For a viscous incompressible flow, given Navier-Stokes equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + Re^{-1} \nabla^2 \vec{u} \longrightarrow (1)$$

Now, taking curl of both sides of (1), we get.

$$\nabla \times \frac{\partial \vec{u}}{\partial t} + \nabla \times (\vec{u} \cdot \nabla)\vec{u} = \nabla \times (-\nabla p) + \nabla \times (Re^{-1}\nabla^2 \vec{u})$$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \nabla \times \left[\frac{1}{2}\nabla(\vec{u}\cdot\vec{u}) - \vec{u} \times (\nabla \times \vec{u})\right] = 0 + Re^{-1} \nabla^2 \vec{\omega}$$

[$\because$ $\nabla \times \nabla \phi = 0$ for any ϕ conservative forces]

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \nabla \times \nabla \left(\frac{u^2}{2}\right) - \nabla \times (\vec{u} \times \vec{\omega}) = Re^{-1} \nabla^2 \vec{\omega}$$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) = Re^{-1} \nabla^2 \vec{\omega}$$

$$\Rightarrow \frac{\partial \vec{\omega}}{\partial t} + \left[(\vec{u}\cdot\nabla)\vec{\omega} - \underbrace{\boxed{(\vec{\omega}\cdot\nabla)\vec{u}}}_{=0} + \underbrace{\boxed{\vec{\omega}(\nabla\cdot\vec{u})}}_{=0} + \underbrace{\boxed{\vec{u}(\nabla\cdot\vec{\omega})}}_{=0}\right] = Re^{-1}\nabla^2\vec{\omega}$$

for 2D (incompressible fluid) $[\nabla \cdot (\nabla \times \vec{u}) = 0]$

as $\vec{\omega} = \nabla \times \vec{u}$ is $\perp^r$ to $\vec{u}$

and $$(\vec{\omega}\cdot\nabla)\vec{u} = (\underbrace{\omega_x \frac{\partial}{\partial x}}_{0} + \underbrace{\omega_y \frac{\partial}{\partial y}}_{0} + \underbrace{\omega_z \frac{\partial}{\partial z}}_{0})\vec{u}$$

$$= 0$$

And we finally get.

$$\dot{\omega} + (u \cdot \nabla)\omega = Re^{-1} \nabla^2 \omega \qquad \text{(shown)}$$

Boundary conditions:

If the velocity field $\vec{u}$ is taken to be sum of a velocity field due to the irrotational flow and a velocity field due to a rotational flow, let ϕ be the potential for the irrotational flow and ψ be the stream function for the rotational flow, so the boundary conditions are-

$$\Delta \phi = 0$$

$$\frac{\partial \phi}{\partial \hat{n}} = B\hat{n} \quad \text{on } \partial\Omega$$

where, $\hat{n}$ is the normal to the boundary