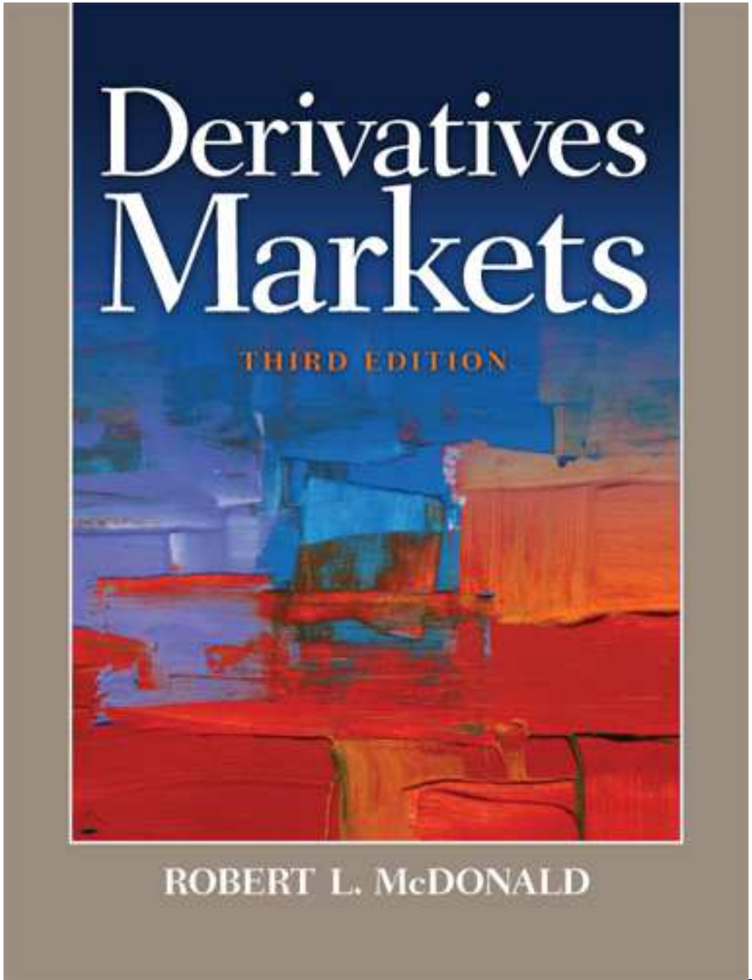


Chapter 13
(Chapter 20 in the textbook)

Brownian Motion and Itô’s Lemma

Points to Note

1. Definition of the Standard Brownian motion. See P.3 – 4.
2. Stochastic Processes which are constructed from the standard Brownian motion. See P.5 – 11.

   1 Arithmetic BM
   2 OU process
   3 GBM

3. Modelling the correlated asset prices through correlated Brownian motions. See P.12 – 16.
4. Ito's lemma: univariate and multivariate versions. See P.17 – 28.
5. Sharpe ratios of two perfectly correlated assets. See P.29 – 31.

Brownian Motion

- A *stochastic process* is a random process that is a function of time.
- Brownian motion is a stochastic process that evolves in continuous time, with movements that are continuous.
  - A random walk can be generated by flipping a coin each period and moving one step, with the direction determined by whether the coin is heads or tails.
  - To generate Brownian motion, we would flip the coins infinitely fast and take infinitesimally small steps at each point.

Brownian Motion (cont'd)

$Z(t) \sim N(0,t)$

- Brownian motion is a continuous stochastic process, $Z(t)$, with the following characteristics:
  - $Z(0) = 0$.
  - $Z(t + s) - Z(t)$ is normally distributed with mean 0 and variance s.
  - $Z(t + s_1) - Z(t)$ is independent of $Z(t) - Z(t - s_2)$, where $s_1, s_2 > 0$. That is, nonoverlapping increments are independently distributed.
  - $Z(t)$ is continuous.

Standard BM

- These properties imply that $Z(t)$ is a martingale: a stochastic process for which

$$E[Z(t+s) \mid \{Z(u),\ 0 \le u \le t\}] = Z(t).$$

Arithmetic Brownian Motion

- With pure Brownian motion, the expected change in Z is 0, and the variance per unit time is 1. We can generalize this to allow an arbitrary variance and a nonzero mean stochastic differential Equation (SDE)

$$dX(t) = \alpha\, dt + \sigma\, dZ(t) \tag{20.8}$$

- This process is called **arithmetic Brownian motion**
  - α is the instantaneous mean per unit time.
  - σ^2 is the instantaneous variance per unit time.
  - The variable $X(t)$ is the sum of the individual changes dX. $X(t)$ is normally distributed, i.e., $X(T) - X(0) \sim N(\alpha T, \sigma^2 T)$.

$$dX(t) = \alpha\, dt + \sigma\, dZ(t)$$

$$\int_0^t dX(s) = \int_0^t \alpha\, ds + \int_0^t \sigma\, dZ(s)$$

0

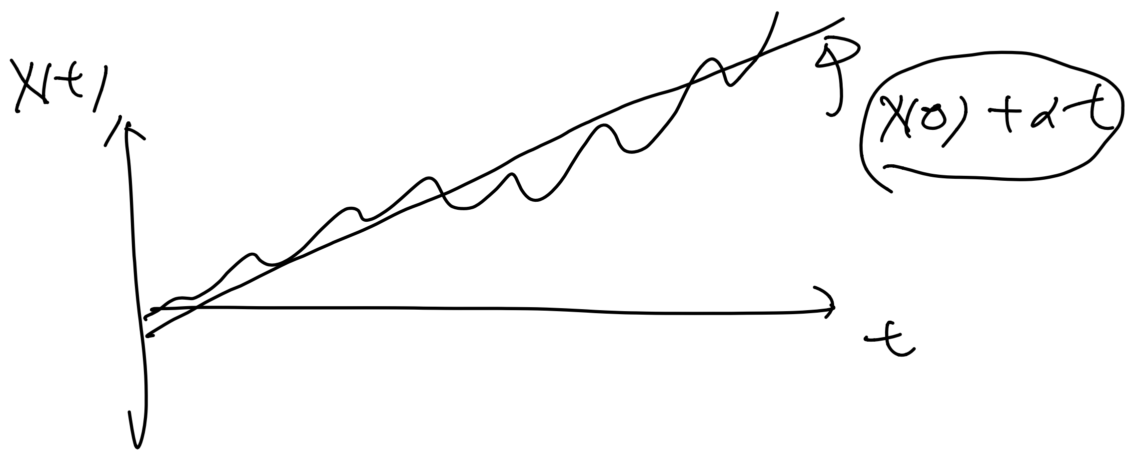
$$X(t) - X(0) = \alpha t + \sigma Z(t) - \cancel{\sigma Z(0)}$$

$$X(t) = X(0) + \alpha t + \sigma Z(t)$$

$$E[X(t)] = X(0) + \alpha t$$

$$\mathrm{Var}(X(t)) = \sigma^2 t \qquad Z(t) \sim N(0, t)$$

$$X(t) \sim N(X(0) + \alpha t,\ \sigma^2 t)$$



Arithmetic Brownian Motion (cont'd)

- An integral representation of equation (20.8) is

$$X(T) = X(0) + \int_0^T \alpha \, dt + \int_0^T \sigma \, dZ(t)$$

- Here are some properties of the process in equation (20.8)
  - $X(t)$ is normally distributed because it is a scaled Brownian process.
  - The random term is multiplied by a scale factor that enables us to specify the variance of the process.
  - The αdt term introduces a nonrandom *drift* into the process.

Arithmetic Brownian Motion (cont'd)

- Arithmetic Brownian motion has several drawbacks
  - There is nothing to prevent X from becoming negative, so it is a poor model for stock prices.
  - The mean and variance of changes in dollar terms are independent of the level of the stock price.
- Both of these criticisms will be eliminated with geometric Brownian motion.

The Ornstein-Uhlenbeck Process

(OU process)

- We can incorporate mean reversion by modifying the drift term

$$dX(t) = \lambda[a - X(t)]\,dt + \sigma\,dZ(t) \qquad (20.9)$$

- This equation is called an **Ornstein-Uhlenbeck process.**
  - The parameter λ measures the speed of the reversion: If λ is large, reversion happens more quickly.
  - In the long run, we expect *X* to revert toward α.
  - As with arithmetic Brownian motion, *X* can still become negative.

$$dX(t) = \lambda(a - X(t))\,dt + \sigma\,dZ(t)$$

$\lambda > 0$

If $X(t) < a$ $\quad\quad$ $\lambda(a - X(t)) > 0$

$\Rightarrow$ $X(t)$ moves up

If $X(t) > a$ $\quad\quad$ $\lambda(a - X(t)) < 0$

$\Rightarrow$ $X(t)$ moves down

Geometric Brownian Motion

- An equation, in which the drift and volatility depend on the stock price, is called an **Itô process**.
  - Suppose we modify arithmetic Brownian motion to make the instantaneous mean and standard deviation proportional to $X(t)$

$$dX(t) = \alpha\, X(t)\, dt + \sigma\, X(t) dZ(t)$$

  - This is an Itô process that can also be written

Instantaneous % return stock

drift volatility

$$\frac{dX(t)}{X(t)} = \alpha\, dt + \sigma\, dZ(t) \qquad (20.11)$$

  - This process is known as **geometric Brownian motion (GBM)**.

Geometric Brownian Motion (cont'd)

- The percentage change in the asset value is normally distributed with instantaneous mean α and instantaneous variance σ^2.

- The integral representation for equation (20.11) is

$$X(T) - X(0) = \int_0^T \alpha\, X(t)dt + \int_0^T \sigma\, X(t)dZ(t)$$

Multiplication Rules

- We can simplify complex terms containing *dt* and *dZ* by using the following “multiplication rules”:

$$dt \times dZ = 0 \tag{20.15a}$$

$$(dt)^2 = 0 \tag{20.15b}$$

$$(dZ)^2 = dt \tag{20.15c}$$

Modeling Correlated Asset Prices

- Suppose that we have m asset processes

$$\frac{dX_i}{X_i} = (\alpha_i - \delta_i)dt + \sigma_i dZ_i \qquad i = 1, \ldots, m$$

The correlation between X_i and X_j will be generated by correlation between $Z_i(t)$ and $Z_j(t)$.

Next, we illustrate how we can create *correlated* diffusion processes by expressing dZ_i and dZ_j as *sums* of *independent* diffusions.

2 Assets:

$$\frac{dX_1(t)}{X_1(t)} = \alpha_1\, dt + \sigma_1\, dZ_1(t)$$

$$\frac{dX_2(t)}{X_2(t)} = \alpha_2\, dt + \sigma_2\, dZ_2(t)$$

$Z_1(t)$ standard BM
$Z_2(t)$ standard BM
} correlated (flexible correlation)

Consider

$$\frac{dX_1(t)}{X_1(t)} = \alpha_1 dt + \sigma_1 dZ(t)$$

$$\frac{dX_2(t)}{X_2(t)} = \alpha_2 dt + \sigma_2 dZ(t)$$

$\mathrm{Corr}(X_1(t), X_2(t)) = 1.$

perfectly correlated.

Given $Z_1(t)$, $Z_2(t)$ are correlated,

Could we find $W_1(t)$ and $W_2(t)$ which are independent BMs to represent $Z_1(t)$ and $Z_2(t)$?

Consider

$$dZ_1(t) = dW_1(t)$$

$$dZ_2(t) = \rho\, dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)$$

$$dZ_1(t) \cdot dZ_2(t) = [dW_1(t)]\,[\rho\, dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)]$$

$$= \rho\,(dW_1(t))^2 + \sqrt{1-\rho^2}\, dW_1(t)\, dW_2(t)$$

Multiplication rules.

$$dW_1(t) \cdot dW_1(t) = dt$$

$$dZ_1(t) \cdot dZ_2(t) = \rho\, dt + \sqrt{1-\rho^2}\, dW_1(t)\, dW_2(t)$$

$$E[dW_1(t)\, dW_2(t)] = E[dW_1(t)]\, E[dW_2(t)] = 0$$

$$\mathrm{Var}[dW_1(t)\, dW_2(t)] = E[(dW_1(t)\, dW_2(t))^2] - (E[dW_1(t)\, dW_2(t)])^2 = 0$$

$$E\left[\left(dW_1(t)\, dW_2(t)\right)^2\right]$$

$$= E\left[\left(dW_1(t)\right)^2\right] E\left[\left(dW_2(t)\right)^2\right]$$

$$= (dt)(dt)$$

$$= (dt)^2.$$

$$dW_1(t) \approx W_1(t+dt) - W_1(t) \sim N(0, dt)$$

$$\Rightarrow \mathrm{Var}(dW_1(t) \cdot dW_2(t)) = (dt)^2 \longrightarrow 0 \text{ as } dt \to 0$$

$$E[dW_1(t)\, dW_2(t)] = E[dW_1(t)]\, E[dW_2(t)]$$

$$= 0$$

$$\Rightarrow dW_1(t) \cdot dW_2(t) = 0$$

$$dZ_1(t)\, dZ_2(t) = \rho\, dt$$

Example

$$\frac{dX_1(t)}{X_1(t)} = 0.05\, dt + 0.8\, dZ_1(t)$$

$$\frac{dX_2(t)}{X_2(t)} = 0.04\, dt + 0.3\, dZ_2(t)$$

$$dZ_1(t)\, dZ_2(t) = 0.3\, dt$$

Let $W_1(t)$, $W_2(t)$ be two independent standard BMs

$$dZ_1(t) = dW_1(t)$$

$$dZ_2(t) = 0.3\, dW_1(t) + \sqrt{1-(0.3)^2}\, dW_2(t)$$

$$\frac{dX_1(t)}{X_1(t)} = 0.05\,dt + 0.8\, dW_1(t)$$

$$\frac{dX_2(t)}{X_2(t)} = 0.04\,dt + (0.3)\left[0.3\, dW_1(t) + \sqrt{1-(0.3)^2}\, dW_2(t)\right]$$

$$= 0.04\,dt + 0.09\, dW_1(t) + (0.3)\sqrt{1-(0.3)^2}\, dW_2(t)$$

Modeling Correlated Asset Prices (cont'd)

$Cov(dZ_1(t), dZ_2(t))$

- With m = 2 as an illustration:

Let $W_1(t)$ and $W_2(t)$ be *independent* Brownian motions and define

$$dZ_1(t) = dW_1(t)$$

$$dZ_2(t) = \rho dW_1(t) + \sqrt{1-\rho^2}\, dW_2(t)$$

$dZ_1(t)\, dZ_2(t)$

$= \rho dt$

  - This is the Cholesky decomposition

Consider

$$dZ_1(t)dZ_2(t) = \rho dW_1(t)^2 + \sqrt{1-\rho^2}\, dW_1(t)dW_2(t)$$

$$= \rho dt + \sqrt{1-\rho^2}\, dW_1(t)dW_2(t)$$

Modeling Correlated Asset Prices (cont'd)

The independence of $W_1(t)$ and $W_2(t)$ implies that

$$E_t\{[W_1(t+s)-W_1(t)][W_2(t+s)-W_2(t)]\}=0$$

Using the differential notation, we can write

$$dW_1(t)\times dW_2(t) = 0$$

Therefore,

$$dZ_1(t)dZ_2(t)=\rho dt$$

Modeling Correlated Asset Prices (cont'd)

In general, we can construct dZ_i, i = 1, …, n, as follows:

$$dZ_i(t) = \sum_{k=1}^{n} \lambda_{i,k} dW_k(t)$$

where we scale the coefficients so that

$$\sum_{k=1}^{n} \lambda_{i,k}^2 = 1$$

Modeling Correlated Asset Prices (cont'd)

Because the Brownian increments are jointly-normally distributed, their sum is normal. We also have

$$Var\left[dZ_i(t)\right] = Var\left(\sum_{k=1}^{n} \lambda_{i,k} dW_k(t)\right) = \sum_{k=1}^{n} \lambda_{i,k}^2 dW_k(t)^2 = dt$$

$$dZ_i(t) dZ_j(t) = \sum_{k=1}^{n} \lambda_{i,k} dW_k(t) \sum_{k=1}^{n} \lambda_{j,k} dW_k(t)$$

$$= \sum_{k=1}^{n} \lambda_{i,k} \lambda_{j,k} dt = \rho_{i,j} dt$$

where

$$\rho_{i,j} = \sum_{k=1}^{n} \lambda_{i,k} \lambda_{j,k}$$

Itô's Lemma

- Suppose that the stock price, $S(t)$, follows the Itô process given by

$$dS(t) = \left[\hat{\alpha}\left[S(t), t\right] - \hat{\delta}\left[S(t), t\right]\right]dt + \hat{\sigma}\left[S(t), t\right]dZ(t)$$

- In this equation, the expected return, α, the dividend yield, δ, and the volatility, σ, can be functions of the stock price and time.

- If

$$\hat{\alpha}\left[S(t), t\right] = \alpha S(t), \quad \hat{\delta}\left[S(t), t\right] = \delta S(t), \quad \hat{\sigma}\left[S(t), t\right] = \sigma S(t),$$

then $S(t)$ follows geometric Brownian motion.

$$dS(t) = \alpha(S(t),t)\,dt + \sigma(S(t),t)\,dZ(t)$$

Eg.

$$\alpha(S(t),t) = \alpha S(t) \quad , \quad \sigma(S(t),t) = \sigma S(t)$$

$\Rightarrow$ GBM.

Given $C(S(t),t)$

Want to find the SDE of $C(S(t),t)$

Taylor series of $C(S(t),t)$

$$C(S(t)+dS(t),\ t+dt) - C(S(t),t)$$

$$= \boxed{\frac{\partial C}{\partial S}(dS(t))} + \boxed{\frac{\partial C}{\partial t}dt} + \boxed{\frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS(t))^2}$$

$$+ \cancel{\frac{1}{2}\left(\frac{\partial^2 C}{\partial t^2}\right)(dt)^2} + \boxed{\frac{\partial^2 C}{\partial S\partial t}\cancel{(dS(t))(dt)}}$$

only keep the terms up to "dt"

$$= \frac{\partial C}{\partial S}\left(\alpha(S(t),t)\,dt + \sigma(S(t),t)\,dZ(t)\right)$$

$$dZ(t) \sim N(0, dt)$$

$$+ \frac{\partial C}{\partial t}\,dt$$

$$+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\boxed{\left(\alpha(S(t),t)\,dt + \sigma(S(t),t)\,dZ(t)\right)^2}$$

$$+ \frac{\partial^2 C}{\partial S \partial t}\left(\alpha(S(t),t)\,\cancel{dt} + \sigma(S(t),t)\,\cancel{dZ(t)}\right)dt$$

$$\rightarrow \cancel{\alpha^2 dt^2} + \cancel{2\alpha\sigma\,dt\,dZ} + \underline{\sigma^2 (dZ(t))^2}$$

$$= \sigma^2 dt$$

Examples

Solve

Given $\frac{dX(t)}{X(t)} = \alpha\, dt + \sigma\, dZ(t)$

Find $d \ln X(t)$?

$C(X(t), t) = \ln X(t)$

$$dC = \frac{\partial C}{\partial X} dX + \frac{\partial C}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C}{\partial X^2}(dX)^2$$

$$\frac{\partial C}{\partial X} = \frac{\partial \ln X(t)}{\partial X} = \frac{1}{X(t)}$$

$$\frac{\partial C}{\partial t} = \frac{\partial \ln X(t)}{\partial t} = 0$$

$$\frac{\partial^2 C}{\partial X^2} = \frac{\partial^2 \ln X(t)}{\partial X^2} = -\frac{1}{X^2(t)}$$

$$dC = \frac{1}{X(t)} dX(t) + \frac{1}{2}\left(-\frac{1}{X^2(t)}\right) X^2(t)\left[\alpha\, dt + \sigma\, dZ(t)\right]^2$$

$$= \left[\alpha\, dt + \sigma\, dZ(t)\right] - \frac{1}{2}\left[\alpha\, dt + \sigma\, dZ(t)\right]^2$$

$$= \alpha\, dt + \sigma\, dZ(t) - \frac{1}{2}\sigma^2 dt$$

$$\boxed{dC = \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma\, dZ(t)}$$

$$C(t) = C(0) + (\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)$$

$$\ln X(t) = \ln X(0) + (\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)$$

$$X(t) = X(0)\, e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma Z(t)}$$

$$= \sigma\sqrt{t}\, Z$$

$$Z \sim N(0,1)$$

$$S_t = S_0 e^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}$$

$$\Leftrightarrow \quad \frac{dS(t)}{S(t)} = (\alpha - \delta)dt + \sigma\, dZ(t)$$

Itô's Lemma (cont'd)

- $C[S(t), t]$ is the value of a derivative claim that is a function of the stock price.
- How can we describe the behavior of this claim in terms of the behavior of S?

Itô's Lemma (cont'd)

- **Itô's Lemma (Proposition 20.1)**
  - If $C[S(t), t]$ is a twice-differentiable function of $S(t)$, then the change in C is

$$dC(S,\ t) = C_S dS + \frac{1}{2} C_{SS} (dS)^2 + C_t dt$$

$$= \left[(\hat{\alpha}(S,t) - \hat{\delta}(S,t))C_S + \frac{1}{2}\hat{\sigma}(S,t)^2 C_{SS} + C_t \right] dt + \sigma(S,t) C_S dZ$$

    - where $C_S = \partial C/\partial S$, $C_{SS} = \partial^2 C/\partial S^2$, and $C_t = \partial C/\partial t$
  - The terms in square brackets are the expected change in the option price.

Itô’s Lemma (cont’d)

- **Proof (Proposition 20.1)**

  Proposition 20.1 can be proved by applying Itô’s lemma and the multiplication rule successively.

Itô's Lemma (cont'd)

- In the case where $S(t)$ follows geometric Brownian motion, we have

$$dC(S,t) = \left[(\alpha - \delta) SC_S + \frac{1}{2}\sigma^2 S^2 C_{SS} + C_t \right] dt + \sigma S C_S dZ$$

Itô's Lemma (cont'd)

Example (The Black-Scholes Assumption of Stock Prices)

The expression for a lognormal stock price is

$$S(t) = S(0)e^{(\alpha-\delta-0.5\sigma^2)t+\sigma Z(t)}$$

The stock price is a function of the Brownian process $Z(t)$. We can use Itô's Lemma to characterize the behavior of the stock as a function of $Z(t)$. We have

$$\frac{\partial S(t)}{\partial t} = \left(\alpha - \delta - \frac{1}{2}\sigma^2\right)S(t); \quad \frac{\partial S(t)}{\partial Z(t)} = \sigma S(t); \quad \frac{\partial^2 S(t)}{\partial Z(t)^2} = \sigma^2 S(t)$$

Itô's Lemma (cont'd)

Itô's Lemma states that $dS(t)$ is given as

$$dS(t) = \frac{\partial S(t)}{\partial t}dt + \frac{\partial S(t)}{\partial Z(t)}dZ(t) + \frac{1}{2}\frac{\partial^2 S(t)}{\partial Z(t)^2}\left[dZ(t)\right]^2$$
$$= (\alpha - \delta)S(t)dt + \sigma S(t)dZ(t)$$

This calculation demonstrates that a variable that follows geometric Brownian motion is lognormally distributed.

Itô's Lemma (cont'd)

Example

Let $Y(t)$ = ln[$S(t)$]. Then

$$d\ln\left[S(t)\right] = \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{dS(t)^2}{S(t)^2} = \frac{dS(t)}{S(t)} - \frac{1}{2}\sigma^2 dt$$

This implies that continuously compounded returns – measured as ln[$S(T)/S(0)$] – are lower than the instantaneous return, $\alpha - \delta$, by the factor $0.5\sigma^2$.

Multivariate Itô's Lemma

- A derivative may have a value depending on more than one price, in which case we can use a multivariate generalization of Itô's Lemma
- **Multivariate Itô's Lemma (Proposition 20.2)**
  - Suppose we have *n* correlated Itô processes

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i dt + \sigma_i dZ_i, \quad i = 1, \ldots, n$$

    - Denote the pairwise correlations as

$$dZ_i \times dZ_j = \rho_{i,j} dt$$

Multivariate Itô's Lemma (cont'd)

- If $C(S_1, \ldots, S_n, t)$ is a twice-differentiable function of the S_i's, we have

$$dC(S_1, \ldots, S_n, t) = \sum_{i=1}^{n} C_{S_i} dS_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} dS_i dS_j C_{S_i S_j} + C_t dt$$

1st order

2nd order term

$\frac{\partial}{\partial t}$

multiplication rule

1) $(dt)^2 = 0$

2) $dt\, dZ_i = 0$

3) $(dZ_i(t))^2 = dt$

4) $dZ_i(t)\, dZ_j(t) = \rho_{ij} dt$

$C(S_1(t), S_2(t), t)$

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_1} dS_1 + \frac{\partial C}{\partial S_2} dS_2$$

$$+ \frac{1}{2} \frac{\partial^2 C}{\partial S_1^2} (dS_1)^2 + \frac{1}{2} \frac{\partial^2 C}{(\partial S_2)^2} (dS_2)^2$$

$$+ \frac{\partial^2 C}{\partial S_1 \partial S_2} (dS_1)(dS_2)$$

Multivariate Itô's Lemma (cont'd)

$\frac{\partial C}{\partial S_1}$ $\frac{\partial C}{\partial S_2}$

Example

Suppose $C(S_1, S_2) = S_1/S_2$. Then by Itô's Lemma we have

$$d\left(\frac{S_1}{S_2}\right) = dS_1\left(\frac{1}{S_2}\right) - dS_2\left(\frac{S_1}{S_2^2}\right) + 0.5\left[2(dS_2)^2\frac{S_1}{S_2^3} - 2dS_1dS_2\frac{1}{S_2^2}\right]$$

$$d\left(\frac{S_1}{S_2}\right)\frac{S_2}{S_1} = \left(\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)dt + \sigma_1 dZ_1 - \sigma_2 dZ_2$$

$\frac{\partial^2 C}{\partial S_1 \partial S_2}$

$\frac{\partial^2 C}{\partial S_1^2}$ $\frac{\partial^2 C}{\partial S_2^2}$

Multivariate Itô's Lemma (cont'd)

From the earlier discussion of correlated Itô's processes, we have

$$d\left(\frac{S_1}{S_2}\right)\frac{S_2}{S_1} = \left(\alpha_1 - \alpha_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)dt + \hat{\sigma}\, dZ$$

where

$$\hat{\sigma} = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \quad \text{and} \quad dZ = \left(\sigma_1 dZ_1 - \sigma_2 dZ_2\right)/\,\hat{\sigma}.$$

Remark:

Even if S_1 and S_2 have equal drifts (i.e., $\alpha_1 = \alpha_2$), the ratio of S_1 and S_2 will have generally have zero drift.

The Sharpe Ratio

- If asset i has (total) expected return α_i, the risk premium is defined as

$$\text{Risk premium}_i = \alpha_i - r$$

  – where r is the risk-free rate.

- The **Sharpe ratio** for asset i is the risk premium, $\alpha_i - r$, per unit of volatility, σ_i

$$\text{Sharpe ratio}_i = \frac{\alpha_i - r}{\sigma_i} \qquad (20.25)$$

Excess return per unit of risk

The Sharpe Ratio (cont'd)

- We can use the Sharpe ratio to compare two perfectly correlated claims, such as a derivative and its underlying asset.
- Two assets that are perfectly correlated must have the same Sharpe ratio, or else there will be an arbitrage opportunity.
  - Consider the processes for two non-dividend paying stocks

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ \qquad (20.26)$$

$$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ \qquad (20.27)$$

  - Because the two stock prices are driven by the same *dZ*, it must be the case that

$$(\alpha_1 - r)/\sigma_1 = (\alpha_2 - r)/\sigma_2$$

Suppose

$$\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2} \quad X$$

Buy 1 Sell 2

The Sharpe Ratio (cont'd)

- The arbitrage is straightforward. Suppose that the Sharpe ratio of asset 1 is greater than that of asset 2. We then
  - Buy $1/(\sigma_1 S_1)$ shares of asset 1. $\frac{1}{\sigma_1 S_1} S_1$ pay $\frac{1}{\sigma_1}$
  - Short $1/(\sigma_2 S_2)$ shares of asset 2. $-\frac{1}{\sigma_2 S_2} S_2$ income $-\frac{1}{\sigma_2}$
  - Invest (or borrow) $1/\sigma_2 - 1/\sigma_1$, by buying (or borrowing) the risk-free bond, which has the rate of return *rdt*.

- The return of the above portfolio is

$$\boxed{\frac{1}{\sigma_1 S_1} dS_1 - \frac{1}{\sigma_2 S_2} dS_2 + \left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right) rdt} = \boxed{\left(\frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2}\right) dt} > 0$$

So, the arbitrage profit is obtained.

Chapter 14
Martingale Pricing Theory

See Section 3.2 of "Mathematical Models of Financial Derivatives", 2nd edition, by Yue Kuen KWOK, Springer Verlag, 2008.

Points to Note

1. What is the definition of the equivalent martingale measure? See P.3 – 6.
2. What is the relationship between the no-arbitrage price of a financial product and the risk-neutral probability? See P.8 – 13.
3. How do we change a measure in the expectation? Use the notation of the Randon-Nikodym derivative. See P.14 – 15.
4. Girsanov Theorem. See P.16 – 19.
5. Converting the dynamic of the asset price processes from the real probability to the risk-neutral probability. See P.20 – 25.
6. Derivation of the BS formula by using the change of numeraire. See P.26 – 38.
7. The Black-Scholes formula for the dividend-paying asset. See P.39 – 47.

Equivalent Martingale Measure and Risk Neutral Valuation

Under the continuous time framework, the investors are allowed to trade continuously in the financial market up to finite time T.

risk-free asset

Consider the securities model, there are

$K + 1$ securities whose price processes are modeled by $M(t)$ and $S_m(t)$ (**non-dividend-paying assets**), where $m = 1, \ldots, K$.

risky asset

The uncertainly of the market is described by the actual (real) probability measure (distribution) P.

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

We use $M(t)$ to denote the money market account process that starts at \$1 and grows at the *deterministic* risk-free interest rate $r(t)$, that is,

$$dM(t) = r(t)M(t)dt \quad \Leftrightarrow \quad \boxed{M(t) = e^{\int_0^t r(s)ds}}.$$

The discounted security price process $S_m^*(t)$ is defined by

$$\boxed{S_m^*(t) = \frac{S_m(t)}{M(t)}, \qquad m = 1, 2, \cdots, K.}$$

$= PV(S_m(t))$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

Suppose that Q_1 and Q_2 are probability measures (distributions) on the sample space Ω, Q_1 and Q_2 are *equivalent* if

$$Q_1(A) = 0 \Leftrightarrow Q_2(A) = 0 \text{ for any } A \subset \Omega.$$

Example 1

X r.v.

$Q_1(X < 20) = 0$, $Q_1(20 \le X \le 80) = 3/4$,

$Q_1(X > 80) = \frac{1}{4}$.

$Q_3(X < 10) = 0$, $Q_3(10 \le X < 19) = \frac{1}{8}$,

$Q_3(X \geq 19) = 7/8$,

Q_1, Q_3

$$Q_1(X < 19) = 0$$

$$Q_2(X < 19) = \frac{1}{8}$$

$$= Q_2(X < 10) + Q_2(10 \le X < 19)$$

$$= \frac{1}{8}$$

$Q_1 \not\Leftrightarrow Q_2$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Definition

A probability measure Q is said to be an <u>*equivalent martingale measure*</u> (or <u>*risk-neutral measure*</u>) to the real probability measure P if it satisfies

i. Q is equivalent to P;

ii. The discounted security price process $S_m^*(t)$, m = 1, 2, …, K, are martingales under Q, that is

$$E_t^Q[S_m^*(u)] = S_m^*(t) \qquad \text{for all } 0 \le t \le u \le T.$$

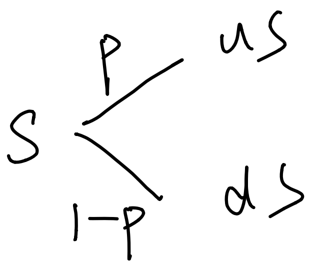
$E_t^Q[PV(S_m(u))] = PV(S_m(t))$

Let $Y(t) = S_m^*(t)$

$$E^Q[\,Y(u) \mid \{Y(s) : 0 \le s \le t\}\,]$$

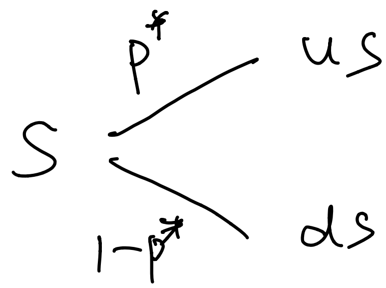
$$= Y(t).$$

Binomial tree



Real prob P

Use replication to find p^* (risk-neutral prob)



$P \Leftrightarrow p^*$

$$p^* = \frac{e^{rh} - d}{u - d}$$

$$S^*(1) = \frac{S(1)}{M(1)} = \frac{S(1)}{e^r}$$

$$E^*(S^*(1)) = S(0) \quad ??$$

$$E^*[S^*(1)] = p^* \frac{uS}{e^r} + (1-p^*)\frac{dS}{e^r}$$

$$= \left(\frac{e^r - d}{u-d}\right)\frac{uS}{e^r} + \left(\frac{u - e^r}{u-d}\right)\frac{dS}{e^r}$$

$$= S$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

In (ii), $E_t^Q(\cdot)$ is defined as the expectation with respect to the probability measure *Q* conditional on the information available up to time *t*.

For simplicity, we would write $E_0^Q(\cdot)$ as $E^Q(\cdot)$.

Remark:

a. Existence of *Q* $\Rightarrow$ Absence of arbitrage.

b. Absence of arbitrage + some technical conditions $\Rightarrow$ Existence of *Q*.

c. Complete market $\Leftrightarrow$ *Q* is unique.

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

Theorem (Risk Neutral Valuation) (Martingale pricing)

Assume that an equivalent martingale measure Q then exists. Let $V(T)$ be the payoff of a contingent claim which can be generated by the $K + 1$ securities, the *no arbitrage price* of the contingent claim at time t, $V(t)$, is given by

$$V(t) = M(t) E_t^Q \left[\frac{V(T)}{M(T)} \right].$$

payoff of the financial product.

$$\frac{V(t)}{M(t)} = E_t^Q \left[\frac{V(T)}{M(T)} \right] \Rightarrow \left\{ \frac{V(t)}{M(t)} : t \geq 0 \right\}$$

Martingale

Zero coupon bond

T-year zero coupon bond face value = \$1

payoff = \$1 $\quad V(T) = 1$

$$\frac{V(0)}{M(0)} = E^Q\left[\frac{1}{e^{rT}}\right] = E^Q[e^{-rT}]$$
$$= e^{-rT}$$

$$V(0) = M(0)e^{-rT} = e^{-rT}$$

Forward contract (Forward price = K \$)

$$K = S_0 e^{rT}$$

Risk-neutral valuation

$$V(T) = S(T) - K$$

$$\frac{V(0)}{M(0)} = E^Q\left[\frac{S(T) - K}{e^{rT}}\right]$$

$$0 = E^Q\left[\frac{S(T)}{e^{rT}}\right] - E^Q\left[Ke^{-rT}\right]$$

$$e^{-rT}K = E_0^Q\left[\frac{S(T)}{e^{rT}}\right]$$

$$= \frac{S(0)}{M(0)} = S(0)$$

$$K = S(0)e^{rT}$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

1. Zero-coupon bond (pays $1 at the maturity T)

   $V(T) = 1$

   By the risk neutral valuation theorem, we have

$$P(0,T) = M(0)E^Q\left[\frac{1}{M(T)}\right]$$

$$= E^Q\left[e^{-\int_0^T r(s)ds}\right].$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

2. Forward price

   The payoff for the long position is

   $V(T) = S(T) - K$ where K is the forward price.

   By the risk neutral valuation theorem, we have

$$V(0) = M(0) E^Q \left[\frac{S(T) - K}{M(T)} \right].$$

   Since there is 0 cost to enter into a forward contract, $V(0) = 0$. This implies

$$E^Q \left[\frac{S(T) - K}{M(T)} \right] = 0.$$

Equivalent Martingale Measure and Risk Neutral Valuation (Cont'd)

So,

$$E^Q\left[\frac{K}{M(T)}\right] = E^Q\left[\frac{S(T)}{M(T)}\right]$$

$$KE^Q\left[\frac{1}{M(T)}\right] = \frac{S(0)}{M(0)}$$

$$KP(0,T) = S(0)$$

$$K = \frac{S(0)}{P(0,T)}.$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

3. Swap

   Consider a swap with the fixed swap price payment dates on t_i, $i = 1, \ldots, n$.

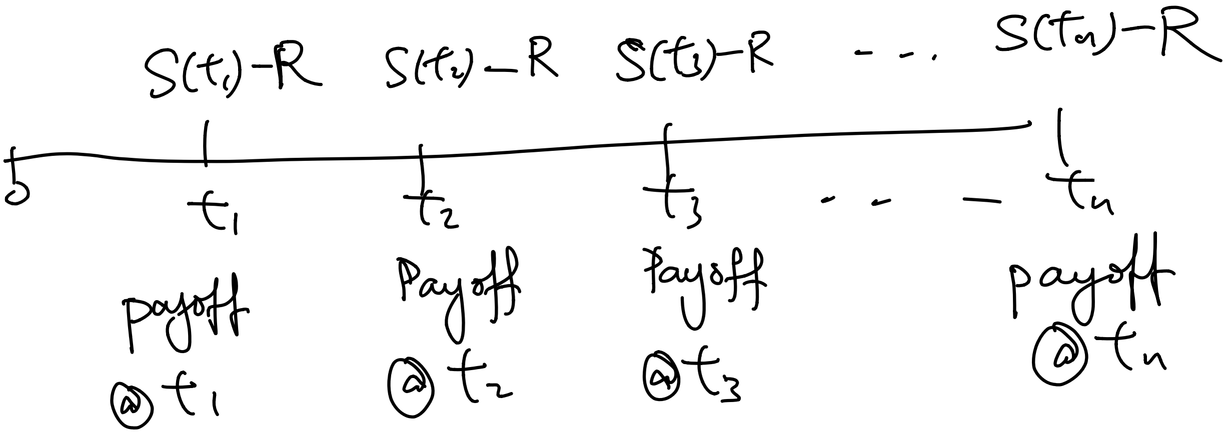
   Let R be the fixed swap price.

   By the risk neutral valuation theorem, the value of the swap from the perspective of long party is given by

$$V(0) = M(0) E^Q \left[\boxed{\sum_{i=1}^{n} \frac{\left(S(t_i) - R\right)}{M(t_i)}} \right].$$

Payoff

   Since the value of the swap is 0 at the inception date ($t = 0$),



$$\frac{V(0)}{M(0)} = E^{Q}\left[\sum_{i=1}^{n} \frac{\text{payoff} @ t_i}{M(t_i)}\right]$$

$$= \sum_{i=1}^{n} E^{Q}\left[\frac{(S(t_i) - R)}{M(t_i)}\right]$$

$$0 = \sum_{i=1}^{n} E^{Q}\left[\frac{S(t_i) - R}{M(t_i)}\right]$$

$$R \sum_{i=1}^{n} E^{Q}\left[\frac{1}{M(t_i)}\right] = \sum_{i=1}^{Q}\left[\frac{S(t_i)}{M(t_i)}\right]$$

Application of the Risk Neutral Valuation Theorem (Cont'd)

$$R\sum_{i=1}^{n} E^{Q}\left[\frac{1}{M(t_i)}\right] = \sum_{i=1}^{n} E^{Q}\left[\frac{S(t_i)}{M(t_i)}\right]$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} S(0)$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} \frac{S(0)P(0,t_i)}{P(0,t_i)}$$

$$R\sum_{i=1}^{n} P(0,t_i) = \sum_{i=1}^{n} F_{0,t_i} P(0,t_i)$$

$$R = \frac{\sum_{i=1}^{n} F_{0,t_i} P(0,t_i)}{\sum_{i=1}^{n} P(0,t_i)}.$$