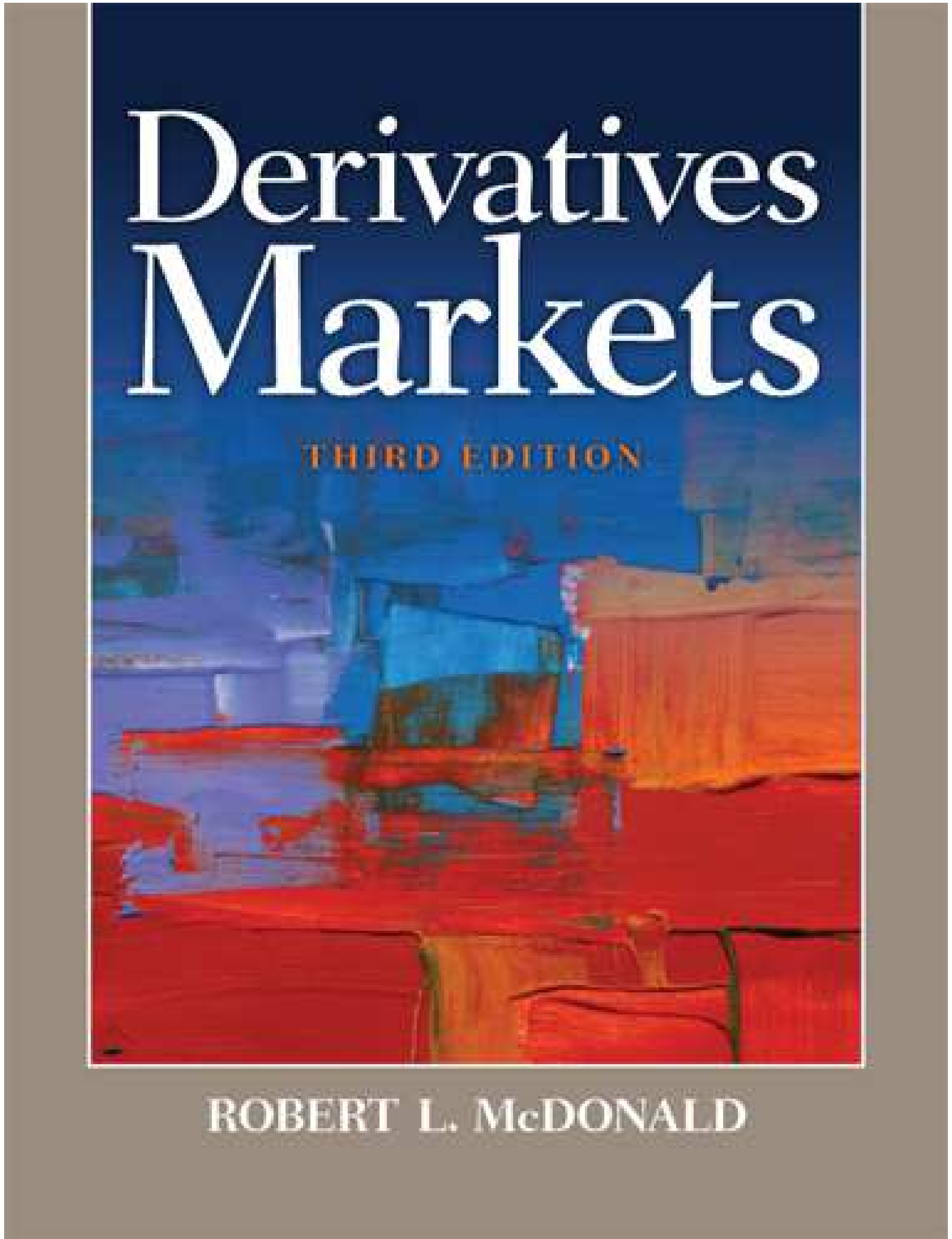

Chapter 12
(Chapter 18 in the textbook)

The Lognormal Distribution

Points to Note

1. Definition of the lognormal distribution. See P.10.
2. Properties of lognormal random variables. See P.10.
3. The expectation and variance of the lognormal random variable. See P.11.
4. The lognormal model of stock prices. See P.14 – 15.
5. Some results of the lognormal distribution. See P.17 – 20.
6. Estimating the parameters of a lognormal distribution. See P.21 – 23.

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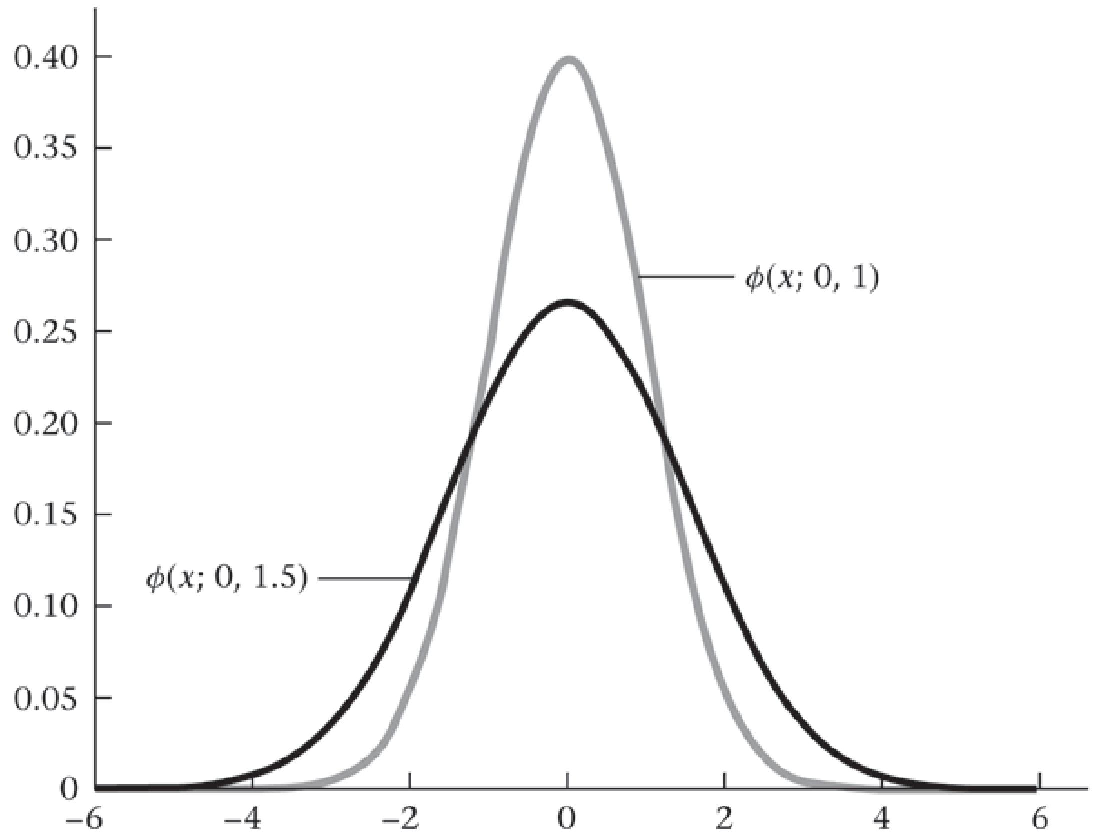
The Normal Distribution

- Normal distribution (or density) $\Phi(x;\mu,\sigma) \equiv \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



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The Normal Distribution (cont'd)

- Normal density is symmetric about the mean μ :

$$\Phi(\mu + a; \mu, \sigma) = \Phi(\mu - a; \mu, \sigma)$$

- If a random variable *x* is normally distributed with mean μ and standard deviation, σ then $x \sim N(\mu, \sigma^2)$
- Use *z* to represent a random variable that has a standard normal distribution: $z \sim N(0,1)$

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The Normal Distribution (cont'd)

- The value of the cumulative normal distribution function $N(a)$ equals to the probability $P(z<a)$ of a number z drawn from the normal distribution to be less than a.

$$N(a) \equiv \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

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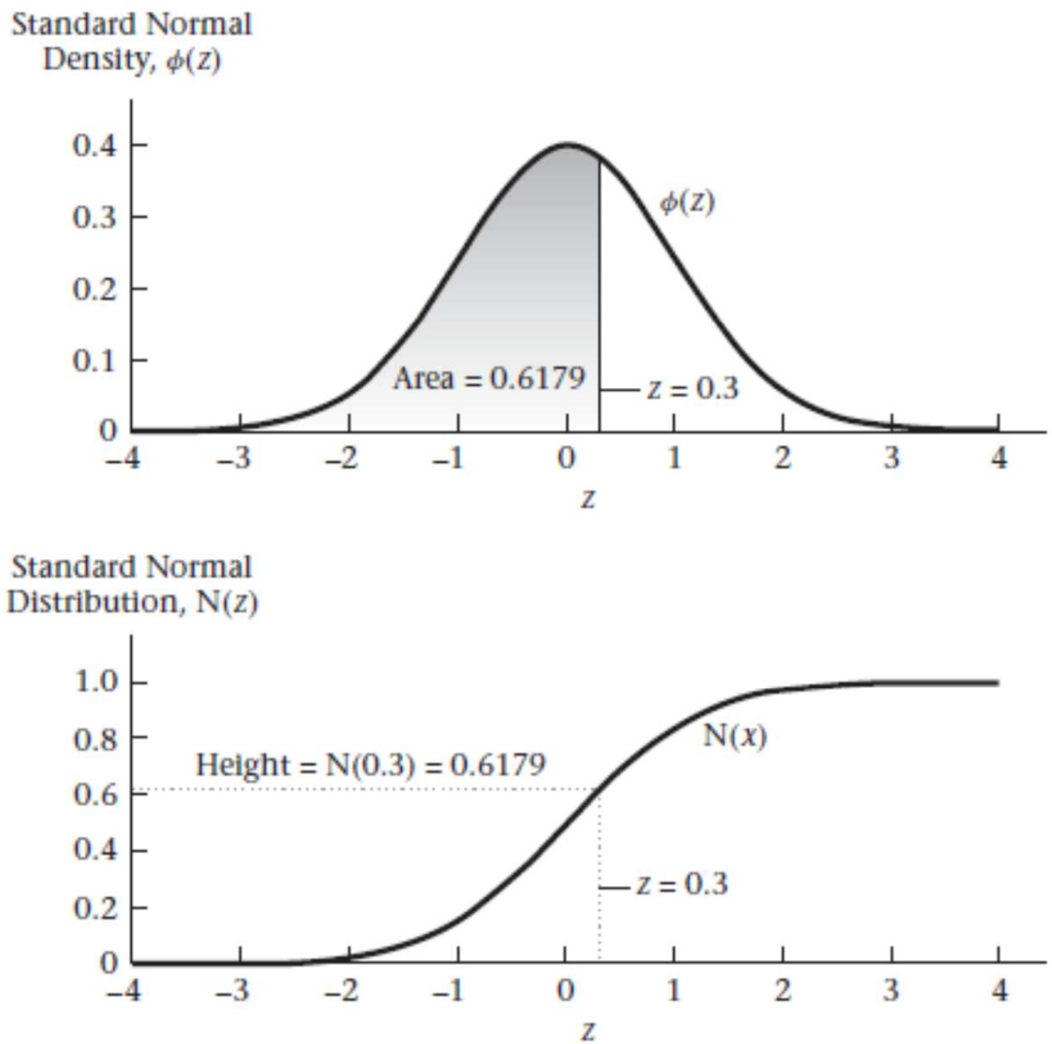
The Normal Distribution (cont'd)

FIGURE 18.2

Top panel: Area under the normal curve to the left of 0.3. *Bottom panel:* Cumulative normal distribution. The height at $x = 0.3$, given by $N(0.3)$, is 0.6179.



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The Normal Distribution (cont'd)

- The probability that a number drawn from the standard normal distribution will be between *a* and –*a*:

$$Prob\ (z < -a) = N(-a)$$

$$Prob\ (z < a) = N(a)$$

therefore

$$Prob\ (-a < z < a) = N(a) - N(-a) =$$

$$N(a) - [1 - N(a)] = 2 \cdot N(a) - 1$$

- Example: $Prob\ (-0.3 < z < 0.3) = 2 \cdot 0.6179 - 1 = 0.2358$.

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The Normal Distribution (cont'd)

- Converting a normal random variable to standard normal
  - If $x \sim N(\mu, \sigma^2)$, then $z \sim N(0,1)$ if $z = \dfrac{x - \mu}{\sigma}$
- Converting a standard normal variable to a normal variable
  - If $z \sim N(0,1)$, then $x \sim N(\mu, \sigma^2)$ if $x = \mu + \sigma z$
- Example 18.2: Suppose $x \sim N(3,25)$ and $z \sim N(0,1)$ then

$$\frac{x-3}{5} \sim N(0,1) \quad \text{and} \quad 3 + 5 \times z \sim N(3,25)$$

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The Normal Distribution (cont'd)

- The sum of normal random variables is also

$$\sum_{i=1}^{n} \omega_i x_i \sim N\left(\sum_{i=1}^{n} \omega_i \mu_i, \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \sigma_{ij}\right)$$

where x_i, $i = 1,\ldots,n$, are n random variables, with mean $E(x_i) = \mu_i$, variance $Var(x_i) = \sigma_i^2$, covariance $Cov(x_i,x_j) = \sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

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The Lognormal Distribution

- A random variable x is **lognormally distributed** if $\ln(x)$ is normally distributed
  - If x is normal, and $\ln(y) = x$ (or $y = e^x$), then y is lognormal.
  - If continuously compounded stock *returns* are *normal* then the stock *price* is *lognormally* distributed.
- Product of lognormal variables is lognormal
  - If x_1 and x_2 are normal, then $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$ are lognormal.
  - The product of y_1 and y_2: $y_1 \text{ x } y_2 = e^{x_1} \text{ x } e^{x_2} = e^{x_1 + x_2}$.
  - Since $x_1 + x_2$ is normal, $e^{x_1 + x_2}$ is lognormal.
  - **Note**: the sum of lognormal variables is NOT lognormal.

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The Lognormal Distribution (cont'd)

- If *ln y* ~ *N*(*m*,*v*²), the lognormal density function of *y* is

$$g(y;m,v) \equiv \frac{1}{yv\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-m]}{v}\right)^2}$$

- If *x* ~ *N*(*m*,*v*²), then

$$E(e^x) = e^{m+\frac{1}{2}v^2}$$

$$\mathrm{Var}(e^x) = e^{2m+v^2}\left(e^{v^2}-1\right)$$

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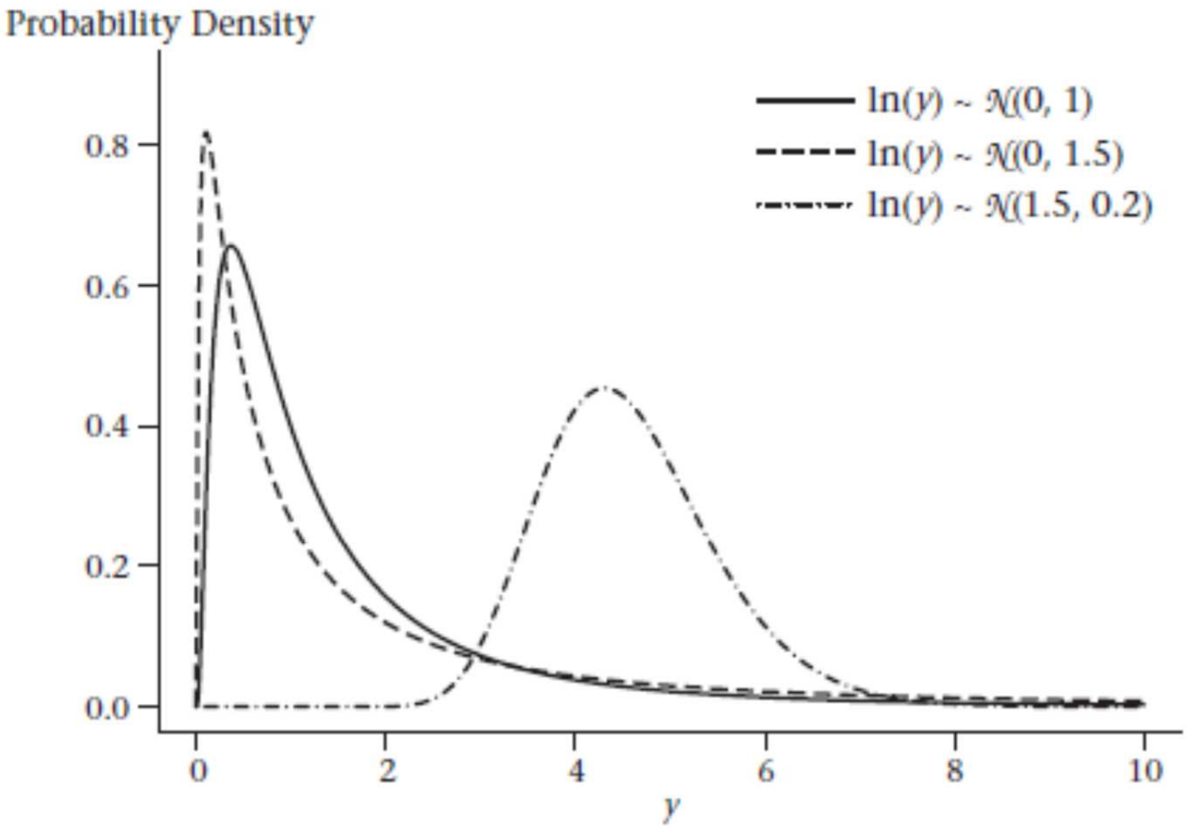
The Lognormal Distribution (cont'd)

FIGURE 18.3

Graph of the lognormal density for y, where $\ln(y) \sim \mathcal{N}(0, 1)$, $\ln(y) \sim \mathcal{N}(0, 1.5)$, and $\ln(y) \sim \mathcal{N}(1.5, 0.2)$.



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A Lognormal Model of Stock Prices

- If the stock price S_t is lognormal, then S_t / $S_0 = e^x$, where x, the continuously compounded return from 0 to t, is normally distributed.
- If $R(t, s)$ is the continuously compounded return from t to s, and, $t_0 < t_1 < t_2$, then $R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$.
- From 0 to T, $E[R(0,T)] = n\alpha_h$, and $Var[R(0,T)] = n\sigma_h^2$, where $\alpha_h = E[R((i - 1)h, ih)]$ and $\sigma_h^2 = Var[R((i - 1)h, ih)]$. Here, $R((i - 1)h, ih)$ are uncorrelated.
- If returns are *i.i.d.*, the mean and variance of the continuously compounded returns are proportional to time.

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A Lognormal Model of Stock Prices (cont'd)

- If we assume that

$$\ln(S_t / S_0) \sim N[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t]$$

then $\ln(S_t / S_0) = (\alpha - \delta - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z$

and therefore $S_t = S_0 e^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}$

$$\begin{aligned} E(S_t) &= E\left(S_0 e^{\left(\alpha-\delta-0.5\sigma^2\right)t+\sigma\sqrt{t}Z}\right) \\ &= S_0 e^{\left(\alpha-\delta-0.5\sigma^2\right)t} E\left(e^{\sigma\sqrt{t}Z}\right) \\ &= S_0 e^{\left(\alpha-\delta-0.5\sigma^2\right)t} e^{0.5\sigma^2 t} \\ &= S_0 e^{(\alpha-\delta)t} \end{aligned}$$

$$\ln E\left(\frac{S_t}{S_0}\right) = (\alpha - \delta)t$$

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A Lognormal Model of Stock Prices (cont'd)

- The expression $\alpha - \delta$ is called the continuously compounded expected rate of stock-price appreciation on the stock.
- The median stock price – the value such that 50% of the time prices will be above or below that value – is obtained by setting $Z = 0$ in S_t. The median is thus

$$E\left(S_t\right)e^{-0.5\sigma^2 t}$$

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A Lognormal Model of Stock Prices (cont'd)

Example

Suppose that the stock price today is \$100, the expected rate of return on the stock is α = 10%/year, and σ = 30%/year. If the stock is lognormally distributed, then we have

$$S_2 = \$100e^{\left(0.1-0.5\times 0.3^2\right)2+\sigma\sqrt{2}Z}$$

Thus,

$$E\left(S_2\right) = \$100e^{(0.1)(2)} = \$122.14.$$

and the median is

$$\$100e^{\left(0.1-0.5\times 0.3^2\right)\times 2} = \$111.63.$$

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Lognormal Probability Calculations

- Probabilities

If

$$\ln(S_t / S_0) \sim N\left[\left(\alpha - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right], \quad \text{or,}$$

$$\ln(S_t) \sim N\left[\ln(S_0) + \left(\alpha - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right]$$

then

$$\Pr(S_t < K) = N\left(-\hat{d}_2\right)$$

where

$$\hat{d}_2 = \frac{\left(\alpha - \delta - 0.5\sigma^2\right)t + \ln(S_0 / K)}{\sigma\sqrt{t}}$$

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Lognormal Probability Calculations (cont'd)

- Given a call option expires in the money, what is the expected stock price?

The **partial** expectation of S_t, conditional on $S_t < K$, is defined as

$$\int_0^K S_t g(S_t;S_0)dS_t = S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln(K)-\left[\ln(S_0)+\left(\alpha-\delta+0.5\sigma^2\right)t\right]}{\sigma\sqrt{t}}\right)$$

$$= S_0 e^{(\alpha-\delta)t} N\left(-\hat{d}_1\right)$$

where $g(S_t;\ S_0)$ is the probability density of S_t conditional on S_0, and $\hat{d}_1$ is the Black-Scholes d_1 with α replacing r.

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Lognormal Probability Calculations (cont'd)

The probability that $S_t < K$ is $N\left(-\hat{d}_2\right)$. Thus, the expectation of S_t conditional on $S_t < K$ is

$$E\left(S_t \mid S_t < K\right) = S_0 e^{(\alpha-\delta)t} \frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)}$$

Similarly, we obtain

$$E\left(S_t \mid S_t > K\right) = S_0 e^{(\alpha-\delta)t} \frac{N\left(\hat{d}_1\right)}{N\left(\hat{d}_2\right)}$$

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Lognormal Probability Calculations (cont'd)

- The Black-Scholes formula—the price of a call option is

$$C(S,K,\sigma,r,t,\delta) = e^{-rt}\int_K^{\infty}(S_t - K)\, g^*(S_t;S_0)\, dS_t$$
$$= e^{-rt}E^*(S_t - K \mid S > K) \times \Pr{}^*(S > K)$$
$$= e^{-\delta t}SN(d_1) - Ke^{-rt}N(d_2)$$

where g^* denote the risk-neutral lognormal probability density, E^* denote the expectation taken with respect to risk-neutral probabilities, and $\Pr^*$ denote those probabilities. Under g^*,

$$\ln(S_t/S_0) \sim N\left[\left(r - \delta - 0.5\sigma^2\right)t, \sigma^2 t\right].$$

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Estimating the Parameters of a Lognormal Distribution

- When stocks are lognormally distributed, the price S_t evolves from the previous price observed at time $t - h$, according to

$$S_t = S_{t-h} e^{(\alpha - \delta - \sigma^2/2)h + \sigma\sqrt{h}z}$$

Thus

$$E\left(\ln\left(\frac{S_t}{S_{t-h}}\right)\right) = (\alpha - \delta - \sigma^2/2)h$$

$$Var\left(\ln\left(\frac{S_t}{S_{t-h}}\right)\right) = \sigma^2 h$$

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Estimating the Parameters of a Lognormal Distribution (cont'd)

TABLE 18.2 Hypothetical weekly stock price observations and corresponding weekly continuously compounded returns, $\ln(S_t/S_{t-1})$

Week	Price ($)	$\ln(S_t/S_{t-1})$
1	100	—
2	105.04	0.0492
3	105.76	0.0068
4	108.93	0.0295
5	102.50	−0.0608
6	104.80	0.0222
7	104.13	−0.0064

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Estimating the Parameters of a Lognormal Distribution (cont'd)

- Example 18.8:
  - The mean of the second column is 0.006745 and the standard deviation is 0.038208.
  - $h = 1/52$.
  - Annualized standard deviation

    $$= 0.038208 \times \sqrt{52} = 0.2755$$

  - Annualized expected return

    $$= 0.006745 \times 52 + 0.5 \times 0.2755^2 = 0.3887$$

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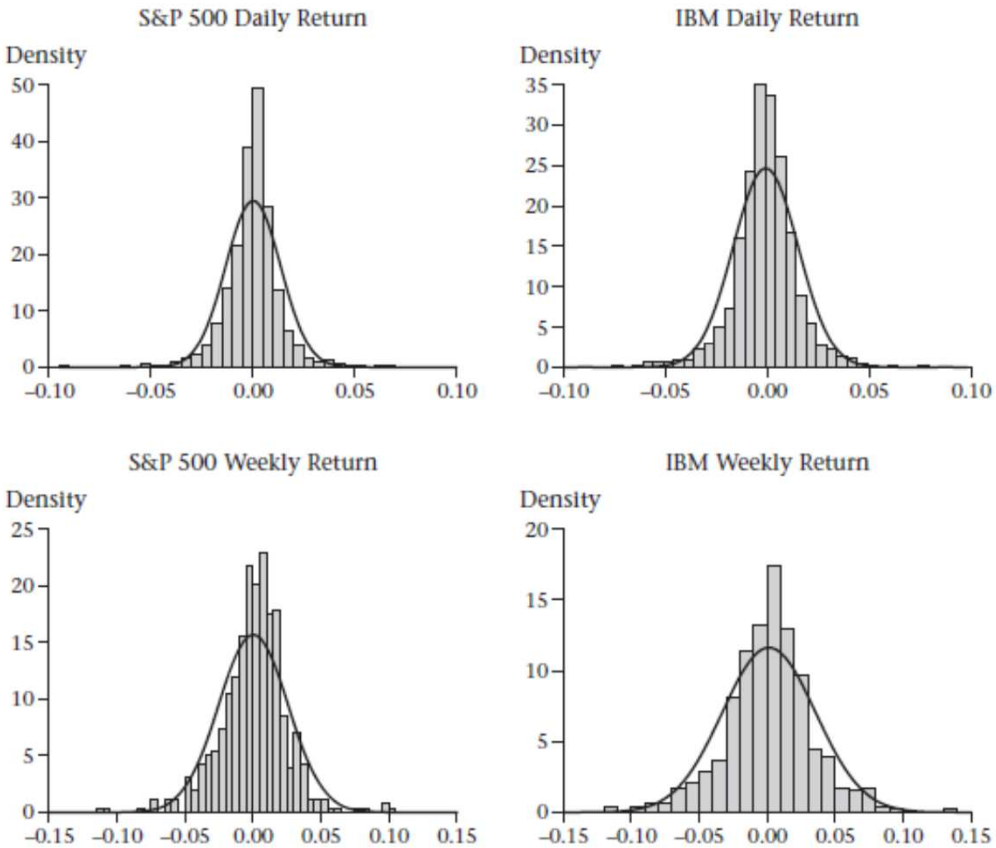
How Are Asset Prices Distributed?

FIGURE 18.4

Histograms for daily and weekly returns on the S&P 500 index and IBM, from July 1, 2001 to July 1, 2011.



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How Are Asset Prices Distributed? (cont’d)

- None of the histograms appears exactly normal. All of the histograms exhibit a peak around 0; the presence or absence of this peakedness is referred to as <u>*kurtosis*</u> (a measure of how sharp the peak of the distribution is).
- The graph displays <u>*leptokurtosis*</u> (small, thin and delicate).
- Kurtosis for the S&P and IBM are 8.03 and 9.54 for daily returns, and 4.68 and 5.21 for weekly returns.

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How Are Asset Prices Distributed? (cont'd)

- Accompanying the peaks are fat tails, large returns that occur more often would be predicted by the lognormal model.
- The normal probability plot can be used for assessing normality. If the data points lie along the straight line in the graph, the data are consistent with a normal distribution. If the data plot is curved, the data are less likely to have come from a normal distribution.

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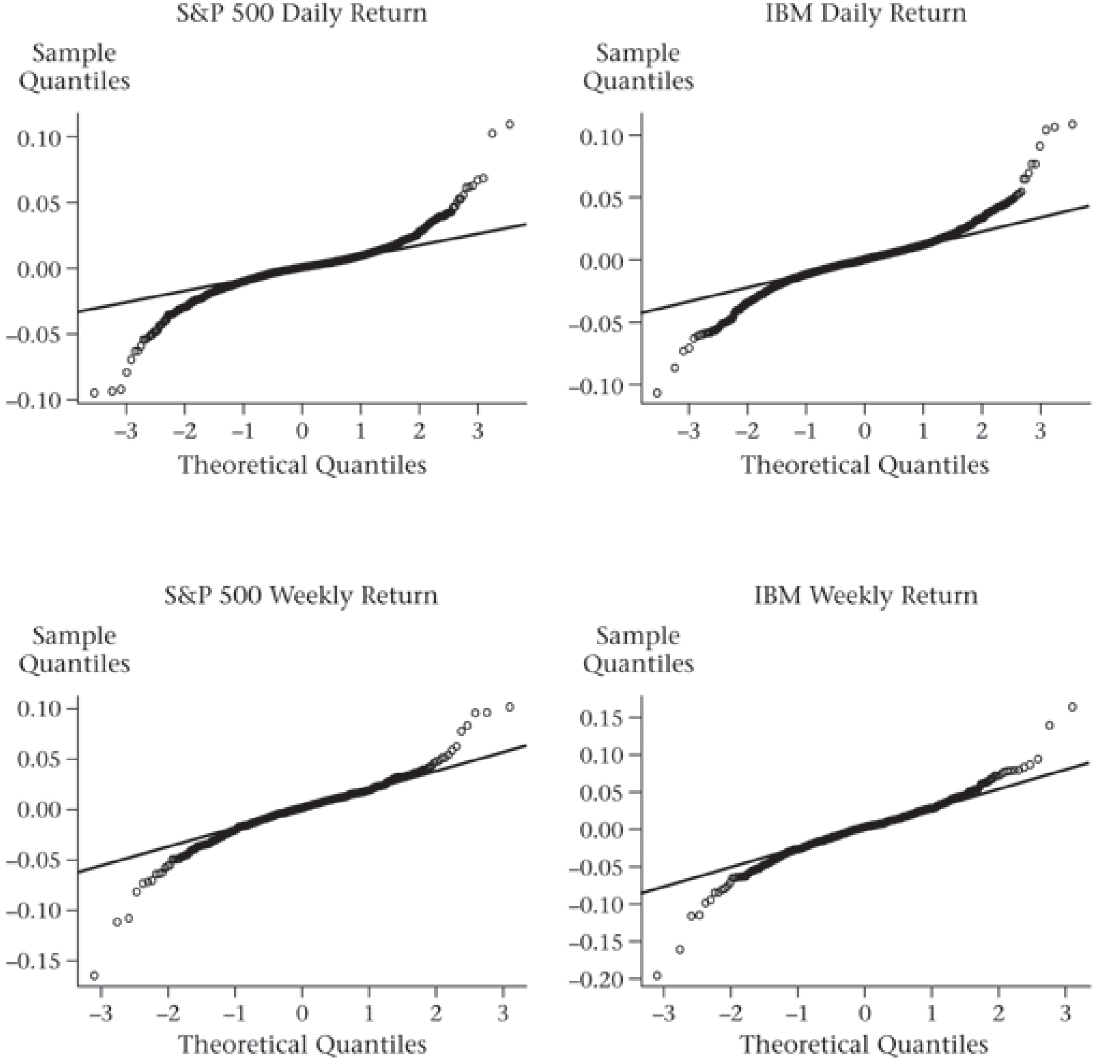
FIGURE 18.5

Normal probability plots for daily and weekly returns on the S&P 500 index and IBM from July 1, 2001 to July 1, 2011.



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