

A Bird's Eye View on Discrete Logarithm Problems (DLP) over Finite Fields

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December 14, 2017

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Outline

Motivation

DLP in Cryptography

Exposition of The Problem

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Motivation

- DLP in Cryptography

- Exposition of The Problem

Previous Work

- Preliminaries

- From Sub-exp to Quasi-poly

- Summary

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Research Plan

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- Key-exchange scheme [Diffie and Hellman, 1976]

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- Encryption algorithm [ElGamal, 1985]

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- Encryption algorithm [ElGamal, 1985]
- Digital Signature Algorithm (DSA) [NIST, 1991]

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Case in General

- Cyclic group: $(G, \cdot) = \langle g \rangle$

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 - $G \rightarrow \mathbb{Z}, h \mapsto x = \log_g h$

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- Generic Algorithms - $O(\exp(\log |G|))$
 - Pohlig-Hellman
 - Collision Making
 - Shanks' baby-step giant-step, Pollard's ρ , etc.

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- Generic Algorithms - $O(\exp(\log |G|))$
 - Pohlig-Hellman
 - Collision Making
 - Shanks' baby-step giant-step, Pollard's ρ , etc.
- Concerning with two class of groups
 1. Elliptic curves
 2. Multiplicative groups of finite fields

Case over Finite Fields

- Given: Q , g and h , where $\langle g \rangle = \mathbb{F}_Q^\times$ containing h
- Find: $\log_g h$

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The \mathcal{L} Notation

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- When $0 < \alpha < 1$, $\mathcal{L}_Q(\alpha)$ is **sub-exponential**

Quasi-poly

$(\log Q)^{O(\log \log Q)}$ is **quasi-polynomial**, which is smaller than any $\mathcal{L}_Q(\alpha)$ for $\alpha > 0$.

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Overview

$\mathcal{L}_Q(\frac{1}{2})$ Index Calculus Method

[Pohlig, 1977] and [Adleman, 1979] independently

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$\mathcal{L}_Q(\frac{1}{3})$ by [Coppersmith, 1984]

- Originally \mathbb{F}_{2^n}
- **Number Field Sieve** by [Gordon, 1993]
- **Function Field Sieve** by [Adleman and Huang, 1999]

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$\mathcal{L}_Q(\frac{1}{4})$ by [Joux, 2013] to **quasi-poly** by [BGJT@EC'14]

- Originated from [Joux and Lercier@EC'06]
- For small char., roughly $p \leq \mathcal{L}_Q(\frac{1}{3})$.
- Heuristics

Smooth - Set A Bound

Definition

Given $B > 0$. $\forall n \in \mathbb{Z}$ is called **B -smooth** if all its prime factors are no larger than B . Thus denote **factor basis** as

$$\mathcal{F}(B) = \{p \in \mathbb{N} : \text{prime and } p \leq B\}$$

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Definition

Given $B \in \mathbb{Z}_{>0}$. $\forall f[X] \in \mathbb{F}_q[X]$ is called **B -smooth** if all its irreducible factors are of degree no higher than B . Thus denote respective **factor basis** as

$$\mathcal{F}_q(B) = \{F[X] \in \mathbb{F}_q[X] : \text{irr. monic and of } \deg \leq B\}$$

Index Calculus Method - Framework of Following Algorithms

- Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$; Find: $\log_g h$

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 1. **Main Phase**

2. Individual Logarithm

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 - **Initialization** Fix parameter B , thus $\mathcal{F}(B)$ is also given.

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- **Smoothness Selection** Random $c \in [1, Q - 2]$ s.t. g^c is B -smooth:

$$g^c = \prod_{p \in \mathcal{F}(B)} p^{v(p,c)}$$

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- **Linea Algebra** Solve system of linear equations.

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2. Individual Logarithm

Find $b \in [1, Q - 2]$ s.t. $g^b h$ is B -smooth. Then factorization $g^b h = \prod p^{v_0(p)}$ implies

$$\log_g h \equiv \sum v_0(p) \log_g p - b \pmod{Q-1}$$

Coppersmith's Method - $\mathbb{F}_Q = \mathbb{F}_q[X]/\langle l_k(X) \rangle = \mathbb{F}(\alpha)$

- Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$ where $Q = p^k$; Find: $\log_g h$.

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Find n satisfying $p^{n-1} < k \leq p^n$, fix $n_1 + n_2 = n$.

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- Relation Collection** Known

$$\left(\prod F(X)^{v(C,F)} \right)^{p^{n_2}} \equiv \prod F(X)^{v(D,F)} \pmod{l_k(X)}$$

$X \leftarrow \alpha$, take \log_g on both sides, and $\log_g F(\alpha) \leftarrow x_F$

$$\sum (p^{n_2} v(C, F) - v(D, F)) x_F \equiv 0 \pmod{Q - 1}$$

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Coppersmith's Method - $\mathbb{F}_Q = \mathbb{F}_q[X]/\langle I_k(X) \rangle = \mathbb{F}(\alpha)$

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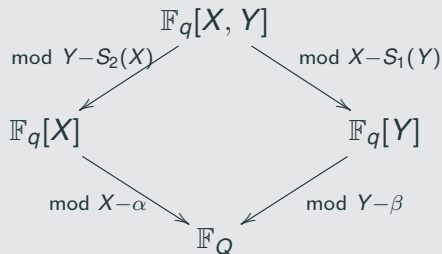
2. Individual Logarithm Phase Descent strategy

Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$ where $Q = q^k$

Find $S_1(X), S_2(X) \in \mathbb{F}_q[X]$ s.t. $\exists I_k(X) | X - S_1(S_2(X))$, a root α . Let $\beta = S_2(\alpha)$, then $\exists J_k(Y) | Y - S_2(S_1(Y))$ with root β .

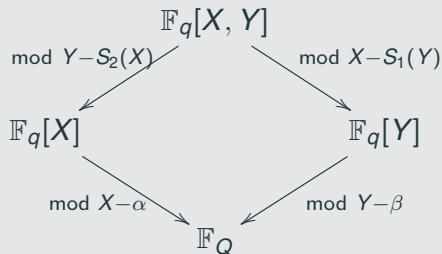
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Start with $G_1(Y)X + G_2(Y)$, we will reach the relation

$$G_1(S_2(\alpha))\alpha + G_2(S_1(\alpha)) = G_1(\beta)S_1(\beta) + G_2(\beta)$$

- Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$ where $Q = q^k$; Find: $\log_g h$
- Process:

1. Main Phase

- Initialization** Fix parameter B then obtain $\mathcal{F}_q(B)$. Find $S_1(X)$, $S_2(X) \in \mathbb{F}_q[X]$ of deg d_1, d_2 satisfying $d_1 d_2 \geq k$ s.t.
 $\exists l_k(X) | X - S_1(S_2(X))$ with a root α
- Smoothness Selection** Random $G_1(X), G_2(X)$ of deg $\leq B$ s.t. both $G_1(S_2(X))X + G_2(S_1(X))$ and $G_1(X)S_1(X) + G_2(X)$ are B -smooth, notice deg $\leq Bd_2 + 1, Bd_1$ respectively.
- Relation Collection** Known
 $G_1(S_2(\alpha))\alpha + G_2(S_1(\alpha)) = G_1(\beta)S_1(\beta) + G_2(\beta)$ and smoothness implies $\prod F(\alpha)^{v(F)} = \prod F(\beta)^{w(F)}$. Then take \log_g and $F(\alpha) \leftarrow x_F, F(\beta) \leftarrow y_F$

$$\sum v(F)x_F \equiv \sum w(F)y_F \pmod{Q-1}$$

- Linear Algebra** Solve system of linear equations.

2. Individual Logarithm Phase Descent strategy.

[Joux, 2013] and [BGJT@EC'14] - Half Relation

Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$ where $Q = q^k$ and $q > k$

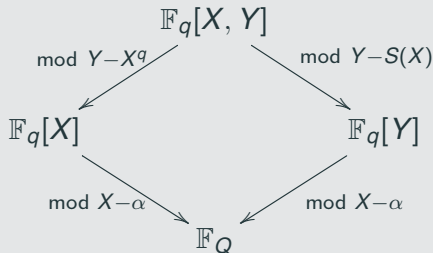
Take advantage of the identity $X^q - X = \prod_{\gamma \in \mathbb{F}_q} (X - \gamma)$.

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Take advantage of the identity $X^q - X = \prod_{\gamma \in \mathbb{F}_q} (X - \gamma)$.

Find $S(X) \in \mathbb{F}_q$ s.t. $\exists l_k(X) | X^q - S(X)$ with a root α .

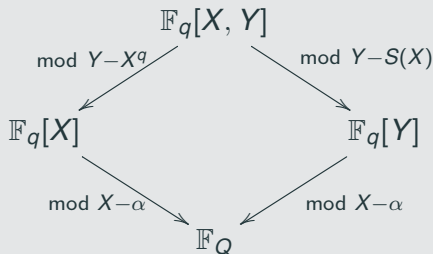


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From $G_1(Y)G_2(X) - G_1(X)G_2(Y)$ we can reach the relation

$$\prod_{\gamma \in \mathbb{F}_q} (G_1(\alpha) - \gamma G_2(\alpha)) = G_1(S(\alpha))G_2(\alpha) - G_1(\alpha)G_2(S(\alpha))$$

- Given: $h \in \mathbb{F}_Q^\times = \langle g \rangle$ where $Q = q^k$ and $q > k$; Find: $\log_g h$
- Process

1. Main Phase

- **Initialization** Fix parameter B then we obtain $\mathcal{F}_q(B)$ Find $S(X) \in \mathbb{F}_q$ of $\deg \leq B$ s.t. $\exists l_k(X) | X^q - S(X)$ with a root α .
- **Smoothness Selection** Random $G_1(X), G_2(X)$ of $\deg \leq B$ s.t. $G_1(S_2(X))G_2(X) - G_1(X)G_2(S(X))$ is B -smooth, notice $\deg \leq B(B+1)$.
- **Relation Collection** Known
$$\prod_{\gamma \in \mathbb{F}_q} (G_1(\alpha) - \gamma G_2(\alpha)) = G_1(S(\alpha))G_2(\alpha) - G_1(\alpha)G_2(S(\alpha))$$
and smoothness implies $\prod (G_1(\alpha) - \gamma G_2(\alpha)) = \prod F(\alpha)^{v(F)}$. Then take \log_g and $G_1(\alpha) - \gamma G_2(\alpha) \leftarrow x_\gamma, F(\alpha) \leftarrow x_F$

$$\sum x_\gamma \equiv \sum v(F)x_F \pmod{Q-1}$$

- **Linear Algebra** Solve system of linear equations.

2. Individual Logarithm Phase Descent strategy.

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Research Plan

1. Main Phase

- Initiation
- Smoothness Selection
- Relation Collection

- Linear Algebra

2. Individual Logarithm Phase

Paradigm with Transitions

1. Main Phase

- **Initiation** Field extension brings freedom in presentation
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Understanding (virtual) logarithms
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Research Plan

Preparations

- **Algorithm**
- **Number theory**
- **Polynomials over finite fields**
- **Lattice**
- **Algebraic geometry**

- **Algorithm**

Algorithmic Cryptanalysis by Joux

Course taken: Computational Number Theory

- **Number theory**

A Classical Introduction to Modern Number Theory

by Kenneth, Ireland and Rosen

- **Polynomials over finite fields** 2017 summer schools

- **Lattice** Lectures

- **Algebraic geometry** Course taken

- **Heuristics:** From assumption to rigorous.
CWZ, 2014: Three Heuristics

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Use other identities in finite fields and number theory.
- **As for other characteristics**
Generalization of FFS and NFS.

Thank you! Any questions?

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