FLUID MECHANICS

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CHAPTER II

VISCOUS FLUIDS

§15. The equations of motion of a viscous fluid

LET us now study the effect of energy dissipation, occurring during the motion of a fluid, on that motion itself. This process is the result of the thermodynamic irreversibility of the motion. This irreversibility always occurs to some extent, and is due to internal friction (viscosity) and thermal conduction.

In order to obtain the equations describing the motion of a viscous fluid, we have to include some additional terms in the equation of motion of an ideal fluid. The equation of continuity, as we see from its derivation, is equally valid for any fluid, whether viscous or not. Euler's equation, on the other hand, requires modification.

We have seen in §7 that Euler's equation can be written in the form

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k},$$

where Π_{ik} is the momentum flux density tensor. The momentum flux given by formula (7.2) represents a completely reversible transfer of momentum, due simply to the mechanical transport of the different particles of fluid from place to place and to the pressure forces acting in the fluid. The viscosity (internal friction) is due to another, irreversible, transfer of momentum from points where the velocity is large to those where it is small.

The equation of motion of a viscous fluid may therefore be obtained by adding to the "ideal" momentum flux (7.2) a term $-\sigma'_{ik}$ which gives the irreversible "viscous" transfer of momentum in the fluid. Thus we write the momentum flux density tensor in a viscous fluid in the form

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma'_{ik} = -\sigma_{ik} + \rho v_i v_k. \tag{15.1}$$

The tensor

$$\sigma_{ik} = -p\delta_{ik} + \sigma'_{ik} \tag{15.2}$$

is called the *stress tensor*, and σ'_{ik} the *viscosity stress tensor*. σ_{ik} gives the part of the momentum flux that is not due to the direct transfer of momentum with the mass of moving fluid.†

The general form of the tensor σ'_{ik} can be established as follows. Processes

† We shall see below that σ'_{ik} contains a term proportional to δ_{ik}, i.e. of the same form as the term $p\delta_{ik}$. When the momentum flux tensor is put in such a form, therefore, we should specify what is meant by the pressure p; see the end of §49.

of internal friction occur in a fluid only when different fluid particles move with different velocities, so that there is a relative motion between various parts of the fluid. Hence σ'_{ik} must depend on the space derivatives of the velocity. If the velocity gradients are small, we may suppose that the momentum transfer due to viscosity depends only on the first derivatives of the velocity. To the same approximation, σ'_{ik} may be supposed a linear function of the derivatives $\partial v_i/\partial x_k$. There can be no terms in σ'_{ik} independent of $\partial v_i/\partial x_k$, since σ'_{ik} must vanish for v = constant. Next, we notice that σ'_{ik} must also vanish when the whole fluid is in uniform rotation, since it is clear that in such a motion no internal friction occurs in the fluid. In uniform rotation with angular velocity $\mathbf{\Omega}$, the velocity $\mathbf{v}$ is equal to the vector product $\mathbf{\Omega}\times\mathbf{r}$. The sums

$$\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}$$

are linear combinations of the derivatives $\partial v_i/\partial x_k$, and vanish when $\mathbf{v} = \mathbf{\Omega}\times\mathbf{r}$. Hence σ'_{ik} must contain just these symmetrical combinations of the derivatives $\partial v_i/\partial x_k$.

The most general tensor of rank two satisfying the above conditions is

$$\sigma'_{ik} = a\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)+b\frac{\partial v_l}{\partial x_l}\delta_{ik},$$

where a and b are independent of the velocity.† It is convenient, however, to write this expression in a slightly different form, in which a and b are replaced by other constants:

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)+\zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}. \tag{15.3}$$

The expression in parentheses has the property of vanishing on contraction with respect to i and k. The constants η and ζ are called *coefficients of viscosity*. As we shall show in §§16 and 49, they are both positive:

$$\eta > 0, \qquad \zeta > 0. \tag{15.4}$$

The equations of motion of a viscous fluid can now be obtained by simply adding the expressions $\partial\sigma'_{ik}/\partial x_k$ to the right-hand side of Euler's equation

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i}.$$

Thus we have

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)$$
$$= -\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right\}+\frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_l}{\partial x_l}\right). \tag{15.5}$$

† In making this statement we use the fact that the fluid is isotropic, as a result of which its properties must be described by scalar quantities only (in this case, a and b).

This is the most general form of the equations of motion of a viscous fluid. The quantities η and ζ are functions of pressure and temperature. In general, p and T, and therefore η and ζ, are not constant throughout the fluid, so that η and ζ cannot be taken outside the gradient operator.

In most cases, however, the viscosity coefficients do not change noticeably in the fluid, and they may be regarded as constant. We then have

$$\frac{\partial \sigma'_{ik}}{\partial x_k} = \eta\left(\frac{\partial^2 v_i}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_k} - \frac{2}{3}\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l}\right) + \zeta\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l}$$

$$= \eta\frac{\partial^2 v_i}{\partial x_k \partial x_k} + (\zeta + \tfrac{1}{3}\eta)\frac{\partial}{\partial x_i}\frac{\partial v_l}{\partial x_l}.$$

But

$$\partial v_l/\partial x_l \equiv \operatorname{div}\mathbf{v}, \qquad \partial^2 v_i/\partial x_k \partial x_k \equiv \triangle v_i.$$

Hence we can write the equation of motion of a viscous fluid, in vector form,

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v}\right] = -\mathbf{grad}\,p + \eta\triangle\mathbf{v} + (\zeta + \tfrac{1}{3}\eta)\mathbf{grad}\operatorname{div}\mathbf{v}. \quad (15.6)$$

If the fluid may be regarded as incompressible, $\operatorname{div}\mathbf{v} = 0$, and the last term on the right of (15.6) is zero. Thus the equation of motion of an incompressible viscous fluid is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -\frac{1}{\rho}\mathbf{grad}\,p + \frac{\eta}{\rho}\triangle\mathbf{v}. \quad (15.7)$$

This is called the *Navier–Stokes equation.* The stress tensor in an incompressible fluid takes the simple form

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right). \quad (15.8)$$

We see that the viscosity of an incompressible fluid is determined by only one coefficient. Since most fluids may be regarded as practically incompressible, it is this viscosity coefficient η which is generally of importance. The ratio

$$\nu = \eta/\rho \quad (15.9)$$

is called the ***kinematic viscosity*** (while η itself is called the *dynamic viscosity*). We give below the values of η and ν for various fluids, at a temperature of 20° C:

	η (g/cm sec)	ν (cm^2/sec)
Water	0·010	0·010
Air	0·00018	0·150
Alcohol	0·018	0·022
Glycerine	8·5	6·8
Mercury	0·0156	0·0012

It may be mentioned that the dynamic viscosity of a gas at a given temperature is independent of the pressure. The kinematic viscosity, however, is inversely proportional to the pressure.

The pressure can be eliminated from the Navier–Stokes equation in the same way as from Euler's equation. Taking the curl of both sides of equation (15.7), we obtain, instead of equation (2.11) as for an ideal fluid,

$$\frac{\partial}{\partial t}(\mathbf{curl\,v}) = \mathbf{curl}\,(\mathbf{v}\times\mathbf{curl\,v})+\nu\triangle(\mathbf{curl\,v}). \tag{15.10}$$

We must also write down the boundary conditions on the equations of motion of a viscous fluid. There are always forces of molecular attraction between a viscous fluid and the surface of a solid body, and these forces have the result that the layer of fluid immediately adjacent to the surface is brought completely to rest, and "adheres" to the surface. Accordingly, the boundary conditions on the equations of motion of a viscous fluid require that the fluid velocity should vanish at fixed solid surfaces:

$$\mathbf{v} = 0. \tag{15.11}$$

It should be emphasised that both the normal and the tangential velocity component must vanish, whereas for an ideal fluid the boundary conditions require only the vanishing of v_n.†

In the general case of a moving surface, the velocity **v** must be equal to the velocity of the surface.

It is easy to write down an expression for the force acting on a solid surface bounding the fluid. The force acting on an element of the surface is just the momentum flux through this element. The momentum flux through the surface element d**f** is

$$\Pi_{ik}\mathrm{d}f_k = (\rho v_i v_k - \sigma_{ik})\mathrm{d}f_k.$$

Writing $\mathrm{d}f_k$ in the form $\mathrm{d}f_k = n_k\,\mathrm{d}f$, where **n** is a unit vector along the normal, and recalling that $\mathbf{v} = 0$ at a solid surface,‡ we find that the force **P** acting on unit surface area is

$$P_i = -\sigma_{ik}n_k = pn_i - \sigma'_{ik}n_k. \tag{15.12}$$

The first term is the ordinary pressure of the fluid, while the second is the force of friction, due to the viscosity, acting on the surface. We must emphasise that **n** in (15.12) is a unit vector along the outward normal to the fluid, i.e. along the inward normal to the solid surface.

If we have a surface of separation between two immiscible fluids, the conditions at the surface are that the velocities of the fluids must be equal

† We may note that, in general, Euler's equations cannot be satisfied with the boundary condition $\mathbf{v} = 0$.

‡ In determining the force acting on the surface, each surface element must be considered in a frame of reference in which it is at rest. The force is equal to the momentum flux only when the surface is fixed.

and the forces which they exert on each other must be equal and opposite. The latter condition is written

$$n_{1,k}\,\sigma_{1,ik}+n_{2,k}\,\sigma_{2,ik} = 0,$$

where the suffixes 1 and 2 refer to the two fluids. The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are in opposite directions, i.e. $n_{1,i} = -n_{2,i} = n_i$, so that we can write

$$n_i\,\sigma_{1,ik} = n_i\,\sigma_{2,ik}. \tag{15.13}$$

At a free surface of the fluid the condition

$$\sigma_{ik}\,n_k \equiv \sigma'_{ik}\,n_k - pn_i = 0 \tag{15.14}$$

must hold.

We give below, for reference, expressions for the components of the stress tensor and the Navier–Stokes equation in cylindrical and spherical co-ordinates. In cylindrical co-ordinates r, ϕ, z the components of the stress tensor are

$$\sigma_{rr} = -p+2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma_{r\phi} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial \phi}+\frac{\partial v_\phi}{\partial r}-\frac{v_\phi}{r}\right),$$

$$\sigma_{\phi\phi} = -p+2\eta\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r}{r}\right), \qquad \sigma_{\phi z} = \eta\left(\frac{\partial v_\phi}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \phi}\right),$$

$$\sigma_{zz} = -p+2\eta\frac{\partial v_z}{\partial z}, \qquad \sigma_{zr} = \eta\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right). \tag{15.15}$$

The three components of the Navier–Stokes equation and the equation of continuity are

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\phi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}-\frac{v_r}{r^2}\right),$$

$$\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi}+v_z\frac{\partial v_\phi}{\partial z}+\frac{v_r v_\phi}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial \phi}+\nu\left(\frac{\partial^2 v_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{\partial^2 v_\phi}{\partial z^2}+\frac{1}{r}\frac{\partial v_\phi}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial \phi}-\frac{v_\phi}{r^2}\right),$$

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi}+v_z\frac{\partial v_z}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right),$$

$$\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r} = 0. \tag{15.16}$$

In spherical co-ordinates r, ϕ, θ we have for the stress tensor

$$
\begin{aligned}
\sigma_{rr} &= -p+2\eta\frac{\partial v_r}{\partial r},\\
\sigma_{\phi\phi} &= -p+2\eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right),\\
\sigma_{\theta\theta} &= -p+2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right),\\
\sigma_{r\theta} &= \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right),\\
\sigma_{\theta\phi} &= \eta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}-\frac{v_\phi\cot\theta}{r}\right),\\
\sigma_{\phi r} &= \eta\left(\frac{\partial v_\phi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r}\right),
\end{aligned}
\tag{15.17}
$$

while the equations of motion are

$$
\begin{aligned}
&\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\theta^2+v_\phi^2}{r}\\
&\quad= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{1}{r}\frac{\partial^2(rv_r)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}-\right.\\
&\qquad\qquad\left.-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2v_r}{r^2}-\frac{2\cot\theta}{r^2}v_\theta\right],\\
&\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\\
&\quad= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left[\frac{1}{r}\frac{\partial^2(rv_\theta)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}+\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta}-\right.\\
&\qquad\qquad\left.-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\sin^2\theta}\right],\\
&\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r v_\phi}{r}+\frac{v_\theta v_\phi\cot\theta}{r}\\
&\quad= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}+\nu\left[\frac{1}{r}\frac{\partial^2(rv_\phi)}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}+\right.\\
&\qquad\qquad\left.+\frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi}{r^2\sin^2\theta}\right],
\end{aligned}
$$

$$
\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{2v_r}{r}+\frac{v_\theta\cot\theta}{r}=0. \tag{15.18}
$$

Finally, we give the equation that must be satisfied by the stream function $\psi(x, y)$ in two-dimensional flow of an incompressible viscous fluid. It is obtained by substituting $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$, $v_z = 0$ in equation (15.10):

$$\frac{\partial}{\partial t}(\triangle\psi) - \frac{\partial\psi}{\partial x}\frac{\partial(\triangle\psi)}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial(\triangle\psi)}{\partial x} - \nu\triangle^2\psi = 0. \tag{15.19}$$

§16. Energy dissipation in an incompressible fluid

The presence of viscosity results in the dissipation of energy, which is finally transformed into heat. The calculation of the energy dissipation is especially simple for an incompressible fluid.

The total kinetic energy of an incompressible fluid is

$$E_{\text{kin}} = \tfrac{1}{2}\rho \int v^2 \,\mathrm{d}V.$$

We take the time derivative of this energy, writing $\partial(\frac{1}{2}\rho v^2)/\partial t = \rho v_i \partial v_i/\partial t$ and substituting for $\partial v_i/\partial t$ the expression for it given by the Navier–Stokes equation:

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma'_{ik}}{\partial x_k}.$$

The result is

$$\begin{aligned}\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) &= -\rho\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} - \mathbf{v}\cdot\mathbf{grad}\,p + v_i\frac{\partial \sigma'_{ik}}{\partial x_k} \\ &= -\rho(\mathbf{v}\cdot\mathbf{grad})\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) + \operatorname{div}(\mathbf{v}\cdot\boldsymbol{\sigma}') - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}.\end{aligned}$$

Here $\mathbf{v}\cdot\boldsymbol{\sigma}'$ denotes the vector whose components are $v_i\sigma'_{ik}$. Since $\operatorname{div}\mathbf{v} = 0$ for an incompressible fluid, we can write the first term on the right as a divergence:

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2) = -\operatorname{div}\left[\rho\mathbf{v}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) - \mathbf{v}\cdot\boldsymbol{\sigma}'\right] - \sigma'_{ik}\frac{\partial v_i}{\partial x_k}. \tag{16.1}$$

The expression in brackets is just the energy flux density in the fluid: the term $\rho\mathbf{v}(\frac{1}{2}v^2 + p/\rho)$ is the energy flux due to the actual transfer of fluid mass, and is the same as the energy flux in an ideal fluid (see (10.5)). The second term, $\mathbf{v}\cdot\boldsymbol{\sigma}'$, is the energy flux due to processes of internal friction. For the presence of viscosity results in a momentum flux σ'_{ik}; a transfer of momentum, however, always involves a transfer of energy, and the energy flux is clearly equal to the scalar product of the momentum flux and the velocity.

If we integrate (16.1) over some volume V, we obtain

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\rho v^2 \,\mathrm{d}V = -\oint \left[\rho\mathbf{v}\left(\tfrac{1}{2}v^2+\frac{p}{\rho}\right)-\mathbf{v}\cdot\boldsymbol{\sigma}'\right]\cdot\mathrm{d}\mathbf{f}-\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\mathrm{d}V. \quad (16.2)$$

The first term on the right gives the rate of change of the kinetic energy of the fluid in V owing to the energy flux through the surface bounding V. The integral in the second term is consequently the decrease per unit time in the kinetic energy owing to dissipation.

If the integration is extended to the whole volume of the fluid, the surface integral vanishes (since the velocity vanishes at infinity†), and we find the energy dissipated per unit time in the whole fluid to be

$$\dot{E}_{\mathrm{kin}} = -\int \sigma'_{ik}\frac{\partial v_i}{\partial x_k}\,\mathrm{d}V.$$

In incompressible fluids, the tensor σ'_{ik} is given by (15.8), so that

$$\sigma'_{ik}\frac{\partial v_i}{\partial x_k} = \eta\frac{\partial v_i}{\partial x_k}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right).$$

It is easy to verify that this expression can be written

$$\tfrac{1}{2}\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2.$$

Thus we have finally for the energy dissipation in an incompressible fluid

$$\dot{E}_{\mathrm{kin}} = -\tfrac{1}{2}\eta\int\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2\mathrm{d}V. \quad (16.3)$$

The dissipation leads to a decrease in the mechanical energy, i.e. we must have $\dot{E}_{\mathrm{kin}} < 0$. The integral in (16.3), however, is always positive. We therefore conclude that the viscosity coefficient η is always positive.

PROBLEM

Transform the integral (16.3) for potential flow into an integral over the surface bounding the region of flow.

SOLUTION. Putting $\partial v_i/\partial x_k = \partial v_k/\partial x_i$ and integrating once by parts, we find

$$\dot{E}_{\mathrm{kin}} = -2\eta\int\left(\frac{\partial v_i}{\partial x_k}\right)^2\mathrm{d}V = -2\eta\int v_i\frac{\partial v_i}{\partial x_k}\mathrm{d}f_k,$$

or

$$\dot{E}_{\mathrm{kin}} = -\eta\int \mathbf{grad}\,v^2\cdot\mathrm{d}\mathbf{f}.$$

† We are considering the motion of the fluid in a system of co-ordinates such that the fluid is at rest at infinity. Here, and in similar cases, we speak, for the sake of definiteness, of an infinite volume of fluid, but this implies no loss of generality. For a fluid enclosed in a finite volume, the surface integral again vanishes, because the normal velocity component at the surface vanishes.

§17. Flow in a pipe

We shall now consider some simple problems of motion of an incompressible viscous fluid.

Let the fluid be enclosed between two parallel planes moving with a constant relative velocity **u**. We take one of these planes as the xz-plane, with the x-axis in the direction of **u**. It is clear that all quantities depend only on y, and that the fluid velocity is everywhere in the x-direction. We have from (15.7) for steady flow

$$\mathrm{d}p/\mathrm{d}y = 0, \qquad \mathrm{d}^2v/\mathrm{d}y^2 = 0.$$

(The equation of continuity is satisfied identically.) Hence p = constant, $v = ay+b$. For $y = 0$ and $y = h$ (h being the distance between the planes) we must have respectively $v = 0$ and $v = u$. Thus

$$v = yu/h. \tag{17.1}$$

The fluid velocity distribution is therefore linear. The mean fluid velocity, defined as

$$\bar{v} = \frac{1}{h}\int_0^h v\,\mathrm{d}y,$$

is

$$\bar{v} = \tfrac{1}{2}u. \tag{17.2}$$

From (15.12) we find that the normal component of the force on either plane is just p, as it should be, while the tangential friction force on the plane $y = 0$ is

$$\sigma_{xy} = \eta\,\mathrm{d}v/\mathrm{d}y = \eta u/h; \tag{17.3}$$

the force on the plane $y = h$ is $-\eta u/h$.

Next, let us consider steady flow between two fixed parallel planes in the presence of a pressure gradient. We choose the co-ordinates as before; the x-axis is in the direction of motion of the fluid. The Navier–Stokes equations give, since the velocity clearly depends only on y,

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = 0.$$

The second equation shows that the pressure is independent of y, i.e. it is constant across the depth of the fluid between the planes. The right-hand side of the first equation is therefore a function of x only, while the left-hand side is a function of y only; this can be true only if both sides are constant. Thus $\mathrm{d}p/\mathrm{d}x$ = constant, i.e. the pressure is a linear function of the co-ordinate x along the direction of flow. For the velocity we now obtain

$$v = \frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x}y^2 + ay + b.$$

The constants a and b are determined from the boundary conditions, $v = 0$ for $y = 0$ and $y = h$. The result is

$$v = -\frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x}[\tfrac{1}{4}h^2-(y-\tfrac{1}{2}h)^2]. \tag{17.4}$$

Thus the velocity varies parabolically across the fluid, reaching its maximum value in the middle. The mean fluid velocity (averaged over the depth of the fluid) is again

$$\bar{v} = \frac{1}{h}\int_0^h v\,\mathrm{d}y;$$

on calculating this, we find

$$\bar{v} = -\frac{h^2}{12\eta}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{17.5}$$

We may also calculate the frictional force $\sigma_{xy} = \eta(\partial v/\partial y)_{y=0}$ acting on one of the fixed planes. Substitution from (17.4) gives

$$\sigma_{xy} = -\tfrac{1}{2}h\,\mathrm{d}p/\mathrm{d}x. \tag{17.6}$$

Finally, let us consider steady flow in a pipe of arbitrary cross-section (the same along the whole length of the pipe, however). We take the axis of the pipe as the x-axis. The fluid velocity is evidently along the x-axis at all points, and is a function of y and z only. The equation of continuity is satisfied identically, while the y and z components of the Navier–Stokes equation again give $\partial p/\partial y = \partial p/\partial z = 0$, i.e. the pressure is constant over the cross-section of the pipe. The x-component of equation (15.7) gives

$$\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta}\frac{\mathrm{d}p}{\mathrm{d}x}. \tag{17.7}$$

Hence we again conclude that $\mathrm{d}p/\mathrm{d}x$ = constant; the pressure gradient may therefore be written $-\Delta p/l$, where Δp is the pressure difference between the ends of the pipe and l is its length.

Thus the velocity distribution for flow in a pipe is determined by a two-dimensional equation of the form $\triangle v$ = constant. This equation has to be solved with the boundary condition $v = 0$ at the circumference of the cross-section of the pipe. We shall solve the equation for a pipe of circular cross-section. Taking the origin at the centre of the circle and using polar co-ordinates, we have by symmetry $v = v(r)$. Using the expression for the Laplacian in polar co-ordinates, we have

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = -\frac{\Delta p}{\eta l}.$$

Integrating, we find

$$v = -\frac{\Delta p}{4\eta l}r^2 + a\log r + b. \tag{17.8}$$

The constant a must be put equal to zero, since the velocity must remain finite at the centre of the pipe. The constant b is determined from the requirement that $v = 0$ for $r = R$, where R is the radius of the pipe. We then find

$$v = \frac{\Delta p}{4\eta l}(R^2 - r^2). \tag{17.9}$$

Thus the velocity distribution across the pipe is parabolic.

It is easy to determine the mass Q of fluid passing each second through any cross-section of the pipe (called the *discharge*). A mass $\rho \cdot 2\pi r v \,\mathrm{d}r$ passes each second through an annular element $2\pi r \,\mathrm{d}r$ of the cross-sectional area. Hence

$$Q = 2\pi\rho \int_0^R rv \,\mathrm{d}r.$$

Using (17.9), we obtain

$$Q = \frac{\pi \Delta p}{8\nu l}R^4. \tag{17.10}$$

The mass of fluid is thus proportional to the fourth power of the radius of the pipe (*Poiseuille's formula*).

PROBLEMS

PROBLEM 1. Determine the flow in a pipe of annular cross-section, the internal and external radii being R_1, R_2.

SOLUTION. Determining the constants a and b in the general solution (17.8) from the conditions that $v = 0$ for $r = R_1$ and $r = R_2$, we find

$$v = \frac{\Delta p}{4\eta l}\left[R_2^2 - r^2 + \frac{R_2^2 - R_1^2}{\log(R_2/R_1)}\log\frac{r}{R_2}\right].$$

The discharge is

$$Q = \frac{\pi \Delta p}{8\nu l}\left[R_2^4 - R_1^4 - \frac{(R_2^2 - R_1^2)^2}{\log(R_2/R_1)}\right].$$

PROBLEM 2. The same as Problem 1, but for a pipe of elliptical cross-section.

SOLUTION. We seek a solution of equation (17.7) in the form $v = Ay^2 + Bz^2 + C$. The constants A, B, C are determined from the requirement that this expression must satisfy the boundary condition $v = 0$ on the circumference of the ellipse (i.e. $Ay^2 + Bz^2 + C = 0$

must be the same as the equation $y^2/a^2+z^2/b^2 = 1$, where a and b are the semi-axes of the ellipse). The result is

$$v = \frac{\Delta p}{2\eta l}\frac{a^2b^2}{a^2+b^2}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right).$$

The discharge is

$$Q = \frac{\pi\Delta p}{4\nu l}\frac{a^3b^3}{a^2+b^2}.$$

PROBLEM 3. The same as Problem 1, but for a pipe whose cross-section is an equilateral triangle of side a.

SOLUTION. The solution of equation (17.7) which vanishes on the bounding triangle is

$$v = \frac{\Delta p}{l}\frac{2}{\sqrt{3}a\eta}h_1h_2h_3,$$

where h_1, h_2, h_3 are the lengths of the perpendiculars from a given point in the triangle to its three sides. For each of the expressions Δh_1, Δh_2, Δh_3 (where $\Delta = \partial^2/\partial x^2+\partial^2/\partial y^2$) is zero; this is seen at once from the fact that each of the perpendiculars h_1, h_2, h_3 may be taken as the axis of y or z, and the result of applying the Laplacian to a co-ordinate is zero. We therefore have

$$\triangle(h_1h_2h_3) = 2(h_1\,\mathbf{grad}\,h_2\cdot\mathbf{grad}\,h_3+h_2\,\mathbf{grad}\,h_3\cdot\mathbf{grad}\,h_1+ \\ +h_3\,\mathbf{grad}\,h_1\cdot\mathbf{grad}\,h_2).$$

But $\mathbf{grad}\,h_1 = \mathbf{n}_1$, $\mathbf{grad}\,h_2 = \mathbf{n}_2$, $\mathbf{grad}\,h_3 = \mathbf{n}_3$, where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors along the perpendiculars h_1, h_2, h_3. Any two of $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are at an angle $2\pi/3$, so that $\mathbf{grad}\,h_1 . \mathbf{grad}\,h_2 = \mathbf{n}_1\cdot\mathbf{n}_2 = \cos(2\pi/3) = -\frac{1}{2}$, and so on. We thus obtain the relation

$$\triangle(h_1h_2h_3) = -(h_1+h_2+h_3) = -\tfrac{1}{2}\sqrt{3}a,$$

and we see that equation (17.7) is satisfied. The discharge is

$$Q = \frac{\sqrt{3}a^4\Delta p}{320\nu l}.$$

PROBLEM 4. A cylinder of radius R_1 moves with velocity u inside a coaxial cylinder of radius R_2, their axes being parallel. Determine the motion of a fluid occupying the space between the cylinders.

SOLUTION. We take cylindrical co-ordinates, with the z-axis along the axis of the cylinders. The velocity is everywhere along the z-axis and depends only on r (as does the pressure): $v_z = v(r)$. We obtain for v the equation

$$\triangle v = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) = 0;$$

the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = v\,\partial v/\partial z$ vanishes identically. Using the boundary conditions $v = u$ for $r = R_1$ and $v = 0$ for $r = R_2$, we find

$$v = u\frac{\log(r/R_2)}{\log(R_1/R_2)}.$$

The frictional force per unit length of either cylinder is $2\pi\eta u/\log(R_2/R_1)$.

PROBLEM 5. A layer of fluid of thickness h is bounded above by a free surface and below by a fixed plane inclined at an angle α to the horizontal. Determine the flow due to gravity.

SOLUTION. We take the fixed plane as the xy-plane, with the x-axis in the direction of flow (Fig. 6). We seek a solution depending only on z. The Navier–Stokes equations with $v_x = v(z)$ in a gravitational field are

$$\eta\frac{\mathrm{d}^2v}{\mathrm{d}z^2}+\rho g\sin\alpha = 0, \qquad \frac{\mathrm{d}p}{\mathrm{d}z}+\rho g\cos\alpha = 0.$$

At the free surface ($z = h$) we must have $\sigma_{xz} = \eta\mathrm{d}v/\mathrm{d}z = 0$, $\sigma_{zz} = -p = -p_0$ (p_0 being the atmospheric pressure). For $z = 0$ we must have $v = 0$. The solution satisfying these conditions is

$$p = p_0+\rho g(h-z)\cos\alpha, \qquad v = \frac{\rho g\sin\alpha}{2\eta}z(2h-z).$$

The discharge, per unit length in the y-direction, is

$$Q = \rho\int_0^h v\,\mathrm{d}z = \frac{\rho g h^3\sin\alpha}{3\nu}.$$

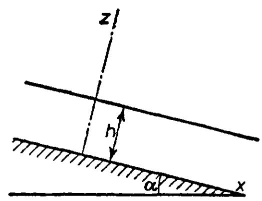

FIG. 6

PROBLEM 6. Determine the way in which the pressure falls along a tube of circular cross-section in which a viscous perfect gas is flowing isothermally (bearing in mind that the dynamic viscosity η of a perfect gas is independent of the pressure).

SOLUTION. Over any short section of the pipe the gas may be supposed incompressible, provided that the pressure gradient is not too great, and we can therefore use formula (17.10), according to which

$$-\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{8\eta Q}{\pi\rho R^4}.$$

Over greater distances, however, ρ varies, and the pressure is not a linear function of x. According to the equation of state, the gas density $\rho = mp/kT$, where m is the mass of a molecule and k is Boltzmann's constant, so that

$$-\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{8\eta QkT}{\pi mR^4}\cdot\frac{1}{p}.$$

(The discharge Q of the gas through the tube is obviously the same, whether or not the gas is incompressible.) From this we find

$$p_2^2-p_1^2 = \frac{16\eta QkT}{\pi mR^4}l,$$

where p_2, p_1 are the pressures at the ends of a section of the tube of length l.

§18. Flow between rotating cylinders

Let us now consider the motion of a fluid between two infinite coaxial cylinders of radii R_1, R_2 ($R_2 > R_1$), rotating about their axis with angular velocities Ω_1, Ω_2. We take cylindrical co-ordinates r, ϕ, z, with the z-axis along the axis of the cylinders. It is evident from symmetry that

$$v_z = v_r = 0, \qquad v_\phi = v(r), \qquad p = p(r).$$

The Navier–Stokes equation in cylindrical co-ordinates gives in this case two equations:

$$\mathrm{d}p/\mathrm{d}r = \rho v^2/r, \tag{18.1}$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r} - \frac{v}{r^2} = 0. \tag{18.2}$$

The latter equation has solutions of the form r^n; substitution gives $n = \pm 1$, so that

$$v = ar + \frac{b}{r}.$$

The constants a and b are found from the boundary conditions, according to which the fluid velocity at the inner and outer cylindrical surfaces must be equal to that of the corresponding cylinder: $v = R_1\Omega_1$ for $r = R_1$, $v = R_2\Omega_2$ for $r = R_2$. As a result we find the velocity distribution to be

$$v = \frac{\Omega_2 R_2{}^2 - \Omega_1 R_1{}^2}{R_2{}^2 - R_1{}^2} r + \frac{(\Omega_1 - \Omega_2) R_1{}^2 R_2{}^2}{R_2{}^2 - R_1{}^2}\frac{1}{r}. \tag{18.3}$$

The pressure distribution is then found from (18.1) by straightforward integration.

For $\Omega_1 = \Omega_2 = \Omega$ we have simply $v = \Omega r$, i.e. the fluid rotates rigidly with the cylinders. When the outer cylinder is absent ($\Omega_2 = 0$, $R_2 = \infty$) we have $v = \Omega_1 R_1{}^2/r$.

Let us also determine the moment of the frictional forces acting on the cylinders. The frictional force acting on unit area of the inner cylinder is along the tangent to the surface and, from (15.12), is equal to the component $\sigma'_{r\phi}$ of the stress tensor. Using formulae (15.15), we find

$$\begin{aligned}[\sigma'_{r\phi}]_{r=R_1} &= \eta\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right]_{r=R_1} \\ &= -2\eta\frac{(\Omega_1 - \Omega_2)R_2{}^2}{R_2{}^2 - R_1{}^2}.\end{aligned}$$

The force acting on unit length of the cylinder is obtained by multiplying by $2\pi R_1$, and the moment M_1 of that force by multiplying the result by R_1. We thus have

$$M_1 = -\frac{4\pi\eta(\Omega_1-\Omega_2)R_1^2R_2^2}{R_2^2-R_1^2}. \tag{18.4}$$

The moment M_2 of the forces acting on the inner cylinder is clearly $-M_1$.†

The following general remark may be made concerning the solutions of the equations of motion of a viscous fluid which we have obtained in §§17 and 18. In all these cases the non-linear term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equations which determine the velocity distribution is identically zero, so that we are actually solving linear equations, a fact which very much simplifies the problem. For this reason all the solutions also satisfy the equations of motion for an incompressible ideal fluid, say in the form (10.2) and (10.3). This is why formulae (17.1) and (18.3) do not contain the viscosity coefficient at all. This coefficient appears only in formulae, such as (17.9), which relate the velocity to the pressure gradient in the fluid, since the presence of a pressure gradient is due to the viscosity; an ideal fluid could flow in a pipe even if there were no pressure gradient.

§19. The law of similarity

In studying the motion of viscous fluids we can obtain a number of important results from simple arguments concerning the dimensions of various physical quantities. Let us consider any particular type of motion, for instance the motion of a body of some definite shape through a fluid. If the body is not a sphere, its direction of motion must also be specified: e.g. the motion of an ellipsoid in the direction of its greatest or least axis. Alternatively, we may be considering flow in a region with boundaries of a definite form (a pipe of given cross-section, etc.).

In such a case we say that bodies of the same shape are *geometrically similar*; they can be obtained from one another by changing all linear dimensions in the same ratio. Hence, if the shape of the body is given, it suffices to specify any one of its linear dimensions (the radius of a sphere or of a cylindrical pipe, one semi-axis of a spheroid of given eccentricity, and so on) in order to determine its dimensions completely.

We shall at present consider steady flow. If, for example, we are discussing flow past a solid body (which case we shall take below, for definiteness), the velocity of the main stream must therefore be constant. We shall suppose the fluid incompressible.

Of the parameters which characterise the fluid itself, only the kinematic

† The solution of the more complex problem of the motion of a viscous fluid in a narrow space between cylinders whose axes are parallel but not coincident may be found in: N. E. Kochin, I. A. Kibel' and N. V. Roze, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, 3rd ed., p. 419, Moscow 1948; A. Sommerfeld, *Mechanics of Deformable Bodies*, §36, Academic Press, New York 1950.

viscosity $\nu = \eta/\rho$ appears in the equations of hydrodynamics (the Navier–Stokes equations); the unknown functions which have to be determined by solving the equations are the velocity $\mathbf{v}$ and the ratio p/ρ of the pressure p to the constant density ρ. Moreover, the flow depends, through the boundary conditions, on the shape and dimensions of the body moving through the fluid and on its velocity. Since the shape of the body is supposed given, its geometrical properties are determined by one linear dimension, which we denote by l. Let the velocity of the main stream be u. Then any flow is specified by three parameters, ν, u and l. These quantities have the following dimensions:

$$\nu = \mathrm{cm}^2/\mathrm{sec}, \qquad l = \mathrm{cm}, \qquad u = \mathrm{cm}/\mathrm{sec}.$$

It is easy to verify that only one dimensionless quantity can be formed from the above three, namely ul/ν. This combination is called the *Reynolds number* and is denoted by R:

$$\mathrm{R} = \rho ul/\eta = ul/\nu. \tag{19.1}$$

Any other dimensionless parameter can be written as a function of R.

We shall now measure lengths in terms of l, and velocities in terms of u, i.e. we introduce the dimensionless quantities $\mathbf{r}/l$, $\mathbf{v}/u$. Since the only dimensionless parameter is the Reynolds number, it is evident that the velocity distribution obtained by solving the equations of incompressible flow is given by a function of the form

$$\mathbf{v} = u\mathbf{f}(\mathbf{r}/l, \mathrm{R}). \tag{19.2}$$

It is seen from this expression that, in two different flows of the same type (for example, flow past spheres of different radii by fluids of different viscosities), the velocities $\mathbf{v}/u$ are the same functions of the ratio $\mathbf{r}/l$ if the Reynolds number is the same for each flow. Flows which can be obtained from one another by simply changing the unit of measurement of co-ordinates and velocities are said to be *similar*. Thus flows of the same type with the same Reynolds number are similar. This is called the *law of similarity* (O. Reynolds 1883).

A formula similar to (19.2) can be written for the pressure distribution in the fluid. To do so, we must construct from the parameters ν, l, u some quantity with the dimensions of pressure divided by density; this quantity can be u^2, for example. Then we can say that $p/\rho u^2$ is a function of the dimensionless variable $\mathbf{r}/l$ and the dimensionless parameter R. Thus

$$p = \rho u^2 f(\mathbf{r}/l, \mathrm{R}). \tag{19.3}$$

Finally, similar considerations can also be applied to quantities which characterise the flow but are not functions of the co-ordinates. Such a quantity is, for instance, the drag force F acting on the body. We can say that the dimensionless ratio of F to some quantity formed from ν, u, l, ρ

and having the dimensions of force must be a function of the Reynolds number alone. Such a combination of ν, u, l, ρ can be $\rho u^2 l^2$, for example. Then

$$F = \rho u^2 l^2 f(\mathrm{R}). \tag{19.4}$$

If the force of gravity has an important effect on the flow, then the latter is determined not by three but by four parameters, l, u, ν and the acceleration g due to gravity. From these parameters we can construct not one but two independent dimensionless quantities. These can be, for instance, the Reynolds number and the *Froude number*, which is

$$\mathrm{F} = u^2/lg. \tag{19.5}$$

In formulae (19.2)–(19.4) the function f will now depend on not one but two parameters (R and F), and two flows will be similar only if both these numbers have the same values.

Finally, we may say a little regarding non-steady flows. A non-steady flow of a given type is characterised not only by the quantities ν, u, l but also by some time interval τ characteristic of the flow, which determines the rate of change of the flow. For instance, in oscillations, according to a given law, of a solid body, of a given shape, immersed in a fluid, τ may be the period of oscillation. From the four quantities ν, u, l, τ we can again construct two independent dimensionless quantities, which may be the Reynolds number and the number

$$\mathrm{S} = u\tau/l, \tag{19.6}$$

sometimes called the *Strouhal number*. Similar motion takes place in these cases only if both these numbers have the same values.

If the oscillations of the fluid occur spontaneously (and not under the action of a given external exciting force), then for motion of a given type S will be a definite function of R:

$$\mathrm{S} = f(\mathrm{R}).$$

§20. Stokes' formula

The Navier–Stokes equation is considerably simplified in the case of flow at small Reynolds numbers. For steady flow of an incompressible fluid, this equation is

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,p + (\eta/\rho)\triangle\mathbf{v}.$$

The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is of the order of magnitude of u^2/l, u and l having the same meaning as in §19. The quantity $(\eta/\rho)\triangle\mathbf{v}$ is of the order of magnitude of $\eta u/\rho l^2$. The ratio of the two is just the Reynolds number. Hence the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ may be neglected if the Reynolds number is small, and the equation of motion reduces to a linear equation

$$\eta\triangle\mathbf{v} - \mathbf{grad}\,p = 0. \tag{20.1}$$

Together with the equation of continuity

$$\operatorname{div} \mathbf{v} = 0 \tag{20.2}$$

it completely determines the motion. It is useful to note also the equation

$$\triangle\, \mathbf{curl}\, \mathbf{v} = 0, \tag{20.3}$$

which is obtained by taking the curl of equation (20.1).

As an example, let us consider rectilinear and uniform motion of a sphere in a viscous fluid. The problem of the motion of a sphere, it is clear, is exactly equivalent to that of flow past a fixed sphere, the fluid having a given velocity $\mathbf{u}$ at infinity. The velocity distribution in the first problem is obtained from that in the second problem by simply subtracting the velocity $\mathbf{u}$; the fluid is then at rest at infinity, while the sphere moves with velocity $-\mathbf{u}$. If we regard the flow as steady, we must, of course, speak of the flow past a fixed sphere, since, when the sphere moves, the velocity of the fluid at any point in space varies with time.

Thus we must have $\mathbf{v} = \mathbf{u}$ at infinity; we write $\mathbf{v} = \mathbf{v}'+\mathbf{u}$, so that $\mathbf{v}'$ is zero at infinity. Since $\operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{v}' = 0$, $\mathbf{v}'$ can be written as the curl of some vector: $\mathbf{v} = \mathbf{curl}\, \mathbf{A}+\mathbf{u}$. The curl of a polar vector is well known to be an axial vector, and *vice versa.* Since the velocity is an ordinary polar vector, $\mathbf{A}$ must be an axial vector. Now $\mathbf{v}$, and therefore $\mathbf{A}$, depend only on the radius vector $\mathbf{r}$ (we take the origin at the centre of the sphere) and on the parameter $\mathbf{u}$; both these vectors are polar. Furthermore, $\mathbf{A}$ must evidently be a linear function of $\mathbf{u}$. The only such axial vector which can be constructed for a completely symmetrical body (the sphere) from two polar vectors is the vector product $\mathbf{r}\times\mathbf{u}$. Hence $\mathbf{A}$ must be of the form $f'(r)\mathbf{n}\times\mathbf{u}$, where $f'(r)$ is a scalar function of r, and $\mathbf{n}$ is a unit vector in the direction of the radius vector. The product $f'(r)\mathbf{n}$ can be written as the gradient, $\mathbf{grad}\, f(r)$, of some function $f(r)$, so that the general form of $\mathbf{A}$ is $\mathbf{grad}\, f\times\mathbf{u}$. Hence we can write the velocity $\mathbf{v}'$ as

$$\mathbf{v}' = \mathbf{curl}\, [\mathbf{grad} f\times\mathbf{u}].$$

Since $\mathbf{u}$ is a constant, $\mathbf{grad}\, f\times\mathbf{u} = \mathbf{curl}\,(f\mathbf{u})$, so that

$$\mathbf{v} = \mathbf{curl}\, \mathbf{curl}\,(f\mathbf{u})+\mathbf{u}. \tag{20.4}$$

To determine the function f, we use equation (20.3). Since

$$\begin{aligned}\mathbf{curl}\, \mathbf{v} = \mathbf{curl}\, \mathbf{curl}\, \mathbf{curl}(f\mathbf{u}) &= (\mathbf{grad}\, \operatorname{div}-\triangle)\mathbf{curl}(f\mathbf{u}) \\ &= -\triangle\, \mathbf{curl}(f\mathbf{u}),\end{aligned}$$

(20.3) takes the form $\triangle^2\, \mathbf{curl}\,(f\mathbf{u}) = 0$, or, since $\mathbf{u} = \text{constant}$,

$$\triangle^2(\mathbf{grad} f\times\mathbf{u}) = (\triangle^2\, \mathbf{grad} f)\times\mathbf{u} = 0.$$

It follows from this that

$$\triangle^2\, \mathbf{grad} f = 0. \tag{20.5}$$

A first integration gives

$$\triangle^2 f = \text{constant}.$$

It is easy to see that the constant must be zero, since the velocity $\mathbf{v}$ must vanish at infinity, and so must its derivatives. The expression $\triangle^2 f$ contains fourth derivatives of f, whilst the velocity is given in terms of the second derivatives of f. Thus we have

$$\triangle^2 f \equiv \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right)\triangle f = 0.$$

Hence

$$\triangle f = 2a/r + A.$$

The constant A must be zero if the velocity is to vanish at infinity. From $\triangle f = 2a/r$ we obtain

$$f = ar + b/r. \tag{20.6}$$

The additive constant is omitted, since it is immaterial (the velocity being given by derivatives of f).

Substituting in (20.4), we have after a simple calculation

$$\mathbf{v} = \mathbf{u} - a\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-\mathbf{u}}{r^3}. \tag{20.7}$$

The constants a and b have to be determined from the boundary conditions: at the surface of the sphere ($r = R$), $\mathbf{v} = 0$, i.e.

$$\mathbf{u} - a\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{R} + b\frac{3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-\mathbf{u}}{R^3} = 0.$$

Since this equation must hold for all $\mathbf{n}$, the coefficients of $\mathbf{u}$ and $\mathbf{n}(\mathbf{u}\cdot\mathbf{n})$ must each vanish:

$$\frac{a}{R} + \frac{b}{R^3} - 1 = 0, \qquad -\frac{a}{R} + \frac{3b}{R^3} = 0.$$

Hence $a = \frac{3}{4}R$, $b = \frac{1}{4}R^3$. Thus we have finally

$$f = \tfrac{3}{4}Rr + \tfrac{1}{4}R^3/r, \tag{20.8}$$

$$\mathbf{v} = -\tfrac{3}{4}R\frac{\mathbf{u}+\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r} - \tfrac{1}{4}R^3\frac{\mathbf{u}-3\mathbf{n}(\mathbf{u}\cdot\mathbf{n})}{r^3} + \mathbf{u}, \tag{20.9}$$

or, in spherical components,

$$\begin{aligned} v_r &= u\cos\theta\left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right], \\ v_\theta &= -u\sin\theta\left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right]. \end{aligned} \tag{20.10}$$

This gives the velocity distribution about the moving sphere. To determine the pressure, we substitute (20.4) in (20.1):

$$\mathbf{grad}\,p = \eta\triangle\mathbf{v} = \eta\triangle\mathbf{curl}\,\mathbf{curl}\,(f\mathbf{u})$$
$$= \eta\triangle(\mathbf{grad}\,\mathrm{div}\,(f\mathbf{u}) - \mathbf{u}\triangle f).$$

But $\triangle^2 f = 0$, and so

$$\mathbf{grad}\,p = \mathbf{grad}[\eta\triangle\mathrm{div}(f\mathbf{u})] = \mathbf{grad}(\eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f).$$

Hence

$$p = \eta\mathbf{u}\cdot\mathbf{grad}\,\triangle f + p_0, \tag{20.11}$$

where p_0 is the fluid pressure at infinity. Substitution for f leads to the final expression

$$p = p_0 - \tfrac{3}{2}\eta\frac{\mathbf{u}\cdot\mathbf{n}}{r^2}R. \tag{20.12}$$

Using the above formulae, we can calculate the force $\mathbf{F}$ exerted on the sphere by the moving fluid (or, what is the same thing, the drag on the sphere as it moves through the fluid). To do so, we take spherical co-ordinates with the polar axis parallel to $\mathbf{u}$; by symmetry, all quantities are functions only of r and of the polar angle θ. The force $\mathbf{F}$ is evidently parallel to the velocity $\mathbf{u}$. The magnitude of this force can be determined from (15.12). Taking from this formula the components, normal and tangential to the surface, of the force on an element of the surface of the sphere, and projecting these components on the direction of $\mathbf{u}$, we find

$$F = \oint(-p\cos\theta + \sigma'_{rr}\cos\theta - \sigma'_{r\theta}\sin\theta)\mathrm{d}f, \tag{20.13}$$

where the integration is taken over the whole surface of the sphere.

Substituting the expressions (20.10) in the formulae

$$\sigma'_{rr} = 2\eta\frac{\partial v_r}{\partial r}, \qquad \sigma'_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

(see (15.17)), we find that at the surface of the sphere

$$\sigma'_{rr} = 0, \qquad \sigma'_{r\theta} = -(3\eta/2R)u\sin\theta,$$

while the pressure (20.12) is $p = p_0 - (3\eta/2R)u\cos\theta$. Hence the integral (20.13) reduces to $F = (3\eta u/2R)\oint\mathrm{d}f$, or, finally,†

$$F = 6\pi R\eta u. \tag{20.14}$$

This formula (called *Stokes' formula*) gives the drag on a sphere moving

† With a view to some later applications, we may mention that, if the calculations are done with formula (20.7) for the velocity (the constants a and b being undetermined), we find

$$F = 8\pi a\eta u. \tag{20.14a}$$

slowly in a fluid. We may notice that the drag is proportional to the first powers of the velocity and linear dimension of the body.†

This dependence of the drag on the velocity and dimension holds for slowly-moving bodies of other shapes also. The direction of the drag on a body of arbitrary shape is not the same as that of the velocity; the general form of the dependence of **F** on **u** can be written

$$F_i = a_{ik} u_k, \tag{20.15}$$

where a_{ik} is a tensor of rank two, independent of the velocity. It is important to note that this tensor is symmetrical ($a_{ik} = a_{ki}$), a result which holds in the linear approximation with respect to the velocity, and is a particular case of a general law valid for slow motion accompanied by dissipative processes.‡

The solution that we have just obtained for flow past a sphere is not valid at great distances from it, even if the Reynolds number is small. In order to see this, we estimate the magnitude of the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$, which we neglected in (20.1). At great distances the velocity is **u**. The derivatives of the velocity at these distances are seen from (20.9) to be of the order of uR/r^2. Thus $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ is of the order of u^2R/r^2. The terms retained in equation (20.1), for example $(1/\rho)\,\mathbf{grad}\,p$, are of the order $\eta Ru/\rho r^3$ (cf. (20.12)). The condition

$$u\eta R/\rho r^3 \gg u^2R/r^2$$

holds only at distances $r \ll \nu/u$, where $\nu = \eta/\rho$. At greater distances, the terms we have omitted cannot legitimately be neglected, and the velocity distribution obtained is incorrect.

To obtain the velocity distribution at great distances from the body, we have to take into account the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ omitted in (20.1). Since the velocity **v** is nearly equal to **u** at these distances, we can put approximately $\mathbf{u}\cdot\mathbf{grad}$ in place of $\mathbf{v}\cdot\mathbf{grad}$. We then find for the velocity at great distances the linear equation

$$(\mathbf{u}\cdot\mathbf{grad})\mathbf{v} = -(1/\rho)\,\mathbf{grad}\,p + \nu\triangle\mathbf{v} \tag{20.16}$$

(C. W. Oseen, 1910).

We shall not pause to give here the solution of this equation for flow

† The drag can also be calculated for a slowly-moving ellipsoid of any shape. The corresponding formulae are given by H. Lamb, *Hydrodynamics*, 6th ed., §339, Cambridge 1932. We give here the limiting expressions for a plane circular disk of radius R moving perpendicular to its plane:

$$F = 16\eta Ru$$

and for a similar disk moving in its plane:

$$F = 32\eta Ru/3.$$

‡ See, for instance, *Statistical Physics*, §120, Pergamon Press, London 1958.

past a sphere,† but merely mention that the velocity distribution thus obtained can be used to derive a more accurate formula for the drag on the sphere, which includes the next term in the expansion of the drag in powers of the Reynolds number uR/ν. This formula is‡

$$F = 6\pi\eta uR\left(1 + \frac{3uR}{8\nu}\right). \tag{20.17}$$

Finally, we may mention that, in solving the problem of flow past an infinite cylinder with the main stream perpendicular to the axis of the cylinder, Oseen's equation has to be used from the start; in this case, equation (20.1) has no solution which satisfies the boundary conditions at the surface of the cylinder and at the same time vanishes at infinity. The drag per unit length of the cylinder is found to be

$$F = \frac{4\pi\eta u}{\frac{1}{2}-\gamma-\log(uR/4\nu)}, \tag{20.18}$$

where $\gamma \doteqdot 0.577$ is Euler's constant.

PROBLEMS

PROBLEM 1. Determine the motion of a fluid occupying the space between two concentric spheres of radii R_1, R_2 ($R_2 > R_1$), rotating uniformly about different diameters with angular velocities Ω_1, Ω_2; the Reynolds numbers $\Omega_1 R_1^2/\nu$, $\Omega_2 R_2^2/\nu$ are small compared with unity.

SOLUTION. On account of the linearity of the equations, the motion between two rotating spheres may be regarded as a superposition of the two motions obtained when one sphere is at rest and the other rotates. We first put $\Omega_2 = 0$, i.e. only the inner sphere is rotating. It is reasonable to suppose that the fluid velocity at every point is along the tangent to a circle in a plane perpendicular to the axis of rotation with its centre on the axis. On account of the axial symmetry, the pressure gradient in this direction is zero. Hence the equation of motion (20.1) becomes $\triangle \mathbf{v} = 0$. The angular velocity vector $\mathbf{\Omega}_1$ is an axial vector. Arguments similar to those given previously show that the velocity can be written as

$$\mathbf{v} = \mathbf{curl}[f(r)\mathbf{\Omega}_1] = \mathbf{grad}f \times \mathbf{\Omega}_1.$$

The equation of motion then gives $\mathbf{grad}\,\triangle f \times \mathbf{\Omega}_1 = 0$. Since the vector $\mathbf{grad}\,\triangle f$ is parallel to the radius vector, and the vector product $\mathbf{r} \times \mathbf{\Omega}_1$ cannot be zero for given $\mathbf{\Omega}_1$ and arbitrary $\mathbf{r}$, we must have $\mathbf{grad}\,\triangle f = 0$, so that

$$\triangle f = \text{constant}.$$

† A detailed account of the calculations for a sphere and a cylinder is given by N. E. KOCHIN, I. A. KIBEL' and N. V. ROZE, *Theoretical Hydromechanics* (*Teoreticheskaya gidromekhanika*), Part 2, 3rd ed., chapter II, §§25–26, Moscow 1948; H. LAMB, *Hydrodynamics*, 6th ed., §§342–3, Cambridge 1932.

‡ At first sight it might appear that OSEEN's equation, which does not correctly give the velocity distribution near the sphere, could not be used to calculate the correction to the drag. In fact, however, the contribution to $\mathbf{F}$ due to the motion of the neighbouring fluid (where $u \ll \nu/r$) must be expanded in powers of the vector $\mathbf{u}$. The first non-zero correction term in $\mathbf{F}$ arising from this contribution is then proportional to $u^2\mathbf{u}$, i.e. is of the second order with respect to the Reynolds number; it therefore does not affect the first-order correction in formula (20.17). Further corrections to Stokes' formula cannot be calculated from Oseen's formula.

Integrating, we find

$$f = ar^2 + \frac{b}{r}, \qquad \mathbf{v} = \left(\frac{b}{r^3} - 2a\right) \mathbf{\Omega}_1 \times \mathbf{r}.$$

The constants a and b are found from the conditions that $\mathbf{v} = 0$ for $r = R_2$ and $\mathbf{v} = \mathbf{u}$ for $r = R_1$, where $\mathbf{u} = \mathbf{\Omega}_1 \times \mathbf{r}$ is the velocity of points on the rotating sphere. The result is

$$\mathbf{v} = \frac{R_1{}^3 R_2{}^3}{R_2{}^3 - R_1{}^3}\left(\frac{1}{r^3} - \frac{1}{R_2{}^3}\right) \mathbf{\Omega}_1 \times \mathbf{r}.$$

The fluid pressure is constant ($p = p_0$). Similarly, we have for the case where the outer sphere rotates and the inner one is at rest ($\Omega_1 = 0$)

$$\mathbf{v} = \frac{R_1{}^3 R_2{}^3}{R_2{}^3 - R_1{}^3}\left(\frac{1}{R_1{}^3} - \frac{1}{r^3}\right) \mathbf{\Omega}_2 \times \mathbf{r}.$$

In the general case where both spheres rotate, we have

$$\mathbf{v} = \frac{R_1{}^3 R_2{}^3}{R_2{}^3 - R_1{}^3}\left\{\left(\frac{1}{r^3} - \frac{1}{R_2{}^3}\right) \mathbf{\Omega}_1 \times \mathbf{r} + \left(\frac{1}{R_1{}^3} - \frac{1}{r^3}\right) \mathbf{\Omega}_2 \times \mathbf{r}\right\}.$$

If the outer sphere is absent ($R_2 = \infty$, $\Omega_2 = 0$), i.e. we have simply a sphere of radius R rotating in an infinite fluid, then

$$\mathbf{v} = (R^3/r^3)\, \mathbf{\Omega} \times \mathbf{r}.$$

Let us calculate the moment of the frictional forces acting on the sphere in this case. If we take spherical co-ordinates with the polar axis parallel to $\mathbf{\Omega}$, we have $v_r = v_\theta = 0$, $v_\phi = v = (R^3\Omega/r^2)\sin\theta$. The frictional force on unit area of the sphere is

$$\sigma'_{r\phi} = \eta\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r=R} = -3\eta\Omega\sin\theta.$$

The total moment on the sphere is

$$M = \int_0^\pi \sigma'_{r\phi} R \sin\theta \cdot 2\pi R^2 \sin\theta \, d\theta,$$

whence we find

$$M = -8\pi\eta R^3 \Omega.$$

If the inner sphere is absent, $\mathbf{v} = \mathbf{\Omega}_2 \times \mathbf{r}$, i.e. the fluid simply rotates rigidly with the sphere surrounding it.

PROBLEM 2. Determine the velocity of a spherical drop of fluid (of viscosity η') moving under gravity in a fluid of viscosity η (W. RYBCZYŃSKI 1911).

SOLUTION. We use a system of co-ordinates in which the drop is at rest. For the fluid outside the drop we again seek a solution of equation (20.5) in the form (20.6), so that the velocity has the form (20.7). For the fluid inside the drop, we have to find a solution which does not have a singularity at $r = 0$ (and the second derivatives of f, which determine the velocity, must also remain finite). This solution is

$$f = \tfrac{1}{4}Ar^2 + \tfrac{1}{8}Br^4,$$

and the corresponding velocity is

$$\mathbf{v} = -A\mathbf{u}+Br^2[\mathbf{n}(\mathbf{u}\cdot\mathbf{n})-2\mathbf{u}].$$

At the surface of the sphere† the following conditions must be satisfied. The normal velocity components outside ($\mathbf{v}_e$) and inside ($\mathbf{v}_i$) the drop must be zero:

$$v_{i,r} = v_{e,r} = 0.$$

The tangential velocity component must be continuous:

$$v_{i,\theta} = v_{e,\theta},$$

as must be the component $\sigma_{r\theta}$ of the stress tensor:

$$\sigma_{i,r\theta} = \sigma_{e,r\theta}.$$

The condition that the stress tensor components σ_{rr} are equal need not be written down; it would determine the required velocity u, which is more simply found in the manner shown below. From the above four conditions we obtain four equations for the constants a, b, A, B, whose solutions are

$$a = R\frac{2\eta+3\eta'}{4(\eta+\eta')}, \qquad b = R^3\frac{\eta'}{4(\eta+\eta')}, \qquad A = -BR^2 = \frac{\eta}{2(\eta+\eta')}.$$

By (20.14a), we have for the drag

$$F = 2\pi u\eta R(2\eta+3\eta')/(\eta+\eta').$$

As $\eta' \to \infty$ (corresponding to a solid sphere) this formula becomes Stokes' formula. In the limit $\eta' \to 0$ (corresponding to a gas bubble) we have $F = 4\pi u\eta R$, i.e. the drag is two-thirds of that on a solid sphere.

Equating F to the force of gravity on the drop, $\frac{4}{3}\pi R^3(\rho-\rho')g$, we find

$$u = \frac{2R^2 g(\rho-\rho')(\eta+\eta')}{3\eta(2\eta+3\eta')}.$$

PROBLEM 3. Two parallel plane circular disks (of radius R) lie one above the other a small distance apart; the space between them is filled with fluid. The disks approach at a constant velocity u, displacing the fluid. Determine the resistance to their motion (O. REYNOLDS).

SOLUTION. We take cylindrical co-ordinates, with the origin at the centre of the lower disk, which we suppose fixed. The flow is axisymmetric and, since the fluid layer is thin, predominantly radial: $v_z \ll v_r$, and also $\partial v_r/\partial r \ll \partial v_r/\partial z$. Hence the equations of motion become

$$\eta\frac{\partial^2 v_r}{\partial z^2} = \frac{\partial p}{\partial r}, \qquad \frac{\partial p}{\partial z} = 0, \tag{1}$$

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{\partial v_z}{\partial z} = 0, \tag{2}$$

† We may neglect the change of shape of the drop in its motion, since this change is of a higher order of smallness. However, it must be borne in mind that, in order that the moving drop should in fact be spherical, the forces due to surface tension at its boundary must exceed the forces due to pressure differences, which tend to make the drop non-spherical. This means that we must have $\eta u/R \ll \alpha/R$, where α is the surface-tension coefficient, or, substituting $u \sim R^2 g\rho/\eta$,

$$R \ll \sqrt{(\alpha/\rho g)}.$$

with the boundary conditions

$$\text{at } z = 0: \qquad v_r = v_z = 0;$$
$$\text{at } z = h: \qquad v_r = 0, \qquad v_z = -u;$$
$$\text{at } r = R: \qquad p = p_0,$$

where h is the distance between the disks, and p_0 the external pressure. From equations (1) we find

$$v_r = \frac{1}{2\eta}\frac{\partial p}{\partial r}z(z-h).$$

Integrating equation (2) with respect to z, we obtain

$$u = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^h r v_r \,\mathrm{d}z = -\frac{h^3}{12\eta r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}p}{\mathrm{d}r}\right),$$

whence

$$p = p_0 + \frac{3\eta u}{h^3}(R^2-r^2).$$

The total resistance to the moving disk is

$$F = 3\pi\eta u R^4/2h^3.$$

§21. The laminar wake

In steady flow of a viscous fluid past a solid body, the flow at great distances behind the body has certain characteristics which can be investigated independently of the particular shape of the body.

Let us denote by $\mathbf{U}$ the constant velocity of the incident current; we take the direction of $\mathbf{U}$ as the x-axis, with the origin somewhere inside the body. The actual fluid velocity at any point may be written $\mathbf{U}+\mathbf{v}$; $\mathbf{v}$ vanishes at infinity.

It is found that, at great distances behind the body, the velocity $\mathbf{v}$ is noticeably different from zero only in a relatively narrow region near the x-axis. This region, called the *laminar wake*,† is reached by fluid particles which move along streamlines passing fairly close to the body. Hence the flow in the wake is essentially rotational. On the other hand, the viscosity has almost no effect at any point on streamlines that do not pass near the body, and the vorticity, which is zero in the incident current, remains practically zero on these streamlines, as it would in an ideal fluid. Thus the flow at great distances from the body may be regarded as potential flow everywhere except in the wake.

We shall now derive formulae relating the properties of the flow in the wake to the forces acting on the body. The total momentum transported by the fluid through any closed surface surrounding the body is equal to the

† In contradistinction to the turbulent wake; see §36.

integral of the momentum flux density tensor over that surface, $\oint \Pi_{ik}\mathrm{d}f_k$. The components of the tensor Π_{ik} are

$$\Pi_{ik} = p\delta_{ik}+\rho(U_i+v_i)(U_k+v_k).$$

We write the pressure in the form $p = p_0+p'$, where p_0 is the pressure at infinity. The integration of the constant term $p_0\delta_{ik}+\rho U_i U_k$ gives zero, since the vector integral $\oint \mathbf{df}$ over a closed surface is zero. The integral $U_i \oint \rho v_k \mathrm{d}f_k$ also vanishes: since the total mass of fluid in the volume considered is constant, the total mass flux $\oint \rho\mathbf{v}\cdot\mathbf{df}$ through the surface surrounding the volume must be zero. Finally, the velocity $\mathbf{v}$ far from the body is small compared with $\mathbf{U}$. Hence, if the surface in question is sufficiently far from the body, we can neglect the term $\rho v_i v_k$ in Π_{ik} as compared with $\rho U_k v_i$. Thus the total momentum flux is

$$\oint (p'\delta_{ik}+\rho U_k v_i)\mathrm{d}f_k.$$

Let us now take the fluid volume concerned to be the volume between two infinite planes x = constant, one of them far in front of the body and the other far behind it. The integral over the infinitely distant "lateral" surface vanishes (since $p' = \mathbf{v} = 0$ at infinity), and it is therefore sufficient to integrate only over the two planes. The momentum flux thus obtained is evidently the difference between the total momentum flux entering through the forward plane and that leaving through the backward plane. This difference, however, is just the quantity of momentum transmitted to the body by the fluid per unit time, i.e. the force $\mathbf{F}$ exerted on the body.

Thus the components of the force $\mathbf{F}$ are

$$F_x = \Big(\iint_{x=x_2} - \iint_{x=x_1}\Big)(p'+\rho U v_x)\mathrm{d}y\,\mathrm{d}z,$$

$$F_y = \Big(\iint_{x=x_2} - \iint_{x=x_1}\Big)\rho U v_y\,\mathrm{d}y\,\mathrm{d}z,$$

$$F_z = \Big(\iint_{x=x_2} - \iint_{x=x_1}\Big)\rho U v_z\,\mathrm{d}y\,\mathrm{d}z,$$

where the integration is taken over the infinite planes $x = x_1$ (far behind the body) and $x = x_2$ (far in front of it). Let us first consider the expression for F_x.

Outside the wake we have potential flow, and therefore Bernoulli's equation

$$p+\tfrac{1}{2}\rho(\mathbf{U}+\mathbf{v})^2 = \text{constant} \equiv p_0+\tfrac{1}{2}\rho U^2$$

holds, or, neglecting the term $\frac{1}{2}\rho v^2$ in comparison with $\rho\mathbf{U}\cdot\mathbf{v}$,

$$p' = -\rho U v_x.$$

We see that in this approximation the integrand in F_x vanishes everywhere outside the wake. In other words, the integral over the plane $x = x_2$ (which lies in front of the body and does not intersect the wake) is zero, and the integral over the plane $x = x_1$ need be taken only over the area covered by the cross-section of the wake. Inside the wake, however, the pressure change p' is of the order of ρv^2, i.e. small compared with $\rho U v_x$. Thus we reach the result that the drag on the body is

$$F_x = -\rho U \iint v_x \,\mathrm{d}y \,\mathrm{d}z, \tag{21.1}$$

where the integration is taken over the cross-sectional area of the wake far behind the body. The velocity v_x in the wake is, of course, negative: the fluid moves more slowly than it would if the body were absent. Attention is called to the fact that the integral in (21.1) gives the amount by which the discharge through the wake falls short of its value in the absence of the body.

Let us now consider the force (whose components are F_y, F_z) which tends to move the body transversely. This force is called the *lift*. Outside the wake, where we have potential flow, we can write $v_y = \partial\phi/\partial y$, $v_z = \partial\phi/\partial z$; the integral over the plane $x = x_2$, which does not meet the wake, is zero:

$$\iint v_y \,\mathrm{d}y \,\mathrm{d}z = \iint \frac{\partial\phi}{\partial y} \mathrm{d}y \,\mathrm{d}z = 0, \qquad \iint \frac{\partial\phi}{\partial z} \mathrm{d}y \,\mathrm{d}z = 0,$$

since $\phi = 0$ at infinity. We therefore find for the lift

$$F_y = -\rho U \iint v_y \,\mathrm{d}y \,\mathrm{d}z, \qquad F_z = -\rho U \iint v_z \,\mathrm{d}y \,\mathrm{d}z. \tag{21.2}$$

The integration in these formulae is again taken only over the cross-sectional area of the wake. If the body has an axis of symmetry (not necessarily complete axial symmetry), and the flow is parallel to this axis, then the flow past the body has an axis of symmetry also. In this case the lift is, of course, zero.

Let us return to the flow in the wake. An estimate of the magnitudes of various terms in the Navier–Stokes equation shows that the term $\nu\triangle\mathbf{v}$ can in general be neglected at distances r from the body such that $rU/\nu \gg 1$ (cf. the derivation of the opposite condition at the beginning of §20); these are the distances at which the flow outside the wake may be regarded as potential flow. It is not possible to neglect that term inside the wake even at these distances, however, since the transverse derivatives $\partial^2\mathbf{v}/\partial y^2$, $\partial^2\mathbf{v}/\partial z^2$ are large compared with $\partial^2\mathbf{v}/\partial x^2$.

The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the Navier–Stokes equation is of the order of magnitude $(U+v)\,\partial v/\partial x \sim Uv/x$ in the wake. The term $\nu\triangle\mathbf{v}$ is of the order of $\nu\partial^2 v/\partial y^2 \sim \nu v/Y^2$, where Y denotes the width of the wake, i.e. the order of magnitude of the distances from the x-axis at which the velocity $\mathbf{v}$ falls off markedly. If these two magnitudes are comparable, we find

$$Y \sim \sqrt{(\nu x/U)}. \tag{21.3}$$

This quantity is in fact small compared with x, by the assumed condition $Ux/\nu \gg 1$. Thus the width of the laminar wake increases as the square root of the distance from the body.

In order to determine how the velocity decreases with increasing x in the wake, we return to formula (21.1). The region of integration has an area of the order of Y^2. Hence the integral can be estimated as $F_x \sim \rho U v Y^2$, and by using the relation (21.3) we obtain

$$v \sim F_x/\rho\nu x. \tag{21.4}$$

PROBLEMS

PROBLEM 1. Determine the flow in the laminar wake when there is both drag and lift.

SOLUTION. Writing the velocity in the Navier–Stokes equation in the form $\mathbf{U}+\mathbf{v}$ and omitting terms quadratic in $\mathbf{v}$ (far from the body) we obtain

$$U\frac{\partial \mathbf{v}}{\partial x} = -\mathbf{grad}\left(\frac{p}{\rho}\right)+\nu\left(\frac{\partial^2 \mathbf{v}}{\partial y^2}+\frac{\partial^2 \mathbf{v}}{\partial z^2}\right);$$

we have also neglected the term $\partial^2\mathbf{v}/\partial x^2$ in $\Delta\mathbf{v}$. We seek a solution in the form $\mathbf{v} = \mathbf{v}_1+\mathbf{v}_2$, where $\mathbf{v}_1$ satisfies

$$U\frac{\partial \mathbf{v}_1}{\partial x} = \nu\left(\frac{\partial^2 \mathbf{v}_1}{\partial y^2}+\frac{\partial^2 \mathbf{v}_1}{\partial z^2}\right).$$

The term $\mathbf{v}_2$, which appears because of the term $-\mathbf{grad}(p/\rho)$ in the original equation, may be taken as the gradient $\mathbf{grad}\ \Phi$ of some scalar. Since the derivatives with respect to x, far from the body, are small in comparison with those with respect to y and z, we may to the same approximation neglect the term $\partial\Phi/\partial x$ in v_x, i.e. take $v_x = v_{1x}$.

Thus we have for v_x the equation

$$U\frac{\partial v_x}{\partial x} = \nu\left(\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right).$$

This equation is formally the same as the two-dimensional equation of heat conduction, with x/U in place of the time, and the viscosity ν in place of the thermometric conductivity. The solution which decreases with increasing y and z (for fixed x) and gives an infinitely narrow wake as $x \to 0$ (in this approximation the dimensions of the body are regarded as small) is (see §51)

$$v_x = -\frac{F_x}{4\pi\rho\nu}\frac{1}{x}e^{-U(y^2+z^2)/4\nu x}. \tag{1}$$

The constant coefficient in this formula is expressed in terms of the drag by means of formula (21.1), in which the integration over y and z may be extended to $\pm\infty$ on account of the rapid decrease of v_x. If we replace the Cartesian co-ordinates by spherical co-ordinates r, θ, ϕ with the polar axis along the x-axis, then the region of the wake ($\sqrt{(y^2+z^2)} \ll x$) corresponds to $\theta \ll 1$. In these co-ordinates formula (1) becomes

$$v_x = -\frac{F_x}{4\pi\rho\nu}\frac{1}{r}e^{-Ur\theta^2/4\nu}. \tag{1'}$$

The term $\partial\Phi/\partial x$ (with Φ given by formula (3) below), which we have omitted, would give a term in v_x which diminishes more rapidly, as $1/r^2$.

v_{1y} and v_{1z} must have the same form as (1). We take the direction of the lift as the y-axis (so that $F_z = 0$). According to (21.2) we have, since $\Phi = 0$ at infinity,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y \,\mathrm{d}y\,\mathrm{d}z = \iint \left(v_{1y} + \frac{\partial \Phi}{\partial y}\right)\mathrm{d}y\,\mathrm{d}z$$

$$= \iint v_{1y}\,\mathrm{d}y\,\mathrm{d}z = -F_y/\rho U,$$

$$\iint v_{1z}\,\mathrm{d}y\,\mathrm{d}z = 0.$$

Determining the constants in v_{1y} and v_{1z} from these conditions, we find

$$v_y = -\frac{F_y}{4\pi\rho\nu}\frac{1}{x}e^{-U(y^2+z^2)/4\nu x} + \frac{\partial \Phi}{\partial y}, \qquad v_z = \frac{\partial \Phi}{\partial z}. \tag{2}$$

To determine the function Φ we proceed as follows. By the equation of continuity,

$$\operatorname{div} \mathbf{v} \approx \partial v_y/\partial y + \partial v_z/\partial z = 0;$$

substituting (2), we have

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = -\frac{\partial v_{1y}}{\partial y}.$$

Differentiating this equation with respect to x and using the equation satisfied by v_{1y}, we obtain

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial \Phi}{\partial x} = -\frac{\partial}{\partial y}\left(\frac{\partial v_{1y}}{\partial x}\right)$$

$$= -\frac{\nu}{U}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial v_{1y}}{\partial y}.$$

Hence

$$\frac{\partial \Phi}{\partial x} = -\frac{\nu}{U}\frac{\partial v_{1y}}{\partial y}.$$

Finally, substituting the expression for v_{1y} and integrating with respect to x, we have

$$\Phi = -\frac{F_y}{2\pi\rho U}\frac{y}{y^2+z^2}[e^{-U(y^2+z^2)/4\nu x} - 1]. \tag{3}$$

The constant of integration is chosen so that Φ remains finite when $y = z = 0$. In spherical co-ordinates (with the azimuthal angle ϕ measured from the xy-plane)

$$\Phi = -\frac{F_y}{2\pi\rho U}\frac{\cos\phi}{r\theta}[e^{-Ur\theta^2/4\nu} - 1]. \tag{3'}$$

It is seen from (2) and (3) that v_y and v_z, unlike v_x, contain terms which decrease only as $1/\theta^2$ as we move away from the "axis" of the wake, as well as those which decrease exponentially with θ (for a given r).

The qualitative results (21.3) and (21.4) are, as we should expect, in agreement with the above formulae. If there is no lift, the flow in the wake is axially symmetrical.

PROBLEM 2. Determine the flow outside the wake far from the body.

SOLUTION. Outside the wake we assume potential flow. Since we are interested only in the terms in the potential Φ which decrease least rapidly with distance, we seek a solution of Laplace's equation $\triangle\Phi = 0$ as a sum of two terms:

$$\Phi = \frac{a}{r} + \frac{\cos\phi}{r} f(\theta),$$

of which the first is centrally symmetric and belongs to the force F_x, while the second is symmetrical about the xy-plane and belongs to the force F_y.

Using the expression for $\triangle\Phi$ in spherical co-ordinates, we obtain for the function $f(\theta)$ the equation

$$\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) - \frac{f}{\sin\theta} = 0.$$

The solution of this equation finite as $\theta \to \pi$ is $f = b\cot\frac{1}{2}\theta$. The coefficient b must be determined so as to give the correct value of F_y. It is simpler, however, to use the fact that in the range $\sqrt{(\nu/Ur)} \ll \theta \ll 1$ this part of Φ must be the same as the expression

$$\Phi = \frac{F_y}{2\pi\rho U}\frac{\cos\phi}{r\theta},$$

obtained from formula (3′), Problem 1, for Φ in the wake. Hence $b = F_y/4\pi\rho U$.

To determine the coefficient a, we notice that the total mass flux through a sphere S of large radius r equals zero, as for any closed surface. The rate of inflow through the part S_0 of S intercepted by the wake is

$$-\iint_{S_0} v_x\, dy\, dz = F_x/\rho U.$$

Hence the same quantity must flow out through the rest of the surface of the sphere, i.e. we must have

$$\oint_{S-S_0} \mathbf{v}\cdot d\mathbf{f} = F_x/\rho U.$$

Since S_0 is small compared with S, we can put

$$\oint_S \mathbf{v}\cdot d\mathbf{f} = \oint_S \mathbf{grad}\,\Phi\cdot d\mathbf{f} = -4\pi a = F_x/\rho U,$$

whence $a = -F_x/4\pi\rho U$.

The complete solution is given by the sum of these two expressions:

$$\Phi = \frac{1}{4\pi\rho U r}(-F_x + F_y\cos\phi\cot\tfrac{1}{2}\theta), \tag{1}$$

which gives the flow everywhere outside the wake far from the body. The potential diminishes with increasing distance as $1/r$; the velocity $\mathbf{v}$, therefore, diminishes as $1/r^2$. If there is no lift, the flow outside the wake is spherically symmetrical.

§22. The viscosity of suspensions

A fluid in which numerous fine solid particles are suspended (forming a *suspension*) may be regarded as an homogeneous medium if we are concerned with phenomena whose characteristic lengths are large compared with the dimensions of the particles. Such a medium has an effective viscosity η which is different from the viscosity η_0 of the original fluid. The value of η can be calculated for the case where the concentration of the suspended particles is small (i.e. their total volume is small in comparison with that of the fluid). The calculations are relatively simple for the case of spherical particles (A. EINSTEIN, 1906).

It is necessary to consider first the effect of a single solid globule, immersed in a fluid, on flow having a constant velocity gradient. Let the unperturbed flow be described by a linear velocity distribution

$$v_{0i} = \alpha_{ik}x_k, \tag{22.1}$$

where α_{ik} is a constant symmetrical tensor. The fluid pressure is constant:

$$p_0 = \text{constant},$$

and in future we shall take p_0 to be zero, i.e. measure only the deviation from this constant value. If the fluid is incompressible (div $\mathbf{v}_0 = 0$), the sum of the diagonal elements of the tensor α_{ik} must be zero:

$$\alpha_{ii} = 0. \tag{22.2}$$

Now let a small sphere of radius R be placed at the origin. We denote the altered fluid velocity by $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$; $\mathbf{v}_1$ must vanish at infinity, but near the sphere $\mathbf{v}_1$ is not small compared with $\mathbf{v}_0$. It is clear from the symmetry of the flow that the sphere remains at rest, so that the boundary condition is $\mathbf{v} = 0$ for $r = R$.

The required solution of the equations of motion (20.1) to (20.3) may be obtained at once from the solution (20.4), with the function f given by (20.6), if we notice that the space derivatives of this solution are themselves solutions. In the present case we desire a solution depending on the components of the tensor α_{ik} as parameters (and not on the vector $\mathbf{u}$ as in §20). Such a solution is

$$\mathbf{v}_1 = \mathbf{curl\,curl}\,[(\boldsymbol{\alpha}\cdot\mathbf{grad})f], \qquad p = \eta_0\alpha_{ik}\partial^2\triangle f/\partial x_i\partial x_k,$$

where $(\boldsymbol{\alpha}\cdot\mathbf{grad})f$ denotes a vector whose components are $\alpha_{ik}\partial f/\partial x_k$. Expanding these expressions and determining the constants a and b in the function $f = ar + b/r$ so as to satisfy the boundary conditions at the surface of the sphere, we obtain the following formulae for the velocity and pressure:

$$v_{1i} = \frac{5}{2}\left(\frac{R^5}{r^4} - \frac{R^3}{r^2}\right)\alpha_{kl}n_in_kn_l - \frac{R^5}{r^4}\alpha_{ik}n_k, \tag{22.3}$$

$$p = -5\eta_0\frac{R^5}{r^3}\alpha_{ik}n_in_k, \tag{22.4}$$

where $\mathbf{n}$ is a unit vector in the direction of the radius vector.

Returning now to the problem of determining the effective viscosity of a suspension, we calculate the mean value (over the volume) of the momentum flux density tensor Π_{ik}, which, in the linear approximation with respect to the velocity, is the same as the stress tensor $-\sigma_{ik}$:

$$\bar{\sigma}_{ik} = (1/V) \int \sigma_{ik} \mathrm{d}V.$$

The integration here may be taken over the volume V of a sphere of large radius, which is then extended to infinity.

First of all, we have the identity

$$\bar{\sigma}_{ik} = \eta_0 \left(\overline{\frac{\partial v_i}{\partial x_k}} + \overline{\frac{\partial v_k}{\partial x_i}} \right) - \bar{p}\delta_{ik} + \\ + \frac{1}{V} \int \left\{ \sigma_{ik} - \eta_0 \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + p\delta_{ik} \right\} \mathrm{d}V. \qquad (22.5)$$

The integrand on the right is zero except within the solid spheres; since the concentration of the suspension is supposed small, the integral may be calculated for a single sphere as if the others were absent, and then multiplied by the concentration c of the suspension (the number of spheres per unit volume). The direct calculation of this integral would require an investigation of internal stresses in the spheres. We can circumvent this difficulty, however, by transforming the volume integral into a surface integral over an infinitely distant sphere, which lies entirely in the fluid. To do so, we note that the equation of motion $\partial \sigma_{il}/\partial x_l = 0$ leads to the identity

$$\sigma_{ik} = \partial(\sigma_{il} x_k)/\partial x_l;$$

hence the transformation of the volume integral into a surface integral gives

$$\sigma_{ik} = \eta_0 \left(\overline{\frac{\partial v_i}{\partial x_k}} + \overline{\frac{\partial v_k}{\partial x_i}} \right) + c \oint \{ \sigma_{il} x_k \mathrm{d}f_l - \eta_0 (v_i \mathrm{d}f_k + v_k \mathrm{d}f_i) \}.$$

We have omitted the term in $\bar{p}$, since the mean pressure is necessarily zero; $\bar{p}$ is a scalar, which must be given by a linear combination of the components α_{ik}, and the only such scalar is $\alpha_{ii} = 0$.

In calculating the integral over a sphere of very large radius, only the terms of order $1/r^2$ need be retained in the expression (22.3) for the velocity. A simple calculation gives the value of the integral as

$$c\eta_0 \cdot 20\pi R^3 \{ 5\alpha_{lm} \overline{n_i n_k n_l n_m} - \alpha_{il} \overline{n_k n_l} \},$$

where the bar denotes an average with respect to directions of the unit vector

n. Effecting the averaging,† we finally have

$$\bar{\sigma}_{ik} = \eta_0\left(\overline{\frac{\partial v_i}{\partial x_k}} + \overline{\frac{\partial v_k}{\partial x_i}}\right) + 5\eta_0\alpha_{ik}\cdot\tfrac{4}{3}\pi R^3 c. \tag{22.6}$$

The ratio of the second term to the first determines the required relative correction to give the effective viscosity of the suspension. If we are interested only in corrections of the first order of smallness, we can take the first term as $2\eta_0\alpha_{ik}$. We then obtain for the effective viscosity of the suspension

$$\eta = \eta_0(1+\tfrac{5}{2}\phi), \tag{22.7}$$

where $\phi = \frac{4}{3}\pi R^3 c$ is the small ratio of the total volume of the spheres to the total volume of the suspension.

§23. Exact solutions of the equations of motion for a viscous fluid

If the non-linear terms in the equations of motion of a viscous fluid do not vanish identically, the solving of these equations offers great difficulties, and exact solutions can be obtained only in a very small number of cases. Furthermore, it has not yet proved possible to carry out a complete investigation of the steady flow of a viscous fluid in all space round a body in the limit of very large Reynolds numbers. Although, as we shall see, such a flow does not in practice remain steady, the solution of the problem would nevertheless be of great methodological interest.‡

We give below examples of exact solutions of the equations of motion for a viscous fluid.

(1) An infinite plane disk immersed in a viscous fluid rotates uniformly about its axis. Determine the motion of the fluid caused by this motion of the disk (T. von Kármán, 1921).

We take cylindrical co-ordinates, with the plane of the disk as the plane $z = 0$. Let the disk rotate about the z-axis with angular velocity Ω. We consider the unbounded volume of fluid on the side $z > 0$. The boundary conditions are

$$\begin{aligned} v_r &= 0, \quad v_\phi = \Omega r, \quad v_z = 0 \quad \text{for} \quad z = 0, \\ v_r &= 0, \quad v_\phi = 0 \qquad\qquad\quad \text{for} \quad z = \infty. \end{aligned}$$

† The required mean values of products of components of the unit vector are symmetrical tensors, which can be formed only from the unit tensor δ_{ik}. We then easily find

$$\overline{n_i n_k} = \delta_{ik},$$

$$\overline{n_i n_k n_l n_m} = \tfrac{1}{15}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}).$$

‡ The "vanishing viscosity" theory of Oseen is concerned with this problem; it is unsatisfactory, since it is based on an unjustified simplification of the Navier–Stokes equations. Prandtl's boundary-layer theory (see §39) does not solve the problem throughout the volume of the fluid.

The axial velocity v_z does not vanish as $z \to \infty$, but tends to a constant negative value determined by the equations of motion. The reason is that, since the fluid moves radially away from the axis of rotation, especially near the disk, there must be a constant vertical flow from infinity in order to satisfy the equation of continuity. We seek a solution of the equations of motion in the form

$$\begin{gathered} v_r = r\Omega F(z_1); \quad v_\phi = r\Omega G(z_1); \quad v_z = \sqrt{(\nu\Omega)}H(z_1); \\ p = -\rho\nu\Omega P(z_1), \quad \text{where} \quad z_1 = \sqrt{(\Omega/\nu)}z. \end{gathered} \tag{23.1}$$

In this velocity distribution, the radial and azimuthal velocities are proportional to the distance from the axis of rotation, while v_z is constant on each horizontal plane.

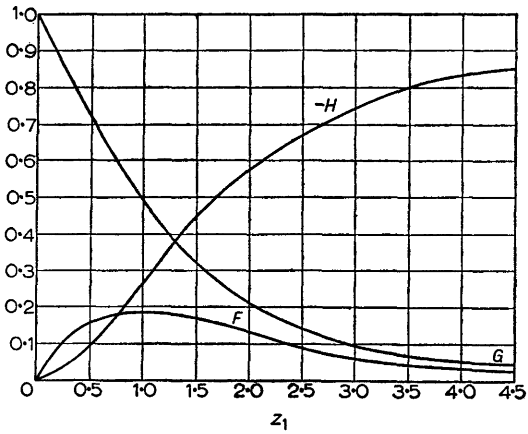

FIG. 7

Substituting in the Navier–Stokes equation and in the equation of continuity, we obtain the following equations for the functions F, G, H and P:

$$\begin{aligned} F^2 - G^2 + F'H &= F'', \qquad & 2FG + G'H &= G'', \\ HH' &= P' + H'', \qquad & 2F + H' &= 0; \end{aligned} \tag{23.2}$$

the prime denotes differentiation with respect to z_1. The boundary conditions are

$$\begin{aligned} &F = 0, \quad G = 1, \quad H = 0 \quad \text{for} \quad z_1 = 0. \\ &F = 0, \quad G = 0 \qquad\qquad\quad \text{for} \quad z_1 = \infty. \end{aligned} \tag{23.3}$$

We have therefore reduced the solution of the problem to the integration of a system of ordinary differential equations in one variable; this can be achieved numerically.† Fig. 7 shows the functions F, G and $-H$ thus obtained.

† The numerical integration has also been carried out for another similar problem, in which the fluid rotates uniformly at infinity and the disc is at rest (U.T. BÖDEWADT, *Zeitschrift für angewandte Mathematik und Mechanik* **20**, 241, 1940).

The limiting value of H as $z_1 \to \infty$ is $-0{\cdot}886$; in other words, the fluid velocity at infinity is $v_z(\infty) = -0{\cdot}886\sqrt{(\nu\Omega)}$.

The frictional force acting on unit area of the disk perpendicularly to the radius is $\sigma_{z\phi} = \eta(\partial v_\phi/\partial z)_{z=0}$. Neglecting edge effects, we may write the moment of the frictional forces acting on a disk of large but finite radius R as

$$M = 2\int_0^R 2\pi r^2 \sigma_{z\phi}\,\mathrm{d}r = \pi R^4 \rho\sqrt{(\nu\Omega^3)}G'(0).$$

The factor 2 in front of the integral appears because the disk has two sides exposed to the fluid. A numerical calculation of the function G leads to the formula

$$M = -1{\cdot}94 R^4 \rho\sqrt{(\nu\Omega^3)}. \tag{23.4}$$

(2) Determine the steady flow between two plane walls meeting at an angle α (Fig. 8 shows a cross-section of the two planes); the fluid flows out from the line of intersection of the planes (G. HAMEL, 1916).

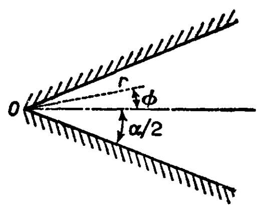

FIG. 8

We take cylindrical co-ordinates r, z, ϕ, with the z-axis along the line of the intersection of the planes (the point O in Fig. 8), and the angle ϕ measured as shown in Fig. 8. The flow is uniform in the z-direction, and we naturally assume it to be entirely radial, i.e.

$$v_\phi = v_z = 0, \qquad v_r = v(r, \phi).$$

The equations (15.16) give

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial^2 v}{\partial \phi} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right), \tag{23.5}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \frac{2\nu}{r^2}\frac{\partial v}{\partial \phi} = 0, \tag{23.6}$$

$$\partial(rv)/\partial r = 0.$$

It is seen from the last of these that rv is a function of ϕ only. Introducing the function

$$u(\phi) = rv/6\nu, \tag{23.7}$$

we obtain from (23.6)

$$\frac{1}{\rho}\frac{\partial p}{\partial \phi} = \frac{12\nu^2}{r^2}\frac{\mathrm{d}u}{\mathrm{d}\phi},$$

whence

$$\frac{p}{\rho} = \frac{12\nu^2}{r^2}u(\phi)+f(r).$$

Substituting this expression in (23.5), we have

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + 4u+6u^2 = \frac{1}{6\nu^2}r^3f'(r),$$

from which we see that, since the left-hand side depends only on ϕ and the right-hand side only on r, each must be a constant, which we denote by $2C_1$. Thus $f'(r) = 12\nu^2C_1/r^3$, whence $f(r) = -6\nu^2C_1/r^2+\text{constant}$, and we have for the pressure

$$\frac{p}{\rho} = \frac{6\nu^2}{r^2}(2u-C_1)+\text{constant}. \qquad (23.8)$$

For $u(\phi)$ we have the equation

$$u''+4u+6u^2 = 2C_1,$$

which, on multiplication by u' and one integration, gives

$$\tfrac{1}{2}u'^2+2u^2+2u^3-2C_1u-2C_2 = 0.$$

Hence we have

$$2\phi = \pm\int\frac{\mathrm{d}u}{\sqrt{(-u^3-u^2+C_1u+C_2)}} + C_3, \qquad (23.9)$$

which gives the required dependence of the velocity on ϕ; the function $u(\phi)$ can be expressed in terms of elliptic functions. The three constants C_1, C_2, C_3 are determined from the boundary conditions

$$u(\pm\tfrac{1}{2}\alpha) = 0 \qquad (23.10)$$

and from the condition that the same mass Q of fluid passes in unit time through any cross-section r = constant:

$$Q = \rho\int_{-\alpha/2}^{\alpha/2} vr\mathrm{d}\phi = 6\nu\rho\int_{-\alpha/2}^{\alpha/2} u\mathrm{d}\phi. \qquad (23.11)$$

Q may be either positive or negative. If $Q > 0$, the line of intersection of the planes is a source, i.e. the fluid emerges from the vertex of the angle: this is called *flow in a diverging channel.* If $Q < 0$, the line of intersection is a sink, and we have *flow in a converging channel.* The ratio $|Q|/\nu\rho$ is dimensionless and plays the part of the Reynolds number in the problem considered.

Let us first discuss converging flow ($Q < 0$). To investigate the solution (23.9)–(23.11) we make the assumptions, which will be justified later, that the flow is symmetrical about the plane $\phi = 0$ (i.e. $u(\phi) = u(-\phi)$), and that the function $u(\phi)$ is everywhere negative (i.e. the velocity is everywhere towards the vertex) and decreases monotonically from $u = 0$ at $\phi = \pm\frac{1}{2}\alpha$ to $u = -u_0 < 0$ at $\phi = 0$, so that u_0 is the maximum value of $|u|$. Then for $u = -u_0$ we must have $du/d\phi = 0$, whence it follows that $u = -u_0$ is a zero of the cubic expression under the radical in the integrand of (23.9). We can therefore write

$$-u^3-u^2+C_1u+C_2 = (u+u_0)\{-u^2-(1-u_0)u+q\},$$

where q is another constant. Thus

$$2\phi = \pm\int\limits_{-u_0}^{u} \frac{du}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}}, \tag{23.12}$$

the constants u_0 and q being determined from the conditions

$$\begin{aligned} \alpha &= \int\limits_{-u_0}^{0} \frac{du}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}}, \\ \tfrac{1}{6}\mathrm{R} &= \int\limits_{-u_0}^{0} \frac{u\,du}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}} \end{aligned} \tag{23.13}$$

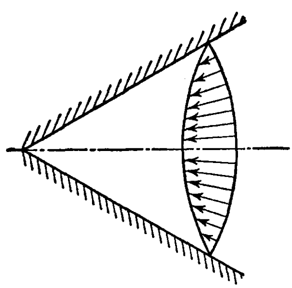

FIG. 9

($\mathrm{R} = |Q|/\nu\rho$); the constant q must be positive, since otherwise these integrals would be complex. The two equations just given may be shown to have solutions u_0 and q for any R and $\alpha < \pi$. In other words, convergent symmetrical flow (Fig. 9) is possible for any aperture angle α and any Reynolds number. Let us consider in more detail the flow for very large R. This corresponds to large u_0. Writing (23.12) (for $\phi > 0$) as

$$2(\tfrac{1}{2}\alpha-\phi) = \int\limits_u^0 \frac{\mathrm{d}u}{\sqrt{[(u+u_0)\{-u^2-(1-u_0)u+q\}]}},$$

we see that the integrand is small throughout the range of integration if $|u|$ is not close to u_0. This means that $|u|$ can differ appreciably from u_0 only for ϕ close to $\pm\frac{1}{2}\alpha$, i.e. in the immediate neighbourhood of the walls.† In other words, we have $u \approx$ constant $= -u_0$ for almost all angles ϕ, and in addition $u_0 = \mathrm{R}/6\alpha$, as we see from equations (23.13). The velocity v itself is $|Q|/\rho\alpha r$, giving a non-viscous potential flow with velocity independent of angle and inversely proportional to r. Thus, for large Reynolds numbers, the flow in a converging channel differs very little from potential flow of an ideal fluid. The effect of the viscosity appears only in a very narrow layer near the walls, where the velocity falls rapidly to zero from the value corresponding to the potential flow (Fig. 10).

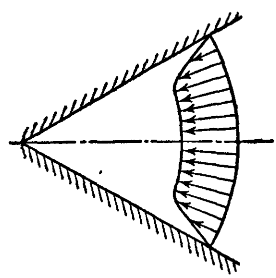

FIG. 10

Now let $Q > 0$, so that we have divergent flow. At first we again suppose that the flow is symmetrical about the plane $\phi = 0$, and that $u(\phi)$ (where now $u > 0$) varies monotonically from zero at $\phi = \pm\frac{1}{2}\alpha$ to $u_0 > 0$ at $\phi = 0$. Instead of (23.13) we now have

$$\alpha = \int\limits_0^{u_0} \frac{\mathrm{d}u}{\sqrt{[(u_0-u)\{u^2+(1+u_0)u+q\}]}},$$
$$\tfrac{1}{6}\mathrm{R} = \int\limits_0^{u_0} \frac{u\,\mathrm{d}u}{\sqrt{[(u_0-u)\{u^2+(1+u_0)u+q\}]}}. \qquad (23.14)$$

† The question may be asked how the integral can cease to be small, even if $u \approx -u_0$. The answer is that, for u_0 very large, one of the roots of $-u^2-(1-u_0)u+q = 0$ is close to $-u_0$, so that the radicand has two almost coincident zeros, the whole integral therefore being "almost divergent" at $u = -u_0$.

If we regard u as given, then α increases monotonically as q decreases, and takes its greatest value for $q = 0$:

$$\alpha_{\max} = \int_0^{u_0} \frac{du}{\sqrt{[u(u_0-u)(u+u_0+1)]}}.$$

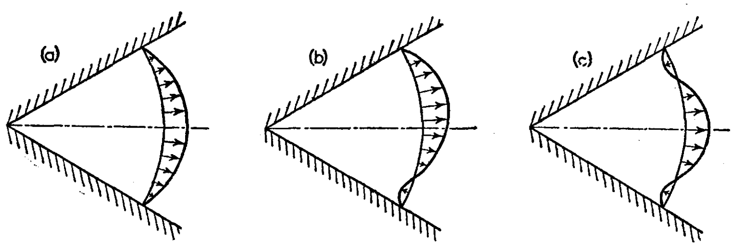

FIG. 11

It is easy to see that for given q, on the other hand, α is a monotonically decreasing function of u_0. Hence it follows that u_0 is a monotonically decreasing function of q for given α, so that its greatest value is for $q = 0$ and is given by the above equation. The maximum $R = R_{\max}$ corresponds to the maximum u_0. Using the substitutions $k^2 = u_0/(1+2u_0)$, $u = u_0 \cos^2 x$, we can write the dependence of $R_{\max}$ on α in the parametric form

$$\alpha = 2\sqrt{(1-2k^2)} \int_0^{\pi/2} \frac{dx}{\sqrt{(1-k^2 \sin^2 x)}},$$

$$R_{\max} = -6\alpha\frac{1-k^2}{1-2k^2} + \frac{12}{\sqrt{(1-2k^2)}} \int_0^{\pi/2} \sqrt{(1-k^2 \sin^2 x)}dx. \tag{23.15}$$

Thus symmetrical flow, everywhere divergent (Fig. 11a), is possible for a given aperture angle only for Reynolds numbers not exceeding a definite value. As $\alpha \to \pi$ ($k \to 0$), $R_{\max} \to 0$; as $\alpha \to 0$ ($k \to 1/\sqrt{2}$), $R_{\max}$ tends to infinity as $18{\cdot}8/\alpha$.

For $R > R_{\max}$ the assumption of symmetrical flow, everywhere divergent, is unjustified, since the conditions (23.14) cannot be satisfied. In the range of angles $-\frac{1}{2}\alpha \leqslant \phi \leqslant \frac{1}{2}\alpha$ the function $u(\phi)$ must now have maxima or minima. The values of $u(\phi)$ corresponding to these extrema must again be zeros of the polynomial under the radical sign. It is therefore clear that the trinomial $u^2+(1+u_0)u+q$ (with $u_0 > 0$, $q > 0$) must have two real negative roots in the range mentioned, so that the radicand can be written $(u_0-u)(u+u_0')(u+u_0'')$, where $u_0 > 0$, $u_0' > 0$, $u_0'' > 0$; we suppose

$u_0' < u_0''$. The function $u(\phi)$ can evidently vary in the range $u_0 \geqslant u \geqslant -u_0'$, $u = u_0$ corresponding to a positive maximum of $u(\phi)$, and $u = -u_0'$ to a negative minimum. Without pausing to make a detailed investigation of the solutions obtained in this way, we may mention that for $R > R_{max}$ a solution appears in which the velocity has one maximum and one minimum, the flow being asymmetric about the plane $\phi = 0$ (Fig. 11b). When R increases further, a symmetrical solution with one minimum and two maxima appears (Fig. 11c), and so on. In all these solutions, therefore, there are regions of both outward and inward flow (though of course the total discharge Q is positive). As $R \to \infty$ the number of alternating minima and maxima increases without limit, so that there is no definite limiting solution. We may emphasise that in divergent flow as $R \to \infty$ the solution does not, therefore, tend to the solution of Euler's equations as it does for convergent flow. Finally, it may be mentioned that, as R increases, the steady divergent flow of the kind described becomes unstable soon after R exceeds R_{max}, and in practice a non-steady or *turbulent* flow occurs (Chapter III).

(3) Determine the flow in a jet emerging from the end of a narrow tube into an infinite space filled with the fluid—the *submerged jet* (L. LANDAU, 1943).

We take spherical co-ordinates r, θ, ϕ, with the polar axis in the direction of the jet at its point of emergence, and with this point as origin. The flow is symmetrical about the polar axis, so that $v_\phi = 0$ and v_θ, v_r are functions of r and θ only. The same total momentum flux (the "momentum of the jet") must pass through any closed surface surrounding the origin (in particular, through an infinitely distant surface). For this to be so, the velocity must be inversely proportional to r, so that

$$v_r = F(\theta)/r, \qquad v_\theta = f(\theta)/r, \tag{23.16}$$

where F and f are some functions of θ only. The equation of continuity is

$$\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) = 0.$$

Hence we find that

$$F(\theta) = -\mathrm{d}f/\mathrm{d}\theta - f\cot\theta. \tag{23.17}$$

The components $\Pi_{r\phi}$, $\Pi_{\theta\phi}$ of the momentum flux density tensor in the jet vanish identically by symmetry. We assume that the components $\Pi_{\theta\theta}$ and $\Pi_{\phi\phi}$ also vanish; this assumption is justified when we obtain a solution satisfying all the necessary conditions. Using the expressions (15.17) for the components of the tensor σ_{ik}, and formulae (23.16), (23.17), we easily see that the relation

$$\sin^2\theta\,\Pi_{r\theta} = \frac{1}{2}\frac{\partial}{\partial\theta}[\sin^2\theta(\Pi_{\phi\phi} - \Pi_{\theta\theta})]$$

holds between the components of the momentum flux density tensor in the jet. Hence it follows that $\Pi_{r\theta} = 0$. Thus only the component Π_{rr} is non-zero, and it varies as $1/r^2$. It is easy to see that the equations of motion $\partial\Pi_{ik}/\partial x_k = 0$ are automatically satisfied.

Next, we write

$$(\Pi_{\theta\theta} - \Pi_{\phi\phi})/\rho = (f^2 + 2\nu f \cot\theta - 2\nu f')/r^2 = 0,$$

or

$$\mathrm{d}(1/f)/\mathrm{d}\theta + (1/f)\cot\theta + 1/2\nu = 0.$$

The solution of this equation is

$$f = -2\nu \sin\theta/(A - \cos\theta), \tag{23.18}$$

and then we have from (23.17)

$$F = 2\nu\left\{\frac{A^2-1}{(A-\cos\theta)^2} - 1\right\}. \tag{23.19}$$

The pressure distribution is found from the equation

$$\Pi_{\theta\theta}/\rho = p/\rho + f(f + 2\nu\cot\theta)/r^2 = 0,$$

which gives

$$p = \frac{4\rho\nu^2(A\cos\theta - 1)}{r^2(A-\cos\theta)^2}. \tag{23.20}$$

The constant A can be found in terms of the momentum of the jet, i.e. the total momentum flux in it. This flux is equal to the integral over the surface of a sphere

$$P = \oint \Pi_{rr}\cos\theta\,\mathrm{d}f = 2\pi\int_0^\pi r^2\,\Pi_{rr}\cos\theta\sin\theta\,\mathrm{d}\theta.$$

The value of Π_{rr} is given by

$$\frac{1}{\rho}\Pi_{rr} = \frac{4\nu^2}{r^2}\left\{\frac{(A^2-1)^2}{(A-\cos\theta)^4} - \frac{A}{A-\cos\theta}\right\},$$

and a calculation of the integral gives

$$P = 16\pi\nu^2\rho A\left\{1 + \frac{4}{3(A^2-1)} - \tfrac{1}{2}A\log\frac{A+1}{A-1}\right\}. \tag{23.21}$$

Formulae (23.16)–(23.21) give the solution of the problem.†

† The solution here obtained is exact for a jet regarded as emerging from a point source. If the finite dimensions of the tube mouth are taken into account, the solution becomes the first term of an expansion in powers of the ratio of these dimensions to the distance r from the mouth of the tube. This is why, if we calculate from the above solution the total mass flux through a closed surface surrounding the origin, the result is zero. A non-zero total mass flux is obtained when further terms in the above-mentioned expansion are considered; see Yu. B. Rumer, *Prikladnaya matematika i mekhanika* **16**, 255, 1952.

The submerged laminar jet with a non-zero angular momentum has bee ndiscussed by L. G. Loĭtsyanskiĭ (*ibid.* **17**, 3, 1953).

The streamlines are determined by the equation $dr/v_r = r d\theta/v_\theta$, integration of which gives $r \sin^2\theta/(A-\cos\theta) = \text{constant}$. Fig. 12 shows the streamlines in the jet (for $A > 1$).

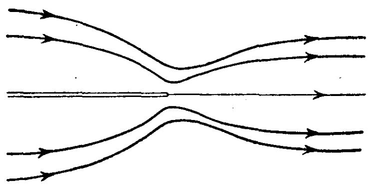

FIG. 12

Let us consider two limiting cases, a weak jet (small momentum P) and a strong jet (large P). As $P \to 0$, the constant A tends to infinity: from (23.21) we have $P = 16\pi\nu^2\rho/A$. For the velocity in this case we have

$$v_\theta = -P\sin\theta/8\pi\nu\rho r, \qquad v_r = P\cos\theta/4\pi\nu\rho r.$$

As $P \to \infty$ (strong jet†), A tends to unity: (23.21) gives $A = 1+\frac{1}{2}\alpha^2$, where $\alpha = 32\pi\nu^2\rho/3P$. For large angles ($\theta \sim 1$), the velocity is given by

$$v_\theta = -(2\nu/r)\cot\tfrac{1}{2}\theta, \qquad v_r = -2\nu/r,$$

but for small angles ($\theta \sim \alpha$) we have

$$v_\theta = -4\nu\theta/(\alpha^2+\theta^2), \qquad v_r = 8\nu\alpha^2/(\alpha^2+\theta^2)^2.$$

§24. Oscillatory motion in a viscous fluid

When a solid body immersed in a viscous fluid oscillates, the flow thereby set up has a number of characteristic properties. In order to study these, it is convenient to begin with a simple but typical example. Let us suppose that an incompressible fluid is bounded by an infinite plane surface which executes a simple harmonic oscillation in its own plane, with frequency ω. We require the resulting motion of the fluid. We take the solid surface as the yz-plane, and the fluid region as $x > 0$; the y-axis is taken in the direction of the oscillation. The velocity u of the oscillating surface is a function of time, of the form $A\cos(\omega t+\alpha)$. It is convenient to write this as the real part of a complex quantity:

$$u = \mathrm{re}(u_0 e^{-i\omega t}),$$

where the constant $u_0 = Ae^{-i\alpha}$ is in general complex, but can always be made real by a proper choice of the origin of time.

† However, it must be borne in mind that the flow in a sufficiently strong jet is actually turbulent (§35).

So long as the calculations involve only linear operations on the velocity u, we may omit the sign re and proceed as if u were complex, taking the real part of the final result. Thus we write

$$u_y = u = u_0\, e^{-i\omega t}. \tag{24.1}$$

The fluid velocity must satisfy the boundary condition $\mathbf{v} = \mathbf{u}$ for $x = 0$, i.e. $v_x = v_z = 0$, $v_y = u$.

It is evident from symmetry that all quantities will depend only on the co-ordinate x and the time t. From the equation of continuity $\operatorname{div} \mathbf{v} = 0$ we therefore have $\partial v_x/\partial x = 0$, whence v_x = constant = zero, from the boundary condition. Since all quantities are independent of the co-ordinates y and z, we have $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = v_x\, \partial \mathbf{v}/\partial x$, and since v_x is zero it follows that $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} = 0$ identically. The equation of motion (15.7) becomes

$$\partial \mathbf{v}/\partial t = -(1/\rho)\mathbf{grad}\, p + \nu \triangle \mathbf{v}. \tag{24.2}$$

This is a linear equation. Its x-component is $\partial p/\partial x = 0$, i.e. p = constant.

It is further evident from symmetry that the velocity $\mathbf{v}$ is everywhere in the y-direction. For $v_y = v$ we have by (24.2)

$$\partial v/\partial t = \nu \partial^2 v/\partial x^2, \tag{24.3}$$

that is, a (one-dimensional) heat conduction equation. We shall look for a solution of this equation which is periodic in x and t, of the form

$$v = u_0\, e^{i(kx-\omega t)},$$

with a complex amplitude u_0, so that $v = u$ for $x = 0$. Substituting in (24.3), we find $i\omega = \nu k^2$, whence

$$k = \sqrt{(i\omega/\nu)} = \pm(i+1)\sqrt{(\omega/2\nu)},$$

so that the velocity v is

$$v = u_0\, e^{-\sqrt{(\omega/2\nu)}x}\, e^{i[\sqrt{(\omega/2\nu)}x-\omega t]}; \tag{24.4}$$

we have taken k to have a positive imaginary part, since otherwise the velocity would increase without limit in the interior of the fluid, which is physically impossible.

The solution obtained represents a transverse wave: its velocity $v_y = v$ is perpendicular to the direction of propagation. The most important property of this wave is that it is rapidly damped in the interior of the fluid: the amplitude decreases exponentially as the distance x from the solid surface increases.†

Thus transverse waves can occur in a viscous fluid, but they are rapidly damped as we move away from the solid surface whose motion generates the waves.

The distance δ over which the amplitude falls off by a factor of e is called the *depth of penetration* of the wave. We see from (24.4) that

$$\delta = \sqrt{(2\nu/\omega)}. \tag{24.5}$$

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† Over a distance of one wavelength the amplitude diminishes by a factor of $e^{2\pi} \approx 540$.

The depth of penetration therefore diminishes with increasing frequency, but increases with the kinematic viscosity of the fluid.

Let us calculate the frictional force acting on unit area of the plane oscillating in the viscous fluid. This force is evidently in the y-direction, and is equal to the component $\sigma_{xy} = \eta \partial v_y / \partial x$ of the stress tensor; the value of the derivative must be taken at the surface itself, i.e. at $x = 0$. Substituting (24.4), we obtain

$$\sigma_{xy} = \sqrt{(\tfrac{1}{2}\omega\eta\rho)}(i-1)u. \tag{24.6}$$

Supposing u_0 real and taking the real part of (24.6), we have

$$\sigma_{xy} = -\sqrt{(\omega\eta\rho)}u_0 \cos(\omega t + \tfrac{1}{4}\pi).$$

The velocity of the oscillating surface, however, is $u = u_0 \cos \omega t$. There is therefore a phase difference between the velocity and the frictional force.†

It is easy to calculate also the (time) average of the energy dissipation in the above problem. This may be done by means of the general formula (16.3); in this particular case, however, it is simpler to calculate the required dissipation directly as the work done by the frictional forces. The energy dissipated per unit time per unit area of the oscillating plane is equal to the mean value of the product of the force σ_{xy} and the velocity $u_y = u$:

$$-\overline{\sigma_{xy}u} = \tfrac{1}{2}u_0^2\sqrt{(\tfrac{1}{2}\omega\eta\rho)}. \tag{24.7}$$

It is proportional to the square root of the frequency of the oscillations, and to the square root of the viscosity.

An explicit solution can also be given of the problem of a fluid set in motion by a plane surface moving in its plane according to any law $u = u(t)$. We shall not pause to give the corresponding calculations here, since the required solution of equation (24.3) is formally identical with that of an analogous problem in the theory of thermal conduction, which we shall discuss in §52 (the solution is formula (52.15)). In particular, the frictional force on unit area of the surface is given by

$$\sigma_{xy} = \sqrt{\frac{\eta\rho}{\pi}} \int_{-\infty} \frac{du(\tau)}{d\tau} \frac{d\tau}{\sqrt{(t-\tau)}}; \tag{24.8}$$

cf. (52.16).

† For oscillations of a half-plane (parallel to its edge) there is an additional frictional force due to edge effects. The problem of the motion of a viscous fluid caused by oscillations of a half-plane, and also the more general problem of the oscillations of a wedge of any angle, can be solved by a class of solutions of the equation $\triangle f + k^2 f = 0$, used by A. Sommerfeld in the theory of diffraction by a wedge; see, for instance, M. von Laue, Interferenz und Beugung elektromagnetischer Wellen (Interference and diffraction of electromagnetic waves), *Handbuch der Experimentalphysik* **18**, 333, Akademische Verlagsgesellschaft, Leipzig 1928.

We give here, for reference, only one result: the increase in the frictional force on a half-plane, arising from the edge effect, can be regarded as the result of increasing the area of the half-plane by moving the edge a distance $\tfrac{1}{2}\delta = \sqrt{(\nu/2\omega)}$.

Let us now consider the general case of an oscillating body of arbitrary shape. In the case of an oscillating plane considered above, the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equation of motion of the fluid was identically zero. This does not happen, of course, for a surface of arbitrary shape. We shall assume, however, that this term is small in comparison with the other terms, so that it may be neglected. The conditions necessary for this procedure to be valid will be examined below.

We shall therefore begin, as before, from the linear equation (24.2). We take the curl of both sides; the term $\mathbf{curl}\,\mathbf{grad}\,p$ vanishes identically, giving

$$\partial(\mathbf{curl}\,\mathbf{v})/\partial t = \nu\triangle\mathbf{curl}\,\mathbf{v}, \tag{24.9}$$

i.e. $\mathbf{curl}\,\mathbf{v}$ satisfies a heat conduction equation. We have seen above, however, that such an equation gives an exponential decrease of the quantity which satisfies it. We can therefore say that the vorticity decreases towards the interior of the fluid. In other words, the motion of the fluid caused by the oscillations of the body is rotational in a certain layer round the body, while at larger distances it rapidly changes to potential flow. The depth of penetration of the rotational flow is of the order of $\delta \sim \sqrt{(\nu/\omega)}$.

Two important limiting cases are possible here: the quantity δ may be either large or small compared with the dimension of the oscillating body. Let l be the order of magnitude of this dimension. We first consider the case $\delta \gg l$; this implies that $l^2\omega \ll \nu$. Besides this condition, we shall also suppose that the Reynolds number is small. If a is the amplitude of the oscillations, the velocity of the body is of the order of $a\omega$. The Reynolds number for the motion in question is therefore $\omega al/\nu$. We therefore suppose that

$$l^2\omega \ll \nu, \qquad \omega al/\nu \ll 1. \tag{24.10}$$

This is the case of low frequencies of oscillation, which in turn means that the velocity varies only slowly with time, and therefore that we can neglect the derivative $\partial\mathbf{v}/\partial t$ in the general equation of motion. The term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$, on the other hand, can be neglected because the Reynolds number is small.

The absence of the term $\partial\mathbf{v}/\partial t$ from the equation of motion means that the flow is steady. Thus, for $\delta \gg l$, the flow can be regarded as steady at any given instant. This means that the flow at any given instant is what it would be if the body were moving uniformly with its instantaneous velocity. If, for example, we are considering the oscillations of a sphere immersed in the fluid, with a frequency satisfying the inequalities (24.10) (l being now the radius of the sphere), then we can say that the drag on the sphere will be that given by Stokes' formula (20.14) for uniform motion of the sphere at small Reynolds numbers.

Let us now consider the opposite case, where $l \gg \delta$. In order that the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ should again be negligible, it is necessary that the amplitude of the oscillations should be small in comparison with the dimensions of the body:

$$l^2\omega \gg \nu, \qquad a \ll l; \tag{24.11}$$

in this case, it should be noticed, the Reynolds number need not be small. The above inequality is obtained by estimating the magnitude of $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$. The operator $(\mathbf{v}\cdot\mathbf{grad})$ denotes differentiation in the direction of the velocity. Near the surface of the body, however, the velocity is nearly tangential. In the tangential direction the velocity changes appreciably only over distances of the order of the dimension of the body. Hence

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \sim v^2/l \sim a^2\omega^2/l,$$

since the velocity itself is of the order of $a\omega$. The derivative $\partial\mathbf{v}/\partial t$, however, is of the order of $v\omega \sim a\omega^2$. Comparing these, we see that

$$(\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \ll \partial\mathbf{v}/\partial t$$

if $a \ll l$. The terms $\partial\mathbf{v}/\partial t$ and $\nu\triangle\mathbf{v}$ are then easily seen to be of the same order.

We may now discuss the nature of the flow round an oscillating body when the conditions (24.11) hold. In a thin layer near the surface of the body the flow is rotational, but in the rest of the fluid we have potential flow.† Hence the flow everywhere except in the layer adjoining the body is given by the equations

$$\mathbf{curl}\,\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{v} = 0. \tag{24.12}$$

Hence it follows that $\triangle\mathbf{v} = 0$, and the Navier–Stokes equation reduces to Euler's equation. The flow is therefore ideal everywhere except in the surface layer. Since this layer is thin, in solving equations (24.12) to determine the flow of the rest of the fluid we should take as boundary conditions those which must be satisfied at the surface of the body, i.e. that the fluid velocity is equal to that of the body. The solutions of the equations of motion for an ideal fluid cannot satisfy these conditions, however. We can require only the fulfilment of the corresponding condition for the fluid velocity component normal to the surface.

Although equations (24.12) are inapplicable in the surface layer of fluid, the velocity distribution obtained by solving them satisfies the necessary boundary condition for the normal velocity component, and the actual variation of this component near the surface therefore has no significant properties. The tangential component would be found, by solving the equations (24.12), to have some value different from the corresponding velocity component of the body, whereas these velocity components should be equal also. Hence the tangential velocity component must change rapidly in the surface layer. The nature of this variation is easily determined. Let us consider any portion of the surface of the body, of dimension large compared

† For oscillations of a plane surface not only **curl v** but also **v** itself decreases exponentially with characteristic distance δ. This is because the oscillating plane does not displace the fluid, and therefore the fluid remote from it remains at rest. For oscillations of bodies of other shapes the fluid is displaced, and therefore executes a motion where the velocity decreases appreciably only over distances of the order of the dimension of the body.

with δ, but small compared with the dimension of the body. Such a portion may be regarded as approximately plane, and therefore we can use the results obtained above for a plane surface. Let the x-axis be directed along the normal to the portion considered, and the y-axis parallel to the tangential velocity component of the surface there. We denote by v_y the tangential component of the fluid velocity relative to the body; v_y must vanish on the surface. Lastly, let $v_0 e^{-i\omega t}$ be the value of v_y found by solving equations (24.12). From the results obtained at the beginning of this section, we can say that in the surface layer the quantity v_y will fall off towards the surface according to the law

$$v_y = v_0 e^{-i\omega t}[1 - e^{-(1-i)x\sqrt{(\omega/2\nu)}}]. \tag{24.13}$$

Finally, the total amount of energy dissipated in unit time will be given by the integral

$$\bar{\dot{E}}_{\text{kin}} = -\tfrac{1}{2}\sqrt{(\tfrac{1}{2}\eta\rho\omega)} \oint |v_0|^2 \, \mathrm{d}f \tag{24.14}$$

taken over the surface of the oscillating body.

In the Problems at the end of this section we calculate the drag on various bodies oscillating in a viscous fluid. Here we shall make the following general remark regarding these forces. Writing the velocity of the body in the complex form $\mathbf{u} = \mathbf{u}_0 e^{-i\omega t}$, we obtain a drag $\mathbf{F}$ proportional to the velocity $\mathbf{u}$, and also complex: $\mathbf{F} = \beta\mathbf{u}$, where $\beta = \beta_1 + i\beta_2$ is a complex constant. This expression can be written as the sum of two terms with real coefficients:

$$\mathbf{F} = (\beta_1 + i\beta_2)\mathbf{u} = \beta_1\mathbf{u} - \beta_2\dot{\mathbf{u}}/\omega, \tag{24.15}$$

one proportional to the velocity $\mathbf{u}$ and the other to the acceleration $\dot{\mathbf{u}}$.

The (time) average of the energy dissipation is given by the mean product of the drag and the velocity, where of course we must first take the real parts of the expressions given above, i.e. $\mathbf{u} = \frac{1}{2}(\mathbf{u}_0 e^{-i\omega t} + \mathbf{u}_0^* e^{i\omega t})$, $\mathbf{F} = \frac{1}{2}(\mathbf{u}_0\beta e^{-i\omega t} + \mathbf{u}_0^*\beta^* e^{i\omega t})$. Noticing that the mean values of $e^{\pm 2i\omega t}$ are zero, we have

$$\overline{\mathbf{F}\mathbf{u}} = \tfrac{1}{4}(\beta + \beta^*)|\mathbf{u}_0|^2 = \tfrac{1}{2}\beta_1|\mathbf{u}_0|^2. \tag{24.16}$$

Thus we see that the energy dissipation arises only from the real part of β; the corresponding part of the drag (24.15), proportional to the velocity, may be called the *dissipative part*. The other part of the drag, proportional to the acceleration and determined by the imaginary part of β, does not involve the dissipation of energy and may be called the *inertial part*.

Similar considerations hold for the moment of the forces on a body executing rotary oscillations in a viscous fluid.

PROBLEMS

PROBLEM 1. Determine the frictional force on each of two parallel solid planes, between which is a layer of viscous fluid, when one of the planes oscillates in its own plane.

SOLUTION. We seek a solution of equation (24.3) in the form†

$$v = (A\sin kx + B\cos kx)e^{-i\omega t},$$

and determine A and B from the conditions $v = u = u_0 e^{-i\omega t}$ for $x = 0$ and $v = 0$ for $x = h$, where h is the distance between the planes. The result is

$$v = u\frac{\sin k(h-x)}{\sin kh}.$$

The frictional force per unit area on the moving plane is

$$P_{1x} = \eta(\partial v/\partial x)_{x=0} = -\eta k u \cot kh,$$

while that on the fixed plane is

$$P_{2x} = -\eta(\partial v/\partial x)_{x=h} = \eta k u \operatorname{cosec} kh,$$

the real parts of all quantities being understood.

PROBLEM 2. Determine the frictional force on an oscillating plane covered by a layer of fluid of thickness h, the upper surface being free.

SOLUTION. The boundary condition at the solid plane is $v = u$ for $x = 0$, and that at the free surface is $\sigma_{xy} = \eta\partial v/\partial x = 0$ for $x = h$. We find the velocity

$$v = u\frac{\cos k(h-x)}{\cos kh}.$$

The frictional force is

$$P_x = \eta(\partial v/\partial x)_{x=0} = \eta k u \tan kh.$$

PROBLEM 3. A plane disk of large radius R executes rotary oscillations of small amplitude about its axis, the angle of rotation being $\theta = \theta_0 \cos \omega t$, where $\theta_0 \ll 1$. Determine the moment of the frictional forces acting on the disk.

SOLUTION. For oscillations of small amplitude the term $(\mathbf{v}\cdot\mathbf{grad})\mathbf{v}$ in the equation of motion is always small compared with $\partial\mathbf{v}/\partial t$, whatever the frequency ω. If $R \gg \delta$, the disk may be regarded as infinite in determining the velocity distribution. We take cylindrical co-ordinates, with the z-axis along the axis of rotation, and seek a solution such that $v_r = v_z = 0$, $v_\phi = v = r\Omega(z, t)$. For the angular velocity $\Omega(z, t)$ of the fluid we obtain the equation

$$\partial\Omega/\partial t = \nu\partial^2\Omega/\partial z^2.$$

The solution of this equation which is $-\omega\theta_0 \sin\omega t$ for $z = 0$ and zero for $z = \infty$ is

$$\Omega = -\omega\theta_0 e^{-z/\delta}\sin(\omega t - z/\delta).$$

The moment of the frictional forces on both sides of the disk is

$$M = 2\int_0^R r\cdot 2\pi r\eta\,(\partial v/\partial z)_{z=0}\,\mathrm{d}r = \omega\theta_0\pi\sqrt{(\omega\rho\eta)}R^4\cos(\omega t - \tfrac{1}{4}\pi).$$

† In all the Problems to this section δ denotes the quantity (24.5):

$$\delta = \sqrt{(2\nu/\omega)}, \quad \text{and} \quad k = (1+i)/\delta.$$

PROBLEM 4. Determine the flow between two parallel planes when there is a pressure gradient which varies harmonically with time.

SOLUTION. We take the xz-plane half-way between the two planes, with the x-axis parallel to the pressure gradient, which we write in the form

$$-(1/\rho)\partial p/\partial x = ae^{-i\omega t}.$$

The velocity is everywhere in the x-direction, and is determined by the equation

$$\partial v/\partial t = ae^{-i\omega t}+\nu\partial^2 v/\partial y^2.$$

The solution of this equation which satisfies the conditions $v = 0$ for $y = \pm\frac{1}{2}h$ is

$$v = \frac{ia}{\omega}e^{-i\omega t}\left[1-\frac{\cos ky}{\cos\frac{1}{2}kh}\right].$$

The mean value of the velocity over a cross-section is

$$\bar{v} = \frac{ia}{\omega}e^{-i\omega t}\left(1-\frac{2}{kh}\tan\tfrac{1}{2}kh\right).$$

For $h/\delta \ll 1$ this becomes

$$\bar{v} \approx ae^{-i\omega t}h^2/12\nu,$$

in agreement with (17.5), while for $h/\delta \gg 1$ we have

$$\bar{v} \approx (ia/\omega)e^{-i\omega t},$$

in accordance with the fact that in this case the velocity must be almost constant over the cross-section, varying only in a narrow surface layer.

PROBLEM 5. Determine the drag on a sphere of radius R which executes translatory oscillations in a fluid.

SOLUTION. We write the velocity of the sphere in the form $\mathbf{u} = \mathbf{u}_0e^{-i\omega t}$. As in §20, we seek the fluid velocity in the form $\mathbf{v} = e^{-i\omega t}\,\mathbf{curl\,curl}\,f\mathbf{u}_0$, where f is a function of r only (the origin is taken at the instantaneous position of the centre of the sphere). Substituting in (24.9) and effecting transformations similar to those of §20, we obtain the equation

$$\triangle^2 f+(i\omega/\nu)\,\triangle f = 0$$

(instead of the equation $\triangle^2 f = 0$ in §20). Hence we have

$$\triangle f = \text{constant}\times e^{ikr}/r,$$

the solution being chosen which decreases exponentially with r. Integrating, we have

$$\mathrm{d}f/\mathrm{d}r = [ae^{ikr}(r-1/ik)+b]/r^2; \tag{1}$$

the function f itself is not needed, since only the derivatives f' and f'' appear in the velocity. The constants a and b are determined from the condition that $\mathbf{v} = \mathbf{u}$ for $r = R$, and are found to be

$$a = -\frac{3R}{2ik}e^{-ikR}, \qquad b = -\tfrac{1}{2}R^3\left(1-\frac{3}{ikR}-\frac{3}{k^2R^2}\right). \tag{2}$$

It may be pointed out that, at large distances ($R \gg \delta$), $a \to 0$ and $b \to -\frac{1}{2}R^3$, the values for potential flow obtained in §10, Problem 2; this is in accordance with what was said in §24.

The drag is calculated from formula (20.13), in which the integration is over the surface of the sphere. The result is

$$F = 6\pi\eta R\left(1+\frac{R}{\delta}\right)u + 3\pi R^2\sqrt{(2\eta\rho/\omega)}\left(1+\frac{2R}{9\delta}\right)\frac{\mathrm{d}u}{\mathrm{d}t}. \qquad (3)$$

For $\omega = 0$ this becomes Stokes' formula, while for large frequencies we have

$$F = \tfrac{2}{3}\pi\rho R^3\frac{\mathrm{d}u}{\mathrm{d}t} + 3\pi R^2\sqrt{(2\eta\rho\omega)}u.$$

The first term in this expression corresponds to the inertial force in potential flow past a sphere (see §11, Problem 1), while the second gives the limit of the dissipative force.

PROBLEM 6. Determine the drag on a sphere moving in an arbitrary manner, the velocity being given by a function $u(t)$.

SOLUTION. We represent $u(t)$ as a Fourier integral:

$$u(t) = \int_{-\infty}^{\infty} u_\omega e^{-i\omega t}\,\mathrm{d}\omega, \qquad u_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau)e^{i\omega\tau}\,\mathrm{d}\tau.$$

Since the equations are linear, the total drag may be written as the integral of the drag forces for velocities which are the separate Fourier components $u_\omega e^{-i\omega t}$; these forces are given by (3) of Problem 5, and are

$$\pi\rho R^3 u_\omega e^{-i\omega t}\left\{\frac{6\nu}{R^2} - \frac{2i\omega}{3} + \frac{3\sqrt{(2\nu)}}{R}(1-i)\sqrt{\omega}\right\}.$$

Noticing that $(\mathrm{d}u/\mathrm{d}t)_\omega = -i\omega u_\omega$, we can rewrite this as

$$\pi\rho R^3 e^{-i\omega t}\left\{\frac{6\nu}{R^2}u_\omega + \tfrac{2}{3}(\dot{u})_\omega + \frac{3\sqrt{(2\nu)}}{R}(\dot{u})_\omega\frac{1+i}{\sqrt{\omega}}\right\}.$$

On integration over ω, the first and second terms give respectively $u(t)$ and $\dot{u}(t)$. To integrate the third term, we notice first of all that for negative ω this term must be written in the complex conjugate form, $(1+i)/\sqrt{\omega}$ being replaced by $(1-i)/\sqrt{|\omega|}$; this is because formula (3) of Problem 5 was derived for a velocity $u = u_0 e^{-i\omega t}$ with $\omega > 0$, and for a velocity $u_0 e^{i\omega t}$ we should obtain the complex conjugate. Instead of an integral over ω from $-\infty$ to $+\infty$, we can therefore take twice the real part of the integral from 0 to ∞. We write

$$2\,\mathrm{re}\left\{(1+i)\int_0^{\infty}\frac{(\dot{u})_\omega e^{-i\omega t}}{\sqrt{\omega}}\mathrm{d}\omega\right\} = \frac{1}{\pi}\mathrm{re}\left\{(1+i)\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau\right\}$$

$$= \frac{1}{\pi}\mathrm{re}\left\{(1+i)\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\dot{u}(\tau)e^{-i\omega(t-\tau)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau + (1+i)\int_t^{\infty}\int_0^{\infty}\frac{\dot{u}(\tau)e^{i\omega(\tau-t)}}{\sqrt{\omega}}\mathrm{d}\omega\,\mathrm{d}\tau\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,\mathrm{re}\left\{\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\mathrm{d}\tau + i\int_t^{\infty}\frac{\dot{u}(\tau)}{\sqrt{(\tau-t)}}\mathrm{d}\tau\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{t}\frac{\dot{u}(\tau)}{\sqrt{(t-\tau)}}\mathrm{d}\tau.$$

Thus we have finally for the drag

$$F = 2\pi\rho R^3 \left\{ \frac{1}{3}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{3\nu u}{R^2} + \frac{3}{R}\sqrt{\frac{\nu}{\pi}} \int_{-\infty}^{t} \frac{\mathrm{d}u}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\sqrt{(t-\tau)}} \right\}. \tag{1}$$

PROBLEM 7. Determine the drag on a sphere which at time $t = 0$ begins to move with a uniform acceleration, $u = \alpha t$.

SOLUTION. Putting, in formula (1) of Problem 6, $u = 0$ for $t < 0$ and $u = \alpha t$ for $t > 0$ we have for $t > 0$

$$F = 2\pi\rho R^3 \alpha \left[\frac{1}{3} + \frac{3\nu t}{R^2} + \frac{6}{R}\sqrt{\frac{t\nu}{\pi}} \right].$$

PROBLEM 8. The same as Problem 7, but for a sphere brought instantaneously into uniform motion.

SOLUTION. We have $u = 0$ for $t < 0$ and $u = u_0$ for $t > 0$. The derivative $\mathrm{d}u/\mathrm{d}t$ is zero except at the instant $t = 0$, when it is infinite, but the time integral of $\mathrm{d}u/\mathrm{d}t$ is finite, and equals u_0. As a result, we have for all $t > 0$

$$F = 6\pi\rho\nu R u_0 \left[1 + \frac{R}{\sqrt{(\pi\nu t)}} \right] + \tfrac{2}{3}\pi\rho R^3 u_0 \delta(t),$$

where $\delta(t)$ is the delta function. For $t \to \infty$ this expression tends asymptotically to the value given by Stokes' formula. The impulsive drag on the sphere at $t = 0$ is obtained by integrating the last term and is $\frac{2}{3}\pi\rho R^3 u_0$.

PROBLEM 9. Determine the moment of the forces on a sphere executing rotary oscillations about a diameter in a viscous fluid.

SOLUTION. For the same reasons as in §20, Problem 1, the pressure-gradient term can be omitted from the equation of motion, so that we have $\partial\mathbf{v}/\partial t = \nu \triangle \mathbf{v}$. We seek a solution in the form $\mathbf{v} = \mathbf{curl}\, f \boldsymbol{\Omega}_0 e^{-i\omega t}$, where $\boldsymbol{\Omega} = \boldsymbol{\Omega}_0 e^{-i\omega t}$ is the angular velocity of rotation of the sphere. We then obtain for f, instead of the equation $\triangle f =$ constant,

$$\triangle f + k^2 f = \text{constant}.$$

Omitting an unimportant constant term in the solution of this equation, we find $f = ae^{ikr}/r$ taking the solution which vanishes at infinity. The constant a is determined from the boundary condition that $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$ at the surface of the sphere. The result is

$$f = \frac{R^3}{1-ikR} e^{ik(r-R)}, \qquad \mathbf{v} = (\boldsymbol{\Omega} \times \mathbf{r}) \left(\frac{R}{r} \right)^3 \frac{1-ikr}{1-ikR} e^{ik(r-R)},$$

where R is the radius of the sphere. A calculation like that in §20, Problem 1, gives the following expression for the moment of the forces exerted on the sphere by the fluid:

$$M = -\frac{8\pi}{3}\eta R^3 \Omega \frac{3 + 6R/\delta + 6(R/\delta)^2 + 2(R/\delta)^3 - 2i(R/\delta)^2(1 + R/\delta)}{1 + 2R/\delta + 2(R/\delta)^2}.$$

For $\omega \to 0$ (i.e. $\delta \to \infty$), we obtain $M = -8\pi\eta R^3 \Omega$, corresponding to uniform rotation of the sphere (see §20, Problem 1). In the opposite limiting case $R/\delta \gg 1$, we find

$$M = \frac{4\sqrt{2}}{3}\pi R^4 \sqrt{(\eta\rho\omega)}(i-1)\Omega.$$

This expression can also be obtained directly: for $\delta \ll R$ each element of the surface of the sphere may be regarded as plane, and the frictional force acting on it is found by substituting $u = \Omega R \sin\theta$ in formula (24.6).

PROBLEM 10. Determine the moment of the forces on a hollow sphere filled with viscous fluid and executing rotary oscillations about a diameter.

SOLUTION. We seek the velocity in the same form as in Problem 9. For f we take the solution $(a/r)\sin kr$, which is finite everywhere within the sphere, including the centre. Determining a from the boundary condition, we have

$$\mathbf{v} = (\mathbf{\Omega}\times\mathbf{r})\left(\frac{R}{r}\right)^3 \frac{kr\cos kr - \sin kr}{kR\cos kR - \sin kR}.$$

A calculation of the moment of the frictional forces gives the expression

$$M = \tfrac{8}{3}\pi\eta R^3\Omega\frac{k^2R^2\sin kR + 3kR\cos kR - 3\sin kR}{kR\cos kR - \sin kR}.$$

The limiting value for $\delta \gg 1$ is of course the same as in the preceding problem. If $R/\delta \ll 1$ we have

$$M = \tfrac{8}{15}\pi\rho\omega R^5\Omega\left(i - \frac{R^2\omega}{35\nu}\right).$$

The first term corresponds to the inertial forces occurring in the rigid rotation of the whole fluid.

§25. Damping of gravity waves

Arguments similar to those given above can be advanced concerning the velocity distribution near the free surface of a fluid. Let us consider oscillatory motion occurring near the surface (for example, gravity waves). We suppose that the conditions (24.11) hold, the dimension l being now replaced by the wavelength λ:

$$\lambda^2\omega \gg \nu, \qquad a \ll \lambda; \tag{25.1}$$

a is the amplitude of the wave, and ω its frequency. Then we can say that the flow is rotational only in a thin surface layer, while throughout the rest of the fluid we have potential flow, just as we should for an ideal fluid.

The motion of a viscous fluid must satisfy the boundary conditions (15.14) at the free surface; these require that certain combinations of the space derivatives of the velocity should vanish. The flow obtained by solving the equations of ideal-fluid dynamics does not satisfy these conditions, however. As in the discussion of v_y in the previous section, we may conclude that the corresponding velocity derivatives decrease rapidly in a thin surface layer. It is important to notice that this does not imply a large velocity gradient as it does near a solid surface.

Let us calculate the energy dissipation in a gravity wave. Here we must consider the dissipation, not of the kinetic energy alone, but of the mechanical energy E_{mech}, which includes both the kinetic energy and the potential energy in the gravitational field. It is clear, however, that the presence or absence of a gravitational field cannot affect the energy dissipation due to processes of internal friction in the fluid. Hence $\dot{E}_{\text{mech}}$ is given by the same formula (16.3):

$$\dot{E}_{\text{mech}} = -\tfrac{1}{2}\eta \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)^2 \mathrm{d}V.$$

In calculating this integral for a gravity wave, it is to be noticed that, since the volume of the surface region of rotational flow is small, while the velocity gradient there is not large, the existence of this region may be ignored, unlike what was possible for oscillations of a solid surface. In other words, the integration is to be taken over the whole volume of fluid, which, as we have seen, moves as if it were an ideal fluid.

The flow in a gravity wave for an ideal fluid, however, has already been determined in §12. Since we have potential flow,

$$\partial v_i/\partial x_k = \partial^2\phi/\partial x_k \partial x_i = \partial v_k/\partial x_i,$$

so that

$$\dot{E}_{\text{mech}} = -2\eta \int \left(\frac{\partial^2\phi}{\partial x_i \partial x_k}\right)^2 \mathrm{d}V.$$

The potential ϕ is of the form

$$\phi = \phi_0 \cos(kx - \omega t + \alpha)e^{-kz}.$$

We are interested, of course, not in the instantaneous value of the energy dissipation, but in its mean value $\bar{\dot{E}}_{\text{mech}}$ with respect to time. Noticing that the mean values of the squared sine and cosine are the same, we find

$$\bar{\dot{E}}_{\text{mech}} = -8\eta k^4 \int \overline{\phi^2}\, \mathrm{d}V. \tag{25.2}$$

The energy E_{mech} itself may be calculated for a gravity wave by using a theorem of mechanics that, in any system executing small oscillations (of small amplitude, that is), the mean kinetic and potential energies are equal. We can therefore write $\bar{E}_{\text{mech}}$ simply as twice the kinetic energy:

$$\bar{E}_{\text{mech}} = \rho \int \overline{v^2}\, \mathrm{d}V = \rho \int \overline{(\partial\phi/\partial x_i)^2}\, \mathrm{d}V,$$

whence

$$\bar{E}_{\text{mech}} = 2\rho k^2 \int \overline{\phi^2}\, \mathrm{d}V. \tag{25.3}$$

The damping of the waves is conveniently characterised by the ***damping coefficient*** γ, defined as

$$\gamma = |\bar{\dot{E}}_{\text{mech}}|/2\bar{E}_{\text{mech}}. \tag{25.4}$$

In the course of time, the energy of the wave decreases according to the law E_{mech} = constant $\times\, e^{-2\gamma t}$; since the energy is proportional to the square of the amplitude, the latter decreases with time as $e^{-\gamma t}$.

Using (25.2), (25.3), we find

$$\gamma = 2\nu k^2. \tag{25.5}$$

Substituting here (12.7), we obtain the damping coefficient for gravity waves in the form

$$\gamma = 2\nu\omega^4/g^2. \tag{25.6}$$

PROBLEMS

PROBLEM 1. Determine the damping coefficient for long gravity waves propagated in a channel of constant cross-section; the frequency is supposed so large that $\sqrt{(\nu/\omega)}$ is small compared with the depth of the fluid in the channel.

SOLUTION. The principal dissipation of energy occurs in the surface layer of fluid, where the velocity changes from zero at the boundary to the value $v = v_0 e^{-i\omega t}$ which it has in the wave. The mean energy dissipation per unit length of the channel is by (24.14) $l|v_0|^2\sqrt{(\eta\rho\omega/8)}$, where l is the perimeter of the part of the channel cross-section occupied by the fluid. The mean energy of the fluid (again per unit length) is $S\rho\overline{v^2} = \frac{1}{2}S\rho|v_0|^2$, where S is the cross-sectional area of the fluid in the channel. The damping coefficient is $\gamma = l\sqrt{(\nu\omega/8S^2)}$. For a channel of rectangular section, therefore,

$$\gamma = \frac{2h+a}{2\sqrt{2ah}}\sqrt{(\nu\omega)},$$

where a is the width and h the depth of the fluid.

PROBLEM 2. Determine the flow in a gravity wave on a very viscous fluid.

SOLUTION. The calculation of the damping coefficient as shown above is valid only when this coefficient is small, so that the motion may be regarded as that of an ideal fluid to a first approximation. For arbitrary viscosity we seek a solution of the equations of motion

$$\left.\begin{aligned}
\frac{\partial v_x}{\partial t} &= \nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x},\\
\frac{\partial v_z}{\partial t} &= \nu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial z^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z}-g,\\
&\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z} = 0
\end{aligned}\right\} \tag{1}$$

which depends on t and x as $e^{-i\omega t+ikx}$, and diminishes in the interior of the fluid ($z < 0$). We find

$$v_x = e^{-i\omega t+ikx}(Ae^{kz}+Be^{mz}), \qquad v_z = e^{-i\omega t+ikx}\left(-iAe^{kz}-\frac{ik}{m}Be^{mz}\right),$$

$$p/\rho = e^{-i\omega t+ikx}\,\omega Ae^{kz}/k-gz, \qquad \text{where } m = \sqrt{(k^2-i\omega/\nu)}.$$

The boundary conditions at the fluid surface are

$$\sigma_{zz} = -p+2\eta\,\partial v_z/\partial z = 0, \qquad \sigma_{xz} = \eta\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right) = 0 \text{ for } z = \zeta.$$

In the second condition we can immediately put $z = 0$ instead of $z = \zeta$. The first condition, however, should be differentiated with respect to t, after which we replace $g\partial\zeta/\partial t$ by gv_z and then put $z = 0$. The condition that the resulting two homogeneous equations for A and B are compatible gives

$$\left(2 - \frac{i\omega}{\nu k^2}\right)^2 + \frac{g}{\nu^2 k^3} = 4\sqrt{\left(1 - \frac{i\omega}{\nu k^2}\right)}. \tag{2}$$

This equation gives ω as a function of the wave number k; ω is complex, its real part giving the frequency of the oscillations and its imaginary part the damping coefficient. The solutions of equation (2) that have a physical meaning are those whose imaginary parts are negative (corresponding to damping of the wave); only two roots of (2) meet this requirement. If $\nu k^2 \ll \sqrt{(gk)}$ (the condition (25.1)), then the damping coefficient is small, and (2) gives approximately $\omega = \pm\sqrt{(gk)} - i.2\nu k^2$, a result which we already know. In the opposite limiting case $\nu k^2 \gg \sqrt{(gk)}$, equation (2) has two purely imaginary roots, corresponding to damped aperiodic flow. One root is $\omega = -ig/2\nu k$, while the other is much larger (of order νk^2), and therefore of no interest, since the corresponding motion is strongly damped.