Derivation of the Black-Scholes Formula

Under the risk-neutral measure Q,

$$S(t) = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)t + \sigma \tilde{Z}(t) \right)$$

where $S(0) = S_0$.

The price of the call option $V(0)$ is given by

$$V(0) = e^{-rT} E_0^Q \left[\max(S(T) - K, 0) \right]$$

$$= e^{-rT} \int_K^\infty (x - K)\, g(x)\, dx$$

where $g(x)$ is the pdf of $S(T)$.

$$V(0) = e^{-rT} \left[\int_K^\infty x\, g(x)\, dx - \int_K^\infty g(x)\, dx \right]$$

$$(I) = \int_K^\infty g(x)\, dx = \Pr(S(T) > K)$$

$$= \Pr\left(\tilde{Z}(t) > \frac{\ln\left(\frac{K}{S_0}\right) - (r - \frac{1}{2}\sigma^2)t}{\sigma} \right)$$

$$= \Pr\left(Z > \frac{\ln\left(\frac{K}{S_0}\right) - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right)$$

$$= \Pr\left(Z < -\left(\frac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right) \right)$$

$$= \Pr\left(Z < \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right)$$

$$= N(d_2)$$

$$(\text{II}) = \int_K^\infty x\, g(x)\, dx$$

$$= \int_K^\infty \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{1}{2\sigma^2 T}\left[\ln x - \quad T + 0.5\sigma^2 T - \ln S_0\right]^2\right) dx$$

$$= \int_{\ln K}^\infty \frac{e^y}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{1}{2\sigma^2 T}\left[y - b\right]^2\right) dy, \quad \text{where } y = \ln x, \; b = T - \tfrac{1}{2}\sigma^2 T + \ln S_0$$

$$= \int_{\ln K}^\infty \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{1}{2\sigma^2 T}\left((y-b)^2 - 2y\sigma^2 T\right)\right) dy$$

Consider.

$$(y-b)^2 - 2y\sigma^2 T$$

$$= y^2 - 2y(b + \sigma^2 T) + b^2$$

$$= (y - (b + \sigma^2 T))^2 + b^2 - (b + \sigma^2 T)^2$$

$$= (y - (b + \sigma^2 T))^2 - 2\sigma^2 T (rT + \ln S_0)$$

Therefore,

$$\int_{\ln K}^\infty x\, g(x)\, dx$$

$$= \int_{\ln K}^\infty e^{rT + \ln S_0} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left[-\frac{1}{2\sigma^2 T}\left(y - (rT + \tfrac{1}{2}\sigma^2 T + \ln S_0)\right)^2\right] dy$$

$$= S_0 e^{rT} N(d_1)$$

Combining (I) and (II), we have.

$$V(0) = e^{-rT}\left[S_0 e^{rT} N(d_1) - K N(d_2)\right]$$

$$= S_0 N(d_1) - K e^{-rT} N(d_2).$$