12 \times \sigma^2_{\text{monthly}}$$

where σ^2 denotes the annual variance.

- The **volatility** of the asset is then given by

$$\sigma = \sigma_{\text{monthly}}\sqrt{12}$$

- To generalize this formula, if we split the year into *n* periods of length *h* (so that $h = 1/n$), the standard deviation over the period of length *h*, σ_h, is

$$\sigma_h = \sigma\sqrt{h} \qquad (10.15)$$

Constructing *u* and *d*

- *In the absence of uncertainty*, the stock price next period must equal the forward price. The formula for the forward price is

$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

Thus, without uncertainly we must have

$$S_{t+h} = F_{t,t+h}$$

The rate of return on the stock must be the risk-free rate.

Constructing *u* and *d* (cont'd)

- *With uncertainty*, the stock price evolution is

$$uS_t = F_{t,t+h}\, e^{+\sigma\sqrt{h}}$$
$$dS_t = F_{t,t+h}\, e^{-\sigma\sqrt{h}} \qquad (10.17)$$

  where σ is the annualized standard deviation of the continuously compounded return, and $\sigma\sqrt{h}$ is standard deviation over a period of length *h*.

- We can also rewrite (10.17) as

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} \qquad (10.9)$$

  – We refer to a tree constructed using equation (10.9) as a "forward tree."

Estimating Historical Volatility

- We need to decide what value to assign to σ, which we cannot observe directly.
- One possibility is to measure σ by computing the standard deviation of continuously compounded historical returns.
  - Volatility computed from historical stock returns is **historical volatility**.
  - For example, calculate the standard deviation with weekly data, then annualize the result by using equation (10.15).
  - For option pricing, it is generally the volatility of the price excluding dividends that matters.

Estimating Historical Volatility (cont'd)

- Example

TABLE 10.1 Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 7/7/2010 to 9/8/2010.

	S&P 500		IBM	
Date	**Price**	$\ln(S_t/S_{t-1})$	**Price**	$\ln(S_t/S_{t-1})$
7/7/2010	1060.27		127	
7/14/2010	1095.17	0.03239	130.72	0.02887
7/21/2010	1069.59	−0.02363	125.27	−0.04259
7/28/2010	1106.13	0.03359	128.43	0.02491
8/4/2010	1127.24	0.01890	131.27	0.02187
8/11/2010	1089.47	−0.03408	129.83	−0.01103
8/18/2010	1094.16	0.00430	129.39	−0.00338
8/25/2010	1055.33	−0.03613	125.27	−0.03238
9/1/2010	1080.29	0.02338	125.77	0.00398
9/8/2010	1098.87	0.01705	126.08	0.00246
Standard deviation	0.02800		0.02486	
Standard deviation $\times \sqrt{52}$	0.20194		0.17926	

One-Period Example with a Forward Tree

- Consider a European call option on a stock, with a \$40 strike and 1 year to expiration. The stock does not pay dividends, and its current price is \$41. Suppose the volatility of the stock is 30%.
- The continuously compounded risk-free interest rate is 8%.
- S = 41, r = 0.08, δ = 0, σ = 0.30, and h = 1.
- Use these inputs to
  - calculate the final stock prices.
  - calculate the final option values.
  - calculate Δ and B.
  - calculate the option price.

One-Period Example with a Forward Tree (cont'd)

- Calculate the final stock prices

$$uS = \$41e^{(0.08-0)\times 1+0.3\times\sqrt{1}} = \$59.954 \qquad\qquad u = \frac{\$59.954}{S} = \frac{59.954}{41} = 1.4623$$

$$\Rightarrow$$

$$dS = \$41e^{(0.08-0)\times 1-0.3\times\sqrt{1}} = \$32.903 \qquad\qquad d = \frac{\$32.903}{S} = \frac{32.903}{41} = 0.8025$$

- Calculate the final option values

$$C_u = \max(uS-K,0) = \max(59.954-40,0) = 19.954$$

$$C_d = \max(dS-K,0) = \max(32.903-40,0) = 0$$

- Calculate Δ and B

$$\Delta = \frac{19.954-0}{\$41\times(1.4632-0.8025)} = 0.7376$$

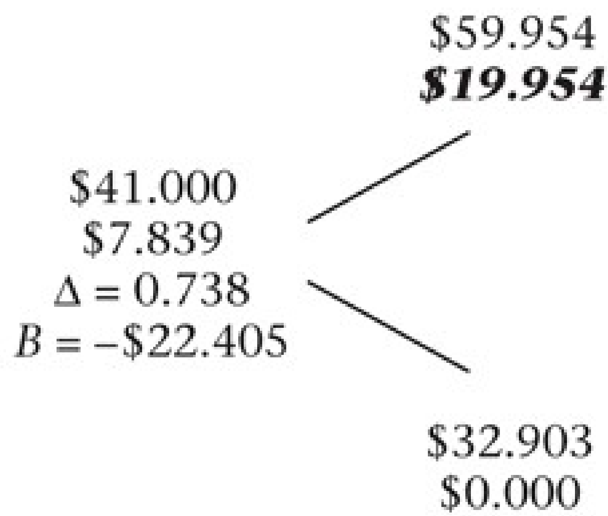
$$B = e^{-0.08}\frac{1.4632\times\$0-0.8025\times\$19.954}{1.4632-0.8025} = -\$22.405$$

- Calculate the option price

$$\Delta S + B = 0.7376\times 41 - \$22.405 = \$7.839$$

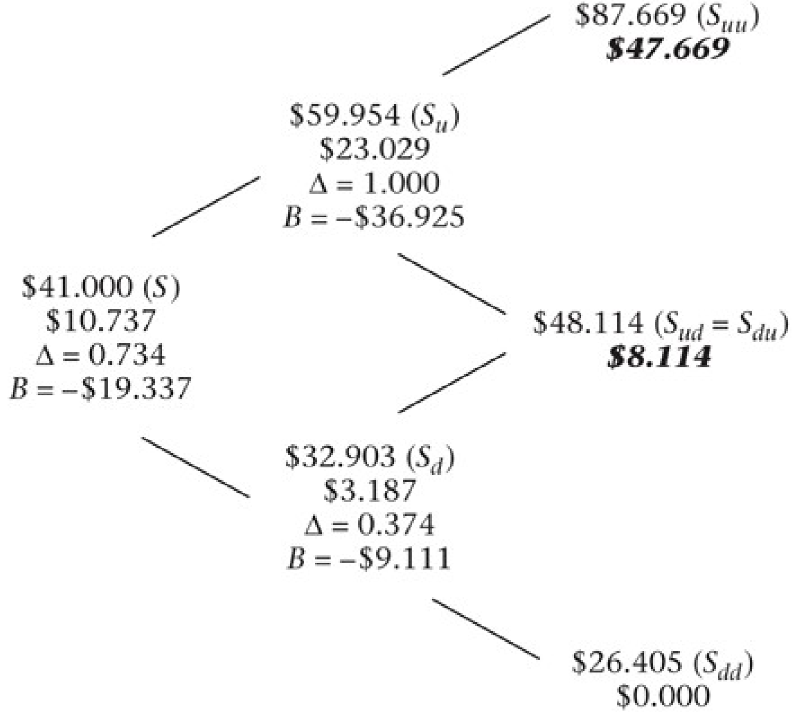
One-Period Example with a Forward Tree (cont'd)

- The following figure depicts the possible stock prices and option prices over 1 year, i.e., a **binomial tree**



A Two-Period European Call

- We can extend the previous example to price a 2-year option, assuming all inputs are the same as before



A Two-period European Call (cont'd)

- Note that an up move by the stock followed by a down move (S_{ud}) generates the same stock price as a down move followed by an up move (S_{du}). This is called a **recombining tree**. Otherwise, we would have a **nonrecombining tree**.

$$S_{ud} = S_{du} = u \times d \times \$41$$
$$= e^{(0.08+0.3)} \times e^{(0.08-0.3)} \$41 = \$48.114$$

Pricing the Call Option

- To price an option with two binomial periods, we work *backward* through the tree
  - *Year 2, Stock Price=$87.669*:
    since we are at expiration, the option value is *max* (0, *S* – *K*) = $47.669.
  - *Year 2, Stock Price=$48.114*:
    similarly, the option value is $8.114.
  - *Year 2, Stock Price=$26.405*:
    since the option is out of the money, the value is 0.

Pricing the Call Option (cont'd)

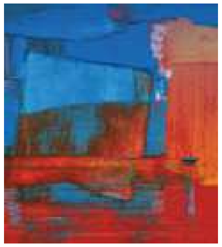
- *Year 1, Stock Price=$59.954:*
  at this node, we compute the option value using equation (10.3), where *uS* is $87.669 and *dS* is $48.114

$$e^{-0.08}\left(\$47.669 \times \frac{e^{0.08}-0.803}{1.462-0.803} + \$8.114 \times \frac{1.462-e^{0.08}}{1.462-0.803}\right) = \$23.029$$

- *Year 1, Stock Price=$32.903:*
  again using equation (10.3), the option value is $3.187.
- *Year 0, Stock Price = $41:*
  similarly, the option value is computed to be $10.737.

Pricing the Call Option (cont'd)

- Notice that
  - The option price is greater for the 2-year than for the 1-year option.
  - The option was priced by working backward through the binomial tree.
  - The option's Δ and B are different at different nodes. At a given point in time, Δ increases to 1 as we go further into the money.
  - Permitting early exercise would make no difference. At every node prior to expiration, the option price is greater than S – K; hence, we would not exercise even if the option had been American.

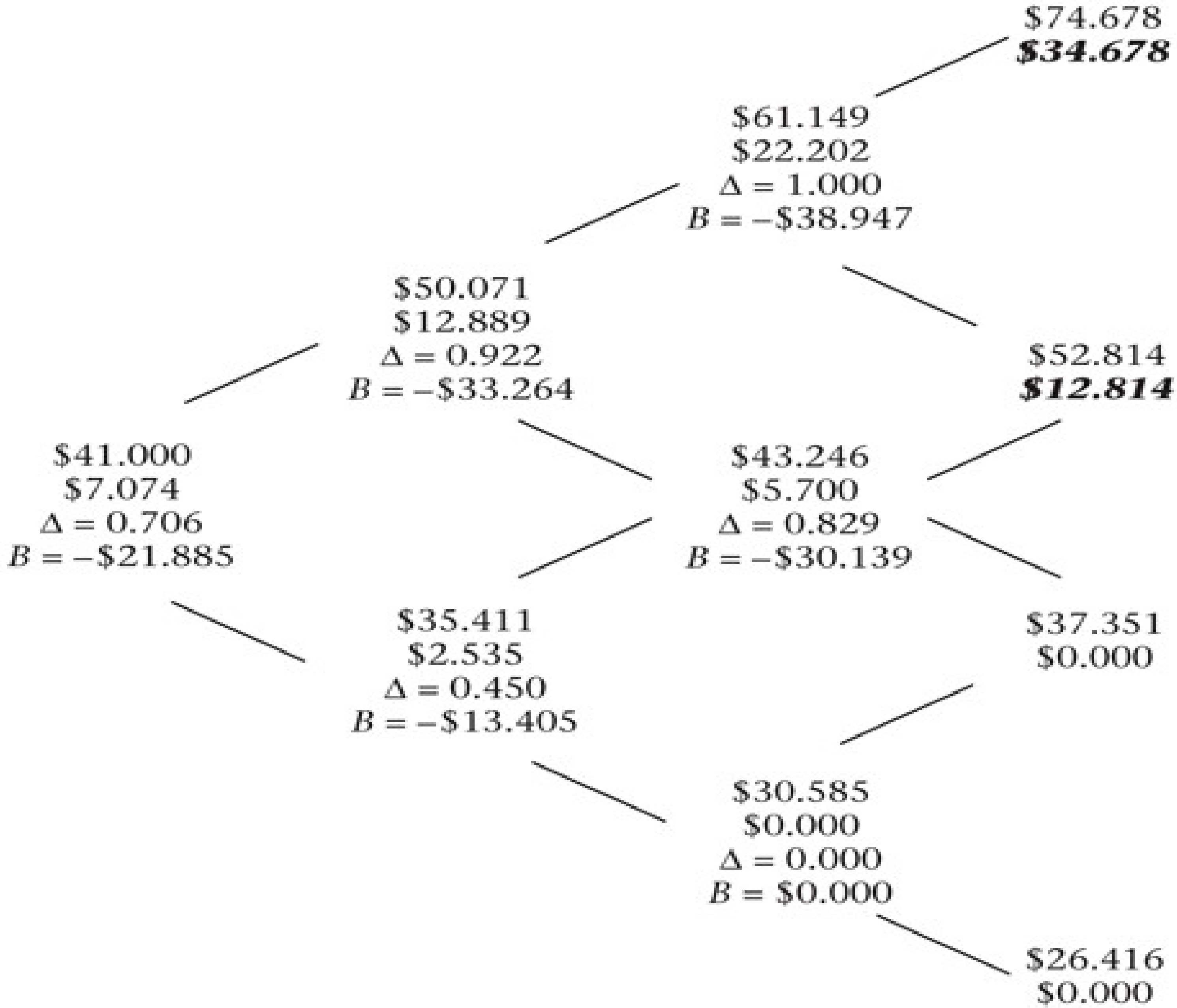


Many Binomial Periods

- Dividing the time to expiration into more periods allows us to generate a more realistic tree with a larger number of different values at expiration.
  - Consider the previous example of the 1-year European call option.
  - Let there be three binomial periods. Since it is a 1-year call, this means that the length of a period is $h = 1/3$.
  - Assume that other inputs are the same as before (so, $r = 0.08$ and $\sigma = 0.3$).

Many Binomial Periods (cont'd)

- The stock price and option price tree for this option.


$74.678
$34.678
$61.149
$22.202
Δ = 1.000
B = –$38.947
$50.071
$12.889
Δ = 0.922
B = –$33.264
$52.814
$12.814
$41.000
$7.074
Δ = 0.706
B = –$21.885
$43.246
$5.700
Δ = 0.829
B = –$30.139
$37.351
$0.000
$35.411
$2.535
Δ = 0.450
B = –$13.405
$30.585
$0.000
Δ = 0.000
B = $0.000
$26.416
$0.000

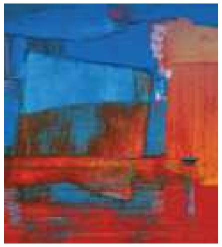
Many Binomial Periods (cont’d)

- Note that since the length of the binomial period is shorter, *u* and *d* are closer to 1 before (*u* = 1.2212 and *d* = 0.8637 as opposed to 1.462 and 0.803 with *h* = 1).
  - The second-period nodes are computed as follows

$$S_u = \$41e^{0.08\times 1/3+0.3\sqrt{1/3}} = \$50.071$$

$$S_d = \$41e^{0.08\times 1/3-0.3\sqrt{1/3}} = \$35.411$$

  - The remaining nodes are computed similarly.
- Analogous to the procedure for pricing the 2-year option, the price of the three-period option is computed by working backward using equation (10.3). The option price is \$7.074.

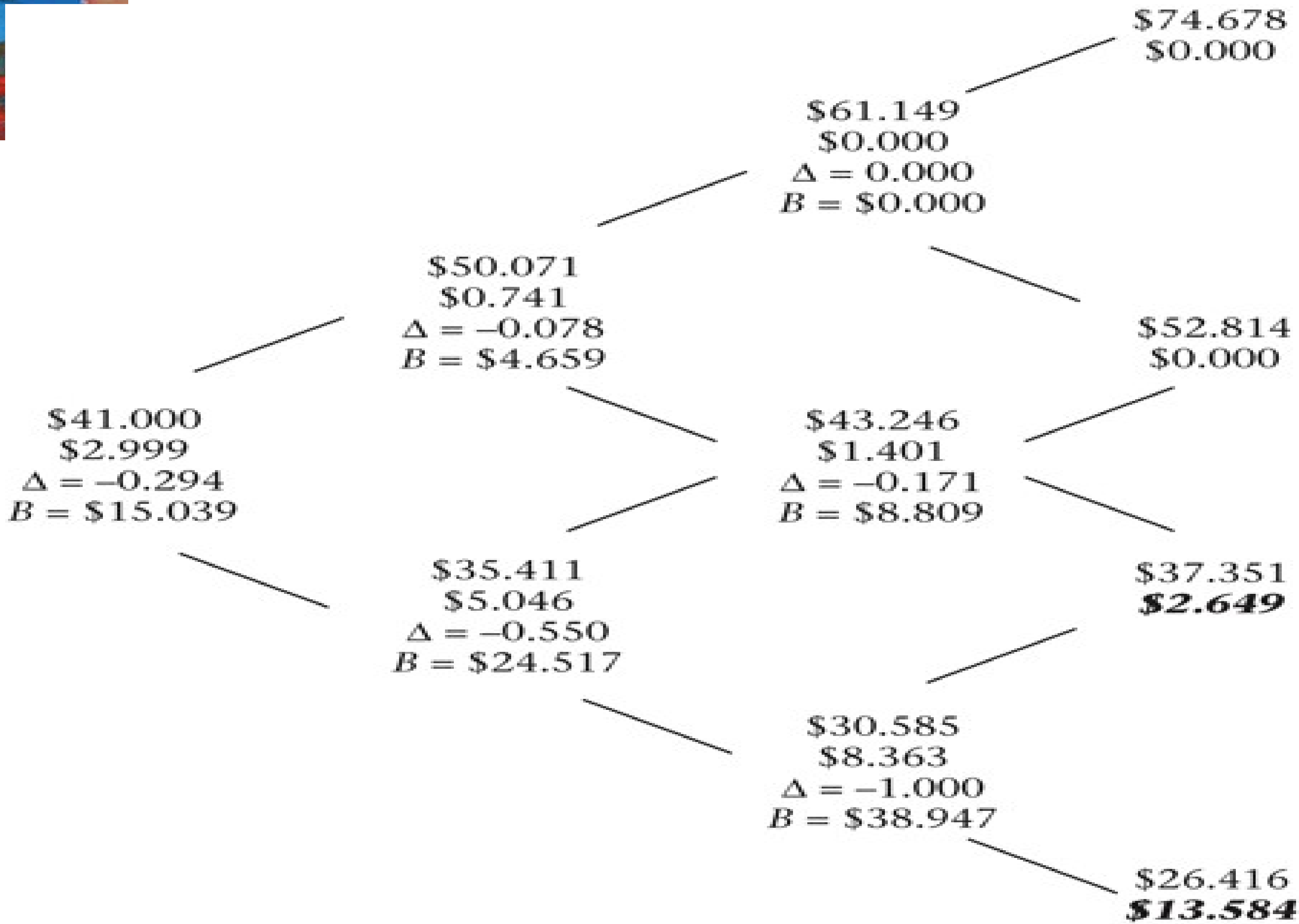


Put Options

- We compute put option prices using the same stock price tree and in almost the same way as call option prices.
- The only difference with a European put option occurs at expiration.
  - Instead of computing the price as max(0, S – K), we use max(0, K – S).

Put Options (cont'd)

- A binomial tree for a European put option with 1-year to expiration.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$2.999
Δ = –0.294
B = $15.039
$43.246
$1.401
Δ = –0.171
B = $8.809
$35.411
$5.046
Δ = –0.550
B = $24.517
$37.351
$2.649
$30.585
$8.363
Δ = –1.000
B = $38.947
$26.416
$13.584

General Formulation

- With loss of generality, consider an European call option on a non-dividend paying asset with the payoff of

$$\max(S_T - K, 0)$$

- Let C be the call option price at time 0.
- In the n-period binomial tree, the risk-neutral probability of having j up-jumps and $(n - j)$ down-jumps is given by

$$C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j}$$

where

$$C_j^n = \frac{n!}{j!(n-j)!}.$$

General Formulation (cont'd)

- The corresponding payoff when *j* up-jumps and *n* – *j* down-jumps occur is seen to be

$$\max\left(u^j d^{n-j} S_0 - K, 0\right)$$

The call value obtained from the *n*-period binomial model is given by

$$C = e^{-rnh} \sum_{j=0}^{n} C_j^n \left(p^*\right)^j \left(1 - p^*\right)^{n-j} \max\left(u^j d^{n-j} S_0 - K, 0\right).$$

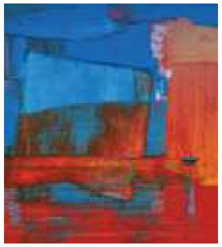
General Formulation (cont'd)

We define k to be the smallest nonnegative integer such that $u^k d^{n-k} S_0 \geq K$, that is

$$k \geq \frac{\ln \dfrac{K}{S_0 d^n}}{\ln \dfrac{u}{d}}.$$

Accordingly, we have

$$\max\left(u^j d^{n-j} S_0 - K, 0\right) = \begin{cases} 0 & \text{when } \quad j < k \\ u^j d^{n-j} S_0 - K & \text{when } \quad j \geq k. \end{cases}$$



General Formulation (cont'd)

The integer *k* gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money. So,

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n,k,\tilde{p}\right) - Ke^{-rT} \Phi\left(n,k,p^*\right).$$

where

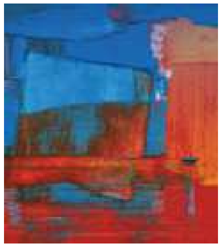
$$\Phi\left(n,k,p\right) = \sum_{j=k}^{n} C_j^n \left(p\right)^j \left(1-p\right)^{n-j}, \quad \tilde{p} = \frac{up^*}{e^{rh}} \quad \text{and} \quad 1-\tilde{p} = \frac{d\left(1-p^*\right)}{e^{rh}}.$$

Note: When $n \to \infty$, the binomial tree model will converge to the Black-Scholes formula (see Binomial to BS.pdf for detail).

General Formulation (cont'd)

The *first term* gives the discounted expectation of the asset price at expiration given that the call expires in-the-money and the *second term* gives the present value of the expected cost incurred by exercising the call, where the expectation is taken under the risk-neutral measure.

The similar formulation can be obtained for the European put option.



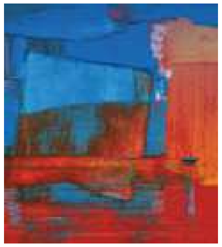
American Options

- The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation (10.3).
- The value of the option if it is exercised is given by *max* (0, *S* – *K*) if it is a call and *max* (0, *K* – *S*) if it is a put.
- For an American put, the value of the option at a node is given by

$$P(S,K,t) = \max\left(K - S, e^{-rh}\left[P(uS,K,t+h)\,p^* + P(dS,K,t+h)(1-p^*)\right]\right)$$

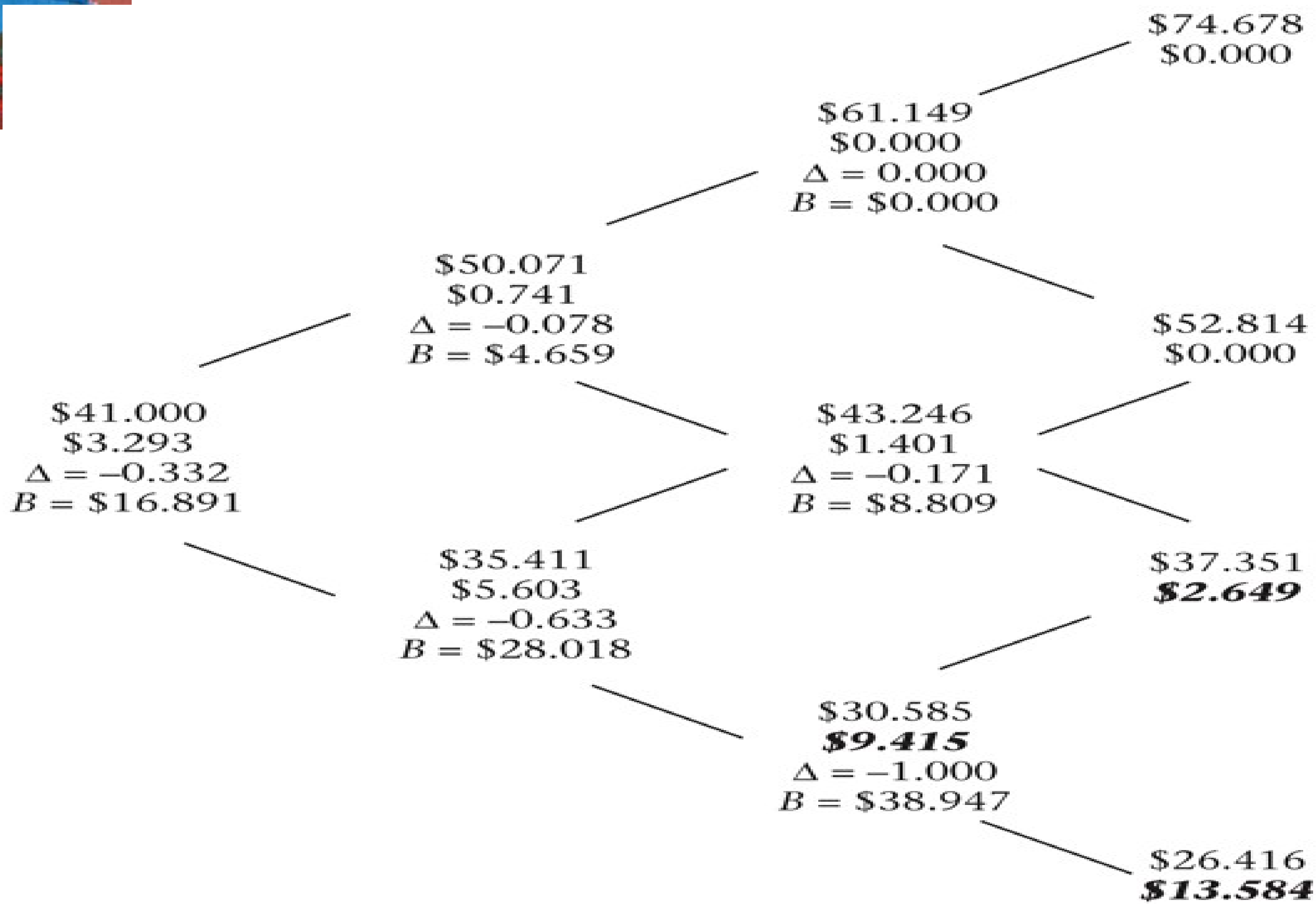
American Options (cont’d)

- The valuation of American options proceeds as follows:
  - At each node, we check for early exercise.
  - If the value of the option is greater when exercised, we assign that the **exercised** value to the node. Otherwise, we assign the value of the option **unexercised**.
  - We work backward through the tree as usual.



American Options (cont'd)

- Consider an American version of the put option valued in the previous example.


$74.678
$0.000
$61.149
$0.000
Δ = 0.000
B = $0.000
$50.071
$0.741
Δ = –0.078
B = $4.659
$52.814
$0.000
$41.000
$3.293
Δ = –0.332
B = $16.891
$43.246
$1.401
Δ = –0.171
B = $8.809
$35.411
$5.603
Δ = –0.633
B = $28.018
$37.351
$2.649
$30.585
$9.415
Δ = –1.000
B = $38.947
$26.416
$13.584

American Options (cont'd)

- The only difference in the binomial tree occurs at the S_{dd} node, where the stock price is \$30.585. The American option at that point is worth \$40 – \$30.585 = \$9.415, its early-exercise value (as opposed to \$8.363 if unexercised). The greater value of the option at that node ripples back through the tree.

Options on Other Assets

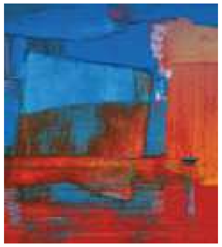
- The binomial model developed thus far can be modified easily to price options on underlying assets other than non-dividend-paying stocks.
- The difference for different underlying assets is the construction of the binomial tree and the risk-neutral probability.
- We examine options on
  - Stock indexes
  - Currencies
  - Futures contracts
  - Commodities
  - Bonds

Options on a Stock Index

- Suppose a stock index pays continuous dividends at the rate δ.
- The procedure for pricing this option is equivalent to that of the first example, which was used for our derivation. Specifically
  - the up and down index moves are given

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

  - the replicating portfolio by equation (10.1) and (10.2).
  - the option price by equation (10.3).
  - the risk-neutral probability by equation (10.5).

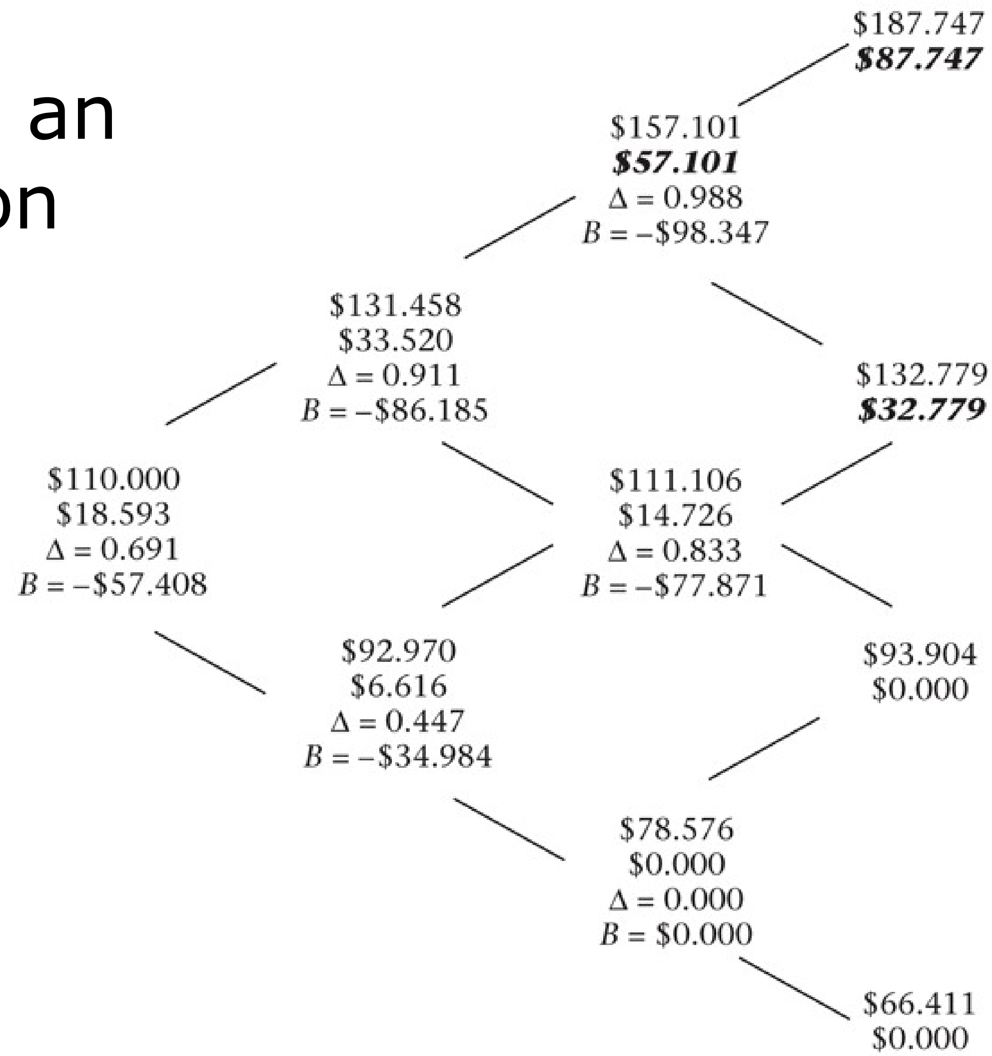


Options on a Stock Index

- Given
  - S = \$110;
  - K = \$100;
  - σ = 0.30;
  - r = 0.05
  - T = 1 year
  - δ = 0.035
  - h = 0.333

Options on a Stock Index (cont'd)

- A binomial tree for an American call option on a stock index:



Options on Currencies

- With a currency with spot price x_0, the forward price is

$$F_{0,h} = x_0 e^{(r-r_f)h}$$

where r_f is the foreign interest rate.

- Thus, we construct the binomial tree using

$$ux = xe^{(r-r_f)h+\sigma\sqrt{h}}$$

$$dx = xe^{(r-r_f)h-\sigma\sqrt{h}}$$

Options on Currencies (cont'd)

- Investing in a "currency" means investing in a money-market fund or fixed income obligation denominated in that currency.
- Taking into account interest on the foreign-currency denominated obligation, the two equations are

$$\Delta \times uxe^{r_f h} + e^{rh} \times B = C_u$$

$$\Delta \times dxe^{r_f h} + e^{rh} \times B = C_d$$

- The risk-neutral probability of an up move is

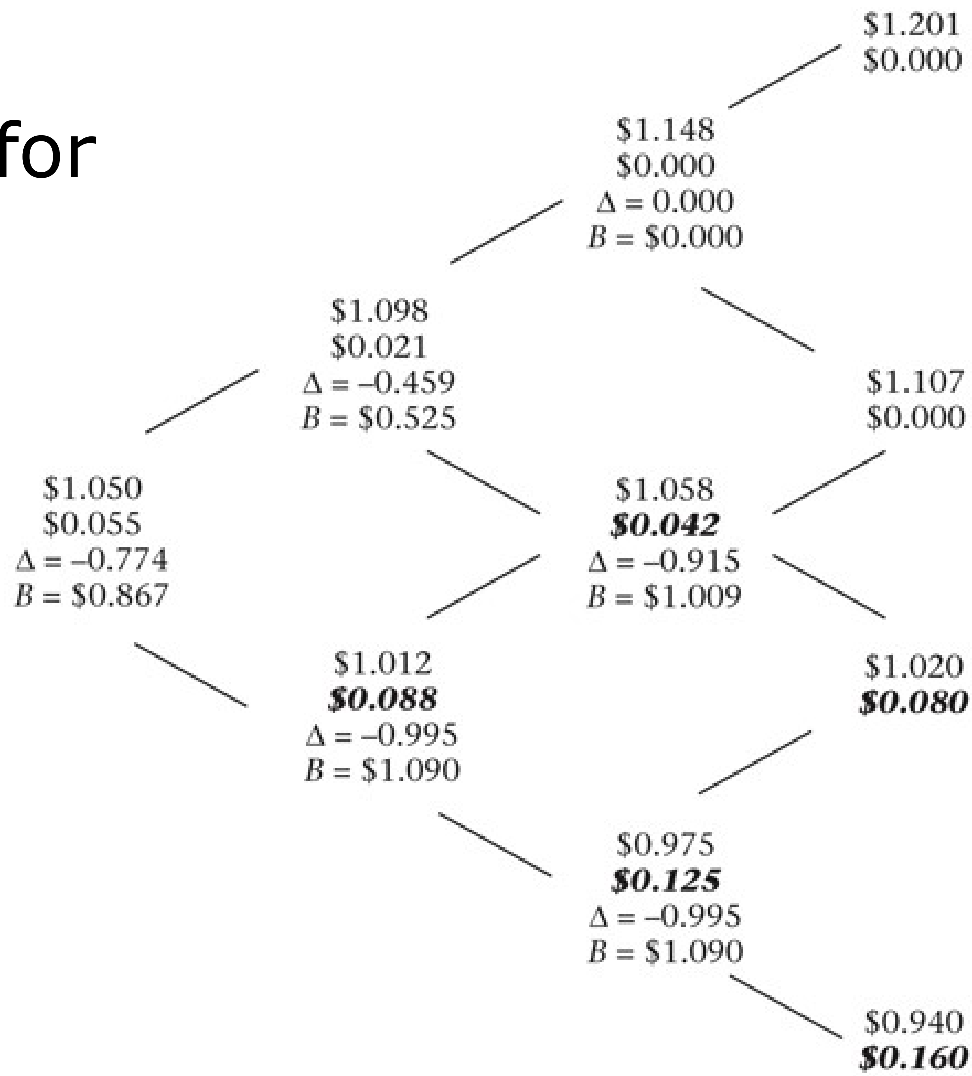
$$p^* = \frac{e^{(r - r_f)h} - d}{u - d} \qquad (10.20)$$

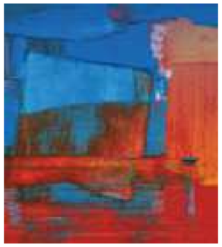
Options on Currencies (cont'd)

- Consider a dollar-denominated American put option on the euro, where
  - The current exchange rate is \$1.05/€ (S);
  - The strike is \$1.10/€ (K);
  - $\sigma = 0.10$;
  - The euro-denominated interest rate is 3.1% (δ);
  - The dollar-denominated rate is 5.5% (r).

Options on Currencies (cont'd)

- The binomial tree for the American put option on the euro





Options on Futures Contracts

- Assume the forward price is the same as the futures price.
- The nodes are constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

- We need to find the number of futures contracts, Δ, and the lending, *B*, that replicates the option.
  - A replicating portfolio must satisfy

$$\Delta \times (uF - F) + e^{rh} \times B = C_u$$
$$\Delta \times (dF - F) + e^{rh} \times B = C_d$$

Options on Futures Contracts (cont'd)

- Solving for Δ and B gives

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d}\right)$$

Δ tells us how many futures contracts to hold to hedge the option, and B is simply the value of the option.

- We can again price the option using equation (10.3).

- The risk-neutral probability of an up move is given by

$$p^* = \frac{1-d}{u-d} \qquad (10.21)$$

Options on Futures Contracts (cont'd)

- The motive for early-exercise of an option on a futures contract is the ability to earn interest on the mark-to-market proceeds.
  - When an option is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price.

Options on Futures Contracts (cont'd)

- A tree for an American
  call option on a
  futures contract