MFE5130 – Financial Derivatives
First Term, 2019 – 20

Problems on Chapters 7 and 8 (Solution)

Question 8.2

a) We first solve for the present value of the cost per three barrels, based on the forward prices:

$$\frac{\$20}{1.06} + \frac{\$21}{(1.065)^2} + \frac{\$22}{(1.07)^3} = 55.3413.$$

We then obtain the swap price per barrel by solving:

$$\frac{x}{1.06} + \frac{x}{(1.065)^2} + \frac{x}{(1.07)^3} = 55.341$$

$$\Leftrightarrow \qquad x = 20.9519$$

b) We first solve for the present value of the cost per two barrels (Year 2 and Year 3):

$$\frac{\$21}{(1.065)^2} + \frac{\$22}{(1.07)^3} = 36.473.$$

We then obtain the swap price per barrel by solving:

$$\frac{x}{(1.065)^2} + \frac{x}{(1.07)^3} = 36.473$$

$$\Leftrightarrow \qquad x = 21.481$$

Question 8.10

Use the following equation for swaps with variable quantity:

$$X = \frac{\sum_{i=1}^{8} Q_{t_i} P_0(0,t_i) F_{0,t_i}}{\sum_{i=1}^{8} Q_{t_i} P_0(0,t_i)}, \text{ where } Q = [1, 2, 1, 2, 1, 2, 1, 2]$$

After plugging in the relevant variables given in the exercise, we obtain a value of \$20.4099 for the swap price.

Question 8.11

We are now asked to invert our equations. The swap prices are given, and we want to back out the forward prices. We do so recursively. For a one-quarter swap, the swap price and the forward price are identical. Given the one-quarter forward price, we can find the second quarter forward price, etc. Doing so yields the following forward prices:

Quarter	Forward price
1	2.25
2	2.60
3	2.20
4	1.90
5	2.20
6	2.50
7	2.15
8	1.80

<u>Question 9.1</u>
This problem requires the application of put-call-parity. We have:

$$P\,(35,\,0.5) = C\,(35,\,0.5) - e^{-\delta T}\,S_0 + e^{-rT}\,35$$

$$\Leftrightarrow \quad P\,(35,\,0.5) = \$2.27 - e^{-0.06\times 0.5}32 + e^{-0.04\times 0.5}35 = \$5.523.$$

<u>Question 9.3</u>

a) We can calculate an initial investment of:

$$-800 + 75 - 45 = -770.$$

This position yields \$815 after one year for sure because either the sold call commitment or the bought put cancel out the stock price. Therefore, we have a one-year rate of return of 0.05844, which is equivalent to a continuously compounded rate of 0.05680.

b) We have a riskless position in (a) that pays more than the risk-free rate. Therefore, we should borrow money at 5 percent and buy a large amount of the aggregate position of (a), yielding a sure return of 0.68 percent.

c) An initial investment of \$775.252 would yield \$815 after one year, invested at the riskless rate of return of 5 percent. Therefore, the difference between call and put prices should be equal to \$24.748.

d) Following the same argument as in (c), we obtain the following results:

Strike Price	Call-Put
780	58.04105
800	39.01646
820	19.99187
840	0.967283

Question 9.7

a) We make use of the version of the put-call-parity that can be applied to currency options. We have:

$$+P(K,T) = -e^{-r_f T} x_0 + C(K,T) + e^{-rT} K$$

$$\Leftrightarrow \quad P(K,T) = -e^{-0.01} 0.009 + 0.0006 + e^{-0.05} 0.009$$

$$= -0.00891045 + 0.0006 + 0.00856106$$

$$= 0.00025.$$

b) The observed option price is too high. Therefore, we sell the put option and synthetically create a long put option, perfectly offsetting the risks involved. We have:

Transaction	$t = 0$	$t = T, x < K$	$t = T, x > K$
Sell Put	0.0004	$-(K - x) = x - K$	0
Buy Call	−0.0006	0	$x - K$
Sell e^{-r_f} Spot	$0.009e^{-0.001} = 0.00891$	$-x$	$-x$
Lend PV(strike)	−0.00856	K	K
Total	0.00015	0	0

We have thus demonstrated the arbitrage opportunity.

c) We can use formula on P.13 of the lecture notes to determine the value of the corresponding yen-denominated at-the-money put, and then use put-call-parity to figure out the price of the yen-denominated at- the-money call option.

$$C_{\$}(x_0, K, T) = x_0 K P_Y\left(\frac{1}{x_0}, \frac{1}{K}, T\right)$$

$$\Leftrightarrow \quad P_Y\left(\frac{1}{x_0}, \frac{1}{K}, T\right) = \frac{C_{\$}(x_0, K, T)}{x_0 K}$$

$$\Leftrightarrow \quad P_Y\left(\frac{1}{0.009}, \frac{1}{0.009}, T\right) = \frac{0.0006}{(0.009)^2} = 7.4074Y$$

Now, we can use put-call-parity, carefully handling the interest rates (since we are now making transactions in yen, the \$-interest rate is the foreign rate):

$$C_Y(K,T) = e^{-r_f T} x_0 + P_Y(K,T) - e^{-rT} K$$

$$\Leftrightarrow \quad C_Y\left(\frac{1}{0.009}, 1\right) = e^{-0.05} \frac{1}{0.009} + 7.4074 - e^{-0.01} \frac{1}{0.009}$$

$$= 105.692 + 7.4074 - 110.0055$$

= 3.093907.

We have used the direct relationship between the yen-denominated dollar put and our answer to (a) and the put-call-parity to find the answer.

Question 9.9

Both of the following equations are violated.

$$C(K_1)-C(K_2)\le K_2-K_1 \quad \text{and} \quad P(K_2)-P(K_1)\le K_2-K_1.$$

We use a call bear spread and a put bull spread to profit from these arbitrage opportunities.

			Expiration or Exercise	
Transaction	$t = 0$	$S_T < 50$	$50 \le S_T \le 55$	$S_T > 55$
Buy 55 strike call	−10	0	0	$S_T - 55$
Sell 50 strike call	+16	0	$50 - S_T$	$50 - S_T$
TOTAL	+6	0	$50 - S_T > -5$	−5

			Expiration or Exercise	
Transaction	$t = 0$	$S_T < 50$	$50 \le S_T \le 55$	$S_T > 55$
Buy 50 strike put	−7	$50 - S_T$	0	0
Sell 55 strike put	14	$S_T - 55$	$S_T - 55$	0
TOTAL	+7	−5	$S_T - 55 > -5$	0

It is important to note that we initially receive more money than our biggest possible negative cash outflow in the future. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

Question 9.13

We have to think carefully about the benefits and costs associated with early exercise. For the American put option, the usual benefit of early exercise is that by exercising we can earn interest on the strike. We lose this interest if we continue to hold the option unexercised. However, when the interest rate is zero, there is no interest benefit from early exercise. There are two benefits to deferring exercise: First, there is a volatility benefit from waiting. Second, if the stock pays dividends, the put is implicitly short the stock without having to pay dividends. By exercising, the put is converted into an actual short position with an obligation to pay the dividends. So with a zero interest rate, we will never use the early exercise feature of the American put option, and the American put is equivalent to a European put.

Mathematical Proof:

From the put-call parity of the European options, we have

$$P(S_t,K,T-t)=K-S_t+PV_{t,T}(K)-K+PV_{t,T}(Div)+C(S_t,K,T-t).$$

If interest rate is 0, we then have

$$P(S_t, K, T-t) = K - S_t + Div + C(S_t, K, T-t) > K - S_t.$$

Since $P_{\text{Amer}}(S_t, K, T-t) > P(S_t, K, T-t) > K - S_t$, so early exercise is not possible.

For the American call option, dividends on the stock are the reason why we want to receive the stock earlier, and we benefit from waiting because we can continue to earn interest on the strike. Now, the interest rate is zero, so we do not have this benefit associated with waiting to exercise. However, we saw that there is a second benefit to waiting: the insurance protection, which will not be affected by the zero interest rate. Finally, we will not exercise the option if it is out-of-the-money. Therefore, there may be circumstances in which we will early exercise, but we will not always early exercise.

Mathematical Proof:

From the put-call parity of the European options with zero interest rate, we have

$$C(S_t, K, T-t) = S_t - K + P(S_t, K, T-t) - Div.$$

So, it is **possible** that

$$C(S_t, K, T-t) < S_t - K \quad \text{if} \quad Div > P(S_t, K, T-t).$$

So early exercise is possible.

Additional Problem 1

From the given put prices, we have

$$\frac{P(K_2) - P(K_1)}{K_2 - K_1} = \frac{10.75 - 7}{55 - 50} = 0.75 \quad \text{and} \quad \frac{P(K_3) - P(K_2)}{K_3 - K_2} = \frac{14.45 - 10.75}{60 - 55} = 0.74,$$

which violates the inequality $\dfrac{P(K_2) - P(K_1)}{K_2 - K_1} \le \dfrac{P(K_3) - P(K_2)}{K_3 - K_2}$ where $K_1 < K_2 < K_3$.

Now, we have

$$\frac{P(55) - P(50)}{55 - 50} > \frac{P(60) - P(55)}{60 - 55}$$

$$2P(55) > P(60) + P(50)$$

$$P(60) + P(50) - 2P(55) < 0$$

It implies that long a symmetric butterfly spread with buying one 50-strike put and one 60-strike put and selling two 55-strike put will have a negative cost.

So, we use a put butterfly spread to profit from these arbitrage opportunities.

Transaction	$t = 0$	$S_T < 50$	$50 \leq S_T \leq 55$	$55 \leq S_T \leq 60$	$S_T > 60$
Buy one 50 strike put	−7	$50 - S_T$	0	0	0
Sell two 55 strike puts	21.50	$2 \times S_T - 110$	$2 \times S_T - 110$	0	0
Buy one 60 strike put	−14.45	$60 - S_T$	$60 - S_T$	$60 - S_T$	0
TOTAL	+0.05	0	$S_T - 50 \geq 0$	$60 - S_T \geq 0$	0

Please note that we initially receive money and have nonnegative future payoffs. Therefore, we have found an arbitrage possibility, independent of the prevailing interest rate.

Additional Problem 2

The prepaid forward price of Stock A is \$22. That is,

$$F_{0,1}^{P}(A) = 22.$$

The prepaid forward price of Stock B is \$26.

$$F_{0,1}^{P}(B) = 26.$$

Option 1 can be regarded as a call option on A and Option 2 can be regarded as a put option on A.

Let $C(A_0, B_0, 1)$ and $P(A_0, B_0, 1)$ be the current prices of Option 1 and Option 2 respectively.

By the generalized form of put-call parity:

$$\begin{aligned} C(A_0, B_0, 1) - P(A_0, B_0, 1) &= F_{0,1}^{P}(A) - F_{0,1}^{P}(B) \\ &= 22 - 26 \\ &= -4. \end{aligned}$$

Therefore, the value of the portfolio is −4.