Part 1 : Soliton equations

The central focus of the first section of this course will be on solitons arising as solutions of intergrable partial differential equations. Solitons are single solitary waves, which appear in physical contexts abundantly, e.g., shallow water waves, fibre optics and even protiens. In partular, our prototypical example is the KdV equation, however, there are many other equations we will introduce as examples.

We start with a historical context for these results, in particular, how this area of study arose from a physical phenomenon which was not accounted by the well established mathematical formalism of the time. We then progress to deal particular solutions given by travelling wave reductions and symmetry reductions. However, perhaps the most important part of the course concerns Lax pairs. This idea that a nonlinear system arises as the compatibility of two linear systems is a very powerful concept.

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1. Introduction : A historical context

Mathematics and Physics have had a long and fruitful relationship. Mathematics can be a language that physicists to describe physical phenomenon. Conversely, physics has a way of bringing relevance and life to mathematics. It is in this context that the area of integrable systems arose. We start this course with a reference to an observation of John Scott Russell, who noted the existence of a solitary wave, or "Great Wave of Translation" in the Union Canal in Scotland:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stoppednot so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."
> - John Scott Russell 1934

This phenomena recieved considerable attention because it did not quite adhere to the established theories for water waves, notably by Newton and Bernoulli. He replicated these "Great Waves of Translation" in experiments using wave tanks, from which, John Scott Russel was able to establish the following key properties of solitons:

(1) The waves are stable, and can travel over very large distances (normal waves tend to either flatten or steepen and topple.
(2) The speed depends on the size of the wave, and its width on the depth of water.
(3) The waves do not merge, larger waves overtake smaller ones rather than combining.
(4) If a wave is too big for the depth of water, it splits into two waves, one big and one small.

His original drawings, based on the experiments using wave tanks have been included in Figure 1.

This was further investigated by the likes of Airy, Stokes, Boussinesq and Rayleigh. There are many who doubted the observations, a major point of contraversy being the stability of these great waves. There documented resistance from Stokes and Airy. Airy, who was seen as an authority on the matter stated:

> We are not disposed to recognize this wave as deserving the epithets "great" or "primary", and we conceive that ever since it was known that the theory of shallow waves of great length

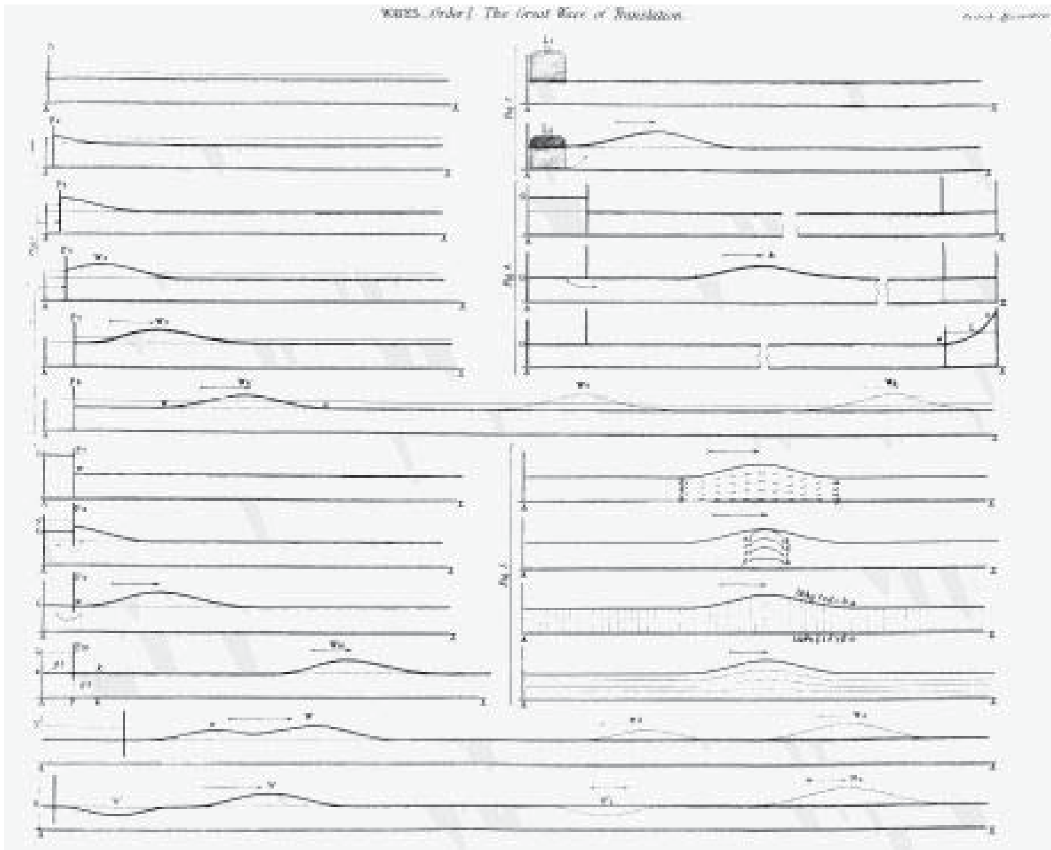

Figure 1. Some of John Scott Russel's original drawings outline-ing the basic properties of solitons.

was contained in the equation

$$\text{(WEQ)} \qquad \frac{\partial^2 u(x,t)}{\partial t^2} = g\kappa \frac{\partial^2 u(x,t)}{\partial t^2}.$$

The theory of the solitary wave has been perfectly well known.

The equation, (WEQ), he refers to is known as the *wave equation*, which was discovered in 1752 in the context of vibrating strings, and is perhaps the first partial differential equation to be written down. In this context, κ is the relative depth of water and g refers to gravity. In fact, it admits solutions that one might naively describe a "Great Wave of Translation", namely

$$u(x,t) = A\mathrm{sech}^2(ax + bt)$$

where

$$a^2 g\kappa = b^2.$$

From what we know of shallow water waves, there are two problems with modelling shallow water waves using (WEQ). The first is that one of the observed speed of the wave does not depend on the amplitude, only the relative water depth. The second problem with this model concerns the observed interaction between two waves.

As a second order *linear* partial differential, two solutions additively superimpose, whereas, the physical observations account for one extra property not listed above is the nonlinear superposition that we will investigate later in our section on inverse scattering. Basically, we will see that when a larger wave overtakes a smaller wave, the interaction "pushes" the smaller wave back.

2. The Korteweg-de Vries equation

We will concerntrate on the most canonical partial differential equation that describes this pheonomenon; the Korteweg-de Vries equation. However, we will introduce other examples to demonstrate more general principles. There was no mathematical model that accounted for this physical phenomenon at the time, hence, this observation attracted considerable attention. The equation that describes shallow water waves in one dimension under gravity may be written as

$$v_t + \sqrt{gh}v_x + \frac{3}{2}\sqrt{\frac{g}{h}}vv_x + \frac{h^2}{6}\sqrt{gh}v_{xxx} = 0, \tag{2.1}$$

where $v = v(x,t)$ is a function defining the elevation in a spatial variable x in the time variable t, $g = 9.8m/s$ is earths gravity, h is the mean water depth in meters and the subscripts denote differentiation. These variables can be made dimensionless and made

$$u = \frac{v}{h}, \qquad t \to \frac{1}{6}\sqrt{\frac{g}{h}}t, \qquad x \to \frac{x}{h} - t,$$

then we obtain what is considered the standard form of KdV

$$u_t + 6uu_x + u_{xxx} = 0. \tag{KdV}$$

We note that through scaling, we associate any equation of the form

$$\alpha u_t + \beta uu_x + \gamma u_{xxx} = 0$$

to be of "KdV type". While we know this as the KdV equation after Korteweg and de Vries, the first time an equation of KdV type was writtten down was actually in the work Bousinesq in 1872, which was over 20 years before Korteweg and de Vries, who discovered (KdV) in 1894. Bousinesq also wrote down a solution that gave the quintessential solitary wave form. These solutions also well with experimental data.

To actually solve (KdV), the intial data is considered to be some function $u(x,0) = u_0(x)$. We say function, $u(x,t)$ decays sufficiently rapidly if all its x-derivatives approach 0 as $|x| \to \infty$. We start our study using a simple lemma.

LEMMA 2.1. *Solutions of the KdV equation that decay sufficiently rapidly are uniquely determined by initial data.*

PROOF. Let u and v be two solutions to (KdV), then if we let $w = u - v$ then w satisfies the equation

$$w_t + w_{xxx} + 6uw_x + 6wv_x = 0.$$

If we multiply by w and integrate once, using integration by parts, and the fact the x-derivatives go to 0 then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} w^2 \mathrm{d}x + 6\int_{-\infty}^{\infty}\left(v_x - \frac{u_x}{2}\right)w^2 \mathrm{d}x = 0.$$

If the derivatives of $u - v$ are bounded, we have

$$M = \sup \left| v_x - \frac{u_x}{2} \right|$$
$$E(t) = \int_{-\infty}^{\infty} w^2 \mathrm{d}x$$

then if M is bounded, we have

$$|E(t)| \leq |E(0)| e^{12Mt}.$$

In particular, if $E(0) = 0$, $w = 0$ and $u = v$ for all times t. □

3. Travelling Wave Solutions

Now that we have that solutions are uniquely determined by the initial data, let us describe a simple solution. This will illustrate what we mean by a travelling wave. Let us consider a solution that encapsulates what we want in a travelling wave.

Definition 3.1. A solitary wave solution of a PDE (in u) is a solution of the form

$$u(x,t) = f(x - ct) = f(z), \tag{3.1}$$

whose transition is from one constant asymptotic state as $z \to \infty$ and another constant asymptotic state as $z \to -\infty$.

We will seek such travelling wave solutions for a number of PDEs.

3.1. The wave equation. The constraint that the function is of one variable, namely $z = x - ct$, means that partial derivatives in t and x become derivatives in the one variable, effectively forming a solution of a PDE from the solution of an ODE. Naturally this means that the speed is given by c, which will depend on the PDE itself. For example, we may take the wave equation.

Proposition 3.2. *The wave equation,*

$$u_{tt} - g\kappa u_{xx} = 0,$$

admits a solitary wave solution of the form

$$u(x) = f(x + \sqrt{g\kappa}t) + g(x - \sqrt{g\kappa}t).$$

for any *functions f and g such that f and g approach one constant asymptotic state as $z \to \infty$ and another constant asymptotic state as $z \to -\infty$.*

Proof. Using the ansatz

$$u(x,t) = f(x - ct),$$

reduces (WEQ) to

$$c^2 f''(z) - g\kappa f''(z) = 0.$$

A solution arises when

$$c^2 + g\kappa = 0.$$

hence, we find

$$u(x) = f(x + \sqrt{g\kappa}t) + g(x - \sqrt{g\kappa}t).$$

solves (WEQ) for *any* functions f and g. Naturally, as solutions are required to admit certain asymptotics to be considered solitary wave solutions. □

While Airy was correct in stating that the sech^2 appears as a form of the solution, it is because the general form of a solution to the wave equation is generic enough to encompass any such solution with a given number of properties.

3.2. The wave equation. We now show that (KdV) does indeed admit the form predicted by John Scott Russel.

PROPOSITION 3.3. *The KdV equation admits the solitary wave solution*

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right).$$

PROOF. If the function $u(x,t)$ satisfies the partial differential equation (KdV), then $f(z)$ satisfies an ordinary differential equation, namely

$$-c\frac{\mathrm{d}f(z)}{\mathrm{d}z} + \frac{\mathrm{d}^3 f(z)}{\mathrm{d}^3 z} + 6f(z)\frac{\mathrm{d}f}{\mathrm{d}z} = 0.$$

This is said to be a way of reducing the partial differential equation to an ordinary differential equation in z. Integrating once, we find

$$-cf(z) + \frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + 3f(z)^2 = A, \tag{3.2}$$

where A is some constant. For applications, we seek solutions that satisfy

$$\lim_{x\to\pm\infty} f(x) = 0, \quad \lim_{x\to\pm\infty} \frac{\mathrm{d}f(x)}{\mathrm{d}x} = 0$$

This points to requiring that $A = 0$. This is a second order nonlinear ordinary differential equation. We test a trial solution,

$$\begin{aligned} f(x) &= \lambda\operatorname{sech}^2(ax). \\ f'(x) &= -\lambda\frac{2a\sinh(ax)}{\cosh^3(ax)}, \\ f''(x) &= -2a\lambda\left(\frac{\cosh(ax)^4}{\cosh(ax)^6} - \frac{3\sinh(ax)^2\cosh^2(ax)}{\cosh^6(ax)}\right) \end{aligned}$$

which we use the identity

$$\cosh^2(ax) + \sinh^2(ax) = 1$$

to see that

$$f''(x) = -2a^2\left(-2\operatorname{sech}^2(ax) + 3\operatorname{sech}^4(ax)\right).$$

which allows us to express the above purely in terms of $f(x)$ as

$$f''(z) + \frac{6a^2}{\lambda}f(z)^2 - 4a^2 f(z) = 0.$$

This is a solution of (KdV) under the conditions

$$\begin{aligned} 4a^2 &= c \\ 6a^2 &= 3\lambda. \end{aligned}$$

Since we may add some displacement arbitrarily, our travelling wave solution in the original variables, x and t, is given by

$$f(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right). \tag{3.3}$$

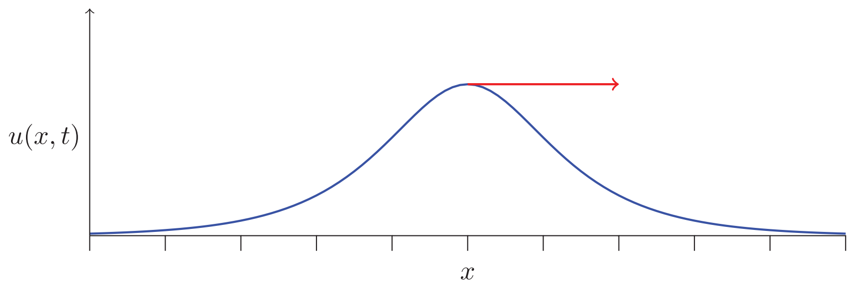

FIGURE 2. The distinctive soliton solution.

We also note

$$\lim_{z\to\infty} \frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(z-x_0)\right) = 0,$$

hence, this is a solitary wave solution. $\square$

This was the original solution of Bousinesq that he wrote down in 1775 (again, note 20 years prior to Korteweg and de Vries). This is shown in Figure 2. Based on this solution, we observe that the speed of a soliton grows asymptotically proportional to the square root of the amplitude. This predicts that a fifth property, that larger waves move faster with the right (observed) speed with respect to the amplitude.

3.3. More general periodic solutions. We can of course take (3.1) to be a more general condition. Suppose we wish to perhaps consider periodic asymptotic behaviour, in which case Lemma 2.1 need not apply. The more general solution may be obtained in terms of elliptic functions in the following way:

If $A \neq 0$, multiply

$$-c\frac{\mathrm{d}f(z)}{\mathrm{d}z} + \frac{\mathrm{d}^3 f(z)}{\mathrm{d}^3 z} + 6f(z)\frac{\mathrm{d}f}{\mathrm{d}z} = A,$$

by $f'(z)$, we see

$$-cf(z)\frac{\mathrm{d}f(z)}{\mathrm{d}z} + \frac{\mathrm{d}f(z)}{\mathrm{d}z}\frac{\mathrm{d}^2 f(z)}{\mathrm{d}^2 z} + 3f(z)^2\frac{\mathrm{d}f(z)}{\mathrm{d}z} = A\frac{\mathrm{d}f(z)}{\mathrm{d}z},$$

$$-\frac{c}{2}f(z)^2 + \frac{1}{2}\left(\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right)^2 + f(z)^3 = Af(z) + B.$$

If we let $f(z) = -2y(z) + \dfrac{c}{6}$ we find

$$(y'(z))^2 = 4y(z)^3 + g_2 y(z) + g_3$$

where the constants g_2 and g_3 are specified by

$$g_2 = -A - \frac{c^2}{12},$$

$$g_3 = \frac{B}{2} + \frac{Ac}{12} + \frac{c^3}{216}.$$

This is the differential equation satisfied by Weierstrass's $\wp$-function.

This is actually more closely related to the work of Korteweg and de Vries who were able to write down the so-called cnoidal solutions, which are expressible in terms of Jacobi's elliptic cn functions. For a more detailed look at the solutions, we point the reader towards Drazin and Johnson (1989).

3.4. Potential and Modified KdV equation. There are serval variants of the KdV equation, which are related via some nice transformations. The simplest of these is called the potential KdV equation. This equation arises by noticing that if we let $u = w_x$, for some function w, then we have

$$w_{x,t} + w_{xxxx} + 6w_x w_{xx} = 0$$

which may be integrated once to obtain the system

$$w_t + w_{xxx} + 3w_x^2 = 0, \tag{3.4}$$

which is called the potential KdV equation, whose solutions are related to those of the KdV by quadratures/differentiation.

THEOREM 3.4. *The travelling wave solutions of* (3.4) *are given by*

$$w(x,t) = \sqrt{c}\tanh\left(\frac{\sqrt{c}}{2}(x-ct)+z_0\right). \tag{3.5}$$

PROOF. The fact this arises as a solution comes from the fact it is the integral of $\sec(z)^2$. That is that we have the solution of the KdV equation,

$$u(x,t) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}\left(x-ct-x_0\right)\right).$$

which we integrate once. Secondly, when we write

$$w(x,t) = \sqrt{c}\tanh(x-ct) = \sqrt{c}\frac{\exp(x-ct)-\exp(-x+ct)}{\exp(x-ct)+\exp(-x+ct)},$$

we see that $w(x,t)$ is asymptotically constant with asymptotic values $\pm\sqrt{c}$. □

Another interesting equation is the modified KdV equation; this is the system

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{mKdV}$$

and it is related to the KdV equation through a remarkable transformation known as the Miura transformation. In this transformation, we let u be a solution of (KdV) then let v be defined by

$$u = v^2 - iv_x,$$

then we see that

$$\begin{aligned}
u_t + u_{xxx} + 6uu_x =& (2vv_t - iv_{t,x}) + 6v_x v_{xx} + 2vv_{xxx} - iv_{xxxx} \\
&+ 6(v^2 - v_x)(2vv_x - iv_{xx}), \\
=& (2v - i\partial_x)v_t + (2v - i\partial_x)v_{xxx} \\
&+ 6(v_x v_{xx} + 2v^3 v_x - iv^2 v_{xx} - 2vv_x^2 + iv_x v_{xx}) \\
=& (2v - \partial_x)u_t + (2v - i\partial_x)v_{xxx} \\
&+ 6(2v - i\partial_x)v^2 v_x, \\
=& (2v - i\partial_x)(v_t + 6v^2 v_x + v_{xxx}).
\end{aligned}$$

This tells us that if (KdV) holds, then so does (mKdV). This nonlinear transformation is called a Miura transformation, which we will see again.

What we saw for the KdV equation was solutions generally expressible in terms of Weirstrass $\wp$-functions. When we do the analogous steps, we find that a travelling wave solution for (mKdV) satisfies

$$-cf'(z) + 6f(z)^2 f'(z) + f'''(z) = 0,$$

which possesses the first integral

$$-cf(z) + 2f(z)^3 + f''(z) = A. \tag{3.6}$$

In terms of applications, let $A = 0$, in which case, if we let $f(z) = \alpha y(z)$ then

$$y''(z) = cy(z) + 2\alpha^2 y(z)^3$$

then we make contact with the differential equation for $y(z) = \mathrm{sn}(z, \kappa)$,

$$y''(z) = (1 + \kappa^2)y(z) - 2\kappa^2 y(z)^3$$

when $\kappa = \sqrt{c-1}$ and hence $\alpha = \kappa/\sqrt{2}$. This means we have a solution in terms of $\mathrm{sn}(x - ct, \sqrt{1-c})$. More generally, multiplying (3.6) by $f'(z)$ gives another integral giving

$$-\frac{c}{2} f(z)^2 + \frac{f(z)^4}{2} + \frac{f'(z)^2}{2} = Af(z) + b,$$

which is also expressible in terms of elliptic functions, however, we do not give a precise form of the solution here.