Midterm, due Friday Oct 23 at 9am

Please note that any requests for an extension on this exam must be accompanied by justification in accordance with university policy (e.g. statement from the Center for Students with Disabilities, a doctor's not explaining the need for an extension, etc.). This is a stricter condition than has been applied to the homework. Also, note the exam is 2 pages (in case you print 2-sided, turn it over before submitting).

On this exam, you are free to use any code we have covered through the course, any function defined in matlab, any library defined in python, and any library in C++'s STL. You are free to use any book, web search, video, and any other resource you feel might be useful. You may use any code provided in class, or in the textbook, or your own homework. You are not permitted to copy solutions or borrow code from one another on this exam, and are not allowed to directly copy and paste code from the internet.

1. **5 points** True or False: When numerically solving a differential equation, it is always better to choose a smaller Δt. Explain why or why not.

2. **10 points** Short answer: Describe the difference between a non-adaptive and adaptive numerical method. Under what conditions are adaptive methods generally useful. Are there any disadvantages to using an adaptive method?

3. **15 points** This problem relates to the Discrete Fourier Transform. You are welcome to use the Fast Fourier Transform if you wish, but the data set is small enough there will be no appreciable difference between the two algorithms.

   (a) Import the data in data.csv, $\{x_i\}$, Fourier transform it, and plot the resulting power spectrum.

   (b) Define the sequence $\{y_i\}$, with $y_i = (x_i + x_{i+1})/2$ and $y_0 = y_N$ (periodic boundary conditions). Fourier transform this data, and comment on similarities or differences with part (a). This averaging procedure is an example of a low-pass filter. Explain why that name is appropriate.

4. **25 points** The logarithm of an $n \times n$ matrix $\mathbf{A}$ can be determined through a Taylor expansion:

$$\log(\mathbf{A}) = -\sum_{k=1}^{\infty} \frac{(\mathbf{I} - \mathbf{A})^k}{k} \tag{1}$$

   with $\mathbf{I}$ the identity matrix.

   (a) What constraints are there on the matrix $\mathbf{A}$ for this sum to converge?

   (b) What is the computational complexity of computing $\log(\mathbf{A})$ for large n using the Taylor expansion?

   (c) Write programs in C++, matlab, and python (all three languages) implementing eq. (1) to compute $\log(\mathbf{A})$. You may not use predefined methods for computing the matrix log. Run this program on the matrix $\mathbf{A}_{ij} = (1 + \delta_{ij})/(n + 1)$ for $n = 10$ and print the result to a file.

   (d) Run your programs in C++, matlab, and python to determine the time it takes (in seconds) to determine $\log(\mathbf{A})$ for $\mathbf{A}_{ij} = (1 + \delta_{ij})/(n + 1)$ with $n \in \{10, 20, 50, 100, 200, 500, 1000, 2000\}$. Plot the time as a function of n on log-log axes all in the same figure (that is, do not submit 3 separate graphs). Also plot your predicted complexity from (a) in the same figure.

5. **45 points** A system of 3 stars of mass M =1 solar mass located at locations $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$ have the potential energy

$$U(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) = -GM^2\left(\frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|} + \frac{1}{|\mathbf{R}_1 - \mathbf{R}_3|} + \frac{1}{|\mathbf{R}_2 - \mathbf{R}_3|}\right) \tag{2}$$

where G is the gravitational constant. Note that this three body problem has no known analytical solution for arbitrary initial conditions, but some initial conditions will give rise to periodic behavior (as you will show below). In this problem, you may use any programming language you like.

(a) Show that if we choose an arbitrary length scale, the system can be nondimensionalized in the form

$$\ddot{\mathbf{r}}_1 = -\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} \quad \ddot{\mathbf{r}}_2 = -\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} \quad \ddot{\mathbf{r}}_3 = -\frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_3|^3} - \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_3|^3}$$

What is a velocity of 1km/s in your units?

(b) Analytically show that if the first star is initially stationary at the origin, and if the second and third stars satisfy $\mathbf{r}_2 = -\mathbf{r}_3$ and with velocities $\mathbf{v}_2 = -\mathbf{v}_3$, that the first star will remain stationary for all time. Is this solution stable if $\mathbf{r}_1$ is perturbed?

(c) Suppose (in two dimensions) that the initial conditions are $\mathbf{r}_1 = (0, 0)$, $\mathbf{r}_2 = (0, 1)$, $\mathbf{r}_3 = (0, -1)$, $\mathbf{v}_1 = (0, 0)$, $\mathbf{v}_2 = (1, 0)$, and $\mathbf{v}_3 = (-1, 0)$ in your nondimensional units. Numerically integrate the equations of motion using either Verlet, RK4, or an adaptive integration scheme. You are *not* required to use all three algorithms, just pick one. Your integration should run to the dimensionless time $T = 50$. You are free to choose your timestep Δt or tolerance ϵ, but be aware the solution *should* be smooth and periodic (so reduce your timestep or debug your code if it is not).

For your solutions to this problem, you should submit (i) the code, (ii) a plot of the Kinetic, Potential, and Total energy of the system as a function of time, and (iii) the trajectories of $\mathbf{r}_i$ on the same plot (that is, $x_1(t)$, $x_2(t)$, and $x_3(t)$ on the x-axis, and $y_1(t)$, $y_2(t)$, and $y_3(t)$ on the y-axis).

(d) Rerun the simulation using the same timestep or tolerance as in (c), but changing the initial position of the first star to $\mathbf{r}_1 = (0.01, 1)$ (with all other initial conditions the same). You should provide code, a plot of the energy, and a plot of the trajectory as in (c). Do you find a periodic orbit? Is that consistent with your answer to (b)?

(e) Rerun the simulation in (d) using a timestep or tolerance smaller by a factor of 10. Is your trajectory with the different timestep identical to the trajectory you found in (d)? If not, explain why that might be. Note that you do not need to correct this discrepancy to receive full credit, simply discuss its origin.