Tutorial - Class Activity

5 Dec, 2018 (Solution)

Problem 1

You are given:

$$\frac{dS_1(t)}{S_1(t)} = \mu_1 dt + \sigma_1 dZ_1(t),$$
$$\frac{dS_2(t)}{S_2(t)} = \mu_2 dt + \sigma_2 dZ_2(t),$$

where μ_1, μ_2, σ_1 and σ_2 are constants, $Z_1(t)$ and $Z_2(t)$ are correlated Brownian motions with $dZ_1(t)dZ_2(t) = \rho dt$.

Let $G(t) = e^{-rt}S_1(t)S_2(t)$.

Find the stochastic differential equation (SDE) of $G(t)$.

Solution

The expression for $G(t)$ is $e^{-rt}S_1(t)S_2(t)$.

The partial derivatives are:

$$G_{S_1} = S_2(t)e^{-rt}, \quad G_{S_2} = S_1(t)e^{-rt}, \quad G_t = -rS_1(t)S_2(t)e^{-rt},$$
$$G_{S_1S_2} = e^{-rt}, G_{S_1S_1} = G_{S_2S_2} = 0.$$

From Itô's lemma, we have:

$$\begin{aligned}
dG(t) &= G_{S_1}dS_1 + G_{S_2}dS_2 + \frac{1}{2}\left(G_{S_1S_1}(dS_1)^2 + 2G_{S_1S_2}(dS_1)(dS_2) + G_{S_2S_2}(dS_2)^2\right) + G_t dt \\
&= S_2(t)e^{-rt}S_1(t)\left(\mu_1 dt + \sigma_1 dZ_1(t)\right) + S_1(t)e^{-rt}S_2(t)\left(\mu_2 dt + \sigma_2 dZ_1(t)\right) + \\
&\quad e^{-rt}S_1(t)\left(\mu_1 dt + \sigma_1 dZ_1(t)\right)S_2(t)\left(\mu_2 dt + \sigma_2 dZ_1(t)\right) - rS_1(t)S_2(t)e^{-rt}dt \\
&= G(t)\left[\left(\mu_1 + \mu_2 - r + \rho\sigma_1\sigma_2\right)dt + \sigma_1 dZ_1(t) + \sigma_2 dZ_1(t)\right].
\end{aligned}$$

Problem 2

Consider two non-dividend-paying assets *X* and *Y*, whose prices are driven by the same standard Brownian motion *Z*(*t*). You are given that the assets *X* and *Y* satisfy the stochastic differential equations:

$$\frac{dX(t)}{X(t)} = 0.09dt + 0.16dZ(t),$$
$$\frac{dY(t)}{Y(t)} = Gdt + HdZ(t),$$

where *G* and *H* are constants.

You are also given:

(i) $d\ln\left[Y(t)\right] = 0.07dt + \sigma dZ(t)$.

(ii) The continuously compounded risk-free interest rate is 5%.

(iii) $\sigma < 0.3$.

Determine the values of *G* and *H*.

Solution

By Itô's Lemma, we have

$$d\ln\left[Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ(t).$$

The arithmetic Brownian motion provided in (i) for $d\ln[Y(t)]$ allows us to find an expression for *H* and *G*:

$$d\ln\left[Y(t)\right] = 0.07dt + \sigma dZ(t) \quad \text{and} \quad d\ln\left[Y(t)\right] = \left(G - 0.5H^2\right)dt + HdZ(t)$$
$$\Rightarrow \quad G - 0.5H^2 = 0.07 \quad \text{and} \quad H = \sigma.$$

Since *X* and *Y* have the same source of randomness, $dZ(t)$, they must have the same Sharpe ratio:

$$\frac{0.09 - 0.05}{0.16} = \frac{G - 0.05}{H}$$
$$0.25 = \frac{G - 0.05}{H}$$
$$0.25 = \frac{0.07 + 0.5H^2 - 0.05}{H}$$
$$0.25H = 0.02 + 0.5H^2$$
$$0.5H^2 - 0.25H + 0.02 = 0$$

We use the quadratic formula to solve for *H*:

$$H = \frac{0.25 \pm \sqrt{(-0.25)^2 - 4(0.5)(0.02)}}{2(0.5)} = 0.1 \quad \text{or} \quad 0.4.$$

Since we are given that $\sigma < 0.3$ and we know that $H = \sigma$, it must be the case that

$$H = 0.1.$$

We can now find the value of *G*:

$$0.25 = \frac{G - 0.05}{H}$$

$$0.25 = \frac{G - 0.05}{0.1}$$

$$G = 0.075.$$

Problem 3

Consider a securities model with the money market account $M(t)$ and a risky asset $S(t)$. Suppose that $M(0) = 1$ and $S(0) = 3$. At $t = 1$, $M(1) = 1.5$ and under the real probability $S(1)$ has three possible values which are are given by the following vector

$$S(1) = \begin{pmatrix} 6 \\ 4.5 \\ 3 \end{pmatrix}.$$

Determine the risk-neutral probability of this security model. Is this risk-neutral probability unique?

Solution

Because the risk-neutral probability is equivalent to the real probability. So, there are three possible states of $S(1)$ under the risk-neutral probability. Let q_1, q_2 and q_3 be the risk-neutral probabilities for $S(1) = 6$, 4.5 and 3 respectively.

By the definition of the risk-neutral probability, we have

$$E^Q\left[\frac{S(1)}{M(1)}\right] = \frac{S(0)}{M(0)} = S(0).$$

So,

$$q_1 \frac{6}{1.5} + q_2 \frac{4.5}{1.5} + q_3 \frac{3}{1.5} = 3,$$

$$q_1 + q_2 + q_3 = 1.$$

or

$$\begin{aligned} 4q_1 + 3q_2 + 2q_3 &= 3, \\ q_1 + q_2 + q_3 &= 1. \end{aligned}$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $q_1 = \lambda$, $q_2 = 1 - 2\lambda$, $q_3 = \lambda$, where λ is a free parameter. In order that all q_i, $i = 1, 2, 3$, are all strictly positive. We must have $0 < \lambda < 1/2$.