We have the continuity eq^n (2-D)

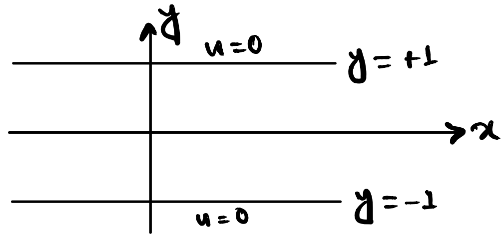
$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \longrightarrow ①$$

And the momentum eq^n. (2-D)

$$\left.\begin{aligned} \frac{\partial u_x}{\partial t} + u_x \cdot \frac{\partial u_x}{\partial x} + u_y \cdot \frac{\partial u_y}{\partial y} &= -\frac{\partial p}{\partial x} + Re^{-1}\left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right] \\ \frac{\partial u_y}{\partial t} + u_x \cdot \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= -\frac{\partial p}{\partial y} + Re^{-1}\left[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right] \end{aligned}\right\} ②$$

Now, from the given boundary conditions we obtain,

- $\underline{u} = 0$ at walls
- flow symmetric around $y = 0$
- periodic boundary conditions on $\underline{u}$ in the x-direction, i.e. flow parallel to the walls.



Therefore, we expect, $u_y = 0$, $\frac{\partial p}{\partial y} = 0$ and $u_x = u_x(y)$

And so ② becomes,

$$\frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} + Re^{-1}\left(\frac{\partial^2 u_x}{\partial y^2}\right)$$

for a fully developed flow - no change in profile in streamwise direction

i.e. $\frac{\partial u_x}{\partial t} = 0$

And, we finally obtain ② as -

$$0 = -\frac{\partial p}{\partial x} + Re^{-1} \cdot \frac{\partial^2 u_x}{\partial y^2} \longrightarrow ③$$

Integrating ③ once, we get.

$$y \cdot \frac{\partial p}{\partial x} = Re^{-1} \cdot \frac{\partial u_x}{\partial y} + c_1 \longrightarrow ④$$

Now, due to symmetry at $y=0$, $\frac{\partial u_x}{\partial y} = 0$

which gives us $c_1 = 0$

And integrating again,

$$\frac{y^2}{2} \cdot \frac{\partial p}{\partial x} = Re^{-1} u_x + c_2 \longrightarrow ⑤$$

Again, at $y = \pm 1$ we have $u_x = 0$

So, $c_2 = \frac{1}{2} \cdot \frac{\partial p}{\partial x}$

$\therefore$ Finally, we obtain,

$$u_x = - Re \frac{(1-y^2)}{2} \cdot \frac{\partial p}{\partial x} \longrightarrow ⑥$$

Now, let. $\frac{\partial p}{\partial x} = -\alpha$

$$\Rightarrow \frac{dp}{dx} = -\alpha$$

$$\Rightarrow p = -\alpha x + A \longrightarrow ⑦$$

at $x=0$, $p = p_0$ which gives us $A = p_0$

So, we obtain a set of solutions for Poiseuille flow –

$$\boxed{\begin{aligned} p &= p_0 - \alpha x \\ u_x &= \alpha Re \frac{(1-y^2)}{2} \\ u_y &= 0 \end{aligned}}$$ for any α.