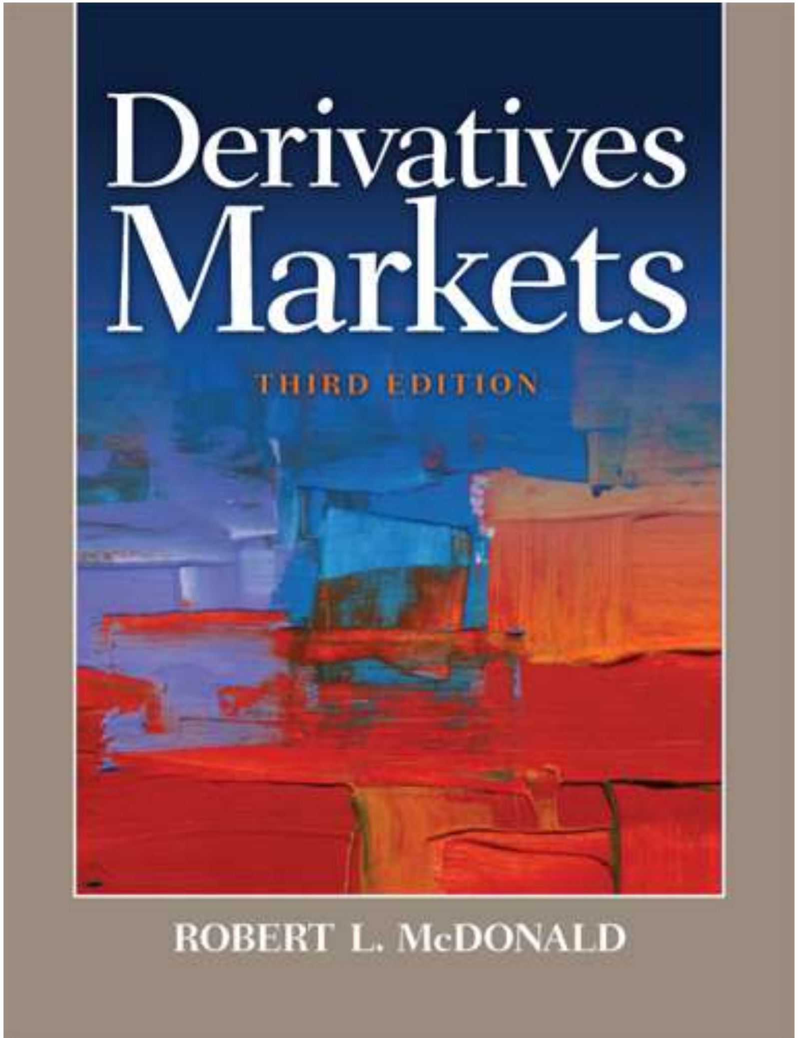



# Chapter 5
## (Chapter 6 in the textbook)

Commodity Forwards and Futures

# Points to Note

1. What are the differences between the commodity and the financial asset? See P.3 to 5.
2. Definitions of backwardation and contango, see P.6 to 10.
3. How do the storage cost and convenience yield determine the forward price? (see $F_{0,T}$ on p.12 and 13)

   $F_{0,T} = S_0 e^{(r+u-y)T}$
4. What does the lease rate mean? See P.15.

   $\boxed{\delta_l = y - u}$
5. What is the relationship among the lease rate, storage cost and convenience yield? See P.16.

Commodity

- storage cost. (u) ~ negative dividend
- convenience yield (y) ~ dividend.

$$F_{0,T} = S_0 e^{(r+u-y)T}$$

: u : cts comp. storage cost

y : convenience yield

vs

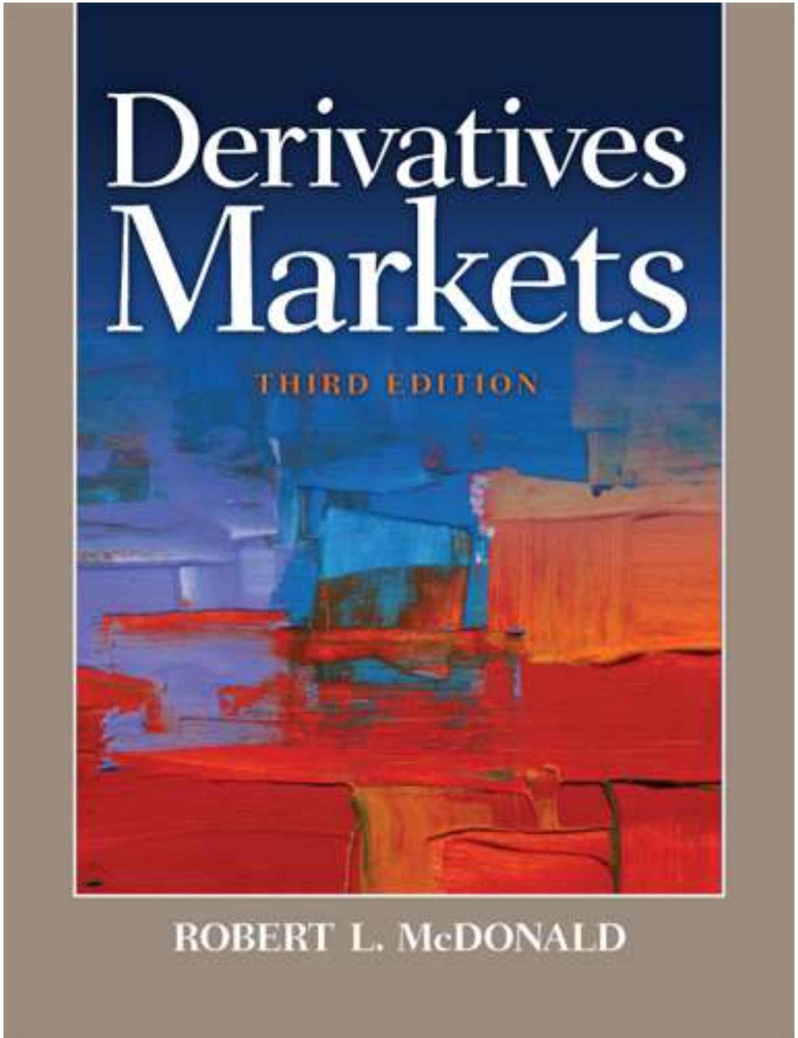
$$F_{0,T} = S_0 e^{(r-\delta)T}$$

$\delta$ : dividend yield

leased market exist ; lease rate ( $\delta_\ell$ )

$$\delta_\ell = y - u$$

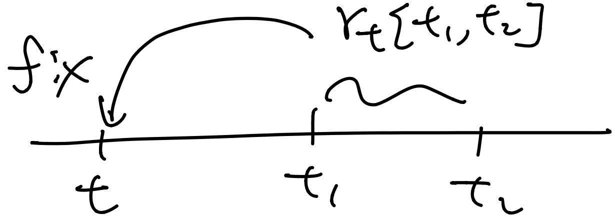
( annualized leased payment per commodity price )




# Chapter 6
## (Chapter 7 in the textbook)

Bond Basics

# Points to Note




1. Definition of $r_t(t_1,t_2)$, see P.4.

2. What is the relationship between the bond price $P(0, n)$ and $r(0,n)$? See P.5.

   yield to maturity

3. How to find YTM from the zero coupon price? See P.7.

4. How to find the implied forward rate? See P.8 – 9.

5. How to find the implied forward zero-coupon price? See P.10.

6. Coupon bonds, see P.12.

7. Bootstrapping zero-coupon price from coupon bonds, see P.14 – 15.

8. Definition of continuously compounded yields $r^{cc}(0, t)$.

**TABLE 7.1** Five ways to present equivalent information about default-free interest rates. All rates but those in the last column are effective annual rates.

| ($t$) Years to Maturity | (1) $r(0,n)$ Zero-Coupon Bond Yield | (2) $P(0,n)$ Zero-Coupon Bond Price | $r_0[t-1,t]$ (3) ?? One-Year Implied Forward Rate | (4) Par Coupon | (5) Continuously Compounded Zero Yield |
|---|---|---|---|---|---|
| 1 | 6.00% | 0.943396 ① | 6.00000% | 6.00000% | 5.82689% |
| 2 | 6.50 | 0.881659 ② | 7.00236 | 6.48423 | 6.29748 |
| 3 | 7.00 | 0.816298 ③ | 8.00705 | 6.95485 | 6.76586 |

$$P(0,n) = \begin{cases} \dfrac{1}{(1+r(0,n))^n} & : \text{discrete} \\ e^{r_{cc}(0,n)\,n} & : \text{continuous} \end{cases}$$

discount factor.

$r_0(t_1,t_2)$ : $\quad (1+r_0[t_1,t_2])^{t_2-t_1} = \dfrac{P(0,t_1)}{P(0,t_2)}$

coupon bond with (par) face value = \$1

$$B_0 = \sum_{i=1}^{n} C\,P(0,t_i) + P(0,t_n)$$

par bond : $B_0 = 1$ (Bond price = face value)

$$1 = \sum_{i=1}^{n} C\,P(0,t_i) + P(0,t_n)$$

$$\Rightarrow C = \frac{1-P(0,t_n)}{\sum_{i=1}^{n} P(0,t_i)}, \quad \text{par coupon}$$

$$F_{0,T} = S_0 e^{rT} = \frac{S_0}{e^{-rT}} = \frac{S_0}{P(0,T)}$$

$$= FV(S_0)$$

$$\frac{1}{P(0,T)} = FV(\$1) \quad , \quad P(0,T) = PV(\$1)$$

## Tutorial - Class Activity

## 15 October, 2019 (Solution)

**Question 1**

$\delta_l$

Suppose that copper costs \$3.00 per pound today and the lease rate for copper is 5%. The continuously compounded interest rate is 10%. The copper price in 1 year is uncertain and copper can be stored costlessly.

a. If you short-sell a pound of copper for 1 year, what payment do you have to make to the copper lender? $aS_1$ $a = ??$

b. Find the equilibrium forward price.

**Solution**

$F_{0,T} = S_0 e^{(r-\delta_l)T}$

(a)

As we need to borrow a pound of copper to sell it short, we must pay the lender the lease rate for the time we borrow the asset, i.e., until expiration of the contract in one year.

Including the leasing cost, the short seller needs to pay back $e^{0.05}$ pounds of copper to the lender at the end of one year.

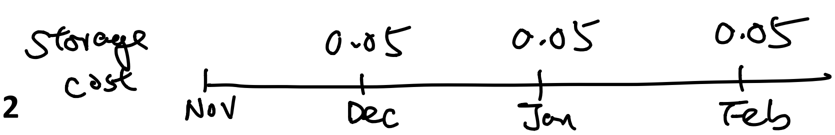
At the end of one year, the amount that the short seller has to pay to the copper lender

$= S_T \times e^{0.05} = 1.05127 \times S_T$.

(b)

The equilibrium forward price is calculated according to our pricing formula:

$F_{0,T} = S_0 \times e^{(r-\delta_l)\times T} = \$3.00 \times e^{(0.10-0.05)\times 1} = \$3.00 \times 1.05127 = \$3.1538.$




**Question 2**

Suppose that the November price of corn is \$2.50/bushel, the effective monthly interest rate is 1%, and the storage cost per bushel are \$0.05/month. Assuming that corn is stored from November to February (i.e. the storage cost is paid on Dec, Jan and Feb).

(a) Find the theoretical forward price of this February forward.

(b) Suppose the February forward price had been \$2.80. What would the arbitrage be?

**Solution**

$F_{0,T} = FV(S_0 + PV(u))$

$= (S_0 + PV(u))(1.01)^3$

| | $t=0$ | $\Delta t$ | $2\Delta t$ | $3\Delta t$ | $4\Delta t$ | … | $N\Delta t$ |
|---|---|---|---|---|---|---|---|
| no. of copper | $-1\text{lb}$ | $-1-\frac{\delta_\ell S_{\Delta t}\Delta t}{S_{\Delta t}} = -(1+\delta_\ell\Delta t)$ | | | | | $(1+\delta_\ell\Delta t)^N$ |
| leased payment | | $\delta_\ell S_{\Delta t}\Delta t$ | | | | | |

$\Delta t \to 0$

$t=0 \longrightarrow t=T$

1 pound of Copper $\longrightarrow$ $e^{\delta_\ell T}$ pound of Copper

Total cost = Repay 1 pound of copper + leased cost

$$= \underbrace{e^{\delta_\ell T}}_{\text{pounds}} \; \underbrace{S_1}_{\text{price of Copper at } t=1}$$

(a)

Payoff of Forward $= S_T - F_{0,T}$

underlying

The future value of the storage cost

$$\$0.05 + (\$0.05 \times 1.01) + (\$0.05 \times 1.01^2) = \$0.1515$$

market $= 2.8$ Cash

Thus, the February forward price will be

$$2.5 \times (1.01)^3 + 0.1515 = 2.7273.$$

(b)

Buy "synthetic" Short "market"

If the February corn forward price is \$2.80, the observed forward price is too expensive relative to our theoretical price of \$2.7273. We will therefore sell the February contract short, and create a synthetic long position, engaging in cash and carry arbitrage:

| Transaction | Nov | Dec | Jan | Feb |
|---|---|---|---|---|
| Short Feb forward | 0 | | | $2.80 - S_T$ |
| Buy spot | −2.50 | | | $S_T$ |
| Borrow purchasing cost | +2.50 | | | −2.57575 |
| Pay storage cost Dec, | | −0.05 | | −0.051005 |
| borrow storage cost | | +0.05 | | |
| Pay storage cost Jan, | | | −0.05 | −0.0505 |
| borrow storage cost | | | +0.05 | |
| Pay storage cost Feb | | | | −0.05 |
| Total | 0 | 0 | 0 | 0.072745 |

NO randomness

We made an arbitrage profit of 0.07 dollar.

**Question 3**

$r_0[1,2]$ $r_0[2,3]$ $r_0[3,4]$ $r_0[0,1]$

**7.4** Suppose you observe the following 1-year implied forward rates: 0.050000 (1-year), 0.034061 (2-year), 0.036012 (3-year), 0.024092 (4-year), 0.001470 (5-year). For each maturity year compute the zero-coupon bond prices, effective annual and continuously compounded zero-coupon bond yields, and the par coupon rate.

$r_0[4,5]$

**Solution**

? ? ✓ ?

| Maturity | Zero-Coupon Bond Yield | Zero-Coupon Bond Price | One-Year Implied Forward Rate | Par Coupon | Cont. Comp. Zero Yield |
|---|---|---|---|---|---|
| 1 | 0.05000 | 0.95238 | 0.05000 | 0.05000 | 0.04879 |

| | Nov | Dec | Jan | Feb |
|---|---|---|---|---|
| Short market forward | 0 | | | $2.8 - S_T$ |
| Buy 1 unit of Corn | $-2.5$ | | | $S_T$ |
| pay Dec Storage cost | | $-0.05$ | | |
| pay Jan storage cost | | | $-0.05$ | |
| Pay Feb Storage cost | | | | $-0.05$ |
| Borrow purchasing Cost | 2.5 | | | |
| Borrow Dec Storage | | 0.05 | | $-0.05(1.01)^2$ |
| Borrow Jan storage | | | 0.05 | $-0.05(1.01)$ |
| Borrow Feb storage | | | | $0.05 - 0.05$ |
| Total | 0 | 0 | 0 | 0.072745 |

| | zero coupon yield | P(0,t) | r0[t-1,t] | Par coupon. | |
|---|---|---|---|---|---|
| 2 | 0.04200 | 0.92101 | 0.03406 | 0.04216 | 0.04114 |
| 3 | 0.04000 | 0.88900 | 0.03601 | 0.04018 | 0.03922 |
| 4 | 0.03600 | 0.86808 | 0.02409 | 0.03634 | 0.03537 |
| 5 | 0.02900 | 0.86681 | 0.00147 | 0.02962 | 0.02859 |

Notes

1. From the one-year implied forward rate, we can find $P(0, t_j)$.
   For example,

$$1 + r_0(0,1) = \frac{P(0,0)}{P(0,1)}$$

P(0,0) = 1

$$= \frac{1}{P(0,1)}$$

$$P(0,1) = \frac{1}{1 + r_0(0,1)}$$

$$= \frac{1}{1 + 0.05}$$

$$= 0.95238.$$

$$1 + r_0(1,2) = \frac{P(0,1)}{P(0,2)}$$

$$= \frac{0.95238}{P(0,2)}$$

$$P(0,2) = \frac{0.95238}{1 + r_0(1,2)}$$

$$= \frac{0.95238}{1 + 0.034061}$$

$$= 0.92101.$$

2. The **zero-coupon bond yield** and **continuous compounding zero-coupon yield** can be calculated by using the following formula:

$$P(0,t) = \frac{1}{\left(1 + r(0,t)\right)^t},$$ Discrete

$$P(0,t) = e^{-r_{CC}(0,t)t}.$$ Continuous

   It is worth to note that the 1-year implied forward rate which matures at the end of the first year is our $r(0, 1)$.

3. The **par coupon** for the $t_n$ maturity bond ($c(t_n)$) can be calculated by the following formula:

$$c(t_n) = \frac{1 - P(0,t_n)}{\sum_{i=1}^{n} P(0,t_i)}.$$