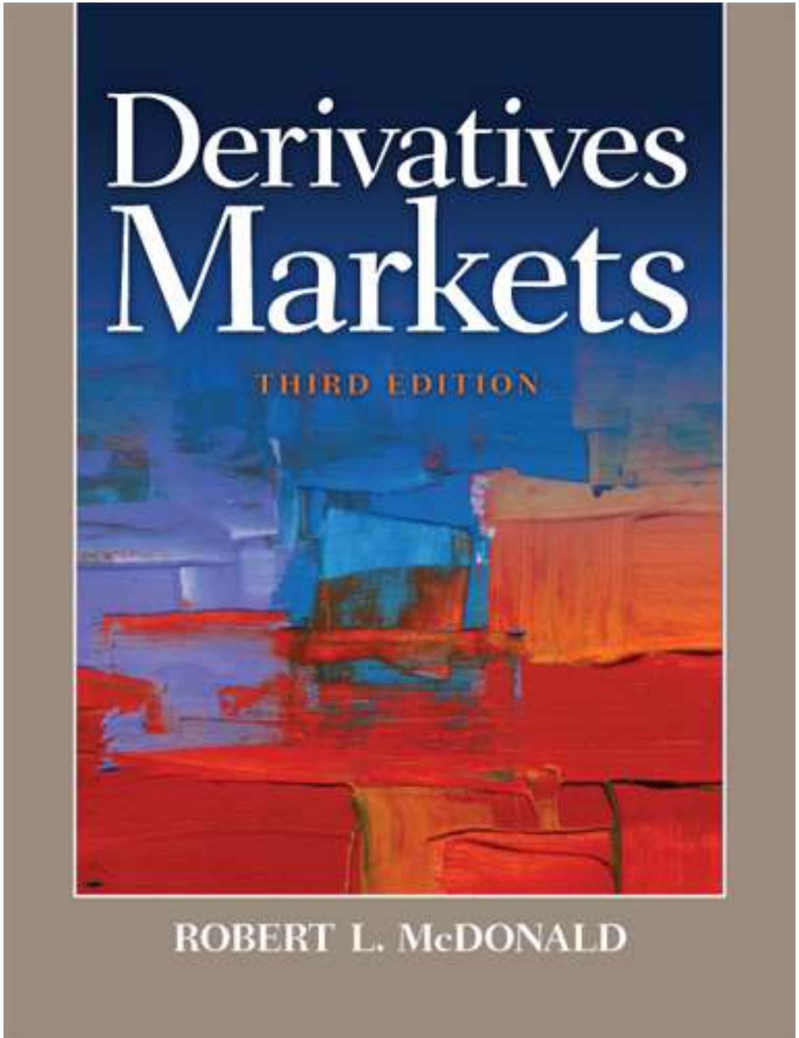


Chapter 10

(Chapter 12 in the textbook)

The Black-Scholes Formula

Points to Note

intrisinic value + time value

1. General understanding of the option price (see P.3 – 4)
2. What is the Black-Scholes formula for the European call and put options? (see P.5 – 7)
3. What are the assumptions of the Black-Scholes formula? (see P.8 – 9)
4. What is the relationship of the binomial model and the Black-Scholes formula? (see P.10 – 11)
5. The Black-Scholes formula for different underlying assets. (see P.12 – 17)
6. Option Greeks (see P.18 – 48)
7. Implied volatility (see P.49 – 54)

Black-Scholes Formula (cont'd)

- Call Option price:

Δ

$$C(S,K,\sigma,r,T,\delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

Δ

- Put Option price:

$$\boxed{+ S(-e^{-\delta T} N(-d_1))}$$

$$P(S,K,\sigma,r,T,\delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

where

$$d_1 = \frac{ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

N(*x*) is the cumulative distribution for standard normal random variable.

Applying the Formula to Other Assets

- Call Options

Let $F_{0,T}^{P}(S)$ and $F_{0,T}^{P}(K)$ be the prepaid forward prices for the stock and strike asset.

$$F_{0,T}^{P}(S) = Se^{-\delta T} \quad \text{and} \quad F_{0,T}^{P}(K) = Ke^{-rT}$$

Using $F_{0,T}^{P}(S)$ and $F_{0,T}^{P}(K)$ to rewrite the call option pricing formula, we have

$$C(F_{0,T}^{P}(S), F_{0,T}^{P}(K), \sigma, T) = F_{0,T}^{P}(S)N(d_1) - F_{0,T}^{P}(K)N(d_2)$$

where

$$d_1 = \frac{ln\left[F_{0,T}^{P}(S) / F_{0,T}^{P}(K)\right] + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

Option on futures (underlying is futures contract)

time to expiration

option : T

futures : T_F

$\}\ T = T_F$

American style

Option delta :

- formula for call, put
- significant of Δ

① replication

② relationship of option and S

$C \quad \Delta \doteq 0.4 \qquad \Delta = \frac{\partial C}{\partial S}$

$S \uparrow \$1, \quad C \uparrow \$0.4$ approximation

$$\Delta \approx \frac{C_{new} - C_{old}}{S_{new} - S_{old}} \uparrow \$1$$

$\Uparrow$ small

NOT usual prob. (in chapter 14)

③ probability $Pr(S_T > K)$ or $Pr(S_T < K)$

option elasticity Ω

$$\Omega = \frac{\%\text{ change of option price}}{\%\text{ change of stock price}} = \frac{S\Delta}{V}$$

① $\sigma_{option} = |\Omega| \times \sigma_{stock}$ ← (Ito's lemma)

② $r_{opt} - r = \Omega \times (r_{stock} - r)$

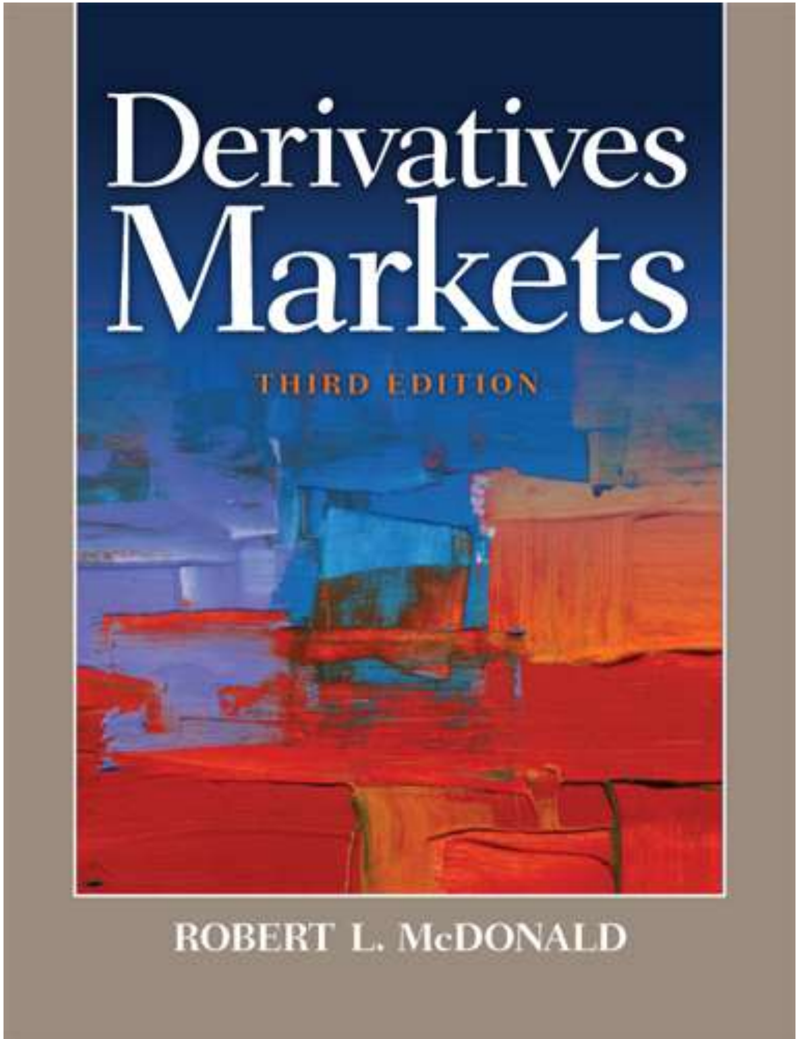
③ $\beta_{opt} = \beta_{asset} \times \Omega$

④ Sharpe ratio$_{call}$ = Sharpe ratio of stock

⑤ $\Omega_{port} = \sum_{i=1}^{n} w_i \Omega_i$

$$w_i = \frac{n_i C_i}{\sum_{j=1}^{N} n_j C_j}$$

refer to lecture notes



Chapter 11

(Chapter 13 in the textbook)

Market-Making and Delta-Hedging

Points to Note

1. The delta-gamma approximation of the option price. See P.10 – 11.
2. How does the delta-hedging work? See P.12 – 19.
3. The relationship between the delta hedging and the Greek letters. See P.18 – 24.
4. The definition of the “Greek” neutral portfolio. See P.26.
5. Construction of the “Greek” neutral portfolio. See P.26 – 31.
6. Determine Greek for the binomial tree. See P.32 – 34.

Delta-Hedging (cont'd)

- Delta hedging for several days

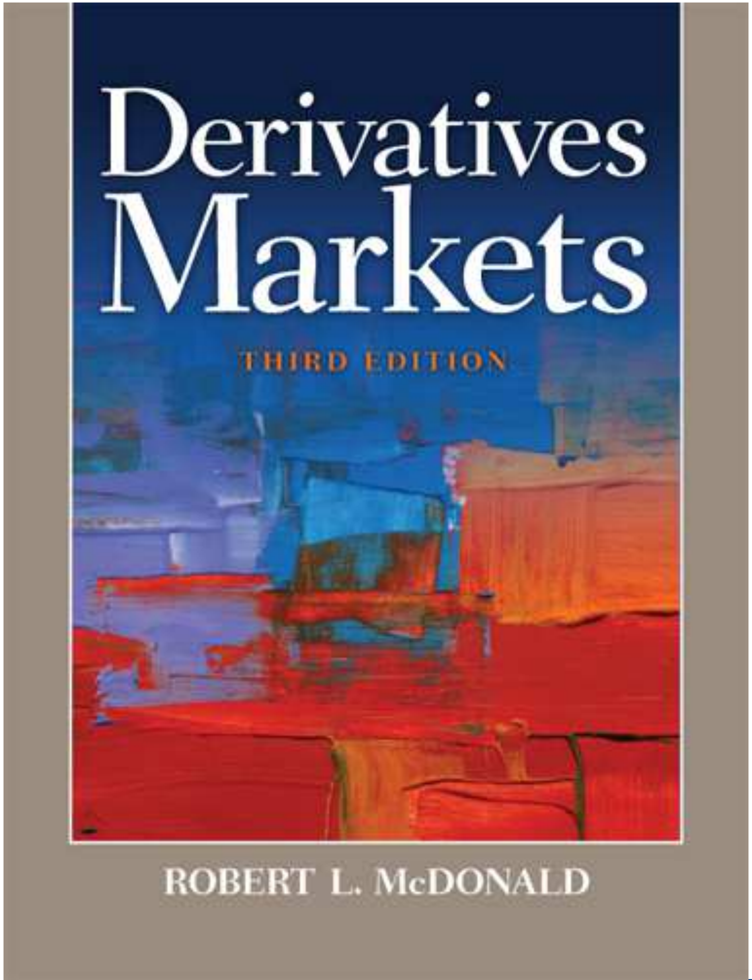
TABLE 13.2 Daily profit calculation over 5 days for a market-maker who delta-hedges a written option on 100 shares.

	Day					
	0	1	2	3	4	5
Stock ($)	40.00	40.50	39.25	38.75	40.00	40.00
Call ($)	278.04	306.21	232.82	205.46	271.04	269.27
100 × delta	58.24	61.42	53.11	49.56	58.06	58.01
Investment ($)	~~2051.58~~ 2051.56	2181.30	1851.65	1715.12	2051.35	2051.29
Interest ($)		−0.45	−0.48	−0.41	−0.38	−0.45
Capital gain ($)		0.95	−3.39	0.81	−3.62	1.77
Daily profit ($)		0.50	−3.87 ?!	0.40	−4.00	1.32

1. Gain on stock

2. Gain on option

3. Interest expenses



Chapter 12
(Chapter 18 in the textbook)

The Lognormal Distribution

Points to Note

1. Definition of the lognormal distribution. See P.10.
2. Properties of lognormal random variables. See P.10.
3. The expectation and variance of the lognormal random variable. See P.11.
4. The lognormal model of stock prices. See P.14 – 15.

   Derive 18-5 formula
5. Some results of the lognormal distribution. See P.17 – 20.
6. Estimating the parameters of a lognormal distribution. See P.21 – 23.

$X \sim LN(\mu, \sigma^2) \Rightarrow \ln X \sim N(\mu, \sigma^2)$

① $E[X]$, $Var(X)$

② $X_1 + X_2 \not\Rightarrow$ lognormal   if X_1 and X_2 are lognormal

$X_1 \cdot X_2 \Rightarrow$ lognormal.

③ $g(x)$ (refer to the lecture notes)

④ partial expectation

$$\int_0^K S_t \, g(S_t; S_0) \, dS_t = S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)$$

Eg $\Pr(X < a)$ $X \sim LN(\mu, \sigma^2)$

$= \Pr(\ln X < \ln a)$

$= \Pr\left(Z < \frac{\ln a - \mu}{\sigma}\right)$

Tutorial - Class Activity (Solution)

26 November, 2019

Problem 1

$K = S$

Assume the Black-Scholes framework. Consider a one-year at-the-money European call option on a stock.

You are given:

(i) The ratio of the call option price to the stock price is less than 10%. $C/S < 10\%$

(ii) The delta of the call option is 0.6. $\Delta = 0.6$ $\quad e^{-\delta T} N(d_1) = 0.6$

(iii) The continuously compounded dividend yield of the stock is 2%. $\delta = 2\%$

(iv) The continuously compounded risk-free interest rate is 4.94%. $r = 4.94\%$

Determine the stock's volatility. $\sigma = ??$

$$d_1 = \frac{\overbrace{\ln(S/K)}^{0 \text{ (ATM)}} + (r - \delta + \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Solution

The value of delta can be used to determine d_1:

$$\begin{aligned}
\Delta_{Call} &= e^{-\delta T} N(d_1) \\
0.6 &= e^{-0.02\times 1} N(d_1) \\
0.6121 &= N(d_1) \\
d_1 &= 0.28.
\end{aligned}$$

The formula for d_1 is used to find a quadratic equation in terms of σ:

$$\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r - \delta + 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
0.28 &= \frac{\ln(S/S) + (0.0494 - 0.02 + 0.5\sigma^2)\times 1}{\sigma\sqrt{1}} \\
0.28 &= \frac{(0.0294 + 0.5\sigma^2)}{\sigma} \\
0.28\sigma &= 0.0294 + 0.5\sigma^2 \\
0.5\sigma^2 &- 0.28\sigma + 0.0294 = 0 \\
\sigma^2 &- 0.56\sigma + 0.0558 = 0.
\end{aligned}$$

The two solutions to the quadratic equation are found below:

$$\sigma = \frac{0.56 \pm \sqrt{(-0.56)^2 - 4(1)(0.0588)}}{2}$$
$$\sigma = 0.14 \quad \text{or} \quad 0.42.$$

The value of $N(d_2)$ depends on the value of σ:

$$\sigma = 0.14 \Rightarrow d_2 = d_1 - \sigma\sqrt{T} = 0.28 - 0.14 = 0.14 \Rightarrow N(d_2) = 0.5557.$$
$$\sigma = 0.42 \Rightarrow d_2 = d_1 - \sigma\sqrt{T} = 0.28 - 0.42 = -0.14 \Rightarrow N(d_2) = 1 - 0.5557 = 0.4443.$$

For an at-the-money option, $K = S$. We are given that the call price is less than 10% of the stock price.

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$
$$\frac{C}{S} = e^{-0.02}N(d_1) - e^{-0.0494}N(d_2)$$
$$e^{-0.02}N(d_1) - e^{-0.0494}N(d_2) < 0.1$$
$$0.6 - e^{-0.0494}N(d_2) < 0.1$$
$$0.5 < e^{-0.0494}N(d_2)$$
$$0.52532 < N(d_2).$$

For the inequality above to be satisfied, it must be the case that:

$$\sigma = 0.14.$$

Problem 2

Assume the Black-Scholes framework. Consider a six-month financial derivative on a stock with the following payoff:

$$\text{Payoff} = \begin{cases} 85 & \text{if } 0 \le S(0.5) \le 85, \\ 170 - S(0.5) & \text{if } S(0.5) > 85. \end{cases}$$

where $S(0.5)$ is the stock price at $t = 0.5$.

You are given that

(i) The current price of the stock is \$90.

(ii) The stock's volatility is 35%.

(iii) The stock's dividend yield is 5%.

(iv) The continuously compounded risk-free interest rate is 8%.

Calculate the elasticity of the given financial derivative.

$\Omega = \frac{S\Delta}{V}$ $\quad \Delta = \frac{\partial\left(85e^{-(0.5)(8\%)} - C(S)\right)}{\partial S}$

$= -\Delta\text{call}$

Solution

The payoff of the financial derivative can be rewritten as

$$85 - \max(S(0.5) - 85, 0).$$

The financial derivative can then be replicated with a portfolio consisting of a zero-coupon bond with the face value of $85 and a short position in a European call option with a strike price of $85.

To find the value of this financial derivative, we must find the value of the call option:

The first step is to calculate d_1 and d_2:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta + 0.5\sigma^2\right)(0.5)}{\sigma\sqrt{0.5}} = \frac{\ln\left(\frac{90}{85}\right) + \left(0.08 - 0.05 + 0.5 \times 0.35^2\right)(0.5)}{0.35\sqrt{0.5}}$$

$$= 0.42$$

$$d_2 = d_1 - \sigma\sqrt{0.5} = 0.42 - 0.35\sqrt{0.5} = 0.17.$$

We have:

$$N(d_1) = N(0.42) = 0.6628,$$
$$N(d_2) = N(0.17) = 0.5675.$$

The value of the European call option is:

$$\begin{aligned} C(90, 85, 0.5) &= Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \\ &= 90 \times e^{-0.05\times0.5} \times 0.6628 - 85 \times e^{-0.08\times0.5} \times 0.5675 \\ &= 11.8331. \end{aligned}$$

The current value of the financial derivative is the price of the zero-coupon bond **minus** the value of the European call option:

$$\boxed{V = 85e^{-0.08\times0.5} - 11.8331 = 69.834.}$$

The delta of the call option is:

$$\Delta_{Call} = e^{-\delta T}N(d_1) = e^{-0.05\times0.5} \times 0.6628 = 0.6464.$$

The delta of the financial derivative is the delta of the zero-coupon bond **minus** the delta of the call option:

$$\boxed{\Delta = 0 - 0.6464 = -0.6464.} \quad = \Delta\text{Bond} - \Delta\text{call}$$

The elasticity of the financial derivative is:

$$\Omega = \frac{S\Delta}{V} = \frac{90(-0.6464)}{69.834} = -0.8331.$$

Problem 3 + 90-day 80-strike Call

The price of a stock is $60. The stock does not pay dividends. The continuously compounded risk-free rate of return is 9%. A $55-strike European put option has a price of $1.60. The delta of the put option is –0.2377. A $60-strike European put option has a price of $3.50. The delta of the put option is –0.4146.

A market-maker enters into a put ratio spread where the market-maker buys 100 of the $55-strike puts and sells 200 of the $60-strike puts. The market-maker delta-hedges the position.

One day later, the price of the stock is $62, the price of the $55-strike European put option is $1.17, and the price of the $60-strike European put option is $2.73. Calculate the overnight profit for the market-maker.

Gain on stock
Gain on option
interest expenses

Solution

From the market-maker's perspective, the initial value of the position to be hedged is:

$$100(1.60) - 200(3.50) = -540.00$$ cost of ratio spread < 0

The delta of the position to be hedged, from the perspective of the market-maker, is: ⇒ income

$$100(-0.2377) - 200(-0.4146) = 59.15$$ Δport

To delta hedge the position, the market-maker shorts 59.15 shares of stock. The value of this position is:

cost $$-59.15(60) = -\$3{,}549.00 < 0$$ ⇒ income

The market-maker receives $540.00 for entering into the put ratio spread. The market-maker also receives $3,549 for the stock that it shorts. The sum of the proceeds is lent at the risk-free rate of return:

$$540 + 3{,}549 = \$4{,}089$$ = short stock + Long ratio spread

The initial position, from the perspective of the market-maker, is:

Component	Value (cash outflow)
Options	–540
Shares	–3,549
Risk-free asset (lend)	4,089
Net	0

After 1 day, the value of the option has changed by:

$$\underbrace{100(1.17 - 1.6)}_{\text{55-strike}} - \overbrace{200(2.73 - 3.5)}^{\text{60-strike put}} = 111.$$

55-strike put (Long)
1.6 → 1.17
60-strike put (short)
3.5 → 2.73

After 1 day, the value of the shares of stock has changed by:

$$\underbrace{-59.15}_{\text{position}}(\underbrace{62}_{\text{new}} - \underbrace{60}_{\text{old}}) = -118.3.$$

After 1 day, the value of the funds that were lent at the risk-free rate has changed by:

$$4{,}089(e^{0.09/365}-1)=1.0084.$$

The sum of these changes is the overnight profit:

Component	Change
Gain on options	111
Gain on stock	–118.3
Interest Gain	1.0084
Overnight profit	–6.2916

The position experienced an overnight loss of \$6.2916.

Problem 4 under risk-neutral, $\ln\left(\frac{S_t}{S_0}\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2)t, \sigma^2 t\right)$

Assume the Black-Scholes framework. You are given:

(i) The current price of a stock is 80.

(ii) The stock's volatility is 25%. σ

(iii) The stock has nonzero dividend yield. $\delta \neq 0$

(iv) The continuously compounded risk-free interest rate is 6%.

(v) $E[S(1)] = 84.2069$.

Calculate $\boxed{E[S(1)|S(1) < 80]}$. $= S_0 e^{(r-\delta)(1)} \dfrac{N(-d_1)}{N(-d_2)}$

Solution

We are given from (v) that

$$E\left[S(1)\right]=S(0)e^{(\overset{r}{\cancel{\alpha}}-\delta)T}=80e^{(\overset{r}{\cancel{\alpha}}-\delta)}=84.2069,\quad\text{or}\quad \overset{r}{\cancel{\alpha}}-\delta=0.0513.$$

Then

$$\hat{d}_1=\frac{\ln\left(\dfrac{S(0)}{K}\right)+\left(\overset{r}{\cancel{\alpha}}-\delta+0.5\sigma^2\right)T}{\sigma\sqrt{T}}=\frac{\ln(1)+\left(0.0513+0.5(0.25)^2\right)1}{0.25\sqrt{1}}=0.33,$$

$$\hat{d}_2=\hat{d}_1-\sigma\sqrt{T}=0.33-0.25\sqrt{1}=0.08,$$

$N\left(-\hat{d}_1\right)=0.3707,$

$N\left(-\hat{d}_2\right)=0.4681,$

And

$$
\begin{aligned}
E\left[S(1)\middle|S(1)<80\right] &= S(0)e^{(\overset{r}{\cancel{\alpha}}-\delta)T}\,\frac{N\left(-\hat{d}_1\right)}{N\left(-\hat{d}_2\right)} \\
&= 84.2069\left(\frac{0.3707}{0.4681}\right) \\
&= 66.6855.
\end{aligned}
$$