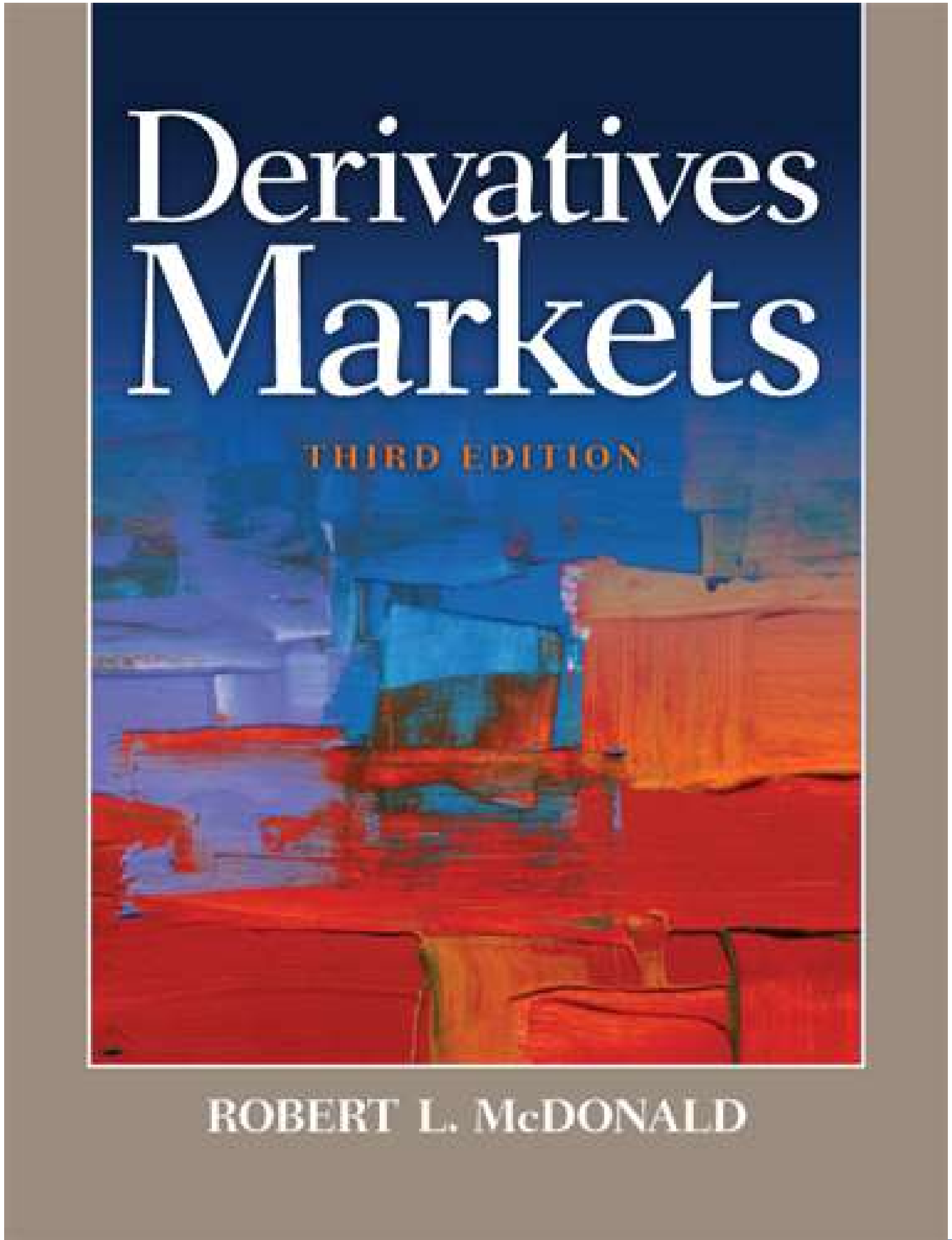


# Chapter 4
## (Chapter 5 in the Textbook)

Financial Forwards and Futures

# Points to note

1. Alternative ways to buy a stock, see P. 4 to 5.
2. Pricing pre-paid forwards, see P. 6 to 17.
3. Pricing forwards on stock, see P. 18 to 20.
4. Creating a synthetic forward, see P. 21 to 25.
5. Synthetic forwards in market-making and arbitrage, see P. 26 – 28.
6. An interpretation of the forward pricing formula, see P. 29 – 30.
7. Future contracts, see P. 31 – 34.
8. Future contracts – marking to market, see P. 35 – 41.
9. Comparing futures and forward prices, see P. 42 – 43.
10. Quanto Index Contracts, see P. 44 – 46.

# Introduction

- Financial futures and forwards
  - On stocks and indexes
  - On currencies
  - On interest rates
- How are they used?
- How are they priced?
- How are they hedged?

# Alternative Ways to Buy a Stock

- Four different payment and receipt timing combinations:
  - Outright purchase: ordinary transaction.
  - Fully leveraged purchase: A purchase in which you borrow the entire purchase price of the security and repay the borrowed amount later.
  - Prepaid forward contract: pay today (not necessarily the stock price), receive the share later.
  - Forward contract: agree on price now, pay/receive later.

# Alternative Ways to Buy a Stock (cont'd)

**TABLE 5.1** Four different ways to buy a share of stock that has price $S_0$ at time 0. At time 0 you agree to a price, which is paid either today or at time $T$. The shares are received either at 0 or $T$. The interest rate is $r$.

| Description | Pay at Time | Receive Security at Time | Payment |
|---|---|---|---|
| Outright purchase | 0 | 0 | $S_0$ at time 0 |
| Fully leveraged purchase | $T$ | 0 | $S_0 e^{rT}$ at time $T$ |
| Prepaid forward contract | 0 | $T$ | ? |
| Forward contract | $T$ | $T$ | $? \times e^{rT}$ |

# Pricing Prepaid Forwards

- If we can find the *prepaid* forward price for an asset bought at time 0 and delivered at time $T$ which is denoted as $F^P_{0,T}$, then we can calculate the price for a forward contract

$$F = \text{Future value of } F^P_{0,T}$$

- Three possible methods to price prepaid forwards:
  - Pricing by analogy.
  - Pricing by discounted present value.
  - Pricing by arbitrage.
- For now, assume that there are no dividends.

# Pricing Prepaid Forwards (cont'd)

- Pricing by analogy
  - In the absence of dividends, the timing of delivery is irrelevant.
  - Price of the prepaid forward contract same as current stock price.
  - $F^{P}{}_{0,T} = S_0$ (where the asset is bought at $t$ = 0, delivered at $t$ = $T$).

# Pricing Prepaid Forwards (cont'd)

- Pricing by discounted present value
  - If expected stock price at time *T* based on information we have at time 0 is $E_0(S_T)$, then

$$F^P_{0,T} = E_0(S_T)e^{-\alpha T}$$

where $\alpha$ is the risk-adjusted discount rate.

  - By the definition of the expected return, we have

$$E_0(S_T) = S_0 e^{\alpha T}$$

  - Combining them,

$$F^P_{0,T} = S_0$$

# Pricing Prepaid Forwards (cont'd)

- Pricing by arbitrage
  - Arbitrage: a situation in which one can generate positive cash flow by simultaneously buying and selling related assets, with no net investment of funds and with no risk ➔ free money!!!
  - An *extremely important pricing principle* is that the price of a derivative should be such that no arbitrage is possible.
    - Suppose that $F^P_{0,T} > S_0$. The arbitrage can be created as in Table 5.2

# Pricing Prepaid Forwards (cont'd)

**TABLE 5.2** Cash flows and transactions to undertake arbitrage when the prepaid forward price, $F_{0,T}^P$, exceeds the stock price, $S_0$.

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Buy stock @ $S_0$ | $-S_0$ | $+S_T$ |
| Sell prepaid forward @ $F_{0,T}^P$ | $+F_{0,T}^P$ | $-S_T$ |
| **Total** | $F_{0,T}^P - S_0$ | 0 |

# Pricing Prepaid Forwards (cont'd)

– Now suppose on the other hand that $F^P_{0,T} < S_0$. The arbitrage can be formed by
  - buying the prepaid forward and shorting the stock, earning

$$S_0 - F^P_{0,T} > 0$$

  - One year from now, we acquire the stock via the prepaid forward and we use the stock to close the short position. So, there is an arbitrage profit.

– Therefore, to preclude the arbitrage, we have

$$F^P_{0,T} = S_0$$

# Pricing Prepaid Forwards (cont'd)

- When a stock pays dividend, the owner of stock receives dividends but the owner of the prepaid forward contact does not.
- It is necessary to adjust the prepaid forward price to reflect dividends that are received by the shareholder but not by the holder of prepaid forward contact.

# Pricing Prepaid Forwards (cont'd)

*Discrete Dividends*

- Suppose a stock is expected to make dividend payments of $D_{t_i}$ at times $t_i$, $i = 1, 2, \ldots, n$.
- A prepaid forward contract will entitle you to receive the stock at time $T$ but without receiving the interim dividends.
- Thus, the price of a prepaid forward contract will be the stock price less the present value of dividends to be paid over the life of the contract

$$F_{0,T}^{P} = S_0 - \sum_{i=1}^{n} PV_{0,t_i}\left(D_{t_i}\right)$$

where $PV_{0,t_i}$ denotes the time 0 present value of a time $t_i$ payment.

# Pricing Prepaid Forwards (cont'd)

<u>*Continuous Dividends*</u>

- Suppose the dividend is being paid continuously at a rate proportional to the level of the stock price; i.e., the dividend yield, is constant.
- **Dividend yield**: the annualized dividend payment divided by the stock price.
- Let $\delta$ be the dividend yield.
- Now suppose we wish to invest today in order to have one share at time *T*. We can buy $e^{-\delta T}$ shares today. Because of **dividend reinvestment**, at time *T*, we will end up with <u>*exactly one share*</u>.

$$F_{0,T}^{P} = S_0 e^{-\delta T}$$

Let $a = e^{-\delta T}$ and $N = T/\Delta t$ .

Let $S_t$ be the stock price at time $t$.

| **Time** | **0** | $\boldsymbol{\Delta t}$ | $\boldsymbol{2\Delta t}$ | **…** | $\boldsymbol{N\Delta t = T}$ |
|---|---|---|---|---|---|
| No. of shares of stock | $a$ | $a+a\delta\Delta t$ $=a(1+\delta\Delta t)$ | $a(1+\delta\Delta t)+a(1+\delta\Delta t)\delta\Delta t$ $=a(1+\delta\Delta t)^2$ | | $a(1+\delta\Delta t)^N$ |
| Value of the stock | $aS_0$ | $aS_{\Delta t}$ + $a\delta S_{\Delta t}\Delta t$ | $a(1+\delta\Delta t)\, S_{2\Delta t}+a(1+\delta\Delta t)\delta$ $\Delta t\, S_{2\Delta t}$ | | $a(1+\delta\Delta t)^N S_T$ |

By taking $\Delta t \rightarrow 0$, we have $(1+\delta\Delta t)^N \rightarrow e^{\delta T}$. So, we end up with 1 unit of stock at time $T$.

# Pricing Prepaid Forwards (cont'd)

- Example 5.1
  - XYZ stock costs \$100 today and is expected to pay a quarterly dividend of \$1.25. If the risk-free rate is 10% compounded continuously, how much does a 1-year prepaid forward cost?

$$F^{p}_{0,1} = \$100 - \Sigma^{4}_{i=1} \$1.25 e^{-0.025i} = \$95.30$$

# Pricing Prepaid Forwards (cont'd)

- Example 5.2
  - The index is \$125 and the dividend yield is 3% continuously compounded. How much does a 1-year prepaid forward cost?
  - $F^{P}_{0,1} = \$125e^{-0.03} = \$121.31$

# Pricing Forwards on Stock

- Forward price is the future value of the *prepaid* forward price

$$F_{0,T} = FV({F^{P}}_{0,T}) = {F^{P}}_{0,T}e^{rT} = \frac{{F^{P}}_{0,T}}{P\left(0,T\right)}$$

where $r$ is the yield to maturity for a default-free zero coupon bond with maturity at time $T$ and $P(t,T)$ is the time $t$ price of a zero-coupon bond maturing at time $T$.

# Pricing Forwards on Stock (cont'd)

- Forward premium
  - Definition
    - Forward premium = $F_{0,T} / S_0$.
    - Annualized forward premium = $(1/T) \ln(F_{0,T} / S_0)$.
  - For the case of continuous dividends, the annualized forward premium is $r - \delta$.
  - The forward price can be used to infer the price of the underlying asset when it is unobserved. For example, the future contract of the S&P 500 index trades at times when the NYSE is not open. The asset price implied by the forward pricing formulas is called the *fair value* of the underlying stock.

# Does the Forward Price Predict the Future Spot Price?

- The forward price is the expected future spot price discounted at the risk premium

$$F_{0,T} = e^{rT} F_{0,T}^{P} = E_0\left(S_T\right) e^{-(\alpha - r)T}$$

where $\alpha$ is the expected return on a stock and $r$ is the risk-free interest rate.

- Therefore, the forward price *systematically errs* in predicting the future stock price. If the asset has a positive risk premium, the future spot price will on average be greater than the forward price.

# Creating a *Synthetic* Forward

- One can offset the risk of a forward by creating a *synthetic* forward to offset a position in the actual forward contract.
- How can one do this? (assume continuous dividends at rate $\delta$)
  - Recall the long forward payoff at expiration: $= S_T - F_{0,\,T}$.
  - Borrow and purchase shares as follows.

# Creating a *Synthetic* Forward (cont'd)

**TABLE 5.3** Demonstration that borrowing $S_0 e^{-\delta T}$ to buy $e^{-\delta T}$ shares of the index replicates the payoff to a forward contract, $S_T - F_{0,T}$.

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Buy $e^{-\delta T}$ units of the index | $-S_0 e^{-\delta T}$ | $+S_T$ |
| Borrow $S_0 e^{-\delta T}$ | $+S_0 e^{-\delta T}$ | $-S_0 e^{(r-\delta)T}$ |
| **Total** | 0 | $S_T - S_0 e^{(r-\delta)T}$ |

Note that the total payoff at expiration is same as forward payoff.

# Creating a *Synthetic* Forward (cont'd)

- The idea of creating synthetic forward leads to following:
  - (Synthetic) Forward = Stock – zero-coupon bond.
  - (Synthetic) Stock = Forward + zero-coupon bond.
  - (Synthetic) Zero-coupon bond = Stock – forward.

# Creating a *Synthetic* Forward (cont'd)

TABLE 5.4 Demonstration that going long a forward contract at the price $F_{0,T} = S_0 e^{(r-\delta)T}$ and lending the present value of the forward price creates a synthetic share of the index at time $T$.

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Long one forward | 0 | $S_T - F_{0,T}$ |
| Lend $S_0 e^{-\delta T}$ | $-S_0 e^{-\delta T}$ | $+S_0 e^{(r-\delta)T}$ |
| **Total** | $-S_0 e^{-\delta T}$ | $S_T$ |

Note that the total payoff at expiration is same as stock payoff.

# Creating a *Synthetic* Forward (cont'd)

**TABLE 5.5** Demonstration that buying $e^{-\delta T}$ shares of the index and shorting a forward creates a synthetic bond.

| | Cash Flows | |
|---|---|---|
| **Transaction** | **Time 0** | **Time $T$ (expiration)** |
| Buy $e^{-\delta T}$ units of the index | $-S_0 e^{-\delta T}$ | $+S_T$ |
| Short one forward | 0 | $F_{0,T} - S_T$ |
| **Total** | $-S_0 e^{-\delta T}$ | $F_{0,T}$ |

The payoff at $T$ is

$$F_{0,T} = S_0 e^{(r-\delta)T} = \left(S_0 e^{-\delta T}\right) e^{rT}$$

It is the payoff of $S_0 e^{-\delta T}$ unit of $T$-year zero coupon bond.

The rate of return ($r$) on this synthetic bond is called <u>*implied repo rate.*</u>

# *Synthetic* Forwards in Market-Making and Arbitrage

- The market-maker can ***offset*** his risk on the short forward position by creating a synthetic long forward position.
- A transaction in which you buy the underlying asset and short the offsetting forward contract is called a ***cash-and-carry***.

**TABLE 5.6** Transactions and cash flows for a cash-and-carry: A market-maker is short a forward contract and long a synthetic forward contract.

| | Cash Flows | |
|---|---|---|
| **Transaction** | **Time 0** | **Time $T$ (expiration)** |
| Buy tailed position in stock, paying $S_0e^{-\delta T}$ | $-S_0e^{-\delta T}$ | $+S_T$ |
| Borrow $S_0e^{-\delta T}$ | $+S_0e^{-\delta T}$ | $-S_0e^{(r-\delta)T}$ |
| Short forward | 0 | $F_{0,T} - S_T$ |
| **Total** | 0 | $F_{0,T} - S_0e^{(r-\delta)T}$ |

 

# *Synthetic* Forwards in Market-Making and Arbitrage (cont'd)

TABLE 5.7 Transactions and cash flows for a reverse cash-and-carry: A market-maker is long a forward contract and short a synthetic forward contract.

| Transaction | Cash Flows: Time 0 | Cash Flows: Time $T$ (expiration) |
|---|---|---|
| Short tailed position in stock, receiving $S_0e^{-\delta T}$ | $+S_0e^{-\delta T}$ | $-S_T$ |
| Lend $S_0e^{-\delta T}$ | $-S_0e^{-\delta T}$ | $+S_0e^{(r-\delta)T}$ |
| Long forward | 0 | $S_T - F_{0,T}$ |
| **Total** | 0 | $S_0e^{(r-\delta)T} - F_{0,T}$ |

# *Synthetic* Forwards in Market-Making and Arbitrage (cont'd)

- If $F_{0,T} \neq S_0 e^{(r-\delta)T}$, then an arbitrageur can make a costless profit.
  - If $F_{0,T} > S_0 e^{(r-\delta)T}$, then an arbitrageur or market-maker can use the strategy in Table 5.6 to make a risk-free profit.
  - On the other hand, if $F_{0,T} < S_0 e^{(r-\delta)T}$, then an arbitrageur or market-maker can use the strategy in Table 5.7 to make a risk-free profit.

# An Interpretation of the Forward Pricing Formula

- Suppose you buy a unit of index that costs $S$ and fund the position by borrowing at the risk-free rate. Over the short time interval $\Delta t$, you will
  - pay $rS\Delta t$ on the borrowed amount.
  - Receive the dividend $\delta S\Delta t$.
- Net cashflow for you to carry a long position in the asset = $(r - \delta)S\Delta t$.
- So, $(r - \delta)$ is the called the *cost of carry*.

# An Interpretation of the Forward Pricing Formula (cont'd)

- The forward contract, unlike the stock, requires no investment and makes no payouts and therefore has a zero cost of carry. Therefore, the forward contract saves our having to pay the cost of carry, we are willing to pay a higher price.

$$\text{Forward price} = \text{Spot price} + \underbrace{\text{Interest to carry the asset} - \text{Asset lease rate}}_{\text{Cost of carry} \times \text{spot price} \times \Delta t}$$

where $\Delta t$ is the maturity of the forward contract.

# Futures Contracts

- Exchange-traded “forward contracts”
- Typical features of futures contracts
  - Standardized, with specified delivery dates, locations, procedures.
  - A clearinghouse
    - Matches buy and sell orders.
    - Keeps track of members’ obligations and payments.
    - After matching the trades, becomes counterparty.
- Open interest: total number of buy/sell pairs.

# Futures Contracts (cont'd)

- Differences from forward contracts
  - Futures contract is settled daily. The determination of who owes what to whom is called *marking-to-market*.
  - Because of daily settlement, futures contract is liquid – it is possible to offset an obligation on a given day by entering into the opposite position.
  - Highly standardized structure ➔ harder to customize.
  - Because of daily settlement, the nature of credit risk is different with the futures contract.
  - There are typically daily *price limits* in futures market. The price limit is a move in the futures price that triggers a temporary halt in trading.

# The S&P 500 Futures Contract

- *Underlying asset:* S&P 500 stock index.
- *Notational value (the dollar value of the assets underlying one contract):*

  $250 x S&P 500 index

- *Cash-settled contract (not physical settlement)*:

  On the expiration day, the S&P 500 futures contract is marked-to-market against the actual cash index. This final settlement against the cash index guarantees that the futures price equals the index value at contract expiration.

# Example: S&P 500 Futures

FIGURE 5.1

Specifications for the S&P 500 index futures contract.

| | |
|---|---|
| Underlying | S&P 500 index |
| Where traded | Chicago Mercantile Exchange |
| Size | $250 × S&P 500 index |
| Months | March, June, September, December |
| Trading ends | Business day prior to determination of settlement price |
| Settlement | Cash-settled, based upon opening price of S&P 500 on third Friday of expiration month |

# Margins and Marking to Market

- Suppose you would enter 8 long S&P 500 futures contracts with the futures price of 1,100.
- Total notional value

  = 8x$250x1100 = $2.2 million.
- A broker executes your buy order by matching with another sell order. The total number of open positions (buy/sell pairs) is called the *open interest* of the contract.

# Margins and Marking to Market (cont'd)

- Both buyers and sellers need to make a deposit, which can earn interest, with the broker. The deposit is called *margin* which is intended to protect the counterparty against your failure to meet your obligations.
- Here, we suppose the margin is *10%* and *weekly settlement*.
- Margin deposit = 10% x $2.2 million

  = $220,000

# Margins and Marking to Market (cont’d)

*Interest rate for the margin deposit:* 6% p.a. (compounded continuously).

Suppose that over the first week, the futures price drops 72.01 points to 1,027.99, our margin balance after 1 week is

$\$220{,}000e^{0.06 \times 1/52} + 8 \times 250 \times (1{,}027.99 - 1{,}100)$

$= \$76{,}233.99$

## TABLE 5.8

Mark-to-market proceeds and margin balance over 10 weeks from long position in 8 S&P 500 futures contracts. The last column does not include additional margin payments. The final row represents expiration of the contract.

| Week | Multiplier ($) | Futures Price | Price Change | Margin Balance($) |
|---|---|---|---|---|
| 0 | 2000.00 | 1100.00 | — | 220,000.00 |
| 1 | 2000.00 | 1027.99 | −72.01 | 76,233.99 |
| 2 | 2000.00 | 1037.88 | 9.89 | 96,102.01 |
| 3 | 2000.00 | 1073.23 | 35.35 | 166,912.96 |
| 4 | 2000.00 | 1048.78 | −24.45 | 118,205.66 |
| 5 | 2000.00 | 1090.32 | 41.54 | 201,422.13 |
| 6 | 2000.00 | 1106.94 | 16.62 | 234,894.67 |
| 7 | 2000.00 | 1110.98 | 4.04 | 243,245.86 |
| 8 | 2000.00 | 1024.74 | −86.24 | 71,046.69 |
| 9 | 2000.00 | 1007.30 | −17.44 | 36,248.72 |
| 10 | 2000.00 | 1011.65 | 4.35 | 44,990.57 |

# Margins and Marking to Market (cont'd)

The 10-week profit on the futures position

$$\$44{,}990.57 - \$220{,}000e^{0.06\times 10/52} = -\$177{,}562.6$$

In the case of forward contract, the 10-week profit is

$$(1{,}011.65 - 1{,}100) \times \$2{,}000 = -\$176{,}700$$

The difference is because the interest is earned on the mark-to-market proceeds in the case of futures contract.

# Margins and Marking to Market (cont'd)

- The decline in the margin balance means the broker has significantly less protection should we default. For this reason, participants are required to maintain the margin at a minimum level, called the *maintenance margin*.
- If the margin balance falls below the maintenance margin, the broker would make a *margin call*, requesting additional margin to bring the margin balance to the level of initial margin.

# Margins and Marking to Market (cont'd)

- If we failed to post additional margin, the broker would close the futures position (long/short position) and return the remaining margin to the corresponding party.

# Comparing Futures and Forward Prices

- *<u>If the interest rate were not random</u>*, then forward and future prices would be the same. (see supplementary "forwardandfutures.pdf")
- If the interest rate is random and positively (negatively) correlated with the future price, the future price will  be higher (less) than the price on an otherwise identical forward contract.

# Comparing Futures and Forward Prices (cont'd)

- In general, the difference between the price of the short-lived forward and futures contract is small. However, for long-lived contracts, the difference can be significant.

# Arbitrage in Practice: S&P 500 Index Arbitrage

- See P.143 to understand how the arbitrage can be achieved.

# Quanto Index Contracts

- Contract specifications of Nikkei 225 index futures contract

FIGURE 5.2

Specifications for the Nikkei 225 index futures contract.

| | |
|---|---|
| Underlying | Nikkei 225 Stock Index |
| Where traded | Chicago Mercantile Exchange |
| Size | $5 × Nikkei 225 Index |
| Months | March, June, September, December |
| Trading ends | Business day prior to determination of settlement price |
| Settlement | Cash-settled, based upon opening Osaka quotation of the Nikkei 225 index on the second Friday of expiration month |

# Quanto Index Contracts (cont'd)

- The Nikkei 225 futures contact can eliminate the exchange rate risk for the US investors by using dollars to settle instead of yen.
- The contract insulates investors from currency risk, permitting them to speculate solely on whether the underlying asset rises or falls. This kind of contract is called *quanto*.