The Hopf model:

when converting cartesian co-ordinates into polar co-ordinates,

we know –

$$x = r\cos\theta \; ; \; y = r\sin\theta \quad \text{and} \quad x^2 + y^2 = r^2$$

Now,
$$\frac{dx}{dt} = ax + y - x(x^2+y^2)$$

$$\Rightarrow \frac{d}{dt}(r\cos\theta) = ar\cos\theta + r\sin\theta - r^3\cos\theta$$

$$\Rightarrow \cos\theta \cdot \frac{dr}{dt} - r\sin\theta \cdot \frac{d\theta}{dt} = ar\cos\theta + r\sin\theta - r^3\cos\theta$$

$$\Rightarrow r\cos\theta\left(\frac{1}{r}\cdot\frac{dr}{dt} - a + r^2\right) = r\sin\theta\left(1+\frac{d\theta}{dt}\right)$$

$$\therefore \frac{1}{r}\cdot\frac{dr}{dt} - a + r^2 = \tan\theta \cdot \left(1 + \frac{d\theta}{dt}\right) \longrightarrow (1)$$

Again,
$$\frac{dy}{dt} = -x + ay - y(x^2+y^2)$$

$$\Rightarrow \frac{d}{dt}(r\sin\theta) = -r\cos\theta + ar\sin\theta - r^3\sin\theta$$

$$\Rightarrow \sin\theta \cdot \frac{dr}{dt} + r\cos\theta \cdot \frac{d\theta}{dt} = ar\sin\theta - r\cos\theta - r^3\sin\theta$$

$$\Rightarrow -r\sin\theta\left(\frac{1}{r}\cdot\frac{dr}{dt} - a + r^2\right) = r\cos\theta\left(1+\frac{d\theta}{dt}\right)$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{dt} - a + r^2 = -\cot\theta\left(1+\frac{d\theta}{dt}\right)$$

$$\Rightarrow \tan\theta\left(1+\frac{d\theta}{dt}\right) = -\cot\theta\left(1+\frac{d\theta}{dt}\right) \quad [\text{from } (1)]$$

$$\Rightarrow \left(1+\frac{d\theta}{dt}\right)(\tan\theta + \cot\theta) = 0$$

$$\Rightarrow \left(1 + d\theta/dt\right) = 0 \quad [\because (\tan\theta + \cot\theta) \neq 0]$$

$$\therefore \boxed{\frac{d\theta}{dt} = -1}$$

using this in ①, we obtain.

$$\frac{1}{r} \cdot \frac{dr}{dt} - a + r^2 = 0$$

$$\therefore \boxed{\frac{dr}{dt} = ar - r^3}$$

So, rewriting the given equations in polar co-ordinates gives us —

$$\begin{cases} \dfrac{dr}{dt} = ar - r^3 ; \\ \text{and } \dfrac{d\theta}{dt} = -1 \end{cases}$$

Now, we can write.

$$\dot{r} = ar - r^3$$

the only critical point of this system is $r^* = 0$; i.e the origin. Since, $\frac{d\theta}{dt} < 0$, the trajectories move clockwise about the origin.

If $a = 0$, then $\dot{r} = -r^3$. Therefore, for non-zero r, we have $\dot{r} < 0$. Hence, there are no closed orbits and all trajectories approach the origin as $t \to \infty$. The origin is a stable focus.

If $a < 0$, then $(a - r^2) < 0$ for all r. As in the previous case $\dot{r} < 0$ for all non-zero "r" values. Again, there are no closed orbits and all trajectories fall into the origin.

Now, if $a > 0$, then $\dot{r} < 0$ for $r \in (\sqrt{a}, \infty)$ and $\dot{r} > 0$ for $r \in (0, \sqrt{a})$. In this case, the origin is unstable focus. and there is a stable orbit at $r = \sqrt{a}$, which means they limit to a circle of radius $\sqrt{a}$.

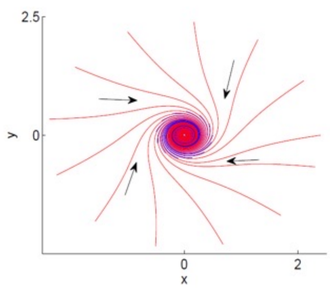

Fig-1

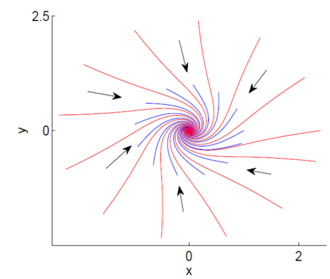

Fig-2

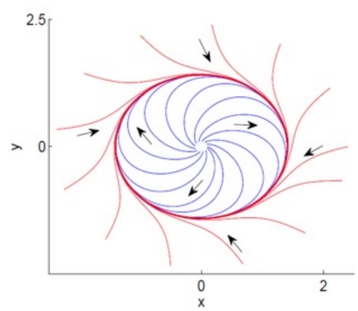

Fig-3

Fig-1 is Phase potrait for $\dot{r} = ar - r^3$ with $a = 0$.
The origin is a stable focus.

Fig-2 is Phase potrait for $\dot{r} = ar - r^3$. with $a < 0$.
The origin is a stable focus.

Fig-3 is phase potrait for $\dot{r} = ar - r^3$, with $a>0$
In this case, the origin is an unstable focus and there is a stable orbit $r=\sqrt{a}$.