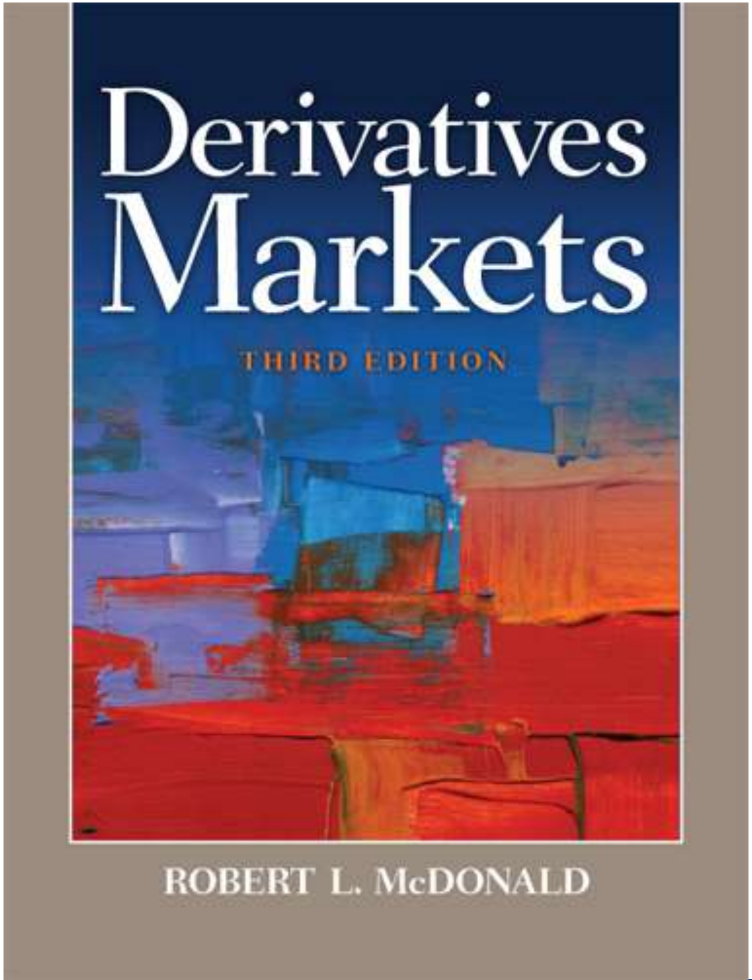


Chapter 10
(Chapter 12 in the textbook)

The Black-Scholes Formula

Points to Note

1. What is the Black-Scholes formula for the European call and put options? (see P.3 – 5)
2. What are the assumptions of the Black-Scholes formula? (see P.6 – 7)
3. What is the relationship of the binomial model and the Black-Scholes formula? (see P.8 – 9)
4. The Black-Scholes formula for different underlying assets. (see P.10 – 15)
5. Option Greeks (see P.16 – 46)
6. Implied volatility (see P.47 – 52)

partial derivative of option price w.r.t different parameters

Black-Scholes Formula

- The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.
- Consider an European call (or put) option written on a stock.
- Assume that the stock pays dividend at the continuous rate δ.

Black-Scholes Formula (cont'd)

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Black-Scholes Formula (cont'd)

$$C = \Delta S + B$$

- Call Option price:

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T}N(d_1)\boxed{-Ke^{-rT}N(d_2)}$$

- Put Option price:

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$$

where

$$d_1 = \frac{ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

N(*x*) is the cumulative distribution for standard normal random variable.

Black-Scholes Assumptions

- Assumptions about stock return distribution:
  - Continuously compounded returns on the stock are normally distributed and independent over time. (We assume there are no "jumps" in the stock price).
  - The volatility of continuously compounded returns is known and constant.
  - Future dividends are known, either as dollar amount or as a fixed dividend yield.

Black-Scholes Assumptions (cont'd)

- Assumptions about the economic environment:
  - The risk-free rate is known and constant.
  - There are no transaction costs or taxes.
  - It is possible to short-sell costlessly and to borrow at the risk-free rate.

Continuous Limits of the Binomial Model

By considering the general formulation of the call option price on a non-dividend stock under the binomial model,

$$C = S_0 \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j} \frac{u^j d^{n-j}}{e^{rnh}} - Ke^{-rnh} \sum_{j=k}^{n} C_j^n \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

$$= S_0 \Phi\left(n,k,\tilde{p}\right) - Ke^{-rT} \Phi\left(n,k,p^*\right).$$

where $C = S_0 N(d_1) - Ke^{-rT} N(d_2)$

$$\Phi\left(n,k,p\right) = \sum_{j=k}^{n} C_j^n \left(p\right)^j \left(1-p\right)^{n-j}, \quad \tilde{p} = \frac{up^*}{e^{rh}} \quad \text{and} \quad 1-\tilde{p} = \frac{d\left(1-p^*\right)}{e^{rh}}.$$

Continuous Limits of the Binomial Model (cont'd)

It can shown that

$$\lim_{n\to\infty}\left[S_0\Phi\left(n,k,\tilde{p}\right)-Ke^{-rT}\Phi\left(n,k,p^*\right)\right]=S_0N\left(d_1\right)-Ke^{-rT}N\left(d_2\right).$$

Recall that $\Phi\left(n,k,p^*\right)$ is the risk neutral probability that the number of upward moves in the asset price is greater than or equal to *k* in the *n*-period binomial model, where p^* is the risk neutral probability of an upward move.

(See Binomial tree to BS.pdf)

BS formula

$$C = S e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

T: time to expiration

$F^P_{0,T}(S) = S e^{-\delta T}$ $F^P_{0,T}(K) = K e^{-rT}$

Applying the Formula to Other Assets

- Call Options

Let $F_{0,T}^{P}(S)$ and $F_{0,T}^{P}(K)$ be the prepaid forward prices for the stock and strike asset.

$$F_{0,T}^{P}(S) = Se^{-\delta T} \quad \text{and} \quad F_{0,T}^{P}(K) = Ke^{-rT}$$

Using $F_{0,T}^{P}(S)$ and $F_{0,T}^{P}(K)$ to rewrite the call option pricing formula, we have

$$C(F_{0,T}^{P}(S), F_{0,T}^{P}(K), \sigma, T) = F_{0,T}^{P}(S)\,N(d_1) - F_{0,T}^{P}(K)\,N(d_2)$$

where

$$d_1 = \frac{ln\left[F_{0,T}^{P}(S) / F_{0,T}^{P}(K)\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Options on Stocks with Discrete Dividends

- The prepaid forward price for stock with discrete dividends is

$$F_{0,T}^{P}(S) = S_0 - PV_{0,T}(Div)$$

- Examples 12.3
  - *S* = \$41, *K* = \$40, σ = 0.3, *r* = 8%, *T* = 0.25, *Div* = \$3 in one month.
  - *PV* (*Div*) = $\$3e^{-0.08/12}$ = \$2.98
  - Use \$41– \$2.98 = \$38.02 as the stock price in BS formula.
  - The BS European call price is \$1.763.

Options on Currencies

- The prepaid forward price for the currency is

$$F_{0,T}^{P}(x) = x_0 e^{-r_f T}$$

$r_f = \delta$

where x is domestic spot rate and r_f is foreign interest rate

$$C(x, K, \sigma, r, T, r_f) = xe^{-r_f T} N(d_1) - Ke^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(x/K) + (r - r_f + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Options on Currencies (cont'd)

- The price of a European put is obtained using parity:

$$P\left(x,K,\sigma,r,T,r_f\right)=C\left(x,K,\sigma,r,T,r_f\right)+Ke^{-rT}-xe^{-r_f T}$$

Options on Currencies (cont'd)

- Example 12.4
  - x_0 = $1.25/€ , K = $1.20, σ = 0.10, r = 1%, T = 1, and r_f = 3%.
  - The price of a dollar-denominated euro call is $0.061407.
  - The price of a dollar-denominated euro put is $0.03641.

Options on Futures

$F_{0,T}^P(F) = Fe^{-rT}$

$= PV(F_{0,T})$

- The prepaid forward price for a futures contract is the PV of the futures price. Therefore

$$C(F, K, \sigma, r, T) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{ln\left[F / K\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

- Example 12.5:
  - Suppose 1-yr. futures price for natural gas is \$6.50/MMBtu, r = 2%.
  - F=\$6.50, K=\$6.50, and δ = 2%.
  - If σ = 0.25, T= 1, call price = put price = \$0.63379.

Option Greeks

- What happens to the option price when *one and only one* input changes?

  stock price

  - Delta (Δ): change in option price when stock price increases by $1.
  - Gamma (Γ): change in delta when ~~option~~ price increases by $1.
  - Vega: change in option price when volatility increases by 1%.
  - Theta (θ): change in option price when time to maturity decreases by 1 day.
  - Rho (ρ): change in option price when interest rate increases by 1%.
  - Psi (Ψ): the change in the option price when there is an increase in the continuous dividend yield of 1% (100 basis points).

(The formulas for option Greeks are given in Appendix B of the textbook)

Option Greeks (cont'd)

- Delta $\frac{\partial V}{\partial S}$
  - Delta is defined as the number of shares in the portfolio that replicates the option. Delta can be interpreted as a share-equivalent of the option.
  - For a call (put) option, delta is positive (negative).
  - An in-the-money option will be more sensitive to the stock price than an out-of-money option. For example, if a call is deep in-the-money, it is like to be exercised and hence the option should behave much like a leveraged position in a full share (Delta approaches 1).
  - Similar analysis can be applied for an out-of-money and at-the-money options.

$$C = S e^{-\delta\tau} N(d_1) - K e^{-r\tau} N(d_2)$$

$$\Delta = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S}\left(S e^{-\delta\tau} N(d_1)\right) - \boxed{\frac{\partial}{\partial S}\left(K e^{-r\tau} N(d_2)\right)} \neq 0$$

$$= e^{-\delta\tau} N(d_1) - 0$$

$$= e^{-\delta\tau} N(d_1)$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$\Delta = \frac{\partial C}{\partial S} = \frac{\partial}{\partial S}\left(S e^{-\delta\tau} N(d_1)\right) - \frac{\partial}{\partial S} K e^{-r\tau} N(d_2)$$

$$= S e^{-\delta\tau} \frac{\partial N(d_1)}{\partial S} + e^{-\delta\tau} N(d_1)$$

$$- K e^{-r\tau} \boxed{\frac{\partial}{\partial S} N(d_2)}$$

$$\frac{\partial N(d_1)}{\partial S} = \frac{\partial}{\partial S} \overbrace{\boxed{\int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}}^{N(d_1)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\partial d_1}{\partial S}$$

$$\Delta_{call} = e^{-\delta \tau} N(d_1)$$

$$C = S \boxed{e^{-\delta \tau} N(d_1)} - K e^{-r\tau} N(d_2)$$

Replication $\quad C = \boxed{\Delta} S + B$

$$P = K e^{-r\tau} N(-d_2) - S e^{-\delta \tau} N(-d_1)$$

<u>by replication</u>

$$\Delta_{put} = - e^{-\delta \tau} N(-d_1)$$

<u>by put-call parity</u>

$$C - P = S e^{-\delta \tau} - K e^{-r\tau}$$

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = e^{-\delta \tau}$$

$$\Delta_{call} - \Delta_{put} = e^{-\delta \tau}$$

$$\Delta_{put} = - e^{-\delta \tau} N(-d_1)$$

<u>by direct calculation</u>

$$\frac{\partial P}{\partial S} = \Delta_{put}$$

Importance of Δ

1) replication

2) Relationship of V and S

3) Δ tells $\widetilde{\Pr}(S_T > K)$

Option Greeks (cont'd)

- As time to expiration increases, the call delta is lower at high stock prices and greater at low stock prices. This reflects the fact that for an option with greater time to expiration, the likelihood is greater that an out-of-the-money option will eventually become in-the-money, and the likelihood is greater that an in-the-money option will eventually become out-of-the-money.

Option Greeks (cont'd)

- The Black-Scholes formula tells us that the call delta is

$$\Delta = e^{-\delta T} N(d_1)$$

Δ put = ?

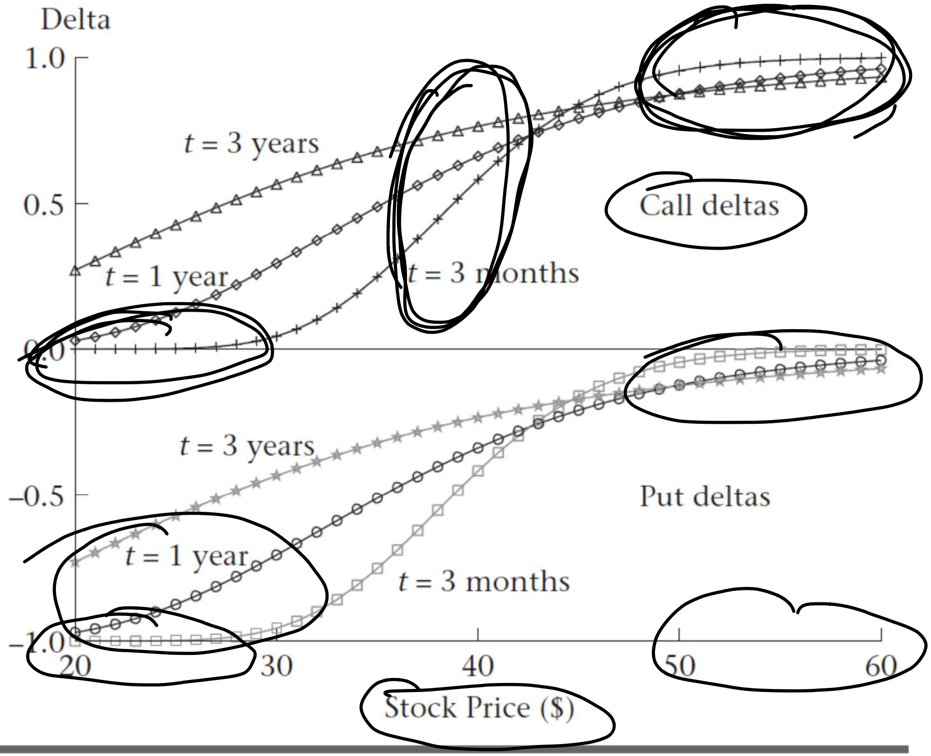
- The Black-Scholes formula also tells that the call can be synthetically created if we hold $e^{-\delta T}N(d_1)$ shares of stock and borrowing $Ke^{-rT}N(d_2)$ dollars.
- We see that delta changes with the stock price, so as the stock price moves, the replicating portfolio changes and must be adjusted dynamically.
- Delta for put option is negative, so a stock price increase reduces the put price.

Option Greeks (cont'd)

FIGURE 12.1

Call (top graph) and put (bottom graph) deltas for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.

t: time to expiration



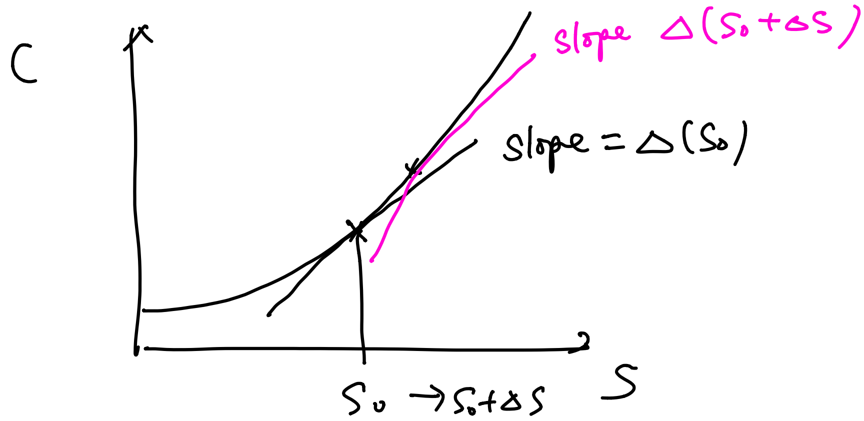
Option Greeks (cont'd)

- Gamma $\frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$
  - Gamma is always positive for a purchased call or put.
  - Because of put-call parity, gamma is the same for a European call and put with the same strike price and time to expiration.
  - Deep in-the-money options have a delta of about 1, and hence, a gamma of about 0. Similarly, deep out-of-money options have a delta of about 0 and hence, a gamma of about 0.

$$C - P = Se^{-\delta T} - Ke^{-rT}$$

$$\Gamma_{call} = \Gamma_{put}$$

option convexity



$\Delta(S_0+\delta S) > \Delta(S_0) \Rightarrow \Gamma > 0$

Option Greeks (cont'd)

- Vega $= \frac{\partial V}{\partial \sigma}$
  - Vega tends to be greater for at-the-money option, and greater for options with moderate than with short times to expiration. The behavior of vega can be different for very long-lived options.
  - Because of put-call parity, vega, like gamma, is the same for calls and puts with the same strike price and time to expiration.

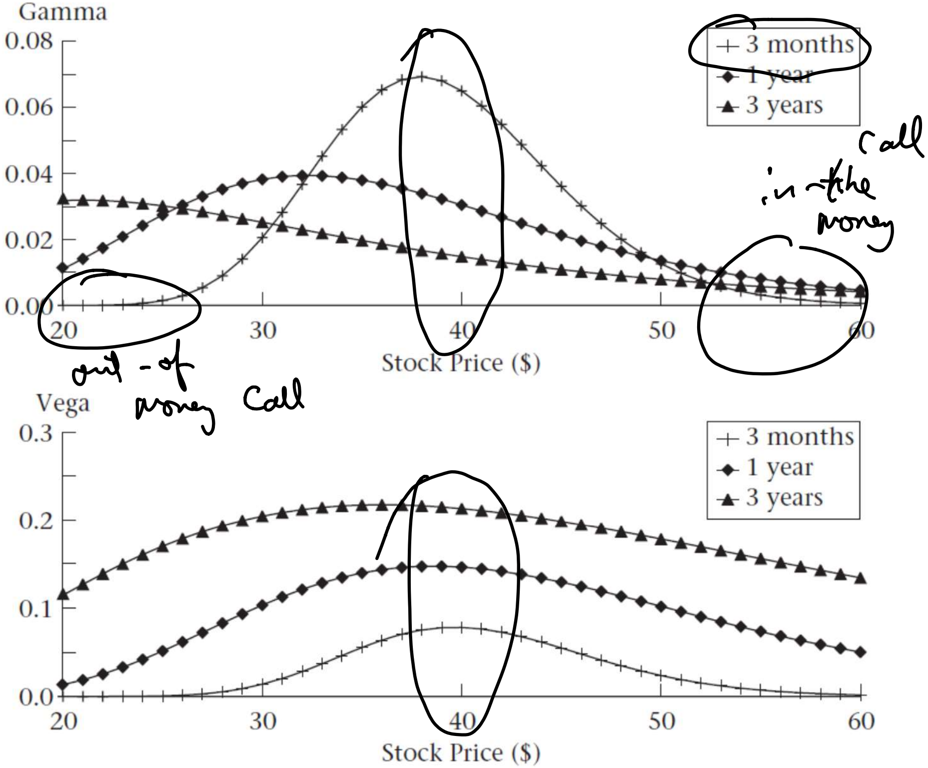
$$\frac{\partial S}{\partial \sigma} = 0 \qquad C - P = Se^{-r\tau} - Ke^{-r\tau}$$

$$\Rightarrow \nu_{call} = \nu_{put}$$

Option Greeks (cont'd)

FIGURE 12.2

Gamma (top panel) and vega (bottom panel) for 40-strike options with different times to expiration. Assumes $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$. Vega is the sensitivity of the option price to a 1 percentage point change in volatility. Otherwise identical calls and puts have the same gamma and vega.



Option Greeks (cont'd)

- Theta $\frac{\partial V}{\partial t}$
  - Options generally – but not always – become less valuable as time to expiration decreases.

Time to expiration $= T - t$

$$C = S e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\Theta_{call} = \frac{\partial C}{\partial t}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

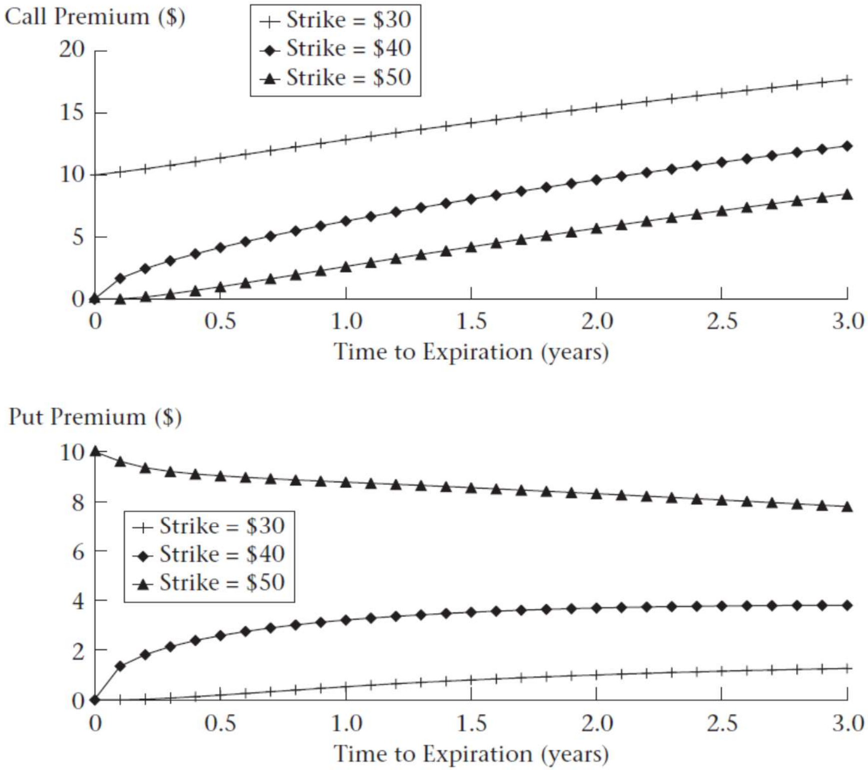
$$\Theta_{call} \neq \Theta_{put}$$

$$\frac{\partial}{\partial t}(C - P) = \frac{\partial}{\partial t}\left(Se^{-\delta(T-t)} - Ke^{-r(T-t)}\right)$$

Option Greeks (cont'd)

FIGURE 12.3

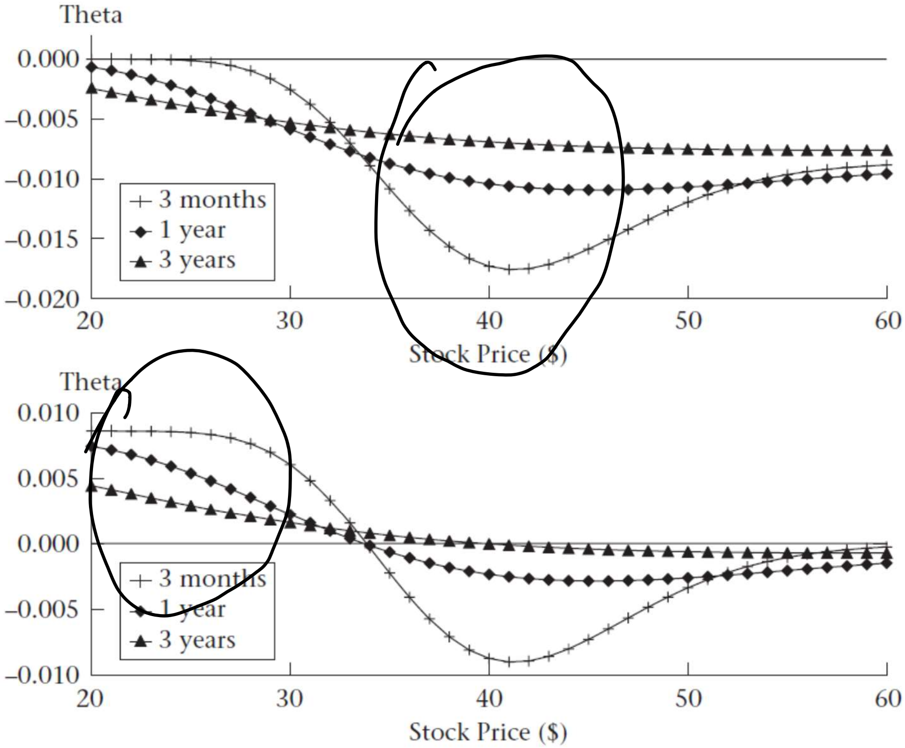
Call (top panel) and put (bottom panel) prices for options with different strikes at different times to expiration. Assumes $S = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Option Greeks (cont'd)

FIGURE 12.4

Theta for calls (top panel) and puts (bottom panel) with different expirations at different stock prices. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Option Greeks (cont'd)

- Rho $\frac{\partial V}{\partial r}$
  - Rho is positive for an ordinary stock call option. Exercising a call ~~entails~~ paying the fixed strike price to receive the stock; a higher interest rate reduces the present value of the strike.
  - Rho is negative for a put. The put entails the owner to receive cash, and the present value of this is lower with a higher interest rates.

r affects PV(?)

$\rho_{call} \overset{?}{=} \rho_{put}$

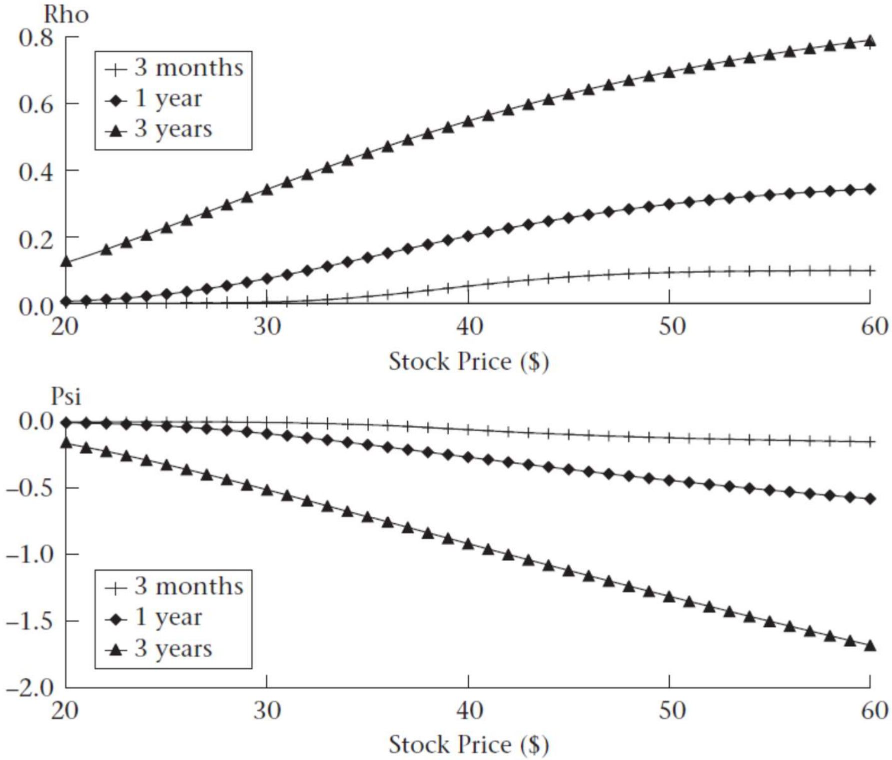
Option Greeks (cont'd)

- Psi $\frac{\partial V}{\partial \rho}$ $\phi_{call} \overset{?}{=} \phi_{put}$
  - Psi is negative for an ordinary stock call option and positive for a put.
  - A call entails the holder to receive the stock, but without receiving the dividends paid on the stock prior to exercising the option. Thus the present value of the stock to be received is lower, the greater the dividend yield.
  - Owning a put entails an obligation to deliver the stock in the future in exchange for cash. The present value of the stock to be delivered goes down when the dividend yield goes up, so the put is more valuable when the dividend is greater. Hence, psi for a put is positive.

Option Greeks (cont'd)

FIGURE 12.5

Rho (top panel) and psi (bottom panel) at different stock prices for call options with different maturities. Assumes $K = \$40$, $\sigma = 30\%$, $r = 8\%$, and $\delta = 0$.



Option Greeks (cont'd)

- Greek measures for portfolios
  - The Greek measure of a portfolio is the sum of the Greeks of the individual portfolio components.
  - For a portfolio containing N options with a single underlying stock, where the quantity of each option is given by n_i, we have

$$\text{Greek}_{\text{portfolio}} = \sum_{i=1}^{N} n_i \text{Greek}_i$$

Here, $n_i > 0$ for **long** position and $n_i < 0$ for **short** position.

$$\Delta_{port} = \frac{\partial\left(n_1 V_1 + n_2 V_2 + n_3 V_3 + \cdots + n_k V_k\right)}{\partial S}$$

$$= \sum_{i=1}^{k} n_i \frac{\partial V_i}{\partial S} = \sum_{i=1}^{n} n_i \Delta V_i$$

Option Greeks (cont'd)

TABLE 12.2 Greeks for a bull spread where $S = \$40$, $\sigma = 0.3$, $r = 0.08$, and $T = 91$ days, with a purchased 40-strike call and a written 45-strike call. The column titled "combined" is the difference between column 1 and column 2.

	40-Strike Call	45-Strike Call	Combined
ω_i	1	−1	—
Price	2.7804	0.9710	1.8094
Delta	0.5824	0.2815	0.3009
Gamma	0.0652	0.0563	0.0088
Vega	0.0780	0.0674	0.0106
Theta	−0.0173	−0.0134	−0.0040
Rho	0.0511	0.0257	0.0255

Option Greeks (cont'd)

- Option elasticity (Ω) Effective Gearing
  - Ω describes the risk of the option relative to the risk of the stock in percentage terms: If stock price (S) changes by 1%, what is the percent change in the value of the option (C)?
  - Dollar risk of the option

    If the stock price changes by ε, the change in the option price is

$$\text{Change in option price} = \text{Change in stock price} \times \text{option delta}$$
$$= \varepsilon \times \Delta$$

$$\Delta = \frac{\partial V}{\partial S}$$

% change of S

$\Rightarrow$? % change of V

Option Greeks (cont'd)

- Option elasticity (Ω)
  - The option elasticity (Ω) is given by

ε

$$\Omega = \frac{\%\text{ change in option price}}{\%\text{ change in stock price}} = \frac{\frac{\varepsilon\Delta}{C}}{\frac{\varepsilon}{S}} = \frac{S\Delta}{C}$$

$$\begin{array}{ccc} S & \longrightarrow & S+\varepsilon \\ \downarrow & & \downarrow \\ C & & C+\delta \end{array}$$

$$\underbrace{C \quad\quad C+\delta}_{\delta}$$

  - The elasticity tells us the percentage change in the option for a 1% change in the stock. It is effectively a measure of the leverage implicit in the option.

Recall $\Delta = \dfrac{\partial C}{\partial S}$

ε is small : change of $C = \Delta \varepsilon$

$$\% \text{ change of } C = \frac{\text{change of } C}{C}$$

$$= \frac{\Delta \varepsilon}{C}$$

Option Greeks (cont'd)

? (HW)

- For a call, $\Omega \geq 1$. A call option is replicated by a levered investment in the stock. A levered position in an asset is always riskier than the underlying asset. Also, the implicit leverage in the option becomes greater as the option is more out-of-the-money. Thus, Ω decreases as the strike price decreases.
- For a put, $\Omega \leq 0$. This occurs because the replicating position for a put option involves shorting the stock.

Option Greeks (cont'd)

Example

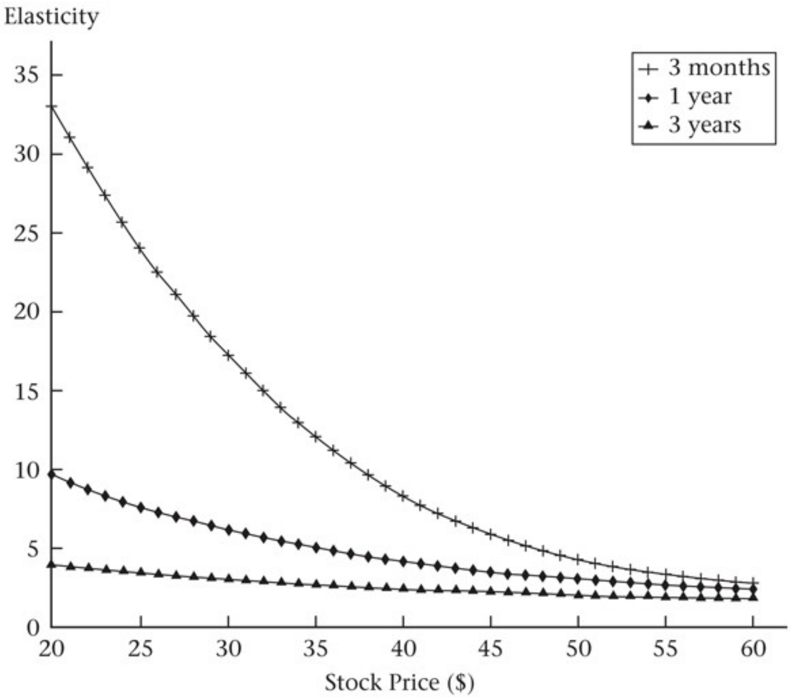
S = \$41, K = \$40, σ = 0.30, r = 0.08, T = 1,
δ = 0

- Elasticity for call: $\Omega = S\,\Delta / C$ = \$41 x 0.6911 / \$6.961 = 4.071.
- Elasticity for put: $\Omega = S\,\Delta / P$ = \$41 x (– 0.3089) / \$2.886 = – 4.389.

Option Greeks (cont'd)

FIGURE 12.6

Elasticity for a call option for different stock prices and times to expiration. Assumes $K = \$40$, $\sigma = 0.3$, $r = 0.08$, and $\delta = 0$.



Option Greeks (cont'd)

- Option elasticity (Ω) (cont'd)
  - The volatility of an option $\sigma_{option} = \sigma_{stock} \times |\Omega|$
  - *The risk premium of an option*

    At a point in time, the option is equivalent to a position in the stock and in bonds; hence, *<u>under discrete compounding</u>*, the return on the option is a weighted average of the return on the stock and the risk-free rate.

Option Greeks (cont'd)

Let α denote the expected rate of return on the stock, γ the expected return on the option, and r the risk-free rate. We have

$$\gamma = \frac{\Delta S}{C(S)}\alpha + \left(1-\frac{\Delta S}{C(S)}\right) r = \Omega\alpha + (1-\Omega) r$$

risk prem of option

$$\gamma - r = (\alpha - r)\times\Omega$$

risk-premium of stock

Although the above derivation is based on the assumption of discrete compounding, the conclusion about the the risk premium $\gamma - r$ on the option equals the risk premium on the stock times Ω is also true for continuous compounding (need to use the knowledge of Ito's formula).

$$\text{return}_{port} = \sum_{i=1}^{n} w_i \, \text{return}_i$$

Option Greeks (cont'd)

In the capital asset pricing model (CAPM), the beta of an asset is proportional to the risk premium. Thus, since the risk premium scales with elasticity, so does beta:

Risk prem of Asset i $\boxed{E[R_i] - R_f}$

$= \beta_i \, [E[R_m] - R_f]$

$$\beta_{\text{Option}} = \beta_{\text{Asset}} \times \Omega$$

In terms of the CAPM, we would say that the option beta goes down as the option becomes more in-the-money. For puts, we conclude that if the stock risk premium is positive, the put has an expected return less than that of the stock.

Risk prem of portfolio

Option Greeks (cont'd)

$\frac{\text{risk premi}}{\text{risk}}$

– *The Sharpe Ratio of an Option*

$$\text{Sharpe ratio for call} = \frac{\Omega(\alpha - r)}{\Omega\sigma} = \boxed{\frac{\alpha - r}{\sigma}}$$

Thus, the Sharpe ratio for a call equals the Sharpe ratio for the underlying asset.

This equivalence of the Sharpe ratios is obvious once we realize that the option is always equivalent to a levered position in the stock, and that leverage per se does not change the Sharpe ratio.

Sharpe ratio = return per unit of risk

Option Greeks (cont'd)

– *The Elasticity and Risk Premium of a Portfolio*

Suppose there is a portfolio of N calls with the same underlying stock, where the ith call has value C_i and delta Δ_i, and where n_i is the quantity of the ith call.

$$\Omega_{\text{portfolio}} = \frac{\dfrac{\sum_{i=1}^{N} n_i \Delta_i}{\sum_{j=1}^{N} n_j C_j}}{1/S} = \sum_{i=1}^{N} \left(\frac{n_i C_i}{\sum_{j=1}^{N} n_j C_j} \right) \frac{S\Delta_i}{C_i} = \sum_{i=1}^{N} \omega_i \Omega_i$$

$$\Omega_{port} = \frac{\%\text{ change of portfolio}}{\%\text{ change of } S} = \frac{\dfrac{\sum_{i=1}^{N} n_i \, \varepsilon \Delta_i}{\sum_{i=1}^{N} n_i C_i}}{\varepsilon / S}$$

Option Greeks (cont'd)

The risk premium of the portfolio is given by

$$\gamma - r = \Omega_{\text{portfolio}}\left(\alpha - r\right)$$

Profit Diagrams Before Maturity

- For purchased call option

TABLE 12.3 Value of 40-strike call option at different stock prices and times to expiration. Assumes $r = 8\%$, $\sigma = 30\%$, $\delta = 0$.

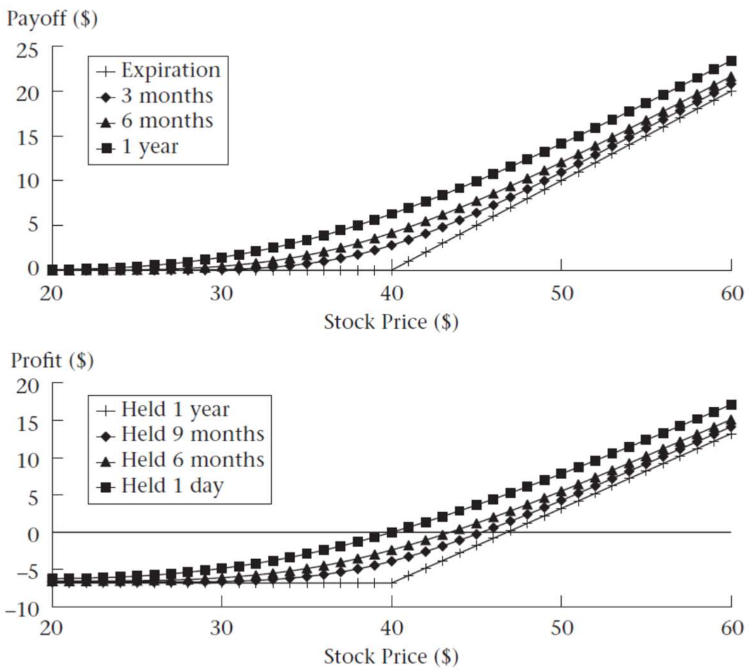
Stock Price ($)	Time to Expiration 12 Months	6 Months	3 Months	0 (Expiration)
36	3.90	2.08	1.00	0
38	5.02	3.02	1.75	0
40	6.28	4.16	2.78	0
42	7.67	5.47	4.07	2
44	9.15	6.95	5.58	4

Profit Diagrams Before Maturity (cont'd)

- For purchased call option

FIGURE 12.7

Payoff and profit diagram for a purchased call option. Top panel shows payoff diagrams for options with different remaining times to expiration. Bottom panel shows profit diagrams for a one-year call bought when the stock price was $40 and then held for different lengths of time. Assumes $K = \$40$, $r = 8\%$, $\delta = 0$, and $\sigma = 30\%$.



Profit Diagrams Before Maturity (cont'd)

- For purchased call option
  - The figure shows that the value of the option prior to expiration is a smoothed version of the value of the option at expiration.
  - The profit of the options are computed by subtracting from the value of the option at each stock price the original cost of the position, plus interest.

Profit Diagrams Before Maturity (cont'd)

- For purchased call option

Example

The 1-year option in the table costs $6.285 at a stock price of $40.

After 6 months, the holding period profit at a price of $40 would be

$$\$4.16 - \$6.285e^{0.08\times 0.5} = -\$2.386$$

Implied Volatility

- Volatility is unobservable.
- Option prices, particularly for near-the-money options, can be quite sensitive to volatility.
- One approach is to compute **historical volatility** using the history of returns.
- A problem with historical volatility is that expected future volatility can be different from historical volatility.
- Alternatively, we can calculate **implied volatility**, which is the volatility that, when put into a pricing formula (typically Black-Scholes), yields the observed option price.

$$C = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) = f(\sigma)$$

$\Uparrow$

from the market

Given $C = 10.5$

Solve $f(\sigma) = 10.5$

$\Rightarrow \hat{\sigma} = ?$

IV

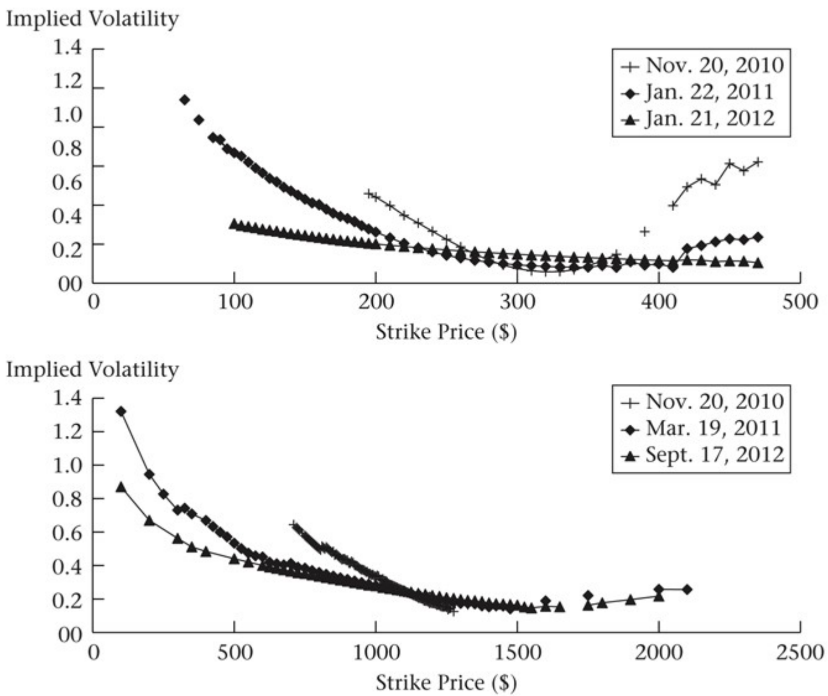
Historical volatility = Realized Vol.

Implied Volatility (cont'd)

- In practice implied volatilities of in-, at-, and out-of-the money options are generally different
  - A **volatility smile** refers to when volatility is symmetric, with volatility lowest for at-the-money options, and high for in-the-money and out-of-the-money options.
  - A difference in volatilities between in-the-money and out-of-the-money options is referred to as a **volatility skew.**

FIGURE 12.9

Implied put volatilities for Apple and the S&P 500 on October 27, 2010. The top panel shows implied volatility curves for Apple and the bottom panel shows the same for the S&P 500 index, each for three different maturities. Closing prices for Apple and the S&P 500 were 307.83 and 1182.45.



Data from OptionMetrics.

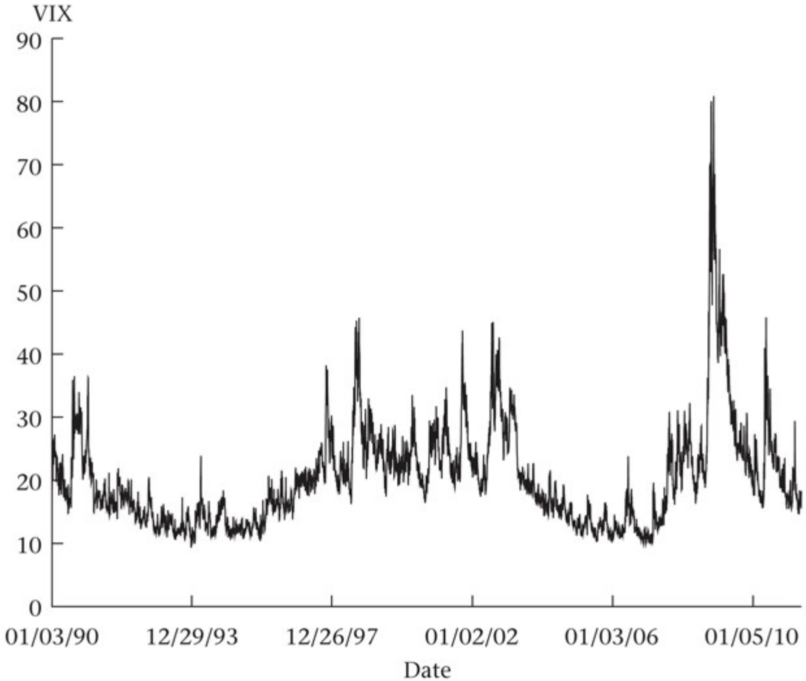
Implied Volatility (cont'd)

- In addition to computing implied volatility using the Black-Scholes, there is also a so-called model-free calculation that is the basis of the S&P 500 VIX volatility index, known as the VIX.
- The VIX calculation uses option prices for all available strike prices to compute a single implied volatility.

Implied Volatility (cont'd)

FIGURE 12.10

S&P 500 implied volatility (VIX) from January 1990 to May 2011.



Data from Yahoo.

Implied Volatility (cont'd)

- Some practical uses of implied volatility include
  - Use the implied volatility from an option with an observable price to calculate the price of another option on the same underlying asset.
  - Use implied volatility as a quick way to describe the level of options prices on a given underlying asset: you could quote option prices in terms of volatility, rather than as a dollar price.
  - Checking the uniformity of implied volatilities across various options on the same underlying assets allows one to verify the validity of the pricing model in pricing those options.