In the binomial option pricing model, the call option price is given by

$$
\begin{aligned}
C &= S_0 \sum_{j=k}^{n} C_j^n (p^*)^j (1-p^*)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n (p^*)^j (1-p^*)^{n-j} \\
&= S_0 \Phi(n,k,\tilde{p}) - Ke^{-rT} \Phi(n,k,p^*)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Phi(n,k,p) = \sum_{j=k}^{n} C_j^n (p)^j (1-p)^{n-j}, \\
&\tilde{p} = \frac{up^*}{e^{rh}} \text{ and} \\
&1-\tilde{p} = \frac{d(1-p^*)}{e^{rh}}.
\end{aligned}
$$

Define $\tilde{J} \sim$ binomial$(n,\tilde{p})$ and $J \sim$ binomial(n,p). So, we have

$$
C = S_0 \Pr(\tilde{J} \geq k) - Ke^{-rT} \Pr(J \geq k).
$$

Approximate the binomial distribution by the normal distribution, we have

$$
\begin{aligned}
C &= S_0 \Pr\left(Z \geq \frac{k - 0.5 - n\tilde{p}}{\sqrt{n\tilde{p}(1-\tilde{p})}}\right) - Ke^{-rT} \Pr\left(Z \geq \frac{k - 0.5 - np^*}{\sqrt{np^*(1-p^*)}}\right) \\
&= S_0 \Pr\left(Z \geq \frac{\ln\left(\frac{K}{S_0}\right) - n\ln d - 2\sigma\sqrt{h}n\tilde{p} - 0.5(2\sigma\sqrt{h})}{2\sigma\sqrt{h}\sqrt{n\tilde{p}(1-\tilde{p})}}\right) - \\
&\quad Ke^{-rT} \Pr\left(Z \geq \frac{\ln\left(\frac{K}{S_0}\right) - n\ln d - 2\sigma\sqrt{h}np^* - 0.5(2\sigma\sqrt{h})}{2\sigma\sqrt{h}\sqrt{np^*(1-p^*)}}\right) \\
&= S_0 \Pr\left(Z \geq \frac{\ln\left(\frac{K}{S_0}\right) - n(rh - \sigma\sqrt{h}) - 2\sigma\sqrt{h}n\tilde{p} - \sigma\sqrt{h}}{2\sigma\sqrt{h}\sqrt{n\tilde{p}(1-\tilde{p})}}\right) - \\
&\quad Ke^{-rT} \Pr\left(Z \geq \frac{\ln\left(\frac{K}{S_0}\right) - n(rh - \sigma\sqrt{h}) - 2\sigma\sqrt{h}np^* - \sigma\sqrt{h}}{2\sigma\sqrt{h}\sqrt{np^*(1-p^*)}}\right) \\
&= S_0 N\left(\frac{\ln\left(\frac{S_0}{K}\right) + n(rh - \sigma\sqrt{h}) + 2\sigma\sqrt{h}n\tilde{p} + \sigma\sqrt{h}}{2\sigma\sqrt{h}\sqrt{n\tilde{p}(1-\tilde{p})}}\right) - \\
&\quad Ke^{-rT} N\left(\frac{\ln\left(\frac{S_0}{K}\right) + n(rh - \sigma\sqrt{h}) + 2\sigma\sqrt{h}np^* + \sigma\sqrt{h}}{2\sigma\sqrt{h}\sqrt{np^*(1-p^*)}}\right) \\
&= S_0 N\left(\frac{\ln\left(\frac{S_0}{K}\right) + n[2\sigma\sqrt{h}\tilde{p} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}]}{2\sigma\sqrt{h}\sqrt{n\tilde{p}(1-\tilde{p})}}\right) - \\
&\quad Ke^{-rT} N\left(\frac{\ln\left(\frac{S_0}{K}\right) + n[2\sigma\sqrt{h}p^* + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}]}{2\sigma\sqrt{h}\sqrt{np^*(1-p^*)}}\right)
\end{aligned}
$$

where Z is a standard normal random variable.

Lemma 1

1.
$$\lim_{n\to\infty} n\left[p^* 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}\right] = \left(r - \frac{\sigma^2}{2}\right)T.$$

2.
$$\lim_{n\to\infty} n\left[\tilde{p} 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}\right] = \left(r + \frac{\sigma^2}{2}\right)T.$$

Proof

(1)

By considering

$$\begin{aligned} p^* &= \frac{e^{rh} - e^{rh-\sigma\sqrt{h}}}{e^{rh+\sigma\sqrt{h}} - e^{rh-\sigma\sqrt{h}}} \\ &= \frac{e^{\sigma\sqrt{h}} - 1}{e^{2\sigma\sqrt{h}} - 1} \end{aligned}$$

We expand the exponential functions into the Taylor series. So, we have

$$\begin{aligned} p^* &= \frac{\sigma\sqrt{h} + \frac{\sigma^2 h}{2} + O(h^{3/2})}{2\sigma\sqrt{h} + 2\sigma^2 h + O(h^{3/2})} \\ &= \left(\sigma\sqrt{h} + \frac{\sigma^2 h}{2} + O(h^{3/2})\right)\frac{1}{2\sigma\sqrt{h}}(1 + \sigma\sqrt{h} + O(h))^{-1} \\ &= \frac{1}{2\sigma\sqrt{h}}\left(\sigma\sqrt{h} + \frac{\sigma^2 h}{2} + +O(h^{3/2})\right)(1 - \sigma\sqrt{h} + O(h)) \\ &= \frac{1}{2\sigma\sqrt{h}}\left[\sigma\sqrt{h} - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] \end{aligned}$$

Therefore,

$$
\begin{aligned}
& \lim_{n\to\infty} n\left[p^* 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}\right] \\
= & \lim_{h\to 0} \frac{T}{h}\left[\frac{1}{2\sigma\sqrt{h}}\left[\sigma\sqrt{h} - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h})\right] + \sigma\sqrt{h} \\
= & \lim_{h\to 0} \frac{T}{h}\left[rh - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] + \sigma\sqrt{h} \\
= & \left(r - \frac{\sigma^2}{2}\right) T.
\end{aligned}
$$

(2)
Since

$$
\begin{aligned}
\tilde{p} &= e^{\sigma\sqrt{h}} p^* \\
&= (1 + \sigma\sqrt{h} + O(h)) p^*,
\end{aligned}
$$

$$
\begin{aligned}
& \lim_{n\to\infty} n\left[\tilde{p} 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}\right] \\
= & \lim_{n\to\infty} n\left[(1 + \sigma\sqrt{h} + O(h)) p^* 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h}) + \frac{\sigma\sqrt{h}}{n}\right] \\
= & \lim_{h\to 0} \frac{T}{h}\left[(1 + \sigma\sqrt{h} + O(h))\frac{1}{2\sigma\sqrt{h}}\left[\sigma\sqrt{h} - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] 2\sigma\sqrt{h} + (rh - \sigma\sqrt{h})\right] + \sigma\sqrt{h} \\
= & \left(r + \frac{\sigma^2}{2}\right) T.
\end{aligned}
$$

Lemma 2

1.
$$\lim_{n\to\infty} np^*(1-p^*)\left(2\sigma\sqrt{h}\right)^2 = \sigma^2 T.$$

2.
$$\lim_{n\to\infty} n\tilde{p}(1-\tilde{p})\left(2\sigma\sqrt{h}\right)^2 = \sigma^2 T.$$

Proof
(1)
From the last proof, we have

$$\begin{aligned} p^* &= \frac{1}{2\sigma\sqrt{h}}\left[\sigma\sqrt{h} - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] \\ &= \frac{1}{2} - O(\sqrt{h}). \end{aligned}$$

$$\begin{aligned} &\lim_{n\to\infty} np^*(1-p^*)\left(2\sigma\sqrt{h}\right)^2 \\ =\ &\lim_{h\to 0}\frac{T}{h}\left(\frac{1}{2} - O(\sqrt{h})\right)\left(\frac{1}{2} - O(\sqrt{h})\right)\left(2\sigma\sqrt{h}\right)^2 \\ =\ &\sigma^2 T. \end{aligned}$$

(2)
We have

$$\begin{aligned} \tilde{p} &= (1+\sigma\sqrt{h}+O(h))p^* \\ &= (1+\sigma\sqrt{h}+O(h))\frac{1}{2\sigma\sqrt{h}}\left[\sigma\sqrt{h} - \frac{\sigma^2 h}{2} + O(h^{3/2})\right] \\ &= \frac{1}{2} - O(\sqrt{h}). \end{aligned}$$

Therefore,

$$
\begin{aligned}
& \lim_{n\to\infty} n\tilde{p}(1-\tilde{p})\left(2\sigma\sqrt{h}\right)^2 \\
= \; & \lim_{h\to 0} \frac{T}{h}\left(\frac{1}{2}-O(\sqrt{h})\right)\left(\frac{1}{2}-O(\sqrt{h})\right)\left(2\sigma\sqrt{h}\right)^2 \\
= \; & \sigma^2 T.
\end{aligned}
$$

By Lemmas 1 and 2, we have the Black-Scholes formula for the call option price.