(a)

Here, $u = \frac{1}{2} K_1 \sum_i r_i^2 + \sum V_{int} (|r_1 - r_2|)$

and $m\ddot{r}_i = -\xi \dot{r}_i - K_1 r_i - \sum_{j \neq 1} u'(|r_{ij}|)\, \hat{r}_{ij} + \vec{F}_i$

Now, $\frac{m}{N} \sum_i \ddot{r}_i = m\ddot{R} = -\xi \sum \dot{r}_i - K_1 \sum r_i + \sum f_i - \sum_i \sum_{j\neq 1} u'(|r_{ij}|)\, \hat{r}_{ij}$

$$= -\xi \dot{R} - K_1 R + f_c - \left[u(r_{12}) (\hat{r}_{12} + \hat{r}_{21}) + u(r_{23}) (\hat{r}_{12} + \hat{r}_{21}) + \ldots \right]$$

$$= -\xi \dot{R} - K_1 R + f_c \quad \left[\because \hat{r}_{ij} = -\hat{r}_{ji} \right]$$

So, probability that $\sum_i \vec{F}_i = \vec{F}_c$

$$P(\vec{F}_c) = \int d^3F_1 d^3F_2 d^3F_3 \ldots d^3F_N \, P(F_1) P(F_2) \ldots P(F_N) \times \delta\left(\sum_i \vec{F}_i - F_c\right)$$

$$\propto \int \frac{d^d k}{2\pi} e^{ikF_c} \int \prod_i d^d F_i \, e^{\left(-\frac{\Delta t}{4\xi k_B T} \sum_{ij} F_{ij}^2 - i \sum_{ij} k_{ij} F_{ij}\right)}$$

$$\propto \int \frac{d^d k}{2\pi} e^{ikF_c} \int \prod_i d^d F_i \cdot \exp\left[-\frac{\Delta t}{4\xi k_B T} \left(F_{ij} - i\frac{2k\xi k_B T}{\Delta t}\right)^2 - \frac{N k^2 \xi k_B T}{\Delta t} \right]$$

Now, integrating over F_{ij} gives us a constant independent of k or F_c

So, we get.

$$\propto \int \frac{d^d K}{2\pi} e^{iKF_c - K^2 \xi K_B T/\Delta t}$$

$$= \int \frac{d^d K}{2\pi} e^{-\frac{N \xi K_B T}{\Delta t}\left(K - i\frac{\Delta t F_c}{2N\xi K_B T}\right)^2 - \frac{\Delta t F_c^2}{2N\xi K_B T}}$$

$$\propto e^{-\Delta t F_c^2/2N\xi K_B T}$$

The effective random force on "R" and "r" are both Gaussian distribution with "0" mean and variance

$$\langle F_c^2 \rangle = \frac{2\xi K_B T}{N \Delta t}$$

So, it gives us

$$\langle F_c \rangle = 0 \quad \text{and}$$

$$\langle F_c(t) F_c(t') \rangle = \frac{2\xi K_B T}{N} \delta(t-t')$$

(showed)

(b)

Laplace transforms $t \to \lambda$

$$\mathcal{L}\{\dot{x}\} = \lambda \tilde{x} - x_0$$

$$\mathcal{L}\{\ddot{x}\} = \lambda^2 \tilde{x} - \lambda \dot{x}_0 + x_0$$

so, taking the Laplace transform of Langevin eq^n

$$\mathcal{L}\{m\ddot{x} + \xi\dot{x} + K_1 x\} = (m\lambda^2 + \lambda\xi + K_1)\tilde{x} = \tilde{F}_c$$

Since, $x_0 = \dot{x}_0 = 0$

Therefore, $\tilde{x} = \dfrac{\tilde{F}_c}{m\lambda^2 + \xi\lambda + k_1}$

and with the convolution theorem

$$x(t) = \int_0^t dt' F_c(t') \, \mathcal{L}^{-1}_{t-t'} \left[\frac{1}{m\lambda^2 + \xi\lambda + k_1} \right]$$

$$= \int_0^t dt' F_c(t') \left[\frac{2m}{\tau_2} e^{-(t-t')/2\tau_1} \sinh\left(\frac{t-t'}{2\tau_2}\right) \right]$$

where, $\tau_1 = m/\xi$ and $\tau_2 = \dfrac{m}{\sqrt{\xi^2 - 4mk_1}}$

if $\xi < 2\sqrt{mk}$; τ_2 becomes imaginary

therefore, within the integration limit $[0,t]$

we need to consider $\dfrac{1}{\tau_2} \sinh\left(\dfrac{t-t'}{2\tau_2}\right)$ as

$$\frac{1}{|\tau_2|} \sinh \frac{(t-t')}{2|\tau_2|}$$

Now, $\langle R^2 \rangle = 3\langle R_x^2 \rangle = 3 \int_0^t dt' dt'' K(t-t') K(t-t'') \langle F_c(t') \cdot F_c(t'') \rangle$

$$= \frac{12 K_B T \xi m^2}{N \tau_2^2} \int_0^t dt' K^2(t-t')$$

with, $K(t-t') = e^{-t/2\tau_1} \sinh\left(\dfrac{t}{2\tau_2}\right)$

$$K(t) = \frac{1}{2}\left[e^{-t\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)} - e^{-t\left(\frac{1}{\tau_1}-\frac{1}{\tau_2}\right)}\right]$$

So, we finally obtain,

$$\langle R^2 \rangle = \frac{3\tau_1\tau_2 k_B T \xi}{m^2}\Big[2e^{-t/\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_1+t/\tau_2} - \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_1-t/\tau_2} - \frac{2\tau_1^2}{\tau_1^2-\tau_2^2}\Big]$$

with $\tau_2 \to 0$, this doesn't diverge. Rather

$$\frac{1}{|\tau_2|}\sinh\left(\frac{t-t'}{2/|\tau_2|}\right) \approx \frac{t-t'}{2}$$